

Let's learn how to solve systems of linear ODE's $\vec{x}' = A\vec{x}$

We'll start by looking for a solution of the form $\vec{x} = \vec{v}e^{\lambda t}$

(\vec{v} is a constant vector)

(e.g., $\vec{x}(t) = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} e^t = \begin{bmatrix} e^t \\ \frac{e^t}{2} \end{bmatrix}$)

Then, $\vec{x}' = \vec{v} \lambda e^{\lambda t}$

And $A\vec{x} = A(\vec{v}e^{\lambda t}) = e^{\lambda t}(A\vec{v})$

For \vec{x}' to equal $A\vec{x}$, we must have

$$e^{\lambda t} \lambda \vec{v} = e^{\lambda t} A\vec{v}$$

i.e. $e^{\lambda t} (A\vec{v} - \lambda \vec{v}) = \vec{0}$

i.e. $A\vec{v} = \lambda \vec{v}$

Since $e^{\lambda t}$ is never zero!

$$A\vec{v} = \lambda\vec{v}$$

That is, λ must be an eigenvalue of A , with a corresponding eigenvector!

Example. Find the general solution to

$$\vec{x}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \vec{x}$$

Find eigenvalues of A !

Find characteristic polynomial, $\det(A - \lambda I)$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} =$$

$$(1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - (-1) \begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} +$$

$$4 \begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

Find roots:

$$-\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

Try Easy roots, say $\lambda = 1$

$$-1 + 2 + 5 - 6 = \underline{0}, \text{ so } \lambda = 1 \text{ is a root}$$

$$(\lambda - 1)(-\lambda^2 + \lambda + 6)$$

Want to find quadratic!

$$(\lambda - 1)(\lambda + 3)(\lambda - 2)$$

Eigenvalues!

$$\lambda = 1$$

$$\lambda = -2$$

$$\lambda = 3$$

Next, find eigenvectors:

$$\boxed{\lambda=1}$$

Solve $[A - \lambda I | 0]$

$$\left[\begin{array}{ccc|c} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array} \right]$$



$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

do rref until you
get a row of
zeros on bottom

Here, we've got

$$x + z = 0$$

$$y - 4z = 0$$

Three unknowns, two equations. So, let's
pick a free variable!

$$\text{Let } z=1, \quad x=-1, \quad y=4$$

$$v = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \begin{matrix} x \\ y \\ z \end{matrix}$$

If you were to pick a different free value,
you'd have a "different picture" for say, but
it would just be a constant multiple.
And remember, for eigenvectors, that is
no issue at all.

$$\lambda_2 = -2$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array} \right]$$

↓

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Ref down to zero row, and 1's on pivot

$$x + z = 0$$

$$y - z = 0$$

$$\text{Set } x = 1, z = -1$$

$$y = -1$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\lambda_3 = 3$$

$$\left[\begin{array}{ccc|c} -2 & -1 & 4 & 6 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array} \right]$$

Ref →

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$x - z = 0$$

$$y - 2z = 0$$

$$\text{Set } x = 1, y = 2, z = 1$$

So linearly independent solutions are:

$$\vec{x}_1 = \vec{v}_1 e^{\lambda_1 t} = e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = \vec{v}_2 e^{\lambda_2 t} = e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{x}_3 = \vec{v}_3 e^{\lambda_3 t} = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

General Solution

$$x(t) = C_1 e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + C_3 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Example with complex eigenvalues

Example. Find the general solution to

$$\vec{x}' = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \vec{x}$$

Eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} 6 - \lambda & -1 \\ 5 & 2 - \lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 12 + 5$$

$$= \lambda^2 - 8\lambda + 17$$

$$= (\lambda - 4)^2 + 1$$

$$\frac{8}{2} = 4^2 = 16$$

$$\boxed{\lambda = 4 \pm i}$$

Eigenvectors

only really need one of the pairs. Pick $\boxed{\lambda = 4 + i}$

$$\left[\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array} \right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

Solution

$$\vec{X}(t) = e^{(4+i)t} \begin{bmatrix} 1 \\ z-i \end{bmatrix} \text{ is a } \overset{\text{Complex}}{\text{solution}}$$

Let's use the real & imaginary parts to find two linearly independent real-valued solutions:

Let's use Euler's identity

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\vec{X}(t) = e^{4t} \cdot e^{it} \begin{bmatrix} 1 \\ z-i \end{bmatrix}$$

$$= e^{4t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ z-i \end{bmatrix}$$

$$= \begin{bmatrix} e^{4t} \cos t + i e^{4t} \sin t \\ z e^{4t} \cos t - i e^{4t} \cos t + z i e^{4t} \sin t + e^{4t} \sin t \end{bmatrix}$$

$$= e^{4t} \underbrace{\begin{bmatrix} \cos t \\ z \cos t + \sin t \end{bmatrix}}_{x_1(t)} + i e^{4t} \underbrace{\begin{bmatrix} \sin t \\ z \sin t - \cos t \end{bmatrix}}_{x_2(t)}$$

Solution

$$\vec{x}(t) = c_1 e^{4t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}$$

