I overview

E.) cross-sections (scattering cross-sections)

(i.) Definition:

$$\sigma(\theta) = \text{differential scattering cross-section} \equiv \frac{\frac{\#\text{ particles scattered}}{\text{solid angle}}}{\frac{\#\text{ incident particles}}{\text{Area}}}$$

Solid Angle $\equiv \Omega$ = surface area of a sphere covered by the detector.
(ie; the detectors area projected onto the surface of a sphere)

A = surface area of detector
r = distance from interaction point to detector

$$\Omega_{max} = \frac{A}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi \text{ steradians}$$

$A_{sphere} = 4\pi r^2 \Rightarrow$ if your detector was a hollow ball.

Units:

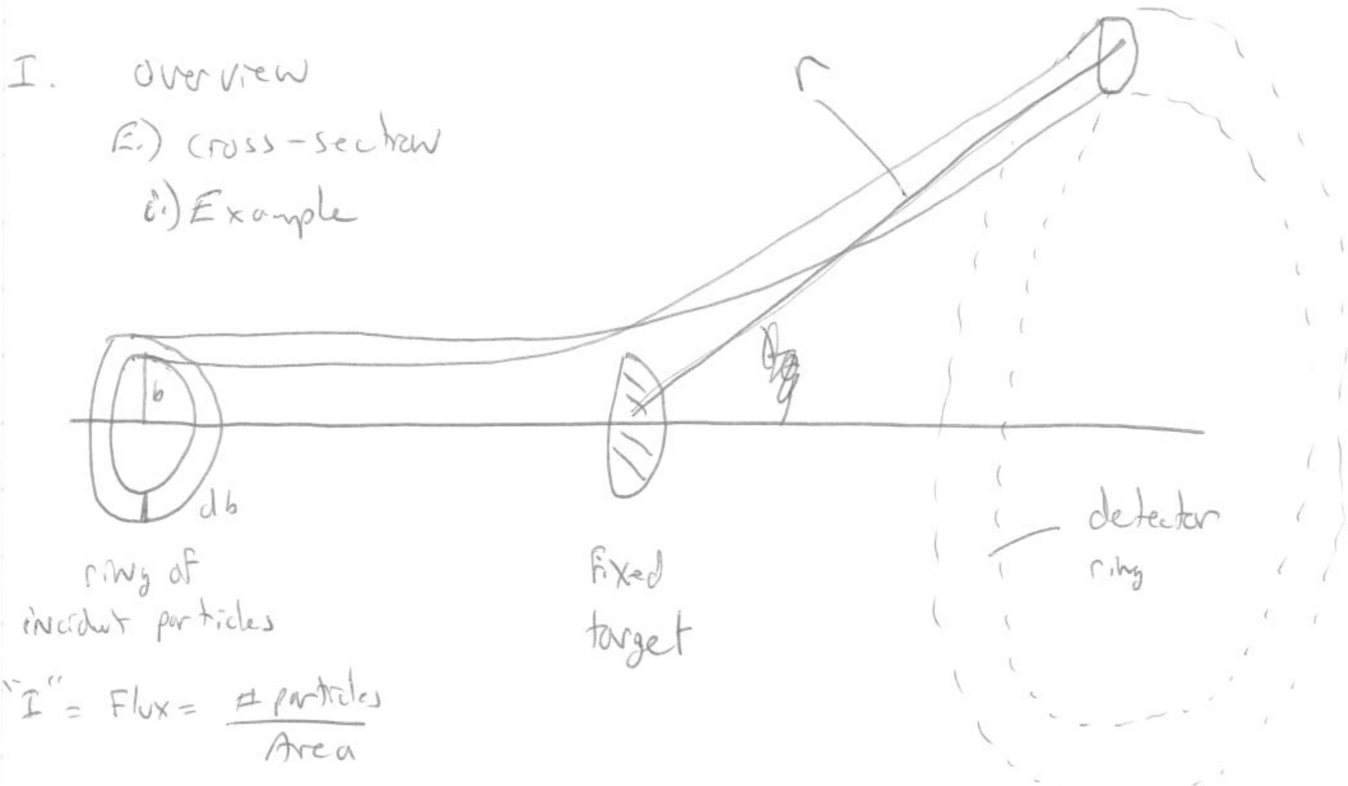
$$\left[\text{Units of } \sigma(\theta)\right] = \frac{\frac{\text{particles}}{\text{steradian}}}{\frac{\text{particles}}{m^2}} = m^2$$

$$1 \text{ barn} = 10^{-28}\ m^2$$

I. overview

E.) cross-section

i.) Example



"I" = Flux = $\frac{\#\text{ particles}}{\text{Area}}$

N_{in} = # particles in = $I \cdot A$ ~~= $I \cdot (2\pi b)(db)$~~

area of ring of incident particles

$dN_{in} = I \, dA = I \, (2\pi b)(db)$

= # particles in a ring of radius b and thickness db

if $dN \equiv$ # particles detected at the detector

then $\sigma(\theta) = \dfrac{\frac{dN}{d\Omega}}{I}$

if $\sigma(\theta)$ & $d\Omega$ are known then

~~$N = \int dN = I$~~

$dN = I\,\sigma(\theta)\, d\Omega = I\,\sigma(\theta)\,\frac{dA}{r^2} = I\sigma(\theta)\frac{r^2 \sin\theta \, d\theta d\phi}{r^2}$

I Overview ~~A~~ E.) Cross-section

~~A~~ i.) definition

if all particles are scattered

Then $\quad dN_{in} = dN$

or $\quad I\,(2\pi b)(db) = I\,\sigma(\theta)\,\sin\theta\, d\theta\, \underbrace{d\phi}_{\text{integrate over } \phi}$

$$= I\,\sigma(\theta)\,\sin\theta\, d\theta\,(2\pi)$$

$\Rightarrow$

$$\sigma(\theta) = \frac{b}{\sin\theta}\,\underbrace{\frac{db}{d\theta}}_{\substack{\text{This term is calculated from} \\ \text{physics description of} \\ \text{process}}} \quad : \text{ classical relation}$$

~~physics~~ ~~descri~~

ii.) example.

Find $\sigma(\theta)$ for an **elastic** collision of 2 impenetrable spheres of radius "a"

~~i.e~~ $\cancel{U(r) = \begin{cases} 0 & r > a \\ \infty & \end{cases}}$

LAB Frame

I overview

i.) X-sect (cross-section)

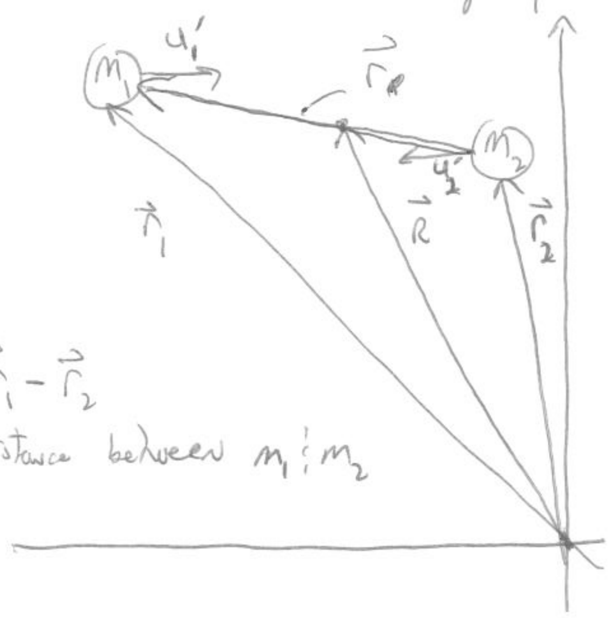
ii.) example

To solve this elastic scattering problem we will describe the scattering process in the center of mass (C.M.) frame and then reduce the 2-body problem to a 1-body problem.

In the C.M. frame

b = impact parameter = smallest distance between m_1 & m_2 during collision

Now Reduce 2-body problem to 1-body problem



$\vec{r}_1$ & $\vec{r}_2$ point to m_1 & m_2

$\vec{R}$ points to the center of Mass

let $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$

$|\vec{r}| \doteq$ distance between m_1 & m_2

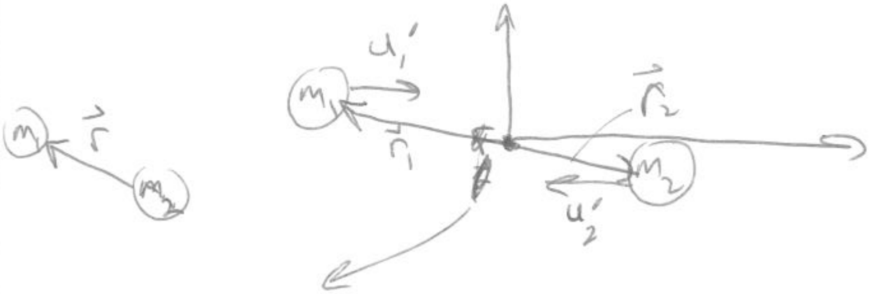
I overview

E.) X-sect

(ii) example

Now simplify the problem by moving the coordinate system to the C.M. ; $\vec{R} = 0$

(Note: your coordinate system is moving at speed v_{cm})



$$\vec{R} = 0 = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$\Rightarrow \quad m_1\vec{r}_1 = -m_2\vec{r}_2$$

~~define a new vector~~ $\cancel{\vec{r} = \vec{r}_1 - \vec{r}_2}$

def of reduced mass comes from ~~stor~~ solving 2 equations:

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$
$$\vec{r}_1 - \vec{r}_2 = \vec{r}$$

$$\vec{r}_1 = \frac{m_2\vec{r}}{m_1+m_2} \quad \text{or} \quad \vec{r}_2 = \frac{m_1(-\vec{r})}{m_1+m_2}$$

$$\mu \equiv \text{reduced mass} = \frac{m_1 m_2}{(m_1+m_2)} \quad \Rightarrow \quad \vec{r}_1 = \frac{\mu}{m_1}\vec{r}$$

$$\vec{r}_2 = \frac{\mu}{m_2}\vec{r}$$

How is this usefull?

you can now recast the problem ~~Rat~~ from a 2 body proble to a 1-body problem

I overview E) x-sect

ii.) example

In general interactions are described in terms of a Lagrangian (Hamiltonian is ~conservation of E)

$$L = T - U$$ where T = kinetic energy of system

U = potential energy of system.

The interaction is then described in terms of a potential (U)

if ~~m_1 = photon , m_2 =~~ Brehmsstrahlung

m_1 = electron and m_2 = proton

then $U(r) = \underbrace{\frac{zZe^2}{r}}_{\text{coulomb}} \underbrace{e^{(-r/a)}}_{\text{Fermi-Thomas}}$ electron screening

$b_{min} = \frac{\hbar}{2\underbrace{Mv}_{p}}$ = minimal impact parameter

$b_{max} \propto \frac{1}{p - p'}$ momentum of e^- before - after

$a \approx 1.4\, a_0 \, z^{-1/3}$

$a_0 = \frac{\hbar^2}{me^2}$ = hydrogen Bohr radius

$$L = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(\vec{r})$$

$$\vec{r}_1 = \frac{\mu}{m_1} \vec{r} \Rightarrow |\dot{\vec{r}}_1|^2 = \left(\frac{\mu}{m_1}\right)^2 |\dot{\vec{r}}|^2 \; ; \; |\dot{\vec{r}}_2|^2 = \left(\frac{\mu}{m_2}\right)^2 |\dot{\vec{r}}|^2$$

$$L = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_1+m_2}\right)^2 |\dot{\vec{r}}|^2 - U(\vec{r})$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} |\dot{\vec{r}}|^2 - U(r) = \underbrace{\frac{1}{2} \mu |\dot{\vec{r}}|^2 - U(r)}_{\text{1-body problem}}$$

I overview F.) X-sect

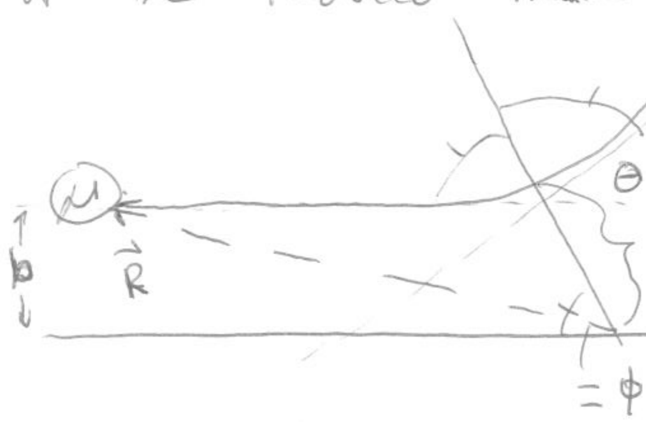
(ii) example

In the Lagrangian formalism, the equations of motion are given by

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$ where q represents one of the coordinates (canonical variables)

To get a X-sect we are interested in finding an expression for $\frac{db}{d\theta}$ (the dependence of the impact parameter on the scattering angle (in this case the CM scattering angle)

so Let's redraw the collision in terms of the reduced mass (μ)



$\Rightarrow \theta + 2\phi = \pi$

$R + a$; $a =$ radius of m_2 = radius of m_1

For headon collision $b = 0$ ~~$\theta = \pi$~~ $\theta = \pi$

For ~~glancing~~ No collision $b > a +$ radius of $m_1 = 2a$ $\theta = 0$

observation : θ gets smaller as b gets bigger

$\Rightarrow \frac{db}{d\theta} < 1$: we will take magnitude in the end

I Overview

ii.) example E.) X-sect

Using ~~a~~ plane polar coordinates (R, ϕ) (not θ yet)

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi$$

$$T = \frac{1}{2}\mu\left(\dot{R}^2 + R^2\dot{\phi}^2\right)$$

$$L = T - U \quad ; \quad U(R) = \begin{cases} 0 & R > a \\ \infty & R \leq a \end{cases}$$

$$L = \frac{1}{2}\mu\left(\dot{R}^2 + R^2\dot{\phi}^2\right) - U(R)$$

Lagranges equation of motion: $\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$

$$0 = \frac{d}{dt}\Big[\underbrace{\mu R^2\dot{\phi}}_{= \text{constant}}\Big]$$

def of constant angular momentum ←

$$l = \text{constant} = \mu R^2\dot{\phi} = \vec{R}\times\vec{p} = \vec{R}\times\mu\vec{v} = R^2\mu\dot{\phi} \;;\; \omega = \frac{v}{r} = \dot{\phi}$$

substitute l into L

$$L = \frac{1}{2}\mu\dot{R}^2 + \frac{\mu}{2}R^2\left(\frac{l}{\mu R^2}\right)^2 - U(r)$$

$$= \frac{1}{2}\mu\dot{R}^2 + \frac{1}{2}\frac{l^2}{\mu R^2} - U(r)$$

I overview E.) X-sect

(ii) example:

summary: we have the Lagrangian

$$L = \frac{1}{2}\mu \dot{R}^2 + \frac{l}{2\mu R^2} - U(r)$$

&

$$l = \mu R^2 \dot{\phi}$$

we want $\frac{db}{d\theta}$; since $R \Leftrightarrow b$

; $\theta \Leftrightarrow \phi$ $(\theta + 2\phi = \pi)$

we should try and find expressions for $d\phi$ in terms of R (b)

$l = \mu R^2 \dot{\phi}$; Trick $\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{dR}\frac{dR}{dt}$

$$\Rightarrow l = \mu R^2 \frac{d\phi}{dR}\frac{dR}{dt} = \mu R^2 \frac{d\phi}{dR}\dot{R}$$

or $$d\phi = \frac{l}{\mu R^2 \dot{R}}\, dR$$

need expression for $\dot{R}$

return to the Lagrangian ~~expt~~ ~~excep~~ except this ~~time~~ write ~~Hamiltonis~~ the Hamiltonian

$$H = T + U = \frac{1}{2}\mu \dot{R}^2 + \frac{l^2}{2\mu R^2} + U(r) = \text{constant} \equiv E$$

I overview E.) x-sect

ii.) example!

solving the Hamiltonian for $\dot{R}$:

$$(\dot{R})^2 = \frac{2}{\mu}\left[E - U(r)\right] - \frac{l^2}{\mu^2 R^2}$$

or $$\dot{R} = \pm\sqrt{\frac{2(E-U(r))}{\mu} - \frac{l^2}{\mu^2 R^2}}$$

units check: $$\left[\frac{l^2}{\mu^2 R^2}\right] = \frac{kg^2\, m^4/s^2}{kg^2\ m^2} = \left(\frac{m}{s}\right)^2 \checkmark$$

$$\left[\frac{2\left[E-U(r)\right]}{\mu}\right] = \frac{N\cdot m}{kg} = \frac{kg\ m^2/s^2}{kg} = \left(\frac{m}{s}\right)^2 \checkmark$$

Now substitute $\dot{R}$ into $d\phi$ equation

$$\int d\phi = \frac{l\ dR}{\mu R^2 \dot{R}} = \int\left(\frac{l}{\mu R^2}\right)\left(\frac{dR}{\pm\sqrt{\frac{2\left[E-U(r)\right]}{\mu} - \frac{l^2}{\mu^2 R^2}}}\right)$$

this is how we take abs of $\frac{db}{d\phi} < 0$

$$\Delta\phi = \int_{r_{min}}^{r_{max}} \frac{l}{\mu R^2}\frac{dR}{\sqrt{\frac{2\left[E-U(r)\right]}{\mu} - \frac{l^2}{\mu^2 R^2}}}$$

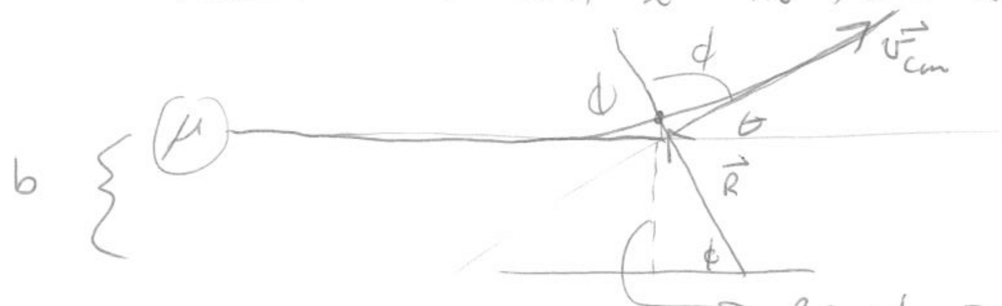
$r_{min} = a \qquad r_{max} = \infty \qquad U(r) = 0 \ ; \ a \le r < \infty$

$$\therefore \Delta\phi = \frac{l}{\cancel{\mu}}\int_a^{\infty}\frac{l\ dR}{R^2\sqrt{2\mu E - \frac{l^2}{R^2}}}$$

I overview E.) x-sect

(ii) example

Trick # 2 : cast ℓ^2 in terms of E



$R \sin\phi \equiv b$

$\vec{\ell} = \vec{r} \times \vec{p} \Rightarrow$

$$|\vec{\ell}| = |\vec{R}|\,|\vec{p}| \sin\phi = R \mu v_{cm} \sin\phi$$

$$E = \text{constant} = \frac{1}{2}\mu v_{cm}^2 \Rightarrow v_{cm} = \sqrt{\frac{2E}{\mu}}$$

or $$\ell = \mu \sqrt{\frac{2E}{\mu}}\, R \sin\phi = \sqrt{2\mu E}\, b$$

$$\therefore \quad \Delta\phi = \int_a^\infty \frac{\ell}{R^2} \frac{dR}{\sqrt{2\mu E - \frac{2\mu E b^2}{R^2}}} = \int_a^\infty \frac{\sqrt{2\mu E}\, b \; dR}{R^2 \sqrt{2\mu E} \left(1 - \frac{b^2}{R^2}\right)^{1/2}}$$

$$= \int_a^\infty \frac{b \; dR}{R\left(R^2 - b^2\right)^{1/2}}$$

units check $\frac{\cancel{[m]}[m]}{[m][m]}$ $\approx$ radians

Integral table $\Rightarrow$

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{-c}} \sin^{-1}\left(\frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}\right)$$

so let $x = R$, $a = 1$, $b = 0$, $c = -b^2$

I overview E.) X-sect

ii.) example

$$\Delta\phi = b \frac{1}{\sqrt{-(-b)^2}} \sin^{-1}\left(\frac{-2b^2}{R\sqrt{0 - 4(1)(-b^2)}} \right)\Bigg|_a^\infty$$

$$= \sin^{-1}(\emptyset) - \sin^{-1}\left(\frac{-b}{a}\right) = \sin^{-1}\left(\frac{b}{a}\right)$$

or $\quad \sin(\underbrace{\Delta\phi}_{\phi}) = \frac{b}{a}$

$$\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{b}{a}$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{b}{a} \Rightarrow b = a\cos\left(\frac{\theta}{2}\right)$$

Finally

$$\sigma(\theta) = \frac{b}{\sin\theta}\frac{db}{d\theta}$$

$$= \frac{a\cos(\theta/2)}{\sin\theta}\, a\left[-\sin\left(\frac{\theta}{2}\right)\right]\frac{1}{2}$$

$$= \frac{a^2}{2}\frac{\cos(\theta/2)\sin(\theta/2)}{\sin\theta}$$

Trig identity:

$$\sin\left(\frac{\theta}{2} + \frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$\sin\theta = 2\cos(\theta/2)\sin(\theta/2)$$

$$\Rightarrow \quad \sigma = \frac{a^2}{2}\left(\frac{1}{2}\right) = \frac{a^2}{4}$$

= cross-section in the C.M. frame for 2 impenetrable balls colliding

I overview

E.) X-sect

iii.) $\sigma(\psi)$: cross-section in LAB frame

The C.M. frame is often chosen to theoretically calculate cross-section, however, the experiments are done in the lab frame. Sometimes you need to compare data with theory that is transformed between 2 frames (Jacobian).

In either frame $N_{out}/_{steradian} = I \int \sigma(\theta)\, d\Omega$ = constant

or $$\underbrace{\sigma(\theta)\, d\Omega}_{\text{In C.M. Fram}} = \underbrace{\sigma(\psi)\, d\Omega'}_{\text{In LAB Frame.}}$$

A non-relativistic transformation

$$\sigma(\theta)\, d\Omega = \sigma(\psi)\, d\Omega'$$

$$\sigma(\theta)\, 2\pi \sin\theta\, d\theta = \sigma(\psi)\, 2\pi \sin\psi\, d\psi$$

$$\Rightarrow \sigma(\psi) = \frac{\sin\theta}{\sin\psi} \underbrace{\frac{d\theta}{d\psi}}\, \sigma(\theta)$$

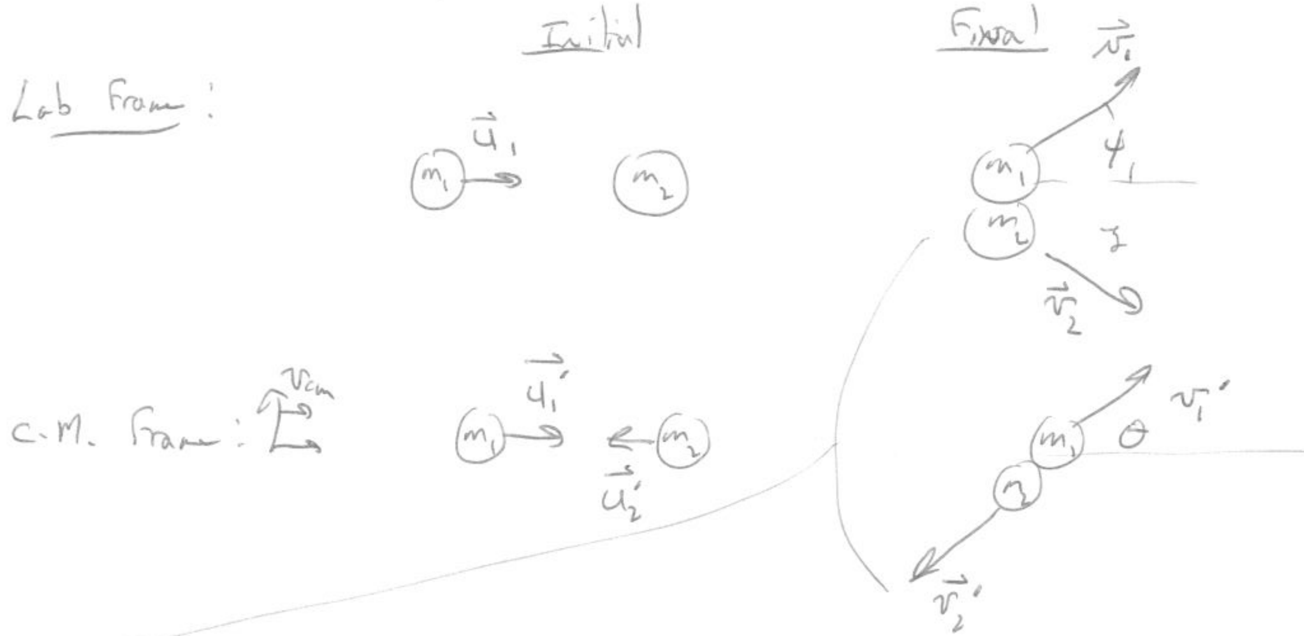
↳ need to know dependence ~~of~~ of θ on ψ

I overview

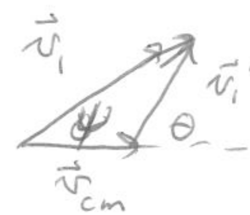
E.) X-sectn

(iii.) $\sigma(\psi)$

Lets return back to our picture of the scattering process



superimpose pictures!



$$v_1' \sin\theta = v_1 \sin\psi$$

$$v_1 \cos\psi = v_{cm} + v_1' \cos\theta$$

trig identities → non-relativistic (Galilean coord. transf.)

$$\tan\psi = \frac{\sin\psi}{\cos\psi} = \frac{v_1'\sin\theta / v_1}{\frac{v_{cm}}{v_1} + \frac{v_1'\cos\theta}{v_1}} = \frac{v_1'\sin\theta}{v_{cm} + v_1'\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta + \left(\frac{v_{cm}}{v_1'}\right)}$$

I overview

iii) $\sigma(\psi)$

For an elastic collision: $u_1' = v_1'$; $u_2' = v_2'$

in c.m. frame; directions change

$$\vec{v}_{cm} = \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2} = \left(\frac{m_1}{m_1+m_2}\right)\vec{u}_1 \quad : \text{Def.}$$

cons. of momentum $\Rightarrow$ $u_1' m_1 = -m_2 u_2'$; P initial in c.M. $= 0$

$$v_1' = u_1' = -\frac{m_2}{m_1} u_2'$$

Galilean coordinate transformation:

$$\vec{u}_1 = \vec{u}_1' + \vec{v}_{cm} = \vec{u}_1' + \left(\frac{m_1}{m_1+m_2}\right) u_1$$

$$\Rightarrow u_1' = \left[1 - \left(\frac{m_1}{m_1+m_2}\right)\right] u_1 = \frac{m_2}{m_1+m_2} u_1$$

$$\therefore \quad v_1' = u_1' = \frac{m_2}{m_1+m_2} u_1$$

$$\Rightarrow \quad \frac{v_{cm}}{v_1'} = \frac{\left(\frac{m_1}{m_1+m_2}\right) u_1}{\left(\frac{m_2}{m_1+m_2}\right) u_1} = \frac{m_1}{m_2}$$

$$\text{or} \quad \tan\psi = \frac{\sin\theta}{\cos\theta + \left(\frac{m_1}{m_2}\right)}$$

I overview

(ii) $J(\psi)$

$$\tan\psi = \frac{\cancel{\cos\theta}\ \sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$

$$\Rightarrow \quad \frac{m_1}{m_2} = \frac{\sin\theta}{\tan\psi} - \cos\theta$$

$$= \frac{\sin\theta\cos\psi - \cos\theta\sin\psi}{\sin\psi}$$

$$= \frac{\sin(\theta-\psi)}{\sin\psi} = \frac{m_1}{m_2} = \text{constant}$$

let $f \equiv \dfrac{\sin(\theta-\psi)}{\sin\psi)} = \text{constant}$

then $df = 0 = \dfrac{\partial f}{\partial\psi}d\psi + \dfrac{\partial f}{\partial\theta}d\theta$: chain rule

or $$\frac{d\theta}{d\psi} = \frac{-\partial f/\partial\psi}{\frac{\partial f}{\partial\theta}}$$

$$\frac{\partial f}{\partial\psi} = \frac{\partial}{\partial\psi}\left[\frac{\sin(\theta-\psi)}{\sin\psi}\right] = -\frac{\cos(\theta-\psi)}{\sin\psi} - \frac{\sin(\theta-\psi)\cos\psi}{\sin\psi}$$

$$\Rightarrow \quad -\partial f/\partial\psi = \frac{\cos(\theta-\psi)}{\sin\psi} + \frac{\sin(\theta-\psi)\cos\psi}{\sin^2\psi}$$

$$\frac{\partial f}{\partial\theta} = \frac{\partial}{\partial\theta}\left[\frac{\sin(\theta-\psi)}{\sin\psi}\right] = \frac{\cos(\theta-\psi)}{\sin\psi}$$

$$\therefore \quad \frac{d\theta}{d\psi} = \frac{\frac{\cos(\theta-\psi)}{\sin\psi} + \frac{\sin(\theta-\psi)}{\sin^2\psi}}{\frac{\cos(\theta-\psi)}{\sin\psi}} = 1 + \frac{\sin(\theta-\psi)\cos(\psi)}{\cos(\theta-\psi)\sin\psi}$$

I overview

iii.) $\sigma(\psi)$

$$\sigma(\psi) = \frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi}\,\sigma(\theta)$$

$$= \frac{\sin\theta}{\sin\psi}\left[1 + \frac{\sin(\theta-\psi)\cos\psi}{\cos(\theta-\psi)\sin\psi}\right]\sigma(\theta)$$

~~measured in lab~~ (sc

given by theorists

ψ: measured in Lab

θ: Not measured

For a usefull equation we need to recast the c.m. angle θ in terms of things we can measure in the lab ~~$(\theta, m_1, m$~~ (ψ, m_1, m_2)

This will be a lot of tedious algebra

lets focus on the numerator of 1 term

$$\sigma(\psi) = \frac{\sin\theta}{\sin\psi}\left[\frac{\cos(\theta-\psi)\sin\psi + \sin(\theta-\psi)\cos\psi}{\cos(\theta-\psi)\sin\psi}\right]\cdot\sigma(\theta)$$

I overview

iii) $\sigma(\psi)$

$$\cos(\theta-\psi)\sin\psi + \sin(\theta-\psi)\cos\psi =$$

$$[\cos\theta\cos\psi + \sin\theta\sin\psi]\sin\psi + [\sin\theta\cos\psi - \cos\theta\sin\psi]\cos\psi$$: Trig ID

$$= \cancel{\cos\theta\cos\psi\sin\psi} + \sin\theta\sin^2\psi - \cancel{\cos\theta\cos\psi\sin\psi} + \sin\theta\cos^2\psi$$

$$= \sin\theta(\sin^2\psi + \cos^2\psi)$$

$$= \sin\theta$$

$$\Rightarrow \sigma(\psi) = \frac{\sin^2\theta}{\sin^2\psi}\frac{1}{\cos(\theta-\psi)\cancel{\sin\psi)}}\sigma(\theta)$$

Now lets work ~~to~~ find expression for $\frac{\sin\theta}{\sin\psi}$

as shown on pg 26-27

$$\frac{m_1}{m_2} = \frac{\sin(\theta-\psi)}{\sin\psi}$$ — lets get this looking like

$$\cos(\psi)\left(\frac{m_1}{m_2}\right) = \cos(\psi)\frac{\sin(\theta-\psi)}{\sin(\psi)}$$: multiply both ~~side~~ sides by $\cos(\psi)$

add $\cos(\theta-\psi)$ to both sides

$$\cos(\theta-\psi) + \cos(\psi)\frac{m_1}{m_2} = \cos(\theta-\psi) + \cos\psi\frac{\sin(\theta-\psi)}{\sin\psi}$$

$$\frac{m_1}{m_2}\cos(\psi) + \cos(\theta-\psi) = \frac{\sin(\psi)\cos(\theta-\psi) + \cos(\psi)\sin(\theta-\psi)}{\sin\psi}$$

$$= \frac{\sin\theta}{\sin\psi}$$

I overview

iii.) $\sigma(\psi)$

Now we have:

$$\sigma(\psi) = \left[\frac{m_1}{m_2}\cos(\psi) + \cos(\theta-\psi)\right]^2 \frac{\sigma(\theta)}{\cos(\theta-\psi)}$$

Now we need to get rid of $\cos(\theta-\psi)$ term and everything is in terms of Lab frame observables

re-using term: $\frac{m_1}{m_2} = \frac{\sin(\theta-\psi)}{\sin(\psi)}$

$$\Rightarrow \quad \sin(\theta-\psi) = \frac{m_1}{m_2}\sin\psi$$

square both sides

$$\sin^2(\theta-\psi) = \left(\frac{m_1}{m_2}\right)^2 \sin^2\psi$$

$$1-\cos^2(\theta-\psi) = \left(\frac{m_1}{m_2}\right)^2 \sin^2\psi$$

$$\Rightarrow \cos^2(\theta-\psi) = 1-\left(\frac{m_1}{m_2}\right)^2 \sin^2\psi$$

Finally!

$$\sigma(\psi) = \frac{\left[\frac{m_1}{m_2}\cos\psi + \sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\psi}\right]^2}{\sqrt{1-\left(\frac{m_1}{m_2}\right)^2\sin^2\psi}}\,\sigma(\theta)$$

we can now transform $\sigma(\theta)$ to $\sigma(\psi)$ and compare with experiment