

Laplace Transforms

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$s = \sigma + j\omega$$

Some Rules:

Linearity

$$1) \mathcal{L}\{a f(t) + b g(t)\} = a F(s) + b G(s)$$

$$2) \mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$3) \mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

4) Laplace Transform Table.

Like how $\frac{d}{dt} \sin = \cos$, Laplace
has similar rules!

Examples:

$$y'' + y = 0, \quad y(0) = 1 \quad y'(0) = 0$$

Find Laplace Transform

$$\mathcal{L}(y'') + \mathcal{L}(y) = 0$$

↓

$$s^2 Y(s) - \overset{1}{s} \cancel{Y(0)} - \overset{0}{\cancel{y'(0)}} + Y(s) = 0$$

$$Y(s)(s^2 + 1) - s = 0$$

$$Y(s) = \frac{s}{s^2 + 1}$$

Now, Let's take the inverse Laplace to get back to t !

From Laplace Table:

$$\frac{s}{s^2 + \beta} = \cos \beta t$$

$$y(t) = \cos t$$

Inverse Laplace

- Inverse Laplace (using the Laplace table)

$$f(s) = \frac{3}{s^2+9}, \quad \mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] \quad \times \text{ From Table}$$
$$= \sin 3t \quad \text{Since } \mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$$

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- Partial Method (using partial fractions)

Example:

⚡ This is trickier!
How to decompose?

$$f(s) = \frac{s+3}{(s+1)(s+2)}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

①

For each distinct linear factor ($s+a$ for example) we will assign:

$$\frac{A}{s+a}$$

②

For each repeated linear factor $(s+a)^n$ for example we will assign:

$$\frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \dots + \frac{A_n}{(s+a)^n}$$

③

For each irreducible factor (such as $s^2 + bs + c$), assign

$$\frac{Bs + C}{s^2 + bs + c}$$

—————→ Multiply through common (initial) denominator & find A, B, C by equating coefficients.

Back to that example from earlier...

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

Multiply out →

$$s+3 = A(s+2) + B(s+1)$$

$$s+3 = As + 2A + Bs + B$$

s terms

$$As + Bs = s \rightarrow A + B = 1$$

Const

$$2A + 1 = 3 \rightarrow A = 2$$

$$\text{So } B = -1$$

$$F(s) = \frac{2}{(s+1)} - \frac{1}{(s+2)}$$

From Table

$$f(t) = 2e^{-t} - e^{-2t}$$

Example:

$$F(s) = \frac{s+2}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

mult. by
out

$$s+2 = A(s+1) + B$$

$$s+2 = As + A + B$$

s terms

$$As = s$$

$$A = 1$$

constants

$$As + B = 2$$

$$B = 2 - s$$

Example 2

$$F(s) = \frac{2s+5}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

multiply
out

$$2s+5 = A(s^2+4) + (Bs+C)(s+1)$$

$$2s+5 = As^2+4A + Bs^2+Bs+Cs+C$$

s^2 terms:

$$0 = A + B$$

s terms

$$2 = B + C$$

const

$$5 = 4A + C$$

$$A = \frac{3}{5}$$

$$B = -\frac{3}{5}$$

$$C = \frac{13}{5}$$

Cover-up Method

$$\frac{3}{(s+2)(s-3)} = \frac{A}{(s+2)} + \frac{B}{(s-3)}$$

$$3 = A(s-3) + B(s+2)$$

To solve for each constant, you're going to want to set s equal to something that will cancel out one of the two.

Say solving for A , you'd set $s = -2$, making $B = 0$. You can then solve for A .

Solving for B , you'd set $s = 3$, making $A = 0$. You can now solve for B .

First Shift Theorem

(or exponential Shift Theorem)

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

For example, in the case of $\mathcal{L}(e^{2t}\cos 3t)$

$$\mathcal{L}\{e^{2t}\cos 3t\} = \frac{s-2}{(s-2)^2+9}$$

(Note: In the original image, 's-2' and '(s-2)' are circled with arrows pointing to 's', indicating the shift.)

This is the Shift! We can call $(s-2)$, s !

Another example:

$$\mathcal{L}\{\sin 5t\} = \frac{5}{s^2+25}$$

$$\mathcal{L}\{e^{-4t}\sin 5t\} = \frac{5}{(s+4)^2+25}$$

(Note: In the original image, an arrow points from the text 'e shifts the s!' to the 's+4' term in the denominator.)

e shifts the s !

Completing the Square

$$as^2 + bs + c$$

$$c = \left(\frac{b}{2}\right)^2$$

$$F(s) = \frac{1}{(s^2 + 2s + 2)} = \frac{1}{(s^2 + 2s + 1) + 1}$$

\downarrow
 $(s+1)^2$

$$F(s) = \frac{1}{(s+1)^2 + 1} = e^{-t} \sin t$$

If we had say $\frac{\tau K_0}{(s+1)^2 + 1}$, factor τK_0 out!

Example:

$$F(s) = \frac{s}{s^2 + 2s + 5} = \frac{s}{(s^2 + 2s + 1) + 5}$$

$$\Rightarrow F(s) = \frac{s}{(s+1)^2 + 5}$$

$(\frac{1}{2}b)^2 \quad \downarrow \quad 1^2$

Can't do first shift here because of numerator!
Here's what we'll do here then...

$$F(s) = \frac{(s+1)-1}{(s+1)^2 + (\sqrt{5})^2} = \frac{s+1}{(s+1)^2 + (\sqrt{5})^2} - \frac{1}{(s+1)^2 + (\sqrt{5})^2}$$

↑ We want b^2 for the form

$$e^{-t} \cos(\sqrt{5}) - \frac{\frac{\sqrt{5}}{\sqrt{5}}}{(s+1)^2 + (\sqrt{5})^2} \leftarrow \text{Want to have B!}$$

$$- \frac{1}{\sqrt{5}} \frac{\sqrt{5}}{(s+1)^2 + (\sqrt{5})^2}$$

$$- \frac{1}{\sqrt{5}} e^{-t} \sin(\sqrt{5})$$