

Fourier Series

We can break any periodic wave into cos and sin waves!

There are four ways to write Fourier Series:

① Full Sin/Cos F-series

$$\underline{f(t)} = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(n\pi t/L) + b_n \sin(n\pi t/L)$$

↑
Fourier Approximation. $f(t)$ is an approximation of the function

1) Find a_0

$$a_0 = \frac{1}{L} \int_{-L}^L f(t) dt$$

2) Find a_n

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos(n\pi t/L) dt$$

3) Find b_n

$$b_n = \frac{1}{L} \int_{-L}^L f(t) \sin(n\pi t/L) dt$$

- One complete period is T . Half of that is L

②

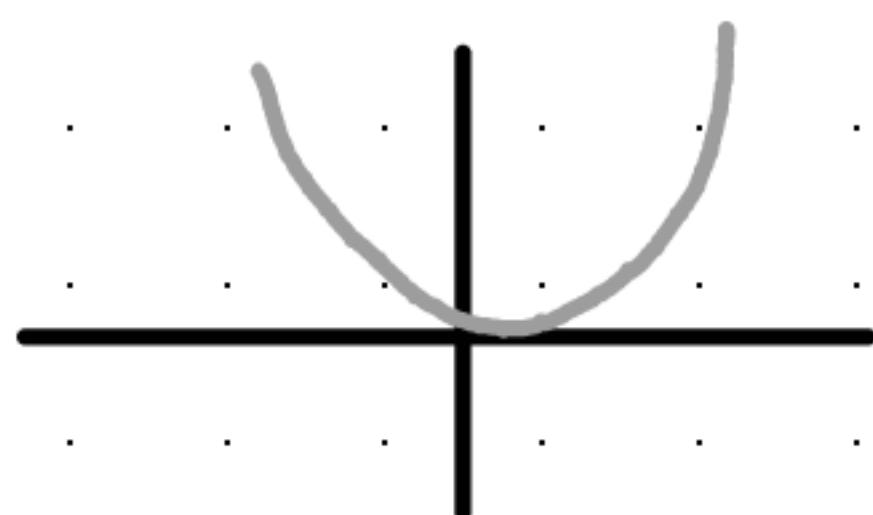
Cosine F-Series (when we have even function)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(n\pi t/L)$$

In an even function, $f(-t) = f(t)$.

ex: $f(t) = t^2$
 $f(-t) = (-t)^2 = t^2$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos(n\pi t/L) dt$$



← Even function

$$\int_{-L}^L = \cancel{\int_{-L}^0} + 2 \int_0^L$$

This is
where that
comes from!

③

Sine f-Series (when we have odd function)

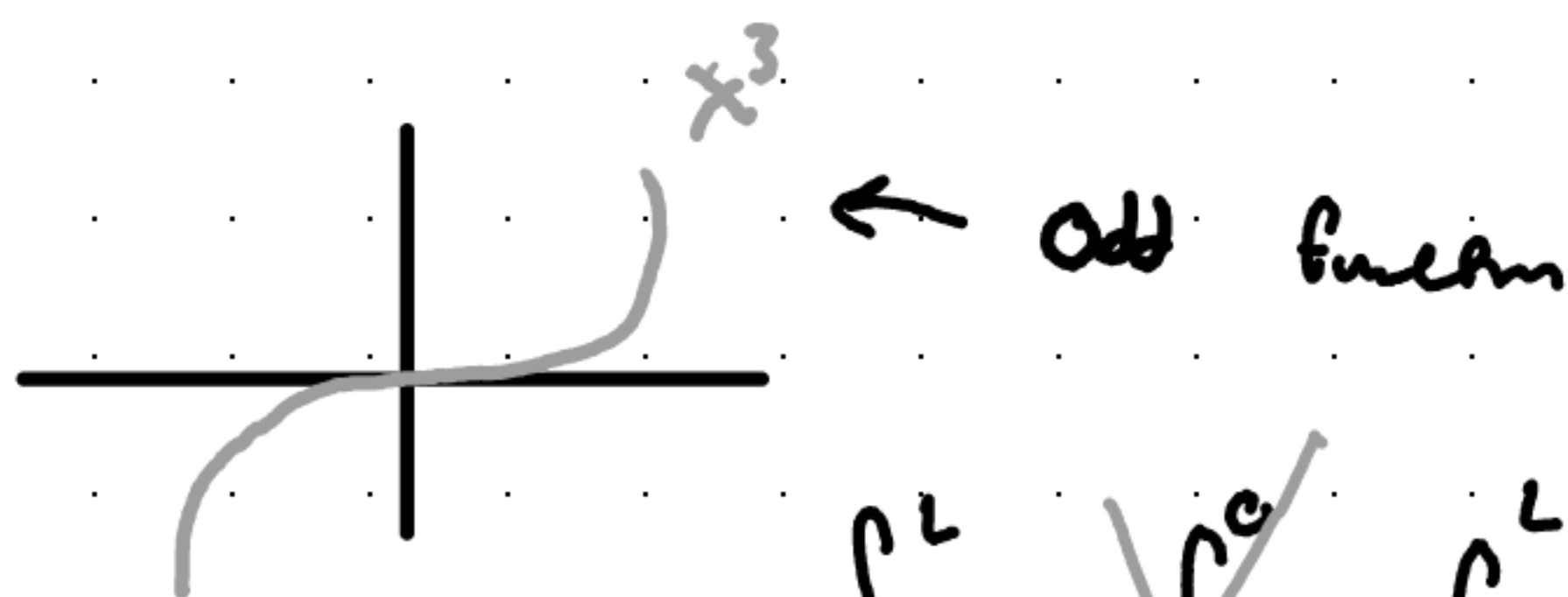
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} b_n \sin(n\pi t/L)$$

In an odd function, $f(-t) = -f(t)$.

ex: $f(t) = t^3$

$$f(-t) = (-t)^3 = -t^3$$

$$b_n = \frac{2}{L} \int_0^L f(t) \sin(n\pi t/L) dt$$



$$\int_{-L}^L = \cancel{\int_{-L}^0} + 2 \int_0^L$$

This is
where that
comes from!

④

Imaginary Case

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{i n \pi t}{L}}$$

$$C_n =$$

Integration By Parts

$$\int u dv = uv - \int v du$$

Often $\begin{cases} u = t^p \\ dv = \sin \dots \text{ or } \cos \dots \end{cases} \xrightarrow{\int} v$

Integrals of Sin and Cos

$$\int \sin(at) dt = -\frac{1}{a} \cos(at)$$

$$\int \cos(at) dt = \frac{1}{a} \sin(at)$$

!! Special Trig To Look out for!!

$$\sin(n\pi) = 0$$

$$\cos(n\pi) = (-1)^n$$

Example One

$$\ddot{y} + \dots = t \quad \xrightarrow{[-\pi, \pi]}$$

$$T = 2\pi$$

1) Check if odd or even
odd : use sine form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(n\pi t/L)$$

$$b_n = \frac{2}{L} \int_0^L f(t) \sin(n\pi t/L) dt$$

$$L = \frac{2\pi}{2} = \pi$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \underbrace{t}_{u} \underbrace{\sin\left(\frac{n\pi t}{\pi}\right)}_{dv} dt$$

IBP

$$\int u dv = uv - \int v du$$

$$u = t \longrightarrow du = dt$$

$$dv = \sin(nt) dt \xrightarrow{\int} v = \int \sin nt dt = -\frac{1}{n} \cos nt$$

$$b_n = \frac{2}{\pi} \left(t \left(-\frac{1}{n} \cos nt \right) \Big|_0^{\pi} \right) - \int_0^{\pi} \left(-\frac{1}{n} \right) \cos nt dt$$

$$b_n = \frac{2}{\pi} \left(-\frac{t}{n} \cos nt \Big|_0^{\pi} \right) + \left(\frac{1}{n^2} \sin(nt) \Big|_0^{\pi} \right)$$

$$b_n = \frac{2}{\pi} \left(-\frac{\pi}{n} \cos n\pi - \frac{0}{n} \cos 0 \right)$$

$$b_n = \frac{2}{\pi} \left(-\frac{\pi}{n} \cos n\pi \right)$$

$$t \quad f(t) = \sum_{h=1}^{\infty} -\frac{2}{h} (-1)^h \sin ht$$

$$\hookrightarrow -\frac{2}{1}(-1)^1 \sin t + \left(-\frac{2}{2}\right)(-1)^2 \sin 2t \dots$$

Example:

$$M=1$$

$$C=2$$

$$K=1$$

Mass damper spring

$$f(t) = t^2 + \pi \quad -\pi < t < \pi$$

$$M\ddot{y} + C\dot{y} + K = f(t)$$

$$\ddot{y} + 2\dot{y} + y = \underset{f(t)}{t^2 + \pi}$$

Use Fourier Series for LHS, and then solve:

1) Check if odd or even

$$f(-t) = -t^2 + \pi = t^2 + \pi \quad (\text{The Same})$$

• Even, will use cosine form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t/L$$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos n\pi t/L dt$$

$$T = 2\pi, L = \pi$$

$$a_0 = \frac{2}{L} \int_0^L f(t) dt$$

$$\uparrow$$
$$\cos(0) = 1$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (t^2 + \pi) dt$$

$$a_0 = \frac{2}{\pi} \left(\frac{t^3}{3} + \pi t \right) \Big|_0^{\pi} \rightarrow \frac{2}{\pi} \left(\frac{\pi^3}{3} + \pi^2 \right)$$
$$= \frac{2}{3} \pi^2 + 2\pi$$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos(n\pi t/L) dt$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (t^2 + \pi) (n\pi t/\pi) dt$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \underbrace{t^2 \cos(nt)}_{I_1} dt + \int_0^{\pi} \underbrace{\pi \cos(nt)}_{I_2} dt$$

$$a_n = \frac{2}{\pi} (I_1 + I_2)$$

$$I_2 = \pi \int_0^{\pi} \cos nt dt \rightarrow \pi \left(\frac{1}{n} \sin nt \Big|_0^{\pi} \right)$$



$$I_1 = \int_0^{\pi} \underbrace{t^2}_u \underbrace{\cos(nt)}_{dv} dt \quad \text{IBP} \quad UV - \int v du$$

$$u = t^2 \quad du = 2t dt$$

$$dv = \cos nt dt \quad v = \frac{1}{n} \sin \pi t$$

$$\frac{t^2}{n} \sin(nt) - \frac{2}{n} \int_0^{\pi} \underbrace{t \sin(nt)}_{dv} dt$$

$$I_1 = \frac{t^2}{n} \sin(nt) - \frac{2}{n} \int_0^{\pi} \underbrace{t}_u \underbrace{\sin(nt)}_{dv} dt$$

$$u = t \quad dv = dt$$

$$dv = \sin(nt) \quad v = -\frac{1}{n} \cos(nt)$$

$$uv - \int v du$$

$$-\frac{t}{n} \cos(nt) + \underbrace{\frac{1}{n} \int_0^{\pi} \cos nt \, dt}_{\frac{1}{n^2} \sin(nt)}$$

$$-\frac{t}{n} \cos(nt) + \frac{1}{n^2} \sin(nt)$$

$$I_1 = \frac{t^2}{n} \sin(nt) - \frac{2}{n} \left(-\frac{t}{n} \cos nt + \frac{1}{n^2} \sin nt \right)$$

$$a_n = \frac{2}{\pi} (I_1 + I_2)$$

Evaluate now! ↘

$$a_n = \frac{2}{\pi} \left(\frac{t^2}{n} \sin(nt) - \frac{2}{n} \left(-\frac{t}{n} \cos nt + \frac{1}{n^2} \sin nt + 0 \right) \right) \Big|_0^{\pi}$$

Evaluate now! ↘

$$a_n = \frac{2}{\pi} \left(\frac{t^2}{n} \sin(n\pi t) - \frac{2}{n} \left(-\frac{t}{n} \cos n\pi t + \frac{1}{n^2} \sin n\pi t + 0 \right) \right) \Big|_0^\pi$$

$$= \frac{2}{\pi} \left(0 - \frac{2}{n} \left(-\frac{\pi}{n} \cos n\pi + \frac{1}{\pi^2} \sin n(\pi) \right) + 0 \right)$$

$$a_n = \frac{4}{n^2} \cos n\pi \rightarrow \frac{4}{n^2} (-1)^n$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t/L)$$

$$f(t) = \frac{\frac{2}{3}\pi^2 + 2\pi}{2} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} (-1)^n \right) \cos\left(\frac{n\pi t}{\pi}\right)$$

$$f(t) = \frac{2}{6}\pi^2 + \pi + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} (-1)^n \cos(n\pi t) \right)$$

Okay! Now solve.

$$\ddot{y} + 2\dot{y} + y = \underbrace{\frac{1}{3}\pi^2 + \pi}_{y_{p_0}} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{4}{n^2}(-1)^n \cos(nt)\right)}_{y_{p_n}}$$

$$y_p = y_{p_0} + y_{p_n}$$

y_{p_0}

$$\ddot{y} + 2\dot{y} + y = \frac{1}{3}\pi^2 + \pi$$

$$y_p = A, \quad \dot{y}_p = 0, \quad \ddot{y}_p = 0$$

$$0 + 2(0) + A = \frac{1}{3}\pi^2 + \pi$$

$$y_p = \frac{1}{3}\pi^2 + \pi$$

y_N

$$\ddot{y} + z\dot{y} + y = \sum_{n=1}^{\infty} \left(\frac{4}{n^2} (-1)^n \cos(nt) \right)$$

A little different! don't expand out series!

Also, Also problem!

$$\cos t = \operatorname{Re} [e^{it}]$$

$$\ddot{y} + z\dot{y} + y = \sum_{n=1}^{\infty} \left(\frac{4}{n^2} (-1)^n \operatorname{Re} [e^{int}] \right)$$

While solving this,
just take it out.
We can put it
back later
(superposition)

Can either
bring in, or
leave out

Sub Back Later!

$$\ddot{Y} + z\dot{Y} + Y = e^{int}$$

$$Y = D e^{int}, \quad \dot{Y} = D i n e^{int}, \quad \ddot{Y} = -D n^2 e^{int}$$

$$\cancel{D} e^{int} (-n^2 + z i n + 1) = \cancel{e^{int}}$$

$$D = \frac{1}{(-n^2 + z i n + 1)}$$

$$y_N = \frac{1}{(-n^2 + z i n + 1)} e^{int}$$

Aux Y_p : $Y_p = \frac{1}{(-n^2 + 2in + 1)} e^{int}$

$$\frac{1}{(-n^2 + 1 + 2in)} \cdot \frac{-n^2 + 1 - 2in}{-n^2 + 1 - 2in} (\cos nt + i \sin nt)$$

$$\frac{-n^2 - 2in + 1}{(n^2 + 1)^2 + 4n^2} \cos nt + i \sin nt$$

$$\operatorname{Re}[Y_p] = \frac{-n^2 + 1}{(n^2 + 1)^2 + 4n^2} \cos nt + \frac{2n}{(n^2 + 1)^2 + 4n^2} \sin nt$$

$$y_{p_n} = \left(-\frac{4}{n} (-1)^n\right) \operatorname{Re}[Y_p]$$

$$y_{p_n} = \left(-\frac{4}{n} (-1)^n\right) \left(\frac{-n^2 + 1}{(n^2 + 1)^2 + 4n^2} \cos nt + \frac{2n}{(n^2 + 1)^2 + 4n^2} \sin nt \right)$$

$$y_p = y_{p_0} + y_{p_n}$$

$$y_p = \frac{1}{3} \pi^2 + \pi + \sum_{n=1}^{\infty} \left(-\frac{4}{n} (-1)^n\right) \left(\frac{-n^2 + 1}{(n^2 + 1)^2 + 4n^2} \cos nt + \frac{2n}{(n^2 + 1)^2 + 4n^2} \sin nt \right)$$

Example Two, Complex Form

$$f(t) = \begin{cases} t - 2\pi & 0 \leq t \leq 2\pi \\ 0 & 2\pi < t < 4\pi \end{cases} \rightarrow 0 \leq t < 4\pi$$

Complex Fourier Form

$$T = 4\pi, L = 2\pi$$

$$f(t) = C_0 + \sum_{n=-\infty}^{n=\infty} C_n e^{in\pi t/L}$$

$$C_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-in\pi t/L} dt \quad C_0 = \frac{1}{2L} \int_{-L}^L f(t) dt$$

$$\begin{aligned} C_0 &= \frac{1}{4\pi} \int_0^{4\pi} f(t) dt = \frac{1}{4\pi} \left(\int_0^{2\pi} t - 2\pi dt + \int_{2\pi}^{4\pi} 0 dt \right) \\ &= \frac{1}{4\pi} \left(\frac{t^2}{2} - 2\pi t \Big|_0^{2\pi} \right) \\ &= \frac{1}{4\pi} \left(\frac{4\pi^2}{2} - 4\pi^2 \right) \\ &= \frac{\pi}{2} - \pi = -\frac{\pi}{2} \end{aligned}$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(t) e^{-\frac{in\pi t}{L}} dt$$

$$c_n = \frac{1}{4\pi} \int_0^{2\pi} (t-2\pi) e^{-\frac{int}{2}} dt + \int_{2\pi}^{4\pi} 0 e^{-\frac{in\pi t}{2\pi}} dt$$

$$\frac{1}{4\pi} \left(\underbrace{\int_0^{2\pi} t e^{\frac{int}{2}} dt}_{I_1} - 2\pi \underbrace{\int_0^{2\pi} e^{-\frac{int}{2}} dt}_{I_2} \right)$$

$$c_n = \frac{1}{4\pi} (I_1 - I_2)$$

$$I_2 = 2\pi \int_0^{2\pi} e^{-\frac{int}{2}} dt$$

$$= 2\pi \left(\frac{2}{-in} e^{-\frac{int}{2}} \right)$$

$$I_1 = \int_0^{2\pi} t e^{\frac{int}{2}} dt$$

$$\int u dv = uv - \int v du$$

$$e^{in\pi} = (-1)^n \quad \cos n\pi = (-1)^n$$

$$u = t \quad dv = e^{-\frac{int}{2}} dt$$

$$du = 1 dt \quad v = -\frac{2}{in} e^{-\frac{int}{2}}$$

$$-\frac{2t}{in} e^{-\frac{int}{2}} - \int -\frac{2}{in} e^{-\frac{int}{2}} dt$$

$$I_1 = -\frac{zt}{in} e^{-\frac{int}{z}} - \int -\frac{z}{in} e^{-\frac{int}{z}} dt$$

$$-\frac{zt}{in} e^{-\frac{int}{z}} + \frac{z}{in} \left(-\frac{z}{in} e^{-\frac{int}{z}} \right)$$

$$-\frac{zt}{in} e^{-\frac{int}{z}} + \frac{z}{in} \left(-\frac{z}{in} e^{-\frac{int}{z}} \right)$$

$$-\frac{zt}{in} e^{-\frac{int}{z}} + \frac{4}{n^2} e^{-\frac{int}{z}}$$

$$C_n = \frac{1}{4\pi} \left(-\frac{zt}{in} e^{-\frac{int}{z}} + \frac{4}{n^2} e^{-\frac{int}{z}} - 2\pi \left(\frac{z}{in} e^{-\frac{int}{z}} \right) \right)$$

$$\frac{e^{-\frac{int}{z}}}{4\pi} \left(-\frac{zt}{in} + \frac{4}{n^2} + \frac{4\pi}{in} \right)$$

$$= \frac{e^{-\frac{int}{z}}}{4\pi} \left(\frac{4\pi - zt}{in} + \frac{4}{n^2} \right) \Big|_0^{2\pi}$$

$$= \frac{e^{-in\pi}}{4\pi} \left(\frac{4}{n^2} \right) - \frac{e^{-in(0)}}{4\pi} \left(\frac{4\pi}{in} + \frac{4}{n^2} \right)$$

$$C_n = \frac{e^{-in\pi}}{4\pi} \left(\frac{4}{n^2} \right) - \frac{e^{-in\pi}}{4\pi} \left(\frac{4\pi}{in} + \frac{4}{n^2} \right)$$

$$= \frac{(-1)^n}{4\pi} \left(\frac{4}{n^2} \right) - \frac{1}{4\pi} \left(\frac{4\pi}{in} + \frac{4}{n^2} \right)$$

$$= \frac{(-1)^n}{\pi n^2} - \frac{1}{in} + \frac{1}{\pi n^2}$$

$$= \frac{(-1)^n + 1}{\pi n^2} - \frac{1}{in}$$

Diff from
Hesslen. will use
this solution

$$f(t) = C_0 + \sum_{n=-\infty}^{n=\infty} C_n e^{in\pi t/L}$$

$$L = 2\pi$$

$$f(t) = \underbrace{-\frac{\pi}{2}}_{y_{f_0}} + \underbrace{\sum_{n=-\infty}^{n=\infty} C_n e^{\frac{int}{2}}}$$

$$y_{f_0} = A$$

$$\dot{y}_{f_0} = 0$$

$$\ddot{y}_{f_0} = 0$$

$$y_{f_0} = -\frac{\pi}{6}$$

$$y_{f_n} = D e^{\frac{int}{\pi}}$$

$$\dot{y}_{f_n} = D \left(\frac{in}{\pi} \right) e^{\frac{int}{\pi}}$$

$$\ddot{y}_{f_n} = -D \left(\frac{n^2}{\pi^2} \right) e^{\frac{int}{\pi}}$$

Skill a few Plugging in Steps...

$$D e^{\frac{i \omega t}{\pi}} \left(-\frac{n^2}{\pi} + \frac{z_{in}}{\pi} + 3 \right) c_n e^{\frac{i \omega t}{\pi}}$$

$$D = \frac{c_n}{\left(-\frac{n^2}{\pi} + \frac{z_{in}}{\pi} + 3 \right)}$$

$$y_{pn} = \sum_{n=-\infty}^{n=\infty} \frac{c_n}{\left(-\frac{n^2}{\pi} + \frac{z_{in}}{\pi} + 3 \right)} e^{\frac{i \omega t}{\pi}}$$

$$y_p = y_{p0} + y_{pn}$$

$$y_p = \frac{-\pi}{6} + \sum_{n=-\infty}^{n=\infty} \frac{c_n}{\left(-\frac{n^2}{\pi} + \frac{z_{in}}{\pi} + 3 \right)} e^{\frac{i \omega t}{\pi}}$$