

## Non Homogeneous Linear Equations (Constant Coefficients)

$$a\ddot{y} + b\dot{y} + cy = G(x)$$

Consider complementary Homogeneous D.E:

$$a\ddot{y} + b\dot{y} + cy = 0$$

and find general solution:  $y_c$

If we can be so lucky as to find one, just one Particular Solution  $y_p$  to  $a\ddot{y} + b\dot{y} + cy = G(x)$  then any other solution  $y$  to  $a\ddot{y} + b\dot{y} + cy = G(x)$  can be written as

$$y = y_p + y_c$$

$$y = G(x) + 0$$

## Method of Undetermined Coefficients

Example: Solve  $\ddot{y} + \dot{y} - 2y = x^2$

First, solve  $\ddot{y} + \dot{y} - 2y = 0$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$\begin{aligned} r &= -2 \\ r &= 1 \end{aligned}$$

$$y_c = C_1 e^{-2x} + C_2 e^x$$

Next, try to find one  $y_p$  to  $\ddot{y} + \dot{y} - 2y = x^2$ .

Guess:

$$y_p = Ax^2 + Bx + C$$

Determine  $A, B, C$

$$\dot{y}_p = 2Ax + B$$

$$\ddot{y}_p = 2A$$

Plug in guess & set equal

$$2A + 2Ax + B - 2(Ax^2 + Bx + C) = x^2$$

$$2A + 2Ax + B - 2(Ax^2 + Bx + C) = x^2$$

$$2A + 2Ax + B - 2Ax^2 + 2Bx + 2C = x^2$$

Solve system of Eq's

$$x^2: -2A = 1$$

$$x: 2A - 2B = 0$$

$$1: 2A + B - C = 0$$

$$A = -\frac{1}{2}$$

$$B = -\frac{1}{2}$$

$$C = -\frac{3}{4}$$

So,  $y_p = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$

General Solution:

$$y = c_1 e^{-2x} + c_2 e^x - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

Example:

Solve  $\ddot{y} - 4y = \underbrace{xe^x}_{y_{r1}} + \underbrace{\cos 2x}_{y_{r2}}$

First,  $\ddot{y} - 4y = 0, \quad r^2 - 4 = 0 \quad r = \pm 2$

$$y_c = C_1 e^{2x} + C_2 e^{-2x}$$

$y_{r1}$

$$xe^x$$

$y_{r2}$

$$\cos 2x$$

## Duplication Issue

Example:  $\ddot{y} + y = \sin x$ ,  $r^2 + 1$ ,  $r = \pm i$

$$y_c = C_1 \sin x + C_2 \cos x$$

Guess for  $\sin x$

$$y_p = A \sin x + B \cos x$$

$$\hookrightarrow y_p = Ax \sin x + Bx \cos x$$

Solve as normal

Dupe!

When this happens, we need to multiply by  $x$ !

Example:

$$\ddot{y} + 2\dot{y} + 5y = 4xe^{-x} \cos 2x + e^{2x}, \quad y(0) = 1, \quad \dot{y}(0) = 2$$

## Variation of Parameters

Solve

$$\ddot{y} + y = \tan x, \quad 0 < x < \frac{\pi}{2}$$

First,  $\ddot{y} + y = 0$ ,  $y_0 = C_1 \cos x + C_2 \sin x$

guess

$$y_p = A \tan x \dots$$

Arand problem...

This will go FOREVER! MUC falls flat here, because no matter how many terms you add, you'll never have a genuinely different derivative.

## Variation of Parameters Method.

Guess:

$$y_p = u_1(x) \cos x + u_2(x) \sin x$$

↑ functions?! ↑

$$\dot{y}_p = \dot{u}_1 \cos x - u_1 \sin x + \dot{u}_2 \sin x + u_2 \cos x$$

TRY TO KEEP THINGS SIMPLE

Suppose:  $\dot{u}_1 \cos x + \dot{u}_2 \sin x = 0$

So,  $\dot{y}_p = -u_1 \sin x + u_2 \cos x$

$$\ddot{y}_p = -\dot{u}_1 \sin x - u_1 \cos x + \dot{u}_2 \cos x - u_2 \sin x$$

Plug into LHS:

$$\ddot{y}_p + y_p = \tan x$$

$$-\dot{u}_1 \sin x - \cancel{u_1 \cos x} + \dot{u}_2 \cos x - \cancel{u_2 \sin x} + \cancel{u_1 \cos x} + \cancel{u_2 \sin x} = \tan x$$
$$-\dot{u}_1 \sin x + \dot{u}_2 \cos x = \tan x$$

Solve for  $u_1$  &  $u_2$ ?



We have two Eqn's

$$\dot{u}_1 \cos x + \dot{u}_2 \sin x = 0$$

$$-\dot{u}_1 \sin x + \dot{u}_2 \cos x = \tan x$$

• Cramer's Rule have.

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix}$$

$$\dot{u}_1 = \cos x - \sec x$$

$$\dot{u}_2 = \sin x$$

Integrate to get  $u_1$  &  $u_2$

$$u_1 = \int (\cos x - \sec x) dx = \sin x - \ln(\sec x + \tan x)$$

$$u_2 = \int \sin x dx = -\cos x$$

$$y_p = [\sin x - \ln(\sec x + \tan x)] \cos x - \cos x \sin x$$

