

Let's learn how to solve systems of linear ODE's $\vec{\dot{x}} = A\vec{x}$

We'll start by looking for a solution of the form $\vec{x} = \vec{v}e^{\lambda t}$

(\vec{v} is a constant vector)

e.g., $\vec{\dot{x}}(t) = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} e^t = \begin{bmatrix} 0^t \\ \frac{0^t}{2} \end{bmatrix}$

Then, $\vec{\dot{x}} = \vec{v} \lambda e^{\lambda t}$

and $A\vec{x} = A(\vec{v}e^{\lambda t}) = e^{\lambda t}(A\vec{v})$

For $\vec{\dot{x}}$ to equal $A\vec{x}$, we must have

$$e^{\lambda t} \lambda \vec{v} = e^{\lambda t} A \vec{v}$$

i.e. $e^{\lambda t} (A \vec{v} - \lambda \vec{v}) = \vec{0}$

i.e. $A \vec{v} = \lambda \vec{v}$

Since $e^{\lambda t}$ is never zero,

$$A\vec{v} = \lambda\vec{v}$$

That is, λ must be an eigenvalue of A , with a corresponding eigenvector!

Example. Find the general solution to

$$\vec{x}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \vec{x}$$

Find eigenvalues of A !

Find characteristic polynomial $\det(A - \lambda I)$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} =$$

$$(1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - (-1) \begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} +$$

$$4 \begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

Find roots:

$$[-x^3 + 2x^2 + 5x - 6]$$

Try Easy roots, say $\lambda = 1$

$$-1 + 2 + 5 - 6 = \underline{0}, \text{ so } \lambda = 1 \text{ is a root}$$

$$(\lambda - 1)(-x^2 + \lambda + 6)$$

$$(\lambda - 1)(\lambda + 3)(\lambda - 2)$$

Hence to find eigenvectors!

Eigenvalues!

$$\boxed{\lambda = 1}$$

$$\boxed{\lambda = -2}$$

$$\boxed{\lambda = 3}$$

Next, find eigenvectors:

$$\lambda = 1$$

Solve $[A - \lambda I | 0]$

$$\left[\begin{array}{ccc|c} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ z & 1 & -2 & 0 \end{array} \right]$$



do rref until you
got a row of
zeros on bottom

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Here, we've got

$$\begin{aligned} x + z &= 0 \\ y - 4z &= 0 \end{aligned}$$

Two unknowns, two

Pick a free variable! So, let's

$$\text{Let } z = 1, x = -1, y = 4$$

$$v = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \begin{matrix} x \\ y \\ z \end{matrix}$$

If you were to pick a different free value, you'd have a "different multiple" for say, but it would just be a constant multiple. And remember, for eigenvectors, that is no issue at all.

$$\boxed{\lambda_2 = -2}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array} \right]$$

↓

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Rref down to zero
row, and 1's col.
Pivot

$$\begin{aligned} x + z &= 0 \\ y - z &= 0 \end{aligned}$$

$$\begin{aligned} \text{Set } x &= 1, z = -1 \\ y &= -1 \end{aligned}$$

$$\boxed{\lambda_3 = 3}$$

$$\left[\begin{array}{ccc|c} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array} \right]$$

Rref

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x - 2z &= 0 \\ y - 2z &= 0 \end{aligned}$$

$$\text{Set } x = 1, y = 2, z = 1$$

So linearly independent Solutions are:

$$\vec{x}_1 = \vec{v}_1 e^{\lambda_1 t} = e^t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = \vec{v}_2 e^{\lambda_2 t} = e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{x}_3 = \vec{v}_3 e^{\lambda_3 t} = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

General Solution

$$x(t) = C_1 e^t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + C_3 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Example with complex eigenvalues

Example. Find the general solution to

$$\vec{x}' = \begin{vmatrix} 6 & -1 \\ 5 & 2 \end{vmatrix} \vec{x}$$

Eigenvalues

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} \\ &= \lambda^2 - 8\lambda + 12 + 5 \\ &= \lambda^2 - 8\lambda + 17 \\ &= (\lambda - 4)^2 + 1\end{aligned}$$

$$\frac{8}{2} = 4^2 = 16$$

$$\boxed{\lambda = 4 \pm i}$$

Eigenvectors

Only really need one of the two. Pick $\boxed{\lambda = 4+i}$

$$\begin{bmatrix} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

Solutions

$$\vec{x}(t) = e^{(4+i)t} \begin{bmatrix} 1 \\ z-i \end{bmatrix} \quad \text{is a } \begin{array}{l} \xrightarrow{\text{Complex}} \\ \text{solution} \end{array}$$

Let's use the real & imaginary parts to find two linearly independent real-valued solutions:

Let's use Euler's identity

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\begin{aligned} \vec{x}(t) &= e^{4t} \cdot e^{it} \begin{bmatrix} 1 \\ z-i \end{bmatrix} \\ &= e^{4t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ z-i \end{bmatrix} \\ &= \begin{bmatrix} e^{4t} \cos t + ie^{4t} \sin t \\ ze^{4t} \cos t - ie^{4t} \cos t + z_1 e^{4t} \sin t + e^{4t} \sin t \end{bmatrix} \\ &= e^{4t} \begin{bmatrix} \cos t \\ z \cos t + \sin t \end{bmatrix} + ie^{4t} \begin{bmatrix} \sin t \\ z \sin t - \cos t \end{bmatrix} \\ &\quad x_1(t) \qquad \qquad \qquad x_2(t) \end{aligned}$$

Solution

$$\vec{x}(t) = C_1 e^{4t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix}$$

