The Uncertainty Principle Revisited

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For bound states governed by power-law potentials $V(r) \sim r^s$, the Heisenberg position-momentum (r,p) inequality $\Delta r \Delta p_r \geq \hbar/2$ is replaced by the equality $\overline{r^2}\,\overline{p_r^2} = \overline{E_{\rm pot}}\,\overline{E_{\rm kin}}\,(s+2)^2/s^2 = (\omega\hbar/2)^2 = E_n^2/\omega^2$ with E_n and $\omega \equiv 2\pi\nu$ the total energy and frequency of the nth state. The lack of properly definable operators casts serious doubts on the physical contents of uncertainty relations originating from operator formalism. For "spreading solutions" (wave packets), approximate equalities $\Delta x \Delta k \cong 1$ and $\Delta E \Delta t \cong h$ hold. Planck's quantum hypothesis $E = h\nu$ and the concept of the wave packet seem to be incompatible.

Key words: Action; Operator Formalism; Uncertainty Principle; Bound systems.

1. The Operator-Commutator Approach

Together with the quantization of energy, Heisenberg's uncertainty principle (UP) is generally considered one of the pillars of quantum mechanics. Bush (1982) even declared: "... since Heisenberg's uncertainty principle guarantees the quantization of energy on an atomic scale, which in turn permits life as we know it, it can, perhaps, be considered the most important principle of physics."

The contention that principles of uncertainty (like in quantum physics), of impossibility (like in thermodynamics), or of impotence (like in special relativity) could play a constructive rôle in science is worth considering from an epistemological point of view. Here we limit the discussion to the derivation, the meaning, and the practical relevance of the UP, as well as to its connection with models of extended particles.

The genuine quantum mechanical derivation of the UP is based on the general commutation relation

$$[F,H] = FG - GF = iA \tag{1}$$

where F, G, and A are Hermitian operators.

Then, for the mean-square deviations from the average, or "variances" ΔF and ΔG:

$$\begin{aligned}(\Delta F)^2 &\equiv \langle\Psi|(F-\langle F\rangle)^2|\Psi\rangle \\ (\Delta G)^2 &\equiv \langle\Psi|(G-\langle G\rangle)^2|\Psi\rangle\end{aligned} \tag{2}$$

it follows that

$$(\Delta F)^2(\Delta G)^2 \geq \frac{|\langle\Psi|A|\Psi\rangle|^2}{4} \tag{3a}$$

or, less rigorously:

$$\Delta F \cdot \Delta G \geq \frac{|\langle\Psi|A|\Psi\rangle|}{2} \tag{3b}$$

where F and G are termed "the uncertainty" in F and G, respectively. For the special choice $F = p_x$ (linear momentum) and $G = x$ ("position"), eq. (1) reduces to:

$$p_x x - x p_x = -iI \tag{4}$$

with I the identity operator. The inequality:

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{5}$$

is the most frequently quoted form of the UP.

The joint measurement of the x and y spin components bears a formal analogy with the position-momentum case. Indeed, the commutation relation for the spin numbers:

$$[S_x, S_y] = \frac{iS_z}{2} \tag{6}$$

together with the matrix elements:

$$\langle m'|S_z|m\rangle = m\delta_{mm'} \tag{7}$$

where $m, m' = \pm\frac{1}{2}$, lead to the inequality

$$\Delta S_x \Delta S_y \geq \frac{\hbar}{2} \tag{8}$$

This result is (rightly) questioned by Sanchez-Ruiz (1993) since S_x and S_y are discrete observables in a finite dimensional Hilbert space so that ΔS_x, ΔS_y, and $[S_x, S_y]$ vanish in eigenstates of S_x or S_y, and therefore no restriction is imposed by the inequalities (3a) and (3b). In both examples the operators and their associated "variances" are time-independent; therefore the inequalities (5) and (8) are said to refer to "simultaneous" values.

The "commutator approach" does not work for angular momentum L_z and (polar) angle φ since the Poisson bracket is always zero

$$\{L_z, \varphi\} = 0 \tag{9}$$

For the quantum phase—heuristically understood as complementary to a well-defined photon number—the quantum mechanical approach is not applicable at all, since "the phase operator" does not exist (Carruthers and Nieto 1968). Although D. T. Pegg and S. M. Barnett (1988) succeeded in constructing a unitary phase operator, we agree with J. M. Lévy-Leblond (1976) that this requires us "to break one of the sacred dogmas of the current exposition of quantum theory, namely, the rule which assigns a Hermitian operator to every physical property". As a matter of fact, this dogma is part of the axiomatic basis of the (one and only) quantum mechanics in use.

The search for a position-momentum type inequality applying to time and energy was prompted by the special theory of relativity where the four parameters x, y, z, t are on equal footing. Although no Hermitian time operator can be associated with the parameter

t, the application of Heisenberg's "equation of motion" for the "observable" A:

$$i\frac{dA}{dt} = [A,H] + i\frac{\partial A}{\partial t} \tag{10}$$

leads formally to the following inequality:

$$\tau_a \Delta E \geq \frac{\hbar}{2} \tag{11}$$

Here it is assumed that A has no explicit time dependence (i.e., $\partial A/\partial t = 0$) and that τ_a is defined (Messiah 1976) as:

$$\tau_a \equiv \frac{\Delta A}{|d\langle A\rangle/dt|} \tag{12}$$

The meaning of τ_a is that of a characteristic time of the evolution of the statistical distribution of A, for example the time needed for the center $\langle A\rangle$ of this distribution to be shifted by ΔA (i.e., its own width). The reasoning implies a non-stationary (or metastable) state, otherwise $d\langle A\rangle/dt = 0$ and $\tau_a = \infty$. However, the difficulty connected with (11) is still with us, since t is a parameter—rather than a dynamic variable of the one-particle system under study—and there is no way to replace A by t in equation (10). This fact is in line with an earlier contention that we do not know anything about the particle under study by choosing the coordinate of an "instant". *Strangely, it was never openly realized that the same difficulty occurs with respect to "position" in free space.* What information do we gain, indeed, about an extended particle whose "center of mass" coordinate is fixed? If the particle were placed in a potential $V(r)$, we have some estimate about its potential energy; but in a potential that varies considerably along a particle diameter, this knowledge is certainly not what we want. If, on the other hand, the potential does not vary sensibly, then it comes close to being constant, and we are left with the dilemma of "position in free space". This sort of enquiry inevitably brings interpretational problems into the discussion. According to Born, particles are material points, and the square of the absolute value of the wave function $|\Psi(x,t)|^2$ is the probability density for finding a particle at a given point in space and in time. According to Schrödinger's interpretation of 1926 (Schrödinger 1926), the electrons are not "small" and their charge fills the whole volume of the atom. Schrödinger devised his equation mainly for studying the electron charge and current distributions in quantum systems (atoms, molecules, crystals, etc.) characterized by a potential well. In external problems, where particles move through macroscopic regions of free space, "non-spreading solutions" of the wave equation are required.

In de Broglie's original theory (de Broglie 1925), the wave function $|\Psi(x,t)|^2$ associated with a point-like particle, moving with velocity v in the laboratory

$$\Psi(x,t) = \Psi_o \exp\left[-i\omega\left(t - \frac{xv}{c^2}\right)\right] \tag{13}$$

was obtained from the "periodic phenomenon"

$$\Psi(0,t_o) = \Psi_o \exp(i\omega_o t_o) \tag{14}$$

by applying the Lorentz transformation for the time parameter t_o

$$t_o = \frac{t - \frac{xv}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \tag{15}$$

The frequency of "the internal periodic phenomenon"—called later "Zitterbewegung"—was related to the rest mass m_o of the particle by:

$$\nu_o = \frac{m_o c^2}{h} \tag{16}$$

while the frequency of the external de Broglie wave was given by:

$$\nu = \nu_o \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{17}$$

Because of its reliance upon special relativity—a local point-event theory *par excellence*—there was no trace and no principle of "uncertainty" in de Broglie's original theory. One unpleasant feature of the theory was the superluminal velocity c^2/v of the wave associated with the particle and describing its behavior. Strangely, this wave should have run away from the particle with a velocity inversely proportional to v!

2. The Wave Packet

In order to overcome the difficulty of superluminal wave velocities, Schrödinger and de Broglie replaced the monochromatic plane wave (13) by a "wave-packet"—a linear superposition of monochromatic waves with frequencies between $\nu^* - \Delta\nu/2$ and $\nu^* + \Delta\nu/2$ centered around ν^* having a group velocity equal to the particle velocity v. No physical mechanism accounting for wave dispersion in an vacuum was ever suggested. Apparently, the very way of constructing the "wave-packet" implies an energy distribution of width ΔE, with a corresponding width Δt in the Fourier transformed time-domain.

The approximate equality:

$$\Delta\nu\Delta t \cong 1 \quad \text{or} \quad \Delta E\Delta t \cong h \tag{18}$$

—a well-known property of Fourier transforms—has to be contrasted with the inequality (11). There is no mystery behind (18) and there is no need for a new principle. The concept of the "wave-packet" has a built-in width ΔE, which is inversely correlated with Δt. This picture, of course, loses its validity for bound states with a discrete energy spectrum. For photons—the "free particles" *par excellence*—the concept of the wave packets and Planck's quantum hypothesis $E = h\nu$ seem to be mutually exclusive.

Since free particles are basically thought as localizable (or even point-like), the initial spread Δx_o in the position of a particle at $t = 0$ is actually ascribed to the wave packet associated with it. Assuming that the free particle has a real Gaussian wave function at $t = 0$ and solving the time-dependent Schrödinger equation, one obtains that after time T has elapsed the wave function is a complex Gaussian (Messiah 1976) with a width Δx given by:

$$(\Delta x)^2 = (\Delta x_o)^2 + \frac{{}^2T^2}{4m_o^2(\Delta x_o)^2} \tag{19}$$

Due to the Gaussian character of the wave function, the Fourier transform of $\Psi(x,t)$ will provide a spread Δp_x in momentum such that:

$$\Delta x \Delta p_x = \frac{\hbar}{2} \tag{20}$$

This equality—within the standard formalism—is a unique property of coherent states (Carruthers and Nieto 1968). These states are called the "closest to classical" states, in contrast with the kq representation of Zak (1972), which corresponds to the "most quantum mechanical" ones. For non-Gaussian wave-packets with a pronounced maximum and a finite spread Δk around the center wave vector k^*, the Fourier transformation predicts qualitatively:

$$\Delta x \cong \frac{1}{\Delta k} \tag{21}$$

Besides the missing factor $\frac{1}{2}$, there is a world of difference between the inequality (5) and the approximate equality (21). The former relationship (5) is derived within the statistical interpretation of the quantum mechanics (QM) of one particle, while (21) is a property of the Fourier transformation. The statistical interpretation of QM, on the other hand, implies the academic "preparation" of a very large number of identical one-particle systems on which "measurements" have to be performed. On top of this, the practically impossible requirement of simultaneous measurements on conjugated, non-commutative "observables" is imposed. Nobody has ever performed such measurements! In general, a thorough measurement of the first observable will so disrupt phase relations that it will serve no physical purpose to subsequently (nay simultaneously!) measure a second observable on the resulting mixture. As remarked by Willis E. Lamb, Jr. (1969), to measure two non-commutative observables F and G simultaneously, one would have to find a potential $V(r,t)$ that was determined by both F and G. This cannot be done in such a way that the desired information emerges from the measurement.

3. Bound Systems

Already in the early years of quantum mechanics it was shown that for the nth state of the harmonic oscillator the following equalities hold:

$$\overline{x^2} = (\Delta x)^2 = \frac{(n+\frac{1}{2})\hbar}{m\omega} = \frac{E_n}{m\omega^2} \tag{22a}$$

$$\overline{p^2} = (\Delta p)^2 = (n+\tfrac{1}{2})m\omega\hbar = E_n m \tag{22b}$$

$$\Delta x \Delta p = (n+\tfrac{1}{2})\hbar = \frac{E_n}{\omega} \tag{22c}$$

As suggested by Fock (1978), relation (22c) is a very general result. If one introduces a quantum number n in the proper way, the formula will be valid not only for an oscillator but for any system in its nth state. This is qualitatively different from the inequality (5) expressing the UP.

For 3-dimensional problems with central potentials of the power-law type:

$$V(r) \sim r^s \tag{23}$$

the conjugated operators r and $p_r = -i(\partial/\partial r + 1/r)$ obey the same commutation relation as x and p_x:

$$[p_r, r] = \frac{\hbar}{i} \tag{24}$$

Therefore (22c) is expected to hold:

$$\overline{r^2}\,\overline{p_r^2} = (\Delta r)^2(\Delta p_r)^2 = \left(\frac{E_n}{\omega}\right)^2 \tag{25}$$

For arbitrary s, the virial theorem (Fock 1978) provides the following relationships between the average potential, kinetic, and total energies of the nth state:

$$\overline{E_{\text{kin}}} = \frac{s}{2}\overline{E_{\text{pot}}} \tag{26a}$$

$$\overline{E_{\text{pot}}} = \frac{2E_n}{s+2} \tag{26b}$$

and (25) takes the form:

$$\overline{r^2}\,\overline{p_r^2} = \overline{E_{\text{pot}}}\,\overline{E_{\text{kin}}}\frac{(s+2)^2}{s^2} = \left(\frac{\omega}{2}\right)^2 = \frac{E_n^2}{\omega^2} \tag{27}$$

Equality (27), valid for bound states governed by power law central potentials provides a *rigorous constraint* on the average kinetic and potential energies of a quantum state. This is a very far cry from the "inaccuracy", "indeterminacy", or "uncertainty" labels attached to inequality (5) (Born et al. 1925). The actual meaning of (25) is that of a constraint imposed on average energy values rather than on variances of r and p_r. The equipartition of average kinetic and potential energies is characteristic of the harmonic oscillator. Remarkably, the equipartition between electric (= potential) and magnetic (= kinetic) parts of the electron self-energy in the torus model of Iida (1974) and Bergman and Wesley (1991) provides the key for the understanding of $\frac{1}{2}$ spin, too. In this model, the electron has a purely electromagnetic nature and (in the first approximation) a radius equal to the Compton length $\hbar/m_o c$. The frequency of the mysterious "Zitterbewegung" is replaced here by the angular velocity of the rotating ring divided by 2π.

Finally, although (27) was derived in the context of Schrödinger's time-independent theory, it could be seen as a dynamic interplay between potential and kinetic energy. This appeals to our intuition, since even in a time-independent formalism, motion is hidden behind the symbol I_z designating angular momentum. The connection between the energy eigenvalues E_n and the quantum action variable

$$J_n = nh = J_n(E_n) \tag{28}$$

exploited in the extended Hamilton-Jacoby theory by Leacock and Padgett (1983a,b; Kim and Choi 1993) compels us to envisage a *limiting energy conversion*, characteristic of every quantized system. Although quantization is a feature of bound systems, it is postulated for free electromagnetic fields, too. The equipartition between electric and magnetic energies in an electromagnetic wave, interpreted as a flux of photons, is, however,

not amenable to intuitive modeling. It appears that electromagnetic waves, propagating independent of a potential gradient, constitute the singular case of pure kinetic energy in physics. This external view is in accordance with their vanishing rest mass and one and only allowed propagation speed.

4. Conclusion

Summing up, there is no need to work with an inequality within the framework of "quantum uncertainty". (In fact, it is always the equality that is used in estimates and more rigorous calculations). This concept invokes the idea to relate Planck's "quantum of action" to the concept of minimal action which we will treat in a subsequent paper. Any "uncertainty" connected with wave packets is constructed by the very model itself and is in contrast with a quantum hypothesis. The equipartition between kinetic and potential energies in bound systems governed by a central power-law potential is a feature based on the virial theorem and finds its correspondence if quantum uncertainties are interpreted as mean values of the involved contributions to the total energy. The "derivation" of the UP via the operator formalism is subject to the severe criticism that the formal definition of position, time, and phase operators is physically meaningless. The physical mechanism behind the historical UP is the ubiquitous conversion of different forms of energy into each other.

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