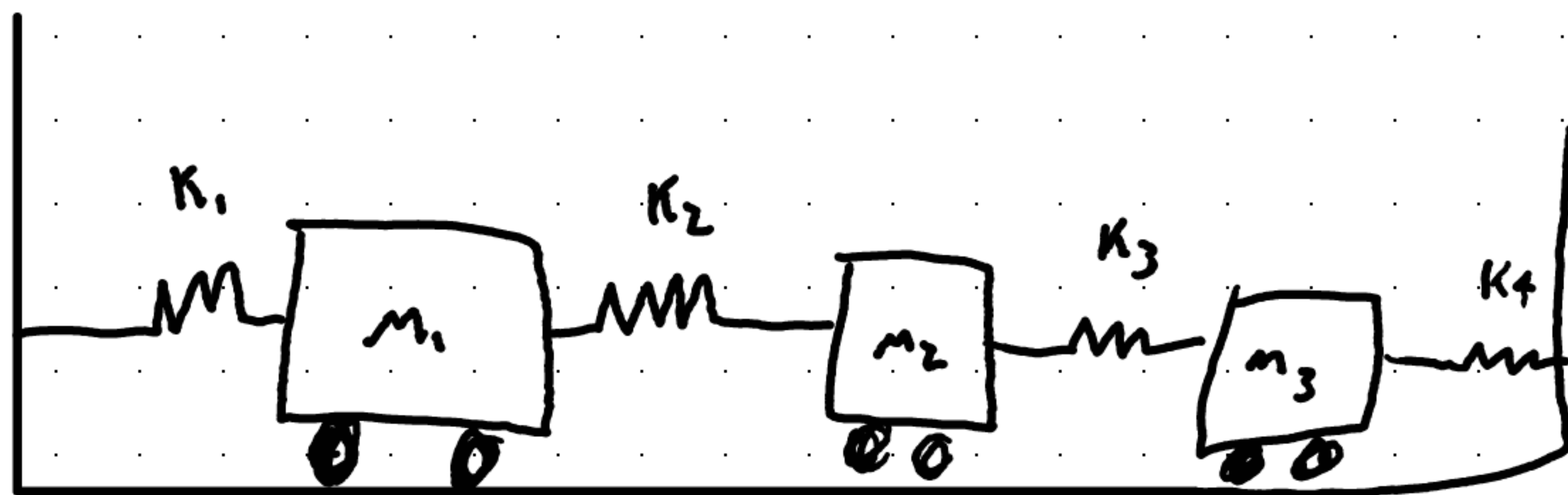


Undamped Spring Mass Systems

Let's consider again several masses & springs.
 For example, suppose m_1, m_2, m_3 are masses connected in series by four springs with spring constants K_1, K_2, K_3, K_4 , sliding on a frictionless plane.
 Let $x_i(t)$ denote the displacement from equilibrium of m_i .



$$\begin{aligned}
 m_1 \ddot{x}_1 &= -K_1 x_1 + K_2 (x_2 - x_1) = -(K_1 + K_2)x_1 + K_2 x_2 \\
 m_2 \ddot{x}_2 &= -K_2 (x_2 - x_1) + K_3 (x_3 - x_2) = K_1 x_1 - (K_2 + K_3)x_2 + K_3 x_3 \\
 m_3 \ddot{x}_3 &= -K_3 (x_3 - x_2) - K_4 x_3 = K_3 x_2 - (K_3 + K_4)x_3
 \end{aligned}$$

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_M \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}'' = \underbrace{\begin{bmatrix} -(K_1 + K_2) & K_2 & 0 \\ K_1 & -(K_2 + K_3) & K_3 \\ 0 & K_3 & -(K_3 + K_4) \end{bmatrix}}_K \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}$$

$$\begin{aligned}
 M \vec{x}'' &= K \vec{x} \\
 \vec{x}'' &= M^{-1} K \vec{x} = A \vec{x}
 \end{aligned}$$

We look for solutions of the form

$$\vec{x}(t) = e^{\alpha t} \vec{v}$$

$$\vec{x}''(t) = \alpha^2 e^{\alpha t} \vec{v}$$

Plug into $\vec{x}'' = A\vec{x}$:

$$\alpha^2 e^{\alpha t} \vec{v} = e^{\alpha t} A \vec{v}$$

$$A \vec{v} = \alpha^2 \vec{v}$$

α^2 is eigenvalue of $A = M^{-1}K$ with corresponding eigenvector \vec{v}

For our examples, the eigenvalues of $A = M^{-1}K$ will either be real & negative, or zero

If real & negative $\lambda = -\omega^2$, If $\alpha^2 = -\omega^2$, $\alpha = \pm i\omega$

$$\vec{x}(t) = e^{i\omega t} \vec{v} = (\cos \omega t + i \sin \omega t) \vec{v}$$

Split real and imaginary

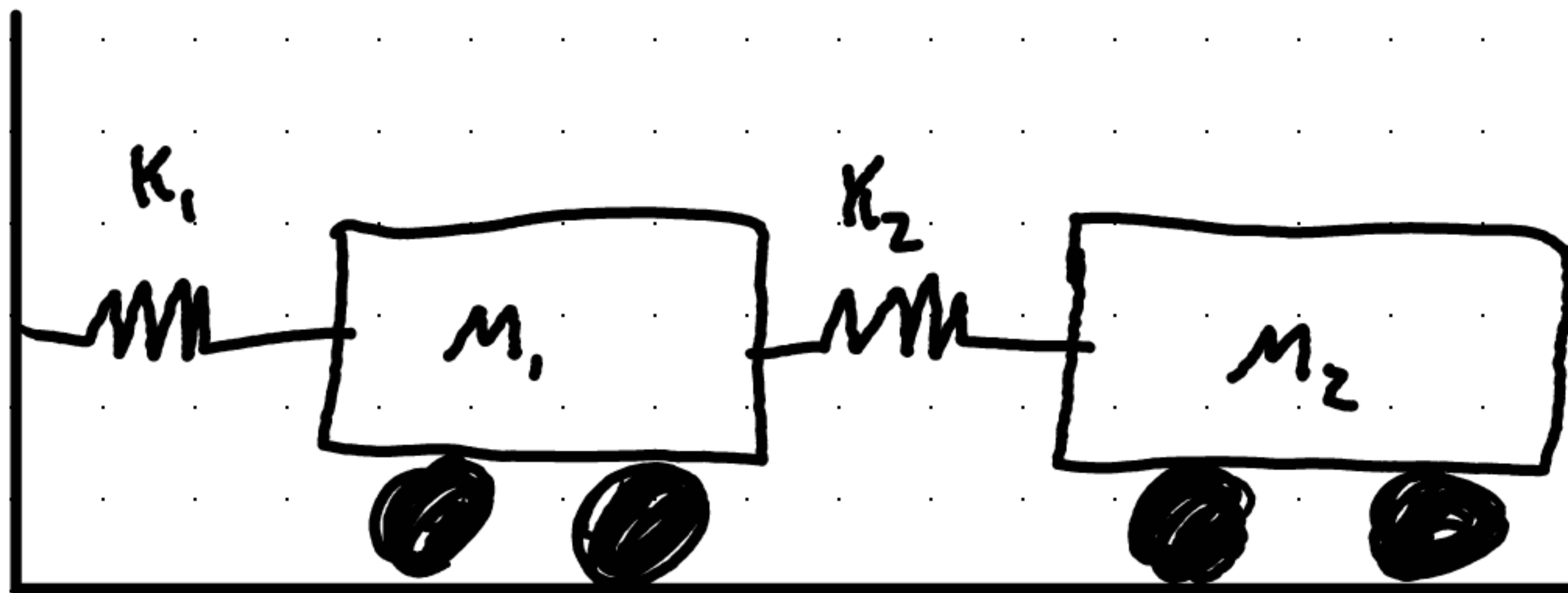
$$\vec{x}_1 = (a, \cos \omega t) \vec{v}$$

$$\vec{x}_2 = (b, \cos \omega t) \vec{v}$$

Ideally, we want a solution for each equation & each order.

If \mathbf{O} is an eigenvalue with corresponding eigenvector \vec{v} , then $\mathbf{x}(t) = \vec{v}(a + bt)$, for any a, b , is a solution.

Example. Consider the Setup in the figure below,
 with $m_1 = 2\text{kg}$, $m_2 = 1\text{kg}$, $K_1 = 4\text{N/m}$ & $K_2 = 2\text{N/m}$



$$m_1 x_1'' = -K_1 x_1 + K_2 (x_2 - x_1) = -(K_1 + K_2)x_1 + K_2 x_2$$

$$m_2 x_2'' = -K_2 (x_2 - x_1) = K_2 x_1 - K_2 x_2$$

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{M} \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -(4+2) & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, \text{ so } A = M^{-1}K = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 2 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3 - \lambda & 1 \\ 2 & -2 - \lambda \end{vmatrix}$$

$$= \lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4)$$

$$\lambda_1 = -1, \lambda_2 = -4$$

...

$$\lambda = -1, v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = -4, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

From earlier

$$\alpha^2 = \lambda$$

$$\alpha^2 = -1, \alpha = \pm i;$$

$$\omega_1 = 1$$

$$\alpha^2 = -4, \alpha = \pm 2i;$$

$$\omega_2 = 2$$

So, general solution:

$$\vec{x}(t) = a_1 \cos(\omega_1 t) \vec{v}_1 + b_1 \sin(\omega_1 t) \vec{v}_1 \\ + a_2 \cos(\omega_2 t) \vec{v}_2 + b_2 \sin(\omega_2 t) \vec{v}_2$$

Two solutions for each eigenvalue!

Fill in the blanks:

$$a_1 \cos t \begin{bmatrix} 1 \\ z \end{bmatrix} + b_1 \sin t \begin{bmatrix} 1 \\ z \end{bmatrix} + a_2 \cos 2t \begin{bmatrix} -1 \\ i \end{bmatrix} + b_2 \sin 2t \begin{bmatrix} -1 \\ i \end{bmatrix}$$

Determine a_1, b_1, a_2, b_2 from IC's!

$$x(0) = a_1 \vec{v}_1 + a_2 \vec{v}_2$$

$$\dot{x}(0) = \omega_1 b_1 \vec{v}_1 + \omega_2 b_2 \vec{v}_2$$

Or....

$$\underbrace{a_1 \cos t}_{\text{Scalar}} \underbrace{\begin{bmatrix} 1 \\ z \end{bmatrix}}_{\text{Vector}} + \underbrace{a_2 \cos 2t}_{\text{Scalar}} \underbrace{\begin{bmatrix} -1 \\ i \end{bmatrix}}_{\text{Vector}} + \underbrace{b_1 \sin t}_{\text{Scalar}} \underbrace{\begin{bmatrix} 1 \\ z \end{bmatrix}}_{\text{Vector}} + \underbrace{b_2 \sin 2t}_{\text{Scalar}} \underbrace{\begin{bmatrix} -1 \\ i \end{bmatrix}}_{\text{Vector}}$$

So...

$$\underbrace{\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}}_E \underbrace{\begin{bmatrix} \cos \omega_1 t & 0 \\ 0 & \cos \omega_2 t \end{bmatrix}}_{C(t)} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} +$$

$$\underbrace{\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}}_E \underbrace{\begin{bmatrix} \sin \omega_1 t & 0 \\ 0 & \sin \omega_2 t \end{bmatrix}}_{S(t)} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and what we'll end is...

$$\vec{x}(0) = E C(0) \vec{a} + E S(0) \vec{b} = E \vec{a}$$

$\hookrightarrow 1$ $\hookrightarrow 0$

So...

$$\vec{a} = E^{-1} \vec{x}(0)$$

and

$$\vec{x}' = E C'(0) \vec{a} + E S'(0) \vec{b} = E S'(0) \vec{b}$$

$\hookrightarrow 0$ $\hookrightarrow 1$

So...

$$\vec{b} = [S'(0)]^{-1} E^{-1} \vec{x}'(0)$$

All to say:

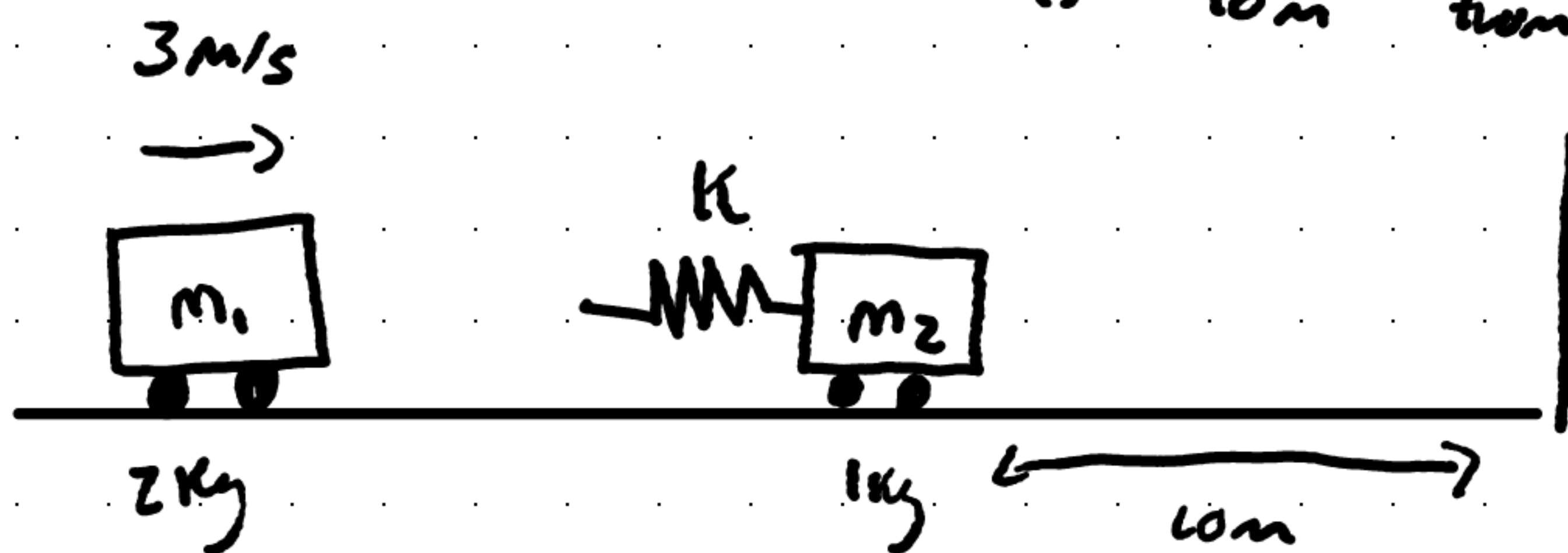
Given IC's

$$\vec{x}(t) = E C(t) E^{-1} \vec{x}_0 + E S(t) [S'(0)]^{-1} E^{-1} \vec{x}'_0$$

Can be useful if you are getting software to solve!

Example:

We have two toy rail cars. Car 1 of mass 2 kg is traveling at 3 m/s towards the second rail car that engages at the moment the cars hit (it connects to the cars) and does not let go. The bumper acts like a spring of Spring constant $K = 2\text{ N/m}$. The second car is 10 m from the wall.



At what time after the cars link does impact with the wall happen? What is the speed of Car 2 when it hits the wall?

Let

$t=0$ correspond to moment car 1 strikes

Car 2's Spring:



We can use our previous example!

Matrix Setup

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} -K & K \\ K & -K \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$A = M^{-1}K = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

0 is the eigenvalue of A , with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So is $\lambda = -3$, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\alpha^2 = -3$$

$$\alpha = \pm \sqrt{3}i$$

General Solution:

So $v e^{\sqrt{3}it}$ is

solution

$$\vec{x}(t) = \vec{v}_1(a_1 + b_1 t) + \vec{v}_2(a_2 \cos \sqrt{3}t + b_2 \sin \sqrt{3}t)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} (a_1 + b_1 t) + \begin{bmatrix} -1 \\ 2 \end{bmatrix} (a_2 \cos \sqrt{3}t + b_2 \sin \sqrt{3}t)$$

$$= \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}_E \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \cos \sqrt{3}t \end{bmatrix}}_{C(t)} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}}_E \underbrace{\begin{bmatrix} t & 0 \\ 0 & \sin \sqrt{3}t \end{bmatrix}}_{S(t)} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\vec{x}(0) = \vec{x}_0, \quad \vec{x}'(0) = \vec{x}'_0$$

$$\vec{x}(0) = E \vec{a} \Rightarrow \vec{a} = E^{-1} \vec{x}_0$$

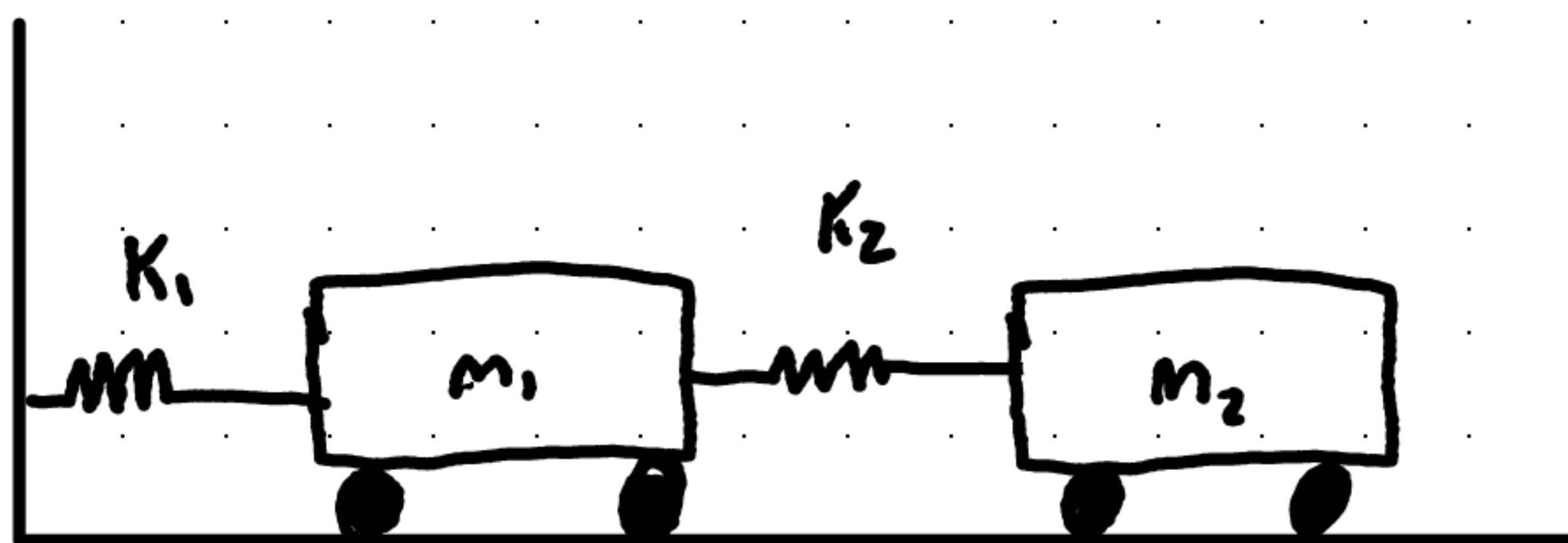
$$\vec{x}'(0) = E(S'(0)) \vec{b} \Rightarrow \vec{b} = [S'(0)]^{-1} E^{-1} \vec{x}'_0$$

$$\vec{x}(t) = E(t) E^{-1} \vec{x}_0 + E(t) S(t) [S'(0)]^{-1} E^{-1} \vec{x}'_0$$

$$\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{x}'(0) = \vec{x}'_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Adding ^{an oscillator} Forcing To Second Order De Systems

Example. Consider again the setup in the figure below, with $m_1 = 2\text{kg}$ & $m_2 = 1\text{kg}$, $K_1 = 4\text{N/m}$, $K_2 = 2\text{N/m}$, but suppose a force of $2\cos 3t$ (not pictured) acts on the second cart.



$$\omega = 3$$

$$\mathbf{x}'' = A\mathbf{\hat{x}} + F\cos 3t,$$

forcing only on second cart

$$\mathbf{\hat{F}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Recall:

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \quad \omega_1 = 1, \quad \omega_2 = 2$$

$$\text{So } \mathbf{\hat{x}}_p = \mathbf{\hat{C}} \cos 3t$$

$$\vec{z} = (A + \omega^2 I)^{-1} (-\vec{F})$$

$$A + \omega^2 I = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix}$$

$$(A + \omega^2 I)^{-1} = \underset{\text{Det}}{\frac{1}{40}} \begin{bmatrix} 7 & -1 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} \frac{7}{40} & -\frac{1}{40} \\ -\frac{1}{20} & \frac{3}{20} \end{bmatrix}$$

$$(A + \omega^2 I)^{-1} (-\vec{F}) = \begin{bmatrix} \frac{7}{40} & -\frac{1}{40} \\ -\frac{1}{20} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ -\frac{3}{10} \end{bmatrix}$$

$$\text{So, } \vec{x}_p = \begin{bmatrix} \frac{1}{20} \\ -\frac{3}{10} \end{bmatrix} \cos 3t$$

$$x = \vec{x}_c + \vec{x}_p$$

↑
Already
found.

----- Remember that

$$\vec{x}_c = E(\omega t) \vec{a} + E_S(\omega t) \vec{b}$$

→
More on
Next Page

Remember That:

$$\vec{x}_c = E(t) \vec{a} + ES(t) \vec{b}$$

§

$$E = \begin{bmatrix} 1 & 1 \\ \vec{v}_1 & \vec{v}_2 \\ 1 & 1 \end{bmatrix},$$

(Eigenvector)
matrix

$$C(t) = \begin{bmatrix} \cos \omega_1 t & 0 \\ 0 & \cos \omega_2 t \end{bmatrix}$$

$$S(t) = \begin{bmatrix} \sin \omega_1 t & 0 \\ 0 & \sin \omega_2 t \end{bmatrix}$$

So the general solution is

$$\vec{x}(t) = \underbrace{EC(t) \vec{a} + ES(t) \vec{b}}_{\vec{x}_c} + \underbrace{\vec{c} \cos \omega t}_{\vec{x}_p}$$

$$\dot{\vec{x}}(t) = \underbrace{EC(t)\vec{a} + ES(t)\vec{b}}_{\vec{x}_c} + \underbrace{\vec{c}\cos\omega t}_{\vec{x}_r}$$

So let's Apply IC's

IC's $x(0) = \vec{x}_0, \quad x'(0) = x'_0:$

$$\vec{x}(0) = E\vec{a} + \vec{c} \Rightarrow \vec{a} = E^{-1}(x_0 - \vec{c})$$

$$x'(0) = ES'(0)\vec{b} \Rightarrow \vec{b} = E^{-1}[S'(0)]^{-1}x'_0$$

This is the only
difference from
before!!