Austin Dollar Homework 4 Q1 9/19/21

rate B of gamma distribution $\Gamma(\text{shape}=3, \text{rate})$

$$f(x|\alpha, B) = \frac{B^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-Bx}$$ maximum likilihood estimator:

PDF: $F(x|\alpha, B) = \frac{B^{\alpha} x^{\alpha-1} e^{-Bx}}{\Gamma\alpha}$

likihood: $L(B\alpha|x) = \prod_{i=1}^{n} \frac{B^{\alpha} x_i^{\alpha-1} e^{-Bx_i}}{\Gamma(\alpha)}$

Log likihood: $\ell(B\alpha|X) = \log\left(\prod_{i=1}^{n} \frac{B^{\alpha} x_i^{\alpha-1} e^{-Bx_i}}{\Gamma(\alpha)}\right)$

$$= \sum_{i=1}^{n} \log\left(\frac{B^{\alpha} x_i^{\alpha-1} e^{-Bx_i}}{\Gamma(\alpha)}\right)$$

$$= \sum_{i=1}^{n} \log(B^{\alpha}) + \sum_{i=1}^{n} \log(x_i^{\alpha-1}) + \sum_{i=1}^{n} \log(e^{-Bx_i}) - \sum_{i=1}^{n} \log(\Gamma(\alpha))$$

$$= \sum_{i=1}^{n} \alpha \log(B) + \sum_{i=1}^{n} (\alpha-1) \log(x_i) + \sum_{i=1}^{n} -Bx_i \log(e) - \sum_{i=1}^{n} \log(\Gamma(\alpha))$$

$$= \sum_{i=1}^{n} \alpha \log(B) + (\alpha-1) \sum_{i=1}^{n} (\log(x_i)) + -B \sum_{i=1}^{n} (x_i) - \log(\Gamma(\alpha))$$

$$= n\alpha \log(B) + (\alpha-1) \sum_{i=1}^{n} \log(x_i) - n \log(\Gamma(\alpha)) - B \sum_{i=1}^{n} x_i$$

Solve for $\frac{\partial \ell}{\partial B} = \emptyset$:

$$\frac{\partial \ell B(\alpha, x)}{\partial B} = \frac{\partial}{\partial B}\left[n\alpha \log(B) + \cancel{(\alpha-1)\log(x_i)} - n\log(\Gamma(\alpha)) - B\sum_{i=1}^{n} x_i\right]$$

$$= \frac{n\alpha}{B} - \sum_{i=1}^{n} x_i = 0$$

$$\frac{n\alpha}{B} = \sum_{i=1}^{n} x_i$$

$$n\alpha = B\left(\sum_{i=1}^{n} x_i\right)$$

$$\frac{n\alpha}{\sum_{i=1}^{n} x_i} = B$$

$$\hat{B} = \frac{3n}{\sum_{i=1}^{n} x_i}$$