

Non Homogeneous Linear Equations (Constant coefficients)

$$a\ddot{y} + b\dot{y} + cy = g(x)$$

Consider complementary Non homogeneous D.E:

$$a\ddot{y} + b\dot{y} + cy = 0$$

and find general Solution: y_c

If we can be so lucky as to find one, just one particular solution y_p to $a\ddot{y} + b\dot{y} + cy = g(x)$ then any other solution y to $a\ddot{y} + b\dot{y} + cy = g(x)$ can be written as

$$y = y_p + y_c$$

$$y = g(x) + 0$$

Method of Undetermined Coefficients

Exampk: Solve $\ddot{y} + \dot{y} - 2y = x^2$

First, solve

$$\ddot{y} + \dot{y} - 2y = 0$$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r = -2$$

$$r = 1$$

$$y_c = C_1 e^{-2x} + C_2 e^x$$

Next, try to find one guess: y_p to $\ddot{y} + \dot{y} - 2y = x^2$

$$y_p = Ax^2 + Bx + C$$

Determine A, B, C

$$y_p = 2Ax + B$$

$$\dot{y}_p = 2A$$

Plug in guess & Set equal

$$2A + 2Ax + B - 2(Ax^2 + Bx + C) = x^2$$

$$ZA + ZAx + B - Z(Ax^2 + Bx + C) = x^2$$

$$ZA + ZAx + B - ZAx^2 - 2Bx - 2C = x^2$$

Solve System of Eqs

$$x^2: -2A = 1$$

$$A = -\frac{1}{2}$$

$$x: ZA - 2B = 0$$

$$B = -\frac{1}{2}$$

$$1: ZA + B - C = 0$$

$$C = -\frac{3}{4}$$

So, $y_p = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$

General Solution:

$$y = c_1 e^{-2x} + c_2 e^x - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

Example:

Solve $\ddot{y} - 4y = \underbrace{xc^x}_{y_p 1} + \underbrace{\cos 2x}_{y_p 2}$

First, $\ddot{y} - 4y = 0, r^2 - 4 = 0 \quad r = \pm 2$

$y_c = C_1 e^{2x} + C_2 e^{-2x}$

$y_p 1$

xc^x

$y_p 2$

$\cos 2x$

Duplication Issue

Example: $\ddot{y} + y = \sin x$, $r^2 + 1$, $r = \pm i$

$$y_c = C_1 \sin x + C_2 \cos x$$

Guess for $\sin x$

$$y_p = \underline{\underline{A \sin x + B \cos x}}$$

$$\hookrightarrow y_p = A x \sin x + B x \cos x$$

Solve as normal

Dupe!

When this happens, we need to multiply by x ,

Example:

$$\ddot{y} + 2\dot{y} + 5y = 4xe^{-x} \cos 2x + e^{2x}, \quad y(0) = 1, \quad \dot{y}(0) = 2$$

Variation Of Parameters

Solve

$$\ddot{y} + y = \tan x, \quad 0 < x < \frac{\pi}{2}$$

First, $\ddot{y} + y = 0, \quad y_0 = C_1 \cos x + C_2 \sin x$

guess

$$y_p = A \tan x \dots$$

Second problem...

This will go FOR EVER! MJC falls flat here,
because no matter how many terms you add,
you'll never have a genuinely different derivative.

Variation of Parameters Method.

guess:

$$y_p = u_1(x) \cos x + u_2(x) \sin x$$

$\underbrace{\quad}_{\text{functions?}} \quad \overbrace{\quad}^{\uparrow}$

$$\dot{y}_p = \dot{u}_1 \cos x - u_1 \sin x + \dot{u}_2 \sin x + u_2 \cos x$$

TRY TO KEEP THINGS SIMPLE

Suppose: $\dot{u}_1 \cos x + \dot{u}_2 \sin x = 0$

so, $\dot{y}_p = -u_1 \sin x + u_2 \cos x$

$$\ddot{y}_p = -\dot{u}_1 \sin x - u_1 \cos x + \dot{u}_2 \cos x - u_2 \sin x$$

Plug into LHS:

$$\ddot{y}_p + y_p = \tan x$$

$$\begin{aligned} & -\dot{u}_1 \sin x - u_1 \cos x + \dot{u}_2 \cos x - u_2 \sin x + u_1 \cos x + \cancel{u_2 \sin x} \\ & -\dot{u}_1 \sin x + \dot{u}_2 \cos x = \tan x \end{aligned}$$

Solve for u_1 & u_2 ?

We have two Eqn's

$$\dot{U}_1 \cos x + \dot{U}_2 \sin x = 0$$

$$-\dot{U}_1 \sin x + \dot{U}_2 \cos x = \tan x$$

Cramer's Rule have.

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix}$$

$$\dot{U}_1 = \cos x - \sin x$$

$$\dot{U}_2 = \sin x$$

Integrate to get U_1 & U_2

$$U_1 = \int (\cos x - \sin x) dx = \sin x - \ln(\sec x + \tan x)$$

$$U_2 = \int \sin x dx = -\cos x$$

$$Y_P = [\sin x - \ln(\sec x + \tan x)] \cos x - \cos x \sin x$$

