Example:

Find the general Solution to

$$x' = \overset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \vec{x}$$

Find Eigenpairs of A:

Eigenvalues: $\{2, -1, -1\}$

Repeated Values!

Simpler $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ Simpler

$\lambda = 2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\lambda = -1$, $\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 3 \\ -3/2 \end{bmatrix}$

(double root)

$\boxed{\text{Algebraic Multiplicity of } 2}$

(double roots)

$\boxed{\text{Geometric multiplicity of } 2}$

(two unique EV's from one value)

General Solution:

$$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Here, we encountered repeated eigenvalues, but we still need 3 Linearly Independent eigenvectors, so all is well. Just need to crunch an extra term or there.

Example:

Find general solution to

$$\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$$

$\lambda = 1, 1$ (repeated Roots) (Algebraic multiplicity of 2)

Computers have issues solving these problems when the geometric multiplicity (# of Eigenvectors) is <u>less</u> than the algebraic multiplicity (# of Eigenvalues)

So, have to do ref

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (Geometric multiplicity of 1)

We can write one solution now:

$\vec{x}_1(t) = e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

We need a <u>Second Linearly Independent</u> Solution

Look for $X_2(t)$

Needed compared to MUC method!

$$X_2(t) = \vec{v}_1 t e^{\lambda t} + \vec{V}_2 e^{\lambda t}$$

LHS: $x_2'(t) = \vec{v}_1 e^{\lambda t} + \vec{v}_1 \lambda t e^{\lambda t} + \vec{v}_2 \lambda e^{\lambda t}$

RHS: $A\vec{x}_2 = t e^{\lambda t} A v_1 + e^{\lambda t} A v_2$

For $\vec{x}_2$ to be a solution, we must have:

$$A\vec{v}_1 = \lambda \vec{v}_1 \qquad A v_2 = \lambda \vec{v}_2 + \vec{v}_1$$

$\vec{v}_1$ is eigenvector for λ

Rearrange $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

$\vec{v}_2$ is called a generalized eigenvector.

For our example:

$\lambda = 1$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $x_1(t) = e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2^{nd} Solution:

$$X_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \vec{v}_2 e^t,$$

V_2 must satisfy $(A-I)\vec{v}_2 = \vec{v}_1$

So,

$$\left[\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right] \overset{REF}{\sim} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

If $\vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$

$a - 2b = 1$

Let's set $b = 0$, so $a = 1$

$$V_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we can plug this into $(A-I)\vec{v}_2 = \vec{v}_1$ and check if desired.

So,

$$X_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t = \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$$

General Solution

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$$

Let's turn things up a little bit more, and make the algebraic multiplicity 3, and the geometric 1.

Example:

Find the general solution to:

$$x' = \begin{bmatrix} -5 & -5 & 9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{bmatrix} \vec{x}$$

There is a single repeated eigenvalue, $\lambda = -1^{\times 3}$ & a single eigenvector $\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$

$$x_1(t) = e^{-t}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

$$x_2(t) = te^{-t}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} + \vec{v}_2 e^{-t}$$

where $(A+I)v_2 = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$

take $v_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$

$$x_3(t) = \frac{1}{2}t^2 e^{-t}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} + te^{-t}\begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} + \boxed{v_3}\, e^{-t}$$

Where

$$(A+I)\boxed{v_3} = v_2$$

Take $\vec{v}_3 = \begin{bmatrix} 5/3 \\ 2/3 \\ 0 \end{bmatrix}$