Fourier Series

We can break any periodic wave into cos and sin waves!

There are four ways to write Fourier Series:

① Full Sin/Cos F-Series

$$\underline{\underline{f(t)}} = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(n\pi t/L) + b_n \sin(n\pi t/L)$$

Fourier Approximation. f(t) is an approximation of the function

1) Find a_0

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt}$$

2) Find a_n

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\pi t/L)\,dt}$$

3) Find b_n

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(n\pi t/L)\,dt}$$

- One complete period is T. Half of that is L

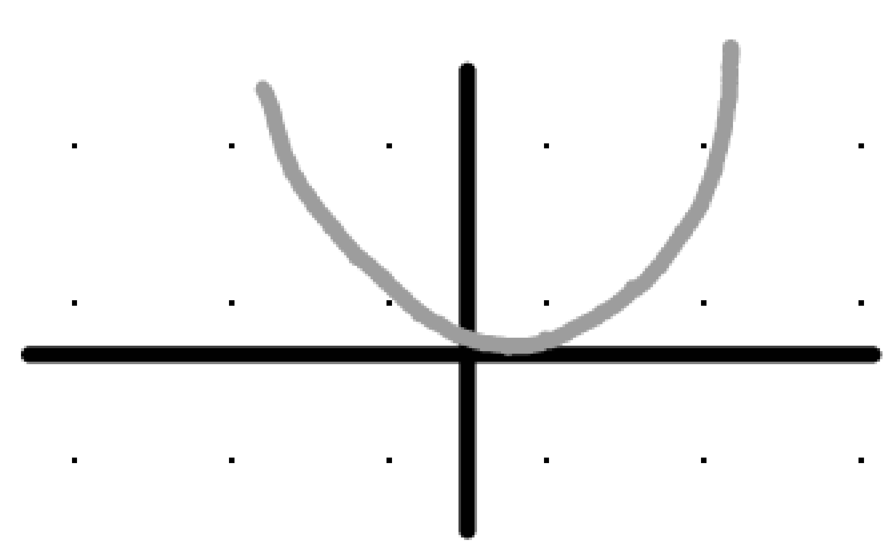
② Cosine F-Series (when we have even function)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(n\pi t/L)$$

In an even function, $f(-t) = f(t)$.

ex: $f(t) = t^2$
$f(-t) = (-t)^2 = t^2$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos(n\pi t/L)\,dt$$



← Even function

$$\int_{-L}^{L} = \cancel{\int_{L}^{0}} + 2\int_0^L$$

This is where that comes from!

(3) Sine f-Series (when we have odd function)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} b_n \sin(n\pi t/L)$$

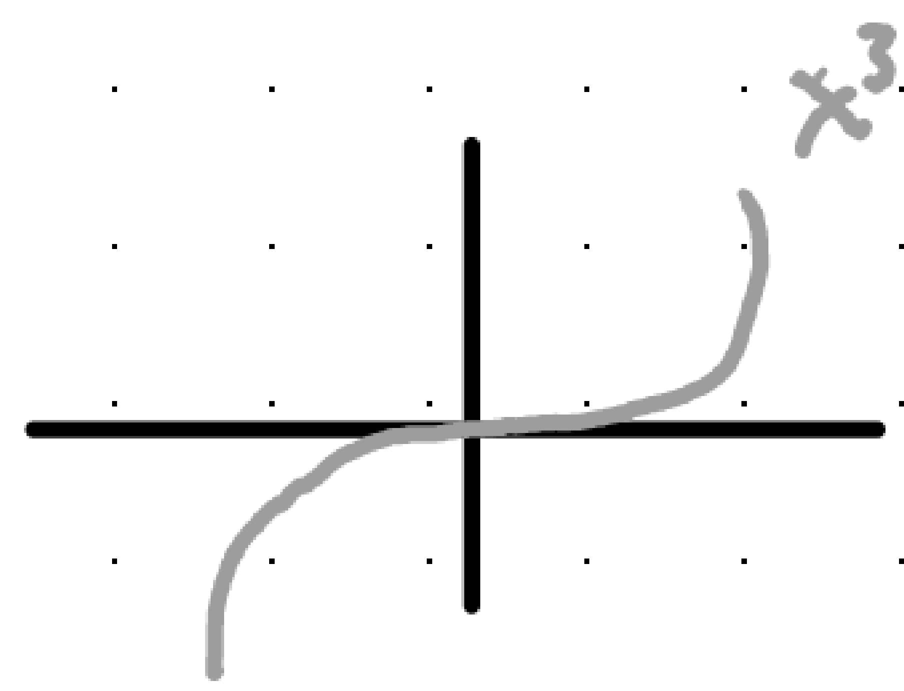
In an odd function, $f(-t) = -f(t)$.

ex: $f(t) = t^3$

$f(-t) = (-t)^3$

$= -t^3$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin(n\pi t/L)\,dt$$



← odd function

$$\int_{-L}^{L} = \cancel{\int_{L}^{0}} + 2\int_0^L$$

This is where that comes from!

4 Imaginary Case

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi t}{L}}$$

$$c_n =$$

Integration By Parts

$$\int u\,dv = uv - \int v\,du$$

Often $\begin{cases} u = t^p \\ dv = \sin\ldots \text{ or } \cos\ldots \xrightarrow{\int} v \end{cases}$

Integrals of Sin and Cos

$$\int \sin(at)\,dt = -\frac{1}{a}\cos(at)$$

$$\int \cos(at)\,dt = \frac{1}{a}\sin(at)$$

!! Special Thing To Look out for !!

$$\sin(n\pi) = 0$$

$$\cos(n\pi) = (-1)^n$$

Example One

$$\ddot{y} + \dots\dots = \underbrace{t}_{\to [-\pi, \pi]} \qquad T = 2\pi$$

1) Check if odd or Even

odd : use Sine form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} b_n \sin(n\pi t/L)$$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin(n\pi t/L)\,dt$$

$$L = \frac{2\pi}{2} = \pi$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{t}_{u} \underbrace{\sin\left(\frac{n\cancel{\pi} t}{\cancel{\pi}}\right)dt}_{dv}$$

IBP

$$\int u\,dv = uv - \int v\,du$$

$$u = t \longrightarrow du = dt$$

$$dv = \sin(nt)dt \xrightarrow{\int} V = \int \sin nt\,dt = -\frac{1}{n}\cos nt$$

$$b_n = \frac{2}{\pi}(t)\left(-\frac{1}{n}\cos nt\Big|_0^{\pi}\right) - \int_0^{\pi}\left(-\frac{1}{n}\right)\cos nt\,dt$$

$$b_n = \frac{2}{\pi}\left(-\frac{t}{n}\cos nt\Big|_0^{\pi}\right) + \left(\frac{1}{n^2}\sin(nt)\Big|_0^{\pi}\right)$$

0

$$b_n = \frac{2}{\pi}\left(-\frac{\pi}{n}\cos n\pi - \frac{0}{n}\cos 0\right)$$

0

$$\boxed{b_n = \frac{2}{\pi}\left(-\frac{\pi}{n}\cos n\pi\right)}$$

$$t \quad f(t) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nt$$

$$\hookrightarrow -\frac{2}{1}(-1)^1 \sin t + \left(-\frac{2}{2}\right)(-1)^2 \sin 2t \ldots\ldots$$

Example: $m=1$ Mass damper spring
$c=2$
$K=1$

$$f(t)=t^2+\pi \qquad -\pi < t < \pi$$

$$m\ddot{y}+c\dot{y}+K=f(t)$$

$$\underbrace{\ddot{y}+2\dot{y}+y=t^2+\pi}_{f(t)}$$

use Fourier Series for LHS, and then Solve:

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1) Check if odd or even

$$f(-t)=-t^2+\pi = t^2+\pi \quad \text{(The Same)}$$

- Even, will use Cosine form

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{n=\infty} a_n\cos(n\pi t/L)$$

$$a_n=\frac{2}{L}\int_0^L f(t)\cos(n\pi t/L)$$

$$T=2\pi,\ L=\pi$$

$$a_0=\frac{2}{L}\int_0^L f(t)\,dt$$

$\cos(0)=1$

$$a_0=\frac{2}{\pi}\int_0^\pi (t^2+\pi)\,dt$$

$$a_0=\frac{2}{\pi}\left(\frac{t^3}{3}+\pi t\right)\Big|_0^\pi\Big) \rightarrow \frac{2}{\pi}\left(\frac{\pi^3}{3}+\pi^2\right)$$

$$=\frac{2}{3}\pi^2+2\pi$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos(n\pi t/L)\,dt$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (t^2+\pi)\ (n\cancel{\pi}t/\cancel{\pi})\,dt$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{t^2\cos(nt)dt}_{I_1} + \int_0^{\pi} \underbrace{\pi\cos(nt)}_{I_2}dt$$

$$a_n = \frac{2}{\pi}(I_1 + I_2)$$

$$I_2 = \pi\int_0^{\pi}\cos nt\,dt \longrightarrow \pi\left(\frac{1}{n}\sin nt\,\Big|_0^{\pi}\right)$$

$$0$$

$$I_1 = \int_0^{\pi} \underbrace{t^2}_{u}\underbrace{\cos(nt)}_{dv}\,dt$$

IBP $uv - \int v\,du$

$u = t^2$ $du = 2t\,dt$

$dv = \cos nt\,dt$ $v = \frac{1}{n}\sin \pi t$

$$\frac{t^2}{n}\sin(nt) - \frac{2}{n}\int_0^{\pi} \underbrace{t}_{u}\ \underbrace{\sin(nt)}_{dv}\,dt$$

$$I_1 = \frac{t^2}{n}\sin(nt) - \frac{2}{n}\int_0^{\pi} \underbrace{t}_{u}\,\underbrace{\sin(nt)}_{dv}\,dt$$

$u = t \qquad du = dt$

$dv = \sin(nt) \qquad v = -\frac{1}{n}\cos(nt)$

$uv - \int v\,du$

$$-\frac{t}{n}\cos(nt) + \underbrace{\frac{1}{n}\int_0^{\pi}\cos nt\,dt}_{\frac{1}{n}\sin(nt)}$$

$$-\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)$$

$$I_1 = \frac{t^2}{n}\sin(nt) - \frac{2}{n}\left(-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right.$$

$$a_n = \frac{2}{\pi}(I_1 + \underbrace{I_2}_{0})$$

Evaluate Now!

$$a_n = \frac{2}{\pi}\left(\frac{t^2}{n}\sin(nt) - \frac{2}{n}\left(-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right. + 0\right)\Bigg|_0^{\pi}$$

Evaluate Now!

$$a_n = \frac{2}{\pi}\left(\frac{t^2}{n}\sin(nt) - \frac{2}{n}\left(-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt + 0\right)\right)\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\left(0 - \frac{2}{n}\left(-\frac{\pi}{n}\cos n\pi + \frac{1}{\pi^2}\sin n(\pi) + 0\right)\right.$$

$$a_n = \frac{4}{n^2}\cos n\pi \longrightarrow \boxed{\frac{4}{n^2}(-1)^n}$$

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(n\pi t/L)}$$

$$f(t) = \frac{\frac{2}{3}\pi^2 + 2\pi}{2} + \sum_{n=1}^{n=\infty}\left(\frac{4}{n^2}(-1)^n\right)\cos\left(\frac{n\pi t}{\pi}\right)$$

$$f(t) = \frac{2}{6}\pi^2 + \pi + \sum_{n=1}^{n=\infty}\left(\frac{4}{n^2}(-1)^n \cos(nt)\right)$$

Okay! Now solve.

Const term $\frac{a_0}{2}$

$$\ddot{y}+2\dot{y}+y=\underbrace{\frac{1}{3}\pi^2+\pi}_{y_{p_0}}+\underbrace{\sum_{n=1}^{n=\infty}\left(\frac{4}{n^2}(-1)^n\cos(nt)\right)}_{y_{p_n}}$$

$$y_p=y_{p_0}+y_{p_n}$$

y_{p_0}

$$\ddot{y}+2\dot{y}+y=\frac{1}{3}\pi^2+\pi$$

$$y_p=A\ ,\ \dot{y}_p=0\ ,\ \ddot{y}_p=0$$

$$0+2(0)+\underbrace{A=\frac{1}{3}\pi^2+\pi}$$

$$y_p=\frac{1}{3}\pi^2+\pi$$

Y_{PN}

$$\ddot{y} + 2\dot{y} + y = \sum_{n=1}^{n=\infty} \left(\frac{4}{n^2} (-1)^n \cos(nt) \right)$$

A little diversion! don't expand out forever!

Also, Aux problem!

$$\cos nt = Re[e^{int}]$$

While solving this, just take it out. We can put it back later (superposition)

$$\ddot{y} + 2\dot{y} + y = \sum_{n=1}^{n=\infty} \left(\underbrace{\frac{4}{n^2}(-1)^n}_{\text{can either bring in, or leave out}} RE[e^{int}] \right)$$

Sub Back Later!

$$\ddot{Y} + 2\dot{Y} + Y = e^{int}$$

$$Y = De^{int}, \quad \dot{Y} = Din e^{int} \quad \ddot{Y} = -Dn^2 e^{int}$$

$$D\cancel{e^{int}}(-n^2 + 2in + 1) = \cancel{e^{int}}$$

$$D = \frac{1}{(-n^2 + 2in + 1)} \qquad Y_{PN} = \frac{1}{(-n^2 + 2in + 1)} e^{int}$$

Aux Y_p: $Y_{pN} = \frac{1}{(-n^2 + 2in + 1)} e^{int}$

$$\frac{1}{(-n^2+1+2in} \cdot \frac{-n^2+1-2in}{-n^2+1-2in} (\cos nt + i \sin nt)$$

$$\frac{-n^2-2in+1}{(n^2+1)^2+4n^2} \cos nt + i \sin nt$$

$$Re[Y_p] = \frac{-n^2+1}{(n^2+1)^2+4n^2} \cos nt + \frac{2n}{(n^2+1)^2+4n^2} \sin nt$$

$$y_{p_n} = \left(-\frac{4}{n}(-1)^n\right) RE[Y_p]$$

$$y_{p_n} = \left(-\frac{4}{n}(-1)^n\right)\left(\frac{-n^2+1}{(n^2+1)^2+4n^2} \cos nt + \frac{2n}{(n^2+1)^2+4n^2} \sin nt\right)$$

$$y_p = y_{p_0} + y_{p_n}$$

$$y_p = \frac{1}{3}\pi^2 + \pi + \sum_{n=1}^{\infty} \left(-\frac{4}{n}(-1)^n\right)\left(\frac{-n^2+1}{(n^2+1)^2+4n^2} \cos nt + \frac{2n}{(n^2+1)^2+4n^2} \sin nt\right)$$

Example Two, Complex Form

$$f(t)\begin{cases} t-2\pi & 0\le t\le 2\pi \\ 0 & 2\pi<t<4\pi \end{cases} \rightarrow 0<t<4\pi$$

Complex Fourier Form

$T=4\pi,\ L=2\pi$

$$f(t)=C_0+\sum_{n=-\infty}^{n=\infty} C_n e^{in\pi t/L}$$

$$c_n=\frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\pi t}\,dt \qquad C_0=\frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$C_0=\frac{1}{4\pi}\int_0^{4\pi} f(t)\,dt = \frac{1}{4\pi}\left(\int_0^{2\pi} t-2\pi\,dt+\cancel{\int_{2\pi}^{4\pi} 0\,dt}\right)$$

$$=\frac{1}{4\pi}\left(\frac{t^2}{2}-2\pi t\,\Big|_0^{2\pi}\right)$$

$$=\frac{1}{4\pi}\left(\frac{4\pi^2}{2}-4\pi^2\right)$$

$$=\frac{\pi}{2}-\pi=\boxed{-\frac{\pi}{2}}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-\frac{in\pi t}{L}}\, dt$$

$$c_n = \frac{1}{4\pi}\int_{0}^{2\pi} (t-2\pi)\, e^{\frac{-int}{2}}\, dt + \cancel{\int_{2\pi}^{4\pi} 0\, e^{\frac{-in\pi t}{2\pi}}\, dt}$$

0

$$\frac{1}{4\pi}\Big(\underbrace{\int_{0}^{2\pi} t\, e^{\frac{int}{2}}\, dt}_{I_1} - 2\pi \underbrace{\int_{0}^{2\pi} e^{\frac{-int}{2}}\, dt}_{I_2}\Big)$$

$$c_n = \frac{1}{4\pi}(I_1 - I_2)$$

$$I_2 = 2\pi \int_{0}^{2\pi} e^{-\frac{int}{2}}\, dt \qquad = 2\pi\left(\frac{2}{-in}\, e^{-\frac{int}{2}}\right)$$

$$I_1 = \int_{0}^{2\pi} t\, e^{\frac{int}{2}}\, dt$$

$$\int u\, dv = uv - \int v\, du$$

$$e^{in\pi} = (-1)^n \qquad \cos n\pi = (-1)^n$$

$$u = t \qquad dv = e^{\frac{-int}{2}}\, dt$$

$$du = 1\, dt \qquad v = -\frac{2}{in}\, e^{\frac{-int}{2}}$$

$$-\frac{2t}{in}\, e^{-\frac{int}{2}} - \int -\frac{2}{in}\, e^{-\frac{int}{2}}\, dt$$

$$I_1 = -\frac{2t}{in} e^{-\frac{int}{2}} - \int -\frac{2}{in} e^{-\frac{int}{2}} dt$$

$$-\frac{2t}{in} e^{-\frac{int}{2}} + \frac{2}{in}\left(-\frac{2}{in} e^{-\frac{int}{2}}\right)$$

$$-\frac{2t}{in} e^{-\frac{int}{2}} + \frac{2}{in}\left(-\frac{2}{in} e^{-\frac{int}{2}}\right)$$

$$-\frac{2t}{in} e^{-\frac{int}{2}} + \frac{4}{n^2} e^{-\frac{int}{2}}$$

$$c_n = \frac{1}{4\pi}\left(-\frac{2t}{in} e^{-\frac{int}{2}} + \frac{4}{n^2} e^{-\frac{int}{2}} - 2\pi\left(\frac{2}{-in} e^{-\frac{int}{2}}\right)\right)$$

$$\frac{e^{-\frac{int}{2}}}{4\pi}\left(-\frac{2t}{in} + \frac{4}{n^2} + \frac{4\pi}{in}\right)$$

$$= \frac{e^{-\frac{int}{2}}}{4\pi}\left(\frac{4\pi - 2t}{in} + \frac{4}{n^2}\right)\Bigg|_0^{2\pi}$$

$$= \frac{e^{-in\pi}}{4\pi}\left(\frac{4}{n^2}\right) - \frac{e^{-in(0)}}{4\pi}\left(\frac{4\pi}{in} + \frac{4}{n^2}\right)$$

$$C_N = \frac{e^{-in\pi}}{4\pi}\left(\frac{4}{n^2}\right) - \frac{e^{-in(0)}}{4\pi}\left(\frac{4\pi}{in} + \frac{4}{n^2}\right)$$

$$= \frac{(-1)^n}{4\pi}\left(\frac{4}{n^2}\right) - \frac{1}{4\pi}\left(\frac{4\pi}{in} + \frac{4}{n^2}\right)$$

$$= \frac{(-1)^n}{\pi n^2} - \frac{1}{in} + \frac{1}{\pi n^2}$$

$$= \boxed{\frac{(-1)^n + 1}{\pi n^2} - \frac{1}{in}}$$

* Different from Hossein. Will use wk solutions

$$f(t) = C_0 + \sum_{n=-\infty}^{n=\infty} C_n e^{in\pi t/L}$$

$L = 2\pi$

$$f(t) = \underbrace{\frac{-\pi}{2}}_{y_{p0}} + \sum_{n=-\infty}^{n=\infty} \underbrace{C_n e^{\frac{int}{2}}}$$

$y_{p0} = A$

$\dot{y}_{p0} = 0$

$\ddot{y}_{p0} = 0$

$y_{p0} = -\frac{\pi}{6}$

$y_{pn} = De^{\frac{int}{\pi}}$

$\dot{y}_{pn} = D\left(\frac{in}{\pi}\right)e^{\frac{int}{\pi}}$

$\ddot{y}_{pn} = -D\left(\frac{n^2}{\pi^2}\right)e^{\frac{int}{\pi}}$

Skip a few plugging in steps....

$$D\cancel{e^{\frac{int}{\pi}}}\left(-\frac{n^2}{\pi}+\frac{2in}{\pi}+3\right) \; C_n \cancel{e^{\frac{int}{\pi}}}$$

$$D = \frac{C_n}{\left(-\frac{n^2}{\pi}+\frac{2in}{\pi}+3\right)}$$

$$y_{pn} = \sum_{n=-\infty}^{n=\infty} \frac{C_n}{\left(-\frac{n^2}{\pi}+\frac{2in}{\pi}+3\right)} e^{\frac{int}{\pi}}$$

$$y_p = y_{p_0} + y_{pn}$$

$$y_p = \frac{-\pi}{6} + \sum_{n=-\infty}^{n=\infty} \frac{C_n}{\left(-\frac{n^2}{\pi}+\frac{2in}{\pi}+3\right)} e^{\frac{int}{\pi}}$$