Let's learn how to solve systems of linear ODE's $\vec{x}' = A\vec{x}$

We'll start by looking for a solution of the form $\vec{x} = \vec{v}e^{\lambda t}$

($\vec{v}$ is a constant vector)

(e.g, $\vec{x}(t) = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} e^t = \begin{bmatrix} e^t \\ \frac{e^t}{2} \end{bmatrix}$

Then, $\vec{x}' = \vec{v}\lambda e^{\lambda t}$

And $A\vec{x} = A(\vec{v}e^{\lambda t}) = e^{\lambda t}(A\vec{v})$

For $\vec{x}'$ to equal $A\vec{x}$, we must have

$$e^{\lambda t}\lambda\vec{v} = e^{\lambda t}A\vec{v}$$

i.e. $e^{\lambda t}(A\vec{v} - \lambda\vec{v}) = \vec{0}$

i.e. $\boxed{A\vec{v} = \lambda\vec{v}}$ since $e^{\lambda t}$ is never zero!

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

That is, λ must be an eigenvalue of A, with a corresponding eigenvector!

Example. Find the general solution to

$$\vec{x}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \vec{x}$$

Find eigenvalues of A!

Find characteristic polynomial $\det(A - \lambda I)$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} =$$

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} +$$

$$4\begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

Find roots:

$$\boxed{-\lambda^3+2\lambda^2+5\lambda-6}$$

Try Easy roots, say $\lambda=1$

$-1+2+5-6=\underset{\sim}{0}$, so $\lambda=1$ is a root

$(\lambda-1)(-\lambda^2+\lambda+6)$ Hard to find quadratic!

$(\lambda-1)(\lambda+3)(\lambda-2)$

Eigenvalues! $\boxed{\lambda=1}$ $\boxed{\lambda=-2}$ $\boxed{\lambda=3}$

Next, find eigenvectors:

$\boxed{\lambda = 1}$ Solve $[A - \lambda I \mid 0]$

$$\left[\begin{array}{ccc|c} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

do ref until you got a row of zeros on bottom

$\downarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here, we've got

$x + z = 0$

$y - 4z = 0$

Three unknowns, two equations. So, let's pick a free variable!

Let $z = 1$, $x = -1$, $y = 4$

$$\begin{array}{cc} V = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

If you were to pick a different free value, you'd have a "different matrix" for say, but it would just be a constant multiple. And remember, for eigenvectors, that is no issue at all.

$\boxed{\lambda_2 = -2}$

$$\left[\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

Ref down to zero row, and 1's on pivot

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + z = 0$
$y - z = 0$

Set $x = 1$, $z = -1$
$y = -1$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$\boxed{\lambda_3 = 3}$

$$\left[\begin{array}{ccc|c} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \xrightarrow{\text{Ref}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$x - z = 0$
$y - 2z = 0$
Set $x = 1$, $y = 2$, $z = 1$

So linearly independent solutions are:

$$\vec{x}_1 = \vec{v}_1 e^{\lambda_1 t} = e^{t}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = \vec{v}_2 e^{\lambda_2 t} = e^{-2t}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{x}_3 = \vec{v}_3 e^{\lambda_3 t} = e^{3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

General Solution

$$x(t) = c_1 e^{t}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Example with complex eigenvalues

Example. Find the general solution to

$$\vec{x}' = \begin{vmatrix} 6 & -1 \\ 5 & 2 \end{vmatrix} \vec{x}$$

Eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 12 + 5$$

$$= \lambda^2 - 8\lambda + 17$$

$$\frac{8}{2} = 4^2 = 16$$

$$= (\lambda - 4)^2 + 1$$

$$\boxed{\lambda = 4 \pm i}$$

Eigenvectors

only really need one of the pairs. pick $\boxed{\lambda = 4 + i}$

$$\left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

Solution

$$\vec{x}(t) = e^{(4+i)t}\begin{bmatrix} 1 \\ 2-i \end{bmatrix} \quad \text{is a } \underline{\text{Solution}} \rightarrow \underline{\text{Complex}}$$

Let's use the real & imaginary parts to find two linearly independent real-valued solutions:

Let's use euler's identity

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\vec{x}(t) = e^{4t}\cdot e^{it}\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$= e^{4t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

$$= \begin{bmatrix} e^{4t}\cos t + ie^{4t}\sin t \\ 2e^{4t}\cos t - ie^{4t}\cos t + 2ie^{4t}\sin t + e^{4}\sin t \end{bmatrix}$$

$$= \underbrace{e^{4t}\begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix}}_{x_1(t)} + \underset{x_2(t)}{ie^{4t}\begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}}$$

Solution

$$\vec{x}(t) = c_1 e^{4t}\begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}$$