

Non-homogeneous Systems

We'll briefly explore a couple of ways to solve non-homogeneous systems.

Integrating Factor / Variation of parameters Method:

If $X(t)$ is a fundamental matrix solution to $\vec{x}' = A\vec{x}$, then:

$$\vec{x}_p = X(t) \int [X(t)]^{-1} f(t) dt$$

is a particular solution to $\vec{x}' = A\vec{x} + \vec{f}(t)$

Undetermined Coefficients Method:

Look for a Particular Solution based on the form of the non-homogeneous term.

Example: Find the general solution of the system:

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Homogeneous Solution: $\vec{x}_h = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Forcing: $\vec{f}(t)$

$$\vec{f}(t) = \begin{bmatrix} e^t \\ t \end{bmatrix} = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Here, I would extract the forcings for simplicity

Okay, let's make a guess now.

Try:

$$\vec{x}_p = e^t \vec{a} + t \vec{b} + \vec{c}$$



Warning!
There is a
Dupe!

So for systems, we add another term.

$$\vec{x}_p = t e^t \vec{a} + e^t \vec{b} + t \vec{c} + \vec{d}$$

$$\vec{x}'_p = t e^t \vec{a} + e^t \vec{b} + t \vec{c} + \vec{d}$$

$$\begin{aligned} \vec{x}'_p &= t e^t \vec{a} + e^t \vec{a} + e^t \vec{b} + \vec{c} \\ &= t e^t \vec{a} + e^t (\vec{a} + \vec{b}) + \vec{c} \end{aligned}$$

$$\vec{x}'_p = A \vec{x}_p$$

$$t e^t \vec{a} + e^t (\vec{a} + \vec{b}) + \vec{c} = t e^t A \vec{a} + e^t A \vec{b} + t A \vec{c} + A \vec{d}$$

So,

$$e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\cancel{t e^t \vec{a}} = \cancel{t e^t} A \vec{a}$$

$$\cancel{e^t} (\vec{a} + \vec{b}) = \cancel{e^t} (A \vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

$$A \vec{c} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$A \vec{d} = \vec{c}$$

System of Equations

$$A\vec{a} = \vec{a}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{a} = \vec{a}$$

$$2a_1 - a_2 = a_1$$

$$3a_1 - 2a_2 = a_2$$

$$A\vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{a} + \vec{b}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{a} + \vec{b}$$

$$2b_1 - b_2 + 1 = a_1 + b_1$$

$$3b_1 - 2b_2 = a_2 + b_2$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{c} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{0}$$

$$2c_1 - c_2 = 0$$

$$3c_1 - 2c_2 + 1 = 0$$

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{d} = \vec{c}$$

$$2d_1 - d_2 = c_1$$

$$3d_1 - 2d_2 = c_2$$

$$a_1 = \frac{3}{2}, \quad a_2 = \frac{3}{2}$$

$$b_1 = \frac{-1}{4}, \quad b_2 = \frac{-3}{4}$$

$$c_1 = 1, \quad c_2 = 2$$

$$d_1 = 0, \quad d_2 = -1$$

So,

$$x_p(t) = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} t e^t + \begin{bmatrix} -1/4 \\ -3/4 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} t e^t - \frac{1}{4} e^t + t \\ \frac{3}{2} t e^t - \frac{3}{4} e^t + 2t - 1 \end{bmatrix}$$

Either Method Can be used, but for certain problems, one can be vastly more efficient than the other. In this case, MUC proved to be way slower than IF/VOP Method. One should not discredit MUC Method based on just this example.