Homework II solution

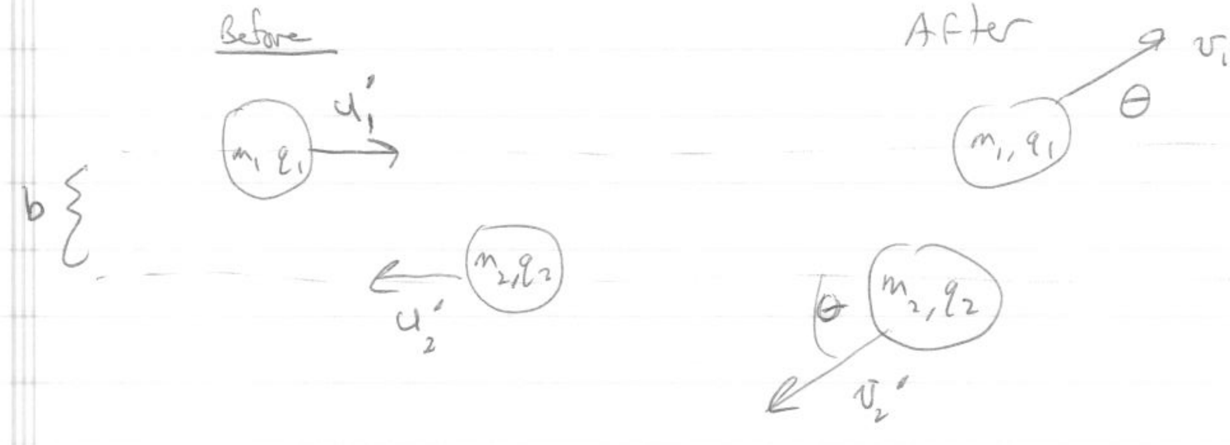
1.) Derive Rutherford scattering formula

Rutherford scattering is the scattering of 2 charged particles which experience the coulomb force (no gravity, strong ~~or~~ or weak force)

$$U(r) = \frac{k}{r} \quad \text{where} \quad k \equiv \frac{q_1 q_2}{4\pi\varepsilon_0}$$

In Lab Frame

In C.M. frame



$b \equiv$ impact parameter = closest distance ~~of~~ m_1 & m_2 ever get to eachother

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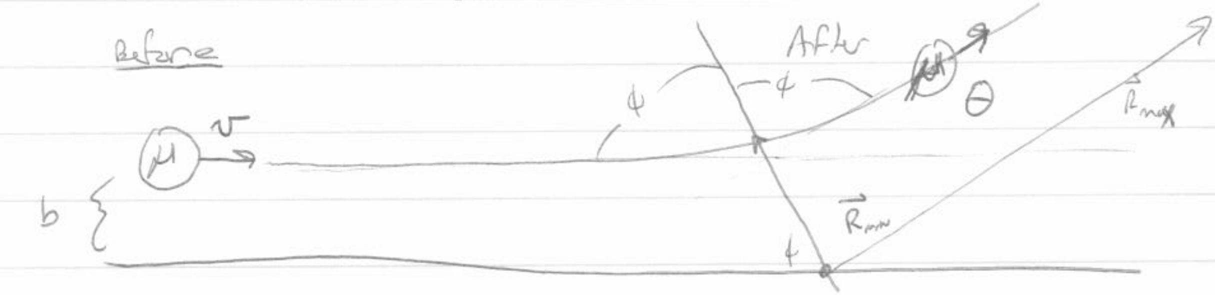
In terms of reduced mass in C.M. frame



$$H = T + U = \frac{1}{2}\mu \dot{R}^2 + \frac{l^2}{2\mu R} + U(r) \equiv E$$

$$\Rightarrow \quad \dot{R} = \pm \sqrt{\frac{2}{\mu}\left(E - U(r)\right) - \frac{l^2}{\mu^2 R^2}}$$

$$l = \mu R^2 \frac{d\phi}{dR}\frac{dR}{dt} \Rightarrow \quad d\phi = \frac{l}{\mu R^2 \dot{R}}\, dR$$

$$\phi = \int d\phi = \int_{R_{min}}^{R_{max}} \frac{l}{\mu R^2} \frac{dR}{\pm\sqrt{\frac{2\left[E - U(r)\right]}{\mu} - \frac{l^2}{\mu^2 R^2}}}$$

$l = b\sqrt{2\mu E}$: cons. of angular momentum

$$\phi = \int_{R_{min}}^{R_{max}} \frac{b\sqrt{2\mu E}}{\mu R^2} \frac{dR}{\sqrt{\frac{2}{\mu}\left[E - \frac{k}{R}\right] - \frac{\left(b\sqrt{2\mu E}\right)^2}{\mu^2 R^2}}}$$

$$= \int_{R_{min}}^{R_{max}} \frac{b\, dR}{R\sqrt{R^2 - \frac{k}{R}\frac{R^2}{E} - b^2}}$$

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1.) Derive Rutherford Scattering formula

$$\phi = \int_{R_{min}}^{R_{max}} \frac{b\,dR}{R\sqrt{\frac{2\mu E R^2}{2\mu E} - \frac{2kR^2\mu^2}{2\mu^2 E R} - \frac{b^2 2\mu E \mu^2 R^2}{2\mu E \mu^2 R^2}}}$$

$$= \int_{R_{min}}^{R_{max}} \frac{b\,dR}{R\sqrt{R^2 - \frac{kR}{E} - b^2}}$$

let $u = \frac{b}{R}$ $\qquad du = -\frac{b}{R^2}\,dR$

$$\phi = \int_{R_{min}}^{R_{max}} \frac{(b/R^2)\,dR}{\sqrt{1 - \frac{k}{RE} - \frac{b^2}{R^2}}} = \int_{R_{min}}^{R_{max}} \frac{-du}{\sqrt{1 - \frac{ku}{bE} - u^2}}$$

if point charges $\Rightarrow$ $R_{min} \sim 0 \Rightarrow u = \infty$

$R_{max} = \infty \Rightarrow u = 0$

$$\phi = \int_{\infty}^{0} \frac{-du}{\sqrt{1 - \frac{ku}{bE} - u^2}} = \int_{0}^{\infty} \frac{du}{\sqrt{1 - \frac{ku}{bE} - u^2}}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{-1}{\sqrt{-a}} \sin^{-1}\left[\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right] + \text{constant}$$

$a = (-1) \qquad b = \left(\frac{-k}{bE}\right) \qquad c = 1$

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1.) Derive Rutherford's scattering formula

$$\phi = \frac{-1}{\sqrt{-(-1)}} \sin^{-1}\left[\frac{2(-1)u + \left(-\frac{k}{bE}\right)}{\sqrt{\left(-\frac{k}{bE}\right)^2 - 4(-1)(1)}}\right]\Bigg|_0^\infty$$

$$= \underbrace{-\sin^{-1}[\infty]}_{\substack{\text{constant} \\ \equiv \frac{\pi}{2}}} + \sin^{-1}\left[\frac{-\frac{k}{bE}}{\pm\sqrt{4 + \left(\frac{k}{bE}\right)^2}}\right]$$

$$\phi = \sin^{-1}\left[\frac{-k}{bE\sqrt{4 + \left(\frac{k}{bE}\right)^2}}\right] + \frac{\pi}{2}$$

$$\sin\left(\phi - \frac{\pi}{2}\right) = \frac{-k/bE2}{\sqrt{1 + \left(\frac{k}{2bE}\right)^2}} = \cancel{-\cos\left(\frac{\theta}{2}\right)}$$

$$\cancel{\theta + 2\phi = \pi}$$

$$\cos(\phi) = \frac{k/2bE}{\sqrt{1 + \left(\frac{k}{2bE}\right)^2}}$$

$$\tan\phi = \frac{1}{\frac{k}{2bE}}$$

$$b = \frac{k}{2E}\tan\phi$$

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1) Derive Rutherford's Scattering Formula

$$2\phi + \theta = \pi \Rightarrow \phi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$b = \frac{k}{2E}\tan\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = \frac{k}{2E}\cot\left(\frac{\theta}{2}\right)$$

we Now have $b = f(\theta)$

so back to the Cross-section equation

$$\sigma(\theta) = \frac{b}{\sin\theta}\frac{db}{d\theta} = \frac{\frac{k}{2E}\cot(\theta/2)}{\sin\theta}\frac{db}{d\theta}$$

$$db = \frac{k}{2E}\, d\left[\cot\left(\frac{\theta}{2}\right)\right] = \frac{k}{2E}\left(\frac{-1}{\sin^2\left(\frac{\theta}{2}\right)}\right)\frac{d\theta}{2}$$

$$\sigma(\theta) = \frac{k}{2E}\,\frac{\cot(\theta/2)}{\sin\theta}\left(\frac{k}{4E}\right)\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}$$

(remember $\frac{db}{d\theta} < 0$ but we want magnitude)

$$\sin(\theta) = \sin\left(\frac{\theta}{2} + \frac{\theta}{2}\right) = 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$\sigma(\theta) = \frac{k^2}{8E^2}\,\frac{\cos(\theta/2)}{\sin(\theta/2)}\,\frac{1}{2\cos(\theta/2)\sin(\theta/2)}\,\frac{1}{\sin^2(\theta/2)}$$

$$= \frac{k^2}{16E^2}\,\frac{1}{\sin^4(\theta/2)}$$