Homework I Solution

1.) Given the Maxwell - Boltzman Distribution

$$N(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$$

a.) Find $\langle v \rangle$

$$\langle v \rangle = \int_0^\infty N(v)\, v\, dv$$

$$= \int_0^\infty 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}\, v\, dv$$

let $t = \frac{mv^2}{2kT}$ $\quad dt = \frac{mv\,dv}{kT}$

$$\Rightarrow \quad v^2 = \left(\frac{2kT}{m}\right) t \qquad \cancel{k}\, v\,dv = \left(\frac{kT}{m}\right) dt$$

$$\langle v \rangle = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty \left(\frac{2kT}{m}\right) t\, e^{-t} \left(\frac{kT}{m}\right) dt$$

$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} 2\left(\frac{kT}{m}\right)^2 \int_0^\infty t e^{-t}\, dt$$

$$\Gamma\left(\frac{n+3}{2}\right) \equiv \int_0^\infty t^{\frac{n+1}{2}} e^{-t}\, dt \Rightarrow n = 1$$

$$\text{or} \quad \Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt$$

$$\langle v \rangle = 4\pi\left(\cancel{\frac{m}{2\pi kT}}\right)\left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \cancel{2}\left(\cancel{\frac{kT}{m}}\right)\left(\frac{kT}{m}\right) \Gamma(2)$$

$$= 4\left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}}\left(\frac{kT}{m}\right)(1)$$

$$= \left(\frac{16\, m\, k^2 T^2}{2\pi kT m^2}\right)^{\frac{1}{2}} = \left(\frac{8\, kT}{\pi m}\right)^{\frac{1}{2}}$$

Homework I Solution

1.) b.) show that the energy fluctuation:

$$\frac{1}{4}m^2 \langle (v^2 - \langle v^2 \rangle)^2 \rangle = \frac{3}{2}(kT)^2$$

$$\langle (v^2 - \langle v^2\rangle)^2 \rangle = \langle v^4 - 2v^2\langle v^2\rangle + (\langle v^2\rangle)^2 \rangle$$

$$= \langle v^4 \rangle - 2(\langle v^2\rangle)^2 + (\langle v^2 \rangle)^2$$

$$= \langle v^4 \rangle - (\langle v^2 \rangle)^2$$

looks like we need a general expression for $\langle v^n \rangle$

$$\langle v^n \rangle = \int_0^\infty N(v)\, v^n\, dv$$

$$= \int_0^\infty 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} v^n\, dv$$

$$= 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty v^{n+1} e^{-mv^2/2kT}\, v\,dv$$

let $t = \frac{mv^2}{2kT}$ $\quad dt = \frac{mv\,dv}{kT}$

$\Rightarrow v^2 = \left(\frac{2kT}{m}\right) t \qquad v\,dv = \left(\frac{kT}{m}\right) dt$

$v = \left(\frac{2kT}{m}\right)^{1/2} t^{1/2} \rightarrow v^{n+1} = \left(\frac{2kT}{m}\right)^{\frac{n+1}{2}} t^{\frac{n+1}{2}}$

$v^n = \left(\frac{2kT}{m}\right)^{n/2} t^{n/2} \qquad v^n v = \left(\frac{2kT}{m}\right)^{n/2} t^{n/2} \left(\frac{2kT}{m}\right)^{1/2} t^{1/2}$

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1) b.) $$\langle v^n \rangle = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \int_0^\infty \left(\frac{2kT}{m}\right)^{\frac{n+1}{2}} t^{\frac{n+1}{2}} e^{-t} \left(\frac{kT}{m}\right) dt$$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{kT}{m}\right) \left(\frac{2kT}{m}\right)^{\frac{n+1}{2}} \underbrace{\int_0^\infty t^{\frac{n+1}{2}} e^{-t}\, dt}_{= \Gamma\left(\frac{n+3}{2}\right)}$$

$$\langle v^n \rangle = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{kT}{m}\right) \left(\frac{2kT}{m}\right)^{\frac{n+1}{2}} \Gamma\left(\frac{n+3}{2}\right)$$

$$\Gamma\left(z + \tfrac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2z-1)}{2^n}\sqrt{\pi} = \frac{(2z-1)!!}{2^n}\sqrt{\pi}$$

$$\langle v^2 \rangle = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{kT}{m}\right) \left(\frac{2kT}{m}\right)^{3/2} \Gamma\left(\tfrac{5}{2}\right)$$

$$\Gamma\left(\tfrac{5}{2}\right) = \Gamma\left(2 + \tfrac{1}{2}\right) = \frac{3!!}{2^2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$

$$\langle v^2 \rangle = \frac{4\pi}{\pi\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{3/2} \left(\frac{kT}{m}\right) \left(\frac{2kT}{m}\right)^{3/2} \frac{3\sqrt{\pi}}{2} = 3\,\frac{kT}{m}$$

$$\langle v^4 \rangle = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{kT}{m}\right) \left(\frac{2kT}{m}\right)^{5/2} \Gamma\left(\tfrac{7}{2}\right)$$

$$\Gamma\left(\tfrac{7}{2}\right) = \Gamma\left(3 + \tfrac{1}{2}\right) = \frac{5!!}{2^3}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$

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1.) b.)

$$\langle v^4 \rangle = \frac{4\pi}{\pi\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{3/2} \left(\frac{kT}{m}\right)\left(\frac{2kT}{m}\right)^{3/2} \left(\frac{2kT}{m}\right) \frac{15\sqrt{\pi}}{8}$$

$$= 15 \left(\frac{kT}{m}\right)^2$$

$$\langle v^4 \rangle - \left(\langle v^2 \rangle\right)^2 = 15\left(\frac{kT}{m}\right)^2 - \left(3\frac{kT}{m}\right)^2$$

$$= 6\left(\frac{kT}{m}\right)^2$$

$$\frac{1}{4}m^2 \langle (v^2 - \langle v^2 \rangle)^2 \rangle = \frac{1}{4} m^2\, 6\left(\frac{kT}{m}\right)^2 = \frac{3}{2}(kT)^2$$