

Example:

Find the general solution to

$$\vec{x}' = \overset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \vec{x}$$

Find Eigenpairs of A :

Eigenvalues: $\{2, -1, -1\}$

Repeated values!

Simpler $\rightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\rightarrow \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ simpler

$\lambda = 2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\lambda = -1$, $\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 3 \\ -3/2 \end{bmatrix}$
(double root)

Algebraic multiplicity
of 2
(double roots)

Geometric multiplicity
of 2

(two unique EV's
from one value)

General solution:

$$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Here, we encountered repeated eigenvalues,
but we still had 3 Linearly Independent
eigenvectors, so all is well. Just need
to include an extra term on there.

Example:

Find general solution to

$$\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$$

$\lambda = 1, 1$ (Repeated Roots) (Algebraic multiplicity of 2)

Computers have issues solving these problems when the geometric multiplicity (C# of Eigenvectors) is less than the algebraic multiplicity

So, have to do ref (C# of Eigenvalues)

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (\text{geometric multiplicity of } 1)$$

We can write one solution now:

$$\vec{x}_1(t) = e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We need a second linearly independent solution

Look for $X_2(t)$

Needed compared to
MVC method!

$$X_2(t) = \vec{v}_1 t e^{\lambda t} + \vec{v}_2 e^{\lambda t}$$

$$\text{LHS: } \dot{X}_2(t) = \vec{v}_1 e^{\lambda t} + \vec{v}_1 \lambda t e^{\lambda t} + \vec{v}_2 \lambda e^{\lambda t}$$

$$\text{RHS: } A \vec{X}_2 = t e^{\lambda t} A \vec{v}_1 + e^{\lambda t} A \vec{v}_2$$

For \vec{X}_2 to be a solution, we must have:

$$A \vec{v}_1 = \lambda \vec{v}_1$$

\vec{v}_1 is eigenvector
for λ

$$A \vec{v}_2 = \lambda \vec{v}_2 + \vec{v}_1$$

Rearrange $(A - \lambda I) \vec{v}_2 = \vec{v}_1$
 \vec{v}_2 is called a generalized
eigenvector.

For an example:

$$\lambda = 1, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, x_1(t) = e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2nd Solution:

$$x_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^t + \vec{v}_2 e^t, \quad v_2 \text{ must satisfy}$$

So,

$$(A - I) \vec{v}_2 = \vec{v}_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & 2 \\ 1 & -2 & 1 \end{array} \right] \xrightarrow{\text{REF}} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

$$\text{If } \vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a - 2b = 1$$

$$\text{Let's set } b = 0, \text{ so } a = 1$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we can plug this into $(A - I) \vec{v}_2 = \vec{v}_1$ and check if desired.

So,

$$x_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t = \begin{bmatrix} 2t + 1 \\ t \end{bmatrix} e^t$$

General Solution:

$$\vec{x}(t) = C_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2t+1 \\ t \end{bmatrix} e^t$$

Let's turn things up a little bit more,
and make the algebraic multiplicity 3, and
the geometric 1.

Example:

Find the general solution to:

$$\mathbf{x}' = \begin{bmatrix} -5 & -5 & 9 \\ 8 & 9 & 18 \\ -2 & -3 & -7 \end{bmatrix} \mathbf{x}$$

There is a single repeated eigenvalue, $\lambda = -1$ of
a single eigenvector $\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$

$$\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_2(t) = t e^{-t} \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} + \vec{v}_2 e^{-t} \quad \text{where } (A + I)\vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_3(t) = \frac{1}{2} t^2 e^{-t} \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} + t e^{-t} \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \end{bmatrix} + \vec{v}_3 e^{-t} \quad \text{take } \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

where

$$(A + I)\vec{v}_3 = \vec{v}_2$$

$$\text{Take } \vec{v}_3 = \begin{bmatrix} 5/3 \\ 2/3 \\ 0 \end{bmatrix}$$