The idea of decomposing a complicated thing into simpler components is both useful and common across many branches of mathematics, science and engineering.

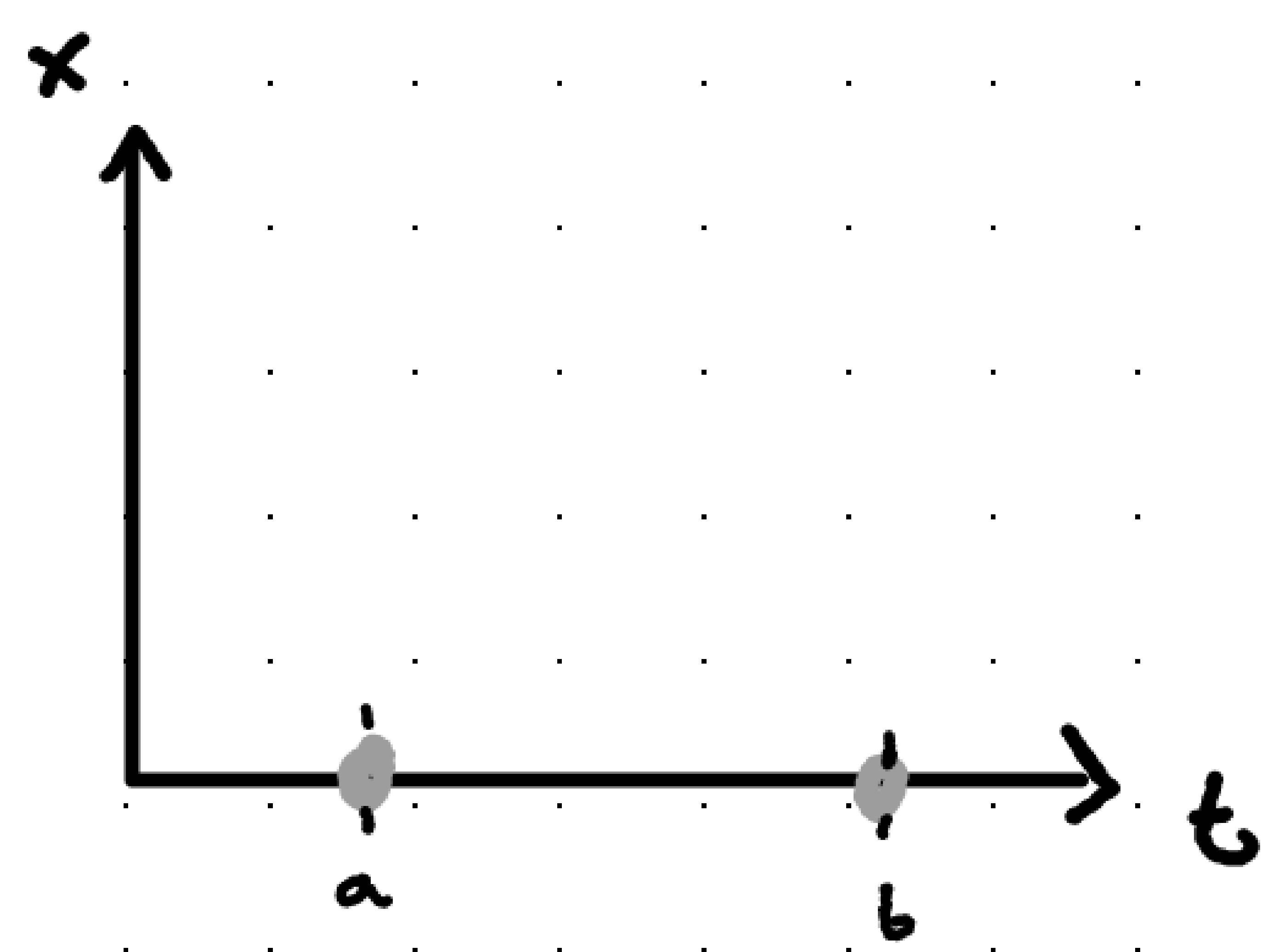
For example, Taylor Series allows functions to be decomposed into infinite sums of power functions, which consequently allow for approximation by polynomials.

We will now explore the Fourier Series, which, roughly speaking, allow a function to be decomposed into sines & cosines.

Boundary Value problems (BVP's)

Consider the BVP

$$\ddot{x} + \lambda x = 0, \quad x(a) = 0, \; x(b) = 0$$



Solving at different points in time is fundamentally different from IC's!

Rewrite:

$$-x'' = \lambda x$$

Denote $\frac{-d^2}{dx^2}$ by L

$$\frac{-d^2 x}{dt^2} = \lambda x$$

$$\boxed{Lx = \lambda x}$$

Like an infinite vector, one value for each point in time.

If this BVP has a non-zero solution $x(t)$, then λ is called an eigenvalue, and $x(t)$ the corresponding Eigenfunction.

Example:

Take $\lambda = 1$, $\alpha = 0$, $\beta = \pi$

$$\ddot{x} + x = 0, \quad x(0) = 0, \quad x(\pi) = 0$$

General Solution

$$x(t) = C_1 \cos t + C_2 \sin t$$

Apply: $\boxed{x(0) = 0}$

$$x(0) = C_1 \cdot 1 + C_2 \cdot 0 = \boxed{C_1 = 0}$$

So, $x(t) = C_2 \sin(t)$

Apply $\boxed{x(\pi) = 0}$

$$x(\pi) = C_2 \sin(\pi) = C_2 \cdot 0 = \boxed{C_2 = 0}$$

True for any C_2

So, $\boxed{x(t) = C_2 \sin t}$ is a solution for any C_2

So, $\boxed{\lambda = 1}$ is an eigenvalue for $\ddot{x} + \lambda x = 0$, $x(0) = x(\pi) = 0$, with corresponding eigenfunction $\sin t$

Example:

Take $\lambda = 2$, $\alpha = 0$, $\beta = \pi$

$$\ddot{x} + 2x = 0, \quad x(0) = 0, \quad x(\pi) = 0$$

$$x(t) = c_1 \cos(\sqrt{2}\,t) + c_2 \sin(\sqrt{2}\,t)$$

Apply: $x(0) = 0$

$$x(0) = c_1 \cdot 1 + c_2 \cdot 0 = \boxed{c_1 = 0}$$

So, $x(t) = c_2 \sin(\sqrt{2}\,t)$

Apply: $x(\pi) = 0$

$$x(\pi) = c_2 \boxed{\sin(\sqrt{2}\,\pi)} = 0 \qquad \boxed{c_2 = 0}$$

This does not equal zero!

So in this case for this solution to be true, c_2 HAS to be zero

So, $x(t) = 0$. No nonzero solution

A value λ is an Eigenvalue for the BVP if you can find a non zero solution (or solutions) to the BVP.

And if you can, the solution is an <u>Eigenfunction</u>

Eigenvalue problems

For basic Fourier series theory, we will need the following three eigenvalue problems:

① $\ddot{x} + \lambda x = 0$, $x(a) = 0$, $x(b) = 0$ Sin

② $\ddot{x} + \lambda x = 0$, $x'(a) = 0$, $x'(b) = 0$ Cosine

③ $\ddot{x} + \lambda x = 0$, $x(a) = x(b)$, $x'(a) = x'(b)$ General Fourier

$x(t) = 0$ is ALWAYS a solution to these problems, but we don't particularly care about that.

Example: 1

Let us find the eigenvalues & eigenfunctions of

$$\ddot{x} + \lambda x = 0, \quad x(0) = 0, \quad x(\pi) = 0$$

General solution

Assuming $\lambda > 0$

$$x(t) = c_1 \cos(\sqrt{\lambda}\, t) + c_2 \sin(\sqrt{\lambda}\, t)$$

$\boxed{\text{Apply } x(0) = 0}$

$$x(0) = c_1 \cdot 1 + c_2 \cdot 0 = 0$$

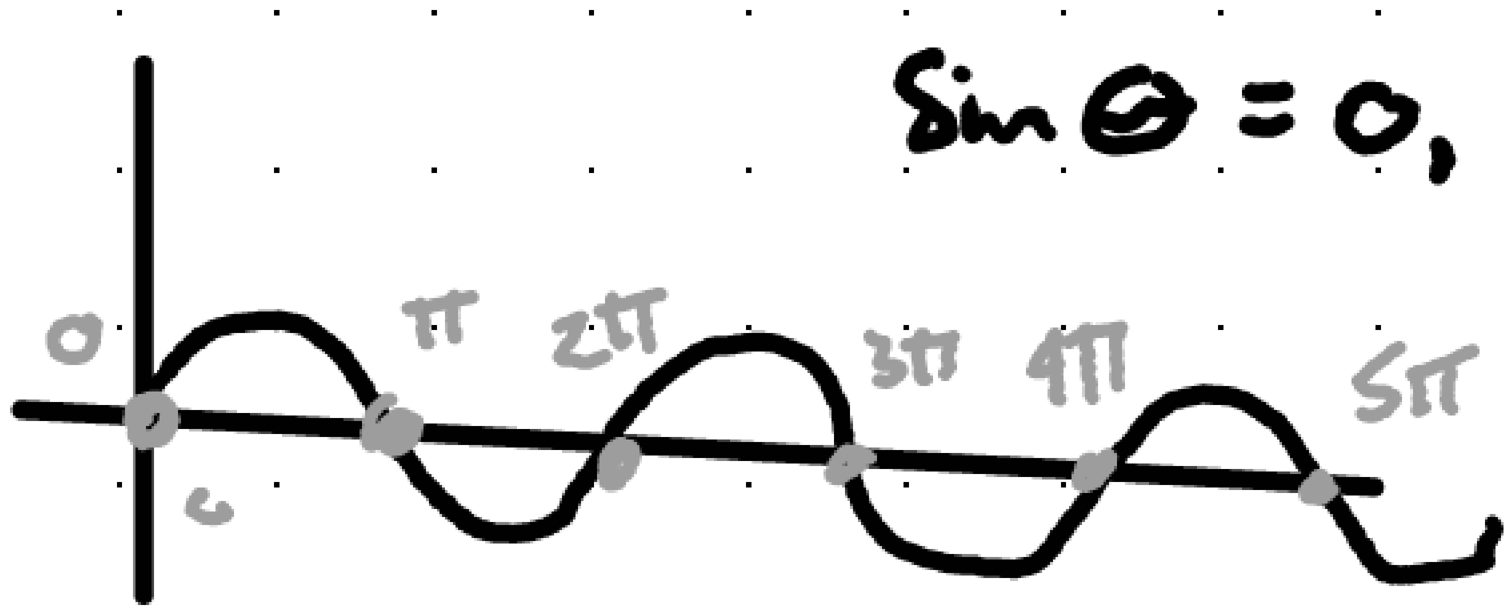
$\boxed{c_1 = 0}$

$\boxed{\text{Apply } x(\pi) = 0}$

$$x(\pi) = c_2 \sin(\sqrt{\lambda}\, \pi) = 0$$

We need to find a value here that makes this equal to zero

For what values of λ is $\sin(\sqrt{\lambda}\, \pi) = 0$?



$\sin\theta = 0$, when

$\theta = n\pi$,

$n =$ any integer

4 could work!

$\sin = 0$ at any multiple of π

So, $\sin(\sqrt{\lambda}\pi)=0$, when $\sqrt{\lambda}=n$,

i.e

$\lambda = n^2$, n being any integer

Eigenvalues: 1, 4, 9, 16.... (n^2)

Eigenfunctions: $\sin t$, $\sin 2t$, $\sin 3t$, $\sin 4t$... $\sin nt$, ...

What if $\lambda=0$?

The DE Becomes: $\ddot{x}+0=0$

General solution is now:

$x(t)=At+B$

$\boxed{x(0)=0} \Rightarrow B=0$, $\boxed{x(\pi)=0} \Rightarrow A=0$

$\lambda=0$ is not an eigenvalue

What if $\lambda<0$?

The DE Becomes: $\ddot{x}-9x=0$, $\ddot{x}=9x$

No negative eigenvalues exist.

$\lambda=n^2$

Example: 2

Let us find the eigenvalues & eigenfunctions of

$$\ddot{x} + \lambda x = 0, \quad \dot{x}(0) = 0, \quad \dot{x}(\pi) = 0$$

General Solutions

$$x(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t$$

$\boxed{n > 0}$

$\boxed{\dot{x}(0) = 0}$

$$\dot{x}(t) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, t + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, t = 0$$

$$\dot{x}(0) = -c_1\sqrt{\lambda}\cdot 0 + c_2\sqrt{\lambda}\cdot 1 = 0$$

$$\boxed{c_2 = 0}$$

So, $x(t) = c_1 \cos\sqrt{\lambda}\, t$

$\boxed{\dot{x}(\pi) = 0}$

$$\dot{x}(t) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, t + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, t = 0$$

$$\dot{x}(\pi) = -c_1\sqrt{\lambda}\,\boxed{\sin(\sqrt{\lambda}\,\pi)} = 0$$

Just as before, this equals zero when $\boxed{\lambda = n^2}$

Eigenvalues are $\lambda_n = n^2$,

Eigenfunctions: $\cos nt$

If $\lambda=0$ De becomes: $x''=0$

General Solution:

$x(t) = At + B$

$\boxed{x'(0)=0}$ $A=0$, $x(t) = B$

$\boxed{x'(\pi)=0}$ True

So, we can have $\lambda=0$. That works no matter the value.

$\lambda=0$ is an Eigenvalue, with any Eigenfunction B, lets say 1

If $\lambda<0$

Turns out, there are no negative Eigenvalues.

So, Eigenvalues are 0, and Integers.
Eigenfunctions are 1 & Cosines

Example:

Let us find the eigenvalues & eigenfunctions of

$$\ddot{x} + \lambda x = 0, \quad x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi)$$

$\lambda > 0$

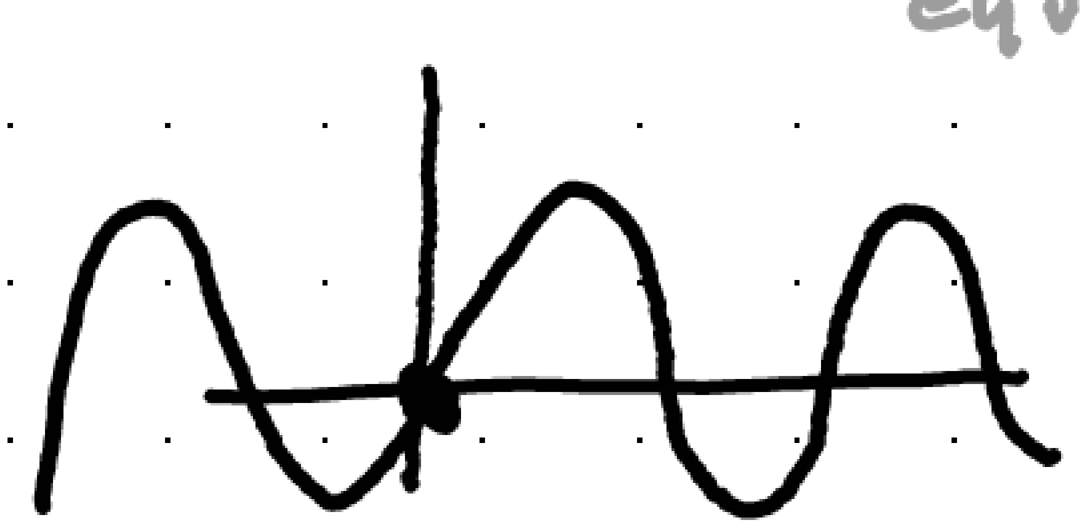
General Solution:

$$x(t) = c_1 \cos\sqrt{\lambda t} + c_2 \sin\sqrt{\lambda t}$$

$\boxed{x(-\pi) = x(\pi)}$ Apply First Boundary Condition

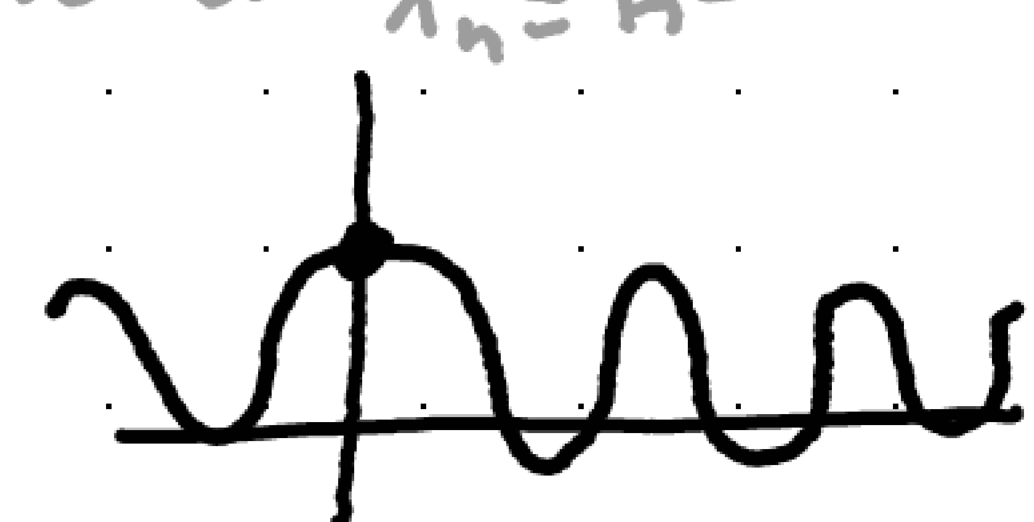
$$= c_1 \underbrace{\cos(-\sqrt{\lambda}\,\pi)}_{\cos(\sqrt{\lambda}\pi)} + c_2 \underbrace{\sin(-\sqrt{\lambda}\,\pi)}_{-\sin\sqrt{\lambda}\pi} = c_1 \cos(\sqrt{\lambda}\,\pi) + c_2 \sin(\sqrt{\lambda}\,\pi)$$

$$= 2c_2 \underbrace{\sin(\sqrt{\lambda}\,\pi)}_{\text{equals zero if } \lambda_n = n^2} = 0$$



Sin (Odd function)

If you replace a positive input with the corresponding negative, you'll get a negative output



Cos (Even function)

If you replace a positive input with the corresponding negative, you'll get the same output

$$\boxed{X'(-\pi) = X'(\pi)}$$ Apply other boundary condition

$$\underbrace{-c_1\sqrt{\lambda}\sin(-\sqrt{\lambda}\pi)}_{c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} + \underbrace{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)}_{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\cos(\sqrt{\lambda}\pi)$$

$$= 2c_1 \underbrace{\sin\sqrt{\lambda}\pi}_{\text{zero if } \lambda = n^2} = 0$$

Remember, we are interested in the non-zero solutions

If $\lambda = 0$

Eigenvalue exists, same as last example
Eigenfunction equals B, i.e 1

$\lambda < 0$

No negative Eigenvalues.

Eigenvalues are $0, n^2$

Eigenfunctions are $1, \cos nt, \sin nt$

Orthogonality of Eigenfunctions

For any of the three BVP's we've considering, suppose that $x_1(t)$ and $x_2 t$ are two eigen-functions for two different eigenvalues λ_1 & λ_2. Then, they are orthogonal in the sense that

$$\boxed{\int_a^b x_1(t)\, x_2(t)\, dt = 0} \qquad \star$$

a & b are boundaries, remember.

Example:

$\ddot{x} + \lambda x = 0$, $x(0) = x(\pi) = 0$

$\lambda_1 = 4$ $\qquad$ $\lambda_2 = 9$ $\qquad$ Found Eigenpairs

$\sin 2t$ $\qquad$ $\sin 3t$

$$\int_0^\pi (\sin 2t \sin 3t)\, dt = 0$$ ← No need to do the integral, we know this is true! Fourier Theory.

So, Let's take stock of our problems so far:

From BVP problem #

$$\int_0^{\pi} \sin(nt)\sin(mt)\,dt = 0 \quad \text{if } (m \neq n)$$ ①

$$\int_0^{\pi} \cos(nt)\cos(nt)\,dt = 0 \quad \text{if } (m \neq n)$$ ②

$$\int_0^{\pi} \cos(nt)\,dt = 0$$ ②

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,dt = 0 \quad \text{if } (m \neq n)$$ ③

$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,dt = 0 \quad \text{if } (m \neq n)$$ ③

$$\int_{-\pi}^{\pi} \sin(nt)\cos(mt)\,dt = 0 \quad \left(\text{even if } m = n\right)$$ ③

$$\int_{-\pi}^{\pi} \sin(nt)\,dt = 0$$ ③

$$\int_{-\pi}^{\pi} \cos(nt)\,dt = 0$$ ③

The Fredholm Alternative

Exactly one of the following statements holds.

Either

$$x'' + \lambda x = 0, \quad x(a)=0,\ x(b)=0$$

has a nonzero solution, or (i.e., λ is an eigenvalue)

$$x'' + \lambda x = f(t), \quad x(a)=0,\ x(b)=0$$

has a unique solution for every function f continuous on $[a, b]$.

If λ is NOT an eigenvalue, that second statement is true. The problem will have a unique solution no matter what $f(t)$ is.

Periodic Functions

As motivation for studying Fourier Series, suppose we have the problem

$$\ddot{x} + \omega_0^2 x = f(t)$$

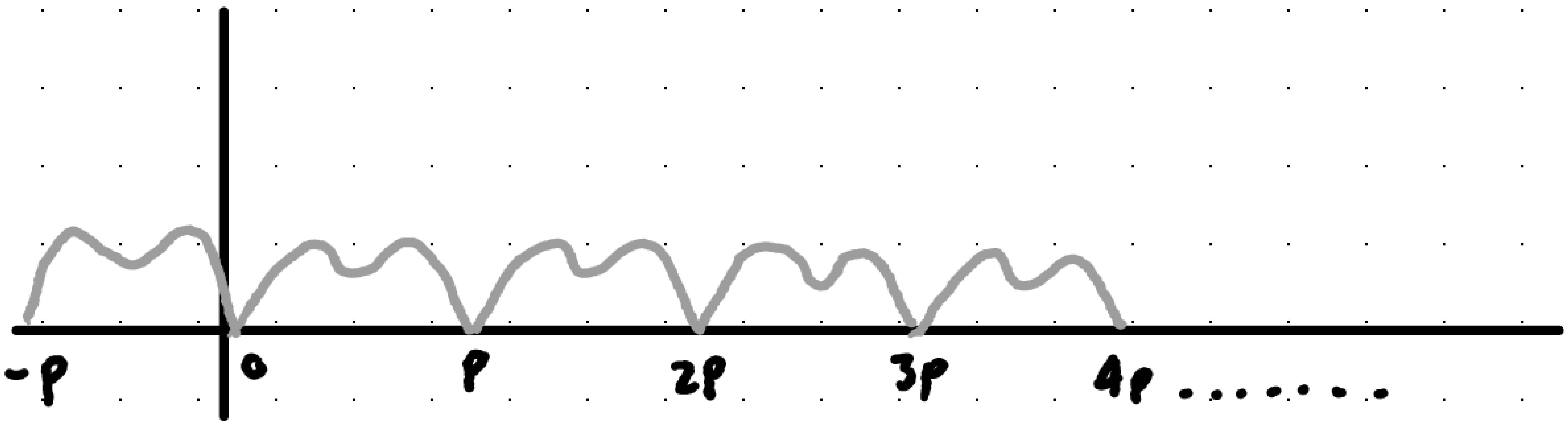
for some periodic function $f(t)$

We've actually solved this before, for $f(t) = F_0 \cos \omega t$

For more general $f(t)$, we could try expressing $f(t)$ in terms of many sines & cosines (via Fourier Series), and then solve many copies of the ODE

Periodic Functions

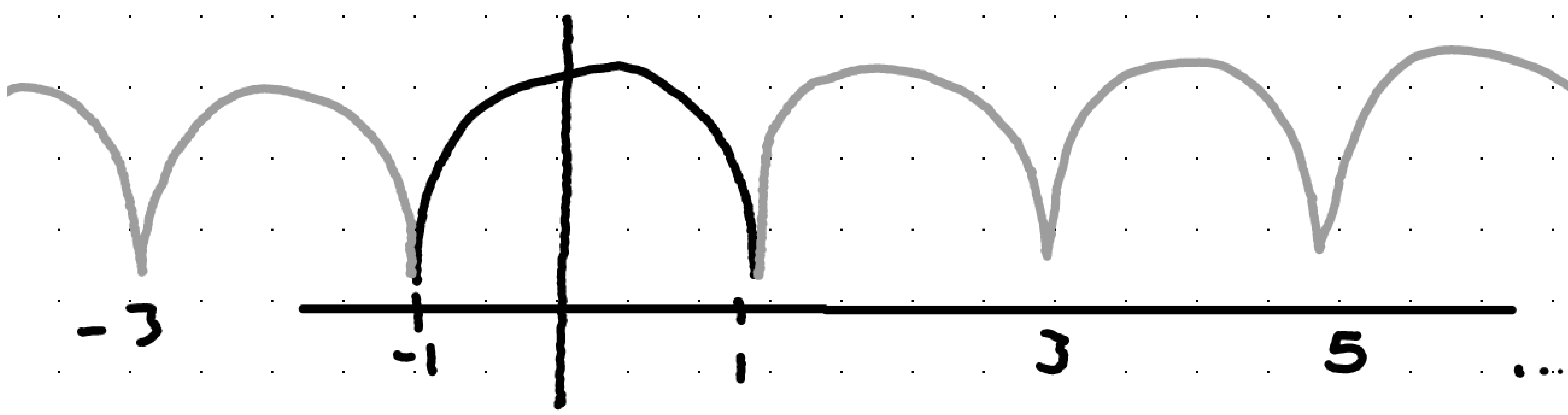
$f(t)$ is P-periodic, with period P, if $f(t+P) = f(t)$



e.g, $\sin t$, $\cos t$ are 2π periodic

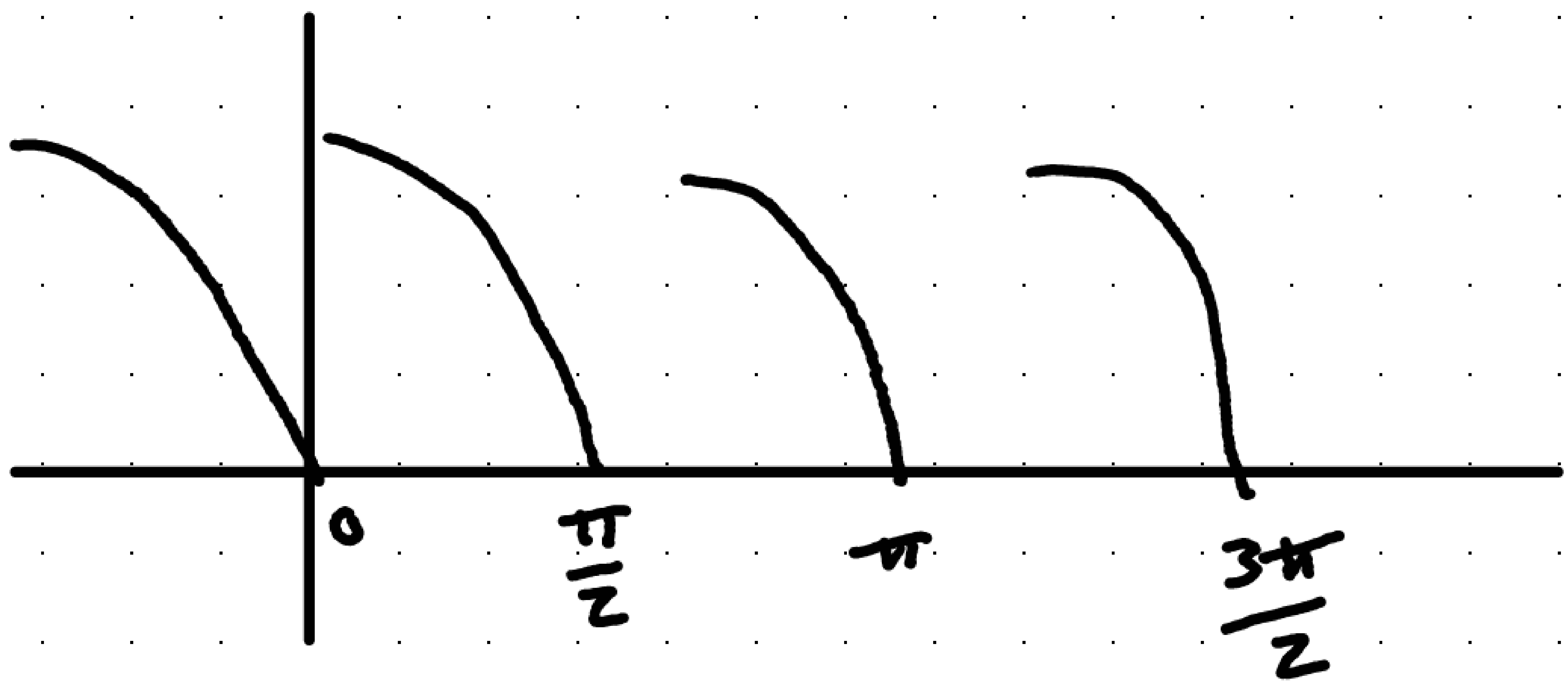
Periodic extensions:

e.g: $f(t) = 1 - t^2$, $-1 \le t \le 1$. Extend to a 2-periodic function



$$g(t) = \begin{cases} f(t) & -1 \le t \le 1 \\ f(t-2) & 1 \le t \le 3 \\ f(t-4) & 3 \le t \le 5 \\ \vdots & \end{cases}$$

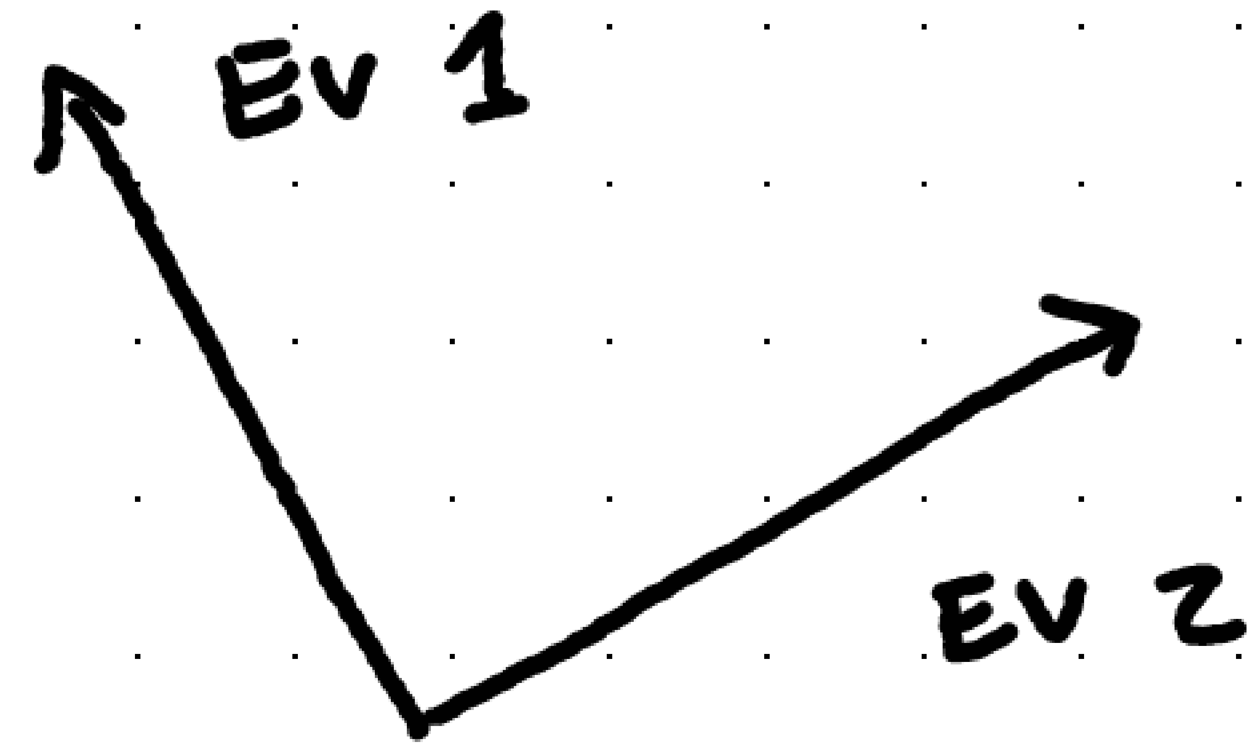
e.g. $f(t) = \cos t$, $0 \le t \le \frac{\pi}{2}$. Extend to a $\frac{\pi}{2}$ periodic function



Inner Product and Eigenvector Decomposition

FACT:

If A is an $n \times n$ symmetric real matrix, then the eigenvalues are real, distinct, so eigenvectors are L.I and mutually orthogonal.



Orthogonal!

These can form a basis for us.

Say the eigenvectors are $\vec{v}_1$ & $\vec{v}_2$. Then $\vec{v}_1 \cdot \vec{v}_2 = 0$ (Assume A is 2×2 here). Given <u>any</u> vector $\vec{v}$, we can express $\vec{v}$ as

$$\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2$$

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2$$

OKay. How do we find a_1 & a_2?

Well:

$=0$, orthogonal

$$\vec{v}\cdot\vec{v}_2 = a_1\vec{v}_1\cdot\vec{v}_2 + a_2\vec{v}_2\cdot\vec{v}_2$$

So,

$$\boxed{a_2 = \frac{\vec{v}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}}$$

and:

$=0$

orthogonal

$$\vec{v}\cdot\vec{v}_1 = a_1\vec{v}_1\cdot\vec{v}_1 + a_2\vec{v}_2\cdot\vec{v}_1$$

So,

$$\boxed{a_1 = \frac{\vec{v}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}}$$

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{Say } v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

What are a_1 & a_2?

$$a_1 = \frac{\langle 2,3\rangle \cdot \langle -1, 1\rangle}{\langle 1, 1\rangle \cdot \langle 1, 1\rangle} = \frac{5}{2}$$

$$a_2 = \frac{\langle 2,3\rangle \cdot \langle -1, 1\rangle}{\langle -1, 1\rangle \cdot \langle -1, 1\rangle} = \frac{1}{2}$$

So, $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ✓

Trigonometric Series

Let's consider our friend eigenvalue problem:

$$x'' + \lambda x = 0, \quad x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi)$$

Eigenvalues: $0, n^2$

Eigenfunctions: $\boxed{1}$, $\sin nt$, $\cos nt$

QUESTION: Given a 2π-periodic function $f(t)$, can we express in terms of these eigenfunctions?

Assume we can write

$$f(t) = \frac{a_0}{2} \cdot \boxed{1} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

weird convention

What are the coefficients, a_0, a_n, b_n???

Consider Eigenfunction: $\cos(mt)$, some m. $m \geq 1$

$$\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt = \boxed{\int_{-\pi}^{\pi} \frac{a_0}{2}\cos(mt)\,dt} + \sum_{n=1}^{\infty}\left[a_n \boxed{\int_{-\pi}^{\pi}\cos(nt)\cos(mt)\,dt} + b_n \boxed{\int_{-\pi}^{\pi}\sin(nt)\cos(mt)\,dt}\right]$$

Integrate cos on these bounds, and you will ALWAYS get $\underline{\underline{0}}$

will also be zero

This is $=0$ if $n \neq m$, but is <u>NOT</u> zero if/when n does $= m$. In fact, $\int_{-\pi}^{\pi} \cos^2(mt)\,dt = \pi$

$= a_m \pi$

So, $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt.$

Similarly (multiply both sides by $\sin(nt)$ & integrate $-\pi$ to π)

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt$$

We can do all of this, because of the fact that the eigenfunctions are <u>orthogonal!</u>

Summary :

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

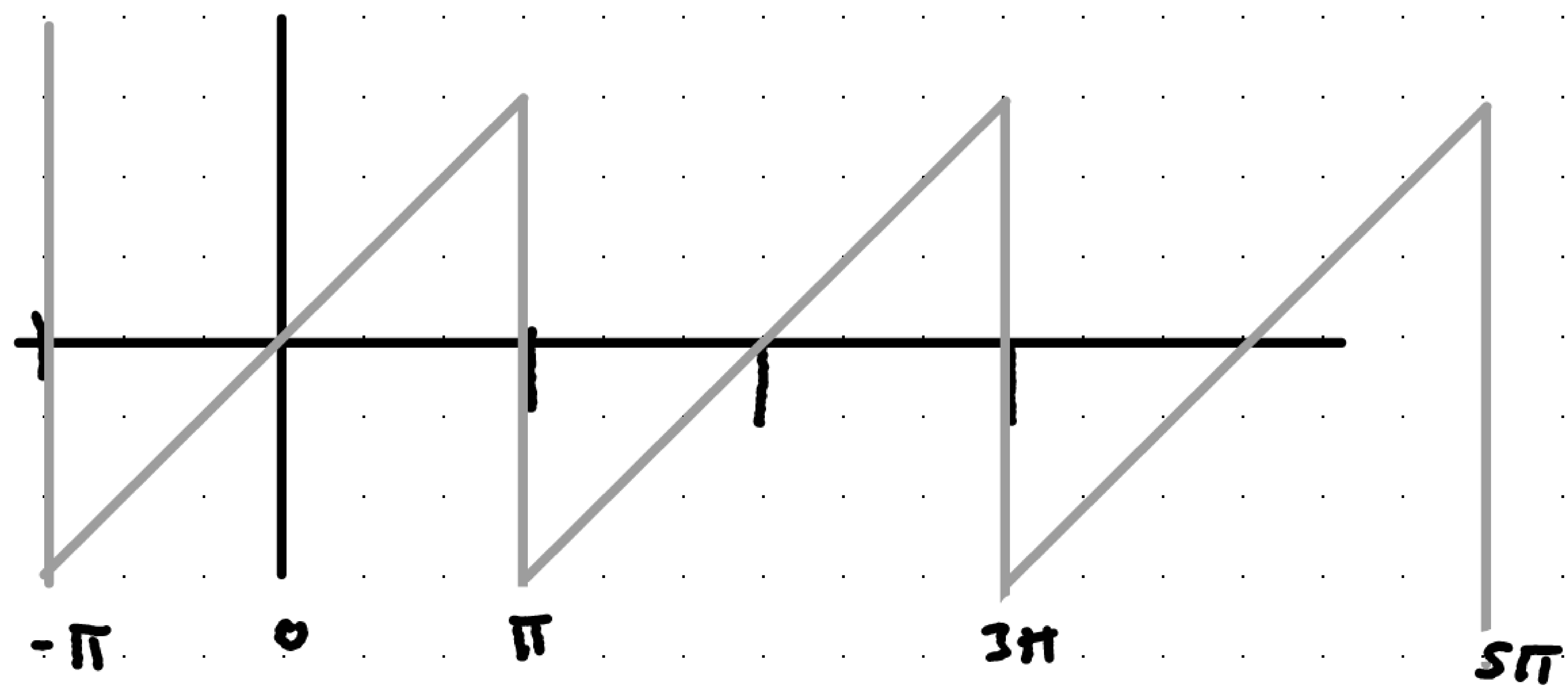
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

Fourier Coefficients for 2π periodic functions

Example.

Take the function $f(t)=t$ for t in $(-\pi, \pi]$.

Extend $f(t)$ periodically and write it as a fourier series.
this function is known as **the Sawtooth**



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\,dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\,dt$$

Let's use these!

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\underbrace{\cos(nt)}_{\text{Even}}dt$$

Odd

Symmetric Interval.

Even x odd = odd

$$= 0!$$

Recall:

cos is an even function

$$\cos(-t) = \cos(t)$$

Sin(nt) is an odd function

$$\sin(-t) = -\sin(t)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\underbrace{\sin(nt)}_{\text{odd}}dt$$

odd

Symmetric Interval

Odd x odd = Even

$$= \frac{2}{\pi}\int_{0}^{\pi} t\sin(nt)\,dt$$

You can multiply by two and chop off the bottom, this makes sense!

IBP ☆

$dv = \sin(nt)dt$

$v = \frac{-1}{n}\cos(nt)$

$u = t$

$du = dt$

$$= \frac{2}{\pi}\left[\frac{-t}{n}\cos(nt)\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos(nt)\,dt\right.$$

This goes to zero

$$= \frac{2}{\pi}\left[-\frac{\pi}{n} \cos(n\pi + 0 \right]$$

$$= \frac{-2}{n} \cos(n\pi)$$

$$= \frac{-2}{n} (-1)^n$$

$$\cos(n\pi) = (-1)^n$$

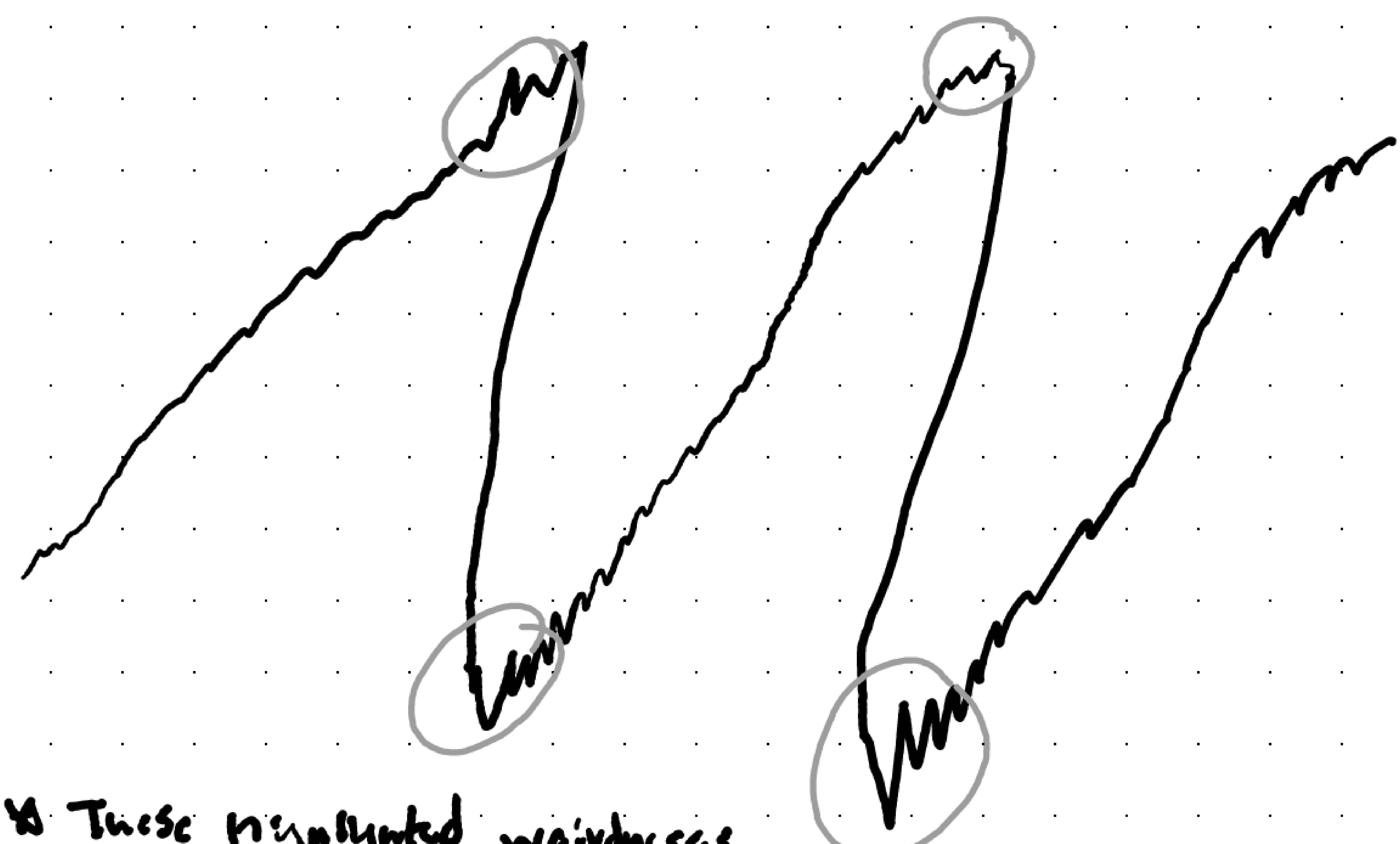
Unit Circle.

So the periodic extension is:

$$\sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(nt)$$

And this approximates our sawtooth wave at high n!

One thing to note, is at the point of discontinuity, in this case π & $-\pi$ on the extension, we observe something known as <u>Gibbs phenomena</u>.



* These highlighted weirdnesses

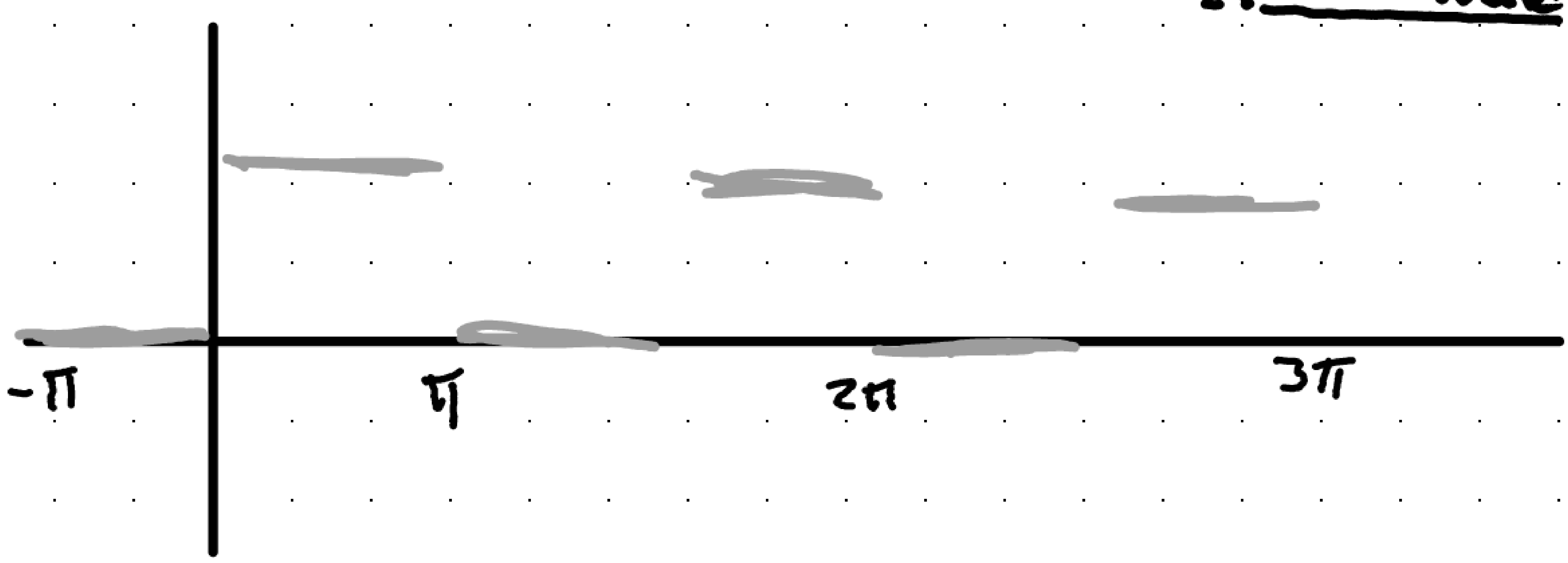
The better the approximation, the less of an effect GIBBS phenomena will have. But, it will always be present.

Example:

Take the function

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \leq 0, \\ \pi & \text{if } 0 < t \leq \pi. \end{cases}$$

Extend $f(t)$ periodically and write it as a Fourier Series. This function or its variants appear often in applications, and the function is called the Square Wave



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\pi^2 = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{\pi}\left[\overbrace{\int_{-\pi}^{0} 0\cdot\cos(nt)\,dt}^{0} + \int_{0}^{\pi} \pi\cdot\cos(nt)\,dt\right]$$

$$= \sin(nt)\cdot\frac{1}{n}\Big|_{0}^{\pi}$$

$$= 0$$

$$b_n = \frac{1}{\cancel{\pi}}\int_{0}^{\pi} \cancel{\pi}\sin(nt)\,dt = -\frac{\cos(nt)}{n}\Big|_{0}^{\pi}$$

↳ Skipping the first interval, I know it is going to be 0

$$= -\frac{\cos(nt)}{n} + \frac{1}{n} = \frac{1}{n}\left[1-(-1)^n\right]$$

~ Hold up... Let's expand this out a bit

$b_1 = [1-(-1)] = 2$

$b_2 = \frac{1}{2}[1-(-1)^2] = 0$

$b_3 = \frac{1}{3}[1-(-1)^3] = \frac{2}{3}$

$b_4 = \sim\sim \quad = 0$

See a pattern!

$$b_n = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Fourier Series for Square Wave:

$$\frac{\pi}{2} + \sum_{n=1}^{\infty} b_n \sin(nt)$$

To accomodate for what we just saw with b_n being equal to zero:

$$\frac{\pi}{2} + \sum_{\substack{n=1, \\ n \text{ is odd}}}^{\infty} \frac{2}{n} \sin(nt)$$