Action and Quantum Mechanics

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Rotational motion is inherent to matter at every level of organization, from electrons up to clusters of galaxies. The prominent feature of bound, periodic systems is the quantization of action integrals in general, and of angular momentum in particular. Every bound system is governed by a permanent interplay between kinetic and potential energy, so that the product (energy converted) × (characteristic time) = action is minimized. The wave (Schrödinger or Dirac) equation determines the discrete frequency spectrum of the system, as in classical wave theory. The genuine, irreducible quantizations are of charge, angular momentum and magnetic flux. The stability of the hydrogen atom against radiative decay is explained by Weber forces which obey Newton's action and reaction principle. De Broglie's monochromatic plane waves $\nabla^2\psi - v_p^{-2}\partial^2\psi/\partial t^2 = 0$ are a consequence of "Lorentz transforming" a simple harmonic oscillation, but are not Lorentz-invariant because their phase velocity, $v_p \equiv c^2/v$ (with v the particle velocity) always exceeds the velocity of light, c. Schrödinger's equation—which is no wave equation at all—cannot be derived in a consistent way from de Broglie's wave mechanics, where $E^2 - c^2\mathbf{p}^2 = (m_o c^2)^2$ has to hold for the momentum $(\mathbf{p})$ and energy (E) of a particle with rest mass m_o. Schrödinger's equation and de Broglie's waves are incompatible in the time domain. As a kinematic concept and interpreted as abstract "probability patterns" in space associated with the motion of hypothetical point masses, de Broglie waves are devoid of physical meaning. The dispersion relation $\omega^2 = (m_o c^2/\hbar)^2 + c^2k^2$ suggests taking $\omega^2 - (ck)^2$ as the proper invariant of a dynamical wave theory instead of the space-time invariant $r^2 - (ct)^2$ from special relativity. Now the waves can be understood in terms of action, which traces the origin of particle motion to the potential, in line with physical causality.

Key Words: Action, Wave Mechanics, Special Relativity, Dispersion Relations, Wave Propagation, Bound Systems, Quantization

1. Minimal action in bound systems

1.1. Action variables

According to the opinion now prevailing, quantum mechanics originated with the hypothesis, made by Planck in 1900:

$$E = h\nu \tag{1}$$

Planck stipulated that emission and absorption of electromagnetic radiation occurs in discrete quanta of energy E given by the frequency of the radiation, ν, multiplied by a universal constant h with the dimensions of action. This was in line with Poincaré's picture of the atom as providing the "vestibule" for the quantization of (free) radiation (Poincaré 1924). It was, however, a less radical assumption than Einstein's dictum (in colorful south German slang): "Beer is not merely sold in pints, it consists of pints".

The Bohr-van Leeuwen theorem (i.e. the absence of classical diamagnetism) and the failure of electrodynamics and of statistical mechanics to explain both spontaneous magnetization (i.e. ferromagnetism) and paramagnetism—in short the failure to account for all three types of magnetism described by Faraday—provided the strongest motivation for Bohr's "old" quantum mechanics. Since the magnetic moments induced by an external magnetic field vanish when the field is turned off, Langevin assumed in 1907 that paramagnetic substances contain permanent moments. Weiss, in 1911, postulated the existence of permanent, elementary (i.e. quantized) magnetic moments in order to describe the behavior of ferromagnets above the Curie-Weiss temperature. Remarkably, in 1913, Bohr explained the discreteness of both electron energy and magnetic moment by postulating the quantization of angular momentum, which turned out to be a particular case of the quantization of action variables J_i:

$$J_i \equiv J_i(E_n) = \oint p_i(q_i, E_n)\, dq_i = n_i h \tag{2}$$

with E_n the energy eigenvalues, and p_i and q_i a pair of conjugated variables. The integral in Eq. (2) is carried out counterclockwise around a closed contour enclosing the two classical turning points and the section of the q_i axis between them. The eigenvalues E_n may be correlated with the value $n_i h$ by inverting Eq. (2).

Energy quantization for bound systems is, in itself, nothing mysterious, and there is no justification for the artificial schism between "classical" and "modern" (i.e. quantum) physics. Indeed, the mechanics of vibrating systems, such as plates, strings, and rods, yielded numerous examples of quantum relationships in the 19th century (Wesley 1983a). The mathematics of "eigenfunctions" and "eigenvalues" is common to a vibrating string and to an electron bound to a proton. In the mathematical sense, the eigenvalue problem is the quantum theory, as recognized first by Schrödinger (Schrödinger 1926).

1.2. Bound systems and uncertainty

The usual quantum mechanical uncertainty statements relate to a conjugate pair of variables. In a mathematical sense, the uncertainty relation is a local statement about, say, momentum and "position". (The notion of "position" is physically relevant only in connection with potential energy). Its global

counterpart may be regarded as a cyclic integral, as in Eq. (2), that can only assume equidistant discrete values (Post 1977). The uncertainty, when viewed in global terms, then reflects a smallest error that can occur in the evaluation of the integral. The process so described represents a counting operation concerning entities enclosed by the cyclic path of the integral. Out of four examples considered by Post (1977), two are highly relevant for the present discussion:

Counting action quanta inside a closed loop:

$$\oint \mathbf{p} \cdot d\mathbf{r} = nh \tag{3}$$

The uncertainty in its evaluation is

$$\Delta p \Delta r = n'h \tag{4}$$

Counting flux quanta in a closed loop:

$$\oint \mathbf{A} \cdot d\mathbf{r} = \frac{nh}{2e} \tag{5}$$

where $\mathbf{A}$ and e denote the vector potential and the (quantized!) electron charge, respectively. The uncertainty in the evaluation of Eq. (5) is:

$$\Delta A \Delta r = \frac{n''h}{2e} \tag{6}$$

Of course, there is a world of difference between the traditional interpretation of Δr in particle position and Δr as an uncertainty in loop position. Moreover, the "uncertainty relation" (4) is always an equality since the error in counting can only be an integer! The basic assumption is that the entity in question (action, flux) is indivisible and is either inside or outside the closed loop.

The "Fock conjecture" (Fock 1978), meanwhile,

$$\Delta x \Delta p = \left(n + \frac{1}{2}\right)\hbar = \frac{E_n}{\omega} \tag{7}$$

with n, E_n, and ω a quantum number, the energy of the system in its nth state, and the characteristic cyclic frequency of the bound system, respectively, has allowed us (see Marquardt and Galeczki 1994) to rewrite the traditional uncertainty relation as an equality:

$$\overline{r^2}\,\overline{p_r^2} = \frac{\overline{E_{\rm pot}}\,\overline{E_{\rm kin}}(s+2)^2}{s^2\omega^2} \tag{8}$$

Equality (8), valid for bound states governed by power law central potentials $V(r) \sim r^s$ provides a rigorous constraint on the average kinetic and potential energies of a stationary quantum state. Remarkably, relation (8) also holds for a superposition of stationary states, since the virial theorem upon which it relies holds for time averages, too (Ma 1991).

$$\langle E_{\rm kin}\rangle_{\rm av} = \frac{s}{2}\langle E_{\rm pot}\rangle_{\rm av} \tag{9}$$

$$\langle E_{\rm pot}\rangle_{\rm av} = \frac{2E_n}{s+2} \tag{10}$$

where $\langle\,\rangle$ denotes the expectation value in the state and $\langle\,\rangle_{\rm av}$ indicates time average.

Eq. (8) is thus a very far cry from the "inaccuracy", "indeterminacy", or "uncertainty" labels attached to inequalities like (4): The actual meaning of (8) is that of a constraint imposed on average energy values rather than on variances of r and p_r.

1.3. Rotating systems and quantized angular momentum

The following considerations stem from the observation that rotational motion is inherent to matter at every level of organization, from electrons up to clusters of galaxies. Before World War I, this led to the so-called Schuster-Wilson hypothesis (Aspden 1980), viz., that a fundamental property exists which causes any rotating body to have a magnetic moment. Strangely, the interpretation of the quantum mechanics of stationary states banished dynamics in general and rotation in particular from both physics and language. According to the established jargon, "the electron does not have angular momentum; it is in a specific angular momentum state". How a "smeared out particle" or a static (spatial) probability distribution function could account for a dynamical property like angular momentum is considered "a wrong question" in orthodox quantum mechanics.

In the history of quantum mechanics since 1925, spin became an abstract mathematical concept related to the irreducible two-dimensional (spinor) representation of the rotation group. In spite of its relation to the rotation group (which, by the way, is a perfectly classical construction with no specifically "quantum" or "special relativistic" features), the quantum mechanical spin of an elementary particle is denied any relationship to rotation in the dynamical sense. For example, according to Landau and Lifshitz (1965), spin is a genuine quantum mechanical property (i.e. it disappears when $\hbar = 0$) and it would be meaningless to represent the intrinsic angular momentum of an elementary particle as a consequence of rotation around its own axis. The origin of this dogma can be traced back to Born's conceptualization of the elementary particles as "material points guided by probability distributions", the unspoken premise being that a point cannot rotate around itself.

Fortunately—the orthodox formalism notwithstanding—attempts have been made to explain "spin" by models of rotating extended particles (Bergman and Wesley 1990). When assessing the relative merits and drawbacks of such models, we must always keep in mind that rotating extended charge models have to be consistent with electrodynamics. This is not an easy task, in view of the notorious failure of both the spherical charge model of the electron and of the Rutherford atom to remain stable under the action of electromagnetic forces alone. Contrary to the widespread lore that the existence of stationary states of the hydrogen atom (either postulated à la Bohr or derived à la Schrödinger) has solved the problem in one stroke. Quantum mechanics has actually evaded rather than tackled the problem of electrodynamical instability.

Schrödinger (1926) attempted to apply the unaltered Maxwell-Lorentz equations inside an atom filled with a time-independent electron charge and current distributions. He imagined the electric and magnetic field inside the atom as having axial symmetry, so that the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ would be directed azimuthally and closed on itself. Such a "rotational atom" would not radiate, since the electromagnetic energy would only circulate inside the atom. Unfortunately, the Schrödinger model of the atom does not fit the spectroscopic data accurately enough, since the suggested charge distribution would require (via the Poisson equation) the addition of an electron self-energy term to the already well-tested Schrödinger eigenvalue equation.

It is our conviction that the stationary states of the hydrogen atom are not static and therefore—in so far as Maxwell's theory is preserved—the moving electron would radiate, even if its trajectory could not be visualized in a "classical" sense. The instability of the Rutherford-Bohr atom is not rooted—as is usually claimed—in the incompatibility between the Larmor radiation formula and Planck's hypothesis (1), since it is not electromagnetic frequency that is quantized. Similarly, in statistical mechanics, the thermal energy per degree of freedom, $k_B T$, is not quantized, the temperature being a continuous intensive variable. As already pointed out, it is action in bound systems which is genuinely quantized. In our view, the instability of the atom against the effect of electromagnetic forces is tied to the violation of Newton's action-reaction principle by these forces (Cleveland 1936).

Paradoxically, it is customary to consider this violation as a requirement for the endowment of the fields with momentum and, in this way, to restore linear momentum conservation to the system consisting of both charges and fields (Panofsky and Phillips 1962). This procedure is consistent with the relational electromagnetism of Sachs (Sachs 1971) and Schwebel (Schwebel 1972) in which fields are always bound to their sources, but incompatible with both the accepted Maxwell equations and the photon concept as quanta of free fields. If we wish to preserve this photon concept, we have no other choice than to change our electrodynamics. The Weber potential (Weber 1848)

$$U = \frac{qq'(1 - v^2/c^2)}{4\pi\varepsilon_o R} \approx \frac{qq'(1 - \frac{1}{2}v^2/c^2)}{4\pi\varepsilon_o R} \qquad (11)$$

between two moving charges q at $\mathbf{r}$ and q' at $\mathbf{r}'$ where $\mathbf{v} = d\mathbf{r}/dt = d(\mathbf{r} - \mathbf{r}')/dt$ is a promising start. It can be shown that:

$$\frac{dU}{dt} = -\mathbf{v} \cdot \mathbf{F}_W \qquad (12)$$

with $\mathbf{F}_W$ the Weber force obeying Newton's third principle. Since the force is derived from a potential, energy is conserved for a system of moving charges under the sole action of electromagnetic forces.

In the Maxwellian electrodynamic case, Newton's principle of equality between action and reaction is clearly violated. For example, the electron moving on a closed orbit "acts" on the "fixed" proton with both electric and magnetic forces, while the proton "reacts" only with an (attractive) electric force. On the other hand, according to Larmor's radiation formula, the electron should continuously lose energy by radiation and finally fall onto the proton. Endowing the radiation with linear momentum neither restores Newton's third principle, nor does it confer any stability to the hydrogen atom. On the other hand, Weber's force, obeying Newton's third principle, does not allow for radiative energy loss from the bound electron-proton system. Since the electromagnetic stability of the atom is assured, one can neglect the velocity-dependent term in Eq. (11) ($v^2/c^2 \approx 2 \times 10^{-6}$) and solve the wave equation for $U \approx qq'/4\pi\varepsilon_o R$. Solving Schrödinger's equation in the approximation $U \approx qq'(1 - \frac{1}{2}v^2/c^2)/4\pi\varepsilon_o R$ as in Eq. (11) provides us with the fine structure of the spectrum. Quantum mechanics and Weber electrodynamics are thus perfectly compatible.

The Weber force $\mathbf{F}_W$ assures the stability of both the hydrogen atom and of the electron itself. The Bergman-Wesley (1990) spinning ring model of the electron is a pure electromagnetic field model with no rotating matter involved. It is essentially a steady-state toroidal field, the discontinuity of the field on the torus and the electromagnetic field energy defining the electric charge e, magnetic moment μ and (rest) mass m. The field is divided into an electrostatic and a magnetostatic field associated with potential and kinetic energy, respectively. Due to the equipartition between electric and magnetic energy, and to the fact that only the magnetostatic energy contributes to the angular momentum of the electron, the puzzle of the gyromagnetic ratio being about twice the classical value is readily explained. The magnitude $h/4\pi$ of the free electron spin angular momentum, however, has to be postulated in order to fit the spectroscopic data. The circumference of the ring equals the Compton length, h/mc, while the tangential velocity of the ring equals the velocity of light, c.

Within the hydrogen atom, both the magnetic and the electric energy contribute to the orbital momentum of the electron. Therefore, the gyromagnetic ratio equals $e/2mc$, as expected classically. We emphasize that neither orbital momentum nor spin alone, but only the total angular momentum, is a constant of motion. Moreover, as shown by Oudet and Lochak (1987), only Dirac's equation is able to correctly describe the angular momentum of an atom and to give the best nomenclature of the magnetic states. Although Dirac's equation rigorously holds for the hydrogen atom only, Oudet and Lochak succeeded in accounting quantitatively for the magnetic moments at 0° K of heavy rare earth metals, Cr compounds, Fe, Co, and Ni, by assuming that "the coherence of the angular states of each electron is stronger than the ensemble angular coherence of the electrons of the atom. The total angular momentum of an electron is considered as an inalienable (= untouchable) property."

The angular momentum of composite systems like protons, neutrons, and nuclei equals the additive sum of their re-

spective components. The magnetic moments of these composite systems, however, require the knowledge of the gyromagnetic ratio of each component. This is known only empirically; the spin (s) and orbital (l) g-factors of protons (p) and neutrons (n) are $g_{s,p} = 5.57$, $g_{l,p} = 1$, $g_{s,n} = -3.83$, $g_{l,n} = 0$, respectively. The g-factors of the hypothetical quarks within the baryons are in principle unknown because, obviously, no free quarks exist.

We conclude this section by mentioning that microparticles with integer spin ("bosons") are, as a rule, composite particles, the notorious exception being the photon. (The status of the W^+, W^-, and Z^0 intermediate bosons of the Salam-Weinberg theory is not clear (Dirac 1981) mainly due to the fact that they are very massive. Moreover, the experimental evidence found by Rubbia and van der Meer (Nobel prize 1983) is indirect, consisting in "single electrons whose trajectory matched the one expected in a W^- particle's decay.") This vector boson, with angular momentum $h/2\pi$, has resisted every attempt at modeling to date. The vector nature of this boson is required, first of all, by the empirical selection rule $\Delta l = \pm 1$, known as "Laporte's rule" (Schiff 1955). This rule, together with angular momentum conservation, requires $j = 1$ for the photon.

1.4. Minimal action and characteristic time

By and large, quantum mechanics deals with two types of problems: internal and external. External problems include scattering, diffraction, and interference phenomena and are not the aim of the present article. Uniform linear motion, the trademark of a "free" system, should rather be considered the exception corresponding to an idealized model. The propagation of photons in the absence of gravity comes closest to Newton's concept of inertial linear motion.

Internal problems deal with stationary states of bound, periodic systems. In such systems, the concept of action explains the existence of zero-point energies and of characteristic eigenfrequencies. This is understandable, recalling that there is no degree of freedom for kinetic energy in the absolute minimum of a potential, and therefore a lower limit of the kinetic energy is compulsory for dynamic systems. The prominent feature of these bound systems is the quantization of action integrals in general, and of angular momentum in particular. Rotation, angular momentum, and magnetic moment are inherent to matter at every level of organization. Although, in some situations, the minimal action h may be negligible, it is always finite on empirical grounds and therefore "the classical limit $\hbar \to 0$" expresses a faulty philosophy. In specific situations, $\hbar$ may be neglected, but it does not vanish and cannot smoothly approach zero. The intrinsic angular momentum of macroscopic bodies averages to negligible values due to their complexity rather than due to their large masses as such. The situation is reminiscent of the vanishing net magnetization of macroscopic ferromagnets due to the complex domains-and-walls structure. It is, however, known that such materials display single domain behaviour below a critical size.

Every bound system seems to be governed by a permanent interplay between kinetic and potential energy (the latter being the trademark of a bound system), the energy conversion proceeding in such a way that the product *energy converted* $\times$ *characteristic time* $\equiv$ *action* is minimized. Physically, the characteristic time, τ_a, ("action time") is a much more relevant quantity than ad hoc defined quantities like the "time required for light to traverse the electron diameter" or the "Planck length". Barut (1992) arrives at a similar conclusion by defining the Rydberg (for the Coulomb problem) as:

$$\mathrm{Ry} \equiv h\nu_o \tag{13}$$

where ν_o denotes the characteristic frequency ($\approx 10^{15}$ s^{-1}) of the system. Solving the Schrödinger equation, he obtains the frequency spectrum

$$\nu_n = \frac{\nu_o}{n^2} \quad \text{or} \quad \nu_n - \nu_m = \nu_o \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \tag{14}$$

The proportionality constant h in (13) is taken from experiment. The wave equation determines all possible frequencies of the system as in classical wave theory.

2. The dynamical foundation of wave mechanics

2.1. Waves and wave packets

De Broglie's and Schrödinger's wave mechanics is based on a formal analogy between the principle of Maupertuis (for particles) and that of Fermat for waves in the geometrical optics approximation (Fermi 1955). The phase $\varphi = \mathbf{k} \cdot \mathbf{r} - \omega t$ of plane waves in optics with $\mathbf{k} = \nabla\varphi$ and $\omega = -\partial\varphi/\partial t$ the wave vector and circular frequency, respectively, finds its correspondence by means of Hamilton's principal function S, which is Lagrange's function integrated over the time interval τ_a of action and thus defines a surface of action in configuration space (Jammer 1966). Here we have $\nabla S = \mathbf{p}$ and $\partial S/\partial t = -E$ for momentum and energy, respectively. Using Planck's relation

$$E = \hbar\omega \tag{15}$$

and the proportionality between the (free) particle's linear momentum $\mathbf{p}$ and the wave vector $\mathbf{k}$ of the associated wave

$$\mathbf{p} = \hbar\mathbf{k} \tag{16}$$

we arrive at the wave-mechanical phase of a free particle

$$\varphi = \frac{(\mathbf{p} \cdot \mathbf{r} - Et)}{\hbar} = \frac{S}{\hbar} \tag{17}$$

Writing relation (1) in the proper frame of the particle with rest mass m_o, where

$$E_o = m_o c^2 \tag{18}$$

holds, de Broglie (1925) introduced a mysterious "periodic phenomenon" with the circular frequency

$$\omega_o = \frac{m_o c^2}{\hbar} \tag{19}$$

and attempted to correlate this particle-internal frequency with the particle-external frequency of the wave having phase

φ as in Eq. (17). (Although—as we shall see in the next section—he followed a defective procedure, his relation (16) survived and has been overwhelmingly confirmed, especially in present-day microelectronics).

In order to overcome the localization problem, recourse is taken to ad hoc wave packets, even in non-dispersive vacuum. The concept of wave packets leads to a definition of group velocity $v_g \equiv \partial\omega/\partial k$ besides the phase velocity $v_p \equiv \omega/k$.

De Broglie (1925) postulated the validity of relation $v_p v_g = c^2$ for both photons and particles with "proper mass" m_o, thus introducing the velocity of light, c, into the dynamics of point masses. This procedure has to be contrasted with the introduction of c into special relativity kinematics via the peculiar velocity measurement prescription involving a clock and light-signals with equal (by decree) to- and fro one-way velocities. To be specific, we therefore distinguish Kaufmann dynamics and Lorentz-Einstein kinematics. Once Kaufmann dynamics is incorporated into the model, the mass-energy equivalence (with E the total energy),

$$E = mc^2 \tag{20}$$

enters as a dynamical concept.

Eq. (20) implies the velocity dependence of masses (Galeczki 1994):

$$m(v) = \frac{m_o}{\left(1 - \dfrac{v^2}{c^2}\right)^{1/2}} \equiv \gamma m_o \tag{21}$$

where v is a time-of-flight velocity to be identified with v_g. Inserting (20) into (21), squaring, rearranging terms and introducing $\mathbf{p} = m(v)\mathbf{v}$ yields

$$E^2 = (m_o c^2)^2 + (c\mathbf{p})^2 \tag{22}$$

the basic equation of Kaufmann mechanics.

Using Eqs. (1) and (16) and the definition of ω_o in Eq. (19), we reformulate Eq. (22) in the form:

$$\omega^2 = \omega_o^2 + c^2 k^2 \tag{23}$$

which bears a mathematical resemblance to the hyperbolic dispersion relations known in "ordinary" wave physics where ω and k are continuous variables. In Eq. (23), however, both ω_o and ω designate discrete frequencies "seen" in the rest system of the mass m_o and by an observer in relative motion.

Replacing the monochromatic "pilot wave" of the particles by a wave packet, we extend Eq. (23) to continuous variables ω and k, according to Schrödinger and de Broglie, and see that it is equivalent to:

$$v_p v_g = \frac{\omega}{k} \cdot \frac{d\omega}{dk} = \frac{d(\omega^2)}{d(k^2)} \equiv c^2 \tag{24}$$

A hyperbolic dispersion relation like Eq. (23) is found in various branches of wave physics, and, remarkably, is a characteristic of a homogeneous isotropic propagation medium (the "vacuum" being trivially homogeneous). In various homogeneous media, the hyperbolic $\omega(k)$ relation holds for different values of v_p and different meanings of the "frequency gap" ω_o. E.g., for the propagation of transverse electromagnetic waves in a dilute plasma, such as the ionosphere, $\omega_p = (ne^2/\varepsilon_o\varepsilon_e m)^{1/2}$ (with n the density of charges, e, and $\varepsilon_o\varepsilon_e$ the permittivity of the medium) is the plasma frequency. This, however, is in conflict with the kinematic interpretation: if Eq. (23) were a consequence of special relativity, c should be the one and only velocity of light in vacuum. For free electromagnetic waves, there is no frequency gap ($m_o = 0$), and Eq. (23) degenerates into a linear relation with $v_p = v_g = c$ for all values of k.

2.2. The kinematic picture of de Broglie waves: Lorentz-transformed oscillations lacking Lorentz invariance

De Broglie's mysterious "periodic phenomenon" associated with m_o was conceived as an oscillation in the particle's rest system:

$$\psi(t_o) = A_o \exp(i\omega_o t_o) \tag{25}$$

with t_o denoting the particle's "proper time".

Special relativity kinematics applies the Lorentz transformation for an observer in relative motion who now claims to "see" a wave instead:

$$\psi(x,t) = A \exp\left\{i\omega\left(t - \frac{vx}{c^2}\right)\right\} \tag{26}$$

with $\omega = \gamma\omega_o$.

In electrodynamics, the Lorentz transformation is known to turn an electric field $\mathbf{E}$ into a magnetic field $\mathbf{B}$. Similarly, here a stationary phenomenon is given a seemingly dynamical quality, and an oscillation becomes a wave. But strictly speaking, this is a pseudo-wave, because the spatial coordinate x is artificially introduced by the transformation. No wonder that the phase of de Broglie's wave (26) is not proportional to the action S of the free particle ($S \neq \hbar\varphi$), and that Eq. (16) is clearly violated, since the phase velocity of the wave (26) equals c^2/v. To restore the equality $S = \hbar\varphi$, de Broglie and Schrödinger had the (unfortunate) idea to replace the monochromatic (plane) wave associated with a free particle with an unstable (in time) "wave packet" centered around some value k_o and to identify the particle velocity v with the group velocity $v_g = c^2/v_p$. The seeming success of this procedure is due to the fortuitous coincidence between $v_g = c^2/v_p$—a formal consequence of the Lorentzian kinematics of a point-like particle—and formula (24) relating v_p and v_g for waves in a dispersive medium. The price of this "success" is the undermining of Planck's formula (16) postulated for photons—quanta of free electromagnetic fields—and extrapolated by de Broglie for free particles with $m_o \neq 0$. Apparently, the only way to save at once all three relations (1), (16), and $S = \hbar\varphi$, is to replace de Broglie's pseudo-wave (26) by a monochromatic wave having its phase velocity equal to the particle velocity v. Such a wave has indeed been proposed by Wesley (1983b) who also remarked that the combination of a harmonic oscillation and a translation cannot produce a wave. The transformed Eq.

(25) actually describes a standing wave, as is easily seen by putting $t_o = x/v$.

The wave ψ of Eq. (26) obeys the wave equation

$$\left(\nabla^2 - \frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0 \tag{27}$$

with $v_p \equiv c^2/v$. We emphasize that Eq. (27), like the wave equation satisfied by the electric field $\mathbf{E}$ in a linear isotropic medium (Post 1962):

$$\left(\nabla^2 - \varepsilon\varepsilon_o\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0 \tag{28}$$

is obviously not Lorentz-invariant.

2.3. Interpretational problems in kinematical wave mechanics

The fact that no quantum mechanical wave equation has been correctly derived might explain the unsettled interpretational problems of quantum mechanics even now, 70 years after its inception. To support this statement, we sketch several approaches to the Schrödinger equation, since the time-independent equation—like de Broglie's relation $\lambda = h/p$—seems to be correct despite its questionable derivation:

Inverting $\psi = a\exp(iS/\hbar)$ and putting $S = -i\hbar\ln(\psi/a)$ into the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\mathbf{r}) = 0 \tag{29}$$

leads to a useless equation for the two unknown functions ψ and a.

Substituting $\mathbf{p} = \nabla S$ and $\partial S/\partial t = -E$ into Eq. (22), we get

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\nabla S)^2 = m_o^2 c^2 \tag{30}$$

which is equivalent to one wave equation of the Klein-Gordon type for ψ and a.

Fermi (1955) assumed the harmonic time dependence

$$\psi = u\exp\left(-\frac{iEt}{\hbar}\right) \tag{31}$$

and, after substituting ψ into the wave equation

$$\nabla^2\psi - \frac{1}{v_p^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{32}$$

he obtained the time-independent equation:

$$\nabla^2 u + \frac{2m}{\hbar^2}(E - V)u = 0 \tag{33}$$

After employing the operator substitution $Eu \to i\hbar\partial\psi/\partial t$, he finally arrived at the time-dependent Schrödinger equation

$$-i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{34}$$

Curiously enough, the Lorentz-invariant Klein-Gordon equation

$$\left[\nabla^2 - \frac{1}{v_p^2}\frac{\partial^2}{\partial t^2} - \left(\frac{m_o c}{\hbar}\right)^2\right]\psi = 0 \tag{35}$$

was obtained from relation (22) which holds for point-like particles rather than for waves. Applying the same operator approach to the classical relation

$$E = \frac{\mathbf{p}^2}{2m} \tag{36}$$

one obtains Schrödinger's time-dependent equation, which is not a wave equation at all! There is no way to transform a hyperbolic equation with partial derivatives, like Klein-Gordon, into a parabolic type equation, like the Schrödinger equation. The "derivation" of Schrödinger's time-independent equation from Eq. (27) is an artifact of the unique exponential time-dependence of the wave functions, as stipulated in the axiomatic formulation of the theory.

In their famous quantum mechanics textbook, Landau and Lifshitz stipulate from the very beginning that the wave function ψ has to obey the operator equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{L}\psi \tag{37}$$

(with $\hat{L}$ a linear operator) rather than a wave equation. Using the expression $\psi = a\exp(iS/\hbar)$—which they call a "quasi-classical wave function"—they obtain the quasi-classical form

$$\frac{\partial\psi}{\partial t} = \frac{i}{\hbar}\frac{\partial S}{\partial t}\psi \tag{38}$$

Further, they assume that a is a slowly varying function of t and identify $\hat{L}$ from Eq. (38) with the Hamiltonian operator $\hat{H}$.

Although aware that Schrödinger's time-dependent equation is of the parabolic type, Bohm, while developing his "causal quantum mechanics" (Bohm and Hiley 1989), took it for granted. He used the already mentioned quasi-classical form of ψ, substituted it in Schrödinger's time-dependent equation, and, after separating the real and the imaginary parts, obtained the following equations:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 a}{a} = 0 \tag{39}$$

$$\frac{\partial(a^2)}{\partial t} + \nabla\cdot\left(a^2\frac{\nabla S}{m}\right) = 0 \tag{40}$$

In Bohm's opinion—to be contrasted with that of Landau-Lifshitz—the classical Hamilton-Jacobi equation is obtained for "slow spatial variation" (as compared to S) of a rather than for $\hbar = 0$. The formal analogy of (40) with the continuity equation suggests that it be regarded as describing the conservation of probability $P \equiv a^2$ in the ensemble of trajectories normal to the wavefront $S = \text{const.}$ with momentum $\mathbf{p} = \nabla S$. In Bohm's and Hiley's words, "we are led to the causal interpretation of the quantum theory if we notice

that the same picture can hold if we do not make the quasi-classical (JWKB) approximation, but simply suppose that in addition to the classical potential V, the particle is acted on by a further quantum potential

$$Q \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2 a}{a} \tag{41}$$

"

The peculiar quantum-mechanical phenomena are all attributed to the existence of Q. Bohm has, however, not solved the equations for specific situations like the double-slit experiment. Bohm traced qualitatively—but never quantitatively—the causal motion of a particle acted upon by the quantum potential. The cause of the fundamental difficulty found in the work of de Broglie and Bohm is their false presumption that the implication of Schrödinger's equation is solely statistical. A related problem which we shall discuss in a forthcoming paper, is the failure in present-day quantum mechanics to distinguish between a given system of N particles and an ensemble of N similar one-particle systems of which each member is independent of the others. This failure leads to a paradoxical prediction when one tries to apply Pauli's exclusion principle to electrons belonging to different atoms. According to this principle and to the (widely accepted) statistical interpretation of quantum mechanics, the intensity of any spectrum line observed in a hydrogen gas would be extremely weak, because only one of the numerous atoms is permitted to be in one energy eigenstate at a given time.

While the action S, which plays the role of a generalized phase $\hbar\varphi$ in the above considerations, more or less tacitly has its share in the wave models, its physical relevance is not fully acknowledged in the kinematical treatment of wave mechanics. From the phase expression $\varphi = \mathbf{k}\cdot\mathbf{r} - \omega t$, kinematics chooses the co-ordinates $(\mathbf{r}, t)$ for a description of physical wave phenomena instead of the wave parameters $(\mathbf{k}, \omega)$. There is no way to construct a dynamical description from time-space geometry.

Summarizing, de Broglie waves, derived via the Lorentz transformation, are devoid of physical meaning, because they lack basic requirements of dynamics. No wonder, they have been given the abstract meaning of probability waves! However, according to Kolmogorov's axiomatic formulation (Kolmogorov 1936, Gnedenko 1969), probabilities cannot interfere. In order to escape the problem of localization, the picture of ad hoc wave packets is adopted whenever suitable. But this is done at the cost of having to understand or postulate dispersion in free space. Yet, as is well known, the application of a de Broglie wavelength $\lambda = h/mv$ does work for the description of results in scattering, diffraction, and interference experiments. In order to bring us closer to a physical picture, wave dynamics should replace Einstein-Lorentz kinematics. This is tantamount to replacing the space-time invariant $r^2 - (ct)^2$ à la special relativity by the invariant $\omega^2 - (ck)^2$ built up from quantities characterizing wave propagation.

2.4. Action dynamics vs Einstein-Lorentz kinematics

Looking back to the analogy between Hamilton's principal function S and the phase $\varphi = \mathbf{k}\cdot\mathbf{r} - \omega t$ in plane waves:

$$\nabla S = \mathbf{p} \text{ and } \frac{\partial S}{\partial t} = -E \quad \Leftrightarrow \quad \mathbf{k} = \nabla\varphi \text{ and } \omega = -\frac{\partial\varphi}{\partial t}$$

we recognize that action S and phase φ are related by

$$S = \hbar\varphi \tag{42}$$

Consequently, surfaces of constant phase correspond to those of constant action. This important correspondence urges us to re-interpret de Broglie waves as spatial patterns of action. Doing this, we have avoided the introduction of space-time coordinates as dynamical quantities, which they are not. They are the mathematical parameters of mechanics, and as such cannot have any physical influence. The phase $\varphi = \mathbf{k}\cdot\mathbf{r} - \omega t$ tells us that kinematics chooses the non-physical part $(\mathbf{r}, t)$ of the variables. Space-time coordinates are not the properties of any physical system, but serve as parameters for the description of true physical properties such as wave vector and frequency. It is those properties which change upon transformation into a moving frame of reference and which represent momentum and energy. If one selects the physical pair $(\omega, \mathbf{k})$ of the variables, the invariant of the Lorentz transformation is replaced by

$$(ck)^2 - \omega^2 = (ck')^2 - \omega'^2 = \text{inv} \tag{43}$$

Wesley (1980) proposed this replacement on account of the description of the Michelson-Morley experiment as a Voigt-Doppler effect in absolute space.

The value of the active interpretation of de Broglie waves in terms of action lies in the fact that the motion of the particle has not to be assumed ad hoc, as in the kinematic description where mathematical coordinates are given an active role. Usually, the motion is taken for granted and its cause is ignored. Léon Brillouin (1970) rightfully puts his finger on this critical point in his book *Relativity Reexamined*: "What do we mean by a given velocity? Who gives us this velocity and how... There is only one occasion when it has a definite meaning; this is in the statement of a problem given by an examiner to some helpless students." In the action picture, the cause of motion, namely the conversion of different forms of energy into each other, is the firm basis of a dynamical description. Action also helps to get rid of the uneasy feeling associated with uniform linear motion and the associated plane wave of infinite extent. Clearly, uniform motion, an idealization holding only in a space with strictly constant potential, is always a more or less valuable approximation, and certainly does not hold in the realm of microphysics where potentials change very rapidly with time and space. If uniform motion were typical of quantum particles, the quantum of action, h, would scarcely show up in an experiment, as action vanishes under the condition that potential energy does not participate in the balance. Also in the macroscopic world, uniform motion is, rather, the exception (if ever realized at all!).

2.5. Discussion and conclusions: matter waves are Action Waves

Nature has chosen the wave as a means of transporting energy with a minimum transport of masses and charges; see electromagnetism, acoustics, and heat transport. It appears that even dc currents in conductors, since Drude's time a symbol of "free" charge transport, can be modeled in terms of wave transport: Graneau (1985) proposed a dc conduction mechanism based on pivoted dipole current elements. No one doubts that the phase of a wave constitutes a physical reality, which itself does not transport mass or energy but is a necessary "ingredient" for transport. Similarly, action, behind the ubiquitous energy conversion observed everywhere in Nature, is a phenomenon that can propagate with a phase velocity, ω/k, and describes the changes of local energy balances with space and time. As any wave is accompanied by energy conversion, action must be thought of as the generalized phase of all wave phenomena. This, in turn, suggests that we should also apply the idea to other dynamical phenomena which at first glance are not wave-like, e.g., the motion of a particle in a varying potential. Hence, we have found a quite natural way to save the basic idea of de Broglie waves without having to grapple with the conceptual difficulties of probability interference and non-causal events in macroscopic regions of space. A particle does not manifest "wave nature" by itself. It is its motion with respect to the local potential that gives rise to wavy disturbances in its surrounding. These wiggles in the potential are detected by experiment; they are an honest-to-goodness physical effect. For a probabilistic Born-de Broglie wave, it is hard to understand how (if at all) the abstract phase velocity v_p of a probability pattern could contribute to the energy of a particle, for if we accept the general agreement that $v_p v_g \equiv c^2$ holds for de Broglie waves, then Eq. (20) reads $E = m v_p v_g$—which would be a rather peculiar mixture of physical and non-physical velocities. The propagation of action, on the other hand, does contribute to the energetic processes. Local disturbances of potentials propagate in space and time and mediate between more or less local changes of kinetic energy and their interaction with the surroundings.

Potentials are the spatial part of energy, and no wave propagation would be possible without them. The wave-external potential for electromagnetic waves remains to be specified. It is certainly not an "ether" with contradictory properties, nor is it empty space represented by the mere mathematical construction of space-time coordinates which are devoid of any physical significance. In our preceding paper (Marquardt and Galeczki 1994), we pointed out that a physically significant coordinate corresponds to a potential with finite spatial extension, and that the purely mathematical "position operator" is meaningless, to say nothing of "position" in the manner of a point in space. Similarly, one has to differentiate between the continuous, algebraic "time parameter"—which can by no means "contract" or "dilate"—and time as "rate-of-physical processes". Cyclic periodic processes allow us to define units of time ("periods"), which are, of course, subject to dynamical changes. In terms of these units, various physical processes could show "slowing down" or "speeding up" without any system-internal change. Time inevitably involves the change of a state in a potential by virtue of kinetic energy (keeping in mind, that uniform motion is always an approximation, and that the state of rest is the consequence of our choice of a reference frame). The same argument applies to our very measurements of "time" which are based on periodic changes of position in a potential, and hence on kinetic energy. The latter can always be associated with a frequency, which is a characteristic property of the system in question. Action S is the principle that combines both the "energy of position" ("energy in space") and the "energy of velocity" ("energy in time"). In order to keep the discussion within the usual framework of the Hamilton formalism, we consider conservative systems only (i.e. without dissipation) whose total energy E does not depend on time and is fully determined by the two basic types of energy, positional and kinetic. Now action waves can be modeled from the gradients. The spatial gradient $\nabla S = \mathbf{p}$ describes the conversion of momentum, i.e. of kinetic energy into or from potential energy, depending on the sign. (This is in accordance with Newton's force principle and the fact that conservative forces can be traced back to the gradient ∇V of a potential). Vice versa, the gradient in time $\partial S/\partial t = -E$ is determined by the negative total energy (note the minus sign) of the system, i.e. equal amounts of energy are converted in equal action time intervals. The significance of the gradients lies in the fact that they tell us about the possibility of action propagation like that of a monochromatic physical wave, with $\mathbf{p}$ determining the "wavelength" and E the "frequency" of this wave. Indeed, by virtue of Eq. (22), identifying momentum and total energy with the gradients ∇S and $\partial S/\partial t$, respectively, we arrive at the Klein-Gordon Eq. (35), this time formulated for S, rather than for an abstract probability amplitude ψ. Now we have a wave equation for action that is compatible with Kaufmann dynamics. The term $(m_o c/\hbar)^2$ that makes it an inhomogeneous wave equation represents the "rock bottom" potential energy that cannot further be converted into kinetic energy. A "monochromatic" action wave in a conservative dynamical system represents perpetual energy conversion in its simplest form, i.e. when those processes occur periodically in space and time. In a forthcoming paper, we extend this model to dissipative processes.

Strangely, it is never asked why different forms of energy can change into one another, or whether our distinction of these different forms is justified or purely phenomenological. Is there basically only one energy showing different aspects? Action provides the unifying aspect, and we may indeed envisage kinetic and potential energy as linked by gradients of one quantity in time and space.

In conclusion, by replacing mathematical coordinates by physically active wave parameters, matter waves can be reinterpreted on a dynamical basis as patterns of action in time and space. Action plays the role of a generalized phase in physics, and as such allows us to describe the cause of

observed phenomena instead of contenting ourselves with a formal coordinate transformation. Action is brought into agreement with causality and the interpretation of experimental observations is rid of the notorious difficulties associated with probabilities. Planck's action quantum relates wave properties that are not a priori quantized (ω, k) to the "steps" Nature has chosen for all dynamical processes.

The gradients which are essential for dynamics are always finite. Localization, e.g., of an electron in a box, is usually treated in a probabilistic manner, indicating that the electron itself is admittedly smaller than the box, yet needs the whole space for its motion. The action concept tells us why: confinement determines the range of action in this problem, and electron motion covers all the space. This is no more probabilistic than the localization of a vibrating string! The action picture also indicates where the often-used quantum box model is grossly oversimplified: neither a classical nor a quantum particle could move in a region of strictly constant potential with infinite gradients on either side.

It is noteworthy that the principle of action helps us bridge the (artificial) gap between classical and quantum physics, since it applies to any physical system, irrespective of its size and nature. Here, we believe, lies its greatest merit.

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