

## Laplace Transformations

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$s = \sigma + j\omega$$

### Some Rules:

Linearity

$$1) \mathcal{L}\{a f(t) + b g(t)\} = a F(s) + b G(s)$$

$$2) \mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$3) \mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

### 4) Laplace Transform Table.

Like how  $\frac{d}{dt} \sin t = \cos t$ , Laplace has similar rules!

## Examples:

$$\ddot{y} + y = 0, \quad y(0) = 1 \quad y'(0) = 0$$

Find Laplace Transform

$$L(\ddot{y}) + L(y) = 0$$



$$s^2 Y(s) - s Y(s) - \cancel{y(0)}^1 + Y(s) = 0$$

$$Y(s)(s^2 + 1) - s = 0$$

$$Y(s) = \frac{s}{s^2 + 1}$$

Now, Let's take the inverse Laplace to get back to t!

From Laplace Table:

$$\frac{s}{s^2 + \beta^2} = \cos \beta t$$

$$y(t) = \cos t$$

## Inverse Laplace

- Inverse Laplace (using the Laplace table)

$$f(s) = \frac{3}{s^2 + 9}, \quad \mathcal{L}^{-1} \left[ \frac{3}{s^2 + 9} \right] \quad x \text{ from Table}$$

$\text{Sinh}(at) = \frac{a}{s^2 + a^2}$

$= \text{Sinh} 3t$

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- Partial Fraction Method (using Partial Fractions)

Example:

$$f(s) = \frac{s+3}{(s+1)(s+2)}$$

→ This is trickier!  
How to decompose?

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

①

For each distinct linear factor ( $s+a$  for example) we will assign:

②

$$\frac{A}{s+a}$$

For even repeated linear factor ( $(s+a)^n$  for example) we will assign:

$$\frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \dots + \frac{A_n}{(s+a)^n}$$

③

For each irreducible factor (such as  $s^2 + 6s + c$ ), assign

$$\frac{Bs + C}{s^2 + 6s + c}$$

→ Multiply through common (initial) denominator & find A, B, C by equating coefficients.

Back to that example

From earlier...

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

Multiply out  $\rightarrow s+3 = A(s+2) + B(s+1)$

$$s+3 = As + 2A + Bs + B$$

$$As + Bs = s \rightarrow A + B = 1$$

$$2A + 1 = 3 \rightarrow A = 2$$

$$\text{So } B = -1$$

$$F(s) = \frac{2}{(s+1)} - \frac{1}{(s+2)}$$

From Table

$$F(t) = 2e^{-t} - e^{-2t}$$

Example:

$$F(s) = \frac{s+2}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

Multiply  
out

$$s+2 = A(s+1) + B$$

$$s+2 = As + A + B$$

s terms

$$As = s \quad A = 1$$

constants

$$A + B = 2 \quad B = 2 - s$$

## Example 2

$$f(s) = \frac{2s+5}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

multiply  
out

$$2s+5 = A(s^2+4) + (Bs+C)(s+1)$$

$$2s+5 = As^2 + 4A + Bs^2 + Bs + Cs + C$$

$s^2$  terms:

$$0 = A + 0$$

$s$  terms

$$A = \frac{3}{5}$$

$$2 = B + C$$

$$B = -\frac{3}{5}$$

const

$$5 = 4A + C$$

$$C = \frac{13}{5}$$

## Cover-up Method

$$\frac{3}{(s+2)(s-3)} = \frac{A}{(s+2)} + \frac{B}{(s-3)}$$

$$3 = A(s-3) + B(s+2)$$

To solve for each constant, you're going to want to set s equal to something that will cancel out one of the terms.

Say solving for A, you'd set  $s = -2$ , making  $B = 0$ . You can then solve for A.

Solving for B, you'd set  $s = 3$ , making  $A = 0$ . You can now solve for B.

## First Shift Theorem

(or Exponential Shift Theorem)

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

for example, in the case of  $\mathcal{L}(e^{2t}\cos 3t)$

$$\mathcal{L}\{e^{2t}\cos 3t\} = \frac{s-2}{(s-2)^2 + 9}$$

This is the  
Shift! We can  
call  $(s-2)$ ,  $s$ !

Another example:

$$\mathcal{L}\{\sin 5t\} = \frac{5}{s^2 + 25}$$

$$\mathcal{L}\{e^{-4t}\sin 5t\} = \frac{5}{(s+4)^2 + 25}$$



e shifts  
the  $s$ !

## Completing the Square

$$as^2 + bs + c$$

$$c = \left(\frac{b}{2}\right)^2$$

$$f(s) = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s^2 + 2s + 1) + 1}$$

↓  
 $(s+1)^2$

$$F(s) = \frac{1}{(s+1)^2 + 1} = e^{-t} \sin t$$

If we had say  $\frac{\tau K_0}{(s+1)^2 + 1}$ , factor  $\tau K_0$  out!

Example:

$$F(s) \frac{s}{s^2 + 2s + 5} = \frac{s}{(s^2 + 2s + 1) + 4}$$

$$\left(\frac{1}{2}b\right)^2 \quad 1^2$$

$$\Rightarrow F(s) = \frac{s}{(s+1)^2 + 4}$$

Can't do first shift here because of numerator!

Here's what we'll do here then...

$$F(s) = \frac{(s+1)-1}{(s+1)^2 + (\sqrt{5})^2} = \frac{s+1}{(s+1)^2 + (\sqrt{5})^2} - \frac{1}{(s+1)^2 + (\sqrt{5})^2}$$

↑ we want  $b^2$  for the term

$$e^{-t} \cos(\sqrt{5}t) - \frac{\frac{\sqrt{5}}{\sqrt{5}}}{(s+1)^2 + (\sqrt{5})^2} \leftarrow \text{Want to have } B!$$

$$- \frac{1}{\sqrt{5}} \frac{\sqrt{5}}{(s+1)^2 + (\sqrt{5})^2}$$

$$- \frac{1}{\sqrt{5}} e^{-t} \sin(\sqrt{5}t)$$