2L Periodic Functions

Say $f(t)$ is $2L$ periodic.

i.e, $f(t+2L) = f(t)$

or really any multiple.

Let $s = \frac{\pi}{L}t$, Let $g(s) = f(t) = f\left(\frac{L}{\pi}s\right)$

Then, $g(s)$ is 2π periodic:

F.S. of $g(s)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(ns) + b_n \sin(ns)$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\,ds$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(s)\cos(ns)\,ds$, $b_n = \ldots$

(Skip some steps; change of variables, u-substitution, rewriting)

F.S for f(t):

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

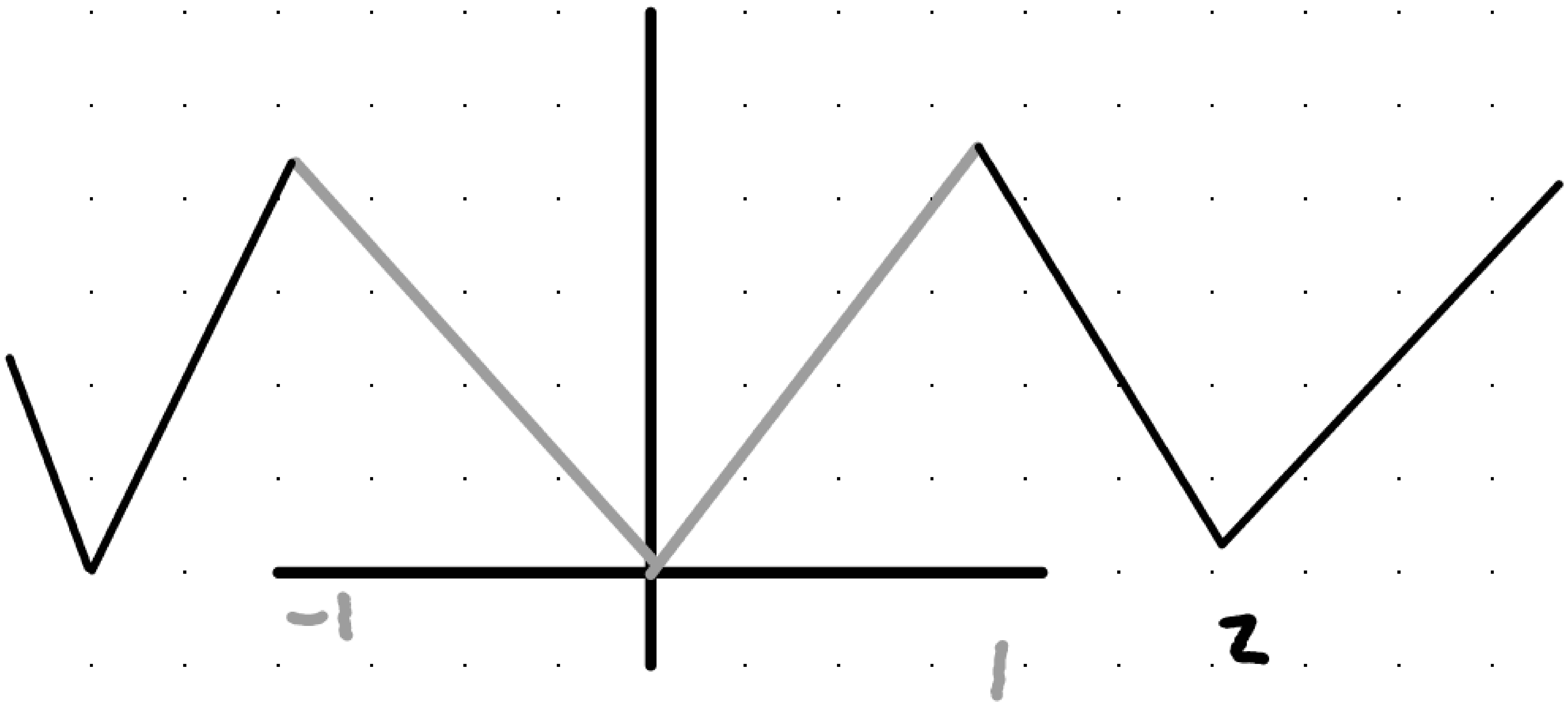
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

Could fill pi back in for L, and the function would still work as it did before!

Example:

Let $f(t) = |t|$ for $-1 < t \leq 1$



extended periodically. Compute the Fourier Series of $f(t)$

2- periodic function, so half period $L = 1$

F.S:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$$

$$a_0 = \int_{-1}^{1} |t| \, dt = 1$$

$$a_n = \int_{-1}^{1} |t| \cos(n\pi t)\, dt$$

$$= \int_{-1}^{0} (-t) \cos(n\pi t)\, dt + \int_{0}^{1} t \cos(n\pi t)\, dt \ldots$$

OR

$\equiv$

Remember this trick?!

$$= \int_{-1}^{1} \underbrace{|t|}_{\text{Even}} \underbrace{\cos(n\pi t)}_{\text{Even}}\, dt = 2\int_{0}^{1} t \cos(n\pi t)\, dt$$

Even

(IBP)

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\boxed{\begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}}$$

Odd, so zero!

$$b_n = \int_{-1}^{1} \underbrace{|t|}_{\text{even}} \underbrace{\sin(n\pi t)}_{\text{odd}}\, dt = 0$$

Odd

F.S for $f(t)$

$$\frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{-4}{n^2\pi^2} \cos n\pi t$$

We could also write the solution like this!
If you want to pick up only odd numbers, this form could work.

$$= \frac{1}{2} + \sum_{K=1}^{\infty} \frac{-4}{\underbrace{(2K-1)^2}_{\substack{2K-1 \\ \text{signals} \\ \text{odd only.}}}\pi^2} \cos((2K-1)\pi t)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\cos \pi t - \frac{4}{9\pi^2}\cos(3\pi t)$$

As a side note, if you have a continuous function like this one, the fourier approximation (if you graph it) will be pretty strong, even with fewer terms.

Differentiation & Integration of Fourier Series

Not only does the fourier series converge nicely, but it is also easy to differentiate and integrate the series.

We can do this by differentiating or integrating term by term.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

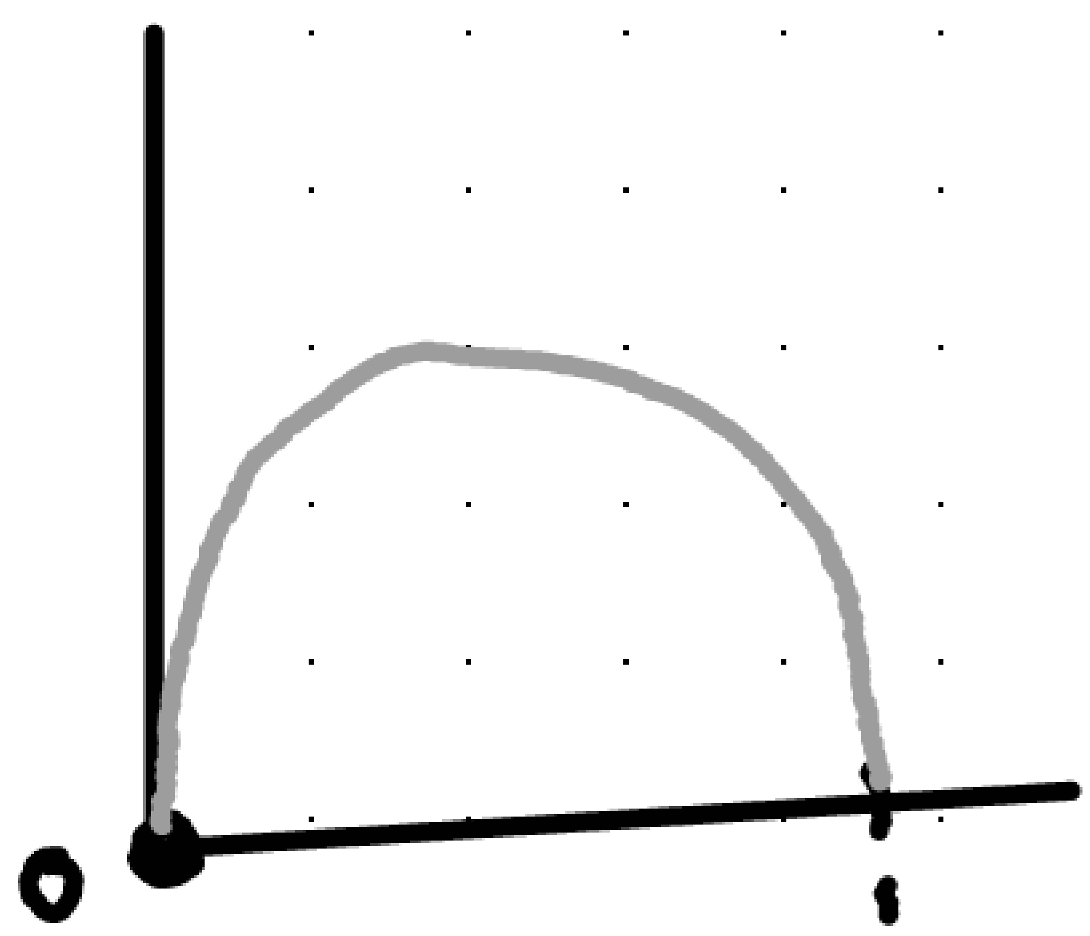
$$\downarrow$$

$$f'(t) = \sum_{n=1}^{\infty} \frac{-a_n n\pi}{L} \sin\left(\frac{n\pi}{L} t\right) + \frac{b_n n\pi}{L} \cos\left(\frac{n\pi}{L} t\right)$$

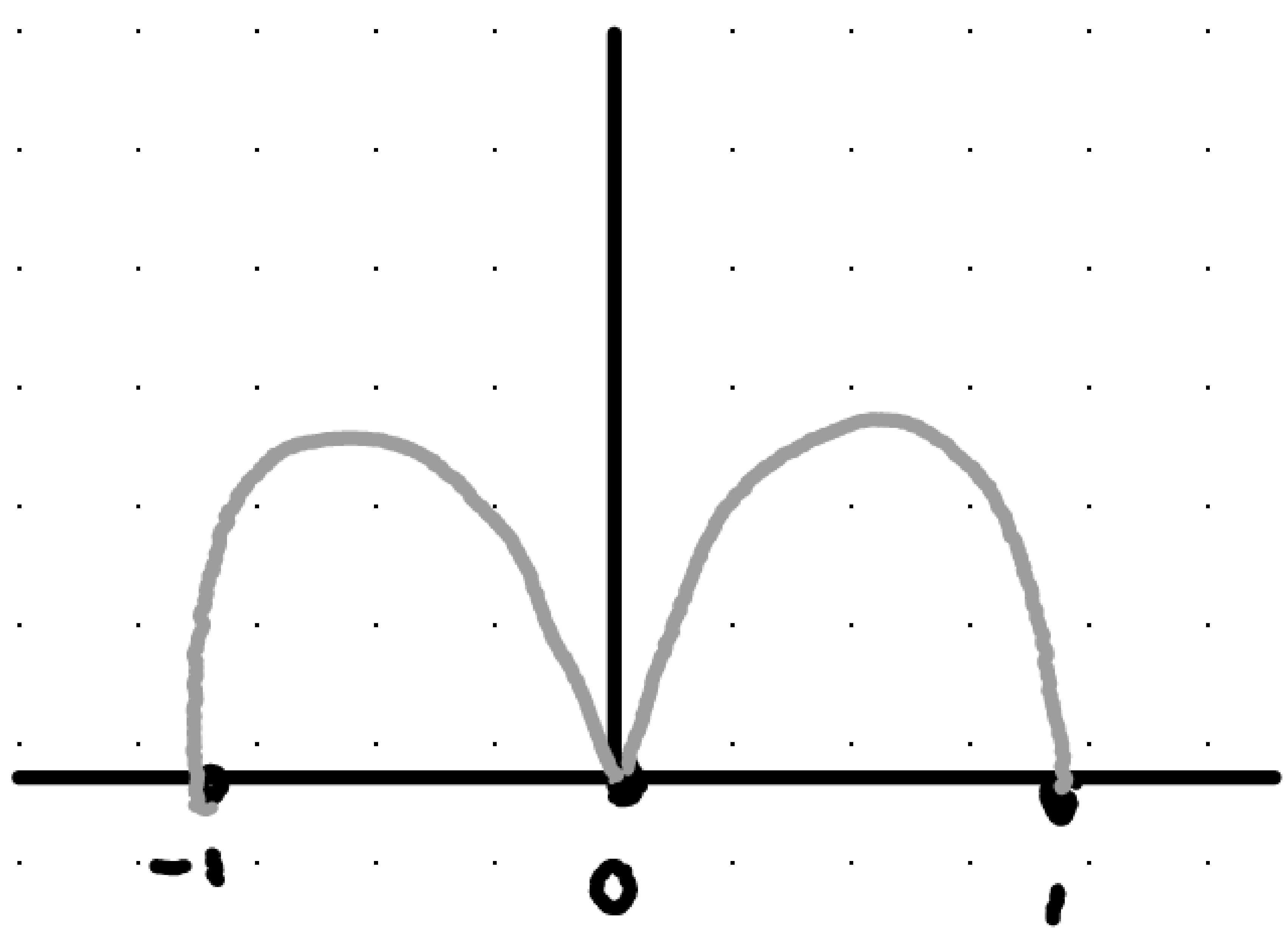
and the same for the integration of course.

Even & Odd Periodic Functions

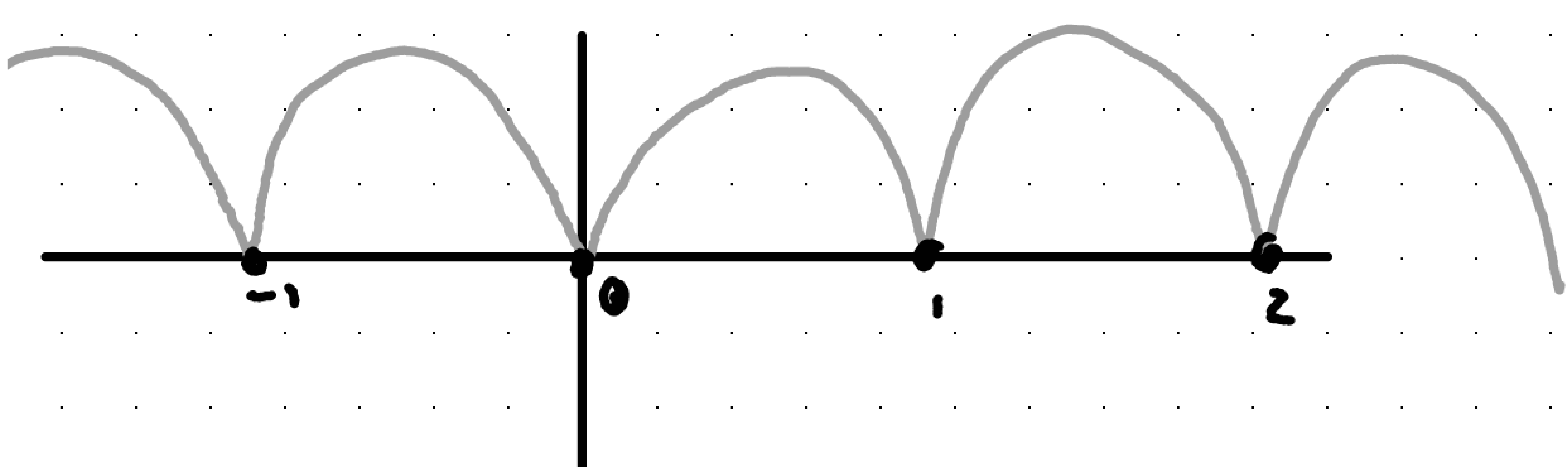
e.g. $f(t) = t(1-t), \quad 0 \leq t \leq 1$



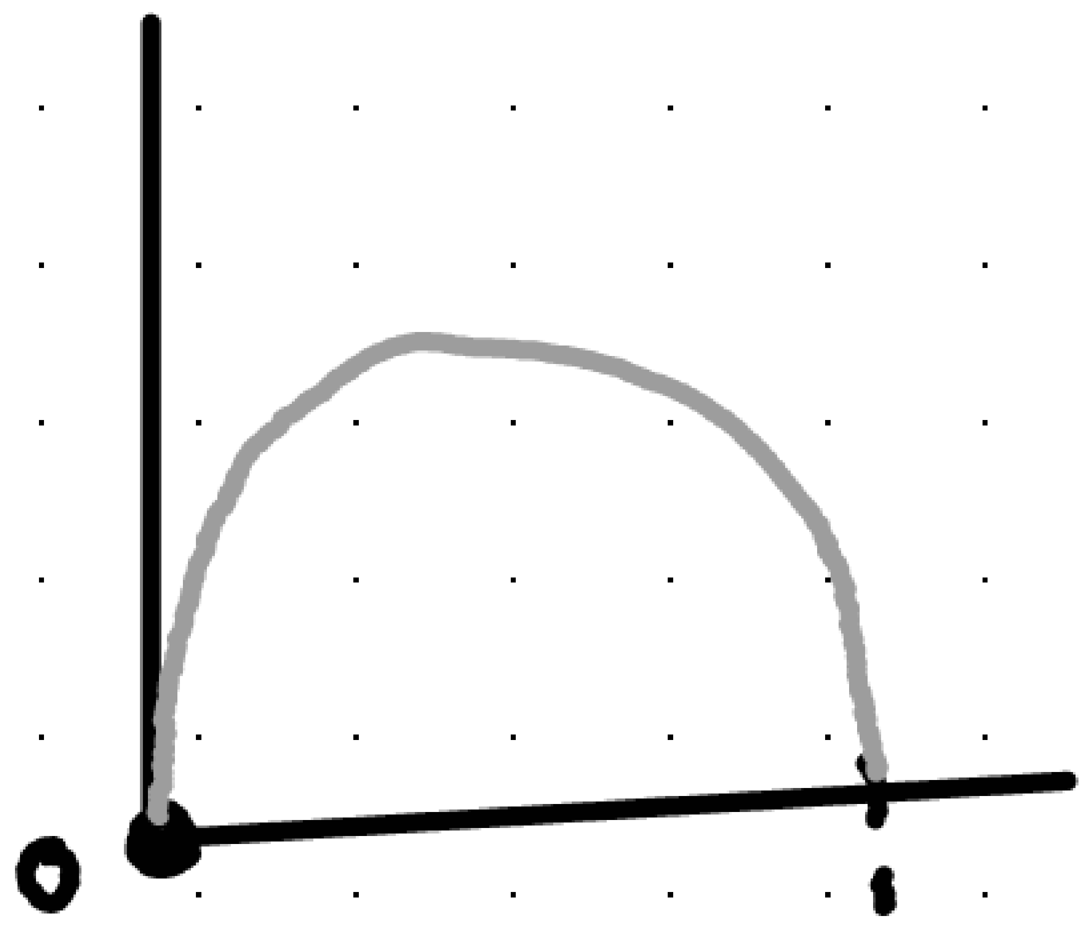
Even Extension:



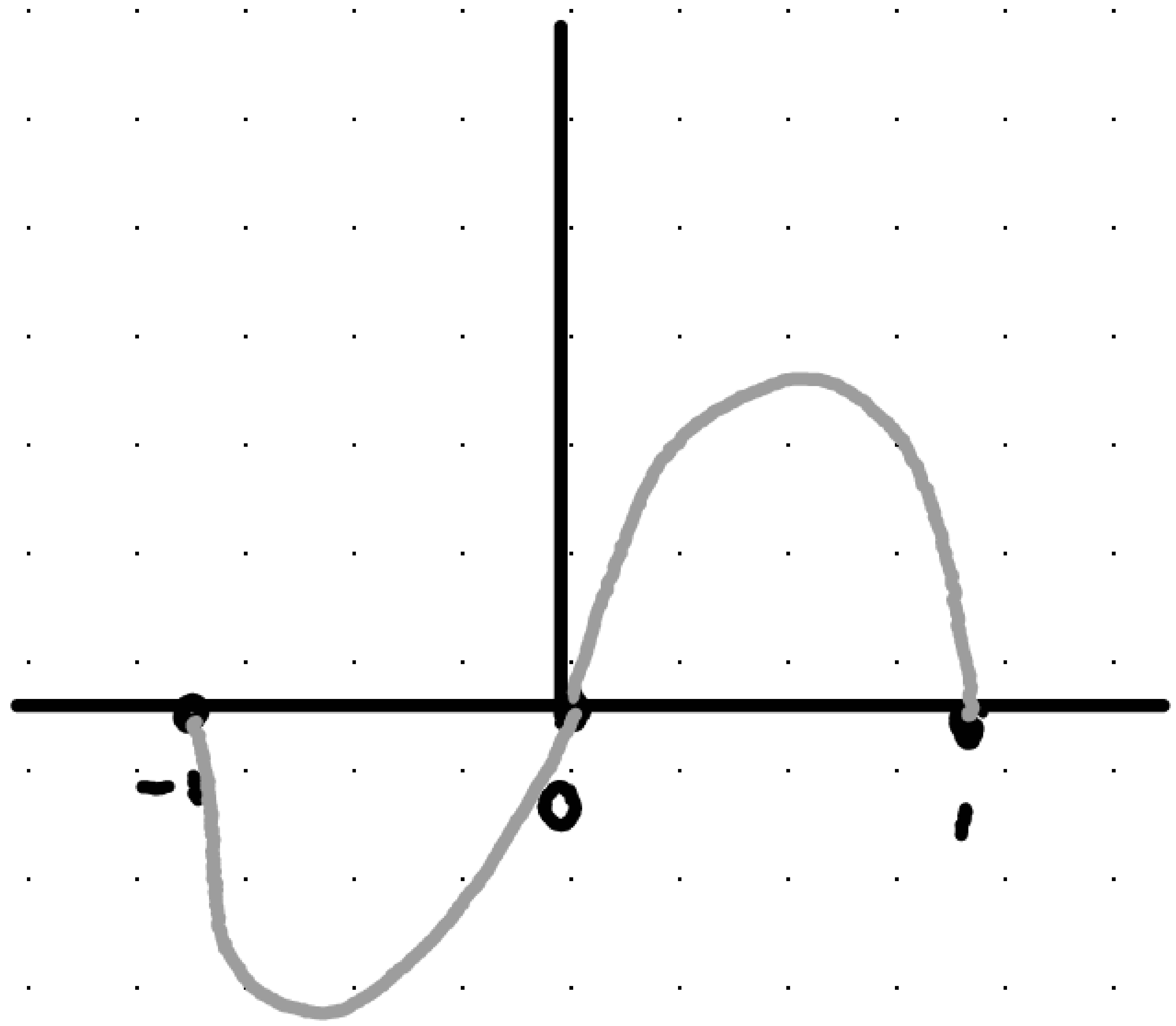
2-periodic extension:



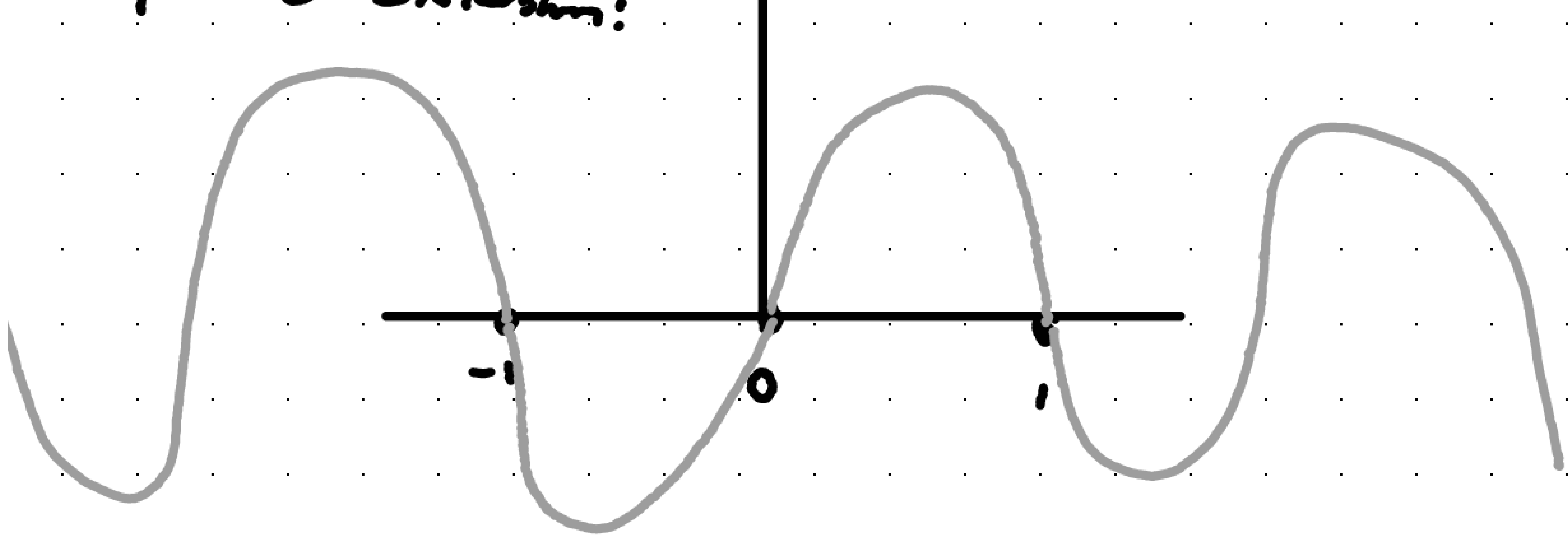
e.g. $f(t) = t(1-t), \quad 0 \le t \le 1$



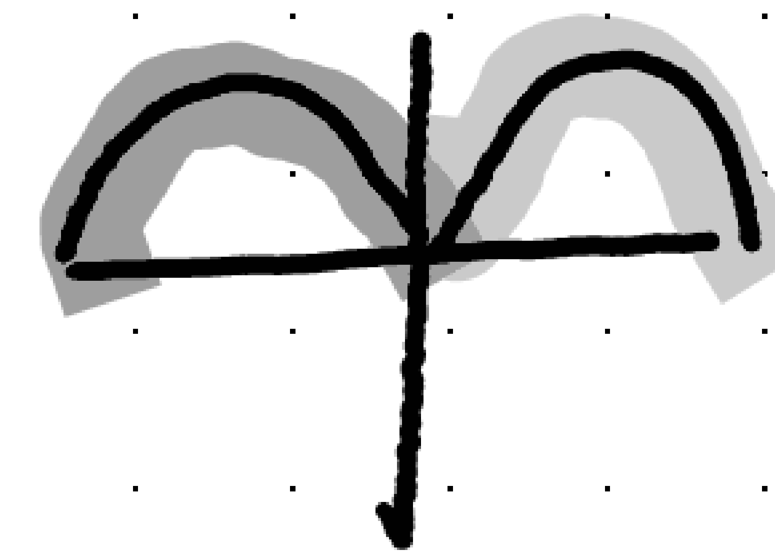
Odd Extension:



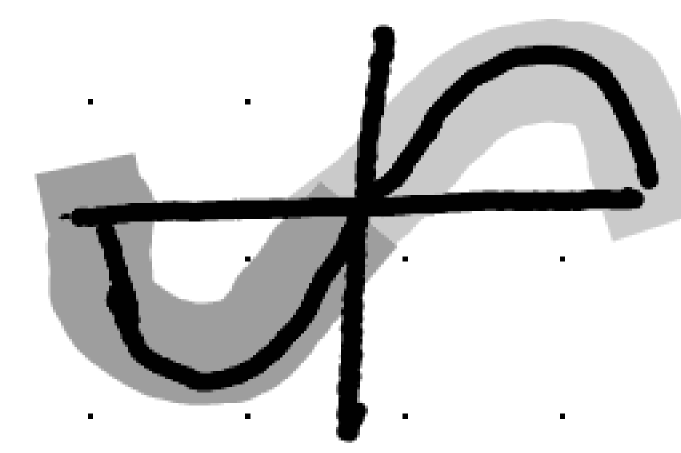
2 periodic Extension:



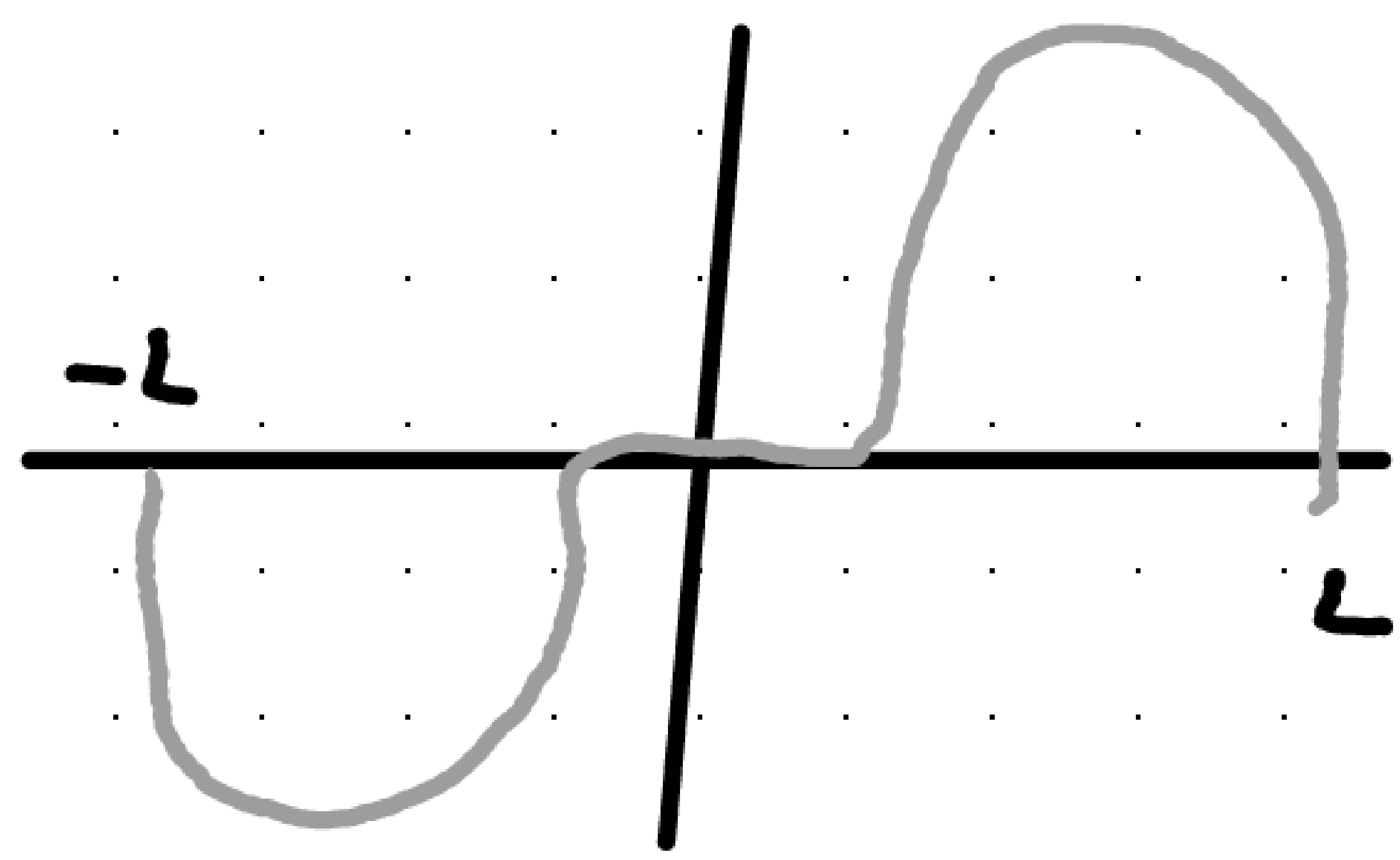
$$F_{even} = \begin{cases} f(t) & 0 \leq t \leq L \\ f(-t) & -L \leq t \leq 0 \end{cases}$$



$$F_{odd} = \begin{cases} f(t) & 0 \leq t \leq L \\ -f(-t) & -L \leq t \leq 0 \end{cases}$$



Suppose $f(t)$ is an odd $2L$ period function.



F.S

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \boxed{0}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(t)}_{\text{odd}} \underbrace{\cos\left(\frac{n\pi t}{L}\right)}_{\text{even}}}_{\text{odd}}\,dt = \boxed{0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(t)}_{\text{odd}} \underbrace{\sin\left(\frac{n\pi t}{L}\right)}_{\text{odd}}}_{\text{Even}}\,dt$$

$$= \boxed{\frac{2}{L}\int_{0}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt}$$

Suppose f is an even 2L-periodic function:

$$a_0 = \frac{2}{L} \int_0^L f(t)\,dt$$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

Fourier Odd Periodic Extension

$$F_{odd}(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} t\right),$$

where

$$b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi}{L} t\right) dt$$

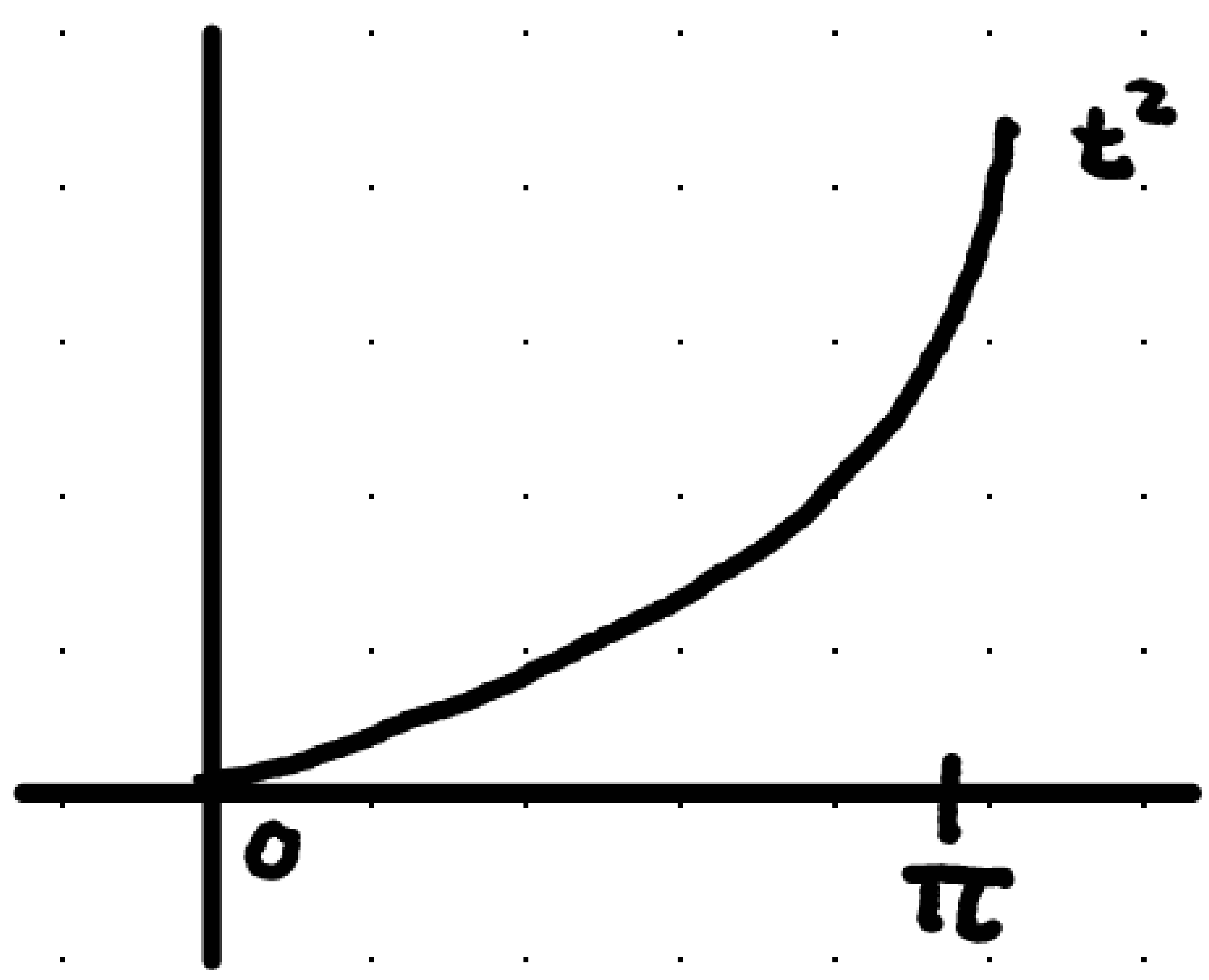
Fourier Even Periodic Extension

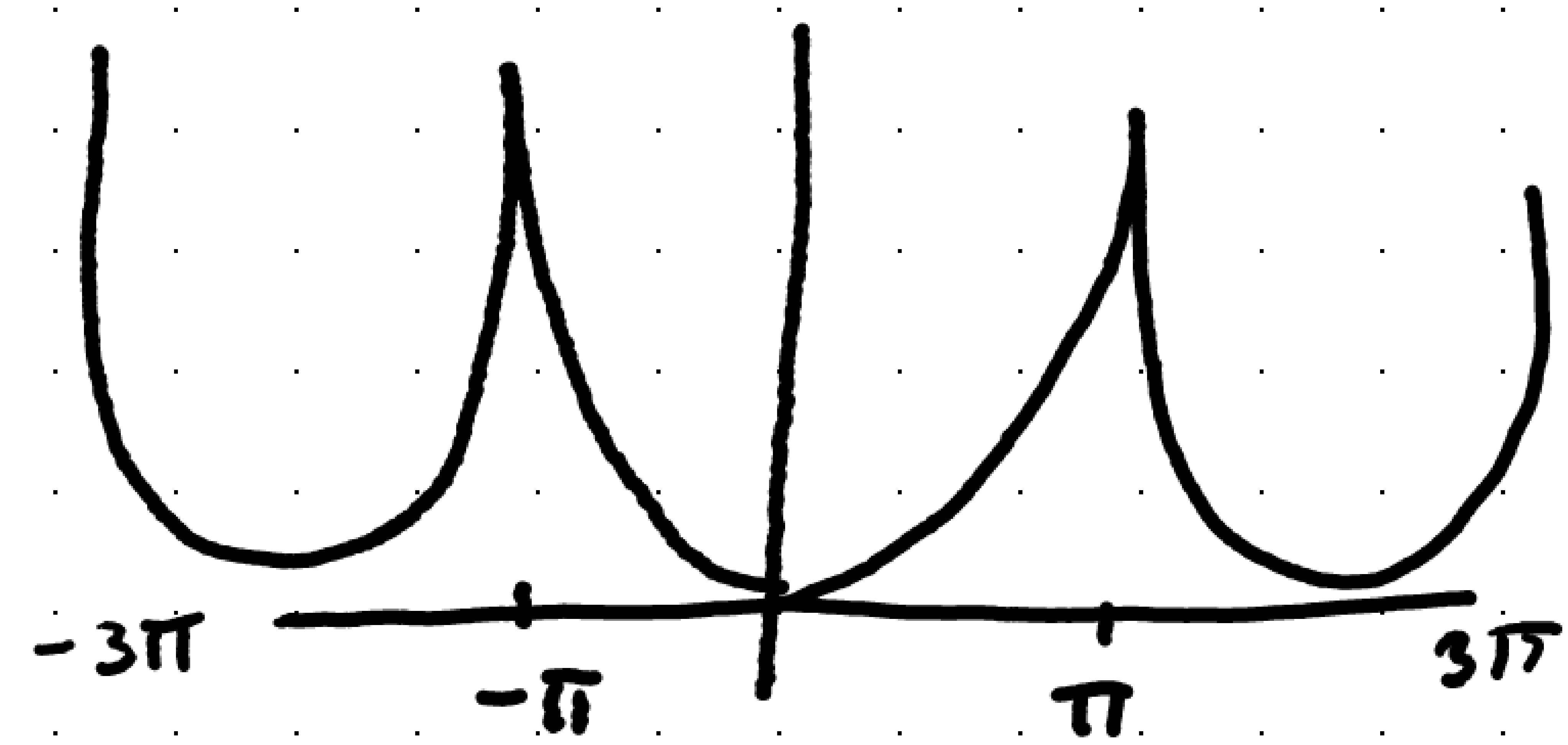
$$F_{even}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right)$$

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi}{L} t\right) dt$$

Example:

Find Fourier Series of the even periodic extension of the function $f(t)=t^2$ for $0 \le t \le \pi$





F.S: Use even periodic extension:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t^2 \cos nt\,dt = \frac{2}{\cancel{\pi}}\left[\frac{2\cancel{\pi}(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

F.S: $\boxed{\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nt}$