Review

Overdamped $c^2 > 4mk$ $\lambda =$ 2 real distinct root

$y_h = Ae^{\lambda t} + Be^{\lambda t}$

Critically Damped

$c^2 = 4mk$ $\lambda =$ Single real repeated root

$y_h = Ae^{\lambda t} + Be^{\lambda t}$

Underdamped

$c^2 < 4mk$ $\lambda = a \pm ib$

$y_h = e^{\alpha}(A\cos\beta t + B\sin\beta t)$

Comes from Eulers formula

$Ae^{(a+ib)t} + Be^{(a-ib)t}$

Example

$$\ddot{y} + 2\dot{y} + y = e^{-t}$$

$$\lambda^2 + 2\lambda + 1 = 0 \qquad \lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$= \frac{-(2) \pm \sqrt{2^2 - 4(1)(1)}}{2(1)}$$

$$y_h = C_1 e^{-1t} + C_2 t e^{-1t}$$

$c^2 = 4mk$ Critically Damped

$\lambda = -1$

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y_p: e^{-t}

Guess Ae^{-t} dupe!
$\downarrow$
tAe^{-t} dupe!
$\downarrow$
t^2Ae^{-t} good.

$y_p = At^2e^{-t}$ ← THIS IS A PRODUCT!!!

$y'_p = A(2te^{-t} + t^2(-e^{-t}))$

$= Ae^{-t}(2t - t^2)$ ∴

$y''_p = A(-e^{-t}(2t - t^2) + e^{-t}(2 - 2t))$

$= Ae^{-t}((-2t + t^2) + (2 - 2t)$

$= Ae^{-t}(t^2 - 4t + 2)$ ∴

$$\ddot{y} + 2\dot{y} + y = e^{-t}$$

$$Ae^{-t}(t^2 - 4t + 2) + 2(Ae^{-t}(2t - t^2)) + At^2e^{-t} = e^{-t}$$

$$Ae^{-t}(t^2 - 4t + 2) + 2(2t - t^2) + t^2) = e^{-t}$$

t^2: $t^2 - 2t^2 + t^2 = 0$

t: $-4t + 4t = 0$

Const: $2 = e^{-t}$

$$A\cancel{e^{-t}}(2) = \cancel{e^{-t}}$$

$$\boxed{A = \frac{1}{2}}$$

$$\boxed{y_p = \frac{1}{2}t^2e^{-t}}$$

$y = y_h + y_p$

$$y = c_1 e^{-1t} + c_2 t e^{-1t} + \frac{1}{2}t^2 e^{-t}$$

Now, Solve initial Conditions,

$y(0) = 0$

$y'(0) = 1$

$y(0) = c_1(1) + 0 + 0$

$c_1 = 0$

$\dot{y} = -e^{-t}\left(c_2 t + \frac{1}{2}t^2\right) + e^{-t}\left(c_2 + t\right)$

$c_2 = 0$

$$y(t) = e^{-t}\left(t + \frac{1}{2}t^2\right)$$

Final Answer!

Example 2:

$$\boxed{\ddot{y} + 2y' + 5y = e^{-t}\cos(2t)}$$

y_h

$$e^{\lambda t}(\lambda^2 + 2\lambda + 5) = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$m=1 \quad c=2 \quad k=5$

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\frac{-2 \pm \sqrt{4-20}}{2}$$

$$\lambda \Rightarrow \frac{-2 \pm 4i}{2} \Rightarrow -1 \pm 2i$$

Underdamped

$\sqrt{-16}$

$i\sqrt{16} \rightarrow 4i$

$$y_h = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

$$y_h = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$$

y_p $\boxed{\ddot{y} + 2y' + 5y = e^{-t}\cos(2t)}$

$y_h = e^{-t}(C_1\cos 2t + C_2\sin 2t)$

Compare

<u>Duplications</u>

$e^{-t}(C_1\cos 2t + C_2\sin 2t) \leftrightarrow e^{-t}\cos 2t$

① Check $\lambda = a \pm ib$

a and b match

Dupe. ☆

OR

Use eulers to convert $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i2t} = \cos 2t + i\sin 2t$

$\cos 2t = RE[e^{i2t}]$

$\ddot{y} + 2y' + 5y = e^{-t} RE[e^{i2t}]$

$\downarrow$

$\ddot{y} + 2\dot{y} + 5y = RE[e^{-t}e^{i2t}]$

$\downarrow$

$\ddot{y} + 2\dot{y} + 5y = RE[e^{(-1+2i)t}]$

We have to factor Out t!!

AND we see λ here as well! We can check for Dupe here as well Dupe.

Aux problem

$\ddot{Y} + 2\dot{Y} + 5Y = e^{(-1+2i)t}$ ... $y_p = Re[Y_p]$

Aux problem Cont....

Let's deal with the dupe now!

$$\ddot{Y} + 2\dot{Y} + 5Y = e^{(-1+2i)t}$$

Lets make our guesses!

Dupe Dealt with!

$$Y_p = Ate^{(-1+2i)t}$$

* Remember, this is a product.....

$$\dot{Y}_p = A\left(e^{(-1+2i)t} + t(-1+2i)e^{(-1+2i)t}\right)$$

$\hookrightarrow Ae^{(-1+2i)t}(1 + t(-1+2i))$

* Remember, this is a product.....

$$\ddot{Y}_p = A\left((-1+2i)e^{(-1+2i)t}[1+t(-1+2i)] + e^{(-1+2i)t}(-1+2i)\right)$$

$\hookrightarrow Ae^{(-1+2i)t}\left((-1+2i)(1+2i)t + 1\right)$

$\hookrightarrow Ae^{(-1+2i)t}(2+(-1+2i)t)$

Let's Sub in!

$$\underbrace{Ae^{(-1+2i)t}(2+(-1+2i)t)}_{\ddot{Y}} + 2\underbrace{\left(Ae^{(-1+2i)t}(1 + t(-1+2i))\right)}_{\dot{Y}} + 5\underbrace{(-te^{(-1+2i)t})}_{Y} = e^{(-1+2i)t}$$

Let's Clean that up a bit....

$$Ae^{(-1+2i)t}\left((-1+2i)(2+(-1+2i)t)+2(1+(-1+2i)t)+5t\right)=e^{(-1+2i)t}$$

$\boxed{\text{A LOT OF Skipped Simplification Steps}}$

$$Ae^{(-1+2i)t}(-2+4i-3t-4i)+(2-2t+4it)+5t=e^{(-1+2i)t}$$

$$\hookrightarrow Ae^{(-1+2i)t}(4i)=e^{(-1+2i)t}$$

$$A(4i)=1$$

$$\boxed{A=\frac{1}{4i}}$$

$$Y_p=\frac{1}{4i}te^{(-1+2i)t}$$

don't forget about Me!

* We need to get rid of i in the denominator and i in the exponent

$$\rightarrow \frac{1}{4i}\times\frac{i}{i}=\frac{-i}{4}$$

✓

$$e^{-t+2it}=e^{-t}e^{2it}=e^{-t}\cos 2t+i\sin 2t$$

✓

So...

$$Y_p = -\frac{i}{4} t\, e^{-t} (\cos 2t + i \sin 2t)$$

We have a problem here!
Be careful when expanding...

$$Y_p = \frac{-te^{-t}}{4} (i\cos 2t - \sin 2t)$$

$$\boxed{y_p = RE[Y_p] = \frac{te^{-t}}{4} \sin 2t}$$

$$y = \underbrace{e^{-t}(c_1\cos 2t + c_2\sin 2t)}_{y_h} + \underbrace{\frac{te^{-t}}{4}\sin 2t}_{y_p}$$