Non Homogeneous Linear Equations (Constant Coefficients)

$$a\ddot{y} + b\dot{y} + cy = G(x)$$

Consider complimentary Homogeneous D.E:

$$a\ddot{y} + b\dot{y} + cy = 0$$

and find general solution: y_c

If we can be so lucky as to find one, just one particular solution y_p to $a\ddot{y} + b\dot{y} + cy = G(x)$ then any other solution y to $a\ddot{y} + b\dot{y} + cy = G(x)$ can be written as

$$\boxed{y = y_p + y_c}$$

$$y = G(x) + 0$$

Method of Undetermined Coefficients

Example: Solve $\ddot{y} + \dot{y} - 2y = x^2$

First, Solve $\ddot{y} + \dot{y} - 2y = 0$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$ $r = -2$ $r = 1$

$$\boxed{y_c = c_1 e^{-2x} + c_2 e^{x}}$$

Next, try to find one y_p to $\ddot{y} + \dot{y} - 2y = x^2$

Guess:

$y_p = Ax^2 + Bx + C$ Determine A, B, C

$\dot{y}_p = 2Ax + B$

$\ddot{y}_p = 2A$

Plug in guess & Set equal

$$2A + 2Ax + B - 2(Ax^2 + Bx + C) = x^2$$

$$2A + 2Ax + B - 2(Ax^2 + Bx + C) = x^2$$

$$2A + 2Ax + B - 2Ax^2 + 2Bx + 2C = x^2$$

Solve System of Eqn's

$$x^2: \quad -2A = 1$$

$$x: \quad 2A - 2B = 0$$

$$1: \quad 2A + B - C = 0$$

$$A = -\frac{1}{2}$$

$$B = -\frac{1}{2}$$

$$C = -\frac{3}{4}$$

So, $\boxed{y_p = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}}$

General Solution:

$$\boxed{y = c_1 e^{-2x} + c_2 e^{x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}}$$

Example:

Solve $$\ddot{y} - 4y = \underbrace{xe^{x}}_{y_p1} + \underbrace{\cos 2x}_{y_p2}$$

First, $$\ddot{y} - 4y = 0, \quad r^2 - 4 = 0 \qquad r = \pm 2$$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

y_p1

xe^{x}

y_p2

$\cos 2x$

Duplication Issue ⚠

Example: $\ddot{y} + y = \sin x$, r^2+1, $r = \pm i$

$y_c = C_1 \sin x + C_2 \cos x$

Guess for Sinx

$y_p = A\sin x + B\cos x$

Dupe!

$\hookrightarrow y_p = Ax\sin x + Bx\cos x$

When this happens, we need to multiply by x!

Solve as normal

Example:

$$\ddot{y} + 2\dot{y} + 5y = 4xe^{-x}\cos 2x + e^{2x}, \quad y(0) = 1, \quad \dot{y}(0) = 2$$

Variation Of Parameters

Solve

$\ddot{y} + y = \tan x, \quad 0 < x < \frac{\pi}{2}$

First, $\ddot{y} + y = 0, \quad y_0 = c_1 \cos x + c_2 \sin x$

Guess

$y_p = A \tan x \ldots$

Aaaand problem...

This will go FOREVER! MUC falls flat here, because no matter how many terms you add, you'll never have a genuinely different derivative.

Variation of Parameters Method.

Guess:

$$y_p = u_1(x)\cos x + u_2(x)\sin x$$

functions?!

$$\dot{y}_p = \dot{u}_1 \cos x - u \sin x + \dot{u}_2 \sin x + u_2 \cos x$$

TRY TO KEEP THINGS SIMPLE

Suppose: $\dot{u}_1 \cos x + \dot{u}_2 \sin x = 0$

So, $\dot{y}_p = -u \sin x + u \cos x$

$$\ddot{y}_p = -\dot{u}_1 \sin x - u_1 \cos x + \dot{u}_2 \cos x - u_2 \sin x$$

Plug into LHS:

$$\ddot{y}_p + y_p = \tan x$$

$$-\dot{u}_1 \sin x - \cancel{u_1 \cos x} + \dot{u}_2 \cos x - \cancel{u_2 \sin x} + \cancel{u_1 \cos x} + \cancel{u_2 \sin x} = \tan x$$

$$-\dot{u}_1 \sin x + \dot{u}_2 \cos x = \tan x$$

Solve for u_1 & u_2?

We have two Eqn's

$$\dot{u}_1 \cos x + \dot{u}_2 \sin x = 0$$
$$-\dot{u}_1 \sin x + \dot{u}_2 \cos x = \tan x$$

• Cramer's Rule here.

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix}$$

$$\dot{u}_1 = \cos x - \sec x$$
$$\dot{u}_2 = \sin x$$

Integrate to get u_1 & u_2

$$u_1 = \int (\cos x - \sec x)\,dx = \sin x - \ln(\sec x + \tan x)$$
$$u_2 = \int \sin x\,dx = -\cos x$$

$$y_p = [\sin x - \ln(\sec x + \tan x)]\cos x - \cos x \sin x$$