Switches

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

* This would be "off" until a!

$$\ddot{y} + 2\dot{y} + y = \underbrace{e^{-2t} + t^2}_{g(t)}$$

In this case, we will start the forcing at a

$$u(t-a)g(t)$$

Now, what if we wanted to turn a forcing off?

turning on　　　　turning off

$$\left(U(t-a) - U(t-b)\right) g(t)$$

$$\left\{\begin{array}{lll} 0 & t < a & \text{off until } a \\ 1 & a < t < b & \text{on from } a \text{ to } b \\ 0 & b < t & \text{off from } b \text{ until } \infty \end{array}\right\}$$

Second Shift Theorem

Laplace Transform

$$g(t)U(t-a) = e^{-as} L[g(t+a)]$$

$$U(t) \left\{\begin{array}{ll} 0 & t < a \\ 1 & t > a \end{array}\right\}$$

Laplace Inversion

$$f(t-a)U(t-a) = e^{-as} F(s)$$

Example:

$$t\,U(t-2) = e^{-2s} g(t+a)$$

$$= e^{-2s} \mathcal{L}\{t + 2)$$

$$= e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

Example:

$$\mathcal{L}(t^2 U(t-3)) = e^{-3s}\,\mathcal{L}\{g(t+3))\}$$

$$= e^{-3s}\,\mathcal{L}\{(t+3)^2\}$$

$$= e^{-3s}\,\mathcal{L}\{t^2 + 6t + 9\}$$

$$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$$

Example

$$Y(s) = \frac{e^{-3s}}{s^2+1} = \underbrace{e^{-3s}}_{e^{-as}} \underbrace{\frac{1}{s^2+1}}_{F(s)}$$

$$F(s) = \frac{1}{s^2+1}$$

$$f(t) = \sin t$$

$$f(t-3)$$

$$\boxed{\sin(t-3)\,U(t-3)}$$

Example

$$\ddot{y} + 4\dot{y} + 4y = e^{-2t} + te^{-2t}$$

$y(0) = 0, \quad \dot{y}(0) = 0$

turns on at $t = 1$
turns off at $t = 3$

$$\ddot{y} + 4\dot{y} + 4y = (\underbrace{e^{-2t} + te^{-2t}}_{g(t)})[U(t-1) - U(t-3)]$$

$\downarrow \mathcal{L}$

$$s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 4(sY(s) - \cancel{Y(0)}^{0}) + 4Y(s) = \ldots$$

$$Y(s)(\underbrace{s^2 + 4s + 4}_{(s+2)^2}) = \mathcal{L}(f(t))$$

$$g(t) = e^{-2t} + te^{-2t}$$

$$U(t-1)g(t) - U(t-3)g(t)$$

* multiply the switch out

$$U(t-1)\,g(t) - U(t-3)\,g(t)$$ * Later

$$\downarrow$$

$$\mathcal{L}\left(\left(e^{-2t} + te^{-2t}\right)U(t-1)\right)$$

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* Using Second Shift theorem

$$\mathcal{L}\{g(t)U(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$$= e^{-s}\,\mathcal{L}\left(g\left(e^{-2(t+1)} + (t+1)\,e^{-2(t+1)}\right)\right)$$

$$= e^{-s}\,\mathcal{L}\left(g\left(e^{-2t-2} + (t+1)\,e^{-2t-2}\right)\right)$$

$$= e^{-s}\,\mathcal{L}\left(e^{-2t-2}\left(1 + (t+1)\right)\right)$$

$$= e^{-s}\,\mathcal{L}\left(e^{-2t-2}(t+2)\right)$$

$$= e^{-s}\,\mathcal{L}\left(e^{-2}e^{-2t}(t+2)\right)$$

$$= e^{-s}e^{-2}\,\mathcal{L}\left(e^{-2t}(t+2)\right)$$

$$= e^{-s}e^{-2}\,\mathcal{L}\left(te^{-2t} + 2e^{-2t}\right) = e^{-s}e^{-2}\left(\frac{1}{(s+2)^2} + 2\left(\frac{1}{s+2}\right)\right)$$

$$= e^{-s}e^{-2}\left(\frac{1}{(s+2)^2} + 2\left(\frac{1}{s+2}\right)\right)$$

$$Y(s)(s+2)^2 = e^{-s}e^{-2}\left(\frac{1}{(s+2)^2} + 2\left(\frac{1}{s+2}\right)\right) - \underbrace{e^{-3s}e^{-6}\left(\frac{1}{(s+2)^2} + \frac{4}{s+2}\right)}_{\text{From switch close. Same process}}$$