Standard Form of Second Order Linear Differential Equation

$$\boxed{P(x)\ddot{y} + Q(x)\dot{y} + R(x)y = G(x)}$$

Special Case: $G(x) = 0$ (Homogenious)

$$\boxed{P(x)\ddot{y} + Q(x)\dot{y} + R(x)y = 0}$$

Theorem:

If $y_1(x)$ and $y_2(x)$ are both Solutions of

$P(x)\ddot{y} + Q(x)\dot{y} + R(x)y = 0$ ← Homogenous

and c_1, c_2 are ~~Constants~~ are any Constants,

then

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a Solution

Two solutions are <u>linearly independent</u> if they are not constant multiples of each other.

<u>Theorem:</u>

If y_1 and y_2 are linear independent solutions of

$$P(x)\ddot{y} + Q(x)\dot{y} + R(x)y = 0$$ Homogeneous

then any solution $y(x)$ can be expressed as

$$y = c_1 y_1 + c_2 y_2$$

for some constants c_1 & c_2

$$\boxed{\text{Constant Coefficients}}$$

$$a\ddot{y} + b\dot{y} + cy = 0$$

(a, b, c are constants, and equation is homogeneous)

Is there some $y(x)$, for which

$$a\ddot{y} + b\dot{y} + cy = 0?$$

<u>Guess</u>: $y(x) = e^{rx}$, for some r

$$\dot{y} = re^{rx}$$

$$\ddot{y} = r^2e^{rx}$$

$$a\ddot{y} + b\dot{y} + cy = (ar^2 + br + c)e^{rx}$$

Can we find r such that:

$$(ar^2 + br + c)e^{rx} = 0?$$

We know that e^{rx} cannot be zero. So it must be the characteristic equation!

$$\boxed{(ar^2 + br + c)}$$

Knowing that, let's set it equal to zero, and solve for r!

Example:

$$\ddot{y} + \dot{y} - 6y = 0$$

make guesses

$(r^2 + r - 6)\cancel{e^{rx}} = 0$

So $r = -3,\ r = 2$

$(r+3)(r-2) = 0$

So let $y_1 = e^{-3x}$

$y_2 = e^{2x}$

(We can actually check our solution if desired!

$\dot{y}_1 = -3e^{-3x},\quad \ddot{y}_1 = 9e^{-3x}$ (plug in)

$9e^{-3x} - 3e^{-3x} - 6e^{-3x} = 0$ (Homogeneous Answer!)

So e^{-3x} is a Solution!)

Note: y_1 and y_2 are L.I $\left(\frac{y_2}{y_1} = \frac{e^{2x}}{e^{-3x}} = e^{5x}\right.$

NOT constant

So,

$$\boxed{y(x) = c_1 e^{-3x} + c_2 e^{2x}}$$

Going From DE problem $\xrightarrow{\text{to}}$ Algebra problem

With guesses!

Getting r:

Best Case Scenario.

The discriminant is > 0 $(b^2 - 4ac)$

Roots are real.

Next Case Scenario

The discriminant is 0 $(b^2 - 4ac = 0)$
or you only have one root!

$y_1 = e^{rx}$ Solution one

$y_2\ xe^{rx}$ Solution Two

Solutions need to be L.I!
Solutions Can't Dupe!

only one real root here!

$$\boxed{y_1 = C_1 e^{rx} + C_2 x e^{rx}}$$

Final Case

The discriminante is negative $(b^2 - 4ac < 0)$

We get two complex roots

$$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$$

... and they are conjugates!

Solutions

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}$$

General Solution

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)}$$

....

... Some euler's idententities later

....

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

Initial and Boundary-Value problems

Example: Solve $\ddot{y} + y = 0$ with the initial Conditions $y(0) = 2$ and $\dot{y}(0) = 3$

Char Eqn: $r^2 + 1 = 0$

$$r = \sqrt{-1} = \pm 1 = 0 \pm 1i$$
$$\alpha \pm \beta i$$

General Solution:

$$y = e^{0x}[C_1 \cos 1x + C_2 \sin 1x]$$

$$\boxed{y = C_1 \cos x + C_2 \sin x}$$

Apply IC's $y(0) = 2$

$$C_1 \cos(0) + C_2 \sin(0) = C_1 \cdot 1 + C_2 \cdot 0 = C_1$$

$C_1 = 2$

and if we do the same for C_2...

$C_2 = 3$

Thus...

$$\boxed{y(x) = 2\cos x + 3\sin x}$$

Example: Solve $\ddot{y} + 2\dot{y} + 1 = 0$ with the boundary conditions $y(0) = 1$, $y(1) = 3$

Char eq:

$$r^2 + 2r + 1 = 0$$
$$(r+1)^2 = 0$$
$$r = -1$$

General solution for one real root

$$y = C_1 e^{-x} + C_2 x e^{-x}$$

Apply BC's

$y(0) = 1$ $\quad y(0) = C_1 e^0 + C_2 0 e^0 = C_1$

So $c = 1$

$y(1) = 3$ $\quad y(1) = C_1 e^{-1} + C_2 \cdot 1 \cdot e^{-1} = \dfrac{1 + C_2}{e}$

$$\frac{1 + C_2}{e} = 3 \qquad C_2 = 3e - 1$$

Thus

$$y = e^{-x} + (3e - 1) x e^{-x}$$