

THE DIFFERENTIAL GEOMETRY DIFFERENTIAL TOPOLOGY DUMP		
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CONTENTS		
<b>Part 1. Combinatorics, Probability Theory</b>	2	32
<b>Part 2. Linear Algebra Review</b>	3	33
<b>Part 3. Analytical Geometry</b>	3	
1. Applications of Vector Algebra to Analytic Geometry	3	
<b>Part 4. Manifolds</b>	4	
2. Inverse Function Theorem	4	
3. Immersions	5	
4. Submersions; Rank Theorem	6	
5. Submanifolds; immersed submanifold, embedded submanifolds, regular submanifolds	9	
6. Curves, Integral Curves, and Flows	9	
7. Push-forward, Pull-back	10	
8. Tensors	11	
<b>Part 5. Lie Groups, Lie Algebra</b>	12	
9. Lie Groups	12	
10. Lie Groups and their Lie Algebras	25	
<b>Part 6. Cohomology; Stoke’s Theorem</b>	25	
11. Stoke’s Theorem	25	
<b>Part 7. Prástaro</b>	26	
<b>Part 8. Connections</b>	26	
12. Connections of Vector Bundles	26	
<b>Part 9. Holonomy</b>	27	
13. Path Groupoid of a smooth manifold; generalization of paths	30	
<b>Part 10. Complex Manifolds</b>	31	
<b>Part 11. Jets, Jet bundles, <math>h</math>-principle, <math>h</math>-Prinzipien</b>	31	
<b>Part 12. Morse Theory</b>	32	
14. Morse Theory introduction from a physicist		32
15. Lagrange multipliers		33
<b>Part 13. Classical Mechanics</b>		33
16. Structure of Galilean Space-Time		34
17. Fundamental Theorems of (Classical) Dynamics		35
18. Rigid Body Dynamics, Rigid Body Motion		39
19. Fluid Mechanics, Fluid Flow		39
20. Viscous Fluids, Navier-Stokes Equation Applications		44
21. Thermodynamics		44
<b>Part 14. Classical Mechanics applications</b>		44
22. Orbital Mechanics		44
<b>Part 15. General Relativity</b>		50
<b>Part 16. WE Heraeus International Winter School on Gravity and Light</b>		50
Introduction (from EY)		50
23. Lecture 1: Topology		50
Topology Tutorial Sheet		50
24. Lecture 2: Topological Manifolds		51
Tutorial Topological manifolds		51
25. Lecture 3: Multilinear Algebra		52
26. Lecture 4: Differentiable Manifolds		54
Tutorial 4 Differentiable Manifolds		56
27. Lecture 5: Tangent Spaces		57
Lecture 6: Fields		59
Lecture 7: Connections		60
Lecture 8: Parallel Transport & Curvature (International Winter School on Gravity and Light 2015)		63
Tutorial 8 Parallel transport & Curvature		65
Lecture 9: Newtonian spacetime is curved!		66
28. Lecture 10: Metric Manifolds		68
29. Symmetry		70
30. Integration		72
31. Lecture 13: Relativistic spacetime		73
32. Lecture 14: <u>Matter</u>		74
33. Lecture 15: Einstein gravity		76
Tutorial 13 Schwarzschild Spacetime		77

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34.  
35.  
36. Canonical Formulation of GR I  
37.  
38.  
39.  
40. Lecture 22: [Black Holes](#)  
References

78 (i) number of distinguishable distributions is  
78  
78 (3)  $A_{r,n} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$   
78  
78 cf. Eq. (5.2) of Feller (1968) [1]  
78 (ii) number of distinguishable distributions in which no cell remains empty is  $\binom{r-1}{n-1}$ .  
80

ABSTRACT. Everything about Differential Geometry, Differential Topology

Part 1. Combinatorics, Probability Theory

**Theorem 1** (4.2. of Feller (1968) [1]). *Let  $r_1, \dots r_k \in \mathbb{Z}$ , s.t.  $r_1 + r_2 + \dots + r_k = n$ ;;  $r_i \geq 0$ .  
Let*

(1) 
$$\frac{N!}{r_1!r_2! \dots r_k!} =$$

*number of ways in which  $n$  elemnts can be divided into  $k$  ordered parts (partitioned into  $k$  subpopulations). cf. Eq. (4.7) of Feller (1968) [1].*

*Note that the order of the subpopulations is essential in the sense that  $(r_1 = 2, r_2 = 3)$  and  $(r_1 = 3, r_2 = 2)$  represent different partitions. However, no attention is paid to the order within the groups.*

*Proof.*

(2) 
$$\binom{n}{r_1} \binom{n-r_1}{r_2} \binom{n-r_1-r_2}{r_3} \dots \binom{n-r_1-\dots-r_{k-2}}{r_{k-1}} = \frac{n!}{r_1!r_2! \dots r_k!}$$

i.e. in order to effect the desired partition, we have to select  $r_1$  elementsout of  $n$ , remaining  $n - r_1$  elements select a second group of size  $r_2$ , etc. After forming the  $(k - 1)$ st group there remains  $n - r_1 - r_2 - \dots - r_{k-1} = r_k$  elements, and these form the last group. □

cf. pp. 37 of Feller (1968) [1] Examples. (g) Bridge. 32 cards are partitioned into 4 equal groups  $\rightarrow 52!/(13!)^4$ .  
Probability each player has an ace (?).  
The 4 aces can be ordered in  $4! = 24$  ways, each order presents 1 possibility of giving 1 ace to each player.  
Remaining 48 cards distributed  $(48!)/(12!)^4$  ways.

$$\rightarrow p = 24 \frac{48!}{(12!)^4} / \frac{52!}{(13!)^4}$$

(h) A throw of 12 dice  $\rightarrow 6^{12}$  different outcomes total. Event each face appears twice can occur in as many ways as 12 dice can be arranged in 6 groups of 2 each.

$$\frac{12!}{(2!)^6} / \frac{52!}{(13!)^4}$$

0.0.1. *Application to Occupancy Problems; binomial coefficients.* cf. Sec. 5 Application to Occupancy Problems of Feller (1968) [1].

Consider randomly placing  $r$  balls intos  $n$  cells.  
Let  $r_k$  = occupancy number = number of balls in  $k$ th cell.  
Every  $n$ -tuple of integers satisfying  $r_1 + r_2 + \dots + r_n = r$ ;  $r_k \geq 0$ . describes a possible configuration of occupancy numbers.  
With indistinguishable balls 2 distributions are distinguishable only if the corresponding  $n$ -tuples  $(r_1, \dots r_n)$  are not identical.

*Proof.* Represent balls by stars, indicate  $n$  cells by  $n$  spaces between  $n + 1$  bars. e.g.  $r = 8$  balls .  
 $n = 6$  cells

$$\begin{array}{ccccccc} 3 & & 10 & & 00 & & 4 \\ | & * & * & | & * & | & | & * & * & * & | \end{array}$$

Such a symbol necessarily starts and ends with a bar, but remaining  $n - 1$  bars and  $r$  starts appear in an arbitrary order. In this way, it becomes apparent that the number of distinguishable distributions equals the number of ways of selecting.  $r$  places out of  $n + r - 1$ ,  $\frac{(n+r-1)!}{(n-1)!r!} = \binom{n-1+r}{r}$

$$\begin{array}{ccc} |||| \dots || & n + 1 \text{ bars} \\ * * * \dots * * & r \text{ stars leave } r - 1 \text{ spaces} \end{array}$$

Condition that no cell be empty imposes the restriction that no 2 bars be adjacent.  $r$  stars leave  $r - 1$  spaces of which  $n - 1$  are to be occupied by bars. Thus  $\binom{r-1}{n-1}$  choices. □

Probability to obtain given occupancy numbers  $r_1, \dots r_n = \frac{r!}{r_1!r_2! \dots r_n!} / n^r$ , with  $r$  balls given by Thm. 4.2. of Feller (1968)  $n$  cells

[1], which is the Maxwell-Boltzmann distribution.

- (a) Bose-Einstein and Fermi-Dirac statistics. Consider  $r$  indistinguishable particles,  $n$  cells, each particle assigned to 1 cell.  
State of the system - random distribution of  $r$  particles in  $n$  cells.  
If  $n$  cells distinguishable,  $n^r$  arrangements equiprobable  $\rightarrow$  Maxwell-Boltzmann statistics.  
Bose-Einstein statistics: only distinguishable arrangements are considered, and each assigned probability  $\frac{1}{A_{r,n}}$

(4) 
$$A_{r,n} = \binom{n+r-1}{r} = \binom{n-1+r}{n-1}$$

cf. Eq. 5.2 of Feller (1968) [1]

Fermi-Dirac statistics.

- (1) impossible for 2 or more particles to be in the same cell.  $\rightarrow r \leq n$ .  
(2) all distinguishable arrangements satisfying the first condition have equal probabilities.  
 $\rightarrow$  an arrangement is completely described by stating which of the  $n$  cells contain a particle  
 $r$  particles  $\rightarrow \binom{n}{r}$  ways  $r$  cells chosen.  
Fermi-Dirac statistics, there are  $\binom{n}{r}$  possible arrangements, prob.  $1/\binom{n}{r}$ .

pp. 39. Feller (1968) [1]. Consider cells themselves indistinguishable! Disregard order among occupancy numbers.  
cf. Feller (1968) [1]

## Part 2. Linear Algebra Review

cf. *Change of Basis*, of Appendix B of John Lee (2012) [3].

**Exercise B.22.** Suppose  $V, W, X$  finite-dim. vector spaces

$S : V \rightarrow W$ ,  $T : W \rightarrow X$

- (a)  $\text{rank } S \leq \dim V$  with  $\text{rank } S = \dim V$  iff  $S$  injective
- (b)  $\text{rank } S \leq \dim W$  with  $\text{rank } S = \dim W$  iff  $S$  surjective
- (c) if  $\dim V = \dim W$  and  $S$  either injective or surjective, then  $S$  isomorphism
- (d)  $\text{rank } TS \leq \text{rank } S$   $\text{rank } TS = \text{rank } S$  iff  $\text{im } S \cap \ker T = 0$
- (e)  $\text{rank } TS \leq \text{rank } T$   $\text{rank } TS = \text{rank } T$  iff  $\text{im } S + \ker T = W$
- (f) if  $S$  isomorphism, then  $\text{rank } TS = \text{rank } T$
- (g) if  $T$  isomorphism, then  $\text{rank } TS = \text{rank } S$

EY : Exercise B.22(d) is useful for showing the chart and atlas of a Grassmannian manifold, found in the More examples, for smooth manifolds.

*Proof.* (a) Recall the **rank-nullity theorem**:

**Theorem 2** (Rank-Nullity Theorem).

$$(5) \quad \dim(\text{im}(S)) + \dim(\ker(S)) = \dim V$$

Now

$$\text{rank}(S) + \dim(\ker(S)) \equiv \dim(\text{im}(S)) + \dim(\ker(S)) = \dim V$$

$$\implies \boxed{\text{rank}(S) \leq \dim V}$$

If  $\text{rank}(S) = \dim V$ ,

then by rank-nullity theorem,  $\dim(\ker(S)) = 0$ , implying that  $\ker S = \{0\}$ .

Suppose  $v_1, v_2 \in V$  and that  $S(v_1) = S(v_2)$ . By linearity of  $S$ ,  $S(v_1) - S(v_2) = S(v_1 - v_2) = 0$ , which implies, since  $\ker S = \{0\}$ , that  $v_1 - v_2 = 0$ .

$\implies v_1 = v_2$ . Then by definition of injectivity,  $S$  injective.

If  $S$  injective, then  $S(v) = 0$  implies  $v = 0$ . Then  $\ker S = \{0\}$ . Then by rank-nullity theorem,  $\text{rank}(S) = \dim V$ .

(b)  $\forall w \in \text{im}(S)$ ,  $w \in W$ . Clearly  $\text{rank } S \leq \dim W$ .

If  $S$  surjective,  $\text{im}(S) = W$ . Then  $\dim(\text{im}(S)) = \text{rank } S = \dim W$ .

If  $\text{rank } S = \dim W = m$ , then  $\text{im}(S)$  has basis  $\{y_i\}_{i=1}^m$ ,  $y_i \in \text{im}(S)$ , so  $\exists x_i \in V$ ,  $i = 1 \dots m$  s.t.  $S(x_i) = y_i$ , with  $\{S(x_i)\}_{i=1}^m$  linearly independent.

Since  $\{S(x_i)\}_{i=1}^m$  linearly independent and  $\dim W = m$ ,  $\{S(x_i)\}_{i=1}^m$  basis for  $W$ .

$\forall w \in W$ ,  $w = \sum_{i=1}^m w^i S(x_i) = S(\sum_{i=1}^m w^i x_i)$ .  $\sum_{i=1}^m w^i x_i \in V$ .  $S$  surjective.

(c)

(d) Now

$$\dim V = \text{rank } TS + \text{nullity } TS$$

$$\dim V = \text{rank } S + \text{nullity } S$$

$\ker S \subseteq \ker TS$ , clearly, so  $\text{nullity } S \leq \text{nullity } TS$

$$\implies \boxed{\text{rank } TS \leq \text{rank } S}$$

If  $\text{rank } TS = \text{rank } S$ ,

then  $\text{nullity } S = \text{nullity } TS$

Suppose  $w \in \text{Im } S \cap \ker T$ ,  $w \neq 0$

Then  $\exists v \in S$ , s.t.  $w = S(v)$  and  $T(w) = 0$

Then  $T(w) = TS(v) = 0$ . So  $v \in \ker TS$

$v \notin \ker S$  since  $w = S(v) \neq 0$

This implies  $\text{nullity } TS > \text{nullity } S$ . Contradiction.

$$\implies \text{Im } S \cap \ker T = 0$$

If  $\text{Im } S \cap \ker T = 0$ ,

Consider  $v \in \ker TS$ . Then  $TS(v) = 0$ .

. Then  $S(v) \in \ker T$

$S(v) = 0$ ; otherwise,  $S(v) \in \text{Im } S$ , contradicting given  $\text{Im } S \cap \ker T = 0$

$v \in \ker S$

$$\ker TS \subseteq \ker S$$

$$\implies \ker TS = \ker S$$

So  $\text{nullity } TS = \text{nullity } S$

$$\implies \text{rank } TS = \text{rank } S$$

(e)

(f)

(g)

□

## Part 3. Analytical Geometry

### 1. APPLICATIONS OF VECTOR ALGEBRA TO ANALYTIC GEOMETRY

cf. Ch. 13 "Applications of Vector Algebra to Analytic Geometry", Apostol (1991) [35]

1.1. **Lines in  $n$ -space.** cf. Sec. 13.2 "Lines in  $n$ -space", Apostol (1991) [35].

**Definition 1** (Line). *Let given pt.  $P$ , given  $A$  nonzero vector.*

*Define line  $L(P; A)$*

$$(6) \quad L(P; A) := \{P + tA | t \in \mathbb{R}\} \equiv L(P; A) = \{P + tA\}$$

1.1.1. *Some simple properties of a straight line.* cf. Sec. 13.3 "Some simple properties of straight line", Apostol (1991) [35].

**Theorem 3** (Thm. 13.4 of Apostol (1991) [35]). *2 distinct pts. determine a line, i.e. if  $P \neq Q$ ,  $\exists!$  line containing both  $P$  and  $Q$ , namely*

$$(7) \quad \{P + t(P - Q)\}$$

*Proof.* Let  $L$  be line parallel through  $P$  to  $Q - P$ , i.e.  $L = L(P; Q - P) = \{P + t(Q - P)\}$ .

$P, Q \in L$  for  $t = 0, 1$ , respectively.

Suppose  $L'$  is another line containing  $P$  and  $Q$ . Since  $P \in L'$ ,  $L' = L'(P, A)$ ,  $Q \in L'$  so  $Q = P + At$ .

$Q - P = tA$ .  $t \neq 0$  since  $Q \neq P$ . So  $A = \frac{Q-P}{t}$ , i.e.  $A$  and  $Q - P$  are parallel.

By Thm. 13.1, since  $L(P; Q - P)$ ,  $L'(P, A)$  through same pt.  $P$  and direction vectors  $Q - P$ ,  $A$  parallel, then  $L = L'$ .

□

1.2. **Planes in Euclidean  $n$ -plane.** cf. Sec. 13.6 "Planes in Euclidean  $n$ -space", Apostol (1991) [35].

**Definition 2.** *Set  $M$  of pts. in  $V_n$  is called a plane if  $\exists$  pt.  $P$  and 2 linearly independent vectors  $A$  and  $B$  s.t.*

$$(8) \quad M = \{P + sA + tB | s, t \in \mathbb{R}\}$$

EY 20211207: Consider the intersection of 2 planes:  $M \cap N$ :

$$M = \{P + sA + tB | s, t \in \mathbb{R}\}$$

$$N = \{Q + uC + vD\}$$

Let  $X \in M$  and  $X \in N$  so  $X = P + sA + tB = Q + uC + vD$ .

Let  $M \cap N \neq \emptyset$ . Let  $X_0 \in M \cap N$ ,  $X_0 = P + s_0A + t_0B = Q + u_0C + v_0D$ .

**Part 4. Manifolds**

Schuller (2015) [31]

(1) (strikethrough)

(2)

(3) transitivity

(4) totality:  $a \leq b$  or  $b \leq a$ ,  $\forall a, b$ **Definition 3** (upper bound). *Schuller, original, version 1:* $u \in P$  **upper bound** to a subset  $T \subseteq P$  if $\forall t \in T : t \leq u$  (EY: in  $t \leq u$ ,  $\leq$  is for partial ordered set)*version 2, possible alternative (bottomline: not right):*(of is it  $u \in P$  **upper bound** to a subset  $T \subseteq P$  if  $\nexists t \in T$  s.t.  $u \leq t$ )

## 2. INVERSE FUNCTION THEOREM

Shastri (2011) had a thorough and lucid and explicit explanation of the Inverse Function Theorem [5]. I will recap it here. The following is also a blend of Wienhard's Handout 4 <https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf>

**Definition 4.** Let  $(X, a)$  metric space.**contraction**  $\phi : X \rightarrow X$  if  $\exists$  constant  $0 < c < 1$  s.t.  $\forall x, y \in X$ 

$$d(\phi(x), \phi(y)) \leq cd(x, y)$$

**Theorem 4** (Contraction Mapping Principle). Let  $(X, d)$  complete metric space.Then  $\forall$  contraction  $\phi : X \rightarrow X$ ,  $\exists ! y \in X$  s.t.  $\phi(y) = y$ ,  $y$  fixed pt.

*Proof.* Recall def. of complete metric space  $X$ ,  $X$  metric space s.t.  $\forall$  Cauchy sequence in  $X$  is convergent in  $X$  (i.e. has limit in  $X$ ).

$$x_1 = \phi(x_0)$$

$$x_2 = \phi(x_1)$$

 $\forall x_0 \in X$ , Define  $\vdots$ 

$$x_j = \phi(x_{j-1})$$

 $\vdots$ 

$$x_n = \phi(x_{n-1})$$

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1})) \leq cd(x_n, x_{n-1}) \leq \cdots \leq c^n d(x_1, x_0)$$

for some  $0 < c < 1$ .

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \leq \sum_{k=n-1}^m c^k d(x_1, x_0)$$

Thus,  $\forall \epsilon > 0$ ,  $\exists n_0 > 0$ , ( $n_0$  large enough) s.t.  $\forall m, n \in \mathbb{N}$  s.t.  $n_0 < n < m$ ,

$$d(x_m, x_n) \leq \sum_{k=n-1}^m c^k d(x_1, x_0) < \epsilon d(x_1, x_0)$$

Thus,  $\{x_n\}$  Cauchy sequence. Since  $X$  complete,  $\exists$  limit pt.  $y \in X$  of  $\{x_n\}$ .

$$\phi(y) = \phi(\lim_n x_n) = \lim_n \phi(x_n) = \lim_n x_{n+1} = y$$

Since by def. of  $y$  limit pt. of  $\{x_n\}$ ,  $\forall \epsilon > 0$ , then  $\{n | |x_n - y| \leq \epsilon, n \in \mathbb{N}\}$  is infinite.Consider  $\delta > \mathbb{N}$ . Consider  $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$  $\exists N_\delta \in \mathbb{N}$  s.t.  $\forall n > N_\delta$ ,  $|x_n - y| < \delta$ ; otherwise,  $\forall N_\delta$ ,  $\exists n > N_\delta$  s.t.  $|x_n - y| \geq \delta$ . Then  $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$  finite.

Contradiction.

 $\phi$  cont. so by def.  $\forall \epsilon > 0$ ,  $\exists \delta > 0$  s.t. if  $|x_n - y| < \delta$ , then  $|\phi(x_n) - \phi(y)| < \epsilon$ .

Pick  $N_\delta$  s.t.  $\forall n > N_\delta$ ,  $|x_n - y| < \delta$ , and so  $|\phi(x_n) - \phi(y)| < \epsilon$ . There are infinitely many  $\phi(x_n)$ 's that satisfy this, and so  $\phi(y)$  is a limit pt.

If  $\exists y_1, y_2 \in X$  s.t.  $\phi(y_1) = y_1$ , then

$$\phi(y_2) = y_2$$

$$d(y_1, y_2) = d(\phi(y_1), \phi(y_2)) \leq cd(y_1, y_2) \text{ with } c < 1$$

so  $c = 1$ 

□

**Theorem 5** (Inverse Function Theorem). Suppose open  $U \subset \mathbb{R}^n$ , let  $C^1 f : U \rightarrow \mathbb{R}^n$ ,  $x_0 \in U$  s.t.  $Df(x_0)$  invertible.Then  $\exists$  open neighborhoods  $V \ni x_0$ ,  $W \ni f(x_0)$  s.t.  $V \subseteq U$  and  $W \subseteq \mathbb{R}^n$ , respectively, and s.t.(i)  $f : V \rightarrow W$  bijection(ii)  $g = f^{-1} : V \rightarrow U$  differentiable, i.e.  $g = f^{-1} : W \rightarrow V$  is  $C^1$ (iii)  $D(f^{-1})$  cont. on  $W$ .(iv)  $Dg(y) = (Df(g(y)))^{-1} \quad \forall y \in W$ Also, notice that  $f(g(y)) = y \forall y \in W$ .*Proof.* Consider  $\tilde{f}(x) = (Df(x_0))^{-1}(f(x + x_0) - f(x_0))$ . Then

$$\tilde{f}(0) = 0 \text{ and}$$

$$D\tilde{f} = (Df(x_0))^{-1}(Df(x + x_0) - 0)$$

$$D\tilde{f}(0) = (Df(x_0))^{-1}Df(x_0) = 1$$

So let  $\tilde{f} \rightarrow f$  (notation) and so assume, without loss of generality, that  $U \ni 0$ ,  $f(0) = 0$ ,  $Df(0) = 1$ Choose  $0 < \epsilon \leq \frac{1}{2}$ . Let  $0 < \delta < 1$  s.t. open ball  $V = B_\delta(0) \subseteq U$ , and  $\|Df(x) - 1\| < \epsilon$ .  $\forall x \in U$ , since  $Df$  cont. at 0.Let  $W = f(V)$ . $\forall y \in W$ , define  $\phi_y : V \rightarrow \mathbb{R}^n$ 

$$\phi_y(x) = x + (y - f(x))$$

$$D(\phi_y)(x) = 1 + -Df(x) \quad \forall x \in V$$

$$\|D(\phi_y)(x)\| = \|1 - Df(x)\| \leq \epsilon < 1$$

 $\forall x_1, x_2 \in V$ , by mean value Thm. (not the equality that is only valid in 1-dim., but the inequality, that's valid for  $\mathbb{R}^d$ ,

$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \|D(\phi_y)(x')\| \|x_1 - x_2\|$$

for some  $x' = cx_2 + (1 - c)x_1$ ,  $c \in [0, 1]$ .  $V$  only needed to be convex set.

$$\implies \|\phi_y(x_1) - \phi_y(x_2)\| \leq \epsilon \|x_1 - x_2\|$$

Then  $\phi_y$  contraction mapping.Suppose  $f(x_1) = f(x_2) = y$ ,  $x_1, x_2 \in V$ .

$$\phi_y(x_1) = x_1$$

$$\phi_y(x_2) = x_2$$

$$\|\phi_y(x_1) - \phi_y(x_2)\| = \|x_1 - x_2\| \leq \epsilon \|x_1 - x_2\| \quad \forall \epsilon > 0 \implies x_1 = x_2$$

 $\implies f|_U$  injective. $W = f(V)$ , so  $f : V \rightarrow W$  surjective.  $f$  bijective.

Fix  $y_0 \in W$ ,  $y_0 = f(x_0)$ ,  $x_0 \in V$ .

Let  $r > 0$  s.t.  $B_r(x_0) \subset V$ .

Consider  $B_{r\epsilon}(y_0)$ . If  $y \in B_{r\epsilon}(y_0)$ .

$$r\epsilon > \|y - y_0\| = \|y - f(x_0)\| = \|\phi_y(x_0) - x_0\| \text{ with}$$

$$\phi_y(x) = x + (y - f(x))$$

If  $x \in B_r(x_0)$ ,

$$\|\phi_y(x) - x_0\| \leq \|\phi_y(x) - \phi_y(x_0)\| + \|\phi_y(x_0) - x_0\| \leq \epsilon\|x - x_0\| + r\epsilon < 2r\epsilon = r$$

Thus  $\phi(B_r(x_0)) = B_{r\epsilon}(y_0)$ .

By contraction mapping principle,  $\exists a \in B_r(x_0)$ , s.t.  $\phi_y(a) = a$ . Then  $\phi_y(a) = a + (y - f(a)) = a \implies f(a) = y$ .

$y \in f(V) = W$ .

So  $B_{r\epsilon}(y_0) \subset W$ .  $W$  open.

Let  $\text{Mat}(n, n) \equiv$  space of all  $n \times n$  matrices;  $\text{Mat}(n, n) = \mathbb{R}^{n^2}$ .

There is a proof of the implicit function theorem and its various forms in Shastri (2011) [5], but I found Wienhard's Handout 4 for Math 327 to be clearer.<sup>1</sup>

**Theorem 6** (Implicit Function Theorem). *Let open  $U \subset \mathbb{R}^{m+n} \equiv \mathbb{R}^m \times \mathbb{R}^n$*

$$C^1 f : U \rightarrow \mathbb{R}^n$$

$$(a, b) \in U \text{ s.t. } f(a, b) = 0 \text{ and } D_y f|_{(a, b)} \text{ invertible.}$$

*Then  $\exists$  open  $V \ni (a, b)$ ,  $V \subset U$*

$$\exists \text{ open neighborhood } W \ni a, W \subseteq \mathbb{R}^m$$

$$\exists! C^1 g : W \rightarrow \mathbb{R}^n \text{ s.t.}$$

$$\{(x, y) \in V | f(x, y) = 0\} = \{(x, g(x)) | x \in W\}$$

Moreover,

$$dg_x = - (d_y f)^{-1}|_{(x, g(x))} d_x f|_{(x, g(x))}$$

and  $g$  smooth if  $f$ .

*Proof.* Define  $F : U \rightarrow \mathbb{R}^{m+n}$

$$F(x, y) = (x, f(x, y))$$

Then  $F(a, b) = (a, 0)$  (given), and

$$DF = \begin{bmatrix} 1 & \\ \frac{\partial f^i(x, y)}{\partial x^j} & \frac{\partial f^i(x, y)}{\partial y^j} \end{bmatrix} \equiv \begin{bmatrix} 1 & \\ D_x f & D_y f \end{bmatrix}$$

$DF(a, b)$  invertible.

By inverse function theorem, since  $DF(a, b)$  invertible at pt.  $(a, b)$ ,

$\exists$  open neighborhoods  $V \ni (a, b) \subseteq \mathbb{R}^m \times \mathbb{R}^n$  s.t.  $F$  diffeomorphism with  $F^{-1} : \widetilde{W} \rightarrow V$ .

$$\widetilde{W} \ni (a, 0) \subseteq \mathbb{R}^m \times \mathbb{R}^n$$

Set  $W = \{x \in \mathbb{R}^m | (x, 0) \in \widetilde{W}\}$ . Then  $\pi_1(\widetilde{W}) = W$  open in  $\mathbb{R}^m$ .

Define  $g : W \rightarrow \mathbb{R}^n$ ,

$$g(x) = \pi_2 \circ F^{-1}(x, 0) \text{ or}$$

$$F^{-1}(x, 0) = (h(x), g(x))$$

Now  $FF^{-1}(x, 0) = (x, 0) = (h(x), f(h(x), g(x)))$  so  $h(x) = x \forall x \in W$ ,  $0 = f(x, g(x))$ .

Then

$$\{(x, y) \in V | f(x, y) = 0\} = \{(x, y) \in V | F(x, y) = (x, 0)\} = \{(x, g(x)) | x \in W, 0 = f(x, g(x))\}$$

<sup>1</sup><https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf>

Since  $\pi$  smooth and  $F^{-1}$  is  $C^1$ ,  $g$  is  $C^1$ .

To reiterate,  $f(x, g(x)) = 0$  on  $W$ .

Using chain rule while differentiating  $f(x, g(x)) = 0$ ,

$$\begin{aligned} \partial_{x^j} f(x, g(x)) &= \frac{\partial f(x, g(x))}{\partial x^k} \frac{\partial x^k}{\partial x^j} + \frac{\partial f(x, g(x))}{\partial y^k} \frac{\partial g^k(x)}{\partial x^j} = D_x f|_{(x, g(x))} + (D_y f)|_{(x, g(x))} \cdot (Dg)_x = 0 \text{ or} \\ (Dg)_x &= - (D_y f)|_{x, g(x)} D_x f|_{(x, g(x))} \end{aligned}$$

□

### 3. IMMERSIONS

**Definition 5** (Immersion). *smooth  $f : M \rightarrow N$ , s.t.  $Df(p) : T_p M \rightarrow T_{f(p)} N$  injective. Then  $f$  **immersion** at  $p$ .*

Absil, Mahony, and Sepulchre [8] pointed out that another definition for a *immersion* can utilize the theorem that rank of  $Df \equiv DF = \dim T_p M$ . Indeed, recall these facts from linear algebra:  
for  $T : V \rightarrow W$ ,

It's always true that  $\text{rank} T \leq V$ , and

$$\text{rank} T \leq W$$

$\text{rank} T = \dim V$  iff  $T$  injective.

$\text{rank} T = \dim W$  iff  $T$  surjective.

$$\begin{array}{ccc} T_x M & \xrightarrow{DF(x)} & T_{F(x)} N = T_y N \\ \uparrow & & \uparrow \\ x \in M & \xrightarrow{F} & y = F(x) \in N \end{array}$$

$$M \xrightarrow{F} N$$

Now

$$\dim T_x M = \dim M$$

$$\dim T_{F(x)} N = \dim N$$

And

$$\text{rank}(DF(x)) \equiv \text{rank of } F$$

I know that the notation above is confusing, but this is what all Differential Geometry books apparently mean when they say "rank of  $F$ ".

Now

$$\text{rank}(DF(x)) = \dim(\text{im}(DF(x))) = \dim T_x M \text{ iff } DF(x) \text{ injective}$$

If  $\forall x \in M$ , this is the case, then  $F$  an **immersion**.

Apply the rank-nullity theorem in this case:

$$\begin{aligned} \text{rank}(DF(x)) + \dim \ker(DF(x)) &= \dim T_x M = \dim M \\ \implies \text{rank}(DF(x)) &= \dim M \leq \dim T_{F(x)} N = \dim N \text{ or } \dim M \leq \dim N \end{aligned}$$

Now

$$\text{rank}(DF(x)) = \dim T_{F(x)}N \text{ iff } DF(x) \text{ surjective}$$

If  $\forall x \in M$ , this is the case, then  $F$  an **submersion** .

$$\text{rank}(DF(x)) = \dim T_{F(x)}N = \dim N \leq \dim M$$

Shastri (2011) has this as the “Injective Form of Implicit Function Theorem”, Thm. 1.4.5, pp. 23 and Guillemin and Pollack (2010) has this as the “Local Immersion Theorem” on pp. 15, Section 3 “The Inverse Function Theorem and Immersions” [4].

**Theorem 7** (Local immersion Theorem i.e. Injective Form of Implicit Function Theorem). *Suppose  $f : M \rightarrow N$  immersion at  $p$ ,  $q = f(p)$ .*

*Then  $\exists$  local coordinates around  $p, q$ ,  $x, y$ , respectively s.t.  $f(x_1 \dots x_m) = (x_1 \dots x_m, 0 \dots 0)$ .*

*Proof.* Choose local parametrizations

$$\begin{array}{ccc} U \subseteq M & \xrightarrow{f} & N \supseteq V \\ \downarrow \phi & & \downarrow \psi \\ \phi(U) & \xrightarrow{f} & \psi(V) \end{array} \quad \begin{array}{l} \phi(p) = x \\ \psi(q) = y \end{array}$$

$D(\psi f \varphi^{-1}) \equiv Df$ .  $Df(p)$  injective (given  $f$  immersion).  $Df(p) \in \text{Mat}(n, m)$

By change of basis in  $\mathbb{R}^n$ , assume  $Df(p) = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$ .

Now define  $G : \phi(U) \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^n$

$$G(x, z) = f(x) + (0, z)$$

Thus,  $DG(x, z) = 1$  and for open  $\phi(U) \times U_2$ ,  $G(\phi(U) \times U_2)$  open.

By inverse function theorem,  $G$  local diffeomorphism of  $\mathbb{R}^n$ , at 0.

Now  $f = G \circ i$ , where  $i$  is canonical immersion.

$$\begin{aligned} G(x, 0) &= f(x) \\ \implies G^{-1}G(x, 0) &= (x, 0) = G^{-1}f(x) \end{aligned}$$

Use  $\psi \circ G$  as the local parametrization of  $N$  around pt.  $q$ . Shrink  $U, V$  so that

$$\begin{array}{ccc} U \subseteq M & \xrightarrow{f} & N \supseteq V \\ \downarrow \phi & & \downarrow \psi \circ G \\ \phi(U) & \xrightarrow{i} & \psi \circ G(V) \end{array}$$

**Theorem 8** (Implicit Function Thm.). *Let open subset  $U \subseteq \mathbb{R}^n \times \mathbb{R}^d$ ,  $(x, y) = (x^1 \dots x^n, y^1 \dots y^k)$  on  $U$ .*

*Suppose smooth  $\Phi : U \rightarrow \mathbb{R}^k$ ,  $(a, b) \in U$ ,  $c = \Phi(a, b)$*

*If  $k \times k$  matrix  $\frac{\partial \Phi^i}{\partial y^j}(a, b)$  nonsingular, then  $\exists$  neighborhoods  $V_0 \subseteq \mathbb{R}^n$  of  $a$  and smooth  $F : V_0 \rightarrow W_0$  s.t.*

$$W_0 \subseteq \mathbb{R}^k \text{ of } b$$

$\Phi^{-1}(c) \cap (V_0 \times W_0)$  is graph of  $F$ , i.e.  
 $\Phi(x, y) = c$  for  $(x, y) \in V_0 \times W_0$  iff  $y = F(x)$ .

#### 4. SUBMERSIONS; RANK THEOREM

cf. pp. 20, Sec. 4 ”Submersions”, Ch. 1 of Guillemin and Pollack (2010) [4].

Consider  $X, Y \in \mathbf{Man}$ , s.t.  $\dim X \geq \dim Y$ .

**Definition 6** (submersion). *If  $f : X \rightarrow Y$ ,  
 if  $Df_x \equiv df_x$  is surjective,  $f \equiv$  **submersion** at  $x$ .*

Recall that,

$$Df_x : T_x X \rightarrow T_{f(x)} Y$$

$$\dim T_x X \geq \dim T_{f(x)} Y$$

$\text{rank } Df_x \leq \dim T_{f(x)} Y$ , in general, while

$\text{rank } Df_x = \dim T_{f(x)} Y$  iff  $Df_x$  surjective

Canonical submersion is standard projection:

If  $\dim X = k$ ,  $k \geq l$ ,

$\dim Y = l$

$$(a_1 \dots a_k) \mapsto (a_1 \dots a_l)$$

**Theorem 9** (Local Submersion Theorem). *Suppose  $f : X \rightarrow Y$  submersion at  $x$ , and  $y = f(x)$ , Then  $\exists$  local coordinates around  $x, y$  s.t.*

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

*i.e.  $f$  locally equivalent to canonical submersion near  $x$*

*Proof.* I’ll have a side-by-side comparison of my notation and the 1 used in Guillemin and Pollack (2010) [4] where I can.

For charts  $(U, \phi), (V, \psi)$  for  $X, Y$ , respectively,  $y = f(x)$  for  $x \in X$ ,

$$\begin{array}{ccc} U \subseteq X & \xrightarrow{f} & Y \supseteq V \\ \downarrow \phi & & \downarrow \psi \circ G \\ \mathbb{R}^k & \xrightarrow{i} & \mathbb{R}^l \end{array} \quad \begin{array}{ccc} x & \xrightarrow{f} & f(x) = y \\ \downarrow \phi & & \downarrow \psi \\ \phi(x) = (a^1 \dots a^k) & \xrightarrow{g} & g(\phi(x)) = g(a^1 \dots a^k) = \psi(y) \end{array}$$

$Dg_x$  surjective, so assume it’s a  $l \times k$  matrix  $\begin{bmatrix} \mathbf{1}_l & 0 \end{bmatrix}$ .

Define

$$G : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^k$$

(9)

$$G(a) \equiv G(a^1 \dots a^k) := (g(a), a_{l+1}, \dots, a_k)$$

Now

□

(10)

$$DG(a) = \begin{bmatrix} \mathbf{1}_l & 0 \\ & \mathbf{1}_{k-l} \end{bmatrix} = \mathbf{1}_k$$

so  $G$  local diffeomorphism (at 0).

So  $\exists G^{-1}$  as local diffeomorphism of some  $U'$  of  $a$  into  $U \subset \mathbb{R}^k$ .

By construction,

(11)

$$g = \mathbb{P}_l \circ G$$



where  $\mathbb{P}_l$  is the *canonical submersion*, the projection operator onto  $\mathbb{R}^l$ .

$$g \circ G^{-1} = \mathbb{P}_l$$

(since  $G$  diffeomorphism)

$$\begin{array}{ccc} U \subseteq X & \xrightarrow{f} & V \subseteq Y \\ \phi^{-1} \circ G^{-1} \uparrow & & \uparrow \psi^{-1} \\ \mathbb{R}^k & \xrightarrow{\mathbb{P}_l} & \mathbb{R}^l \end{array} \quad \text{for}$$

$$\begin{array}{ccc} \phi^{-1} \circ G^{-1}(a) \equiv \phi^{-1} \circ G^{-1}(a^1 \dots a^k) = x & \xrightarrow{f} & f(x) = y = \psi^{-1}(a^1 \dots a^l) \\ \phi^{-1} \circ G^{-1} \uparrow & & \uparrow \psi^{-1} \\ (a^1 \dots a^k) & \xrightarrow{\mathbb{P}_l} & (a^1 \dots a^l) \end{array}$$

$$\implies$$

□

”An obvious corollary worth noting is that if  $f$  is a submersion at  $x$ , then it is actually a submersion in a whole neighborhood of  $x$ .” Guillemin and Pollack (2010) [4]

Suppose  $f$  submersion at  $x \in f^{-1}(y)$ .

By local submersion theorem

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

Choose  $y = (0, \dots, 0)$ .

Then, near  $x$ ,  $f^{-1}(y) = \{(0, \dots, 0, x_{l+1} \dots x_k)\}$  i.e. let  $V \ni x$  neighborhood of  $x$ , define  $(x_1 \dots x_k)$  on  $V$ .

Then  $f^{-1}(y) \cap V = \{(0 \dots 0, x_{l+1}, \dots x_k) | x_1 = 0, \dots x_l = 0\}$ .

Thus  $x_{l+1}, \dots x_k$  form a coordinate system on open set  $f^{-1}(y) \cap V \subseteq f^{-1}(y)$ .

Indeed,

$$\begin{array}{ccc} U \subseteq X & \xrightarrow{f} & V \subseteq Y \\ \downarrow \phi & & \downarrow \psi \\ \mathbb{R}^k & \xrightarrow{\mathbb{P}_l} & \mathbb{R}^l \end{array} \quad \begin{array}{ccc} x & \xrightarrow{f} & f(x) = y \\ \downarrow \phi & & \downarrow \psi \\ \phi(x) = (x^1 \dots x^k) & \xrightarrow{\mathbb{P}_l} & (x^1 \dots x^l) \end{array}$$

and now

$$\begin{array}{ccc} f^{-1}(y) & \xleftarrow{f^{-1}} & y \\ \uparrow \phi^{-1} & & \downarrow \psi \\ \{(0, \dots, 0, x^1 \dots x^k)\} & \xleftarrow{\mathbb{P}_l^{-1}} & (0 \dots 0) \end{array}$$

**4.1. Rank Theorem.** Lee (2012) [3] in pp. 85, Ch. 4 Submersions, Immersions, and Embeddings, combines Theorems 7, 9 (local immersion and local submersion theorems, respectively) into the ”Rank Theorem” (cf. Thm 4.12 ”Rank Theorem” of Lee (2012)):

**Theorem 10** (Rank Theorem). *Suppose smooth manifolds  $M, N$ ,  $\dim M = m$ ,  $\dim N = n$ , smooth map  $F : M \rightarrow N$ ,  $F$  has constant rank  $r$ .*

$\forall p \in M$ ,  $\exists$  smooth charts  $(U, \varphi)$  for  $M$ , centered at  $p$ ,  $(V, \psi)$  for  $N$ , centered at  $F(p)$ , s.t.

$$F(U) \subseteq V$$

in which  $F$  has coordinate representation of form

$$(12) \quad \widehat{F}(x^1 \dots x^r, x^{r+1} \dots x^m) = (x^1 \dots x^r, 0 \dots 0)$$

Particularly, if  $F$  smooth submersion,

$$\widehat{F}(x^1 \dots x^n, x^{n+1} \dots x^m) = (x^1 \dots x^n)$$

and if  $F$  smooth immersion

$$\widehat{F}(x^1 \dots x^m) = (x^1 \dots x^m, 0 \dots 0)$$

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\psi F \varphi^{-1}} & \mathbb{R}^n \\ \uparrow \varphi & & \uparrow \psi \\ U \subset M & \xrightarrow{F} & V \subset N \end{array}$$

Also remember that  $DF(p) : T_p M \rightarrow T_{F(p)} N$

*Proof.*  $DF(p)$  has rank  $r$  (given). Then  $DF(p)$  is some  $r \times r$  submatrix of a  $n \times m$  matrix s.t.  $\det DF(p)$  nonzero.

By change of basis in  $\mathbb{R}^n$ , or reordering coordinates, assume  $DF(p)$  is upper left submatrix  $\left( \frac{\partial F^i}{\partial x^j} \right) \quad \forall i, j = 1, \dots r$ .

Relabel standard coordinate as

$$(x, y) = (x^1 \dots x^r, y^1 \dots y^{m-r}) \in \mathbb{R}^m$$

$$(v, w) = (v^1 \dots v^r, w^1 \dots w^{n-r}) \in \mathbb{R}^n$$

By initial translations of coordinates, assume without loss of generality  $p = (0, 0)$ ,  $F(p) = (0, 0)$

Suppose

$$F(x, y) = (Q(x, y), R(x, y))$$

for some smooth maps  $Q : U \rightarrow \mathbb{R}^r$ ,  $R : U \rightarrow \mathbb{R}^{n-r}$

Define

$$\varphi : U \rightarrow \mathbb{R}^m$$

$$\varphi(x, y) = (Q(x, y), y)$$

so

$$D\varphi(0, 0) = \begin{pmatrix} \frac{\partial Q^i}{\partial x^j}(0, 0) & \frac{\partial Q^i}{\partial y^j}(0, 0) \\ 0 & \delta_j^i \end{pmatrix}$$

$D\varphi(0, 0)$  nonsingular, since  $\det \frac{\partial Q^i}{\partial x^j} \neq 0$  (by hypothesis).

By inverse function thm.,  $\exists$  connected neighborhoods  $U_0$  of  $(0, 0)$ ,  $\tilde{U}_0$  of  $\varphi(0, 0) = (0, 0)$  s.t.

$$\varphi : U_0 \rightarrow \tilde{U}_0$$

is a diffeomorphism.

By shrinking  $U_0, \tilde{U}_0$ , assume  $\tilde{U}_0$  open cube.

Write  $\varphi^{-1}(x, y) = (A(x, y), B(x, y))$ , for some smooth functions  $A : \tilde{U}_0 \rightarrow \mathbb{R}^r$  ,  
 $B : \tilde{U}_0 \rightarrow \mathbb{R}^{m-r}$

$$(x, y) = \varphi(A(x, y), B(x, y)) = (Q(A(x, y), B(x, y)), B(x, y))$$

$$\begin{aligned} & B(x, y) = y \\ \implies & \varphi^{-1}(x, y) = (A(x, y), y) \end{aligned}$$

$$\varphi\varphi^{-1} = 1 \implies x = Q(A(x, y), y)$$

Recall that we had hypotehsized that

$$F(x, y) = (Q(x, y), R(x, y))$$

Then

$$F \circ \varphi^{-1}(x, y) = F(A(x, y), y) = (Q(A(x, y), y), R(A(x, y), y)) = (x, R(A(x, y), y))$$

and so

$$F \circ \varphi^{-1}(x, y) = (x, \tilde{R}(x, y))$$

where  $\tilde{R} : \tilde{U}_0 \rightarrow \mathbb{R}^{n-r}$

$$\tilde{R}(x, y) = R(A(x, y), y)$$

Compute

$$D(F \circ \varphi^{-1})(x, y) = \begin{pmatrix} \delta_j^i & 0 \\ \frac{\partial \tilde{R}^i}{\partial x^j}(x, y) & \frac{\partial \tilde{R}^i}{\partial y^j}(x, y) \end{pmatrix}$$

Since composing with a diffeomorphism doesn't change rank of map,  $D(F \circ \varphi^{-1})$  has rank  $r$  everywhere in  $\tilde{U}_0$ .

$\begin{pmatrix} \delta_j^i \\ \frac{\partial \tilde{R}^i}{\partial x^j}(x, y) \end{pmatrix} j = 1 \dots r$  are linearly independent, so  $\frac{\partial \tilde{R}^i}{\partial y^j}(x, y) = 0$  on  $\tilde{U}_0$ , so  $\tilde{R}^i$  independent of  $y^j$ .

Let  $S(x) = \tilde{R}(x, 0)$ , then

$$F \circ \varphi^{-1}(x, y) = (x, S(x))$$

Let open  $V_0 \subseteq V$ ,  $(0, 0) \in V$  be an open subset  $V_0 = \{(v, w) \in V : (v, 0) \in \tilde{U}_0\}$ .  
 Then  $V_0$  is a neighborhood of  $(0, 0)$ .

Because  $\tilde{U}_0$  is a cube,  $F \circ \varphi^{-1}(x, y) = (x, S(x))$ ,

$$F \circ \varphi^{-1}(\tilde{U}_0) \subseteq V_0$$

so  $F(U_0) \subseteq V_0$ .

Define  $\psi : V_0 \rightarrow \mathbb{R}^n$

$$\psi(v, w) = (v, w - S(v))$$

Because  $\psi^{-1}(s, t) = (s, t + S(s))$ ,

it is a diffeomorphism.

Thus  $(V_0, \psi)$  is a smooth chart.

$$\psi \circ F(\varphi^{-1}(x, y)) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0)$$

**Definition 7** (regular value). *For smooth  $f : X \rightarrow Y$ ,  $X, Y \in \mathbf{Man}$ ,*

*$y \in Y$  is a **regular value** for  $f$  if  $Df_x : T_x X \rightarrow T_y Y$  surjective  $\forall x$  s.t.  $f(x) = y$ .*

*$y \in Y$  **critical value** if  $y$  not a regular value of  $f$ .*

Absil, Mahony, and Sepulchre [8] pointed out that another definition for a *regular value* can utilize the theorem that rank of  $Df \equiv DF = \dim T_p N = \dim N$ , iff  $DF(p)$  surjective, for  $p \in M$ ,  $F : M \rightarrow N$ . Then

**regular value**  $y \in N$ , of  $F$ , if rank of  $F \equiv \text{rank}(DF(x)) = \dim N$ ,  $\forall x \in F^{-1}(y)$ , for  $F : M \rightarrow N$ .

**Theorem 11** (Preimage theorem). *If  $y$  regular value of  $f : X \rightarrow Y$ ,  $f^{-1}(y)$  is a submanifold of  $X$ , with  $\dim f^{-1}(y) = \dim X - \dim Y$*

*Proof.* Given  $y$  is a regular value of  $f : X \rightarrow Y$ ,

$\forall x \in f^{-1}(y)$ ,  $Df_x : T_x X \rightarrow T_y Y$  is surjective. By local submersion theorem,

$$f(x^1 \dots x^k) = (x^1 \dots x^l) = y$$

Since  $x \in f^{-1}(y)$ ,  $(x^1 \dots x^k) = (y^1 \dots y^l, x^{l+1} \dots x^k)$ .

For this chart for  $(U, \varphi)$ ,  $U \ni x$ , consider  $(U \cap f^{-1}(y), \psi)$  with  $\psi(x) = (x^{l+1} \dots x^k) \quad \forall x \in U \cap f^{-1}(y)$ .

$\forall f^{-1}(y)$  submanifold with  $\dim f^{-1}(y) = k - l = \dim X - \dim Y$ . □

*Examples for emphasis*

If  $\dim X > \dim Y$ ,

if  $y \in Y$ , regular value of  $f : X \rightarrow Y$ ,

$f$  submersion,  $\forall x \in f^{-1}(y)$

If  $\dim X = \dim Y$ ,

$f$  local diffeomorphism  $\forall x \in f^{-1}(y)$

If  $\dim X < \dim Y$ ,  $\forall y \in f(X)$  is a critical value.

**Example:**  $O(n)$  as a submanifold of  $\text{Mat}(n, n)$

Given  $\text{Mat}(n, n) \equiv M(n) = \{n \times n \text{ matrices}\}$  is a manifold; in fact  $\text{Mat}(n, n) \cong \mathbb{R}^{n^2}$ ,

Consider  $O(n) = \{A \in \text{Mat}(n, n) | AA^T = 1\}$ .

$$(13) \quad AA^T \in \text{Sym}(n) \equiv S(n) = \{S \in \text{Mat}(n, n) | S^T = S\} = \{ \text{symmetric } n \times n \text{ matrices} \}$$

$\text{Sym}(n)$  submanifold of  $\text{Mat}(n, n)$ ,  $\text{Sym}(n)$  diffeomorphic to  $\mathbb{R}^k$  (i.e.  $\text{Sym}(n) \cong \mathbb{R}^k$ ),  $k := \frac{n(n+1)}{2}$ .

$$f : \text{Mat}(n, n) \rightarrow \text{Sym}(n)$$

$$f(A) = AA^T$$

Notice  $f$  is smooth,

$$f^{-1}(1) = O(n)$$

$$Df_A(B) = \lim_{s \rightarrow 0} \frac{f(A + sB) - f(A)}{s} = \lim_{s \rightarrow 0} \frac{(A + sB)(A^T + sB^T) - AA^T}{s} = AB^T + BA^T$$

If  $Df_A : T_A \text{Mat}(n, n) \rightarrow T_{f(A)} \text{Sym}(n)$  surjective when  $A \in f^{-1}(1) = O(n)$  (???).

**Proposition 1.** *If smooth  $g_1 \dots g_l \in C^\infty(X)$  on  $X$  are independent  $\forall x \in X$ , s.t.  $g_i(x) = 0$ ,  $\forall i = 1 \dots l$ , then  $Z = \{x \in X | g_1(x) = \dots = g_l(x) = 0\}$  is a submanifold of  $X$  s.t.  $\dim Z = \dim X - l$ .*

*Take note that  $g_1 \dots g_l$  are independent at  $x$  means, really, that  $D(g_1)_x \dots D(g_l)_x$  are linearly independent on  $T_x X$ .*

*Proof.* Suppose smooth  $g_1 \dots g_l \in C^\infty(X)$  on manifold  $X$  s.t.  $\dim X = k \geq l$ .

Consider  $g = (g_1 \dots g_l) : X \rightarrow \mathbb{R}^l$ ,  $Z \equiv g^{-1}(0)$ .

Since  $\forall g_i$  smooth,  $D(g_i)_x : T_x X \rightarrow \mathbb{R}$  linear.

Now for

$$Dg_x = (D(g_1)_x \dots D(g_l)_x) : T_x X \rightarrow \mathbb{R}^l$$

□

By rank-nullity theorem (linear algebra),  $Dg_x$  surjective iff  $\text{rank } Dg_x = l$  i.e.  $l$  functionals  $D(g_1)_x \dots D(g_l)_x$  are linearly independent on  $T_x X$ .

"We express this condition by saying the  $l$  functions  $g_1 \dots g_l$  are independent at  $x$ ." (Guillemin and Pollack (2010) [4])

□



## 5. SUBMANIFOLDS; IMMERSED SUBMANIFOLD, EMBEDDED SUBMANIFOLDS, REGULAR SUBMANIFOLDS

**Definition 8** (Embedded Submanifold).

Recall immersion:

$F : M \rightarrow N$  immersion iff  $DF$  injective, i.e. iff  $\text{rank} DF = \dim M$ .

Consider manifolds  $M \subseteq N$ .

Consider inclusion map  $i : M \rightarrow N$ .

$$i : x \mapsto x$$

If  $i$  immersion,  $Di(x) = \frac{\partial y^i}{\partial x^j} = \delta_j^i$  if  $y^i = x^i$ ,  $\forall i = 1, \dots, \dim M$ .

**Definition 9** (immersed submanifold). ***immersed submanifold**  $M \subseteq N$  if inclusion  $i : M \rightarrow N$  is an immersion.*

cf. 3.3 Embedded Submanifolds of Absil, Mahony, and Sepulchre [8], also Ch. 5 Submanifolds, pp. 108, **Immersed Submanifolds** of John Lee (2012) [3].

Immersed submanifolds often arise as images of immersions.

**Proposition 2** (Images of Immersions as submanifolds). *Suppose smooth manifold  $M$ , smooth manifold with or without boundaries  $N$ ,*

*injective, smooth immersion  $F : M \rightarrow N$  ( $F$  injective itself, not just immersion)*

*Let  $S = F(M)$ .*

*Then  $S$  has unique topology and smooth structure of smooth submanifolds of  $N$  s.t.  $F : M \rightarrow S$  diffeomorphism.*

cf. Prop. 5.18 of John Lee (2012) [3].

*Proof.* Define topology of  $S$ : set  $U \subseteq S$  open iff  $F^{-1}(U) \subseteq M$  open ( $F^{-1}(U \cap V) = F^{-1}(U) \cap F^{-1}(V)$ ,  $F^{-1}(U \cup V) = F^{-1}(U) \cup F^{-1}(V)$ ).

Define smooth structure of  $S$ :  $\{F(U), \varphi \circ F^{-1}|(U, \varphi) \in \text{atlas for } M, \text{ i.e. } (U, \varphi) \text{ any smooth chart of } M\}$ .

”smooth compatibility condition”:

$$(\varphi_2 \circ F^{-1})(\varphi_1 F^{-1})^{-1} = \varphi_2 \circ F^{-1} F \varphi_1^{-1} = \varphi_2 \varphi_1^{-1}$$

since  $\varphi_2 \varphi_1^{-1}$  diffeomorphism ( $\varphi_2 \varphi_1^{-1}$  bijection and it and inverse is differentiable)

$F$  diffeomorphism onto  $F(M)$ .

and these are the only topology and smooth structure on  $S$  with this property:

$$S \xrightarrow{F^{-1}} M \xrightarrow{F} N = S \hookrightarrow M$$

and  $F^{-1}$  diffeomorphism,  $F$  smooth immersion, so  $i : S \rightarrow M$  smooth immersion.

## 6. CURVES, INTEGRAL CURVES, AND FLOWS

cf. John Lee (2012) [3], Ch. 9, deals with time-dependent vector fields and I don’t see other texts or references handling such an important, but overlooked, case.

**6.1. Curves in Euclidean space.** cf. 4.1 Jeffrey Lee (2009) [2]

If  $C$  is 1-dim. submanifold of  $\mathbb{R}^n$ ,  $p \in C$ ,  $\exists$  chart  $(V, y)$  of  $C$ ,  $p \in V$  s.t.  $y(V)$  is a connected open interval  $I \subset \mathbb{R}$ , inverse map  $y^{-1} : I \rightarrow V \subset M$  is a local parametrization.

idea is to extract information that’s appropriately independent of parametrization.

If  $\gamma : I \rightarrow \mathbb{R}^n$ ,  $c : J \rightarrow \mathbb{R}^n$  curves with same image,

$c$  is a **positive reparametrization** of  $\gamma$  if  $\exists$  smooth  $h : J \rightarrow I$  with  $h' > 0$  s.t.  $c = \gamma \circ h$ ,

in this case,  $\gamma, c$  have same **sense** and same **orientation**.

Assume  $\gamma : I \rightarrow \mathbb{R}^n$  has  $\|\gamma'\| > 0$ ,

Such a curve is **regular**, i.e. curve is an immersion (Recall, an immersion would be smooth  $\gamma : I \rightarrow \mathbb{R}^n$ , s.t.  $D\gamma(p) : T_p I \rightarrow T_{\gamma(p)} \mathbb{R}^n$  injective).

**Definition 10** (unit tangent field in Euclidean space, 4.3 Lee (2009) [2]). *If  $\gamma : I \rightarrow \mathbb{R}^n$  regular curve, then  $\mathbf{T}(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$  defines **unit tangent field** along  $\gamma$  ( $\|T\| = 1$ )*

**length of a curve** defined on closed interval  $\gamma : [t_1, t_2] \rightarrow \mathbb{R}^n$ ,

$$L = \int_{t_1}^{t_2} \|\gamma'(t)\| dt$$

Define **arc length function** for curve  $\gamma : I \rightarrow \mathbb{R}^n$  by choosing  $t_0 \in I$ ,

$$s = h(t) := \int_{t_0}^t \|\gamma'(\tau)\| d\tau$$

If curve is smooth and regular, then  $h' = \|\gamma'(\tau)\| > 0$ , so by inverse function theorem,  $\exists$  smooth  $h^{-1}$  (since  $h'$  invertible (with  $1/h'$ )).

If  $c(s) = \gamma h^{-1}(s)$ , then  $\|c'\|(s) := \|c'(s)\| = 1$ ,  $\forall s$

$$c'(s) = (\gamma(h^{-1}(s)))' = \gamma'(h^{-1}(s)) \cdot \frac{dh^{-1}}{ds}(s)$$

$$\|c'(s)\| = \|\gamma'(t)\| \left\| \frac{dh^{-1}}{ds}(s) \right\| = h' \cdot \frac{1}{h'} = 1$$

Curves parametrized by arc length are **unit speed curves**.

For a unit speed curve,  $\frac{dc}{ds}(s) = \mathbf{T}(s)$ .

**Definition 11** (curvature vector, curvature function, principal normal in Euclidean space, 4.4 Lee (2009) [2]). *Let  $c : I \rightarrow \mathbb{R}^n$  be a unit speed curve. vector valued function*

$$\kappa(s) := \frac{d\mathbf{T}}{ds}(s)$$

*is called the **curvature vector**.*

$$\kappa(s) := \|\kappa(s)\| = \left\| \frac{d\mathbf{T}}{ds}(s) \right\|$$

*is called the **curvature function**.*

□ *If  $\kappa(s) > 0$ , then define principal normal*

$$\mathbf{N}(s) = \left\| \frac{d\mathbf{T}}{ds}(s) \right\|^{-1} \frac{d\mathbf{T}}{ds}(s)$$

*s.t.  $\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$*

**6.2. Integral Curves.** cf. Ch. 9 of Lee (2012) [3]

integral curves - smooth curves whose velocity at each point is equal to the value of the smooth vector field there.

Let smooth manifold  $M$  with or without boundary.

If smooth curve  $\gamma : J \rightarrow M$ , then  $\forall t \in J$ , velocity vector  $\gamma'(t)$  is a vector in  $T_{\gamma(t)} M$ .

**Definition 12** (Integral Curve  $\Gamma$ ). *Integral curve:  $\mathbf{v}, \gamma$ .*

*If vector field  $\mathbf{v}$  on  $M$ ,*

***integral curve** of  $\mathbf{v}$  is differentiable curve  $\gamma : J \rightarrow M$ , whose velocity  $\forall p \in M$ ,  $V_{\gamma(t)}$*

$$\gamma'(t) = V_{\gamma(t)}, \forall t \in J$$

Finding integral curves boils downs to solving a system of ordinary differential equations in a smooth chart.

Suppose smooth vector field  $V$  on  $M$ , smooth curve  $\gamma : J \rightarrow M$

On smooth coordinate domain  $U \subseteq M$ ,

Write  $\gamma$  in local coordinates as

$$\begin{aligned}\gamma(t) &= (\gamma^1(t), \dots, \gamma^n(t)) \\ \gamma^i(t) &= x^i \circ \gamma(t) \text{ where } x^i : U \subseteq M \rightarrow \mathbb{R}\end{aligned}$$

Then condition for  $\gamma$  to be an integral curve.

$$\gamma'(t) \equiv \dot{\gamma}(t) = V_{\gamma(t)}$$

can be written as

$$\dot{\gamma}^i(t) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)} = V^i(\gamma(t)) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)}$$

which reduces to an autonomous system of ordinary differential equations (ODEs)

$$\begin{aligned}(14) \quad \dot{\gamma}^1(t) &= V^1(\gamma^1(t), \dots, \gamma^n(t)) \\ &\vdots \\ \dot{\gamma}^n(t) &= V^n(\gamma^1(t), \dots, \gamma^n(t))\end{aligned}$$

20200306 EY: I use this notation which is more useful for me, e.g. taking the double derivative of the curve  $\gamma$ :

$$\begin{aligned}\dot{\gamma}(t) &= v(\gamma(t)) \in T_{\gamma(t)}U \\ \dot{\gamma}(t) &= v^i(\gamma^1(t), \dots, \gamma^n(t)) \\ \frac{d}{dt} \dot{\gamma}^i(t) &= \frac{d}{dt} v^i(\gamma^1(t), \dots, \gamma^n(t)) = \frac{d}{dt} v^i(x^1, \dots, x^n) \\ &= \frac{\partial v^i}{\partial x^j}(x) \dot{x}^j = \dot{\gamma}^j \frac{\partial \dot{\gamma}^i}{\partial x^j}\end{aligned}$$

## 7. PUSH-FORWARD, PULL-BACK

**7.1. Push-forward in coordinates,  $f_* : T_p M \rightarrow T_{f(p)} N$ .** Using Nakahara (2003) [7]'s notation.

Consider a smooth mapping  $f : M \rightarrow N$ ,  $\dim M = m, \dim N = n$ .

$$f_* : T_p M \rightarrow T_{f(p)} N$$

With

$$\begin{aligned}(U, \varphi) \subset M \quad \varphi : U \rightarrow \mathbb{R}^m \quad \varphi^i \equiv x^i : U \rightarrow \mathbb{R} \\ (V, \psi) \subset N \quad \psi : V \rightarrow \mathbb{R}^n \quad \psi^j \equiv y^j : V \rightarrow \mathbb{R}\end{aligned}$$

$$\begin{aligned}\text{Now } X \in T_p M \text{ , so we can write } X &= X^i \frac{\partial}{\partial x^i}, \text{ and } f_* X \in T_{f(p)} N, \text{ so we can write } f_* X = Y^j \frac{\partial}{\partial y^j}. \\ Y \in T_{f(p)} N \quad Y &= Y^j \frac{\partial}{\partial y^j}\end{aligned}$$

Let  $g \in \mathcal{F}(N)$

$$g : N \rightarrow \mathbb{R}$$

Now  $g \circ \psi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$  or  $g \circ \psi^{-1} : \psi(N) \rightarrow \mathbb{R}$ .

For notation, let  $g \circ \psi^{-1}(y) \equiv g(y)$ , in that we treat  $y$  to be a point with coordinates in  $\mathbb{R}^n$  notation-wise, and not as  $y^j : V \rightarrow \mathbb{R}$ .

Since  $g \circ f : M \rightarrow \mathbb{R}$ , so  $g \circ f \in \mathcal{F}(M)$ .

$$(f_* X)[g] \equiv X[g \circ f]$$

$$\begin{aligned}(15) \quad (f_* X)[g] &= (Y^j \frac{\partial}{\partial y^j})[g] = Y^j \frac{\partial g}{\partial y^j} = X^i \frac{\partial}{\partial x^i} [g \circ f] = X^i \frac{\partial}{\partial x^i} g(f^j(x)) = X^i \frac{\partial g}{\partial y^j}(y) \frac{\partial f^j}{\partial x^i}(x) = X^i \frac{\partial f^j}{\partial x^i}(x) \frac{\partial}{\partial y^j} g \\ Y^j &= X^i \frac{\partial f^j}{\partial x^i} = X^i \frac{\partial y^j}{\partial x^i} = F^j_i(x) X^i\end{aligned}$$

where  $f : M \rightarrow N$ , but  $\psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ , but write this notation as  $f^j(x) = y^j$ .

Also, notice that for Eq. 15,

$$\begin{aligned}g \circ f : M &\rightarrow \mathbb{R} \\ g \circ \psi^{-1} \circ \psi \circ f \circ \varphi^{-1} \text{ and } g \circ f \circ \varphi^{-1} : \varphi(U) &\subset \mathbb{R}^m \rightarrow \mathbb{R} \\ (g \circ \psi^{-1})(\psi \circ f \circ \varphi^1)(x) &\equiv g(y(x))\end{aligned}$$

**7.2. Pullback in coordinates,  $f^* : T_{f(p)}^* N \rightarrow T_p^* M$ .** For pullback  $f^* : T_{f(p)}^* N \rightarrow T_p^* M$ ,

since  $\omega \in T_{f(p)}^* N$ , then write as  $\omega_j dy^j$

since  $f^* \omega \in T_p^* M$ , so  $f^* \omega = \psi_i dx^i$

$$\begin{aligned}\langle f^* \omega, X \rangle &\equiv f^* \omega(X) = \langle \psi_i dx^i, X^j \frac{\partial}{\partial x^j} \rangle \equiv \psi_i dx^i (X^j \frac{\partial}{\partial x^j}) = \psi_i X^i = \langle \omega, f_* X \rangle \equiv \omega(f_* X) = \\ &= \langle \omega_j dy^j, Y^i \frac{\partial}{\partial y^i} \rangle = \omega_j dy^j (Y^i \frac{\partial}{\partial y^i}) = \omega_j dy^j \left( X^i \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k} \right) = \omega_j \frac{\partial y^i}{\partial x^i} X^i \\ &\implies \psi_i = \omega_j \frac{\partial y^j}{\partial x^i}\end{aligned}$$

Thus

**Theorem 12** (Push-forward in coordinates).

$$\text{For smooth } f : M \rightarrow N, \quad \begin{aligned} \dim M &= m \\ \dim N &= n \end{aligned}$$

$$\begin{aligned}X &= X^i \frac{\partial}{\partial x^i} \in T_p M \\ Y &= Y^j \frac{\partial}{\partial y^j} \in T_{f(p)} N\end{aligned} \quad \text{then}$$

$$(16) \quad \boxed{\begin{aligned} f_* : T_p M &\rightarrow T_{f(p)} N \\ f_*(X) &= Y \\ Y^j &= X^i \frac{\partial y^j}{\partial x^i} \end{aligned} \quad s.t.}$$

where  $f_*$  is the **push-forward**, and where colloquially,  $y^j = y^j(x)$ , but actually  $y^j \equiv \psi \circ f \circ \varphi^{-1}$

And

**Theorem 13** (Pullback in coordinates). *For smooth  $f : M \rightarrow N$ ,*

$$\begin{aligned} \dim M &= m \\ \dim N &= n \end{aligned}$$

$\omega \in T_{f(p)}^* N$ ,  $\psi = \psi_i dx^i \in T_p^* M$ , then

$$(17) \quad \boxed{\begin{array}{l} f^* : T_{f(p)}^* N \rightarrow T_p^* M \\ f^* \omega = \psi_i dx^i \\ \psi_i = \omega_j \frac{\partial y^j}{\partial x^i} \end{array}} \quad \text{where}$$

## 8. TENSORS

I'll go through Ch.7 *Tensors* of Jeffrey Lee (2009) [2].

**Definition 13** (7.1[2]). *Let  $V, W$  be modules over commutative ring  $R$ , with unity. Then, algebraic  $W$ -valued tensor on  $V$  is multilinear map.*

$$(18) \quad \tau : V_1 \times V_2 \times \cdots \times V_m \rightarrow W$$

where  $V_i = \{V, V^*\} \quad \forall i = 1, 2, \dots, m$ .

If for  $r, s$  s.t.  $r + s = m$ , there are  $r$   $V_i = V^*$ ,  $s$   $V_i = V$ , tensor is  $r$ -contravariant,  $s$ -covariant; also say tensor of total type  $\binom{r}{s}$ .

EY : 20170404 Note that

$$\begin{aligned} &(\tau_\beta^{i\alpha} \frac{\partial}{\partial x^i} \text{ or } \tau_\beta^{i\alpha} e_i)(\omega_j dx^j \text{ or } \omega_j e^j \in V^*) \\ &(\tau_{i\alpha}^\beta dx^i \text{ or } \tau_{i\alpha}^\beta e^i)(X^j \frac{\partial}{\partial x^j} \text{ or } X^j e_j \in V) \end{aligned}$$

$\exists$  natural map  $V \rightarrow V^{**}$ ,  $\tilde{v} : \alpha \mapsto \alpha(v)$ . If this map is an isomorphism,  $V$  is **reflexive** module, and identify  $V$  with  $V^{**}$ .

**Exercise 7.5.** Given vector bundle  $\pi : E \rightarrow M$ , open  $U \subset M$ , consider sections of  $\pi$  on  $U$ , i.e. cont.  $s : U \rightarrow E$ , where  $(\pi \circ s)(u) = u, \quad \forall u \in U$ .

Consider  $E^* \ni \omega = \omega_i e^i$ .

$\forall s \in \Gamma(E), \omega(s) = \omega_i(s(x))^i, \quad \forall x \in U \subset M$ . So define  $\tilde{s} : \omega, x \mapsto \omega(s(x)), \quad \forall x \in U$ .

If  $\tilde{s} = 0, \tilde{s}(\omega, x) = \omega(s(x)) = 0 \quad \forall \omega \in E^*, \forall x \in U$ , and so  $s = 0$ . (Let  $\omega_i = \delta_{iJ}$  for some  $J$ , and so  $s^J(x) = 0 \quad \forall J$ ).

$s = 0$ . So  $\ker(s \mapsto \tilde{s}) = \{0\}$  (so condition for injectivity is fulfilled).

Since  $\tilde{s} : \omega, x \mapsto \omega(s(x)), \forall \omega \in E^*, \forall x \in U, s \mapsto \tilde{s}$  is surjective.

$s \mapsto \tilde{s}$  is an isomorphism so  $\Gamma(E)$  is a *reflexive* module.

**Proposition 3.** *For  $R$  a ring (special case),  $\exists$  module homomorphism:*

*tensor product space  $\rightarrow$  tensor, as a multilinear map, i.e.  $\exists$*

$$(19) \quad (\otimes_{i=1}^r V) \otimes (\otimes_{j=1}^s V^*) \rightarrow T_s^r(V; R) \\ u_1 \otimes \cdots \otimes u_r \otimes \beta^1 \otimes \cdots \otimes \beta^s \in (\otimes^r V) \otimes (\otimes^s V^*) \mapsto (\alpha^1 \dots \alpha^r, v_1 \dots v_s) \mapsto \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s)$$

Indeed, consider

$$(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in \underbrace{V^* \times \cdots \times V^*}_r \times \underbrace{V \times \cdots \times V}_s \mapsto \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s)$$

and so for

$$\begin{aligned} \alpha^i &= \alpha_\mu^i e^\mu, & i &= 1, 2, \dots, r, \mu = 1, 2, \dots, \dim V^* & \alpha^i(u_i) &= \alpha_\mu^i u_i^\mu \\ v_i &= v_\mu^i e_\mu, & i &= 1, 2, \dots, s, \mu = 1, 2, \dots, \dim V & \beta^i(v_i) &= \beta_\mu^i v_\mu^i \end{aligned}$$

So that

$$\begin{aligned} \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s) &= \alpha_{\alpha_1}^1 u_1^{\alpha_1} \dots \alpha_{\alpha_r}^r u_r^{\alpha_r} \beta_{\mu_1}^1 v_1^{\mu_1} \dots \beta_{\mu_s}^s v_s^{\mu_s} = \\ &= (u_1^{\alpha_1} \dots u_r^{\alpha_r} \beta_{\mu_1}^1 \dots \beta_{\mu_s}^s)(\alpha_{\alpha_1}^1 \dots \alpha_{\alpha_r}^r v_1^{\mu_1} \dots v_s^{\mu_s}) \end{aligned}$$

Identify  $u_1 \otimes \cdots \otimes u_r \otimes \beta^1 \otimes \cdots \otimes \beta^s$  with this multilinear map.

**Proposition 4.** *If  $V$  is finite-dim. vector space, or if  $V = \Gamma(E)$ , for vector bundle  $E \rightarrow M$ , map*

$$(20) \quad (\otimes_{i=1}^r V) \otimes (\otimes_{j=1}^s V^*) \rightarrow T_s^r(V; R)$$

*is an isomorphism.*

**Definition 14.** *tensor that can be written as*

$$(21) \quad u_1 \otimes \cdots \otimes u_r \otimes \beta^1 \otimes \cdots \otimes \beta^s \equiv u_1 \otimes \cdots \otimes \beta^s$$

is **simple** or **decomposable**.

Now well that not *all* tensors are simple.

**Definition 15** (7.7[2], tensor product).  $\forall S \in T_{s_1}^{r_1}(V), \forall T \in T_{s_2}^{r_2}(V)$ , define tensor product

$$(22) \quad S \otimes T \in T_{s_1+s_2}^{r_1+r_2}(V) \\ S \otimes T(\theta^1 \dots \theta^{r_1+r_2}, v_1 \dots v_{s_1+s_2}) := S(\theta^1 \dots \theta^{r_1}, v_1 \dots v_{s_1}) T(\theta^{r_1+1} \dots \theta^{r_1+r_2}, v_{s_1+1} \dots v_{s_1+s_2})$$

**Proposition 5** (7.8[2]).

$$\tau^{i_1 \dots i_r}_{j_1 \dots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} = \tau(e^{i_1} \dots e^{i_r}, e_{j_1} \dots e_{j_s}) e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} = \tau$$

So  $\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} | i_1 \dots i_r, j_1 \dots j_s \in 1 \dots n\}$  spans  $T_s^r(V; R)$

**Exercise 7.11.** Let basis for  $V$   $e_1 \dots e_n$ , corresponding dual basis for  $V^*$   $e^1 \dots e^n$

Let basis for  $V$   $\bar{e}_1 \dots \bar{e}_n$ , corresponding dual basis for  $V^*$   $\bar{e}^1 \dots \bar{e}^n$

s.t.

$$\begin{aligned} \bar{e}_i &= C_i^k e_k \\ \bar{e}^i &= (C^{-1})^i_k e^k \end{aligned}$$

EY:20170404, keep in mind that

$$\begin{aligned} Ax &= e_i A_i^k e^k (x^j e_j) = e_i A_i^j x^j = A_i^j x^j e_i \\ Ae_j &= e_k A_i^k e^i (e_j) = A_j^k e_k = \bar{e}_j \\ \bar{\tau}_{jk}^i \bar{e}_i \otimes \bar{e}^j \otimes \bar{e}^k &= \bar{\tau}_{jk}^i C_i^l e_l (C^{-1})_m^j e^m (C^{-1})_n^k e^n = \bar{\tau}_{jk}^i C_i^l (C^{-1})_m^j (C^{-1})_n^k = \tau_{mn}^l \\ \bar{\tau}_{jk}^i &= C_k^c C_j^b (C^{-1})_a^i \tau_{bc}^a \end{aligned}$$

On Remark 7.13 of Jeffrey Lee (2009) [2]: first, egregious typo for  $L(V, V)$ ; it should be  $L(V, W)$ . Onward, for  $L(V, W)$ , consider  $W \otimes V^* \ni w \otimes \alpha$  s.t.

$$(w \otimes \alpha)(v) = \alpha(v)w \in W, \forall v \in V, \text{ so } w \otimes \alpha \in L(V, W)$$

Now consider (category of) left  $R$ -module,

$$(23) \quad {}_R \mathbf{Mod} \ni {}_{\text{Mat}_{\mathbb{K}}(N, M)} \mathbb{K}^N$$

where

$$\begin{aligned} V &= \mathbb{K}^N \\ W &= \mathbb{K}^M \end{aligned}$$

For  $A \in \text{Mat}_{\mathbb{K}}(N, M)$ ,  $x \in \mathbb{K}^N$ ,

$$e_i A^i_{\phantom{i}\mu} e^\mu (x^\nu e_\nu) = Ax = e_i A^i_\mu x^\mu, \quad i = 1, 2, \dots M, \mu = 1, 2, \dots N$$

$$A \in \text{Mat}_{\mathbb{K}}(N, M) \cong W \otimes V^* \cong L(V, W)$$

Consider

$$\alpha \in (\mathbb{K}^N)^* = V^* \qquad \alpha = \alpha_\mu e^\mu$$

$$w \in \mathbb{K}^M = W \qquad w = w^i e_i$$

$$\alpha \otimes w = w \otimes \alpha = w^i \alpha_\mu e_i \otimes e^\mu$$

(remember, isomoprhism between  $\text{Mat}_{\mathbb{K}}(N, M)$  and  $W \otimes V^*$  guaranteed, if  $V, W$  are free  $R$ -modules,  $R = \mathbb{K}$ ).

Let  $V, W$  be left  $R$ -modules, i.e.  $V, W \in {}_R\mathbf{Mod}$ .

$$V^* \in \mathbf{Mod}_R$$

For  $V^* \otimes W \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$

$$\alpha \in V^*, w \in W$$

$$(\alpha \otimes w)(v) = \alpha(v)w, \text{ for } v \in V \in {}_R\mathbf{Mod}$$

But  $(w \otimes \alpha)(v) = w\alpha(v)$ .

Note  $\alpha(v) \in R$ .

Let  $V, W$  be right  $R$ -modules, i.e.  $V, W \in \mathbf{Mod}_R$ .

$$V^* \in {}_R\mathbf{Mod}$$

For  $W \otimes V^* \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$ .

$$\alpha \in V^*, w \in W$$

$$(v)(w \otimes \alpha) = w\alpha(v), \text{ with } \alpha(v) \in R, v \in V$$

So  $W \otimes V^* \cong L(V, W)$ , for  $V, W \in \mathbf{Mod}_R$

**Definition 16** (7.20[2], **contraction**). *Let  $(e_1, \dots e_n)$  basis for  $V$ ,  $(e^1 \dots e^n)$  dual basis.*

*If  $\tau \in T^r_s(V)$ , then for  $k \leq r$ ,  $l \leq s$ , define*

$$C_l^k \tau \in T^{r-1}_{s-1}(V)$$

$$C_l^k \tau(\theta^1 \dots \theta^{r-1}, w_1 \dots w_{s-1}) :=$$

$$(24) \qquad \sum_{a=1}^n \tau(\theta^1 \dots \underbrace{e^a}_{kth \text{ position}} \dots \theta^{r-1}, w_1 \dots \underbrace{e_a}_{ith \text{ position}} \dots w_{s-1})$$

$C_l^k$  is called **contraction**, for some single  $1 \leq k \leq r$ , some single  $1 \leq l \leq s$ ,

$$C_l^k : T^r_s(V) \rightarrow T^{r-1}_{s-1}(V)$$

s.t.

$$(C_l^k \tau)^{\widehat{i_1 \dots i_k \dots i_r}}_{\widehat{j_1 \dots j_l \dots j_s}} := \tau^{i_1 \dots a \dots i_r}_{j_1 \dots a \dots j_s}$$

Universal mapping properties can be invoked to give a basis free definition of contraction (EY : 20170405???)

IN general,

$$\forall v_1 \dots v_s \in V, \forall \alpha^1 \dots \alpha^r \in V^*$$

so that

$$v_j = v_j^\mu e_\mu \quad j = 1 \dots s, \quad \mu = 1, \dots \dim V$$

$$\alpha^i = \alpha^i_\mu e^\mu \quad i = 1 \dots r, \quad \mu = 1 \dots \dim V^*$$

then  $\forall \tau \in T^r_s(V)$ ,

$$\begin{aligned} \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) &= \tau(\alpha^1_{\mu_1} e^{\mu_1} \dots \alpha^r_{\mu_r} e^{\mu_r}, v_1^{\nu_1} e_{\nu_1} \dots v_s^{\nu_s} e_{\nu_s}) = \\ &= \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v_1^{\nu_1} \dots v_s^{\nu_s} \tau(e^{\mu_1} \dots e^{\mu_r}, e_{\nu_1} \dots e_{\nu_s}) = \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v_1^{\nu_1} \dots v_s^{\nu_s} \tau^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \end{aligned}$$

which is equivalent to

$$\begin{array}{ccc} \tau \in T^r_s(V) & \xrightarrow{\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes} & \alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes \tau \\ & & \downarrow \\ & C^1_{s+1} C^2_{s+2} \dots C^r_{r+s} C^r_1 C^{r+1}_2 \dots C^{r+s}_s & \\ & & \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in R \end{array}$$

where I've tried to express the right- $R$ -module, "right action" on  $\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \in V^* \otimes \dots \otimes V$ .

Conlon (2008) [17]

Part 5. Lie Groups, Lie Algebra

9. LIE GROUPS

- : Lie Groups
- : Groups
- : Ring
- : group algebra
- : Group Ring
- : Representation Theory
- : Modules
- :  $kG$ -modules

From Sec. 8.1 "Noncommutative Rings" of Rotman (2010) [10]:

**Definition 17.** *ring  $R$  - additive abelian group equipped with multiplication  $R \times R \rightarrow R$  s.t.  $\forall a, b \in R$*   
 $(a, b) \mapsto ab$

- (i)  $a(bc) = (ab)c$
- (ii)  $a(b + c) = ab + ac, (b + c)a = ba + ca$
- (iii)  $\exists 1 \in R$  s.t.  $\forall a \in R, 1a = a = a1$

Example 8.1[10]

- (ii) group algebra  $kG$ ,  $k$  commutative ring,  $G$  group, "its additive abelian group is free  $k$ -module having basis labeled by elements of  $G$ ,  
i.e.  $\forall a \in kG, a = \sum_{g \in G} a_g g, a_g \in k, \quad \forall g \in G, a_g \neq 0$  for only finitely many  $g \in G$ .

$$\begin{array}{ll} \text{define (ring) multiplication } kG \times kG \rightarrow kG & \forall a, b \in kG, \quad a = \sum_{g \in G} a_g g \quad \text{to be} \\ ab = ab & b = \sum_{h \in G} b_h g \end{array}$$

$$\left( \sum_{g \in G} a_g g \right) \left( \sum_{h \in G} b_h h \right) = \sum_{z \in G} \left( \sum_{gh=z} a_g b_h \right) z$$

**Definition 18.** Given  $R$  ring, left  $R$ -module is (additive) abelian group  $M$  equipped with

scalar multiplication  $R \times M \rightarrow M$  s.t.  $\forall m, m' \in M, \forall r, r', 1 \in R$

$$(r, m) \mapsto rm$$

- (i)  $r(m + m') = rm + rm'$
- (ii)  $(r + r')m = rm + r'm$
- (iii)  $(rr')m = r(r'm)$
- (iv)  $1m = m$

EY : 20150922 Example : for  $kG$ -module  $V^\sigma$ , for  $r \in kG$ , so  $r = \sum_{g \in G} a_g g$

$$\begin{array}{ccc} R \times M \rightarrow M & & kG \times V \rightarrow V \\ (r, m) \mapsto rm & \implies & (r, v) \mapsto tv \end{array}$$

For some representation  $\sigma : G \rightarrow GL(V)$ ,

$$rv = \sum_{g \in G} a_g g \cdot v = \sum_{g \in G} a_g \sigma_g(v)$$

So a  $kG$ -module needs to be associated with some chosen representation.

Note for  $V$  as an additive abelian group,  $\forall u, v, w \in V$ ,

$$\begin{aligned} v + w &= w + v, (u + v) + w = u + (v + w) \\ v + 0 &= v \quad \forall v \in V \text{ for } 0 \in V \\ v + (-v) &= 0 \quad \forall v \in V \end{aligned}$$

So a vector space can be an additive abelian group.

Note that

$$\begin{aligned} r(v + w) &= \left( \sum_{g \in G} a_g g \right) (v + w) = \left( \sum_{g \in G} a_g \sigma_g \right) (v + w) = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} a_g \sigma_g(w) = rv + rw \\ (r + r')v &= \left( \sum_{g \in G} a_g g + b_g g \right) v = \sum_{g \in G} (a_g \sigma_g + b_g \sigma_g) v = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} b_g \sigma_g(v) = rv + r'v \\ (rr')v &= \left( \sum_{g \in G} a_g g \sum_{h \in G} b_h h \right) v = \left( \sum_{z \in G} \sum_{gh=z} a_g b_h z \right) v = \sum_{z \in G} \sum_{gh \in z} a_g b_h \sigma_z(v) = \sum_{g \in G} \sum_{h \in G} a_g b_h \sigma_g \sigma_h(v) \end{aligned}$$

since  $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g \sigma_h = \sigma_{gh}$  ( $\sigma$  homomorphism)

$$1v = \sigma(1)v = 1v = v$$

From Sec. 8.3 “Semisimple Ring” of Rotman (2010) [10]:

**Definition 19.**  $k$ -representation of group  $G$  is homomorphism

$$\sigma : G \rightarrow GL(V)$$

where  $V$  is vector field over field  $k$

**Proposition 6** (8.37 Rotman (2010)[10]).  $\forall k$ -representation  $\sigma : G \rightarrow GL(V)$  equips  $V$  with structure of left  $kG$ -module, denote module by  $V^\sigma$ .

Conversely,  $\forall$  left  $kG$ -module  $V$  determines  $k$ -representation  $\sigma : G \rightarrow GL(V)$

*Proof.* Given  $\sigma : G \rightarrow GL(V)$ ,

$$\sigma_g =: \sigma(g) : V \rightarrow V$$

define

$$kG \times V \rightarrow V$$

$$\left( \sum_{g \in G} a_g g \right) v = \sum_{g \in G} a_g \sigma_g(v)$$

$$v, w \in V$$

Let  $r, r', 1 \in kG$

$$r = \sum_{g \in G} a_g g$$

$$r(v + w) = \left( \sum_{g \in G} a_g g \right) (v + w) = \left( \sum_{g \in G} a_g \sigma_g \right) (v + w) = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} a_g \sigma_g(w) = rv + rw$$

$$(r + r')v = \left( \sum_{g \in G} a_g g + b_g g \right) v = \sum_{g \in G} (a_g \sigma_g + b_g \sigma_g) v = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} b_g \sigma_g(v) = rv + r'v$$

$$(rr')v = \left( \sum_{g \in G} a_g g \sum_{h \in G} b_h h \right) v = \left( \sum_{z \in G} \sum_{gh=z} a_g b_h z \right) v = \sum_{z \in G} \sum_{gh \in z} a_g b_h \sigma_z(v) = \sum_{g \in G} \sum_{h \in G} a_g b_h \sigma_g \sigma_h(v)$$

since  $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g \sigma_h = \sigma_{gh}$  ( $\sigma$  homomorphism)

$$1v = \sigma(1)v = 1v = v$$

Conversely, assume  $V$  left  $kG$ -module.

If  $g \in G$ , then  $v \mapsto gv$  defines  $T_g : V \rightarrow V$ .  $T_g$  nonsingular since  $\exists T_g^{-1} = T_{g^{-1}}$

Define  $\sigma : G \rightarrow GL(V)$

$$\sigma : g \mapsto T_g$$

$\sigma$   $k$ -representation

$$\sigma(gh) = T_{gh} = T_g T_h = \sigma(g)\sigma(h)$$

$$\sigma(gh)(v) = T_{gh}v = ghv = T_g T_h v = \sigma(g)\sigma(h)v \quad \forall v \in V$$

□

**Proposition 7.** Let group  $G$ , let  $\sigma, \tau : G \rightarrow GL(V)$  be  $k$ -representations, field  $k$ .

If  $V^\sigma, V^\tau$  corresponding  $kG$ -modules in Prop. 6 (Prop. 8.37 in Rotman (2010) [10]), then

$V^\sigma \simeq V^\tau$  as  $kG$ -modules iff  $\exists$  nonsingular  $\varphi : V \rightarrow V$  s.t.

$$\varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$$

*Proof.* If  $\varphi : V^\tau \rightarrow V^\sigma$   $kG$ -isomorphism, then  $\varphi : V \rightarrow V$  isomorphism s.t.

$$\varphi(\sum a_g gv) = (\sum a_g g)\varphi(v) \quad \forall v \in V, \forall g \in G$$

in  $V^\tau$ ,  $kG \times V \rightarrow V$

$$gv = \tau(g)(v)$$

in  $V^\sigma$ ,  $kG \times V \rightarrow V$  scalar multiplication

$$gv = \sigma(g)(v)$$

$$\implies \forall g \in G, v \in V, \quad \varphi(\tau(g)(v)) = \sigma(g)(\varphi(v))$$

I think

$$\varphi(gv) = \varphi(\tau(g)(v)) = g\varphi(v) = \sigma(g)\varphi(v)$$

$$\implies \varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$$

Conversely, if  $\exists$  nonsingular  $\varphi : V \rightarrow V$  s.t.  $\varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$

$$\varphi\tau(g)v = \varphi(\tau(g)v) = \sigma(g)\varphi(v) \quad \forall g \in G, \forall v \in V$$

Consider scalar multiplication

$$kG \times V \rightarrow V$$

$$\sum_{g \in G} a_g g(v) = \sum_{g \in G} a_g \tau_g(v)$$

$$\varphi \left( \sum_{g \in G} a_g \tau_g(v) \right) = \varphi \left( \sum_{g \in G} a_g \tau(g)v \right) = \sum_{g \in G} a_g \sigma(g) \sigma(g) \varphi(v) = \left( \sum_{g \in G} a_g g \right) \varphi(v)$$

Admittedly, after this exposition from Rotman (2010) [10], I still didn't understand how  $kG$ -modules relate to representation theory and group rings. I turned to Baker (2011) [11], which we'll do right now. Note that I found a lot of links to online resources on representation theory from Khovanov's webpage <http://www.math.columbia.edu/~khovanov/resource/>.

Note,

**Definition 20.** *vector subspace  $W \subseteq V$  is called a*

*$G$ -submodule,  $G$ -subspace, EY : 20150922 “invariant” subspace?*

*if  $\forall g \in G$ , for representation  $\rho : G \rightarrow GL_k(V)$ ,  $\rho_g(w) \in W$ ,  $\forall w \in W$ ,  $\forall g \in G$  i.e. closed under “action of elements of  $G$ ” with  $\rho_g =: \rho(g) : V \rightarrow V$*

Given basis  $\mathbf{v} = \{v_1 \dots v_n\}$  for  $V$ ,  $\dim_k V = n$ ,  $\forall g \in G$ ,

$$\rho_g v_j = \rho(g)v_j = r_{kj}(g)v_k$$

for, indeed,

$$\rho_g x^j v_j = \rho(g)x^j v_j = x^j \rho(g)v_j = x^j r_{kj}(g)v_k = r_{kj} x^j v_k$$

so that

$$\begin{aligned} \rho : G &\rightarrow GL_k(V) \\ \rho(g) &= [r_{ij}(g)] \end{aligned}$$

Example 2.1 (Baker (2011) [11]): Let  $\rho : G \rightarrow GL_k(V)$  where  $\dim_k V = 1$

$$\forall v \in V, v \neq 0, \forall g \in G, \lambda_g \in k \text{ s.t. } g \cdot v = \rho_g(v) = \lambda_g v$$

$$\rho(hg)v = \rho_h \rho_g v = \lambda_{hg} v = \lambda_h \lambda_g v \implies \lambda_{hg} = \lambda_h \lambda_g$$

$\implies \exists$  homomorphism  $\Lambda : G \rightarrow k^\times$

$$\Lambda(g) = \lambda_g$$

From Sec. 2.2 “ $G$ -homomorphisms and irreducible representations” of Baker (2011) [11], suppose  $\begin{matrix} \rho : G \rightarrow GL_k(V) \\ \sigma : G \rightarrow GL_k(W) \end{matrix}$  are 2

representations

Many names for the same thing:  $G$ -equivalent,  $G$ -linear,  $G$ -homomorphism, EY : 20150922  $kG$ -isomorphic?

If  $\forall g \in G$ ,

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V \\ \tau_g \downarrow & & \downarrow \sigma_g \\ V & \xrightarrow{\varphi} & V \end{array} \iff V^\tau \xrightarrow{\varphi} V^\sigma$$

Indeed, define

$$\varphi : V^\tau \rightarrow V^\sigma$$

$$\varphi(v + w) = \varphi(v) + \varphi(w)$$

$$\varphi(rv) = \varphi\left(\sum_{g \in G} a_g g \cdot v\right) = \varphi\left(\sum_{g \in G} a_g \tau_g(v)\right) = \sum_{g \in G} a_g \varphi(\tau_g(v)) = \sum_{g \in G} a_g \sigma_g \cdot \varphi(v) = r\varphi(v)$$

EY : 20150922 So  $\varphi$  is a  $kG$ -isomorphism between left  $kG$  modules  $V^\tau$  and  $V^\sigma$  if it's bijective and is “linear” in “scalars”  $r \in kG$ , i.e.  $\varphi(rv) = r\varphi(v)$ .  $\square$

Define action of  $G$  on  $\text{Hom}_k(V, W)$  ( $\text{Hom}_k(V, W)$  is the vector space of  $k$ -linear transformations  $V \rightarrow W$ )

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ v & \xmapsto{f} & f(v) \end{array}$$

$$\forall f \in \text{Hom}_k(V, W), f : V \rightarrow W$$

$$f(v) \in W$$

Consider

$$G \times \text{Hom}_k(V, W) \rightarrow \text{Hom}_k(V, W)$$

$$(g \cdot f) \mapsto (\sigma_g f) \circ \rho_{g^{-1}} \text{ i.e. } (g \cdot f)(v) = \sigma_g f(\rho_{g^{-1}} v) \quad (f \in \text{Hom}_k(V, W))$$

Let  $g, h \in G$ ,

$$(gh \cdot f)(v) = g \cdot \sigma_h f(\rho_{h^{-1}} v) = \sigma_g \sigma_h f \rho_{h^{-1}} \rho_{g^{-1}}(v) = (\sigma_{gh} f \rho_{(gh)^{-1}})(v)$$

Thus,  $G \times \text{Hom}_k(V, W) \rightarrow \text{Hom}_k(V, W)$  is thus another  $G$ -representation of  $G$ .

$$(g \cdot f) \mapsto (\sigma_g f) \circ \rho_{g^{-1}}$$

For  $k$ -representation  $\rho$ , if the only  $G$ -subspaces of  $V$  are  $\{0\}$ ,  $V$ ,  $\rho$  **irreducible** or **simple**.

$$\rho_g(\{0\}) = \{0\}$$

$$\rho_g(V) = V$$

given subrepresentation  $W \subseteq V$ ,  $V/W$  admits linear action of  $G$ ,  $\bar{\rho}_W : G \rightarrow GL_k(V/W)$  quotient representation

$$\bar{\rho}_W(g)(v + W) = \rho(g)(v) + W$$

if  $v' - v \in W$

$$\rho(g)(v') + W = \rho(g)(v + (v' - v)) + W = (\rho(g)(v) + \rho(g)(v' - v)) + W = \rho(g)(v) + W$$

**Proposition 8** (2.7 Baker (2011)[11]). *if  $f : V \rightarrow W$   $G$ -homomorphism, then*

(a) *kerf is  $G$ -subspace of  $V$*

(b) *imf is  $G$ -subspace of  $W$*



$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \rho_g \downarrow & & \downarrow \sigma_g \\ V & \xrightarrow{f} & W \end{array}$$

*Proof.* Recall

(a) Let  $v \in \ker f$ . Then  $\forall g \in G$ ,

$$f(\rho_g v) = \sigma_g f(v) = 0$$

so  $\rho_g v \in \ker f$ ,  $\forall g \in G$ . So  $\ker f$  is  $G$ -subspace of  $V$

(b) Let  $w \in \operatorname{im} f$ . So  $w = f(u)$  for some  $u \in V$

$$\sigma_g w = \sigma_g f(u) = f(\rho_g u) \in \operatorname{im} f$$

So  $\operatorname{im} f$  is  $G$ -subspace of  $W$

**Theorem 14** (Schur's Lemma). *Let  $\rho : G \rightarrow GL_{\mathbb{C}}(V)$  be irreducible representations of  $G$  over field  $k = \mathbb{C}$ ; let  $f : V \rightarrow W$  be  $G$ -linear map.*

- (a) *if  $f \neq 0$ ,  $f$  isomorphism. True  $\forall k$  field, not just  $\mathbb{C}$*
- (b) *if  $V = W$ ,  $\rho = \sigma$ , then for some  $\lambda \in \mathbb{C}$ ,  $f$  given by  $f(v) = \lambda v$  ( $v \in V$ ) (true for algebraically closed fields)*

*Proof.* (a) By Prop. 8,  $\ker f \subseteq V$ ,  $\operatorname{im} f \subseteq W$  are  $G$ -subspaces.  
For  $\rho$ , only  $G$ -subspaces are 0 or  $V$ , so if  $\ker f = V$ ,  $f = 0$ . If  $\ker f = 0$ ,  $f$  injective.  
For  $\sigma$ , only  $G$ -subspaces are 0 or  $V$ , so  $\operatorname{im} f = 0$ ,  $f = 0$ . If  $\operatorname{im} f = V$ ,  $f$  surjective.  
 $\implies f$  isomorphism.  
(b) Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $f$ ,  $f(v_0) = \lambda v_0$  eigenvector,  $v_0 \neq 0$ .  
Let linear  $f_{\lambda} : V \rightarrow V$  s.t.

$$f_{\lambda}(v) = f(v) - \lambda v \quad (v \in V)$$

$$\forall g \in G$$

$$\rho_g f_{\lambda}(v) = \rho_g f(v) - \rho_g \lambda v = f(\rho_g v) - \lambda \rho_g v = f_{\lambda}(\rho_g v)$$

So  $f_{\lambda}$  is  $G$ -linear, for

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ \rho_g \downarrow & & \downarrow \rho_g \\ V & \xrightarrow{f} & V \end{array}$$

Since  $f_{\lambda}(v_0) = 0$ , by Prop. 8,  $\ker f_{\lambda} = V$ , (for  $\ker f_{\lambda} \neq 0$  and so  $\ker f_{\lambda} = V$ )  
By rank-nullity theorem,  $\dim V = \dim \ker f_{\lambda} + \dim \operatorname{im} f_{\lambda}$ .  
So  $\operatorname{im} f_{\lambda} = 0$ , and so  $f_{\lambda}(v) = 0$  ( $\forall v \in V$ )  $\implies f(v) = \lambda v$

Schur's lemma, at least the first part, implies that the left  $kG$ -modules associated with representations  $\rho, \sigma$  are  $kG$ -isomorphic, i.e.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \rho_g \downarrow & & \downarrow \sigma_g \\ V & \xrightarrow{f} & W \end{array} \iff V^{\rho} \stackrel{f}{\simeq} V^{\sigma}$$

with  $f$  being an isomorphism between  $V^{\rho}$  and  $V^{\sigma}$  s.t.

$$f(v + w) = f(v) + f(w) \quad \forall v, w \in (V^{\sigma}, +)$$

$$f(rv) = rf(v) \quad \forall r = \sum_{g \in G} a_g g \in kG$$

Kosmann-Schwarzbach's **Groups and Symmetries**[12] is a very lucid text that's mathematically rigorous enough and practical for physicists. It's really good and very clear. Let's follow its development for  $SU(2)$ ,  $SO(3)$ ,  $SL(2, \mathbb{C})$  and corresponding Lie algebras  $\mathfrak{su}(2)$ ,  $\mathfrak{so}(3)$ ,  $\mathfrak{sl}(2, \mathbb{C})$ .  
□

From Chapter 2 "Representations of Finite Groups" of Kosmann-Schwarzbach (2010) [12]

**Definition 21** (2.1 Kosmann-Schwarzbach (2010)[12]). *On  $L^2(G)$ , scalar product defined by*

$$\langle f_1 | f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

$f_1, f_2 \in \mathcal{F}(G) \equiv \mathbb{C}[G]$  vector space of functions on  $G$  taking values on  $\mathbb{C}$

**Definition 22** (2.3 Kosmann-Schwarzbach (2010)[12]). *Let  $(E, \rho)$  be representation of  $G$*

$$\begin{aligned} \text{character of } \rho &\equiv \chi_{\rho} : G \rightarrow \mathbb{C} \\ \chi_{\rho}(g) &= \operatorname{tr}(\rho(g)) = \sum_{i=1}^n (\rho(g))_{ii} \end{aligned}$$

*Note: equivalent representations have same character  
each conjugacy class of  $G$ , function  $\chi_{\rho}$  is constant*

Looking at Def. 21

$$\langle \chi_{\rho_1} | \chi_{\rho_2} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2}(g)$$

since  $\overline{\chi_{\rho_1}(g)} = \chi_{\rho_1}(g^{-1})$  by unitarity of representation with respect to scalar product  $\langle, \rangle$

**Proposition 9** (2.7 Kosmann-Schwarzbach (2010)[12]). *Let  $(E_1, \rho_1)$  be representations of  $G$ , let linear  $u : E_1 \rightarrow E_2$ .  
( $E_2, \rho_2$ )*

*Then  $\exists$  linear  $T_u$  s.t.*

$$\square \quad (25)$$

$$\begin{aligned} T_u &: E_1 \rightarrow E_2 \\ T_u &= \frac{1}{|G|} \sum_{g \in G} \rho_2(g) u \rho_1(g)^{-1} \end{aligned}$$

*so that  $\rho_2(g) T_u = T_u \rho_1(g) \quad \forall g \in G$*

*Proof.*

$$\rho_2(g)T_u = \frac{1}{|G|} \sum_{h \in G} \rho_2(gh)u\rho_1(h^{-1}) = \frac{1}{|G|} \sum_{k \in G} \rho_2(k)u\rho_1(k^{-1}g) = T_u\rho_1(g)$$

□

Thus, diagrammatically, we have that

$$\begin{array}{ccc} E_1 & \xrightarrow{u} & E_2 \\ \Downarrow \rho_1(g) & & \Downarrow \rho_2(g) \\ E_1 & \xrightarrow{T_u} & E_2 \end{array} \implies \begin{array}{ccc} E_1 & \xrightarrow{T_u} & E_2 \\ \downarrow \rho_1(g) & & \downarrow \rho_2(g) \\ E_1 & \xrightarrow{T_u} & E_2 \end{array}$$

From Definition 1.12 of Kosmann-Schwarzbach [12], “representations  $\rho_1$  and  $\rho_2$  are called **equivalent** if there is a bijective intertwining operator for  $\rho_1$  and  $\rho_2$ .” So I will interpret this as if an intertwining operator is not bijective, then the representations  $\rho_1, \rho_2$  are not equivalent.

**Proposition 10** (2.8 Kosmann-Schwarzbach (2010)[12]). *Let  $(E_1, \rho_1)$  be irreducible representations of  $G$ , let linear  $u : E_1 \rightarrow E_2$ ,  $(E_2, \rho_2)$*

*define  $T_u$  by  $T_u = \frac{1}{|G|} \sum_{g \in G} \rho_2(g)u\rho_1(g)^{-1}$  by Eq. 25.*

- (i) *If  $\rho_1, \rho_2$  inequivalent, then  $T_u = 0$*
- (ii) *If  $E_1 = E_2 = E$  and  $\rho_1 = \rho_2 = \rho$ , then*

$$T_u = \frac{\text{tr}(u)}{\dim E} 1_E$$

*Proof.* (i) if  $\rho_1, \rho_2$  are inequivalent, by definition,  $T_u$  is not isomorphic. Then by Schur’s lemma (first part),  $T_u = 0$   
(ii) By Schur’s lemma,  $T_u(v) = \lambda v \quad \forall v \in E = E_1 = E_2$ . So  $T_u = \lambda 1_E$ .  $\text{tr} T_u = \lambda \dim E$  or  $\lambda = \frac{\text{tr} T_u}{\dim E}$ . Thus,  $T_u = \frac{\text{tr} T_u}{\dim E} 1_E$  □

Let  $(e_1 \dots e_n)$  basis of  $E$

$(f_1 \dots f_p)$  basis of  $F$

$$\forall u \in \mathcal{L}(E, F), \quad \begin{array}{l} u : E \rightarrow F \\ u(x) = u(x^j e_j) = x^j u(e_j) = x^j u^i_j f_i \quad \text{for } x = x^j e_j \in E \\ u = u^i_j e^j \otimes f_i \quad \quad \quad y = y^i f_i \in F \end{array}$$

For

$$\begin{aligned} T : E^* \otimes F &\rightarrow \mathcal{L}(E, F) \\ T(\xi \otimes y) &= u^i_j e^j \otimes f_i \text{ i.e. set } T(\xi \otimes y) \text{ to this } u \\ T(\xi \otimes y) &= T(\xi_l e^l \otimes y^k f_k) = \xi_l y^k T(e^l \otimes f_k) = (\xi_l y^k T_{kj}^{li}) e^j \otimes f_i \implies \xi_l y^k T_{kj}^{li} = u^i_j \end{aligned}$$

*Exercises.* Exercises of Ch. 2 Representations of Finite Groups [12]

**Exercise 2.6.** [12] *The dual representation.*

Let  $(E, \pi)$  representation of group  $G$ .  
 $\forall g \in G, \xi \in E^*, x \in E$ , set  $\langle \pi^*(g)(\xi), x \rangle = \langle \xi, \pi(g^{-1})(x) \rangle$

(a) *dual* (or *contragredient*) of  $\pi$ ,  $\pi^* : G \rightarrow \text{End}(E^*)$ ,  $\pi^*$  is a representation, since

$$\begin{aligned} \langle \pi^*(gh)(\xi), x \rangle &= \langle \xi, \pi((gh)^{-1})(x) \rangle = \langle \xi, \pi(h^{-1}g^{-1})(x) \rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x) \rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x)) \rangle = \\ &= \langle \pi^*(h)(\xi), \pi(g^{-1})(x) \rangle = \langle \pi^*(g)\pi^*(h)(\xi), x \rangle \end{aligned}$$

since this is true,  $\forall x \in E, \forall \xi \in E^*, \pi^*(gh) = \pi^*(g)\pi^*(h)$ .

dual  $\pi^*$  of  $\pi$  is a representation.

(b) Consider  $G \times \mathcal{L}(E, F) \rightarrow \mathcal{L}(E, F)$ .

$$g \cdot u = \rho(g) \circ u \circ \pi(g^{-1})$$

Define

$$\begin{aligned} \sigma : G &\rightarrow \text{End}(\mathcal{L}(E, F)) \\ \sigma(g) : \mathcal{L}(E, F) &\rightarrow \mathcal{L}(E, F) \\ \sigma(g)(u) &= \rho(g) \circ u \circ \pi(g^{-1}) \end{aligned}$$

Let  $(e_1 \dots e_n)$  be a basis of  $E$ . Let  $\xi = \xi_i e^i \in E^*, x = x^j e_j \in E$ .

Consider the isomorphism  $T : E^* \otimes F \rightarrow \mathcal{L}(E, F)$  defined as<sup>2</sup>

$$T : E^* \otimes F \rightarrow \mathcal{L}(E, F) = \text{Hom}(E, F)$$

$$\xi \otimes y \mapsto (x \mapsto \xi(x)y)$$

Choose bases  $(e_1 \dots e_n)$  of  $E$   
 $(e^1 \dots e^n)$  of  $E^*$ . Then  
 $(f_1 \dots f_p)$  of  $F$

$$T(e^j \otimes f_i)(x) = T(e^j \otimes f_i)(x^k e_k) = \delta_k^j x^k f_i = x^j f_i$$

$$T(e^j \otimes f_i)(e_k) = \delta_k^j f_i$$

Consider

$$\begin{aligned} u &\in \mathcal{L}(E, F) \\ u : E &\rightarrow F \\ u(x) &= u(x^j e_j) = x^j u(e_j) = x^j u^i_j f_i \\ u(e_j) &= u^i_j f_i \text{ i.e. } u : e_j \rightarrow u^i_j f_i \end{aligned}$$

Then  $\forall u \in \mathcal{L}(E, F)$ ,

$$T(u^i_j e^j \otimes f_i)(e_k) = u^i_j \delta_k^j f_i = u^i_k f_i = u(e_k) \implies u = T(u^i_j e^j \otimes f_i)$$

so  $T$  is surjective.

With  $T(\xi \otimes y) = T(\xi' \otimes y')$ ,

$$T(\xi \otimes y)(x) = T(\xi' \otimes y')(x)$$

$$\xi(x)y = \xi'(x)y' \implies \xi(x)y - \xi'(x)y' = 0$$

which implies that  $\xi \otimes y = \xi' \otimes y'$ . So  $T$  is injective. Or, one could consider that  $T^{-1} : \mathcal{L}(E, F) \rightarrow E^* \otimes F$ ,  
 $T^{-1} : u \mapsto u^i_j e^j \otimes f_i$ , which is the inverse of  $T$ .

<sup>2</sup>Mathematics stackexchage Isomorphism between Hom and tensor product [duplicate] <http://math.stackexchange.com/questions/428185/isomorphism-between-hom-and-tensor-product>  
<http://math.stackexchange.com/questions/57189/understanding-isomorphic-equivalences-of-tensor-product>

**Remark 1.**

$$E^* \otimes F \xrightarrow{T} \mathcal{L}(E, F) = \text{Hom}(E, F)$$

$$(\xi, y) \mapsto (x \mapsto \xi(x)y)$$

and so  $(e^j \otimes f_i) \mapsto (x \mapsto e^j(x)f_i = x^j f_i)$   
 So  $E^* \otimes F$  is isomorphic to  $\mathcal{L}(E, F) = \text{Hom}(E, F)$

For representation  $\pi$ ,

$$\begin{aligned} \pi : G &\rightarrow \text{End}(E) \\ \pi(g) : E &\rightarrow E \\ \pi(g)(x) &= \pi(g)(x^j e_j) = x^j \pi(g)(e_j) = x^j \pi(g)^i_j e_i = (\pi(g)^i_j x^j e_i \end{aligned}$$

Consider this matrix formulation:

$$\begin{aligned} \pi^*(g)(\xi) &= \pi^*(g)(\xi_i e^i) = \xi_i \pi^*(g)(e^i) = \xi_i (\pi^*(g))^i_j e^j \\ \implies \langle \pi^*(g)(\xi), x \rangle &= \xi_i (\pi^*(g))^i_j x^j \end{aligned}$$

and

$$\langle \xi, \pi(g^{-1})(x) \rangle = \xi_i \pi(g^{-1})^i_j x^j$$

so that

$$\langle \pi^*(g)(\xi), x \rangle = \langle \xi, \pi(g^{-1})(x) \rangle \implies \pi(g^{-1})^i_j = (\pi^*(g))^i_j$$

Thus, given a choice of basis for  $E$ , the dual of  $\pi$ ,  $\pi^*(g)^i_j$ , and  $\pi(g^{-1})^i_j$  are formally equal.

So for a choice of basis of  $E$  and of  $F$ ,

$$(\pi^* \otimes \rho)(g)(\xi, y) = (\pi^*(g) \otimes \rho(g))(\xi, y) = \pi^*(g)\xi \otimes \rho(g)y = \xi_l \pi(g^{-1})^l_j e^j \otimes \rho(g)^i_k y^k f_i = \rho(g)^i_k y^k \xi_l \pi(g^{-1})^l_j e^j \otimes f_i$$

Applying  $T$ ,

$$T(\pi^* \otimes \rho)(g)(\xi, \rho) = \rho(g)^i_k y^k \xi_l \pi(g^{-1})^l_j = \rho(g)T(\xi, y)\pi(g^{-1})$$

$$\begin{array}{ccc} E^* \otimes F & \xrightarrow{T} & \mathcal{L}(E, F) \\ \downarrow (\pi^* \otimes \rho)(g) & & \downarrow \sigma(g) \\ E^* \otimes F & \xrightarrow{T} & \mathcal{L}(E, F) \end{array} \quad \begin{array}{ccc} (\xi, y) & \xrightarrow{T} & (x \mapsto \xi(x)y) = y^i \xi_j \\ \downarrow (\pi^* \otimes \rho)(g) & & \downarrow \sigma(g) \\ \pi^*(g)(\xi) \otimes \rho(g)y & \xrightarrow{T} & \rho(g)y^i \xi_j \pi(g^{-1}) = \rho(g)T(\xi, y)\pi(g^{-1}) \end{array}$$

Thus

Thus, representation  $\sigma(g)$  is equivalent to representation  $(\pi^* \otimes \rho)$ , a tensor product of representations.

**Exercise 2.15.** Representation of  $GL(2, \mathbb{C})$  on the polynomials of degree 2

Let group  $G$ , let representation  $\rho$  of  $G$  on  $V = \mathbb{C}^n$ , i.e.  $\rho : G \rightarrow \text{End}(V)$

Let  $P^{(k)}(V)$  vector space of complex polynomials on  $V$  that are homogeneous of degree  $k$ .

For  $f \in P^{(k)}(V)$ , the general form is

$$f = \sum_{\substack{i_1+i_2+\dots+i_n=k \\ 0 \leq i_j \leq k}} a_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

Given

$$\binom{n+k}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \dots + \binom{n+k-1}{k-1} = \sum_{i=0}^n \binom{k-1+i}{k-1}$$

$\binom{k+n-1}{n-1}$  is number of monomials of degree  $k$ .

<sup>3</sup>Polynomials. Math 4800/6080 Project Course <http://www.math.utah.edu/~bertram/4800/PolyIntroduction.pdf>

So  $\dim P^{(k)}(V) = \binom{k+n-1}{n-1}$ . This is a very lucid and elementary exposition on the basics of polynomials which I found was useful for the basic facts I forgot<sup>3</sup>.

So we have the graded algebra

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V)$$

$$\rho^{(k)} : G \rightarrow \text{End}(P^{(k)}(V))$$

$$\rho^{(k)}(g) : P^{(k)}(V) \rightarrow P^{(k)}(V)$$

$$\rho^{(k)}(g)(f) = f \circ \rho(g^{-1})$$

This is a representation of  $G$  since

(a)

$$\begin{aligned} \rho^{(k)}(gh)(f) &= f \circ \rho((gh)^{-1}) = f \circ \rho(h^{-1}g^{-1}) = f \circ \rho(h^{-1}\rho(g^{-1})) \implies \rho^{(k)}(gh) = \rho^{(k)}(g)\rho^{(k)}(h) \\ \rho^{(k)}(g)\rho^{(k)}(h)(f) &= \rho^{(k)}(g)(f \circ \rho(h^{-1})) = f \circ \rho(h^{-1}) \circ \rho(g^{-1}) \end{aligned}$$

(b) Choose basis  $(e_1 \dots e_n)$  of  $V$ ,  $x = x^j e_j \in V$ ,  $\rho : G \rightarrow \text{End}(V)$ , and so  $\rho(g)(x) = \rho(g)(x^j e_j) = x^j \rho(g)(e_j) = x^j (\rho(g))^i_j e_i$ .

With  $\xi(e_i) = \xi_i \implies \langle \xi, \rho(g^{-1})x \rangle = \xi_i x^j (\rho(g^{-1}))^i_j$

$\forall \xi \in V^*$ ,  $\xi = \xi_i e^i$ ,

$$\rho^*(g)(\xi) = \rho^*(g)(\xi_i e^i) = \xi_i \rho^*(g)^i_j e^j$$

$$\implies \langle \rho^*(g)(\xi), x \rangle = \xi_i x^j (\rho^*(g))^i_j \implies (\rho^*(g))^i_j = (\rho(g^{-1}))^i_j$$

So  $\forall f \in P^{(1)}(V)$ ,  $x \in V$ ,  $\rho(g^{-1})x = x^j (\rho(g^{-1}))^i_j e_i$ . So  $f \circ \rho(g^{-1})(x) = \sum_{i=1}^n a_i (\rho(g^{-1}))^i_j x^j = \sum_{i=1}^n a_i (\rho^*(g))^i_j x^j$   
 $\implies \rho^{(1)}(g)(f) = f \circ \rho^*(g)$

(c) Suppose  $G = GL(2, \mathbb{C})$ ,  $V = \mathbb{C}^2$ ,  $\rho$  fundamental representation  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $g^{-1} = \frac{1}{\det g} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$  for  $\det g = ad - bc$ .

Let  $k = 2$ ,  $\dim P^{(2)}(\mathbb{C}^2) = \binom{2+2-1}{2-1} = \binom{3}{1} = 3$

$\forall f \in P^{(2)}(\mathbb{C}^2)$ ,  $f(x, y) = Ax^2 + 2Bxy + Cy^2$

Let

$$P^{(2)}(\mathbb{C}^2) \rightarrow \mathbb{C}^3$$

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 \mapsto \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$$

Call this transformation  $T$ ,  $T : P^{(2)}(\mathbb{C}^2) \rightarrow \mathbb{C}^3$ .

$\forall \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$ ,  $f(x, y) = Ax^2 + 2Bxy + Cy^2$  and  $Tf(x, y) = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ .  $T$  surjective.

Suppose  $Tf(x, y) = Tf'(x, y)$ ,

$$\implies Ax^2 + 2Bxy + Cy^2 = A'x^2 + 2B'xy + C'y^2$$

$$\implies (A - A')x^2 + 2(B - B')xy + (C - C')y^2 = 0$$

Then since the monomials form a basis, and its basis elements are independent (by definition), then  $A = A'$ ,  $B = B'$ ,  $C = C'$ .  $T$  injective. So  $T$  is bijective, an isomorphism.

(This is all in `groups.sage`)

```

sage: P2CC.<x,y> = PolynomialRing(CC,2) # this declares a PolynomialRing of field of complex numbers,
# of order 2 (i.e. only 2 variables for a polynomial, such as x, y)
sage: A = var('A')
sage: assume(A, 'complex')
sage: B = var('B')
sage: assume(B, 'complex')
sage: C = var('C')
sage: assume(C, 'complex')
sage: f(x,y) = A*x**2 + 2*B*x*y + C*y**2

```

```

sage: a = var('a')
sage: assume(a, 'complex')
sage: b = var('b')
sage: assume(b, 'complex')
sage: c = var('c')
sage: assume(c, 'complex')
sage: d = var('d')
sage: assume(d, 'complex')
sage: g = Matrix([[a,b],[c,d]] )
sage: X = Matrix([[x],[y]])
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand()
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(x^2).full_simplify()
(C*c^2 - 2*B*c*d + A*d^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(x*y).full_simplify()
-2*(C*a*c + A*b*d - (b*c + a*d)*B)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(y^2).full_simplify()
(C*a^2 - 2*B*a*b + A*b^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)

```

So

$$\rho^{(2)}(g)(f)(x,y) = f \circ \rho(g^{-1})(x,y) = \frac{Cc^2 - 2Bcd + Ad^2}{(ad - bc)^2}x^2 + -2\frac{(Cac + Abd - (bc + ad)B)}{(ad - bc)^2}xy + \frac{Ca^2 - 2Bab + Ab^2}{(ad - bc)^2}y^2$$

So define  $\tilde{\rho}: G \rightarrow \text{End}(\mathbb{C}^3)$ .  $\tilde{\rho}$  is a representation, for

$$\forall v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3, \quad \tilde{\rho}(gh)(v) = T \circ f \circ \rho((gh)^{-1}) = T \circ f \circ \rho(h^{-1}g^{-1}) = T \circ f \circ \rho(h^{-1})\rho(g^{-1})$$

$$\text{Now } \tilde{\rho}(h)(v) = T \circ f \circ \rho(h^{-1})$$

$$\implies \tilde{\rho}(g)\tilde{\rho}(h)(v) = T \circ (f \circ \rho(h^{-1})) \circ \rho(g^{-1}) = T \circ f \circ \rho(h^{-1})\rho(g^{-1}) \text{ and so}$$

$$\tilde{\rho}(gh) = \tilde{\rho}(g)\tilde{\rho}(h)$$

And so

$$\tilde{\rho}^*(g)(v) = Tf\rho(g^{-1})$$

and consider this commutation diagram, that (helped me at least and) clarifies the relationships:

$$\begin{array}{ccc} P^{(2)}(\mathbb{C}^2) & \xrightarrow{T} & \mathbb{C}^3 \\ \rho^{(2)}(g) \downarrow & & \downarrow \tilde{\rho}(g) \\ P^{(2)}(\mathbb{C}^2) & \xrightarrow{T} & \mathbb{C}^3 \end{array} \quad \begin{array}{ccc} & \xrightarrow{T} & \begin{pmatrix} A \\ B \\ C \end{pmatrix} \\ f \downarrow & & \downarrow \tilde{\rho}(g) \\ \rho^{(2)}(g) \downarrow & & \downarrow \tilde{\rho}(g) \\ f \circ \rho(g^{-1}) & \xrightarrow{T} & \begin{pmatrix} D \\ E \\ F \end{pmatrix} \end{array}$$

with

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \frac{Cc^2 - 2Bcd + Ad^2}{(ad - bc)^2} \\ -2\frac{(Cac + Abd - (bc + ad)B)}{(ad - bc)^2} \\ \frac{Ca^2 - 2Bab + Ab^2}{(ad - bc)^2} \end{pmatrix}$$

Now define the dual  $\tilde{\rho}^*$  as such:

$$\tilde{\rho}^*(g): (\mathbb{C}^3)^* \rightarrow (\mathbb{C}^3)^*$$

$$\tilde{\rho}^*(g) = \tilde{\rho}(g^{-1})$$

$$\forall \xi \in (\mathbb{C}^3)^*$$

$$\tilde{\rho}^*(g)\xi = \xi_i(\tilde{\rho}^*(g))^i_j e^j = \xi_i(\tilde{\rho}(g^{-1}))^i_j e^j$$

$$\text{So for } v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3, f = T^{-1}v = Ax^2 + 2Bxy + Cy^2 \in P^2(\mathbb{C}^2),$$

$$\tilde{\rho}(g^{-1})(v) = T \circ (f\rho(g)) = \begin{bmatrix} Aa^2 + 2Bac + Cc^2 \\ Aab + Bbc + Bad + Ccd \\ Ab^2 + 2Bbd + Cd^2 \end{bmatrix}$$

which was found using Sage Math:

```

sage: f((g*X)[0,0],(g*X)[1,0])
(a*x + b*y)^2*A + 2*(a*x + b*y)*(c*x + d*y)*B + (c*x + d*y)^2*C
sage: f((g*X)[0,0],(g*X)[1,0]).expand()
A*a^2*x^2 + 2*B*a*c*x^2 + C*c^2*x^2 + 2*A*a*b*x*y + 2*B*b*c*x*y + 2*B*a*d*x*y + 2*C*c*d*x*y + A*b^2*y^2 + 2*B*b*d*y^2 + C*d^2*y^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x^2)
A*a^2 + 2*B*a*c + C*c^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x*y)
2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(y^2)
A*b^2 + 2*B*b*d + C*d^2

```

or

```

sage: T( f((g*X)[0,0],(g*X)[1,0]).expand() )
[A*a^2 + 2*B*a*c + C*c^2,
2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d,
A*b^2 + 2*B*b*d + C*d^2]

```

So then

$$\tilde{\rho}(g^{-1}) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad + bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

So then

$$\tilde{\rho}^*(g) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad + bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

and operate on row vectors  $\xi \in (\mathbb{C}^3)^*$  with  $\tilde{\rho}^*(g)$  from the row vector's right.

More: Let  $G = SU(2)$ . Then  $U = e^{i\phi} \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$

$$\tilde{\rho}: SU(2) \rightarrow \text{End}(\mathbb{C}^3)$$

$$\tilde{\rho}(U): \mathbb{C}^3 \rightarrow \mathbb{C}^3$$

$$\tilde{\rho}(U)(v) = e^{-2i\varphi} \begin{bmatrix} A\bar{a}^2 + 2B\bar{a}\bar{b} + C\bar{b}^2 \\ -A\bar{a}\bar{b} + B + C\bar{a}\bar{b} \\ Ab^2 - 2Bab + Ca^2 \end{bmatrix}$$

$$\implies \tilde{\rho}(U) = e^{-2i\varphi} \begin{bmatrix} -\bar{a}^2 & 2\bar{a}\bar{b} & \bar{b}^2 \\ -\bar{a}\bar{b} & 1 & \bar{a}\bar{b} \\ b^2 & -2ab & a^2 \end{bmatrix}$$

cf. Ch. 5 Lie Groups of Jeffrey Lee (2009) [2]

**Definition 23** (Lie Group). ***Lie Group**  $G :=$  smooth manifold  $G$  is a **Lie Group** if  $G$  is a group (abstract group), s.t.*

*multiplication map  $\mu: G \times G \rightarrow G$*

$$\mu(g, h) = gh$$

*inverse map  $\text{inv}: G \rightarrow G$*

$$\text{inv}(g) = g^{-1}$$

*are  $C^\infty$  maps.*

*If group is abelian, use additive notation  $g + h$  for group operation.*

**Definition 24** ( $GL(n, \mathbb{R})$ ).  $GL(n, \mathbb{R}) :=$  group of all invertible real  $n \times n$  matrices.

*global chart on  $GL(n, \mathbb{R}) = \{x_j^i\}$ ,  $n^2$  functions  $x_j^i$ , where if  $A \in GL(n, \mathbb{R})$ , then  $x_j^i(A)$  is  $ij$ th entry of  $A$ .*

*Claim:  $GL(n, \mathbb{R})$  is a Lie group.*

*Proof.* multiplication is clearly smooth:  $(AB)_{ij} = A_{ik}B_{kj}$ ,

$$\frac{\partial}{\partial x_m^l}(x_k^i(A)x_j^k(B)) = \delta_l^i \delta_k^m x_j^k(B) + x_k^i(A) \delta_l^k \delta_j^m$$

inversion map; appeal to formula for  $A^{-1}$ ,  $A^{-1} = \text{adj}(A)/\det(A)$ ,  $\text{adj}(A) \equiv$  adjoint matrix (whose entries are cofactors).

$\implies A^{-1}$  depends smoothly on entries of  $A$ .

Similarly,  $GL(n, \mathbb{C})$ , group of invertible  $n \times n$  complex matrices, is a Lie group.

**Exercise 5.5.** Let subgroup  $H$  of  $G$ , consider cosets  $gH$ ,  $g \in G$ .

Recall  $G$  is disjoint union of cosets of  $H$ .

*Claim:* if  $H$  open, so are all its cosets. And  $H$  closed.

*Proof.* cf. [stackexchange: Open subgroups of a topological group are closed](#)

$gH = \{gh|h \in H\}$  is an open neighborhood of  $g$  (since  $1 \in H$ , and mapping  $h \mapsto gh$  sends open sets to open sets, since its inverse,  $gh \mapsto h$ , is  $C^\infty$  (so continuous)).

$$gH \rightarrow H$$

$$H \rightarrow gH$$

$$gh \xrightarrow{g^{-1}} h = \mu(g^{-1}, gh) \quad h \xrightarrow{g} gh = \mu(g, h)$$

Then  $\forall$  coset  $gH$ ,  $gH$  is open.

Suppose  $g' \in H^c \equiv G - H \equiv G \setminus H$ .

Consider  $h \in H$ , if  $g'h \in H$ , then  $g' = (g'h)h^{-1} \in H$  (recall  $h^{-1} \in H$ , and  $H$  is a subgroup).

Contradiction.

$\implies \forall g' \in H^c$ ,  $\exists$  open neighborhood  $g'H \subset H^c$ , so  $H^c$  open (by definition). Then  $H$  closed.

cf. Thm. 5.6 in Jeffrey Lee (2009) [2].

**Theorem 15.** *If  $G$  connected Lie group,  $U$  neighborhood of identity element  $e$ , then  $U$  generates the group, i.e.  $\forall g \in G$ ,  $g$  is a product of elements of  $U$ .*

*Proof.* Note  $V = \text{inv}(U) \cap U$  is an open neighborhood of  $e$ . Note  $\text{inv}(V) = V$ .  $\text{inv}(V) \equiv V^{-1} = \{V^{-1}|v \in V\}$ . We say that  $V$  is *symmetric*.

*Claim:*  $V$  generates  $G$ .

$\forall$  open  $W_1$ , open  $W_2 \subset G$ ,

$W_1 W_2 = \{w_1 w_2 | w_1 \in W_1, w_2 \in W_2\}$  is an open set being a union of open sets  $\bigcup_{g \in W_1} g W_2$ .

Thus, inductively defined sets

$$V^n = V V^{n-1}, \quad n = 1, 2, 3, \dots$$

are open.

$$e \in V \subset V^2 \subset \dots V^n \subset \dots$$

It's easy to check that each  $V^n$  is symmetric.

$$\text{inv}(V) = V$$

$$\text{inv}(V^2) = \text{inv}\left(\bigcup_{v \in V} vV\right) = V \text{inv}(V) = V = V^2$$

$$\text{inv}(V^{n+1}) = \text{inv}\left(\bigcup_{v \in V} vV^n\right) = V \text{inv}(V^n) = V V^n = V^{n+1}$$

so  $V^\infty := \bigcup_{n=1}^\infty V^n$  is symmetric.

$V^\infty$  closed under inversion, also multiplication. Thus  $V^\infty$  is an open subgroup.

From Exercise 5.5, Jeffrey Lee (2009) [2], i.e. Exercise [9](#),  $V^\infty$  also closed, since  $G$  is connected,  $V^\infty = G$ . (a topological space  $X$  is **connected** iff the only open and closed (clopen) sets are  $\emptyset$  and  $X$ ).

□

**Definition 25.** *Identity component of  $G, G_0$ .*

$G_0 :=$  connected component of Lie group  $G$  that contains identity;

$G_0$  is a Lie group, and is generated by any open neighborhood of the identity.

**Definition 26.** *For Lie group  $G$ , fixed element  $g \in G$ ,*

□ *left translation (by  $g$ )  $L_g: G \rightarrow G$ ,  $L_g x = gx$ ,  $\forall x \in G$*

*right translation (by  $g$ )  $R_g: G \rightarrow G$ ,  $R_g x = xg$ ,  $\forall x \in G$*

$L_g, R_g$  are diffeomorphisms with  $L_g^{-1} = L_{g^{-1}}$ ,  $R_g^{-1} = R_{g^{-1}}$ .

**Definition 27** (Product Lie group). *If  $G, H$  are Lie groups, then product manifold  $G \times H$  is a Lie group, where multiplication*

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

*Lie group  $G \times H$  is called **product Lie group***

e.g. product group  $S^1 \times S^1 \equiv$  2-torus group.

Generally, higher torus groups  $T^n = S^1 \times \dots \times S^1$  ( $n$  factors).

**Definition 28** (Lie subgroup of  $G, H$ ). *Let  $H$  be an abstract subgroup of Lie group  $G$ .*

*If  $H$  is a Lie group s.t. inclusion map  $i: H \rightarrow G \equiv H \hookrightarrow G$  is an immersion, then  $H$  is a*

**Lie subgroup** of  $G$ .

Recall  $i: H \rightarrow G$  immersion iff  $Di$  injective, i.e. iff  $\text{rank } Di = \dim H$

cf. Prop. 5.9 in Jeffrey Lee (2009) [2].

**Proposition 11.** *If  $H$  abstract subgroup of Lie group  $G$ , that's also a regular submanifold  $\equiv$  embedded submanifold, then  $H$  closed Lie subgroup.*

□



Recall that  
 $embedded\ submanifold \equiv regular\ submanifold$

Each name is used frequently and we shouldn't be biased against one or the other; we'll have to refer to both, to emphasize they're *exactly the same*.

embedded submanifold  $\equiv$  regular submanifold is an immersed submanifold s.t. inclusion map  $i$  is a topological embedding, i.e. embedded submanifold  $\equiv$  regular submanifold  $S \subset M$ , immersed submanifold  $S$  if  $i : S \rightarrow M \equiv S \hookrightarrow M$  is an immersion, i.e.  $Di$  injective, i.e.  $\text{rank} Di \equiv \dim S$ . topological embedding  $:=$  homeomorphism onto its image, i.e. injective cont. map  $f : X \rightarrow Y$ ,  $X, Y$  topological spaces, is a **topological embedding** if  $f$  is a homeomorphism between  $X$  and  $f(X)$ .  $f$  homeomorphism is a bijection, continuous, and  $f^{-1}$  continuous.

e.g.  $\forall$  embedding  $f : M \rightarrow N$ ,  $f(M) \subset N$  naturally has the structure of an embedding submanifold  $\equiv$  regular submanifold. *Useful, intrinsic definition* of **embedded submanifold**  $\equiv$  regular submanifold. Let manifold  $M$ ,  $\dim M = n$ , let  $k \in \mathbb{Z}^+$ , s.t.  $0 \leq k \leq n$ . A  $k$ -dim. embedded submanifold  $\equiv$  regular submanifold  $S$  is subset  $S \subset M$  s.t.  $\forall p \in S$ ,  $\exists$  chart  $(U \subset M, \varphi : U \rightarrow \mathbb{R}^n \ni 0)$ , s.t.  $\varphi(S \cap U)$  is the intersection of a  $k$ -dim. plane with  $\varphi(U)$ . (pairs  $(S \cap U, \varphi|_{S \cap U})$  form an atlas for differential structure on  $S$ ). Proof 1:

*Proof.*  $H$  subgroup of  $G$ , so multiplication map  $H \times H \rightarrow H$  inversion map  $H \rightarrow H$  are restrictions of multiplication and inversion maps on  $G$ .

Since  $H$  regular submanifold, maps are smooth. Recall  $H$  regular submanifold iff  $H$  immersive submanifold (i.e.  $H \hookrightarrow G$  is an immersion) and  $H$  topological subspace of  $G$ , i.e. submanifold topology on  $H$  is same as subspace topology.

Claim:  $H$  closed.  
Let  $x_0 \in \overline{H}$   
Let  $(U, x)$  be a chart adapted to  $H$ , whose domain contains  $e$ .  
Let

$$\delta : G \times G \rightarrow G$$
$$\delta(g_1, g_2) = g_1^{-1}g_2$$

Choose open set  $V$  s.t.  $e \in V \subset \overline{V} \subset U$ .  
By continuity map  $\delta$ , find open neighborhood  $O$  of identity  $e$  s.t.  $O \times O \subset \delta^{-1}(V)$   
If  $\{h_i\}$  sequence in  $H$  converging to  $x_0 \in \overline{H}$ , then  $x_0^{-1}h_i \rightarrow e$  and  $x_0^{-1}h_i \in O$  for all sufficiently large  $i$ .  
Since  $h_j^{-1}h_i = (x_0^{-1}h_j)^{-1}x_0^{-1}h_i$ ,  $h_j^{-1}h_i \in V$  for sufficiently large  $i, j$ .

For any sufficiently large fixed  $j$ ,

$$\lim_{i \rightarrow 0} h_j^{-1}h_i = h_j^{-1}x_0 \in \overline{V} \subset U$$

Since  $U$  is domain of a single-slice chart,  $U \subset H$  closed in  $U$ .

Thus, since  $\forall h_j^{-1}h_i \in U \cap H$ ,  $h_j^{-1}x_0 \in U \cap H \subset H$ ,  $\forall$  sufficiently large  $j$ .  
 $\implies x_0 \in H$ , and since  $x_0$  arbitrary, done.

Proof 2:  
cf. 9.2 The Closed Subgroup Theorem I of 427 Notes<sup>4</sup>

*Proof.* **Claim:** Since  $H$  is an embedded submanifold  $\equiv$  regular submanifold,  $\exists$  neighborhood  $U$  of  $1$ ,  $1 \in G$ , s.t.  $U \cap H$  closed in  $U$ .

Let  $x_0 \in \overline{H}$ ,  $\overline{H} \equiv$  closure of  $x_0$ .  
Then  $x_0U^{-1} \subseteq G$  is a neighborhood of  $x_0$  in  $G$  (since  $1 \in U^{-1}$ ,  $x_01 = x_0 \in x_0U^{-1}$ )  
 $\implies x_0U^{-1} \cap H \neq \emptyset$

$\forall x \in x_0U^{-1} \cap H$ ,  $x = x_0U^{-1}$  for some  $u \in U$ . Thus,  $x^{-1}x_0 = u \in U$ .

Now  
 $L_{x^{-1}} : G \rightarrow G$  is a homeomorphism, so  $L_{x^{-1}}(H) = H$ . By continuity,  $L_{x^{-1}}(\overline{H}) = \overline{H}$ . Thus  $x^{-1}x_0 \in \overline{H}$ .  
**Claim:**  $x^{-1}x_0 \in H \cap U$ .  
Since  $x^{-1}x_0 \in \overline{H} \cap U$ ,  $\exists$  sequence  $\{h_i\} \subset H \cap U$  s.t.  $h_0 \rightarrow x^{-1}x_0$ .  
But recall  $H \cap U$  closed in  $U$ , so  $x^{-1}x_0 \in H \cap U$ .

$$\implies x_0 \in xH = H, \quad \overline{H} \subseteq H$$

Thus  $H$  closed.

**Claim:** If  $H$  abstract subgroup of Lie group  $G$ , that's also an embedded submanifold  $\equiv$  regular submanifold, then  $H$  is a Lie subgroup.

Recall that by definition, Lie group has group multiplication and inverse map to be  $C^\infty$ . Then, just show group multiplication is  $C^\infty$ , first.

Since  $G$  is a Lie group, then

$$\mu : G \times G \rightarrow G$$
$$\mu(x, y) = xy$$

is  $C^\infty$  (by definition).

Then  $\mu : G \times G \rightarrow G$  cont.  
Consider subgroup  $H \subseteq G$  and  $\mu : H \times H \rightarrow H$ .  
Since  $H \times H \subseteq G \times G$ ,  $\forall (x, y) \in H \times H$  (fix  $(x, y) \in H \times H$ ),  $\forall$  neighborhood  $V$  of  $\mu(x, y) = xy$ ,  $V \subset G$ ,  $\exists$  neighborhood  $U$  of  $(x, y)$  s.t.  $\mu(U) \subseteq V$  (by  $\mu : G \times G \rightarrow G$  cont.).  
Since  $H$  embedded submanifold  $\equiv$  regular submanifold of  $G$ ,  
 $\exists$  neighborhood  $V' \subseteq V$  of  $xy \in G$ , coordinate map  $\varphi : V' \rightarrow \mathbb{R}^n$  ( $n = \dim G$ ) s.t.

$$\varphi(H \cap V') = \varphi(V') \cap (\mathbb{R}^k \times \{0\})$$

where  $k = \dim H$   
(since  $H$  is a  $k$ -dim. embedded submanifold  $\equiv$  regular submanifold,  $H \subseteq G$ , s.t.  $\forall p \in H$ ,  $\exists$  chart  $(V \subset G, \varphi : U \rightarrow \mathbb{R}^n \ni 0)$ , s.t.  $\varphi(U \cap V) = \varphi(V) \cap (\mathbb{R}^k \times \{0\})$ ).  
Now

$$\varphi \circ \mu : \mu^{-1}(V') \cap U \rightarrow \mathbb{R}^n \text{ is } C^\infty, \text{ and } \varphi \circ \mu(\mu^{-1}(V') \cap U) \subseteq \mathbb{R}^k \times \{0\}$$

Let projection  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be the standard projection,

$$\pi \circ \varphi \circ \mu : \mu^{-1}(V') \cap U \rightarrow \mathbb{R}^k \text{ is } C^\infty$$

□

<sup>4</sup><https://faculty.math.illinois.edu/~lerman/519/s12/427notes.pdf>



$\implies \mu$  is  $C^\infty$

□ For Lie group

From Chapter 4 “Lie Groups and Lie Algebras” of Kosmann-Schwarzbach (2010) [12]

While Proposition 2.6 of Kosmann-Schwarzbach (2010) [12] states that

$$\det(\exp(X)) = \exp(\operatorname{tr} X)$$

here are some other resources online that gave further discussion on the characteristic polynomial,  $\det(A - \lambda 1)$  and the different terms of it, called Newton identities:

- [http://scipp.ucsc.edu/~haber/ph116A/charpoly\\_11.pdf](http://scipp.ucsc.edu/~haber/ph116A/charpoly_11.pdf)
- <http://math.stackexchange.com/questions/1126114/how-to-find-this-lie-algebra-proof-that-mathfraksl-is-trace-zero-matrice>
- <http://mathoverflow.net/questions/131746/derivative-of-a-determinant-of-a-matrix-field>

**Theorem 16** (5.1 [12]). Consider  $\mathfrak{g} = \{X = \gamma'(0) | \gamma : 1 \rightarrow G \text{ of class } C^1, \gamma(0) = 1\}$

Let Lie group  $G$

- (i)  $\mathfrak{g}$  vector subspace of  $\mathfrak{gl}(n, \mathbb{R})$
- (ii)  $X \in \mathfrak{g}$  iff  $\forall t \in \mathbb{R}, \exp(tX) \in G$
- (iii) if  $X \in \mathfrak{g}$ , if  $g \in G$ , then  $gXg^{-1} \in \mathfrak{g}$
- (iv)  $\mathfrak{g}$  closed under matrix commutator, i.e. if  $X, Y \in \mathfrak{g}$ ,  $[X, Y] \in \mathfrak{g}$

*Proof.* (i)

- (ii) If  $\exp(tX) \in G$ , then  $X \frac{d}{dt} \exp(tX) \big|_{t=0} \in \mathfrak{g}$  (by def.)

If  $X \in \mathfrak{g}$ , then by def.,  $X = \frac{d}{dt} \gamma(t) \big|_{t=0}$  with  $\gamma(t) \in G$ .

Now Taylor expand;  $\forall k \in \mathbb{Z}^+$

$$\begin{aligned} \gamma\left(\frac{t}{k}\right) &= 1 + \frac{t}{k}X + O\left(\frac{1}{k^2}\right) = \exp\left(\frac{t}{k}X + O\left(\frac{1}{k^2}\right)\right) \\ \implies \left(\gamma\left(\frac{t}{k}\right)\right)^k &= \exp(tX) \\ \gamma\left(\frac{t}{k}\right) &\in G \quad \forall k \in \mathbb{Z}^+ \end{aligned}$$

$G$  closed subgroup, so  $\lim_{k \rightarrow \infty} (\gamma(\frac{t}{k}))^k = \exp(tX) \in G$

(iii)

(iv)

**Definition 29** (Lie algebra). Lie algebra  $\mathfrak{g}$ , tangent space to  $G$  at 1, i.e.  $\mathfrak{g} := T_1G$  is called Lie algebra of Lie group  $G$ .

$$\mathfrak{g} := \{X = \gamma'(0) | \gamma : 1 \rightarrow G \text{ of class } C^1, \gamma(0) = 1\} = T_1G$$

This is based on Proposition 5.3 of Kosmann-Schwarzbach (2010) [12].

For Lie group

$$U(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1\}$$

If  $X \in \mathfrak{u}(n)$ , then  $\exp(tX) \in U(n)$ . Then

$$\exp(tX) \exp(tX)^\dagger = (1 + tX + O(t^2))(1 + tX^\dagger + O(t^2)) = 1 + t(X + X^\dagger) + O(t^2) = 1 \forall t \in \mathbb{R} \implies X + X^\dagger = 0$$

i.e.  $X \in \mathfrak{u}(n)$  is an anti-Hermitian complex  $n \times n$  matrix.

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0\}$$

*Physicists:*  $X = iA$  and so  $A - A^\dagger$ .  $A \in \mathfrak{u}(n)$  is a Hermitian complex  $n \times n$  matrix.

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n, \mathbb{C}) | A - A^\dagger = 0\}$$

Regardless,  $\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2 = 2n^2 - n^2$

$$SU(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1\}$$

Then

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0, \operatorname{tr} X = 0\}$$

is the Lie algebra of traceless anti-Hermitian complex  $n \times n$  matrices, and that

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

In summary,

$$\begin{array}{ccc} \mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0\} & & \mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0, \operatorname{tr} X = 0\} \\ \exp(tX) \downarrow & & \exp(tX) \downarrow \\ U(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1\} & & SU(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1\} \end{array}$$

$$\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2 \quad \dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

From Chapter 5 “Lie Groups  $SU(2)$  and  $SO(3)$ ” of Kosmann-Schwarzbach (2010) [12],

9.0.1. *Bases of  $\mathfrak{su}(2)$ , Subsection 1.1 of Chapter 5 of Kosmann-Schwarzbach (2010) [12].* Recall that

$$\begin{array}{ccc} \mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0, \operatorname{tr} X = 0\} & & \\ \exp(tX) \downarrow & & \\ SU(n) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1\} & & \end{array}$$

□

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

and so

$$\begin{array}{ccc} \mathfrak{su}(2) = \{X \in \mathfrak{gl}(2, \mathbb{C}) | X + X^\dagger = 0, \operatorname{tr} X = 0\} & & \\ \exp(tX) \downarrow & & \\ SU(2) = \{U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1\} & & \end{array}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(2) = 3$$

Also, recall that  $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{C})$  is a vector subspace (16) and that  $X \in \mathfrak{g}$  iff  $\forall t \in \mathbb{R}, \exp(tX) \in G$ .

if  $X \in \mathfrak{g}$ , if  $g \in G$ , then  $gXg^{-1} \in \mathfrak{g}$

$\mathfrak{g}$  closed under  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$

$$(X, Y) \mapsto [X, Y]$$

and so with  $\mathfrak{g}$  as a vector space, we can have a choice of bases.

(a) 
$$\begin{aligned} \xi_1 &= \frac{i}{2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ \xi_2 &= \frac{1}{2} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \\ \xi_3 &= \frac{i}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{aligned}$$
satisfying

$$[\xi_k, \xi_l] = \epsilon_{klm} \xi_m$$

(b) *Physics*

$$\begin{aligned} \sigma_1 &= -2i\xi_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ \sigma_2 &= 2i\xi_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \\ \sigma_3 &= -2i\xi_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \end{aligned}$$

satisfying

$$[\sigma_k, \sigma_l] = 2i\epsilon_{klm} \sigma_m$$

EY : 20151001 Sage Math 6.8 doesn’t run on Mac OSX El Capitan: I suspect that it’s because in Mac OSX El Capitan, `/usr` cannot be modified anymore, even in an Administrator account. The TUG group for MacTeX had a clear, thorough, and useful (i.e. copy UNIX commands, paste, and run examples) explanation of what was going on:

<http://tug.org/mactex/elcapitan.html>

So keep in mind that my code for Sage Math is for Sage Math 6.8 that doesn’t run on Mac OSX El Capitan. I’ll also use `sympy` in Python as an alternative and in parallel.

One can check in `sympy` the traceless anti-Hermitian (or Hermitian) property of the bases and Pauli matrices, and the commutation relations (see `groups.py`):

```
import itertools
from itertools import product, permutations
```

```
import sympy
from sympy import I, LeviCivita
from sympy import Rational as Rat
```

```
from sympy.physics.matrices import msigma # <class 'sympy.matrices.dense.MutableDenseMatrix'>
```

```
def commute(A,B):
    """
    commute = commute(A,B)
    commute takes the commutator of A and B
    """
    return (A*B - B*A)
```

```
def xi(i):
    """
    xi = xi(i)
    xi is a function that returns the independent basis for
```

Lie algebra  $\mathfrak{su}(2) \backslash \text{equiv } \mathfrak{su}(2, \mathbb{C})$  of Lie group  $\text{SU}(2)$  of traceless anti-Hermitian matrices, based on `msigma` of `sympy`  
cf. [http://docs.sympy.org/dev/\\_modules/sympy/physics/matrices.html#msigma](http://docs.sympy.org/dev/_modules/sympy/physics/matrices.html#msigma)  
"""

```
if i not in [1,2,3]:
    raise IndexError("Invalid Pauli index")
elif i==1:
    return I/Rat(2)*msigma(1)
elif i==2:
    return -I/Rat(2)*msigma(2)
elif i==3:
    return I/Rat(2)*msigma(3)
```

```
## check anti-Hermitian property and commutation relations with xi
# xi is indeed anti-Hermitian
xi(1) == -xi(1).adjoint() # True
xi(2) == -xi(2).adjoint() # True
xi(3) == -xi(3).adjoint() # True
```

# xi obeys the commutation relations

```
for i,j in product([1,2,3],repeat=2): print i,j
```

```
for i,j in product([1,2,3],repeat=2): print i,j, "\tCommutator:", commute(xi(i),xi(j))
```

## check traceless Hermitian property and commutation relations with Pauli matrices  
# Pauli matrices i.e. `msigam` is indeed traceless Hermitian

```
msigma(1) == msigma(1).adjoint() # True
msigma(2) == msigma(2).adjoint() # True
msigma(3) == msigma(3).adjoint() # True
```

```
msigma(1).trace() == 0 # True
msigma(2).trace() == 0 # True
msigma(3).trace() == 0 # True
```

```
# Pauli matrices obey commutation relation
print "For Pauli matrices, the commutation relations are:"
for i,j in product([1,2,3],repeat=2): print i,j, "\tCommutator:", commute(msigma(i),msigma(j))
```

```
for i,j,k in permutations([1,2,3],3): print "Commute:", i,j,k, msigma(i), msigma(j), \ \
": and is 2*i of", msigma(k), commute(msigma(i),msigma(j)) == 2*I*msigma(k)*LeviCivita(i,j,k)
```

And finally the traceless property of the Pauli matrices:

```
>>> msigma(1).trace()
0
>>> msigma(2).trace()
0
>>> msigma(3).trace()
0
```

**9.1. Rotation Group  $SO(N)$  and Lie Algebra  $\mathfrak{so}(3)$ .** cf. Kambe (2009) [22], Appendix C  
 $g \in SO(N)$  represented by  $N \times N$  orthogonal matrix (i.e.  $gg^T = 1$ ) s.t.  $\det g = 1$ .

Let curve  $\xi(t)$  on  $SO(N)$ , initially from identity 1, with tangent vector  $\mathbf{a}$  at 1.  
 $\implies \xi(t) = 1 + t\mathbf{a} + O(t^2)$  for an infinitesimal parameter  $t$ .  
 $\mathbf{a} = \xi'(0)$  is an element of tangent space  $T_1SO(N)$  i.e. the Lie algebra  $\mathfrak{so}(N)$ , i.e.  $T_1SO(N) = \mathfrak{so}(N)$  by orthogonality condition  $\xi(t)\xi^T(t) = 1$ , then

$$(26) \qquad (1 + t\mathbf{a} + \dots)(1 + t\mathbf{a}^T + \dots) = 1 \implies \mathbf{a} + \mathbf{a}^T = 0 \text{ or } \mathbf{a} = \mathbf{a}^T$$

$\implies \mathbf{a}$  skew-symmetric.

9.1.1. **so(3).**  $\dim(\mathfrak{so}(3)) = 3$ , vector space of Lie algebra  $\mathfrak{so}(3)$  represented by skew-symmetric basis  $(E_1, E_2, E_3)$

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[E_i, E_j] = \epsilon_{ijk} E_k \text{ where } [E_i, E_j] = E_i E_j - E_j E_i$$

9.2. **Spin.** Let's follow the development by Baez and Muniain (1994) on pp. 175 of the Section II.1 “Lie Groups”, the second (II) chapter on “Symmetry” [9].

Let  $V = \mathbb{C}^2$ ,  $G = SU(2)$ . Then consider the graded algebra of polynomials on  $V = \mathbb{C}^2 \ni (x, y)$

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V) = \bigoplus_{\substack{j=0 \\ 2j \in \mathbb{Z}}}^{\infty} P^{(2j)}(V) = \bigoplus_{\substack{j=0 \\ j \in \mathbb{Z}}}^{\infty} P^{(2j)}(V) \oplus \bigoplus_{\substack{j=1/2 \\ 2j \text{ odd}}}^{\infty} P^{(2j)}(V)$$

$$P^{(2j)}(V) \equiv \text{vector space of complex polynomials of degree } 2j$$

and recall this representation on  $P^{(2j)}(V)$

$$\rho^{(2j)} : G \rightarrow \text{End}(P^{(2j)}(V))$$

$$\rho^{(2j)} : P^{(2j)}(V) \rightarrow P^{(2j)}(V)$$

$$\rho^{(2j)}(g)(f) = f \circ \rho(g^{-1}) \text{ where } \rho \text{ is the fundamental representation of } G = SU(2)$$

$$\rho^{(2j)}(g)(f)(v) = f \circ \rho(g^{-1})(v) \quad \forall f \in P^{(2j)}(V), \forall v \in V = \mathbb{C}^2$$

Note,

$$\dim P^{(2j)} = \binom{2j+2-1}{2-1} = 2j+1$$

**Exercise 21.** [9] *spin-0* Consider the trivial representation  $\tau$ :

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{T} & P^{(0)}(V) \\ \tau(g) \downarrow & & \downarrow \rho^{(0)}(g) \\ \mathbb{C} & \xrightarrow{T} & P^{(0)}(V) \end{array}$$

$$\tau : G \rightarrow \text{End}(\mathbb{C})$$

$$\tau(g) : \mathbb{C} \rightarrow \mathbb{C}$$

$$\tau(g) = 1_{\mathbb{C}}$$

Clearly,  $P^{(0)}(V) = \mathbb{C}$ , since  $P^{(0)}(V)$  consists of polynomials of constants in  $\mathbb{C}$ .

Consider  $c_0 \in \mathbb{C}$ ,  $f = k_0 \in P^{(0)}(V)$

$$\rho^{(0)}(g)(f) = f \circ \rho(g^{-1}) = k_0$$

$$\implies \rho^{(0)}(g)T(c_0) = T \circ \tau(g)c_0 = T(c_0). \text{ Let } T = 1_{\mathbb{C}} = 1_{P^{(0)}(V)}$$

So  $\rho^{(0)}(g) = \tau(g) = 1$ .  $T = 1$ . So representations  $\rho^{(0)}$  and trivial representation  $\tau$  on  $G$  are equivalent.

**Exercise 22.** [9] *spin- $\frac{1}{2}$*  For spin- $\frac{1}{2}$ ,  $j = \frac{1}{2}$ ,  $2j = 1$ .

$$\forall f \in P^{(1)}(V), V = \mathbb{C}^2. \text{ So in general form, } f(x, y) = ax + by \in P^{(1)}(V), \begin{pmatrix} x \\ y \end{pmatrix} \in V = \mathbb{C}^2$$

$$\begin{array}{l} \text{Recall the fundamental representation } \rho : G \rightarrow GL(2, \mathbb{C}) \equiv GL(\mathbb{C}^2) \\ \rho(g) : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \\ \rho(g) = g \end{array}$$

So consider  $T$  such that

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{T} & P^{(1)}(V) \\ \rho(g) \downarrow & & \downarrow \rho^{(1)}(g) \\ \mathbb{C}^2 & \xrightarrow{T} & P^{(1)}(V) \end{array}$$

Consider  $\forall v \in \mathbb{C}^2$ ,  $v = \begin{pmatrix} x \\ y \end{pmatrix}$ , then

$$\rho(g)v = gv = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

```
sage: g*X
[a*x + b*y]
[c*x + d*y]
```

For notation, let  $U \in G = SU(2)$  s.t.  $UU^\dagger = 1$ .

Consider  $(\rho^{(2j)}(U)(f))(x) = f(U^{-1}x)$ ,  $\forall x \in \mathbb{C}^2$ .

Choose  $f(x, y) = x$ . So for  $f(x, y) = Ax + By$ ,  $A = 1, B = 0$ . Choose  $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$  so  $U^{-1} = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$ . Then

$$U^{-1}x = \begin{pmatrix} \bar{a}x - by \\ \bar{b}x + ay \end{pmatrix}$$

So

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = \bar{a}x - by$$

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = \bar{b}x + ay \text{ for } f(x, y) = y$$

Let  $f(x, y) = Ax + By$

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = (A\bar{a} + B\bar{b})x + (Ba - Ab)y = (\bar{a}x - by)A + (\bar{b}x + ay)B = (A\bar{a} + B\bar{b})x + (Ba - Ab)y$$

which was calculated with the assistance of Sage Math:

```
sage: U_try1 = Matrix( [[a.conjugate(), -b], [b.conjugate(), a ] ] )
sage: f1( U_try1*X).coefficient(x)
A*conjugate(a) + B*conjugate(b)
sage: f1( U_try1*X).coefficient(y)
B*a - A*b
```

Treating  $P^{(1)}(\mathbb{C}^2)$  as a vector space, in its matrix formulation, then  $f(x, y) = Ax + By \in P^{(1)}(\mathbb{C}^2)$  is treated as  $\begin{bmatrix} A \\ B \end{bmatrix}$ , then

$(\rho^{(1)}(U)f)$  is

$$\implies \begin{bmatrix} \bar{a} & \bar{b} \\ -b & a \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A\bar{a} + B\bar{b} \\ -Ab + Ba \end{bmatrix}$$

so conclude in general that  $\rho^{(1)}(U) = (U^\dagger)^T$ .

Now, as Kosmann-Schwarzbach (2010) [12] says, on pp. 13, Chapter 2 Representations of Finite Groups, “Two representations  $(E_1, \rho_1)$  and  $(E_2, \rho_2)$  are equivalent if and only if there is a basis  $B_1$  of  $E_1$  and a basis  $B_2$  of  $E_2$  such that for every  $g \in G$ , the matrix of  $\rho_1(g)$  in the basis  $B_1$  is equal to the matrix of  $\rho_2(g)$  in the basis  $B_2$ . In particular, if the representations  $(E_1, \rho_1)$  and  $(E_2, \rho_2)$  are equivalent, then  $E_1$  is isomorphic to  $E_2$ .” So we need a change of basis between  $\rho(U) = U$  and  $\rho^{(1)}(U)$ . What's the linear transformation  $T$  s.t.

$$T^{-1}\rho^{(1)}(U)T = U?$$

By intuition,

$$T = \sigma_x \sigma_z \equiv \sigma_1 \sigma_3$$

where  $\sigma_i$ 's are Pauli matrices.

Indeed,

```
sage: Paulimat[3] * Paulimat[1]*U_try*Paulimat[1] * Paulimat[3]
[conjugate(a) conjugate(b)]
[ -b          a]
```

Then  $\rho^{(1)}(U) \circ T = TU$ , so this  $T = \sigma_1 \sigma_3$  is an “intertwining operator” between  $\rho^{(1)}(U)$  and fundamental representation  $\rho(U) = U$ , with  $T = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ , and  $T^{-1} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ .

$T$  is an isomorphism between  $\mathbb{C}^2$  and  $P^{(1)}(\mathbb{C}^2)$ . So fundamental representation  $\rho$  of  $G = SU(2)$  is equivalent to  $\rho^{(1)}(U)$  on  $P^{(1)}(\mathbb{C}^2)$ .

**Exercise 23.** [9] (Also from Exercise 2.6 of Kosmann-Schwarzbach (201) [12])

Let  $(E, \pi)$  representation of group  $G$ .

$\forall g \in G, \xi \in E^*, x \in E$ , set  $\langle \pi^*(g)(\xi), x \rangle = \langle \xi, \pi(g^{-1})(x) \rangle$

*dual* (or *contragredient*) of  $\pi, \pi^* : G \rightarrow \text{End}(E^*)$ ,  $\pi^*$  is a representation, since

$$\begin{aligned} \langle \pi^*(gh)(\xi), x \rangle &= \langle \xi, \pi((gh)^{-1})(x) \rangle = \langle \xi, \pi(h^{-1}g^{-1})(x) \rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x) \rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x)) \rangle = \\ &= \langle \pi^*(h)(\xi), \pi(g^{-1})(x) \rangle = \langle \pi^*(g)\pi^*(h)(\xi), x \rangle \end{aligned}$$

since this is true,  $\forall x \in E, \forall \xi \in E^*, \pi^*(gh) = \pi^*(g)\pi^*(h)$ .

dual  $\pi^*$  of  $\pi$  is a representation.

**9.3. Adjoint Representation.** I will first follow Sec. 7.3 The Adjoint Representation of Ch. 4 Lie Groups and Lie Algebras of Kosmann-Schwarzbach (201) [12]).

The *conjugation action*  $\mathcal{C}_g : G \rightarrow G$  is defined as

$$\mathcal{C}_g : G \rightarrow G$$

$$\mathcal{C}_g : h \mapsto ghg^{-1}$$

So

$$\mathcal{C} : G \rightarrow \text{Aut}(G)$$

$$\mathcal{C}g = \mathcal{C}_g$$

Now define the *adjoint action* of  $g$  as the differential or push forward of  $\mathcal{C}_g$ :

$$\text{Ad}_g := D_1\mathcal{C}_g \equiv (\mathcal{C}_g)_{*1} \equiv (\mathcal{C}_g)_*|_{g=1} \quad (\text{adjoint action of } g)$$

Now  $\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$ , so  $\text{Ad} : G \rightarrow \text{End}(\mathfrak{g})$

$$\text{Ad}(g) \equiv \text{Ad}_g$$

Note  $\mathcal{C}_{gg'} = \mathcal{C}_g\mathcal{C}_{g'} \equiv \mathcal{C}(gg') = \mathcal{C}(g) \circ \mathcal{C}(g')$  and so

$$\xrightarrow{D_1} \text{Ad}_{gg'} = \text{Ad}_g \circ \text{Ad}_{g'}$$

Kosmann-Schwarzbach (201) [12]) claims, because  $\text{Ad}_g = 1_{\mathfrak{g}}$  when  $g = 1$ ,

$\text{Ad} : G \rightarrow GL(\mathfrak{g})$  is a representation of  $G$  on  $\mathfrak{g}$ . (EY : 20160505 ???)

$\text{Ad} : g \mapsto \text{Ad}_g$

**Definition 30.** *representation*  $\text{Ad}$  of  $G$  on  $V = \mathfrak{g}$  is called *adjoint representation of Lie group*  $G$ .

Denote adjoint representation of Lie algebra  $\mathfrak{g}$ ,  $\text{ad}$ .

By definition,  $\text{Ad}_{\exp(tX)} = \exp(t\text{ad}_X)$

cf. Prop. 7.8 of Kosmann-Schwarzbach (201) [12])

**Proposition 12.** (1) *Let  $A$  invertible matrix,  $A \in \text{Lie group } G$ .*

*Let  $X$  matrix s.t.  $X \in \mathfrak{g}$ . Then*

$$\text{Ad}_A(X) = AXA^{-1}$$

(2) *Let  $X, Y \in \mathfrak{g}$ . Then*

$$\text{ad}_X(Y) = [X, Y]$$

(3) *Let  $X, Y \in \mathfrak{g}$ . Then*

$$\text{ad}_{[X, Y]} = [\text{ad}_X, \text{ad}_Y]$$

*Proof.* (1) By def.,  $\forall B \in G, \mathcal{C}_A(B) = ABA^{-1}$ , and thus

$$\text{Ad}_A(X) = \left. \frac{d}{dt} A \exp(tX) A^{-1} \right|_{t=0} = AXA^{-1}$$

(2)

$$\begin{aligned} \text{ad}_X(Y) &= \left. \frac{d}{dt} \text{Ad}_{\exp(tX)}(Y) \right|_{t=0} = \left. \frac{d}{dt} \exp(tX) Y \exp(tX) \right|_{t=0} = \\ &= XY - YX = [X, Y] \end{aligned}$$

(3) Use Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \text{ or}$$

$$[[A, B], C] = [A, [B, C]] - [B, [A, C]]$$

$\text{ad}_{[X, Y]}C = [[X, Y], C] = [X, [Y, C]] - [Y, [X, C]] = [X, \text{ad}_Y C] - [Y, \text{ad}_X C]$  and that

$$\text{ad}_X \text{ad}_Y C = [X, [Y, C]] \implies \text{ad}_{[X, Y]}C = [\text{ad}_X, \text{ad}_Y]C$$

□

**9.4. Invariant Vector Fields, Left-invariant Vector Field, Right-invariant Vector Field, Invariant Vector Fields.**

cf. pp. 26, Sec. 1.7 Lie Group and Invariant Vector Fields, Kambe (2009) [22]

Consider group  $G$  of smooth transformations (maps) of a manifold  $M$  into itself, i.e. Lie group.

Given fixed element  $h \in G$ , then

$$L_h : g \mapsto hg \text{ i.e. } L_h(g) = hg$$

(27)

$$R_h : g \mapsto gh \text{ i.e. } R_h(g) = gh$$

where  $L_h \equiv$  left translation of the group

$R_h \equiv$  right translation of the group

Suppose  $g_t$  is a curve on  $G$  described in terms of parameter  $t$ .

Left translation of  $g_t$  by  $g_{\Delta t}$  for infinitesimal  $\Delta t$  given by  $g_{\Delta t} \circ g_t$ .

$$(28) \quad \dot{g}_t = \lim_{\Delta t \rightarrow 0} \frac{g_{t+\Delta t} - g_t}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{(g_{\Delta t} - 1)g_t}{\Delta t} = X \circ g_t \text{ with } X := \lim_{\Delta t \rightarrow 0} \frac{g_{\Delta t} - 1}{\Delta t} \in T_1 G$$

Thus, left translation  $L_{g_{\Delta t}}$  leads to **right**-invariant vector field.

Similarly, right translation  $R_{g_{\Delta t}}$  leads to **left**-invariant vector field.

In summary,

$\dot{g}_t$  is said to be tangent vector at  $g_t$ .

Vector field  $X^L, (X^R)$  on  $G$  is left invariant (right invariant) if it's invariant under all left-translations (right translations) respectively, i.e.  $\forall g, h \in G$ , if

$$(29) \quad \begin{aligned} (L_h)_* X_g^L &= X_{hg}^L \\ (R_h)_* X_g^R &= X_{hg}^R \end{aligned}$$

Given tangent vector  $X$  to  $G$ , at  $e$ , one may left-translate (right translate)  $X$  to every pt.  $g \in G$  as

$$(30) \quad \begin{aligned} X_g^L &= (L_g)_*X = g \circ X = gX \\ X_g^R &= (R_g)_*X = X \circ g = Xg \end{aligned}$$

Since

$$(R_h)_*X_g^R = (R_h)_*(R_g)_*X = X \cdot gh = X_{gh}^R$$

Then  $(R_h)_*$  operation does yield a right-invariant field.

Hence  $(R_h)_*$  transformation in Eq. **30** gives a right-invariant field generated by  $X$ .

In other words, in summary

$$(31) \quad \begin{aligned} &L_{g_{\Delta t}} : g_t \mapsto g_{\Delta t}g_t = g_{t+\Delta t} \\ &\text{Left translation of } g_t \text{ by } g_{\Delta t}: \dot{g}_t = Xg_t \text{ so } X \in T_1G \text{ is right-invariant vector field} \\ &\text{i.e. } X_g^R = (R_g)_*X = Xg \end{aligned}$$

Consider curves  $\xi_t : \mathbb{R} \rightarrow G$  with tangent  $\dot{\xi}_0 = X$  at  $t = 0$ .

$$\xi : t \mapsto \xi(t), \quad t \in \mathbb{R}$$

Left-invariant field  $X_s^L = \frac{d}{dt}(g \circ \xi_t) \quad \forall g_s \in G, (s \text{ a parameter})$

Right-invariant field  $X_s^R = \frac{d}{dt}(\xi_t \circ g_s)$

Lie group  $G$  acts as a group of linear transformations on its own Lie algebra  $\mathfrak{g}$ , namely

$\forall g \in G, \exists$  operator  $\text{Ad}_g$  s.t.

$$(32) \quad \text{Ad}_g Y := (L_g)_* \circ (R_{g^{-1}})_* Y = gYg^{-1}, \quad \forall Y \in \mathfrak{g}$$

Operator  $\text{Ad}_g$  transforms  $Y \in \mathfrak{g}$  into  $\text{Ad}_g Y \in \mathfrak{g}$ , linearly.

Set of all such  $\text{Ad}_g$ , i.e.  $\text{Ad}(G)$  is called the adjoint representation of  $G$ , an adjoint group.

Setting  $g$  to be the inverse  $\xi_t^{-1} := (\xi_t)^{-1}$ ,

adjoint transformation  $\text{Ad}_{\xi_t^{-1}} Y$  is a function of  $t$ .

Its derivative with respect to  $t$  is a linear transformation from  $Y$  to  $\text{ad}_X Y$ :

$$(33) \quad \begin{aligned} \text{ad}_X Y &= \left. \frac{d}{dt} \xi_t^{-1} Y \xi_t \right|_{t=0} := [X, Y] \\ \text{ad}_X : \mathfrak{g} &\rightarrow \mathfrak{g} \end{aligned}$$

$\text{ad}_X$  is a linear transformation  $\mathfrak{g} \rightarrow \mathfrak{g}$ .

## 10. LIE GROUPS AND THEIR LIE ALGEBRAS

**Lie groups and their Lie algebras - Lec 13 - Frederic Schuller**, Schuller (2015) [31]

### 10.1. Chapter 4. Lie Theory, Schuller (2015) [31].

#### 10.1.1. 4.1 Lie groups, Schuller (2015) [31].

**Definition 31** (Lie group, Schuller (2015) [31]). *A lie group  $(G, \cdot)$  is*

## Part 6. Cohomology; Stoke's Theorem

### 11. STOKE'S THEOREM

**Theorem 17** (Stoke's Theorem). *Let  $M$  be oriented, smooth  $n$ -manifold with boundary, let  $\omega$  be a compactly supported smooth  $(n-1)$ -form on  $M$ , or if  $\omega \in A_c^{n-1}(M)$ , Then*

$$(34) \quad \int_M d\omega = \int_{\partial M} \omega$$

If  $\partial M = \emptyset$ , then  $\int_{\partial M} \omega = 0$

$\int_{\partial M} \omega$  interpreted as  $\int_{\partial M} i_{\partial M}^* \omega = \int_{\partial M} i^* \omega$  so

$$(35) \quad \int_M d\omega = \int_{\partial M} i^*(\omega)$$

where inclusion  $i : \partial M \hookrightarrow M$

*Proof.* Begin with very special case:

Suppose  $M = \mathbb{H}^n$  (upper half space),  $\partial M = \mathbb{R}^{n-1}$

$\omega$  has compact support, so  $\exists R > 0$  s.t.  $\text{supp } \omega \subseteq \text{rectangle } A = [-R, R] \times \cdots \times [-R, R] \times [0, R]$ .

$$\forall \omega \in A_c^{n-1}(\mathbb{H}^n)$$

$$(36) \quad \omega = \sum_{j=1}^n (-1)^{j-1} f_j dx^1 \wedge \cdots \wedge \widehat{dx}^j \wedge \cdots \wedge dx^n \equiv \sum_{i=1}^n \omega_i dx^1 \wedge \cdots \wedge \widehat{dx}^i \wedge \cdots \wedge dx^n$$

with Conlon (2008) [17] and John Lee (2012) [3]'s notation, respectively, and where  $f_j$  has compact support.

$$i^* \omega = (f_1 \circ i) dx^2 \wedge \cdots \wedge dx^n \in A_c^{n-1}(\partial \mathbb{H}^n)$$

$$\begin{aligned} d\omega &= \sum_{i=1}^n d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx}^i \wedge \cdots \wedge dx^n = \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx}^i \wedge \cdots \wedge dx^n = \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

i.e. (for another notation)

$$d\omega = \left( \sum_{j=1}^n \frac{\partial f_j}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n \in A_c^n(\mathbb{H}^n)$$

$$d\omega = \left( \sum_{j=1}^n \frac{\partial f_j}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n \in A_c^n(\mathbb{H}^n)$$

$$\int_{\mathbb{H}^n} d\omega = \sum_{i=1}^n (-1)^{i-1} \int_A \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n = \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R dx^1 \cdots dx^n \frac{\partial \omega_i}{\partial x^i}(x)$$

We can change order of integration in each term so to do  $x^i$  integration first.

By fundamental thm. of calculus, terms for which  $i \neq n$  reduce to

$$\begin{aligned} \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^1 \cdots dx^n &= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^i dx^1 \cdots \widehat{dx}^i \cdots dx^n = \\ &= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R [\omega_i(x)]_{x^i=-R}^{x^i=R} dx^1 \cdots \widehat{dx}^i \cdots dx^n = 0 \end{aligned}$$

because we’ve chosen  $R$  large enough that  $\omega = 0$  when  $x^i = \pm R$ .

Part 7. Prástaro

Prástaro (1996) [13]

11.0.1. *Affine Spaces.* cf. Sec. 1.2 - *Affine Spaces* of Prástaro (1996) [13]

**Definition 32** (affine space).

$$(37) \quad \begin{aligned} & \text{affine space} && (M, \mathbf{M}, \alpha) \\ & && \text{with} \\ & M \equiv && \text{set (set of pts.)} \\ & \mathbf{M} \equiv && \text{vector space (space of free vectors)} \\ & \alpha \equiv \mathbf{M} \times M \rightarrow M \equiv && \text{translation operator} \\ & \alpha : (v, p) \mapsto p' \equiv p + v \end{aligned}$$

Note:  $\alpha$  is a **transitive** action and without fixed pts. (free).

$$\begin{aligned} & \text{i.e. } \forall p \in M, \\ & \forall \text{ pt. } O \in M, \alpha : (v, O) \mapsto O' \equiv O + v, \alpha(\cdot, O) \equiv \alpha_O \equiv \alpha(O). \quad \alpha_O(v) = O' = O + \mathbf{v} \quad \forall O' \in M, \exists \mathbf{v} \in \mathbf{M} \text{ s.t. } O' = O + \mathbf{v} \\ & \implies M \equiv \mathbf{M}. \\ & \forall (O, \{e_i\})_{1 \leq i \leq n}, \text{ where } \{e_i\} \text{ basis of } \mathbf{M}, M \equiv \mathbf{M} = \mathbb{R}^n \text{ so isomorphism } M \simeq \mathbb{R}^n \end{aligned}$$

i.e.  $\alpha$  is **without fixed pts.**, meaning,  
Given pointed space  $(M, O)$ , where base pt.  $O \in M$ , we can associate  $\forall p \in M$ , vector  $\mathbf{x} \in \mathbf{M}$ , by 1-to-1 mapping  $M \rightarrow \mathbf{M}$ .  
So for

$$\begin{aligned} \alpha : \mathbf{M} \times M &\rightarrow M \\ \alpha(\mathbf{x}, p) &= p' = p + \mathbf{x} \end{aligned}$$

Consider

$$\alpha(\mathbf{x}, O) = p = \alpha_O(\mathbf{x}) = p \implies \exists \alpha_O^{-1}(p) = \mathbf{x} \in \mathbf{M}$$

- (1) tangent space of  $M$  in  $p \in M$  is vector space  $T_p M \equiv (\mathbf{M}, p) \cong M$

(2) If  $\mathbf{M}$  Euclidean space, affine space  $(M, \mathbf{M}, \alpha)$  is Euclidean

(3) Call dim. of affine space  $(M, \mathbf{M}, \alpha)$ , dim. of  $\mathbf{M} \equiv \dim \mathbf{M}$

$\{\mathbf{e}_i\}$  basis of  $\mathbf{M}$
- Definition 33.**  $(O, \{e_i\}) \equiv$  affine frame.  
 $\forall$  affine frame  $(O, \{e_i\})$ ,  $\exists$  coordinate system  $x^\alpha : M \rightarrow \mathbb{R}$ ,  
where  $x^\alpha(p)$  is  $\alpha$ th component, in basis  $\{e_i\}$ , of vector  $p - O$
- Proposition 13** (1.6, Prástaro (1996) [13]).  $\forall O \in M$ , we have canonical identification  $M \equiv \mathbf{M}$ , since
- $$\begin{aligned} \alpha_O^{-1} : M &\rightarrow \mathbf{M} && \alpha_O : \mathbf{M} \rightarrow M \\ \alpha_O^{-1}(p) &= \mathbf{x} && \alpha_O : \mathbf{x} = \alpha(\mathbf{x}, O) = p \end{aligned}$$
- Furthermore,  
 $\forall$  **affine frame**  $(O, \{\mathbf{e}_i\})_{1 \leq i \leq d}$ , where  $\{\mathbf{e}_i\}$  basis of  $\mathbf{M}$ ,  
 $\exists$  isomorphism  $M \cong \mathbb{R}^d$ ,  
Then,  $\forall (O, \{\mathbf{e}_i\})_{1 \leq i \leq d}$ ,  
 $\exists$  coordinate system  $x^\alpha : M \rightarrow \mathbb{R}$ ,  
where  $x^\alpha(p) = \alpha$ th component, in basis  $\{\mathbf{e}_i\}$ , of vector  $p - O$ .
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- Theorem 18** (1.4 Prástaro (1996) [13]). *Let  $(x^\alpha), (\bar{a}^\alpha)$  2 coordinate systems correspond to affine frames  $(O, \{e_i\}), (\bar{O}, \{\bar{e}_i\})$ , respectively.*
- $$(38) \quad \bar{x}^\alpha = A^\alpha_\beta x^\beta + y^\alpha$$

$$\text{where}$$
- $$y^\alpha \in \mathbb{R}^n, \quad A^\alpha_\beta \in GL(n; \mathbb{R})$$
- Definition 34** (1.10 Prástaro (1996) [13]).
- $$(39) \quad A(n) \equiv GL(n, \mathbb{R}) \times \mathbb{R}^n$$
- affine group of dim.  $n$
- Theorem 19** (1.5). *symmetry group of  $n$ -dim. affine space, called affine group  $A(M)$  of  $M$ .  $\exists$  isomorphism,*
- $$(40) \quad A(M) \simeq A(n), \quad f \mapsto (f^\alpha_\beta, y^\alpha); \quad f^\alpha \equiv x^\alpha \circ f = f^\alpha_\beta x^\beta + y^\alpha$$
- cf. Eq. 1.4 Prástaro (1996) [13]
- Definition 35** (metric). *Let smooth manifold  $M$ ,  $\dim M = n$ ,  $\forall p \in M$ ,  $\exists$  vector space  $T_p M$ , and so for*
- $$(41) \quad \begin{aligned} & g_p(T_p M)^2 \rightarrow \mathbb{R} \\ & g_p : (X_p, Y_p) \mapsto g_p(X_p, Y_p) \in \mathbb{R} \end{aligned}$$
- with  $g_p$  being bilinear, symmetric (in  $X_p, Y_p$ ), nondegenerate (i.e. if  $g_p(X_p, Y_p) = 0$ , then  $X_p$  or  $Y_p = 0$ )  
Note that
- $$g \in \Gamma((TM \otimes TM)^*)$$
- $$\begin{aligned} \text{and that for } X &= X^i \frac{\partial}{\partial x^i} \text{ so} \\ Y &= Y^i \frac{\partial}{\partial x^i} \end{aligned}$$
- $$g(X, Y) = g_{ij} X^i Y^j$$
- Now for
- $$\begin{aligned} F : M &\rightarrow N && DF \equiv F_* : T_p M \rightarrow T_{F(p)} N \\ F : x &\mapsto y = y(x) && DF : X_p \mapsto (DF)(X^j \frac{\partial}{\partial x^j}) = X^j \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} \end{aligned}$$
- $$\begin{aligned} (F^* g')(X, Y) &= (F^* g')(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}) = (F^* g')_{ij} X^i Y^j = g'(F_* X, F_* Y) = \\ &= \end{aligned}$$
- Part 8. Connections
12. CONNECTIONS OF VECTOR BUNDLES
- [34]
- Definition 36** (Connection in a vector bundle). **connection in a vector bundle**  $\pi : E \rightarrow M$  over  $C^\infty$  manifold  $M$ , is a bilinear map
- $$\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$$
- satisfying
- (i)  $\nabla_f X s = f \nabla_X s$

(ii)  $\nabla_X (f s) = f \nabla_X s + (X f) s$  where  $f \in C^\infty(M)$ ,  $X \in \mathfrak{X}(M)$ ,  $s \in \Gamma(E)$

$\nabla_X s$  is covariant derivative of  $s$  relative to  $X$  (for Morita (2001)[34])



Claim: *Any vector bundle admits a connection.*

e.g. product bundle  $M \times \mathbb{R}^n$ . Let  $x_1, \dots, x_n$  be canonical coordinates in  $\mathbb{R}^n$ . Take frame field  $(s_1, \dots, s_n)$ , where  $s_i(p) = \frac{\partial}{\partial x^i}$ . Set  $\nabla_X s_i = 0$  ( $i = 1, \dots, m$ )  $\forall$  vector space  $X$ ,  
 $\forall s = \sum_i a_i s_i, \forall X \in \mathfrak{X}(M)$ , set

$$\nabla_X s = \sum_{i=1}^n (X a_i) s_i$$

For this connection  $\nabla_X s$  is the partial derivative in direction of  $X$  if  $s$  is considered  $\mathbb{R}^n$ -valued function on  $M$ . Call it **trivial connection** in product bundle.

Indeed,

$$\begin{aligned} \nabla_X s &= \nabla_{X^i \frac{\partial}{\partial x^i}} \left( s^m \frac{\partial}{\partial x^m} \right) = X^i \nabla_{\frac{\partial}{\partial x^i}} \left( s^m \frac{\partial}{\partial x^m} \right) = X^i \left( \nabla_{\frac{\partial}{\partial x^i}} s^m \right) \frac{\partial}{\partial x^m} + X^i s^m \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^m} = \\ &= X^i \left( \frac{\partial}{\partial x^i} s^m \right) \frac{\partial}{\partial x^m} + X^i s^m \Gamma_{mi}^q \frac{\partial}{\partial x^q} = X^i \frac{\partial s^m}{\partial x^i} \frac{\partial}{\partial x^m} + 0 \end{aligned}$$

if  $\Gamma_{mi}^q = 0$  at a chosen point  $p$ .

For arbitrary vector bundle  $\pi : E \rightarrow M$ , take locally finite open covering  $\{U_\alpha\}_{\alpha \in A}$  s.t.  $\pi^{-1}(U_\alpha)$  trivial. Denote  $\nabla^\alpha$  trivial connection  $\forall \pi^{-1}(U_\alpha)$ . Let  $\{f_\alpha\}$  be a partition of unity for covering  $U_\alpha$ , define

$$\nabla_X s := \sum_\alpha f_\alpha \nabla_X^\alpha s$$

Verify this defines connection in  $E$ :

$$\begin{aligned} \nabla_X (gs) &= \sum_\alpha f_\alpha \nabla_X^\alpha (gs) = \sum_\alpha f_\alpha \left[ X^i \left( \frac{\partial g}{\partial x^i} \right) s + X^i g \frac{\partial s^m}{\partial x^i} \frac{\partial}{\partial x^m} \right] = \\ &= g \sum_\alpha f_\alpha \nabla_X^\alpha s + \sum_\alpha f_\alpha (Xg) s \end{aligned}$$

**Proposition 14** (5.18 Morita (2001)[34]). *Let  $\nabla_i$  ( $1 \leq i \leq k$ ) be  $k$  connections in a given vector bundle. Then  $\forall$  linear combination  $\sum_{i=1}^k t_i \nabla_i$ , where  $t_1 + \dots + t_k = 1$  is a connection.*

*Proof.* TODO: Ex. 5.5

## Part 9. Holonomy

**Definition 37** (Conlon, 10.1.2). *If  $X, Y \in \mathfrak{X}(M)$ ,  $M \subset \mathbb{R}^m$ , **Levi-Civita connection** on  $M \subset \mathbb{R}^m$*

$$(42) \quad \begin{aligned} \nabla : \mathfrak{X}(M) : \mathfrak{X}(M) &\rightarrow \mathfrak{X}(M) \\ \nabla_X Y &:= p(D_X Y) \end{aligned}$$

with

$$\begin{aligned} D_X Y &:= \sum_{j=1}^m X(Y^j) \frac{\partial}{\partial x^j} = \sum_{i,j=1}^m X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} & \forall X &= \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}, \\ \forall Y &= \sum_{i=1}^m Y^i \frac{\partial}{\partial x^i} \end{aligned}$$

$$\nabla_{fX} Y = f(D_{fX} Y) = p(f D_X Y) = f p D_X Y = f \nabla_X Y$$

$$\nabla_X fY = p(D_X fY) = p \left( \sum_{i,j=1}^m \left( X^i f \frac{\partial Y^j}{\partial x^i} + X^i Y^j \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j} \right) = f \nabla_X Y + p \sum_{j=1}^m X(f) Y^j \frac{\partial}{\partial x^j} = f \nabla_X Y + X(f) p(Y)$$

**Definition 38** (Conlon, 10.1.4; Christoffel symbols).

$$(43) \quad \begin{aligned} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} &= \Gamma_{ij}^k \frac{\partial}{\partial x^k} & (\text{Conlon's notation}) \\ \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} &= \Gamma_{ij}^k \frac{\partial}{\partial x^k} & (F. Schuller's notation) \end{aligned}$$

**Definition 39** (torsion).

$$(44) \quad \begin{aligned} T : \mathfrak{X}(M) \times \mathfrak{X}(M) &\rightarrow \mathfrak{X}(M) \\ T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned}$$

If  $T = 0$ ,  $\nabla$  torsion-free or symmetric.

$$T(fX, Y) = f \nabla_X Y - (f \nabla_Y X + Y(f)X) - \{(fXY - (Y(f)X + fYX))\} = fT(X, Y)$$

$$T(X, fY) = f \nabla_X Y + X(f)Y - f \nabla_Y X - \{((X(f)Y + fXY) - fYX)\} = fT(X, Y)$$

Thus,  $T(X, Y)$   $C^\infty(M)$ -bilinear.

$T \in \tau_1^2(M)$ .

$T(v, w) \in T_x M$  defined,  $\forall v, w \in T_x M, \forall x \in M$ .

Thus, torsion is a **tensor**.

**Exercise 10.1.7 Conlon (2008)[17]** . .

If  $T(X, Y) = 0$ ,

$$T(e_i, e_j) = \Gamma_{ji}^k e_k - \Gamma_{ij}^k e_k - 0 = 0 \implies \Gamma_{ji}^k = \Gamma_{ij}^k$$

If  $\Gamma_{ij}^k = \Gamma_{ji}^k$ ,  $T(e_i, e_j) = 0$ .

**Exercise 10.1.8, Conlon (2008)[17]**.

If  $M \subset \mathbb{R}^m$  smoothly embedded submanifold,  $\forall \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \in T_x M$ , spanning  $T_x M$ , consider  $\frac{\partial}{\partial x^j} = X_j^k \frac{\partial}{\partial \tilde{x}^k}, \frac{\partial}{\partial x^i} = X_i^k(\tilde{x}) \frac{\partial}{\partial \tilde{x}^k}$

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} &= p D_{X_j^k \frac{\partial}{\partial \tilde{x}^k}} X_i^l \frac{\partial}{\partial \tilde{x}^l} = p \left( X_j^k \frac{\partial X_i^l}{\partial \tilde{x}^k} \frac{\partial}{\partial \tilde{x}^l} \right) = X_j^k p \left( \frac{\partial X_i^l}{\partial \tilde{x}^k} \frac{\partial}{\partial \tilde{x}^l} \right) \\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} &= X_i^k p \left( \frac{\partial X_j^l}{\partial \tilde{x}^k} \frac{\partial}{\partial \tilde{x}^l} \right) \end{aligned}$$

If  $X \in \mathfrak{X}(M)$ , smooth  $s : [a, b] \rightarrow M$ , then  $\forall s(t)$ ,

$$X'_{s(t)} = \nabla_{\dot{s}(t)} X \in T_{s(t)} M$$

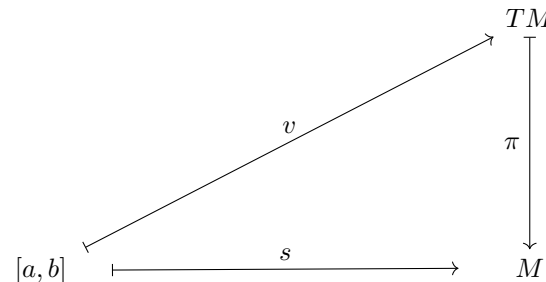
In fact, it's often natural to consider fields  $X_{s(t)}$  along  $s$ , parametrized by parameter  $t$ , allowing

$$X_{s(t_1)} \neq X_{s(t_2)}$$

each of  $s(t_1) = s(t_2)$ .

**Definition 40** (10.1.9). *Let smooth  $s : [a, b] \rightarrow M$ .*

*Vector field along  $s$  is smooth  $v : [a, b] \rightarrow TM$  s.t.*



commutes.

Note that  $v \in \mathfrak{X}(s) \subset \mathfrak{X}(M)$

e.g.  $(Y|s)(t) = Y_{s(t)}$ , restriction of  $Y \in \mathfrak{X}(M)$  to  $s$ .

e.g.  $\dot{s}(t) \in \mathfrak{X}(M)$ .

$\forall v, w \in \mathfrak{X}(s)$ ,  $v + w \in \mathfrak{X}(s)$ ,

$$(fv + gv)(t) := (f(s(t)) + g(s(t)))v(t) = f(s(t))v(t) + g(s(t))v(t) = (f + g)v(t)$$

Likewise,

$$f(v + w) = fv + fw$$

$\mathfrak{X}(s)$  is a real vector space and  $C^\infty[a, b]$ -module.

**Definition 41** (10.1.10). *Let conection  $\nabla$  on  $M$ .*

***Associated covariant derivative*** is operator

$$\frac{\nabla}{dt} \mathfrak{X}(s) \rightarrow \mathfrak{X}(s)$$

$\forall$  smooth  $s$  on  $M$ , s.t.

(1)  $\frac{\nabla}{dt}$   $\mathbb{R}$ -linear

(2)  $\left(\frac{\nabla}{dt}\right)(fv) = \frac{df}{dt}v + f\frac{\nabla}{dt}v$ ,  $\forall f \in C^\infty[a, b]$ ,  $\forall v \in \mathfrak{X}(s)$

(3) If  $Y \in \mathfrak{X}(M)$ , then

$$\frac{\nabla}{dt}(Y|s)(t) = \nabla_{\dot{s}(t)}Y \in T_{s(t)}M, \quad a \leq t \leq b$$

**Theorem 20** (Conlon Thm. 10.1.11[17]).  $\forall$  connection  $\nabla$  on  $M$ ,  $\exists!$  associated covariant derivative  $\frac{\nabla}{dt}$

*Proof.* Consider arbitrary coordinate chart  $(U, x^1 \dots x^n)$ .

Consider smooth curve  $s : [a, b] \rightarrow U$ .

Let  $v \in \mathfrak{X}(s)$ ,  $v(t) = v^i(t)\frac{\partial}{\partial x^i}$ ;  $\dot{s}(t) = s^j\frac{\partial}{\partial x^j}$ .

$$\frac{\nabla v}{dt} = \frac{dv^i(t)}{dt} \frac{\partial}{\partial x^i} + v^i(t) \frac{\nabla}{dt} \frac{\partial}{\partial x^i} = \frac{dv^i}{dt} \frac{\partial}{\partial x^i} + v^i \nabla_{\dot{s}(t)} \frac{\partial}{\partial x^i} = \dot{v}^i \frac{\partial}{\partial x^i} + v^i \dot{s}^j \Gamma_{ij}^k \frac{\partial}{\partial x^k} = (\dot{v}^k + v^i \dot{s}^j \Gamma_{ij}^k) \frac{\partial}{\partial x^k}$$

This is an explicit, local formula in terms of connection, proving uniqueness.

Existence:  $\forall$  coordinate chart  $(U, x^1 \dots x^n)$ ,  $(\dot{v}^k + v^i \dot{s}^j \Gamma_{ij}^k) \frac{\partial}{\partial x^k} =: \frac{\nabla v}{dt}$ .

$$\frac{\nabla}{dt}(fv) = \dot{f}v^k + f\dot{v}^k + fv^i \dot{s}^j = \dot{f}v + f \frac{\nabla v}{dt}$$

If  $f$  constant, then  $\frac{\nabla}{dt}$  is  $\mathbb{R}$ -linear.

**Definition 42** (10.1.12 Conlon (2008)[17]). *Let  $(M, \nabla)$ . Let  $v \in \mathfrak{X}(s)$  for smooth  $s : [a, b] \rightarrow M$ .*

*If  $\frac{\nabla v}{dt} \equiv 0$  on  $s$ , then  $v$  is **parallel** along  $s$ .*

**Theorem 21** (10.1.13). *Let  $(M, \nabla)$ , smooth  $s : [a, b] \rightarrow M$ ,  $c \in [a, b]$ ,  $v_0 \in T_{s(c)}M$ .*

*Then  $\exists!$  parallel field  $v \in \mathfrak{X}(s)$  s.t.  $v(c) = v_0$ .*

*$v$  parallel transport along  $s$ .*

*Proof.*

$$\dot{s}(t) = \dot{s}^j(t)e_j$$

$$v(t) = v^i(t)e_i$$

$$v_0 = a^i e_i$$

$$0 = \left( \frac{dv^k}{dt}(t) + v^i(t) \dot{s}^j(t) \Gamma_{ij}^k(s(t)) \right) e_k$$

or equivalently

$$(45) \quad \frac{dv^k}{dt} = -v^i \dot{s}^j \Gamma_{ij}^k, \quad 1 \leq k \leq n \quad (10.1)$$

with initial conditions  $v^k(c) = a^k$ ,  $1 \leq k \leq n$ .

By existence and uniqueness of solutions of O.D.E.

$\exists \epsilon > 0$  s.t.  $\exists!$  solutions  $v^k(t)$ . For  $c - \epsilon < t < c + \epsilon$ .

In fact, these ODEs being linear in  $v^k$ , by ODE theory (Appendix C, Thm. C.4.1).

$\nexists$  restriction on  $\epsilon$ , so  $\exists! v^k(t) \quad \forall t \in [a, b]$ ,  $1 \leq k \leq n$

□

**12.1. Principal bundle, vector bundle case for parallel transport.** Recall the 2 different forms or viewpoints for Lie-algebra valued 1-forms, or vector-valued 1-forms, or sections of 1-form-valued endomorphisms:

$$\omega_{i\mu}^k dx^\mu \equiv \omega_{i\cdot}^k \in \Omega^1(M, \mathfrak{gl}(n, \mathbb{F})) = \Gamma(\mathfrak{gl}(n, \mathbb{R} \otimes T^*M|_U))$$

for  $i, k = 1 \dots n = \dim E$ .

$$\mu = 1 \dots d = \dim E$$

Now

$$D_X \mu = X^\mu D_{\frac{\partial}{\partial x^\mu}} \mu = X^\mu \left[ \left( \frac{\partial}{\partial x^\mu} \mu^k \right) e_k + \mu^i \omega_{i\mu}^k e_k \right] = (X(\mu^k) + \mu^i \omega_i^k(X)) e_k = (d\mu^k(X) + \mu^i \omega_i^k(X)) e_k$$

So then define

$$(46) \quad \begin{aligned} D : \Gamma(E) &\rightarrow \Gamma(E) \otimes \Gamma(T^*M) \\ D\mu &= D(\mu^i e_i) = e_k (d\mu^k + \mu^i \omega_i^k) \equiv (d + A)\mu \end{aligned}$$

Also,  $D$  can be defined for this case:

$$D : \Gamma(\text{End}(E)) \rightarrow \Gamma(\text{End}E) \otimes \Gamma(T^*M)$$

Let  $\sigma = \sigma_j^i e_i \otimes e^j \in \Gamma(\text{End}(E))$

$$(47) \quad \begin{aligned} D\sigma &= D(\sigma_j^i e_i) \otimes e^j + \sigma_j^i e_i \otimes D^* e^j = (d\sigma_j^k + \sigma^i A_i^k) e_k \otimes e^j + \sigma_j^i e_i \otimes (A^*)_k^j e^k = \\ &= (d\sigma_j^k + \sigma_j^i A_i^k) e_k \otimes e^j + \sigma_i^k e_j \otimes (-A^i_j) e^j = (d\sigma_j^k + [A, \sigma]_j^k) e_k \otimes e^j \end{aligned}$$

□

cf. Def. 4.1.4 of Jost (2011), pp. 138.

For  $\mu \in \Gamma(E)$ , smooth  $s : [a, b] \rightarrow M$ ,  $X(t) = \dot{s}(t)$ ,

$$(48) \quad D_{\dot{s}(t)} \mu = \dot{s}^\mu D_{\frac{\partial}{\partial x^\mu}} \mu = \dot{s}^\mu \left[ \frac{\partial \mu^k}{\partial x^\mu} e_k + \mu^i \omega_{i\mu}^k e_k \right] = \left[ \dot{s}^\mu \frac{\partial \mu^k}{\partial x^\mu} + \dot{s}^\mu \mu^i \omega_{i\mu}^k \right] e_k = \frac{d}{dt} \mu(s(t)) + \mu^i \dot{s}^\mu \omega_{i\mu}^k e_k$$

Let  $D_{\dot{s}(t)} \mu = 0$ . Then,

$$(49) \quad \frac{d}{dt} \mu(s(t)) = -\mu^i \dot{s}^\mu \omega_{i\mu}^k e_k$$

Recall, given vector bundle  $E \xrightarrow{\pi} N$ , given  $\varphi : M \rightarrow N$ , then pullback

$$(50) \quad \varphi^* E \rightarrow M$$

i.e.

$$\begin{array}{ccc}
\varphi^*E & \xleftarrow{\varphi^*} & E \\
\downarrow \psi & & \downarrow \pi \\
M & \xrightarrow{\varphi} & N
\end{array}
\quad
\begin{array}{ccc}
(\varphi^*E)_x & = & E_{\varphi(x)} \\
\uparrow & & \\
x & \in & M
\end{array}$$

i.e. if  $s \in \Gamma(E)$ ,

$$\varphi^*s = s \circ \varphi \in \Gamma(\varphi^*E)$$

Thus,

$$\begin{array}{ccc}
\gamma^*E & \xleftarrow{\gamma^*} & E \\
\downarrow & & \downarrow \pi \\
[a, b] & \xrightarrow{\gamma} & M
\end{array}
\quad
\begin{array}{ccc}
(\gamma^*E)_c & = & E_{\gamma(c)} \\
\uparrow & & \\
c & \in & [a, b]
\end{array}$$

For

$$\begin{aligned}
\dot{v}^k &= -v^i \dot{s}^j \Gamma_{ij}^k \\
v^k(c) &= v_0^k \quad 1 \leq k \leq m \\
\dot{v} &= -v^i \dot{s}^j \Gamma_{ij}
\end{aligned}$$

$$(v + w) = -(v^i + w^i) \dot{s}^j \Gamma_{ij}(v + w)(c) = v(c) + w(c) = v_0 + w_0$$

so  $v + w \in \mathfrak{X}(s)$  is parallel transport of  $v_0 + w_0$ .

Likewise,  $\forall a \in \mathbb{F}$ ,  $av \in \mathfrak{X}(s)$  is the parallel transport of  $av_0$ .

$$\dot{\mu}^k = -\mu^i \dot{s}^\mu \omega_{i\mu}^k = -\mu^i \omega_i^k(\dot{s}^\mu)$$

Suppose  $\gamma^*E$  trivialized over  $[a, b]$ .

Closed interval is contractible, so this is always possible.

For chart  $(U, \varphi)$ ,

$$\begin{array}{ccc}
\gamma^*E & \xleftarrow{\gamma^*} & E \\
\downarrow & & \downarrow \pi \\
[a, b] & \xrightarrow{\gamma} & M
\end{array}
\quad
\begin{array}{ccc}
E|_U & \xrightarrow{\psi} & U \times V \\
\pi^{-1} \uparrow & \nearrow & \\
U \subset M & & 
\end{array}$$

Consider

$$\begin{aligned}
\varphi : [a, b] \times V &\rightarrow \gamma^*E \\
\varphi(t, \cdot) &= \gamma^* \circ \psi^{-1}(\gamma(t), \cdot)
\end{aligned}$$

$$\forall \mu \in \Gamma(E|_{x \in M}),$$

$$\mu = \mu^i e_i.$$

$$\varphi(t, e_i) = \epsilon_i \text{ is a basis for } \gamma^*E.$$

$$\forall \sigma \in \Gamma(\gamma^*E),$$

$$\sigma = \sigma^i \epsilon_i, \quad \sigma^i : [a, b] \rightarrow \mathbb{F}$$

$$\nabla_{\frac{\partial}{\partial x^\mu}} \sigma = \frac{\partial \sigma^k}{\partial x^\mu} \epsilon_k + \omega_{j\mu}^k \sigma^j \epsilon_k = \left( \frac{\partial \sigma^k}{\partial x^\mu} + \omega_{j\mu}^k \sigma^j \right) \epsilon_k$$

$$\nabla \sigma = \epsilon_k \otimes (d\sigma^k + \omega_{j\mu}^k dx^\mu \sigma^j) = \epsilon_k \otimes (d\sigma^k + \omega_j^k \sigma^j)$$

$$\nabla_{\frac{d}{dt}} \sigma = \epsilon_k \otimes \left( \frac{d\sigma^k}{dt} + \omega_{j\mu}^k \dot{x}^\mu \sigma^j \right)$$

Now

$$\frac{d}{dt} = \dot{x}^\nu \frac{\partial}{\partial x^\nu}$$

Then  $\sigma$  parallel along  $\gamma$  if

$$\frac{d\sigma^k}{dt} + \omega_{j\mu}^k \dot{x}^\mu \sigma^j = 0$$

**Definition 43** (3.1.4 [18]). *Parallel transport along  $\gamma$  is*

$$\begin{aligned}
(51) \quad &P_\gamma : E_{\gamma(a)} \rightarrow E_{\gamma(b)} \\
&P_\gamma(v) \mapsto \sigma(b)
\end{aligned}$$

where  $\sigma \in \Gamma(\gamma^*E)$ ,  $\sigma$  unique and s.t.  $\sigma(a) = v$ .

**Lemma 1** (10.1.16[17]). *holonomy*

$$h_s : T_x M \rightarrow T_{x_0} M$$

if  $\nabla$  around piecewise smooth loop  $s$  is a linear transformation.

**Lemma 2** (10.1.18 Conlon (2008)[17]). *Let piecewise smooth loop  $s : [a, b] \rightarrow M$  at  $x_0$ .*

*Let weak reparametrization  $\tilde{s} = s \circ r : [c, d] \rightarrow M$ .*

*If reparametrization is orientation-preserving, then  $h_{\tilde{s}} = h_s$ ,*

*If reparametrization is orientation-reversing, then  $h_{\tilde{s}} = h_s^{-1}$ ,*

*Proof.* Without loss of generality, assume smooth  $s, r$

$$\tilde{s}(\tau) = s(r(\tau))$$

$$\tilde{v}(\tau) = v(r(\tau))$$

$$\tilde{u}^j(\tau) = \frac{dt}{d\tau}(\tau) u^j(r(\tau))$$

$$\frac{d\tilde{v}^k}{d\tau}(\tau) = \frac{dr}{d\tau}(\tau) \frac{dv^k}{dt}(r(\tau))$$

$$\frac{d\tilde{v}^k}{d\tau} = -\tilde{v}^i \tilde{u}^j \Gamma_{ij}^k$$

since

$$\frac{dv^k}{dt} = -v^i u^j \Gamma_{ij}^k; \quad 1 \leq k \leq n$$

$$v^k(c) = a^k; \quad 1 \leq k \leq a$$

$$\frac{dr}{d\tau} \frac{dv^k}{dt} = -v^i \frac{dr}{d\tau} u^j \Gamma_{ij}^k = \frac{d\tilde{v}^k}{d\tau} = -\tilde{v}^i \tilde{u}^j \Gamma_{ij}^k$$

Thus, if  $r(c) = a$ ,  $r(d) = b$

$$h_{\tilde{s}}(v_0) = \tilde{v}(d) = v(b) = h_s(v_0)$$

If  $r(c) = a$ ,  $r(d) = b$ , then

$$\tilde{v}(c) = v(b) = h_s(v_0)$$

and

$$h_{\tilde{s}}(h_s(v_0)) = h_{\tilde{s}}(v(b)) = \tilde{v}(d) = v(a) = v_0$$

At this point, I will switch to my notation because it clarified to me, at least, what was going on, in that a holonomy  $h_s$  is *invariant* under orientation-preserving reparametrization, and its inverse is well-defined.

For  $\tilde{s} = s \circ t : [c, d] \rightarrow M$ ,  
piecewise smooth  $t$  is reparametrized, i.e.

$$(52) \quad t : [c, d] \rightarrow [a, b]$$

Now,

$$\begin{aligned} \frac{d}{d\tau} \tilde{s}(\tau) &= \frac{d}{d\tau} \tilde{s}(t(\tau)) = \dot{s}(t) \frac{dt}{d\tau}(\tau) \equiv \dot{s} \frac{dt}{d\tau} \\ v^k(t) &= v^k(t(\tau)) = v^k(\tau) \\ \frac{dv^k}{d\tau}(t(\tau)) &= \frac{dv^k}{dt} \frac{dt}{d\tau} = \frac{dt}{d\tau} (-v^i(\tau) \dot{s}^j(t) \Gamma_{ij}^k) = -v^i(\tau) \frac{d\tilde{s}^j}{d\tau} \Gamma_{ij}^k \end{aligned}$$

Consider

$$h_s(v_0) = v(b)$$

$$\begin{aligned} \text{If } t(c) &= a, \\ t(d) &= b \end{aligned}$$

$$h_{\tilde{s}}(v_0) = \tilde{v}(d) = v(t(d)) = v(b) = h_s(v_0)$$

$$\begin{aligned} \text{If } t(c) &= b, \\ t(d) &= a \end{aligned}$$

$$\begin{aligned} h_{\tilde{s}}(h_s(v_0)) &= h_{\tilde{s}}(v(b)) = h_{\tilde{s}}(v(t(c))) = h_{\tilde{s}}(\tilde{v}(c)) = \\ &= \tilde{v}(d) = v(t(d)) = v(a) = v_0 \end{aligned}$$

Thus,

$$\boxed{h_{\tilde{s}} = h_s^{-1}}$$

□

I am working through Conlon (2008) [17], Clarke and Santoro (2012) [18], and Schreiber and Waldorf (2007)[19], concurrently, for holonomy.

### 13. PATH GROUPOID OF A SMOOTH MANIFOLD; GENERALIZATION OF PATHS

cf. Schreiber and Waldorf (2007)[19].

**Definition 44** (path). ***path** is a smooth map  $\gamma : [0, 1] \rightarrow M$ , between 2 pts.  $x, y \in M$ , which has a sitting instant; i.e. number  $0 < \epsilon < \frac{1}{2}$  s.t.*

$$(53) \quad \gamma(t) = \begin{cases} x & \text{for } 0 \leq t < \epsilon \\ y & \text{for } 1 - \epsilon < t \leq 1 \end{cases}$$

Denote the set of such paths by  $PM$ ,

$$(54) \quad PM \equiv \{\gamma \in \Gamma(M) \mid \text{smooth } \gamma : [0, 1] \rightarrow M \text{ s.t. } \exists 0 < \epsilon < \frac{1}{2} \text{ s.t. } \begin{cases} x & \text{for } 0 \leq t < \epsilon \\ y & \text{for } 1 - \epsilon < t \leq 1 \end{cases}\}$$

cf. Def. 2.1. of Schreiber and Waldorf (2007)[19]

Define *composition*:

Given paths  $\gamma_1, \gamma_2$ ;  $\gamma_1(0) = x$ ,  $\gamma_2(0) = y$ ,

$$\gamma_1(1) = y \quad \gamma_2(1) = z$$

define composition to be path

$$(55) \quad \begin{aligned} &\gamma_2 \circ \gamma_1 \\ (\gamma_2 \circ \gamma_1)(t) &:= \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases} \end{aligned}$$

$\gamma_2 \circ \gamma_1$  smooth since  $\gamma_1, \gamma_2$  both constant near gluing pt., due to sitting instants  $\epsilon_1, \epsilon_2$ , respectively.

Define *inverse*:

$$(56) \quad \begin{aligned} &\gamma^{-1} : [0, 1] \rightarrow M \\ \gamma^{-1}(t) &:= \gamma(1 - t) \end{aligned}$$

$$\text{(so that } \gamma^1(t) = \begin{cases} y & \text{for } 1 - \epsilon < 1 - t \leq 1 \text{ or } 0 \leq t < \epsilon \\ x & \text{for } 0 \leq 1 - t < \epsilon \text{ or } 1 - \epsilon < t \leq 1 \end{cases})$$

**Definition 45** (thin homotopy equivalent). *2 paths  $\gamma_1, \gamma_2$  s.t.  $\gamma_1(0) = \gamma_2(0) = x$ ,  $\gamma_1, \gamma_2$  are thin homotopy equivalent,*

$$\gamma_1(1) = \gamma_2(1) = y$$

*if  $\exists$  smooth  $h : [0, 1] \times [0, 1] \rightarrow M$  s.t.*

$$(1) \quad \exists 0 < \epsilon < \frac{1}{2} \text{ with}$$

$$(a) \quad h(s, t) = x \text{ for } 0 \leq t < \epsilon$$

$$h(s, t) = y \text{ for } 1 - \epsilon < t \leq 1$$

$$(b)$$

$$(c) \quad h(s, t) = \gamma_1(t) \text{ for } 0 \leq s < \epsilon$$

$$h(s, t) = \gamma_2(t) \text{ for } 1 - \epsilon < s \leq 1$$

$$(2) \quad \text{differential of } h \text{ has at most rank 1 everywhere, i.e.}$$

$$(57) \quad \text{rank}(dh|_{(s,t)}) \leq 1 \quad \forall (s, t) \in [0, 1] \times [0, 1]$$

cf. Def. 2.2. of Schreiber and Waldorf (2007)[19]

$h(s, t) = \gamma_1(t)$  for  $0 \leq s < \epsilon$  is the homotopy from  $\gamma_1$  to  $\gamma_2$ , i.e.  $h(0, t) = \gamma_1(t)$

$h(s, t) = \gamma_2(t)$  for  $1 - \epsilon < s \leq 1$   $h(1, t) = \gamma_2(t)$

and define an equivalence relation on  $PM$ .

Note that for  $h : [0, 1] \times [0, 1] \rightarrow M$ ,

$$(Dh)|_{(s,t)} = \left[ \frac{\partial h^i}{\partial s}, \frac{\partial h^i}{\partial t} \right]$$

$$(58) \quad P^1M \equiv \text{set of thin homotopy classes of paths, i.e.}$$

$$P^1M = \{[\gamma] \mid \gamma_1 \in PM, \text{ if } \exists \text{ smooth } h : [0, 1] \times [0, 1] \rightarrow M \text{ s.t. } h \text{ thin homotopy of } \gamma_1 \text{ and } \gamma_2, \gamma_1 \sim \gamma_2\}$$

$\text{pr} : PM \rightarrow P^1M$  is projection to classes.

Denote thin homotopy class of path  $\gamma$ ,  $\gamma(0) = x$ , by  $\overline{\gamma}$ , or  $[\gamma]$ .

$$\gamma(1) = y$$

**13.1. Reparametrization of thin homotopies.** Let  $\beta : [0, 1] \rightarrow [0, 1]$ ,  $\beta(0) = 0$ ,  
 $\beta(1) = 1$

Then  $\forall$  path  $\gamma$ ,  $\gamma(0) = x$ ,  $\gamma \circ \beta$  is also a path  $\gamma \circ \beta(0) = x$  and

$$\gamma(1) = y$$

$$\gamma \circ \beta(1) = y$$

$$(59) \quad h(s, t) := \gamma(t\beta(1 - s) + \beta(t)\beta(s)$$

defines a homotopy from  $\gamma$  to  $\gamma \circ \beta$ .

$$\gamma_1 \circ \gamma_2 \in PM \xrightarrow{\text{pr}} [\gamma_1 \circ \gamma_2] = [\gamma_1][\gamma_2] \in P^1M$$

Composition of thin homotopy classes of paths obeys following rules:

**Lemma 3.**  $\forall$  path  $\gamma$ ,  $\gamma(0) = x$

$$\gamma(1) = y$$

$$(1) \quad \overline{\gamma \circ id_x} = \overline{\gamma} = \overline{id_y} \circ \overline{\gamma} \equiv [\gamma]1_x = [\gamma] = 1_y[\gamma]$$

$$(2) \quad \text{for paths } \gamma'; \gamma'(0) = y, \quad \gamma''(0) = z$$

$$\gamma'(1) = z \quad \gamma''(1) = w$$

$$(60) \quad (\overline{\gamma''} \circ \overline{\gamma'}) \circ \overline{\gamma} = \overline{\gamma''} \circ (\overline{\gamma'} \circ \overline{\gamma}) \equiv ([\gamma''][\gamma'])[\gamma] = [\gamma'']([\gamma'][\gamma])$$

$$(3) \quad \overline{\gamma} \circ \overline{\gamma}^{-1} = \overline{id_y} \text{ and } \overline{\gamma}^{-1} \circ \overline{\gamma} = \overline{id_x} \equiv [\gamma][\gamma^{-1}] = 1_y \text{ and } [\gamma^{-1}][\gamma] = 1_x$$

cf. Lemma 2.3. of Schreiber and Waldorf (2007)[19]

**Definition 46** (path groupoid).  $\forall$  smooth manifold  $M$ ,

consider category whose set of objects is  $M$ ,

whose set of morphisms is  $P^1M$ , where class  $[\gamma]$ ,  $[\gamma](0) = x$  is a morphism from  $x$  to  $y$  and

$$[\gamma](1) = y$$

composition  $[\gamma_1][\gamma_2] = [\gamma_1 \circ \gamma_2] \in P^1M$  Lemma 3 are axioms of a category, 3rd. property says  $\forall$  morphism is invertible.

Hence, we've defined a groupoid, called **path groupoid** of  $M$ ,  $\mathcal{P}_1(M)$ .

So

$$\text{Obj}(\mathcal{P}_1(M)) = M$$

$$\text{Mor}(\mathcal{P}_1(M)) = P^1M$$

$\forall$  smooth  $f : M \rightarrow N$ , denote functor  $f_*$

$$(61) \quad f_* : \mathcal{P}_1(M) \rightarrow \mathcal{P}_1(N)$$

with

$$f_*(x) = f(x)$$

$$(f_*)([\gamma]) := [f \circ \gamma]$$

If  $\gamma \sim \gamma'$ , for  $f \circ \gamma$ ,  $f \circ \gamma'$ ,

$$f \circ h(s, t) \text{ with } f \circ h(0, t) = f \circ \gamma(t),$$

$$f \circ h(1, t) = f \circ \gamma'(t)$$

so  $f \circ h$  is a thin homotopy between  $f \circ \gamma$ ,  $f \circ \gamma'$  and so  $[f \circ \gamma]$  well-defined.

## Part 10. Complex Manifolds

EY : 20170123 I don't see many good books on Complex Manifolds for physicists other than Nakahara's. I will supplement this section on Complex Manifolds with external links to the notes of other courses that I found useful to myself.

[Complex Manifolds - Lecture Notes](#) Koppensteiner (2010) [14]

[Lectures on Riemannian Geometry, Part II: Complex Manifolds by Stefan Vandoren](#)

Vandoren (2008) [15]

## Part 11. Jets, Jet bundles, $h$ -principle, $h$ -Prinzipien

cf. Eliashberg and Misahchev (2002) [20]

cf. Ch. 1 Jets and Holonomy, Sec. 1.1 Maps and sections of Eliashberg and Misahchev (2002) [20].

Visualize  $f : \mathbb{R}^n \rightarrow \mathbb{R}^q$  as graph  $\Gamma_f \subset \mathbb{R}^n \times \mathbb{R}^q$ .

Consider this graph as image of  $\mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^q$ , i.e.

$$x \mapsto (x, f(x))$$

$\mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^q$  is called section (by mathematicians),

$$x \mapsto (x, f(x))$$

is called *field* or  $\mathbb{R}^q$ -valued field (by physicists).

cf. Ch. 1 Jets and Holonomy, Sec. 1.2 Coordinate definition of jets of Eliashberg and Misahchev (2002) [20].

**Definition 47** ( $r$ -jet). Given (smooth)  $f : \mathbb{R}^n \rightarrow \mathbb{R}^q$ , given  $x \in \mathbb{R}^n$ .

$r$ -jet of  $f$  at  $x$  - sequence of derivatives of  $f$ , up to order  $r$ ,  $\equiv$

$$(62) \quad J_f^r(x) = (f(x), f'(x) \dots f^{(r)}(x))$$

$f^{(q)}$  consists of all partial derivatives  $D^\alpha f$ ,  $\alpha = (\alpha_1 \dots \alpha_n)$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n = s$ , ordered lexicographically.

e.g.  $q = 1$ ,

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

1-jet of  $f$  at  $x = J_f^1(x) = (f(x), f^{(1)}(x))$ .

$$f^{(1)}(x) = \{D^\alpha f | \alpha = (\alpha_1 \dots \alpha_n), |\alpha| = \alpha_1 + \dots + \alpha_n = 1\} = \left( \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \dots, \frac{\partial f}{\partial x^n} \right)$$

Let  $d_r = d(n, r)$  = number of all partial derivatives  $D^\alpha$  of order  $r$  of function  $\mathbb{R}^n \rightarrow \mathbb{R}$ .

Consider  $r$ -jet  $J_f^r(x)$  of map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^q$  as pt. of space  $\mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \dots \times \mathbb{R}^{qd_r} = \mathbb{R}^{qN_r}$ , where  $N_r = N(n, r) =$

$1 + d_1 + d_2 + \dots + d_r$ , i.e.

$$J_f^r(x) = (f(x), f^{(1)}(x), \dots, f^{(r)}(x)) \in \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \dots \times \mathbb{R}^{qd_r} = \mathbb{R}^{qN_r}$$

### Exercise 1.

Given order  $r$ , consider  $n$ -tuple of (positive) integers  $(r_1, r_2 \dots r_n)$  s.t.  $r_1 + r_2 + \dots + r_n = r$ , and  $r_k \geq 0$ .

Imagine  $r_k$  = occupancy number, num ber of balls in  $k$ th cell.  $(r_1 \dots r_n)$  describes a positive ocnfiguration of occupancy numbers, with indistinguishable balls; 2 distributions are distinguishable only if corresponding  $n$ -tuples  $(r_1 \dots r_n)$  not identical.

Represent balls by stars, and indicate  $n$  cells by  $n$  spaces between  $n + 1$  bars.

With  $n+1$  bars,  $r$  stars, 2 bars are fixed.  $n-1$  bars and  $r$  stars to arrange linearly, so a total of  $n-1+r$  objects to arrange.  $r$  stars indistinguishable amongst themselves, so choose  $r$  out of  $n-1+r$  to be stars.

$$(63) \quad \implies d_r = d(n, r) = \binom{n-1+r}{r}$$

Use *induction* (cf. **Ch. 4 Binomial Coefficients**).

$$N_0 = N(n, 0) = \binom{n-1+0}{0} = 1$$

$$N_1 = N(n, 1) = 1 + \binom{n-1+1}{1} = 1 + n = \frac{(n+1)!}{n!1!}$$

Induction step:

$$N_{r-1} = N(n, r-1) = \sum_{k=1}^{r-1} d_k + 1 = \binom{n+r-1}{r-1}$$

and so

$$N_r = N(n, r) = \sum_{k=1}^r d_k + 1 = \sum_{k=1}^r \binom{n-1+k}{k} + 1 = \sum_{k=1}^{r-1} \binom{n-1+k}{k} + \binom{n-1+r}{r} + 1 =$$

$$= \binom{n+r-1}{r-1} + \binom{n-1+r}{r} = \frac{(n+r-1)!}{(r-1)!n!} + \frac{(n-1+r)!}{r!(n-1)!} = \frac{(n+r)!}{n!r!} = \binom{n+r}{r}$$

$$\begin{array}{ccc} \mathbb{R}^{qN_r} & & J_f^r(x) \\ \uparrow J_f^r & & \uparrow J_f^r \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^q \\ & & \downarrow \\ & & x \end{array} \quad \begin{array}{ccc} & & f \\ & & \downarrow \\ & & f(x) \end{array}$$

**Definition 48** (space of  $r$ -jets). *space of  $r$ -jects of maps  $\mathbb{R}^n \rightarrow \mathbb{R}^q$  or space of  $r$ -jets of sections  $\mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^q \equiv$*

$$(64) \quad J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{qN_r} \equiv \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \dots \times \mathbb{R}^{qd_r}$$

*e.g.  $J^1(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^q \times M_{q \times n}$ , where  $M_{q \times n} = \mathbb{R}^{qn}$  is the space of  $(q \times n)$ -matrices.*

## Part 12. Morse Theory

### 14. MORSE THEORY INTRODUCTION FROM A PHYSICIST

I needed some physical motivation to understand Morse theory, and so I looked at Hori, et. al. [16].  
cf. pp. 43, Sec. 3.4 Morse Theory, from Ch. 3. Differential and Algebraic Topology of Hori, et. al. [16].  
Consider smooth  $f: M \rightarrow \mathbb{R}$ , with non-degenerate critical points.

If no critical values of  $f$  between  $a$  and  $b$  ( $a < b$ ), then subspace on which  $f$  takes values less than  $a$  is deformation retract of subspace where  $f$  less than  $b$ , i.e.

$$\{x \in M | f(x) < b\} \times [0, 1] \xrightarrow{F} \{x \in M | f(x) < b\}$$

$\forall x \in M$  s.t.  $f(x) < b$ ,

$$F(x, 0) = x$$

$$F(x, 1) \in \{x \in M | f(x) < a\} \quad \text{and} \quad F(a', 1) = a' \quad \forall a' \in M \text{ s.t. } f(a') < a$$

To show this, consider  $-\nabla f / |\nabla f|^2$

Morse lemma:  $\forall$  critical pt.  $p$  s.t.  $\exists$  choice of coordinates s.t.

$$(65) \quad f = -(x_1^2 + x_2^2 + \dots + x_\mu^2) + x_{\mu+1}^2 + \dots + x_n^2$$

where  $f(p) = 0$  and  $p$  is at origin of these coordinates.

- difference between

$$f^{-1}(\{x \leq -\epsilon\}), f^{-1}(\{x \leq +\epsilon\})$$

can be determined by local analysis and only depends on  $\mu$ ,  $\mu \equiv$  “Morse index” = number of negative eigenvalues of Hessian of  $f$  at critical pt.

Answer:

$f^{-1}(\{x \leq +\epsilon\})$  can be obtained from  $f^{-1}(\{x \leq -\epsilon\})$  by “attaching  $\mu$ -cell” along boundary  $f^{-1}(0)$

- “attaching  $\mu$ -cell to  $X$  mean, take  $\mu$ -ball  $B_\mu = \{|x| \leq 1\}$  in  $\mu$ -dim. space, identity pts. on boundary  $S^{\mu-1}$  with pts. in the space  $X$ , through cont.  $f: S^{\mu-1} \rightarrow X$ , i.e. take

$$X \coprod B_\mu$$

with  $x \sim f(x) \quad \forall x \in \partial B_\mu = S^{\mu-1}$ .

- find homology of  $M$ ,  
 $f$  defines chain complex  $C_f^*$ ,  $k$ th graded piece  $C^{\alpha_k}$ ,  $\alpha_k$  is number of critical pts. with index  $k$ .

$$(66) \quad \begin{aligned} \partial: C_p^k &\rightarrow C_p^{k-1} \\ \partial x_a &= \sum_b \Delta_{a,b} x_b \end{aligned}$$

where  $\Delta_{a,b} :=$  signed number of lines of gradient flow from  $x_a$  to  $x_b$ ,  $b$  labels pts. of index  $k-1$ .

Gradient flow line is path  $x(t)$  s.t.  $\dot{x} = \nabla(f)$ , with  $x(-\infty) = x_a$

$$x(+\infty) = x_b$$

- To define this number ( $\Delta_{a,b}?$ ), construct moduli space of such lines of flow (???)  
by intersecting outward and inward flowing path spaces from each critical point, and then show this moduli space is oriented, 0-dim. manifold (pts. with signs)
- $\partial^2 = 0$  proof  
 $\partial$ , boundary of space of paths connecting critical points, whose index differs by 2 = union over compositions of paths between critical pts. whose index differs by 1.  
 $\implies$  coefficients of  $\partial^2$  are sums of signs of pts. in 0-dim. space, which is boundary of 1-dim. space.  
These signs must therefore add to 0, so  $\partial^2 = 0$ .

Hori, et. al. [16] is good for physics, but there isn’t much thorough, step-by-step explanations of the math. I will look at Hirsch (1997) [6] and Shastri (2011) [5] at the same time.

**14.1. Introduction, definitions of Morse Functions, for Morse Theory.** cf. Ch. 6, Morse Theory of Hirsch (1997) [6], Section 1. Morse Functions, pp. 143-

Recall for  $TM$ ,  $T_x M \xrightarrow{\varphi} \mathbb{R}^n$ .

Cotangent bundle  $T^*M$  defined likewise:

$$T_x^* M \xrightarrow{\varphi^*} \text{dual vector space } (\mathbb{R}^n)^* = L(\mathbb{R}^n, \mathbb{R})$$

i.e.

$$T^*M = \bigcup_{x \in M} (M_x^*) \quad M_x^* = L(M_x, \mathbb{R})$$

If chart  $(\varphi, U)$  on  $M$ , natural chart on  $T^*M$  is

$$T^*U \rightarrow \varphi(U) \times (\mathbb{R}^n)^*$$

$$\lambda \in M_x^* \mapsto (\varphi(x), \lambda \varphi_x^{-1})$$



Projection map

$$p : T^* \rightarrow M$$
$$M_x^* \mapsto x$$

Let  $C^{r+1}$  map,  $1 \leq r \leq \omega$ ,  $f : M \rightarrow \mathbb{R}$ ,  $\forall x \in M$ , linear map  $T_x f : M_x \rightarrow \mathbb{R}$  belongs to  $M_x^*$

$$T_x f = Df_x \in M_x^*$$

Then

$$Df : M \rightarrow T^* M$$
$$x \mapsto Df_x = Df(x)$$

is  $C^r$  section of  $T^* M$ .

**Definition 49. critical point**  $x$  of  $f$  is zero of  $Df$ , i.e.

$$(67) \qquad Df(x) = 0$$

of vector space  $M_x^*$ .

Thus, set of critical pts. of  $f$  is counter-image of submanifold  $Z^* \subset T^* M$  of zeros.  
Note  $Z^* \approx M$ , codim. of  $Z^*$  is  $n = \dim M$ .

**Definition 50. Morse function**  $f$  if  $\forall$  critical pts. of  $f$  are nondegenerate.

Note set of critical pts. closed discrete subset of  $M$ .

Let open  $U \subset \mathbb{R}^n$ , let  $C^2$  map  $g : U \rightarrow \mathbb{R}$ ,  
critical pt.  $p \in U$  nondegenerate iff

- linear  $D(Dg)(p) : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^*$  bijective
- identify  $L(\mathbb{R}^n, (\mathbb{R}^n)^*)$  with space of bilinear maps  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\implies$  equivalent to condition that symmetric bilinear  $D^2g(p) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  non-degenerate
- $n \times n$  Hessian matrix

$$\left[ \frac{\partial^2 g}{\partial x^i \partial x^j}(p) \right]$$

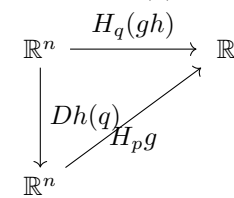
has rank  $n$

Hessian of  $g$  at critical pt.  $p$  is quadratic form  $H_p f$  associated to bilinear form  $D^2g(p)$

$$\implies H_p f(y) = D^2g(p)(y, y) = \sum_{i,j} \frac{\partial^2 g}{\partial x^i \partial x^j}(p) y^i y^j$$

Let open  $V \subset \mathbb{R}^n$ , suppose  $C^2$  diffeomorphism  $h : V \rightarrow U$ .

Let  $q = h^{-1}(p)$ , so  $q$  is critical pt. of  $gh : V \rightarrow \mathbb{R}$ .



(quadratic) form  $(H_p f)$  invariant under diffeomorphisms.

Let  $C^2$   $f : M \rightarrow \mathbb{R}$ .

$\forall$  critical pt.  $x$  of  $f$ , define

Hessian quadratic form

$$H_x f : M_x \rightarrow \mathbb{R}$$

$$H_x f : M_x \xrightarrow{D\varphi_x} \mathbb{R}^n \xrightarrow{H_{\varphi(x)}(f\varphi^{-1})} \mathbb{R}$$

where  $\varphi$  is any chart at  $x$ .

Thus, critical pt. of a  $C^2$  real-valued function nondegenerate iff associated Hessian quadratic form is nondegenerate.

Let  $Q$  nondegenerate quadratic form on vector space  $E$ .

$Q$  negative definite on subspace  $F \subset E$  if  $Q(x) < 0$  whenever  $x \in F$  nonzero.

Index of  $Q \equiv \text{Ind} Q$ , is largest possible dim. of subspace on which  $Q$  is negative definite.

cf. 1.1. Morse's Lemma of Ch. 6, pp. 145, Morse Theory of Hirsch (1997) [6]

**Lemma 4** (Morse's Lemma). *Let  $p \in M$  be nondegenerate critical pt. of index  $k$  of  $C^{r+2}$  map  $f : M \rightarrow \mathbb{R}$ ,  $1 \leq r \leq \omega$ . Then  $\exists C^r$  chart  $(\varphi, U)$  at  $p$  s.t.*

$$(68) \qquad f\varphi^{-1}(u_1 \dots u_n) = f(p) - \sum_{i=1}^k u_i^2 + \sum_{i=k+1}^n u_i^2$$

Let  ${}^T Q \equiv Q^T$  denote tranpose of matrix  $Q$ .

**Lemma 5.** *Let  $A = \text{diag}\{a_1, \dots, a_n\}$  diagonal  $n \times n$  matrix, with diagonal entries  $\pm 1$ . Then  $\exists$  neighborhood  $N$  of  $A$  in vector space of symmetric  $n \times n$  matrices,  $C^\infty$  map*

$$(69) \qquad P : N \rightarrow GL(n, \mathbb{R})$$

s.t.  $P(A) = I$ , and if  $P(B) = Q$ , then  $Q^T B Q = A$

*Proof.* Let  $B = [b_{ij}]$  be symmetri matrix near  $A$  s.t.  $b_{11} \neq 0$  and  $b_{11}$  has same sign as  $a_1$ .

Consider  $x = T y$  where

$$x_1 = \left[ y_1 - \frac{b_{12}}{b_{11}} y_2 - \dots - \frac{b_{1n}}{b_{11}} y_n \right] / \sqrt{|b_n|}$$
$$x_k = y_k \text{ for } k = 2, \dots, n$$

□

15. LAGRANGE MULTIPLIERS

From *wikipedia:Lagrange multiplier*, [https://en.wikipedia.org/wiki/Lagrange\\_multiplier](https://en.wikipedia.org/wiki/Lagrange_multiplier), find local minima (maxima), pt.  $a \in N$ , s.t.  $\exists$  neighborhood  $U$  s.t.  $f(x) \geq f(a)$  ( $f(x) \leq f(a)$ )  $\forall x \in U$ .

For  $f : U \rightarrow \mathbb{R}$ , open  $U \subset \mathbb{R}^n$ , find  $x \in U$  s.t.  $D_x f \equiv Df(x) = 0$ , check if Hessian  $H_x f < 0$ .

Maxima may not exit since  $U$  open.

References:

[Relative Extrema and Lagrange Multipliers](#)

Other interesting links:

[The Lagrange Multiplier Rule on Manifolds and Optimal Control of nonlinear systems](#)

Part 13. Classical Mechanics

**Problem 1.4 The altitude angle of the Sun, Japan Physics Olympiad (2014).** cf. Japan Physics Olympiad, (2014) [37].

**Solution 1.4 The altitude angle of the Sun, Japan Physics Olympiad (2014).**

$NC \equiv$  Niigata City,

$AM \equiv$  Amagi-san in Izu.



$A \in \text{Mat}_{\mathbb{R}}(3,3)$ ,  $\mathbf{v}, \mathbf{a}, \mathbf{l} \in \mathbb{R}^3$ ,  $k, s \in \mathbb{R}$ .

By galilean transformation (Def. 52),  $t_2 - t_1 = t'_2 - t'_1$ , so

$$t_2 - t_1 = \langle \mathbf{l}, \mathbf{x}'_2 \rangle + kt'_2 + s - \langle \mathbf{l}, \mathbf{x}'_1 \rangle - kt'_1 - s = \langle \mathbf{l}, \mathbf{x}'_2 - \mathbf{x}'_1 \rangle + k(t'_2 - t'_1) = t'_2 - t'_1$$

Since each of the 3 components of  $\mathbf{x}_2 - \mathbf{x}_1$  are arbitrary,  $\mathbf{l} = 0$ ,  $k = 1$  clearly. Or, let  $t = 2 = t_1$ . Then clearly  $\mathbf{l} = 0$ .

By galilean transformation (Def. 52),  $|\mathbf{x}_2 - \mathbf{x}_1| = |\mathbf{x}'_2 - \mathbf{x}'_1|$

Let  $t_2 = t_1$ .

$$|\mathbf{x}_2 - \mathbf{x}_1| = |A\mathbf{x}'_2 + \mathbf{v}t'_2 + \mathbf{a} - A\mathbf{x}'_1 - \mathbf{v}t'_1 - \mathbf{a}| = |A(\mathbf{x}'_2 - \mathbf{x}'_1) + \mathbf{v}(t'_2 - t'_1)| = |A(\mathbf{x}'_2 - \mathbf{x}'_1)| = |\mathbf{x}'_2 - \mathbf{x}'_1|$$

Now let  $\mathbf{y} = \mathbf{x}'_2 - \mathbf{x}'_1$ .

$$|A\mathbf{y}|^2 = y^T A^T A y = y^T y$$

only if  $A^T A = 1$ . So  $A$  orthonormal.

Thus we've found the *general form of Galilean transformations*:

$$(75) \quad \boxed{\begin{aligned} \mathbf{x} &= A\mathbf{x}' + \mathbf{v}t' + \mathbf{a}, & A \in O(3), \mathbf{v}, \mathbf{a} \in \mathbb{R}^3, s \in \mathbb{R} \\ t &= t' + s \end{aligned}}$$

Since orthogonal matrices form a 3 parameter family,  $\dim A + \dim \mathbf{v} + \dim \mathbf{a} + \dim s = 3 + 3 + 3 + 1 = 10$  independent parameters. Recall that  $g_1(\mathbf{r}, t) = (\mathbf{r} + \mathbf{v}t, t)$ ,  $g_2(\mathbf{r}, t) = (\mathbf{r} + \mathbf{a}, t + s)$ ,  $g_3(\mathbf{r}, t) = (A\mathbf{r}, t)$ , then  $g = g_1 g_2 g_3$ .

□

Introduce  $E^3$  a "fixed" reference frame by fixing pt.  $o \in E^3$  and choosing 3 mutually perpendicular axes.

*inertial frame* - frame moving uniformly and rectilinearly with respect to fixed pt.  $o \in E^3$ ;  $\forall$  element of Galilean group transforms fixed frame into inertial frame.

action of galilean group on  $E^3 \times \mathbb{R}$  can be extended to action on  $E^3 \times \cdots \times E^3 \times \mathbb{R}$  by the rule:

if  $g : (s, t) \rightarrow (s', t')$ , then

$$g(s_1, \dots, s_n, t) = g(s'_1, \dots, s'_n, t')$$

Galileo-Newton principle of relativity asserts that Newton's equations invariant under Galilean transformation group in inertial reference frame.

$M = 4$ -dim. affine manifold (spacetime). Recall definition of an *affine manifold* from [wikipedia](#).

Affine manifold  $M$  is real manifold with charts  $\psi_i : U_i \rightarrow \mathbb{R}^n$ , s.t.  $\psi_i \circ \psi_j^{-1} \in \text{Aff}(\mathbb{R}^n) \quad \forall i, j$  where  $\text{Aff}(\mathbb{R}^n)$  denotes Lie group of affine transformations;

Recall the definition of a *Lie group of affine transformations*,  $\text{Aff}(\mathbb{R}^n)$ , the "affine structure", <sup>5</sup>,

**Definition 53** (Lie group of affine transformations, affine structure).

$$(76) \quad \text{Aff}(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL(n, \mathbb{R}), b \in \mathbb{R}^n \right\}$$

s.t.  $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$  acts on real affine space  $\widetilde{\mathbb{R}^n}$  by

$$(77) \quad \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Av + b \\ 1 \end{pmatrix}$$

where  $A \in GL(n, \mathbb{R})$ ,  $b \in \mathbb{R}^n$ , where  $(v, 1)^T \in \widetilde{\mathbb{R}^n}$

<sup>5</sup>"Affine structures on Lie algebras" by Michel Goze and Elisabeth Remm

If changing notation, and keeping in mind that  $n = 4$ ,  $\psi_j \equiv y(v_{(y)}) \rightarrow \mathbb{R}^n$ ,

$$\psi_i \equiv x : U_{(x)} \rightarrow \mathbb{R}^n$$

$$x(p) \in \mathbb{R}^n, x_i(p) \in \mathbb{R}$$

Rewrite the Eq. 75 in matrix form:

$$\mathbf{x} = A\mathbf{x}' + \mathbf{v}t + \mathbf{b}$$

$$t = t' + s, A \in O(3), \mathbf{v}, \mathbf{b} \in \mathbb{R}^3, s \in \mathbb{R}$$

$$\begin{pmatrix} A & \mathbf{v} & \mathbf{b} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}' \\ t' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & \mathbf{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}' \\ t' \end{pmatrix} + \begin{pmatrix} \mathbf{b} \\ s \end{pmatrix} = \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix}$$

Consider  $y$  being standard Euclidean coordinates in fixed frame  $o$ .  $x$  then is coordiantes in any other inertial frame.

$T = 1$ -dim. affine space since affine structure  $(T, \mathbf{T}, \beta)$  is s.t.  $T \in \mathbb{R}$ ,  $\mathbf{T} \in \mathbb{R}$ ,  $\beta : \mathbf{T} \times T \rightarrow T$

$$\beta(s, t') = t' + s = t$$

Holm (2011) [21]

## 17. FUNDAMENTAL THEOREMS OF (CLASSICAL) DYNAMICS

cf. Sec. 3.4 - *Fundamental Theorems of Dynamics* of Prástaro (1996) [13].

cf. Thm. 3.20 of Prástaro (1996) [13]

**Theorem 22** (Momentum Theorem). *Variation of the free part of momentum of the observed motion of 1 body, in time interval  $\Delta t \equiv [0, t]$  is equal to the corresponding impulse:*

$$(78) \quad I[0, t] \equiv \int_0^t F dt \equiv \left( \int_0^t F^j dt \right) \mathbf{e}_j$$

where  $\{\mathbf{e}_j\}_{1 \leq j \leq 3}$  is a fixed basis of  $\mathbf{S}$

$$(79) \quad \bar{p}_\psi(t) - \bar{p}_\psi(0) = I[0, t]$$

*Proof.*

$$\bar{p}_\psi = \mu \ddot{n}_\psi = \bar{f}_\psi \implies \dot{\bar{p}}_\psi = \dot{p}^j \mathbf{e}_j = F^j \mathbf{e}_j \implies \int_{[0, t]} \dot{p}^j dt = \int_{[0, t]} F^j dt$$

□

17.1. **Coordinate transformation.** For  $(U, \varphi), (V, \psi)$  charts on  $M$ , s.t.  $\varphi : U \rightarrow \mathbb{R}^n$ ,  $\varphi^i \equiv x^i$ , then consider

$$\psi : V \rightarrow \mathbb{R}^n \quad \psi^j \equiv y^j$$

$$\psi(p) = y = \psi \circ \varphi^{-1} \circ \varphi(p) = \psi \circ \varphi^{-1}(x) = y(x)$$

Consider  $f : M \rightarrow M$  that is a coordinate transformation (bijective). Under this coordinate transformation, transformation of the components of the same vector  $X = Y$  is the push-forward:

$$(80) \quad Y^j = X^i \frac{\partial y^j}{\partial x^i}$$

Correspondingly, basis transformation described by

$$(81) \quad \frac{\partial}{\partial x^j} = \frac{\partial y^k}{\partial x^j} \frac{\partial}{\partial y^k} \text{ or } \frac{\partial}{\partial y^k} = \left[ \left( \frac{\partial y}{\partial x} \right)^{-1} \right]_k^j \frac{\partial}{\partial x^j} = \left( \frac{\partial x^j}{\partial y^k} \right) \frac{\partial}{\partial x^j}$$

Then we can show that indeed  $Y = X$  under coordinate transformations:

$$(82) \quad Y = X = Y^k \frac{\partial}{\partial y^k} = X^j \frac{\partial}{\partial x^j}$$

Also, for  $a_i = b_j \frac{\partial y^j}{\partial x^i}$ , the pull back describing how 1-form components transform, consider the inner product:

$$(83) \quad b(Y) = b_i dy^i \left( Y^k \frac{\partial}{\partial y^k} \right) = b_i Y^i = b_i X^k \frac{\partial y^i}{\partial x^k} = b_i \frac{\partial y^i}{\partial x^k} X^k = a_k X^k = a(X)$$

$\implies$  inner product preserved under coordinate transformation, i.e. invariance of inner product.

17.2. **Transformation of reference frames.** cf. Sec. 4.2.1 "Transformation of reference frames", Kambe (2009) [22].

Let us take pt.  $\mathbf{y}_b$  fixed to body, located at  $\mathbf{x} = X^1 \mathbf{e}_1 + X^2 \mathbf{e}_2 + X^3 \mathbf{e}_3 = X^i \mathbf{e}_i$  at  $t = 0$ , where  $\mathbf{e}_i$  ( $i = 1, 2, 3$ ) are orthonormal basis fixed to space  $F \equiv$  nonrotating **fixed** space (inertial space)

By transformation matrix  $A = (A^i_j) \in \text{SO}(3)$ , initial pt.  $\mathbf{x} = (X^i)$  mapped to current pt.  $\mathbf{y}_b$  at time  $t$ .

$$(84) \quad \mathbf{y}_b(t) = A(t)\mathbf{x} = y^i(t)\mathbf{e}_i, \quad y^i(t) = A^i_j(t)X^j$$

cf. Eq. (4.14) of Kambe (2009) [22].  
i.e.

$$\mathbf{y}_b(t) = A(t)\mathbf{x} = A^i_k e_i \otimes e^k x^l e_l = A^i_k X^k e_i = y^i(t) e_i$$

EY: Observe that  $A(t)$ ,  $t \in [0, t]$  is a curve in  $\text{SO}(3)$ .

In other words, we considered a point in the fixed frame  $F$  given by frame  $\{\mathbf{e}_i\}$ :

$$\mathbf{x} = X^i \mathbf{e}_i \text{ at } t = 0$$

Then, we considered an "active" rotation  $A \in \text{SO}(3)$  of this point, so

$$(85) \quad \mathbf{y}(t) = A(t)\mathbf{x} = A^i_j(t)X^j \mathbf{e}_i = y^i(t)\mathbf{e}_i$$

Indeed, consider  $A = \begin{bmatrix} c & -s \\ s & c \\ & & 1 \end{bmatrix}$ . Then  $A$  would rotate points in the  $x - y$  plane, such as  $[1, 0, 0]$ ,  $[0, 1, 0]$  counter-clockwise in the positive  $z$  axis direction.

On the other hand, relative to body frame  $F_B$ , which is the frame instantaneously fixed to the moving body. For the same pt.,  $\mathbf{y}_b(t)$ , fixed to body, is

$$\mathbf{Y}(= \mathbf{y}_b) = Y^1 \mathbf{b}_1 + Y^2 \mathbf{b}_2 + Y^3 \mathbf{b}_3 = Y^i \mathbf{b}_i$$

where  $\mathbf{b}_i$  ( $i = 1, 2, 3$ ) are orthonormal basis fixed to body which coincided with  $\mathbf{e}_i$  ( $i = 1, 2, 3$ ) at  $t = 0$ .

From property of a rigid body, it's required that  $Y^i = X^i$  which don't change with  $t$ .

According to Sec. 1.5.1(b) (Kambe (2009) [22]), basis transformation written as (1.40) Kambe(2009) [22], where

$$(86) \quad \mathbf{b}_i = \mathbf{b}_i(t) = \mathbf{e}_j B^j_i(t) \\ \text{Then } \mathbf{Y} = Y^i \mathbf{b}_i = Y^i \mathbf{e}_j B^j_i = \mathbf{e}_j B^j_i Y^i$$

(4.10) Kambe (2009) [22], where  $B = (B_k^j) \in \text{SO}(3)$ , i.e.  $BB^T = 1$ .

Note that

$$(87) \quad \mathbf{Y} = Y^i \mathbf{b}_i = \mathbf{e}_j B^j_i Y^i = A^j_k X^k \mathbf{e}_j$$

implies that  $B : F' \rightarrow F$  where  $F'$  rotating frame,  $F$  fixed frame, i.e.  $B$  is a coordinate transformation from coordinates in the  $F'$  rotating frame into coordinates in the  $F$  fixed frame.

Indeed, consider

$$B = \begin{bmatrix} c & -s \\ s & c \\ & & 1 \end{bmatrix}. \text{ Then } B \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [c, s, 0] (\text{in } \{\mathbf{e}_j\} \text{ frame})$$

Now recall that  $\mathbf{x} = X^i \mathbf{e}_i$  at  $t = 0$ .  $\mathbf{Y} = Y^i \mathbf{b}_i$  and  $\{\mathbf{b}_i\}$  frame axes had coincided with  $\{\mathbf{e}_i\}$  frame axes at  $t = 0$ .  $Y^i$ 's are constant, so  $Y^i = X^i$

So for

$$(88) \quad \mathbf{Y} = Y^i \mathbf{b}_i = \mathbf{e}_j B^j_i Y^i = A^j_k(t) X^k \mathbf{e}_j \implies B^j_i Y^i = A^j_k(t) X^k = A^j_i Y^i \text{ so } \boxed{B = A} \text{ since } X^i \text{ arbitrary}$$

For the basis transformation derivation, recall the push forward transformation (cf. Kambe (2009) [22]),

$$\begin{aligned} \phi : M &\rightarrow N \\ \phi_* : T_x M &\rightarrow T_y N \\ y = \phi(x) &= F(x) \end{aligned}$$

$$Y = \phi_* X = F_* X = \left. \frac{d}{dt} (F(p(t))) \right|_{t=0}$$

$$Y^k = (\phi_* X)^k = \left( \frac{\partial F^k}{\partial x^j} \right) X^j$$

$$Y = \phi_* X = \phi_* \left[ X^j \frac{\partial}{\partial x^j} \right] = X^j \phi_* \left[ \frac{\partial}{\partial x^j} \right] = X^j \frac{\partial y^k}{\partial x^j} \frac{\partial}{\partial y^k} = Y^k \frac{\partial}{\partial y^k}$$

$$Y^k = \frac{\partial y^k}{\partial x^j} X^j$$

17.2.1. *Direction Cosine Matrix (DCM)*. cf. pp. 134, 4.1 "The Independent Coordinates of a Rigid Body", Goldstein, Poole, Safko (2001) [24]

Consider frame  $F$ ,  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ ,  
frame  $F'$ ,  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$

If

$$\mathbf{e}'_i = C_{ij}\mathbf{e}_j$$

and so then

$$\mathbf{e}'_i \cdot \mathbf{e}_k = C_{ij}\delta_{jk} = C_{ik}$$

$$\implies \mathbf{e}'_i = (\mathbf{e}'_i \cdot \mathbf{e}_j)\mathbf{e}_j$$

So for  $\mathbf{r}' = y^i \mathbf{e}'_i = x^j \mathbf{e}_j$ ,

$$(89) \quad \implies y^k = x^j (\mathbf{e}'_k \cdot \mathbf{e}_j) \text{ or } y^i = (\mathbf{e}'_i \cdot \mathbf{e}_j)x^j$$

So  $\cos \theta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$  completely specifies the orientation of  $F'$  relative to  $F$ .

Now use this fact, that  $\mathbf{e}_m \cdot \mathbf{e}'_m = \delta_{mm'}$ , to show that

$$\mathbf{e}_m \cdot \mathbf{e}'_m = \delta_{mm'} = (\mathbf{e}_m \cdot \mathbf{e}'_l)\mathbf{e}'_l \cdot (\mathbf{e}_{m'} \cdot \mathbf{e}'_{l'})\mathbf{e}'_{l'} = (\mathbf{e}_m \cdot \mathbf{e}'_l)(\mathbf{e}_{m'} \cdot \mathbf{e}'_{l'})\delta_{ll'} = (\mathbf{e}'_l \cdot \mathbf{e}_m)(\mathbf{e}'_l \cdot \mathbf{e}_{m'}) = \cos \theta_{lm} \cos \theta_{lm'}$$

So  $\cos \theta_{ij}$  are 9 coordinates that can't be generalized coordinates.

Be careful about the interpretation of the DCM. Note that we had derived the following from Eq. 89:

$$(90) \quad \boxed{y^i = C_{ij}x^j} \text{ with } \mathbf{e}'_i = C_{ij}\mathbf{e}_j \text{ and } \boxed{C_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j}$$

So this is a transformation of the same vector between two different frames, not an "active" rotation of the vector itself. Nevertheless,  $C_{ij} : F \rightarrow F'$ , a mapping from frame  $F$  to  $F'$ .

For example, consider

$$\mathbf{e}'_x = \cos(\omega t)\mathbf{e}_x - \sin(\omega t)\mathbf{e}_y$$

$$\mathbf{e}'_y = \sin(\omega t)\mathbf{e}_x + \cos(\omega t)\mathbf{e}_y$$

This isn't an active rotation in the negative z, clockwise direction, but this is the same vector with coordinates in a frame that's rotating in the counterclockwise, positive z direction.

17.3. **Euler Angles.** cf. pp. 150, 4.4 "Euler Angles", Goldstein, Poole, Safko (2001) [24]

rotation 1: about vertical axis, gives **heading** or **yaw** angle

rotation 2: around perpendicular axis fixed in vehicle and normal to figure axis; it's measured by the *pitch* or *attitude* angle

rotation 3: about figure axis of vehicle, *roll* or *bank* angle.

$\implies$  xyz - convention, or *Tait-Bryan* angles.

**Solution Exercise 24, Ch. 4 Goldstein, Poole, Safko (2001) [24].**

Recall that

$$\mathbf{a}_r = \mathbf{a}_i - 2\omega \times \mathbf{v}_r - \omega \times (\omega \times \mathbf{r}) - \dot{\omega} \times \mathbf{r}$$

Let  $\dot{\omega} = 0$ .

Then consider, multiplying by  $m$ ,

$$m\mathbf{a}_r = m\mathbf{a}_i - 2m\omega \times \mathbf{v}_r - m\omega \times (\omega \times \mathbf{r}) \text{ or}$$

$$\mathbf{F}_r = \mathbf{F}_i - 2m\omega \times \mathbf{v}_r - m\omega \times (\omega \times \mathbf{r})$$

where  $\mathbf{F}_r$  is the apparent force experienced by the bug in the rotating frame, and  $\mathbf{F}_i$  are the forces in the inertial reference frame.

By the geometry of the wheel and the problem, and taking the instantaneous coordinate axis  $\mathbf{e}_r$  to be same for the rotating and inertial frames,

$$-\omega \times (\omega \times \mathbf{r}) = \omega^2 R \mathbf{e}_r$$

$$-2\omega \times \mathbf{v}_r = 2\omega v_r (- + \mathbf{e}_y)$$

where the direction of the last expression for the Coriolis force is negative when the apparent velocity is in the positive radial direction and vice versa.

Suppose that the frictional force acts in the opposite direction of the force. In the rotating frame, consider the sum of the 2 fictitious forces, the Coriolis force and centripetal force, which happen to be orthogonal to each other:

$$\|\mathbf{F}_{\text{fic}}\| = \|(-2\omega \times \mathbf{v}_r + \omega \times (\omega \times \mathbf{r}))m\| = m\sqrt{(2\omega v_r)^2 + (\omega^2 R)^2} = m\omega\sqrt{(2v_r)^2 + (\omega R)^2}$$

When  $F_{\text{fric}} \geq F_{\text{fic}}$ , then

$$(\mu g)^2 \geq \omega^2(4v_r^2 + \omega^2 R^2) \text{ or } \left(\frac{\mu g}{\omega}\right)^2 \geq 4v_r^2 + \omega^2 R^2 \text{ or } \omega^2 R^2 \leq \left(\frac{\mu g}{\omega}\right)^2 - 4v_r^2$$

With  $v_r = 0.5 \text{ cm/s}^2$  while  $g = 9.8 \text{ m/s}^2$ , the Coriolis force is negligible compared to max. frictional force.

17.4. **Right Invariance, Left Invariance, Angular Velocity.** For an infinitesimal time increment  $\delta t$ , motion of the body from position  $g_t$  is described by the left translation  $g_{t+\delta t} = g_{\delta t}g_t$ .

This is interpreted as an infinitesimal rotation  $\delta t \bar{\Omega}$  at  $g_t$ , matrix  $\bar{\Omega}$  defined by

$$(91) \quad \begin{aligned} A(t + \delta t) - A(t) &= A(\delta t)A(t) - A(t) = (A(\delta t) - 1)A(t) = \delta t \bar{\Omega} A(t) \\ A(t + \delta t)^i_k &= (A(\delta t)A(t))^i_k = A(\delta t)^i_l A(t)^l_k \end{aligned}$$

cf. 4.16 Kambe (2009) [22]

$\bar{\Omega} = \frac{A(\delta t) - 1}{\delta t}$  is the tangent vector at 1.

Recall that, in general, mathematically,

$$\dot{g}_t = X \circ g_t, \text{ where } X := \lim_{\Delta t \rightarrow 0} \frac{(g_{\Delta t} - 1)}{\Delta t}$$

$$(92) \quad \dot{g}_t := \frac{dg_t}{dt} = \frac{dA}{dt} = \bar{\Omega} g_t$$

so

$$(93) \quad \bar{\Omega} = \dot{g}_t \circ (g_t)^{-1}$$

In my notation, for  $O \in SO(N)$ ,

$$(94) \quad \hat{\omega} = \dot{O} O^T$$

17.4.1. *2 Physical Interpretations of rotations: frame transformation and "active" rotation.* Consider  $O \in SO(N)$ ,  $N = 3$ .

$$O : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

$O$  can be interpreted as either

$O : \text{rotating body frame} \rightarrow \text{lab frame}$ , i.e.

$O : K \rightarrow K'$  (notation used in Hand and Finch (1998) [23] where  $K$  is the body frame,  $K'$  is the fixed frame)

$O : \searrow \mapsto \mathbf{r}'$

or  $O$  interpreted as "active" rotation of vector in  $\mathbb{R}^N$ .



Consider this example:

(95)

$$\begin{aligned} O &= \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & \\ & & 1 \end{bmatrix} & \dot{O} &= \omega \begin{bmatrix} -s & -c \\ c & -s \\ & & \end{bmatrix} \\ \dot{O}O^T &= \omega \begin{bmatrix} -s & -c \\ c & -s \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \\ & & 1 \end{bmatrix} = \omega \begin{bmatrix} & -1 \\ 1 & \\ & & \end{bmatrix} = \widehat{\omega} \end{aligned}$$

s.t.  $\omega = \omega \mathbf{e}_z$ .  
 $\dot{O}O^T$  is the angular velocity of the rotating frame, or angular velocity of how much the vector in  $\mathbb{R}^N$  is rotated by.

17.4.2. *Instantaneous Angular Velocity.* Given  $\mathbf{r} = (r^1, r^2, r^3) \in M$ , an infinitesimal transformation of  $\mathbf{r}$  by  $\xi(t) = 1 + t\mathbf{a} + O(t^2)$ .

$$\begin{aligned} \mathbf{r} &\mapsto \mathbf{r}' = \xi(t)\mathbf{r} = \mathbf{r} + t\mathbf{a}\mathbf{r} \\ \mathbf{a}\mathbf{r} &= \begin{bmatrix} & -a^3 & a^2 \\ a^3 & & -a^1 \\ -a^2 & a^1 & \end{bmatrix} \begin{bmatrix} r^1 \\ r^2 \\ r^3 \end{bmatrix} = \begin{bmatrix} -a^3r^2 + a^2r^3 \\ a^3r^1 - a^1r^3 \\ -a^2r^1 + a^1r^2 \end{bmatrix} = (\widehat{\mathbf{a}} \times \mathbf{r}) = \epsilon_{ijk}a^jr^k\mathbf{e}_i \end{aligned}$$

So multiplication by a skew symmetric matrix is equivalent to taking the cross product with the axial vector representation.  
Now  $\bar{\Omega}$  shown to be skew-symmetric (from a Taylor series expansion). See either Appendix C.3 of Kambe (2009) [22] or Eq. 26.  
Make the following notation change from Kambe’s (2009) [22]:

(96)

$$\widehat{\Omega} \equiv \omega$$

where  $\omega$  is an axial vector of the angular velocity of body rotation relative to the *fixed* reference frame  $F$  (because in body frame,  $F_B$ , there’s no rotation!).  
**Solution Question 9:  $U$  Matrix, pp. 265, Ch. 7 ”Rotating Coordinate Systems”, Hand and Finch (1998) [23].**  
Find the form of  $\mathbf{U}$  for a fixed rotation  $\theta$  about the  $X$  axis:

$$O = \begin{bmatrix} & c & -s \\ & s & c \end{bmatrix}$$

Then

$$\begin{aligned} \dot{O} &= \omega \begin{bmatrix} & -s & -c \\ & c & -s \end{bmatrix} \\ \dot{O}O^T &= \omega \begin{bmatrix} & -s & -c \\ c & & -s \end{bmatrix} \begin{bmatrix} & c & s \\ & -s & c \end{bmatrix} = \omega \begin{bmatrix} & & -1 \\ 1 & & \\ & & \end{bmatrix} \end{aligned}$$

Thus

$$\dot{O}O^T = \widehat{\omega} \mapsto \omega = \omega \mathbf{e}_x$$

So we get the form for  $\omega$ . Now consider

$$\widehat{\omega}_b = O^T \widehat{\omega} O = \begin{bmatrix} & c & s \\ -s & & c \end{bmatrix} \omega \begin{bmatrix} & & -1 \\ 1 & & \end{bmatrix} O = \omega \begin{bmatrix} & s & -c \\ c & & s \end{bmatrix} \begin{bmatrix} & c & s \\ & s & c \end{bmatrix} = \omega \begin{bmatrix} & & -1 \\ & 1 & \end{bmatrix}$$

$\widehat{\omega}_b$  is  $\widehat{\omega}$  expressed in body coordinates.  $\widehat{\omega} = \widehat{\omega}_b$  in this particular case because  $\widehat{\omega}_b$  is the angular velocity measured in the inertial frame, but expressed in the body frame coordinates.

See <https://physics.stackexchange.com/questions/88398/why-is-body-frame-angular-velocity-nonzero/88401#88401>

For  $g_t$  operating on  $\mathbf{x}$ , let’s follow Kambe’s development in pp. 133-134, Eqns. (4.17)-(4.19) (Kambe (2009) [22]). So then

$$g_t\mathbf{x} = \mathbf{y}(t), \quad g_{t+\delta t}\mathbf{x} = \mathbf{y}(t + \delta t)$$

Then

$$\begin{aligned} \mathbf{y}(t + \delta t) &= \mathbf{y}(t) + \frac{d\mathbf{y}}{dt}(t)\delta t + O((\delta t)^2) = \mathbf{y}(t) + (\dot{g}_t\mathbf{x})\delta t + \cdots = \mathbf{y}(t) + (\bar{\Omega}g_t\mathbf{x})\delta t + \cdots = \\ &= g_t\mathbf{x} + (\omega \times (g_t\mathbf{x}))\delta t \end{aligned}$$

Let

$$\mathbf{v}_y = \frac{d\mathbf{y}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\mathbf{y}(t + \delta t) - \mathbf{y}(t)}{\delta t} = \omega \times (g_t\mathbf{x}) = \omega \times \mathbf{y}$$

This means that point  $\mathbf{y}(t)$  is moving with velocity  $\mathbf{v}_y = \omega \times \mathbf{y}$ , i.e.  $\omega$  *angular velocity* in fixed space.  
But typically,  $\mathbf{x} = \mathbf{x}(t)$ , i.e. the position on the body is time dependent, it just doesn’t ”stay on the body”.

Recall that  $\mathbf{x} = \mathbf{x}(t)$  are coordinates in body frame  $F_B$ .  $\mathbf{y}(t)$  are coordinates in fixed reference frame  $F$ .

$$\mathbf{y}(t) = g_t\mathbf{x}(t)$$

Compare this notation to the one in Hand and Finch (1998) [23] in Eq. (7.18):

$$\mathbf{r}' = \widetilde{U}\mathbf{r} \equiv U\mathbf{r}$$

where the rotated or body frame is  $K$ , and fixed, inertial reference frame  $K'$ , and I modify the notation for  $U$  to be an element in  $SO(3)$ .

So instead of how Kambe assumed above that the position is fixed in the body frame, assume that it’s also time-dependent in the rotating frame. So use chain rule:

$$\frac{d\mathbf{y}}{dt}(t) = \frac{d}{dt}(g_t\mathbf{x}) = \dot{g}_t\mathbf{x} + g_t\dot{\mathbf{x}}(t) = \dot{g}_t\mathbf{x} + g_t\mathbf{v}_x = \bar{\Omega}g_t\mathbf{x} + g_t\mathbf{v}_x$$

(97)

$$\implies \mathbf{v}_y = \omega \times \mathbf{y} + g_t\mathbf{v}_x$$

Follow the development in Hand and Finch (1998) [23]:

(98)

$$\begin{aligned} \mathbf{v}|'_{\text{space}} &= \dot{U}\mathbf{r} + U\dot{\mathbf{r}}|_{\text{body}} = \dot{U}\mathbf{r} + U\mathbf{v}|_{\text{body}} = \dot{U}U^{-1}U\mathbf{r} + U\mathbf{v}|_{\text{body}} = \dot{U}U^{-1}\mathbf{r}' + U\mathbf{v}|_{\text{body}} = \\ &= \omega \times \mathbf{r} + \mathbf{v}|_{\text{body}} \end{aligned}$$

What we wrote is essentially a summary of Eqns. (7.28), (7.29) Hand and Finch (1998) [23].  
Compare this, namely Eq. 97, with wikipedia (”[Rotating reference frame](#)”), where the velocities in the 2 reference frames are related by:

(99)

$$\mathbf{v}_i = \mathbf{v}_r + \boldsymbol{\Omega} \times \mathbf{r}$$

where  $i \equiv$  inertial frame of reference,  $r \equiv$  rotating reference frame.

Now

$$\mathbf{v}_y = \omega \times \mathbf{y} + g_t\mathbf{v}_x$$

$$\mathbf{a}_y = \frac{d}{dt}\mathbf{v}_y$$

$$\frac{d}{dt}\omega \times \mathbf{y} = \frac{d}{dt}\epsilon_{ijk}\omega_jy_k = \epsilon_{ijk}\dot{\omega}_jy_k + \epsilon_{ijk}\omega_j\dot{y}_k$$

$$\text{For } \dot{y}_k, \text{ consider } \frac{d\mathbf{y}}{dt} = \omega \times \mathbf{y} + g_t\mathbf{v}_x$$

$$\text{so } \omega \times \frac{d\mathbf{y}}{dt} = \omega \times (\omega \times \mathbf{y}) + \omega \times g_t\mathbf{v}_x$$

$$\implies \frac{d}{dt}(\omega \times \mathbf{y}) = \dot{\omega} \times \mathbf{y} + \omega \times (\omega \times \mathbf{y}) + \omega \times g_t\mathbf{v}_x$$



$$\begin{aligned} \frac{d}{dt}(g_t \mathbf{v}_{\mathbf{x}}) &= \dot{g}_t \mathbf{v}_{\mathbf{x}} + g_t \mathbf{a}_{\mathbf{x}} = \omega \times g_t \mathbf{v}_{\mathbf{x}} + g_t \mathbf{a}_{\mathbf{x}} \\ (100) \quad \implies \mathbf{a}_{\mathbf{y}} &= \omega \times (\omega \times \mathbf{y}) + 2\omega \times g_t \mathbf{v}_{\mathbf{x}} + \dot{\omega} \times \mathbf{y} + g_t \mathbf{a}_{\mathbf{x}} \end{aligned}$$

Compare this to Hand and Finch (1998) [23], Eq. 7.32:

$$(101) \quad \mathbf{a}|_{\text{space}} = \omega \times (\omega \times \mathbf{r}) + 2\omega \times \mathbf{v}|_{\text{body}} + \dot{\omega} \times \mathbf{r} + \mathbf{a}|_{\text{body}}$$

cf. pp. 134 Kambe (2009) [22]

Now compare this, namely Eq. 100, with wikipedia ("Rotating reference frame"), where the accelerations in the 2 reference frames are related by:

$$(102) \quad \mathbf{a}_r = \mathbf{a}_i - 2\boldsymbol{\Omega} \times \mathbf{v}_r - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) - \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r}$$

where  $\mathbf{a}_r := \left( \frac{d^2 \mathbf{r}}{dt^2} \right)_r$  is the apparent acceleration in the rotating reference frame.

**Solution Ch. 7, Angular Velocity, Problem 1, (Locomotive) Hand and Finch (1998) [23].**

- (a)
- (b)
- (c)

$$\begin{aligned} \hat{\omega} &= \dot{O}O^T \\ O &= O_2 O_1 \text{ and } O^T = O_1^T O_2^T \\ \dot{O} &= \dot{O}_2 O_1 + O_2 \dot{O}_1 \\ \dot{O}O^T &= \dot{O}_2 O_2^T + O_2 \dot{O}_1 O_1^T O_2^T = \hat{\omega}_2 + O_2 \hat{\omega}_1 O_2^T = \hat{\omega} \end{aligned}$$

Corresponding to right-invariant  $\dot{g}_t = \omega g_t$  of the tangent vector, the angular momentum is written in the right-invariant way as

$$(103) \quad M_t = J \dot{g}_t = \overline{M} g_t, \quad \overline{M} = J \omega$$

where  $J$  is an inertia operator and time dependent in fixed space. cf. Eqn. (4.20), pp. 134, Kambe (2009) [22].

Make the following notation change:

$$(104) \quad L_t = I \dot{g}_t = \overline{L} g_t, \quad \overline{L} = I \hat{\omega}$$

where  $I$  is an inertia operator.

In mechanics, angular momentum is defined by

$$M_t = \int (\mathbf{y} \times \mathbf{v}_{\mathbf{y}}) \rho d^3 \mathbf{y} = \int (g_t \mathbf{x} \times (\hat{\omega} g_t \mathbf{x})) g_t (\rho d^3 \mathbf{x})$$

and so  $M_t := \overline{M} g_t$ .

Relative to body frame  $F_B$ , the same velocity (relative to fixed space) is represented as  $\frac{d\mathbf{y}_b}{dt} = v_b^i \mathbf{b}_i$ , where  $\mathbf{v}_b = \hat{\Omega}_b \times Y$ .

$\hat{\Omega}_b$  is angular velocity relative to body frame. Now  $\overline{\Omega}_b$  derived by *left translation* of tangent vector.

$$(105) \quad \overline{\Omega}_b = g_t^{-1} \circ \dot{g}_t = L_{g_t^{-1}} \dot{g}_t = g_t^{-1} \overline{\Omega} g_t$$

as a change of notation, I will also write

$$\omega_b = g_t^{-1} \omega g_t$$

An important point is that tangent vector  $\dot{g}_t$  is represented by **left-invariant** form

$$\hat{\omega}_b = g_t^{-1} \dot{g}_t \xrightarrow{L_{g_t}} L_{g_t} \hat{\omega}_b = g_t \hat{\omega}_b = \dot{g}_t \text{ or } \dot{g}_t = g_t \hat{\omega}_b$$

by  $\omega_b$  at 1, whereas it's also represented by right-invariant form  $\dot{g}_t = \hat{\omega} g_t$  with  $\hat{\omega}$  at 1 in Eqn. (4.17) of Kambe (2009) [22],

where, recall  $\dot{g}_t = \frac{dg}{dt} = \frac{dA}{dt}$ .

Indeed, note that for

$$\mathbf{b}_i(t) = \mathbf{e}_j B^j{}_i(t) \text{ then}$$

$$\mathbf{b}_i(t + \delta t) = \mathbf{e}_j B^j{}_i(t + \delta t) = \mathbf{e}_j B^j{}_k(t) B^k{}_i(\delta t)$$

where the last step is because since  $\mathbf{e}_j$  on the left forces an infinitesimal change to be from the right.

Then

$$B(t + \delta t) - B(t) = B(t)B(\delta t) - B(t) = B(t)(B(\delta t) - 1) \text{ so } \dot{B}(t) = B(t)Y_1$$

where  $Y_1 \in \mathfrak{so}(N)$  and  $Y_1 := \lim_{\delta t \rightarrow 0} \frac{B(\delta t) - 1}{\delta t}$ .

Also, recall that

$$\mathbf{Y} = Y^i \mathbf{b}_i = \mathbf{e}_j B^j{}_i Y^i = X^j \mathbf{e}_j \text{ implies that } B : F' \rightarrow F$$

where  $F'$  is a rotating frame,  $F$  is a fixed frame, i.e.  $B$  is a coordinate frame transformation.

Due to right-translation property of  $B(t)$ , then  $\frac{dB}{dt} = B(t)\hat{\Omega}_b$ . Thus, we obtain a left-invariant vector field.

From Eq. 88,  $A = B$  but

$$(106) \quad \begin{aligned} \frac{dA}{dt} &= \hat{\Omega} A \\ \frac{dA}{dt} &= \frac{dB}{dt} = B(t)\hat{\Omega}_b = \hat{\Omega} A(t) \text{ or } \hat{\Omega}_b = A^{-1} \hat{\Omega} A \end{aligned}$$

## 18. RIGID BODY DYNAMICS, RIGID BODY MOTION

**18.1. Euler's equations of motion, Euler's rotation equations.** In the body frame, inertia tensor  $J \equiv I$  is time-independent.

$$\frac{d}{dt} \hat{\Omega} - \text{ad}_{\hat{\Omega}}^* \hat{\Omega} = 0$$

Use (Eq. (4.36) Kambe (2009) [22])

## 19. FLUID MECHANICS, FLUID FLOW

19.0.1. *Courses to consider.* MIT 16.01, 16.02

19.0.2. *Videos to watch.* Engineering MAE 130A. Intro to Fluid Mechanics. Lecture 01. UCI. Follow the rest of the videos.

19.1. **Buoyancy.** From Problem 1.3 of the Japan Physics Olympiad (2014) [37], Ch. 1, General Physics, Elementary Problems.

**Problem 1.3.** The part of the iceberg above the sea.

$$\rho_{sw} = 1024 \text{ kg/m}^3$$

$$\rho_{ice} = 917 \text{ kg/m}^3$$

with  $sw \equiv$  sea water.

Find the ratio of the volume of the part of the iceberg above the sea to whole volume of the iceberg.

**Solution 1.3.**

**The buoyancy force exerted on the iceberg is equal to the weight of the seawater displaced by the iceberg.**

$$\rho_{sw} g V_{is} = \rho_{ice} g V_{ice} \implies \frac{\rho_{sw}}{\rho_{ice}} = \frac{V_{ice}}{V_{is}}$$

with iceberg in sea  $\equiv$  is.

iceberg above sea  $\equiv$  as

$$V_{as}/V_{ice} = \frac{V_{ice} - V_{is}}{V_{ice}} = 1 - \frac{V_{is}}{V_{ice}} = 1 - \frac{\rho_{ice}}{\rho_{sw}} = 1 - \frac{917 \text{ kg/m}^3}{1024 \text{ kg/m}^3} = \frac{1024 - 917}{1024} = \frac{107}{1024}$$

**The buoyancy on a body equals the resultant force due to the pressure exerted by the surrounding fluid**, i.e. **The buoyancy force exerted on a body (iceberg) is equal to the resultant force due to pressure exerted by the surrounding fluid (seawater)**.

Pressure on body of volume  $V$  due to its surrounding fluid (whose density is  $\rho$ ) acts perpendicularly to the boundary surface between the body and the fluid.

Since fluid pressure at a deep location is greater than that at shallow location, resultant force due to pressure on boundary surface points upward. This resultant force is the buoyancy,  $F$ , acting on body.

Consider region of fluid with same volume  $V$  as body.

buoyancy  $F \equiv$  force exerted vertically on body by its surrounding fluid

$$=$$

$$\underbrace{\hspace{1.5cm}}_{\text{balance gravitational force acting on } V} \rho g V$$

For a body floating in a fluid, the magnitude of the buoyancy force acting on the body is equal to the magnitude of the gravitational force on the fluid displaced by the part of the body submerged in the fluid.

19.2. **Fluids as a State of Matter.** cf. Pert (2013) [25], Ch. 1 Introduction, 1.1 Fluids as a State of Matter.

fluid - isotropic, (no directional preference; identical values of a property in all directions),  
locally homogeneous, macroscopic material,  
whose particles are free to move within constraints established by the dynamical laws of continuum physics.  
*requirement that fluid be a continuum* - if volume of fluid successively subdivided into smaller elements, each element will remain structurally similar to its parent, and this subdivision can be carried out down to infinitesimal volumes.

**fluid particle** - fictitious particle fixed within fluid continuum and moving with flow velocity;  
represents average over large number of microscopic particles.

**fluid point** - fixed in fluid moving with flow velocity;  
- fluid particle always situated at same fluid point.

**infinitesimal volume** - within continuum of fluid, large compared with microscopic scales, but small compared with macroscopic ones.

fluid is made up of discrete microscopic particles, namely molecules, distributed randomly, with  
distribution of velocities characteristic of fluid in termal equilibrium, typically Maxwell-Boltzmann distribution in a gas.

At typical experimental densities,  
intermolecular separation extremely small and very much less than lab scale, i.e.  $l_s \ll L_0$   
possible to average over small volumes which contain very large number of particles, yet very small on lab scale,  $N - V$  in  $V_0$ ,  
 $V_0^{1/3} \ll L_0$   
 $\implies$  allow us to recover continuum approximation  
obtain terms which characterize fluid as bulk material

Typical average quantities:  
Density,  $\rho$ : number of mass particles per unit volume.  
temperature,  $\tau$ : average energy of random motion per particle in thermal equilibrium  
pressure  $p$  average momentum flow associated with random motion per unit area.  
flow velocity  $u$  - mean velocity of molecules averaging out random motion.

role of collisions amongst particles important in defining irreversibility through loss of correlation between particles.

*mean free path*  $l_0$  - particles collide on average after distance equal to  $l_0$  and time after collision interval

since fluid mechanics assumes fluid particles in thermal equilibrium and randomly distributed, this condition requires that  
spatial and temporal averages be taken to include large number of collisions; i.e. lab scale length  $L_0$  large compared with  $l_0$ ,  
and time to collision interval

effects of collisions on fluid transport (momentum and energy) are averaged over thermal distribution to yield bulk properties  
of material, namely viscosity and thermal conduction, respectively.

theory: flow of basic fluid may be calculated using Newtonian mechanics, classical thermodynamics, viscosity and thermal conductivity values.  
Conditions under which this theory may be applied:

- lab-scale lengths  $L_0$  must be large compared with intermolecule separation and mean free path  $l_0$
- characteristic lab times  $T_0$  must be large compared with collision interval
- fluid locally in thermal equilibrium

cf. 1.2 The Fundamental Equations for Flows of a Dissipationless Fluid. Pert (2013) [25]

**inviscid limit** - viscosity and thermal conduction are neglected

**adiabatic flow** - specific entropy of fluid particle constant in time

**isentropic flow** - specific entropy of each fluid particle has same initial value

many flows are both isentropic and adiabatic, e.g. ideal steady fluid flow, whose specific entropy an entry is everywhere constant.

*Eulerian frame* - lab frame, coordinates fixed in space and time, derivatives are usual partial derivatives

$$\left. \frac{\partial}{\partial t} \right|_{\mathbf{r}}, \nabla|_t$$

Consider coordinate system  $(t, x^1, x^2, x^3)$  and so each vector field  $\mathbf{e}_j$  forming basis of tangent space at each pt. expressed as  
linear combination of coordinate vector fields

$$\left. \frac{\partial}{\partial t} \right|_{\mathbf{r}}, \left. \frac{\partial}{\partial x^1} \right|_t, \left. \frac{\partial}{\partial x^2} \right|_t, \left. \frac{\partial}{\partial x^3} \right|_t,$$

in Lagrangian frame, spatial variation seen by particle due to its motion is absorbed into time derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

Rather, consider total time derivative of integral of  $n$ -form:

$$\frac{d}{dt} \int_{V_0} \omega = \int_{V_0} \mathcal{L}_{\frac{\partial}{\partial t} + \mathbf{v}} \omega$$

Volume form  $\Omega^n(M) = \bigwedge^n(T^*M)$

$$\text{vol}^n \in \Gamma(\Omega^n(M)) = \Gamma(\bigwedge^n T^*M)$$

i.e.  $\text{vol}^n$  is  $n$ -form, a section of the line bundle,  $\Omega^n(M) = \bigwedge^n T^*M$

Now

$$\text{vol}^n = \frac{\sqrt{g}}{n!} \epsilon_{i_1, \dots, i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

Consider

$$\text{vol}^{n+1} = \frac{\sqrt{g}}{(n+1)!} \epsilon_{i_0 i_1, \dots, i_n} dx^{i_0} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

19.3. **Lagrangian frame.** cf. Pert (2013) [25] 1.3 Lagrangian Frame.

**Lagrangian frame** - frame of the moving particle;

Lagrangian reference frame considers fluid from point of view of an observer on a fluid particle.

fluid particle may be conveniently identified by coordinate set, which is fixed on the particle,

$\mathbf{\Lambda} = (\lambda, \mu, \nu)$  i.e. triad of numbers.

e.g.  $\mathbf{\Lambda}$  maybe initial position of particle  $\mathbf{r}_0 = (x_0, y_0, z_0)$ , i.e. fluid particle labeled by some (time-independent) vector field  $\mathbf{x}_0 \equiv \mathbf{\Lambda}$ .

$\implies$  position, velocity, thermodynamic state of particles are therefore functions of time alone.

$\implies$  leads to simple set of kinematic and dynamic relations governing motion of particle

$$\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t)) = (x^1(t), x^2(t), x^3(t))$$

$$\dot{\mathbf{r}} \equiv \dot{\mathbf{r}}(t) \equiv \frac{d\mathbf{r}}{dt} \equiv \frac{d\mathbf{r}}{dt}(t) = \mathbf{v} \equiv \mathbf{v}(t)$$

or

$$(107) \quad \begin{aligned} \frac{d\mathbf{r}}{dt} &= \mathbf{v} \\ \frac{d\mathbf{v}}{dt} &= \frac{\mathbf{F}}{m} \end{aligned}$$

cf. Eq. (1.2) of Pert (2013) [25], i.e.

flow described by function  $\mathbf{X}(\mathbf{x}_0, t)$  giving position of particle labeled at  $\mathbf{x}_0$  at time  $t$ .

19.4. **Momentum conservation in Lagrangian Frame-Euler's Equation.** Recall that a connection  $\nabla$  on smooth manifold  $(M, \mathcal{O}, \mathcal{A})$  is a map that takes a pair consisting of a vector (field)  $X$  and  $(p, q)$ -tensor field  $T$  and sends them to a  $(p, q)$ -tensor (field)  $\nabla_X T$ , i.e.

$$\nabla : \mathfrak{X}(M) \times (\otimes^p TM \otimes^q T^* M) \rightarrow \otimes^p TM \otimes^q T^* M$$

s.t.

$$\begin{aligned} \nabla_X f &= Xf \quad \forall f \in C^\infty(M) \\ \nabla_X (T + S) &= \nabla_X T + \nabla_X S \\ \nabla_X (T(\omega, Y)) &= (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y) \\ \nabla_{fX+Z} T &= f\nabla_X T + \nabla_Z T, \quad f \in C^\infty(M) \quad \text{”Leibnitz” rule} \end{aligned}$$

cf. Schuller, International Winter School on Gravity and Light held central lectures.

Remember that in the Lagrangian frame, we are an observer following an individual fluid parcel or i.e. ”fluid particle” as it moves through space and time. This fluid particle will obey Newton's 2nd. law:  $\nabla_u u = \frac{F}{m} \iff m \cdot \mathbf{a} = F$ .

$$\begin{aligned} \nabla_u u &= \nabla_{u^\mu \frac{\partial}{\partial x^\mu}} (u^\nu \frac{\partial}{\partial x^\nu}) = u^\mu \nabla_{\frac{\partial}{\partial x^\mu}} (u^\nu \frac{\partial}{\partial x^\nu}) = u^\mu \left( \nabla_{\frac{\partial}{\partial x^\mu}} u^\nu \right) \frac{\partial}{\partial x^\nu} + u^\mu u^\nu \nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu} = \\ &= u^\mu \frac{\partial u^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + u^\mu u^\nu \Gamma_{\mu\nu}^\gamma \frac{\partial}{\partial x^\gamma} \end{aligned}$$

Let  $u^0 = 1$ .

$$\nabla_u u = \frac{\partial u^a}{\partial t} \frac{\partial}{\partial x^a} + u^b \frac{\partial u^a}{\partial x^b} \frac{\partial}{\partial x^a} + u^\mu u^\nu \Gamma_{\mu\nu}^a \frac{\partial}{\partial x^a}$$

where  $\Gamma_{ba}^0 = 0$  since  $\nabla dt = 0$  and we choose a stratified atlas  $\mathcal{A}_{\text{stratified}}$ .

Then let

$$\int \rho \text{vol}^n \nabla_u u = F_{\text{tot}} = \int f^i \text{vol}^n \otimes e_i$$

From Sec. 2.4 Equations of Motion Neglecting Viscosity, pp. 38, Sabersky, et. al. (1998) [26], the principal forces are those due to pressures acting on surfaces and to forces acting directly on a mass of the particle:

$$\begin{aligned} \int_{\partial V} p dS^i \otimes e_i &= \int_V \mathbf{d}(p dS^i \otimes e_i) = \int_V \frac{\partial p}{\partial x^j} dx^j \wedge \left( \frac{\sqrt{g}}{(n-1)!} \epsilon_{i_2, \dots, i_n}^i dx^{i_2} \wedge \dots \wedge dx^{i_n} \right) \otimes e_i = \\ &= \int_V \frac{\partial p}{\partial x^j} g^{ij} \text{vol}^n \otimes e_i = \int_V (\text{grad} p)^i \text{vol}^n \otimes e_i \\ \implies \int \rho \text{vol}^n \nabla_u u &= - \int_V (\text{grad} p)^i \text{vol}^n \otimes e_i + \int_V f^i \text{vol}^n \otimes e_i \\ \implies \rho (\nabla_u u)^i &= -(\text{grad} p)^i + f^i \\ (\nabla_u u)^i &= -\frac{1}{\rho} (\text{grad} p)^i + f^i \end{aligned} \quad (108)$$

describes the motion of a perfect fluid.

Assuming a non-rotating, inertial frame (so that we can set the Christoffel symbols to 0),

$$(109) \quad \frac{\partial}{\partial t} u^i + u^b \frac{\partial u^i}{\partial x^b} = -\frac{1}{\rho} (\text{grad} p)^i + f^i$$

which describes the motion of the perfect fluid in an inertial ref. frame. Compare this to Eqns. (2.10a- 2.10c) of Sabersky (1998) [26]. Sabersky says that no assumptions were made about density so these results apply to compressible fluid as well as incompressible ones. Also Eqns. **108**, **109** are called either Euler's Equation by Pert (2013) [25] or Eulerian equation by Sabersky, et. al. (1998) [26]

19.5. **Mass Conservation for Fluid Flow, Continuum media.** The mass of fluid in some volume  $V_0 \subset N$  is  $\int_{V_0} \rho \text{vol}^n$ , where  $\rho$  is fluid density,  $\rho \in C^\infty(N)$ .

The total mass of fluid flowing out of volume  $V_0$  is

$$\begin{aligned} \frac{d}{dt} \int_{V_0} \rho \text{vol}^n &= \int_{V_0} \mathcal{L}_{\frac{\partial}{\partial t} + \mathbf{u}} (\rho \text{vol}^n) = \int_{V_0} \frac{\partial}{\partial t} \rho \text{vol}^n + \int_{V_0} \mathcal{L}_{\mathbf{u}} \rho \text{vol}^n \\ \int_{V_0} \mathcal{L}_{\mathbf{u}} \rho \text{vol}^n &= \int_{V_0} di_{\mathbf{u}} \rho \text{vol}^n + i_{\mathbf{u}} d\rho \text{vol}^n = \int_{V_0} di_{\mathbf{u}} \rho \text{vol}^n + 0 = \int_{V_0} di_{\mathbf{u}} \rho \text{vol}^n = \int_{\partial V_0} i_{\mathbf{u}} \rho \text{vol}^n \end{aligned}$$

Now

$$\begin{aligned} i_{\mathbf{u}} \text{vol}^n &= i_{\mathbf{u}} \frac{\sqrt{g}}{n!} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n} \\ i_{\mathbf{u}} dx^{i_1} \wedge \dots \wedge dx^{i_n} &= u^{i_1} dx^{i_2} \wedge \dots \wedge dx^{i_n} - dx^{i_1} \wedge u^{i_2} dx^{i_3} \wedge \dots \wedge dx^{i_n} + \dots + (-1)^{n+1} dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}} u^{i_n} = \epsilon_{j_1 \dots j_n}^{i_1 \dots i_n} u^{j_1} dx^{j_2} \wedge \dots \wedge dx^{j_n} \\ \implies i_{\mathbf{u}} \text{vol}^n &= \frac{\sqrt{g}}{(n-1)!} \epsilon_{j_1 \dots j_n} u^{j_1} dx^{j_2} \wedge \dots \wedge dx^{j_n} \end{aligned} \quad (110)$$

We can also rewrite Eq. **110** to be a ”surface differential”:

$$(111) \quad \begin{aligned} \int_{V_0} \mathbf{d} i_{\mathbf{u}} \rho \text{vol}^n &= \int_{\partial V_0} i_{\mathbf{u}} \rho \text{vol}^n = \\ \int_{\partial V_0} \rho \frac{\sqrt{g}}{(n-1)!} u^{j_1} \epsilon_{j_1 j_2 \dots j_n} dx^{j_2} \wedge \dots \wedge dx^{j_n} &\equiv \int_{\partial V_0} \rho \mathbf{u} \cdot d\mathbf{S} \equiv \int_{\partial V_0} \rho \langle \mathbf{u}, d\mathbf{S} \rangle \end{aligned}$$

If  $\sqrt{g} = 1$ ,  $n = 2$ ,

$$i_{\mathbf{u}} \text{vol}^2 = (u^1 dx^2 - u^2 dx^1) = u \cdot n_1 dx^2 + u \cdot n_2 dx^1 = u \cdot n dS$$

with  $n_1 = e_1$  and  $n_2 = -e_2$ .

Now

$$\begin{aligned} di_u \rho \text{vol}^n &= \\ &= \frac{\partial(\sqrt{g} \rho u^{j_1})}{\partial x^k} \frac{\epsilon_{j_1 \dots j_n}}{(n-1)!} dx^k \wedge dx^{j_2} \wedge \dots \wedge dx^{j_n} = \frac{\partial(\sqrt{g} \rho u^k)}{\partial x^k} \frac{\epsilon_{j_1 \dots j_n}}{n!} dx^{j_1} \wedge \dots \wedge dx^{j_n} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} \rho u^k)}{\partial x^k} \text{vol}^n = \\ &= \frac{\partial(\rho u^k)}{\partial x^k} \text{vol}^n + \rho u^k \frac{\partial \ln \sqrt{g}}{\partial x^k} \text{vol}^n = \text{div}(\rho u) \text{vol}^n + \rho u^k \frac{\partial \ln \sqrt{g}}{\partial x^k} \text{vol}^n \end{aligned}$$

Now if  $\sqrt{g} = 1$ , then

$$\frac{d}{dt} \int_{V_0} \rho \text{vol}^n = \int_{V_0} \frac{\partial \rho}{\partial t} \text{vol}^n + \int_{V_0} di_u \rho \text{vol}^n = \int_{V_0} \frac{\partial \rho}{\partial t} \text{vol}^n + \int_{V_0} \text{div}(\rho u) \text{vol}^n \implies \frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0$$

which is the so-called mass continuity equation.  $j = \rho u$  is the mass flux density.

Thus,

$$\begin{aligned} & \text{(mass conservation)} \\ m &= m(t) := \int_{V_0} \rho \text{vol}^n, \quad V_0 \subset N \\ (112) \quad \dot{m} &\equiv \frac{d}{dt} m(t) = \int_{V_0} \left( \frac{\partial \rho}{\partial t} \text{vol}^n + \mathbf{d}_{\mathbf{u}} \rho \text{vol}^n \right) = \boxed{\int_{V_0} \frac{\partial \rho}{\partial t} \text{vol}^n + \int_{\partial V_0} \rho \mathbf{u} \cdot d\mathbf{S}} \\ & \text{if } \sqrt{g} = 1, \text{ and } \dot{m} = 0, \text{ then} \\ & \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0 \end{aligned}$$

Sec. 1.4 Eulerian Frame, 1.4.1 Conservation of Mass-Equation of Continuity, pp. 8 of Pert (2013) [25] says that Eq. 112 is for the Eulerian Frame.

From [wikipedia](#), recall that the *Eulerian specification* focuses on specific locations in space through which fluid flow as time passes. Visualize by sitting on bank of river and watch water pass fixed location. In the Eulerian specification of field, field represented as a function of position  $\mathbf{x}$ , time  $t$ , e.g.  $\mathbf{u}(\mathbf{x}, t) \equiv$  flow velocity.

TODO: 20190804 Frankel (2012) [33] in pp. 138 and onwards, for Sec. 4.3. Differentiation of Integrals posed the rightful question, "How does one compute the rate of change of an integral when the domain of integration is also changing?" Revisit the derivation from a Lie derivative and 1-parameter flow point of view.

Force should not be represented by a vector but rather by a 1-form. Then,

$$f \in \Omega^1(N), \quad f = f(t, \mathbf{x}) = f_j dx^j \quad j = 1, \dots, n, \quad \mathbf{x} = (x^1, \dots, x^n), \quad \dim N = n$$

Indeed, the reason for  $f$  to be a 1-form is that we integrate differential forms, we don't integrate vectors.

$$(113) \quad W = \int_C f \quad \text{(line integral)}$$

If  $f$  conservative,  $\exists$  scalar  $U = U(x); x \in N$  s.t.  $f = -dU$ .

Let  $\mathbf{u} = u^j(t, \mathbf{x}) \frac{\partial}{\partial x^j} \in \mathfrak{X}(N)$  generate flow  $\phi_t$ .

The Lie derivative along vector field  $\mathbf{u}$ ,  $\mathcal{L}_{\mathbf{u}}$ , can be calculated in at least 2 ways: from the definition:

$$\mathcal{L}_{\mathbf{u}} \omega = \frac{d}{dt} \Big|_{t=0} \phi_t^* \omega$$

or Cartan's magic formula:

$$\begin{aligned} \mathcal{L}_{\mathbf{u}} \text{vol}^n &= (i_{\mathbf{u}} \mathbf{d} + \mathbf{d} i_{\mathbf{u}}) \text{vol}^n = 0 + \mathbf{d} i_{\mathbf{u}} \text{vol}^n = \mathbf{d} i_{\mathbf{u}} \left( \frac{\sqrt{g}}{n!} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n} \right) = \\ (114) \quad &= \mathbf{d} \left( \frac{\sqrt{g}}{(n-1)!} \epsilon_{j_1 \dots j_n} u^{j_1} dx^{j_2} \wedge \dots \wedge dx^{j_n} \right) = \frac{\partial(\sqrt{g} u^{j_1})}{\partial x^k} \frac{\epsilon_{j_1 \dots j_n}}{(n-1)!} dx^k \wedge dx^{j_2} \wedge \dots \wedge dx^{j_n} = \\ &= \frac{\partial(\sqrt{g} u^{j_1})}{\partial x^k} \frac{\epsilon_{j_1 \dots j_n}}{(n-1)!} \underbrace{\epsilon_{k_1, \dots, k_n}^{kj_2 \dots j_n}}_{=1} dx^{k_1} \wedge \dots \wedge dx^{k_n} \left( \frac{\sqrt{g}}{\sqrt{g}} \right) = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} u^k)}{\partial x^k} \text{vol}^n = \\ &= (\text{div} \mathbf{u}) \text{vol}^n \end{aligned}$$

cf. <https://math.stackexchange.com/questions/2566381/lie-derivative-of-volume-form?rq=1>

Note that

$$\text{div} : \mathfrak{X}(N) \rightarrow C^\infty(N)$$

$$\mathcal{L}_{\mathbf{u}} : \Omega^n(N) \rightarrow \Omega^n(N)$$

Frankel (2012) [33] on pp. lvii of the "Elasticity and Stresses" section offered this analogy: "While work in particle mechanics pairs a force covector ( $f_i$ ) with a contravariant tangent vector ( $dx^i/dt$ ) to a curve, work done by traction in elasticity pairs the contravariant stress force 2-form  $\mathcal{S}$  with the covector valued deformation 1-form  $\mathcal{E}$ , to yield a scalar valued 3-form.

Instead of pushing along a curve (line) particle to do work, what does it mean to do work on a moving fluid element? One could possibly try to consider the deformation of a volume element under fluid flow. Frankel (2012) [33] in Sec. 4.2. The Lie Derivative of a Form, on pp. 132, asks "If a flow deforms some attribute, say volume, how does one measure the deformation?  $\mathcal{L}_{\mathbf{u}} \text{vol}^n$ " is the  $n$ -form that reads off the rate of change of volume of a parallelopiped spanned by  $n$  vectors that are pushed forward by the flow  $\phi_t$ ." So, "in other words,  $\mathcal{L}_{\mathbf{u}} \text{vol}^n$  measures how volumes are changing under the flow  $\phi_t$  generated by  $\mathbf{X}$ " on pp. 133 on Frankel (2012) [33].

Then rate of work done on a fluid element could be the following:

$$W = \int_V \mathcal{L}_{\mathbf{u}}(p \text{vol}^n) = \int_V \mathbf{d} i_{\mathbf{u}} p \text{vol}^n = \int_{\partial V} i_{\mathbf{u}} p \text{vol}^n = \int_{\partial V} p \mathbf{u} \cdot d\mathbf{S}$$

TODO: work on a fluid element?

## 19.6. Bernoulli's equation.

19.6.1. *Simple cases.* cf. Ch. 3 The Bernoulli Equation, Sec. 3.2 "A Simple Form of the Bernoulli Equation", pp. 90, Sabersky, et. al. (1998) [26]

Start from Eq. 108. So

$$(\nabla_u u)^i = \frac{-1}{\rho} (\text{grad} p)^i + f^i$$

Suppose  $f^i = 0$  (no external forces),  $\frac{\partial u^i}{\partial t} = 0$  (steady flow). Suppose  $\Gamma_{\mu\nu}^a = 0$  (Cartesian coordinates? TODO: find out the conditions when this is true).

Suppose the flow is irrotational (TODO: find out the condition for irrotational flow; is it when having streamlines or ?). For the curl, recall that an external derivative  $\mathbf{d}$  takes a 1-form into a 2-form. So consider

$$\mathbf{u} \in \mathfrak{X}(V), \quad \mathbf{u} \xrightarrow{b} u_i dx^i = g_{ij} u^j dx^i$$

$$d\mathbf{u}^b \in \Omega^2(V), \quad d\mathbf{u}^b = \frac{\partial u_j}{\partial x^i} dx^i \wedge dx^j = \frac{\partial(g_{ij} u^j)}{\partial x^a} dx^a \wedge dx^i$$

If we have the condition that the flow is irrotational, i.e.

$$(115) \quad d\mathbf{u}^b = 0$$

then

$$d\mathbf{u}^b(\mathbf{u}) = \frac{\partial(u_i)}{\partial x^a} u^a dx^i - \frac{\partial(u_i)}{\partial x^a} dx^a u^i = 0 \implies u^a \frac{\partial(u_i)}{\partial x^a} dx^i = u^i \frac{\partial u_i}{\partial x^a} dx^a$$

Take the exterior derivative  $\mathbf{d}$  of the inner product of a vector with itself in the most general case on a metric manifold:

$$\begin{aligned} d(u_i u^i) &= d(g_{ij} u^j u^i) = \frac{\partial g_{ij}}{\partial x^k} u^j u^i dx^k + g_{ij} \frac{\partial u^j}{\partial x^k} u^i dx^k + g_{ij} u^j \frac{\partial u^i}{\partial x^k} dx^k = \\ &= 2u_j \frac{\partial u^j}{\partial x^k} dx^k + u^i u^j \frac{\partial g_{ij}}{\partial x^k} dx^k \end{aligned}$$

Since the Christoffel symbols are the connection coefficients of a *metric-compatible* affine connection (i.e. they can be used to construct a connection  $\nabla$  s.t.  $\nabla g = 0$ ), then <sup>6</sup>

$$(116) \quad \nabla_i g_{jk} = \frac{\partial g_{jk}}{\partial x^i} - \Gamma_{ji}^s g_{sk} - \Gamma_{ki}^s g_{js} = 0$$

and so

$$(117) \quad u^i u^j \frac{\partial g_{ij}}{\partial x^k} = u^i u^j [\Gamma_{ik}^s g_{sj} + \Gamma_{jk}^s g_{is}] = u^i u_s \Gamma_{ik}^s + u_s u^j \Gamma_{jk}^s = 2u_s u^i \Gamma_{ik}^s$$

Then

$$(118) \quad \frac{1}{2} \mathbf{d}(u^2) = u_j \frac{\partial u^j}{\partial x^k} dx^k + u_s u^i \Gamma_{ik}^s dx^k$$

Eq. 118 only depended upon the fact that the connection  $\nabla$  is *metric-compatible*. So if we suppose that the Christoffel symbols  $\Gamma_{ik}^s$  are 0, and have irrotational flow, as well,

$$\mathbf{d}(u^2/2) = u_j \frac{\partial u^j}{\partial x^k} dx^k + 0 = u^a \frac{\partial u_i}{\partial x^a} dx^i$$

Now consider the RHS of Eq. 108, with  $f_i = 0$ ,

$$\frac{-1}{\rho} (\text{grad} p)^i \xrightarrow{b} \frac{-1}{\rho} (\text{grad} p)^i g_{ij} dx^j = \frac{-1}{\rho} \frac{\partial p}{\partial x^k} g^{ki} g_{ij} dx^j = \frac{-1}{\rho} \frac{\partial p}{\partial x^j} dx^j = \frac{-1}{\rho} \mathbf{d}p$$

So if the fluid is incompressible, then  $\rho$  is constant and so we can write

$$(119) \quad \mathbf{d}\left(\frac{u^2}{2} + \frac{p}{\rho}\right) = 0$$

for any irrotational, incompressible, fluid flow with no (external) body forces.

#### 19.6.2. Venturi tube example. cf. Example 3.1 in Sabersky, et. al. (1998) [26]

Consider flow of liquid through the tube shown. Determine pressure difference between points 1 and 2 as a function of flow rate  $Q$ , density  $\rho$ . Velocity assumed constant over cross sections, and pressure will also be assumed uniform over each of these 2 areas  $A_1, A_2$ . Also assume there's no viscosity.

**Solution Example 3.1 in Sabersky, et. al. (1998) [26].**

Since density is constant, continuity equation requires that for  $Q$  the volume flow rate,

$$\text{flow rate } Q = u_1 A_1 = u_2 A_2$$

$$\begin{aligned} \frac{u_1^2}{2} + \frac{p_1}{\rho} &= \frac{u_2^2}{2} + \frac{p_2}{\rho} \implies \frac{Q^2}{2A_1^2} + \frac{Q^2}{-2A_2^2} = \frac{-(p_1 - p_2)}{\rho} \quad \text{or } p_1 - p_2 = \frac{+\rho Q^2}{2A_2^2} \left(1 - \frac{A_2^2}{A_1^2}\right) \\ \frac{Q^2}{2} \left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right) &= \frac{-(p_1 - p_2)}{\rho} \end{aligned}$$

<sup>6</sup>"Prove Christoffel Symbol Identity", Physics StackExchange

#### 19.6.3. Conservative body forces with Bernoulli equation. cf. Sec. 3.3 in Sabersky, et. al. (1998) [26]

Consider  $U = U(x)$ , force potential.

$$\begin{aligned} -\mathbf{d}U &= \frac{-\partial U}{\partial x^i} dx^i = f_i dx^i \\ \mathbf{d}\left(\frac{u^2}{2} + \frac{p}{\rho} + U\right) &= 0 \end{aligned}$$

#### 19.6.4. Barotropic Fluids ( $\rho = \rho(p)$ ) with Bernoulli equation. cf. Sec. 3.4 in Sabersky, et. al. (1998) [26]

*Barotropic fluid*: density is single-valued function of  $p$ . In the absence of heat transfer and absence of entropy change, by thermodynamics this can be shown to be barotropic.

$$\mathbf{d}\left(\frac{u^2}{2}\right) = -\frac{1}{\rho} \mathbf{d}p - \mathbf{d}U \text{ or } \mathbf{d}\left(\frac{u^2}{2} + U\right) + \frac{1}{\rho} \mathbf{d}p$$

Suppose  $i = i(p)$  s.t.  $di = \frac{\partial i}{\partial p} dp = \frac{1}{\rho(p)} = \frac{di}{dp} dp$ . So

$$\begin{aligned} \frac{di}{dp} &= \frac{1}{\rho(p)} \text{ or } i = \int \frac{1}{\rho} dp + \text{const} \\ \implies \mathbf{d}\left(\frac{u^2}{2} + U + i\right) &= 0 \end{aligned}$$

In special case of isentropic flow of perfect gas,  $\frac{P}{\rho^\nu} = \text{const.}$  and  $i = \int \frac{dp}{p^{1/\nu}} = \frac{kp^{1-1/\nu}}{1-1/\nu} = \text{enthalpy.}$

#### 19.6.5. Nonsteady flow $\frac{\partial u}{\partial t}$ with Bernoulli equation. First, use the musical isomorphism to transform the (1,0) tensor:

$$\frac{\partial u^i}{\partial t} \frac{\partial}{\partial x^i} \xrightarrow{b} g_{ij} \frac{\partial u^i}{\partial t} dx^j$$

Along a curve  $\gamma = \gamma^i(t) = x^i(t)$  s.t. . Then

$$\begin{aligned} \dot{\gamma} &= \dot{\gamma}^i(t) \frac{\partial}{\partial x^i} = u^i(t, \gamma(t)) \frac{\partial}{\partial x^i} \\ \int_{\gamma} g_{ij} \frac{\partial u^i}{\partial t} dx^j &= \int g_{ij} \frac{\partial u^i}{\partial t} \dot{\gamma}^j(t) dt = \int g_{ij} \frac{\partial u^i}{\partial t} u^j dt = \int \frac{\partial \|u\|}{\partial t} dt \end{aligned}$$

The last step can be made only if the metric  $g$  is time-independent.

cf. Example 3.2 in Sabersky, et. al. (1998) [26].

Consider the *efflux* of liquid of density  $\rho$  through an orifice of area  $A_0$  at the bottom of a large tank of cross-sectional area  $A_t$ .

The depth of liquid at the given instant is  $y$ .

pressure exerted on the liquid in the tank is maintained at  $p_t$ ,

pressure outside the orifice is the atmospheric pressure  $p_a$ .

The action of gravity is to be considered, but frictional effects may be neglected. Velocity of efflux  $u_j$  is to be determined for these conditions.

efflux - something given off in or as if in a stream (Webster dictionary)

**Solution Example 3.2 in Sabersky, et. al. (1998) [26].**

In Sec. 3.8(b) flows through orifices discussed in great detail. For this example, assume the orifice is carefully designed so streamlines at orifice exit are parallel and thus orifice area  $A_0$  and cross-sectional area of jet  $A_j$  are equal there.



cf. Example 3.3 in Sabersky, et. al. (1998) [26]. Example of a problem in which nonsteady terms cannot be neglected.

Incomplete:

20. VISCOUS FLUIDS, NAVIER-STOKES EQUATION APPLICATIONS

cf. Ch. 2, Viscous Fluids, Sec. 15. The equations of motions of a viscous fluid, Landau and Lifshitz (1987) [27]  
Recall Navier-Stokes equation for *compressible, viscous* fluid flow:

(120)

$$\rho \left( \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right) = - \frac{\partial p}{\partial x^i} + (\lambda + \mu) \frac{\partial}{\partial x^i} \operatorname{div} u + \mu \Delta u^i$$

Compare this with the expression in Eq. 15.6, Landau and Lifshitz (1987) [27]

(121)

$$\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathbf{grad}) \mathbf{v} \right] = - \mathbf{grad} p + \eta \Delta \mathbf{v} + \left( \zeta + \frac{1}{3} \eta \right) \mathbf{grad} \operatorname{div} \mathbf{v}$$

$\mu \equiv \eta$ , fluid isotropic, so properties described by scalar quantities.  
 $\mu \equiv \eta$  viscosity coefficients,  
 $\lambda + \mu \equiv \zeta + \frac{1}{3} \eta$ ,  $\zeta \equiv$  second viscosity.

20.1. Flow in a pipe.

21. THERMODYNAMICS

Let  $\Sigma$  be a (topological) manifold.  
Suppose  $U$  is a global coordinate on  $\Sigma$ :

(122)

(First Law of Thermodynamics (energy conservation))

$$dU = Q - W \text{ or } Q = dU + W$$

where  $dU, Q, W \in \Omega^1(\Sigma)$  (i.e.  $dU, Q, W$  are 1-forms over manifold  $\Sigma$ ).

Consider a path in  $\Sigma$ ,  $\gamma, \gamma : \mathbb{R} \rightarrow \Sigma$ ,  
and using a chart  $(U, S^1, \dots, S^n)$  (e.g.  $n = 1$ ,  $S^1 = v$  for volume)

$$\gamma(t) = (U(t), S^1(t), \dots, S^n(t))$$
$$\dot{\gamma} \in \mathfrak{X}(\Sigma), \dot{\gamma} = \dot{U} \frac{\partial}{\partial U} + \dot{S}^i \frac{\partial}{\partial S^i}$$

Now

$$dU(\dot{\gamma}) = \dot{\gamma}(U) = \dot{U} \frac{\partial}{\partial U} U + 0 = \dot{U}$$
$$Q(\dot{\gamma}) = Q(t) dt \left( \dot{\gamma} \frac{\partial}{\partial t} \right) = Q(t) \dot{\gamma}$$
$$\implies \dot{U} = Q(t) \dot{\gamma} - W(t) \dot{\gamma}$$

Recall that for enthalpy  $H, H := U + pV, H = H(\sigma, p)$   
TODO 20190804 Derive and check convection form of enthalpy against both Kittle and Kroemer plus thermodynamics and Sonntag, et. al.

$$dH = Q + V dp$$
$$W = p dV = p dV + V dp - V dp = d(pV) - V dp =$$
$$dE = Q - W + \mu dH$$
$$dE = Q - W + \hat{h} dN$$

$U \rightarrow E$  notation is to promote the internal energy  $U$  to include kinetic and potential energies, so that possibly,  $E = U + \text{K.E.} + \text{P.E.}$ , or, i.e.,  $E$  includes internal energy and mechanical energy.

$$G = \mu N$$
$$G = U - \tau \sigma + pV \xrightarrow{U \rightarrow E} G = E - \tau \sigma + pV \implies \tau \check{h}$$

$$dE = Q + W + \mu dN = Q + W + (\check{h} - \tau \check{\sigma}) dN$$

$$Q = \tau d\sigma = \tau d(N \check{\sigma}) = \tau N d\check{\sigma} + \tau \check{\sigma} dN$$

$\tau N d\check{\sigma}$  is the entropy change due to change in entropy per particle; i.e. **conduction term**  
 $\tau \check{\sigma} dN$  is entropy change due to change in number of particles, i.e. **convection term**

$$dE = Q + W + \check{h} dN - \tau \check{\sigma} dN = \tau N d\check{\sigma} + \tau \check{\sigma} dN + W + \check{h} dN - \tau \check{\sigma} dN = \tau N d\check{\sigma} + W + \check{h} dN$$

$m(t) = MN$   
Assume only 1 chemical species:

$$\check{Q} := \tau N d\check{\sigma} \qquad \hat{h} := \frac{H}{m} = \frac{H}{MN}$$
$$\implies dE = \check{Q} + W + \check{h} dN \frac{M}{M} = \check{Q} + W + \hat{h} dm$$

The Gibbs free energy equilibrium is given by,

$$dG = \mu dN - \sigma d\tau + V dp$$

For a throttling valve (don't all valves throttle?) the pressure drop is accounted for by the Gibbs free energy, *not* by an isenthalpy condition.

Part 14. Classical Mechanics applications

cf. Arnold, Kozlov, Neishtadt (2006) [29].  
If known forces  $\mathbf{F}_1 \dots \mathbf{F}_n$  acts on points, then

(123)

$$\sum_{i=1}^n \langle m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i, \xi_i \rangle = 0$$

cf. Eq. (1.26) of Arnold, Kozlov, Neishtadt (2006) [29], where  $\xi_1, \dots \xi_n$  are arbitrary tangent vectors to  $M, \xi_i, \dots \xi_n \in TM$ .  
 $\sum_{i=1}^n \langle m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i, \xi_i \rangle$  called "general equation of dynamics" or d'Alembert-Lagrange principle.

22. ORBITAL MECHANICS

References: Arnold, Kozlov, Neishtadt (2006) [29] Choquet-Bruhat (2009) [32], pp. 39, 3. General relativity Bate, Mueller, White (1971) [36], Sec. 1.2, pp. 5, [https://en.wikipedia.org/wiki/N-body\\_problem](https://en.wikipedia.org/wiki/N-body_problem) Abraham and Marsden (2008) [30], Ch. 9 Prástaro (1996) [13], Remark 3.20 many-bodies dynamics, pp. 346



22.1. **Conic sections.** The following is an amalgamation or summary of the development in Ch. 13, Sections 13.18-22, from *Let  $X = aN$ . Plug this into Eq. 127.* pp. 497, Apostol (1991) [35]

**Definition 54** (General definition of conic sections (vector-formulated definition)). *Given line  $L$ , called the directrix, pt.  $F$ , the focus, such that  $F \notin L$ ,*

*eccentric  $e > 0$ ,*

*$d =$  distance of  $L$  from  $F$ , then*

*the **conic section**  $C = \{X\}$  (set of all points) such that*

$$(124) \quad \|X - F\| = ed(X, L)$$

*Now,*

*$e < 1$  ellipse,*

*$e = 1$  parabola,*

*$e > 1$  hyperbola.*

*Further, if  $N$  unit normal to  $L$ , then  $\forall P \in L$ ,*

$$(125) \quad d(X, L) = |(X - P) \cdot N|$$

*If  $(X - P) \cdot N \gtrless 0$ ,  $X \in$  positive half plane (negative half-plane).*

**Theorem 23** (Further conic section development). *Building upon Def. 54,*

*If  $F \in$  positive half plane determined by  $N$ , then  $F - dN \in L$ .*

*If  $F \in$  negative half plane determined by  $N$ , then  $F + dN \in L$ .*

*Choose this convenient, specific point:  $F + dN = P \in L \implies \|X - F\| = e|(X - (F + dN)) \cdot N|$*

*If  $F = 0$ ,  $\|X\| = e|X \cdot N - d| = r = e(d - r \cos \theta)$ . Then*

$$(126) \quad \frac{ed}{r} = 1 + e \cos \theta \text{ or } \boxed{r = \frac{ed}{1 + e \cos \theta}} \text{ for } F = 0$$

*Consider symmetry about the origin (i.e. of  $X, \exists -X$ ).*

*As a caveat, let  $A = e(d + F \cdot N)$ . Then*

$$\|X - F\| = e|(X - (F + dN)) \cdot N| = ed(X, L) = e|X \cdot N - \frac{A}{e}|$$

*Then*

$$(127) \quad \|X - F\|^2 = \|X\|^2 - 2X \cdot F + \|F\|^2 = e^2((X \cdot N)^2 - 2\frac{A}{e}X \cdot N + \left(\frac{A}{e}\right)^2)$$

*Using the symmetry about the origin,  $X \rightarrow -X$ ,*

$$\implies X \cdot F = AeX \cdot N \text{ or } X \cdot (F - AeN) = 0$$

*Then  $F = AeN$ . Note that  $A = e(F \cdot N \pm d)$  (for  $F$  in negative (positive) half-plane).*

*Then*

$$Ae = e^2(Ae + d) \text{ or } \frac{A}{e} = Ae + d \text{ or } A\left(\frac{1}{e} - e\right) = d \text{ or } A = \frac{ed}{1 - e^2}$$

$$\implies \|X - AeN\| = e|(X - (AeN + dN)) \cdot N| = e|X \cdot N - (Ae + d)|$$

$$(128) \quad \begin{aligned} a^2 - 2aAe + A^2e^2 &= e^2\left(a^2 - \frac{2A}{e}a + \left(\frac{A}{e}\right)^2\right) = e^2a^2 - 2Aae + A^2 \implies a = A \\ &\implies a = \frac{ed}{1 - e^2} \end{aligned}$$

*Thus, we've shown that  $A$  is equal to the semi-major axis  $a$  and that*

$$(129) \quad \boxed{F = aeN}$$

*Thus, the semi-major axis is obtained when  $X = \pm aN$  (vertices), so that  $\|X\|^2 + (ae)^2 = e^2(X \cdot N)^2 + a^2$  is satisfied.*

*Likewise, the semi-minor axis is obtained if  $X = \pm bN_\perp$ , so that*

$$(130) \quad b^2 + (ae)^2 = 0 + a^2 \implies \boxed{b = a\sqrt{1 - e^2}}$$

We can then rewrite Eq. 126, with Eq. 128 as

$$\boxed{r = \frac{a(1 - e^2)}{1 + e \cos \theta}}$$

With Eq. 127, and using the symmetry of  $X$  about the origin, then

$$\|X\|^2 + \|F\|^2 = e^2(X \cdot N)^2 + a^2$$

Choose  $\mathbf{N} = \mathbf{e}_x$ . With  $\|X\|^2 = x^2 + y^2$ , and  $\|F\| = Ae = ae$ , then

$$(131) \quad x^2 + y^2 + a^2e^2 = e^2x^2 + a^2 \text{ or } x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

cf. 13.23 "Cartesian equations for the conic sections", Apostol (1991) [35].

22.1.1. *Ellipse.* If  $e < 1$ , then from Eq. 131,

$$\xrightarrow{\text{if } e < 1} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Observe that what Greiner (2004) [28] in pp. 249, Ch. "Planetary Motions", (a) Ellipse, denotes  $c$  is what we and Apostol denote as  $\|F\| = ae$ , the distance from the origin to the focus. Observe that

$$c^2 + b^2 = a^2e^2 + a^2(1 - e^2) = a^2$$

22.1.2. *Hyperbola.* If  $e > 1$ , then from Eq. 131,

$$\xrightarrow{\text{if } e > 1} \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Let  $b = |a|\sqrt{e^2 - 1}$ . The foci of a hyperbola are at  $(c, 0)$ ,  $(-c, 0)$  where  $c = |a|e = \sqrt{a^2 + b^2}$ .

22.1.3. *Parabola.* cf. pp. 507 of 13.23 "Cartesian equations for the conic sections", Apostol (1991) [35]

Go back to Eq. 124, where

$$\|X - F\| = d(X, L)$$

Let  $L$  be a vertical line at  $x = \mp c$ , place focus  $F = (\pm c, 0)$ .

If  $X = (x, y)$ ,  $X - F = (x \mp c, y)$ , so

$$\|X - F\|^2 = (x \mp c)^2 + y^2 = |x \pm c|^2 = x^2 \pm 2xc + c^2 = x^2 \mp 2xc + c^2 + y^2 \implies y^2 = \pm 4xc$$

So midway between the focus and directrix, at the origin, is the vertex of the parabola.

22.2. **Central Force Problem via Energy Methods.**

22.2.1. *Reduction to the Equivalent One-Body Problem.* cf. pp. 70 of Goldstein, Poole, Safko (2001) [24]

Let  $\mathbf{R}$  = radius vector to the center of mass.

Let  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$  for the relative position of two masses. Then the Lagrangian

$$(132) \quad \mathcal{L} = T(\dot{\mathbf{R}}, \dot{\mathbf{r}}) - U(\mathbf{r}, \dot{\mathbf{r}}, \dots)$$

cf. Eq. (3.1) of Goldstein, Poole, Safko (2001) [24]

The kinetic energy  $T$  is the sum of the kinetic energy of the center of mass motion plus the kinetic energy of motion, about the center of mass  $T'$ .

$$T = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + T' \text{ with } T' = \frac{1}{2}m_1\dot{\mathbf{r}}_1'^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2'^2$$

$\mathbf{r}'_1, \mathbf{r}'_2$  are radii vectors relative to the center of mass, since

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}$$

so

$$(133) \quad \begin{aligned} \mathbf{r}_1 - \mathbf{R} &= \frac{m_2(\mathbf{r}_1 - \mathbf{r}_2)}{M} = -\frac{m_2}{M}\mathbf{r} \equiv \mathbf{r}'_1 \\ \mathbf{r}'_2 \equiv \mathbf{r}_2 - \mathbf{R} &= \frac{m_1(\mathbf{r}_2 - \mathbf{r}_1)}{M} = \frac{m_1}{M}\mathbf{r} \end{aligned}$$

cf. Eqn. (3.2) of Goldstein, Poole, Safko (2001) [24].

Thus  $T' = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\dot{\mathbf{r}}^2$

So

$$(134) \quad \mathcal{L} = \frac{M}{2}\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1m_2}{M}\dot{\mathbf{r}}^2 - U(\mathbf{r}, \dot{\mathbf{r}}, \dots)$$

cf. Eqn. (3.3) of Goldstein, Poole, Safko (2001) [24].

3 coordinates  $\mathbf{R}$  are cyclic, so CM is at rest or moving uniformly. Drop the first term. The rest of the Lagrangian is exactly what would be expected if we had a fixed center of force with a single particle at distance  $\mathbf{r}$  from it.

Let

$$\mu = \frac{m_1m_2}{m_1 + m_2} \equiv \text{reduced mass}$$

22.2.2. *Equations of Motion from Lagrangian and First Integrals.* cf. Sec. 3.2 "The Equations of Motions and First Integrals, Chapter 3 The Central Force Problem, pp. 72 of Goldstein, Poole, Safko (2001) [24].

Goldstein, Poole, Safko argues: assume potential is of form  $V(r)$  and so  $V(r)$  has only radial distance, so the problem has spherical symmetry. Then, Goldstein, et. al. argues  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$  conserved. Thus  $\mathbf{r}$  and  $\mathbf{p} \perp \mathbf{L}$  always.

So

$$(135) \quad \mathcal{L} = T - V = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

cf. Eqn. (3.6) Goldstein, Poole, Safko (2001) [24].

Now

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$$

Since  $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$ , then

$$(136) \quad \dot{p}_\theta = \frac{d}{dt}(\mu r^2 \dot{\theta}) = 0 = \frac{\partial \mathcal{L}}{\partial \theta}$$

cf. Eqn. (3.7) Goldstein, Poole, Safko (2001) [24].

So

$$(137) \quad \mu r^2 \dot{\theta} = l$$

cf. Eqn. (3.8) Goldstein, Poole, Safko (2001) [24], where  $l$  is the constant magnitude of the angular momentum.

So then  $\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$  where  $\frac{1}{2}r^2\dot{\theta}$  is the areal velocity, the area swept out by radius vector per unit time.

Goldstein argues that  $\frac{1}{2}r^2\dot{\theta}$  follows from  $dA = \frac{1}{2}r(rd\theta)$ , so  $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$ . Thus, angular momentum conservation is equal to areal velocity constant. Do keep in mind that this argument is based upon the diagram in Goldstein, Poole, Safko (2001) [24] and presumably the small angle approximation (perhaps using  $\sin \theta \approx \theta$  for small  $\theta$ ). Thus, Kepler's 2nd. law: the radius vector sweeps out equal areas in equal times.

For the  $r$  coordinate:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r} \implies \frac{d}{dt}(\mu \dot{r}) = \mu r \dot{\theta}^2 - \frac{\partial V}{\partial r} \text{ or } \mu \ddot{r} - \mu r \dot{\theta}^2 = -\frac{\partial V}{\partial r}$$

Denote

$$(138) \quad f(r) = \mu \ddot{r} - \mu r \dot{\theta}^2$$

cf. Eqn. (3.11) of Goldstein, Poole, Safko (2001) [24].

Since  $\mu r^2 \dot{\theta} = l$ , then

$$(139) \quad f(r) = \mu \ddot{r} - \frac{l^2}{\mu r^3}$$

cf. Eqn. (3.12) of Goldstein, Poole, Safko (2001) [24].

Since

$$(140) \quad \mu \ddot{r} = \frac{-\partial}{\partial r} \left( V + \frac{1}{2} \frac{l^2}{\mu r^2} \right)$$

cf. Eqn. (3.14) of Goldstein, Poole, Safko (2001) [24] and since  $\mu \ddot{r} \dot{r} = \frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right)$  and  $\frac{d}{dt}g(r) = \frac{\partial g}{\partial r}\dot{r}$ , then

$$\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right) = \frac{-d}{dt}\left(V + \frac{1}{2}\frac{l^2}{\mu r^2}\right) \text{ or } \frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + V\right) = 0$$

and so

$$(141) \quad \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + V = \text{constant}$$

cf. Eqn. (3.15) of Goldstein, Poole, Safko (2001) [24], with

$$\frac{1}{2}\frac{l^2}{\mu r^2} = \frac{1}{2\mu r^2}\mu^3 r^4 \dot{\theta}^2 = \frac{\mu r^2 \dot{\theta}^2}{2}$$

and so

$$(142) \quad \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 + V = E = T + V$$

cf. Eqn. (3.13) of Goldstein, Poole, Safko (2001) [24].

For  $E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + V$ , then

$$(143) \quad \dot{r} = \sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}$$

cf. Eqn. (3.16) of Goldstein, Poole, Safko (2001) [24], or

$$(144) \quad dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}}$$

cf. Eqn. (3.17) of Goldstein, Poole, Safko (2001) [24].

Let

$$t = \int_{r_0}^r \frac{dr}{\sqrt{\frac{2}{\mu} \left( E - V - \frac{l^2}{2\mu r^2} \right)}}$$

with initial value  $r_0$ .

From  $l = \mu r^2 \dot{\theta}$ ,

$$(145) \quad d\theta = \frac{ldt}{\mu r^2}$$

cf. Eqn. (3.19) of Goldstein, Poole, Safko (2001) [24] so for  $\theta_0 \equiv$  initial value of  $\theta$ .

$$(146) \quad \theta = l \int_0^t \frac{dt'}{\mu r^2(t')} + \theta_0$$

cf. Eqn. (3.20) of Goldstein, Poole, Safko (2001) [24].

In Sec. 3.3, "The Equivalent 1-Dimensional Problem and Classification of Orbits" of Goldstein, Poole, Safko (2001) [24], consider  $V' = V + \frac{1}{2} \frac{l^2}{\mu r^2}$ , so that  $F' = \frac{-\partial V'}{\partial r} = f(r) + \frac{l^2}{\mu r^3}$ .

Let  $V = \frac{-k}{r}$ , e.g.  $V = \frac{-GMm}{r}$  with  $k = GMm$ .

Then consider  $V' = \frac{-k}{r} + \frac{l^2}{2\mu r^2}$  and its plot.

Compare this, Thm. 23, against (26.6) of Greiner (2004) [28]:

$$(147) \quad r = \frac{ed}{1 + e \cos \theta}, \quad e = \frac{C\hat{L}_0^2}{GM}, \quad d = 1/C$$

cf. Eq. (26.32) of Greiner (2004) [28].

Investigate which physical quantities (e.g.  $E, L$ ) eccentricity  $e$  depends on.

$$\begin{aligned} u &= \frac{GM}{\widehat{L}_0^2} + C \cos(\theta - \theta_0) \implies \frac{du}{d\theta} = -C \sin(\theta - \theta_0) \\ \frac{1}{2} m L_0^2 \left[ C^2 \sin^2(\theta - \theta_0) + \left( \frac{GM}{L_0^2} \right)^2 + 2C \frac{GM}{L_0^2} \cos(\theta - \theta_0) + C^2 \cos^2(\theta - \theta_0) \right] &= \\ \implies \frac{1}{2} m \hat{L}_0^2 \left[ C^2 + \left( \frac{GM}{\hat{L}_0^2} \right)^2 + 2C \frac{GM}{\hat{L}_0^2} \cos(\theta - \theta_0) \right] &= \\ = E - V \left( \frac{1}{u} \right) = E + GMmu = E + \frac{(GM)^2}{\hat{L}_0^2} m + C GMm \cos(\theta - \theta_0) & \\ \implies \frac{1}{2} m \hat{L}_0^2 C^2 = E + \frac{1}{2} \frac{(GM)^2}{\widehat{L}_0^2} m & \end{aligned}$$

Solve for the dummy constant  $C$ :

$$(148) \quad C = \sqrt{\frac{2E}{m\hat{L}_0^2} + \frac{(GM)^2}{\hat{L}_0^4}}$$

cf. Eq. (26.36) of Greiner (2004) [28].

From Eq. (26.32) of Greiner (2004) [28] or Eq. 147,

$$e = \frac{C\hat{L}_0^2}{GM} = \sqrt{\frac{2E}{m\hat{L}_0^2} + \frac{(GM)^2}{\hat{L}_0^4}} \left( \frac{\hat{L}_0^2}{GM} \right)$$

So

$$(149) \quad e = \sqrt{\frac{2E\hat{L}_0^2}{m(GM)^2} + 1}$$

cf. Eq. (26.37) of Greiner (2004) [28].

Hence, shape of the path depends on total energy  $E$ , angular momentum  $L_0 = m\hat{L}_0$  of moving body (recall that  $\hat{L} := \mathbf{r} \times \mathbf{v}$ ).

It holds that for a

parabola $e = 1$ ,	$E = 0$	
ellipse $0 < e < 1$ ,	$E < 0$ ,	$\frac{(GM)^2 m}{-2\hat{L}_0^2} < E < 0$
circle $e = 0$	$E = \frac{-(GM)^2 m}{2\hat{L}_0^2}$	
hyperbola $e = 1$ ,	$E > 0$	

The general form of initial energy  $E_0$  is, recall, give by

$$E_0 = \frac{1}{2} m v^2 + V(r) = \frac{1}{2} m v^2 - \frac{GMm}{r_0} = m \left( \frac{v^2}{2} - \frac{GM}{r_0} \right)$$

Notice that for the case of the parabola that  $E_0 = 0$  when

$$(150) \quad v = \sqrt{\frac{2GM}{r_0}}$$

Eq. 150 is the expression for the **escape velocity**.

Knowing what we know now, let's rewrite some of the expressions from before:

$$\begin{aligned} r(\theta) &= \frac{\hat{L}_0^2/GM}{1 + \frac{C\hat{L}_0^2}{GM} \cos(\theta - \theta_0)} \\ \hat{L}_0^2 &:= r_0^2 \dot{\theta}_0 \\ C &= \sqrt{\frac{2E}{m\hat{L}_0^2} + \frac{(GM)^2}{\hat{L}_0^4}} \end{aligned}$$

22.3. **Path parameters (i.e. orbital parameters).** cf. pp. 262, Greiner (2004) [28].

$$a = \frac{1}{2} (r(\theta = 0) + r(\theta = \pi)) = \frac{1}{2} \left[ \frac{\hat{L}_0^2/GM}{1 + e} + \frac{\hat{L}_0^2/GM}{1 - e} \right] = \frac{\hat{L}_0^2/GM}{1 - e^2}$$

The addition above for  $a$  is needed because we center origin at a focus, for this expression.

$$(151) \quad a = \frac{\hat{L}_0^2/GM}{\frac{-2E\hat{L}_0^2}{m(GM)^2}} = \frac{GMm}{-2E}$$

cf. Eq. (26.39) of Greiner (2004) [28].

$$(152) \quad b = \sqrt{a^2 - f^2} = \sqrt{a^2 - e^2 a^2} = a \sqrt{1 - e^2} = a \sqrt{\frac{-2E\hat{L}_0^2}{m(GM)^2}} = \hat{L}_0 \sqrt{\frac{-m}{2E}}$$

cf. Eq. (26.40) of Greiner (2004) [28].

We can also calculate the perigee (point of closest approach to mass  $M$ ):

$$(153) \quad r(\theta = 0) = \frac{\widehat{L}_0^2/GM}{1+e} = \frac{\widehat{L}_0^2/GM}{1 + \sqrt{\frac{2E\widehat{L}_0^2}{m(GM)^2} + 1}}$$

Given Eq. 151 for  $a$ , plug in for  $E$ :

$$(154) \quad a = \frac{-GMm}{2E} = -\frac{GMm}{2\left(\frac{1}{2}mv_0^2 - \frac{GMm}{r}\right)} = \frac{-GM}{v_0^2 - \frac{2GM}{r}} = \frac{-GM}{v_0^2\left(1 - \frac{2GM}{rv_0^2}\right)}$$

Notice when the denominator is 0. It is when  $v_0$  is the the escape velocity (cf. Eq. 150).

22.3.1. *Eccentricity from  $r_0$ ,  $\mathbf{v}_0$* . Consider being given  $\mathbf{r}$ ,  $\mathbf{v}$ , with  $\mathbf{r}$  in a coordinate frame with its origin at the center of mass (or, if  $M$  large enough, from the focus where  $M$  is centered at).

The first step is to use a coordinate *frame* (set of orthonormal unit vectors) with  $\mathbf{r}$ ,  $\mathbf{v}$  forming a *plane*. Note this important fact from geometry: a plane is spanned by 2 non-collinear vectors. So  $\mathbf{r}$ ,  $\mathbf{v}$  better not be parallel to each other.

Let

$$\begin{aligned} \mathbf{v} &= v^x \mathbf{e}_x + v^y \mathbf{e}_y + v^z \mathbf{e}_z = v^r \mathbf{e}_r + v^\varphi \mathbf{e}_\varphi \\ \mathbf{r} &= x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z = r \mathbf{e}_r \end{aligned}$$

Now

$$(\mathbf{r} \times \mathbf{v})_i = \epsilon_{jki} r_j v_k$$

To get the vector *orthogonal* to the plane that is formed by  $\mathbf{r}$ ,  $\mathbf{v}$ ,

$$\begin{aligned} ((\mathbf{r} \times \mathbf{v}) \times \mathbf{r})_m &= \epsilon_{ijm} \epsilon_{jki} r_j v_k r_l = \epsilon_{ilm} \epsilon_{ijk} r_j v_k r_l = \\ &= r^2 v_m - v_k r_k r_m = r^2 v_m - v_k r_k r_m \end{aligned} \quad \text{or } ((\mathbf{r} \times \mathbf{v}) \times \mathbf{r}) = r^2 \mathbf{v} - (\mathbf{v} \cdot \mathbf{r}) \mathbf{r}$$

Then

$$\widehat{L}_0 = r^2 \dot{\theta} = r \left( \mathbf{v} \cdot \frac{((\mathbf{r} \times \mathbf{v}) \times \mathbf{r})}{|(\mathbf{r} \times \mathbf{v}) \times \mathbf{r}|} \right) = r \left( \frac{r^2 v^2 - (\mathbf{v} \cdot \mathbf{r})^2}{|r^2 \mathbf{v} - (\mathbf{v} \cdot \mathbf{r}) \mathbf{r}|} \right)$$

Thus, we can find the following:

$$e^2 = 1 + \frac{2E\widehat{L}_0^2}{m(GM)^2} = 1 + \frac{2\left(\frac{1}{2}mv_0^2 - \frac{GMm}{r}\right)\widehat{L}_0^2}{m(GM)^2} = 1 + \frac{(v_0^2 - \frac{2GMm}{r})\widehat{L}_0^2}{(GM)^2}$$

$$a = \frac{-GMm}{2E} = -\frac{GMm}{2\left(\frac{1}{2}mv_0^2 - \frac{GMm}{r}\right)} = \frac{-GM}{v_0^2 - \frac{2GM}{r}} = \frac{-GM}{v_0^2\left(1 - \frac{2GM}{rv_0^2}\right)}$$

TODO: Calculate period of revolution.

Here are some calculations (forms) that can help with further calculations:

$$(155) \quad \widehat{L}_0^2 = b^2 \left( \frac{2E}{-m} \right) \quad \text{or} \quad \frac{\widehat{L}_0^2}{GM} = b^2 \left( \frac{2E}{-GMm} \right) = \frac{b^2}{a} = a(1 - e^2)$$

using Eq. 130, for  $e^2 = 1 - \left(\frac{b}{a}\right)^2$ .

22.4. **Path in coordinates.** For  $\mathbf{r}$  centered at focus  $F$  and  $\mathbf{r}$ ,  $\mathbf{v}$  expressed in coordinates in the plane of the conic section path, i.e. in the plane formed by  $\mathbf{r}$ ,  $\mathbf{v}$ , with the axes expressed as  $\frac{\partial}{\partial x}$ ,  $\frac{\partial}{\partial y}$ ,

$$(156) \quad \mathbf{r} = r \cos \theta \frac{\partial}{\partial x} + r \sin \theta \frac{\partial}{\partial y}$$

Compare this to Eq. (2.5-1) of Sec. 2.5.1 "Expressing  $r$  and  $v$  in the Perifocal System" of Bate, Mueller, White (1971) [36].

Recall the equation for the conic section in polar coordinates, with the coordinates centered at focus  $F$ :

$$(157) \quad r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

Compare this to Eq. (1.5-4) of Bate, Mueller, White (1971) [36].

Now

$$(158) \quad \mathbf{v} = \dot{\mathbf{r}} = (\dot{r} \cos \theta - r \dot{\theta} \sin \theta) \frac{\partial}{\partial x} + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta) \frac{\partial}{\partial y}$$

Compare this to the equation on pp. 72, of Sec. 2.5.1 "Expressing  $r$  and  $v$  in the Perifocal System" of Bate, Mueller, White (1971) [36].

Now

$$\dot{r} = \frac{a(1 - e^2)}{(1 + e \cos \theta)^2} e \sin \theta \dot{\theta} = \frac{r^2}{a(1 - e^2)} e \sin \theta \frac{\widehat{L}_0}{r^2} = \frac{\widehat{L}_0 e \sin \theta}{a(1 - e^2)} = \frac{\widehat{L}_0 e \sin \theta}{\frac{-GMm}{2E} \left( \frac{L_0^2}{\left(\frac{(GM)^2 m}{-2E}\right)} \right)} =$$

$$(159) \quad \implies \dot{r} = \frac{GM}{\widehat{L}_0} e \sin \theta$$

Compare this to Eq. (2.5-2) of Sec. 2.5.1 "Expressing  $r$  and  $v$  in the Perifocal System" of Bate, Mueller, White (1971) [36].

Then

$$\implies \frac{a(1 - e^2)}{(1 + e \cos \theta)^2} \dot{\theta} = \frac{GM}{\widehat{L}_0} \quad \text{or}$$

$$(160) \quad r \dot{\theta} = \frac{GM}{\widehat{L}_0} (1 + e \cos \theta)$$

Compare this to Eq. (2.5-3) on pp. 73 of Sec. 2.5.1 "Expressing  $r$  and  $v$  in the Perifocal System" of Bate, Mueller, White (1971) [36].

Then

$$\mathbf{v} = \frac{GM}{\widehat{L}_0} \left[ (e \sin \theta \cos \theta - (1 + e \cos \theta) \sin \theta) \frac{\partial}{\partial x} + (e \sin^2 \theta + (1 + e \cos \theta) \cos \theta) \frac{\partial}{\partial y} \right] =$$

$$(161) \quad \mathbf{v} = \frac{GM}{\widehat{L}_0} \left[ \sin \theta \frac{\partial}{\partial x} + (e + \cos \theta) \frac{\partial}{\partial y} \right] = \frac{GM}{\widehat{L}_0} \left[ \frac{y}{r} \frac{\partial}{\partial x} + \left( e + \frac{x}{r} \right) \frac{\partial}{\partial y} \right]$$

Compare this to Eq. (2.5-4) on pp. 73 of Sec. 2.5.1 "Expressing  $r$  and  $v$  in the Perifocal System" of Bate, Mueller, White (1971) [36].

Consider this ansatz from subsection 4.4.3 "The  $f$  and  $g$  expressions from Ch. 4 section on "Position and velocity - A function of time" in Bate, Mueller, White (1971) [36].

Because  $\mathbf{r}$ ,  $\mathbf{v}$  lie in the plane (i.e. coplanar) containing the entire conic section trajectory, then make the following ansatz:

$$(162) \quad \begin{aligned} \mathbf{r} &= f \mathbf{r}_0 + g \mathbf{v}_0 \\ \dot{\mathbf{r}} &=: \mathbf{v} = \dot{f} \mathbf{r}_0 + \dot{g} \mathbf{v}_0 \end{aligned}$$

Solve for  $f$  and  $g$  by taking the appropriate cross products:

$$(163) \quad \begin{aligned} \mathbf{r} \times \mathbf{v}_0 &= f \mathbf{r}_0 \times \mathbf{v}_0 \\ \mathbf{r} \times \mathbf{r}_0 &= g \mathbf{v}_0 \times \mathbf{r} \end{aligned}$$

**22.5. Eccentricity anomaly; Eccentricity anomaly coordinates.** Let the point of interest be P. Choose the origin for the coordinates  $(x, y)$  to have *its origin* at the *center of the ellipse*.

Let point C be the point at the center of the ellipse (origin), let point A be the projection of point P onto the  $x$ -axis, point F be the point at the focus.

Recall that the length from the center C to the focus,  $\overline{CF}$ , is  $ae$ , where  $a$  is the semi-major axis, and eccentricity is  $e$ .

The eccentricity anomaly  $E$  (it's just an angle) is given by

$$(164) \quad \cos E = \frac{x}{a}$$

Remember  $E$  is *not* the angle formed by triangle CAP, but rather, first take the line through P and A. This line is perpendicular to the  $x$ -axis. Then intersect that line with a circle, centered at C, of radius  $a$ . Then, the line from C to that intersection forms an angle  $E$  with the  $x$ -axis.

By using the ellipse equation and trigonometry,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies \cos^2 E + \frac{y^2}{b^2} = 1 \text{ or } \frac{y^2}{b^2} = \sin^2 E \text{ or}$$

$$(165) \quad \sin E = \frac{y}{b}$$

Now

$$\overline{AF} = \overline{CF} - \overline{CA} = ae - a \cos E$$

Use the right triangle PAF and the hypotenuse being  $r$  (distance from the focus F to point P) to find

$$\overline{AF} = r \cos(\pi - \theta) = -r \cos \theta$$

So

$$\cos \theta = \frac{a(\cos E - e)}{r}$$

Then, plug into the conic section equation, Eq. 126,

$$\begin{aligned} r &= \frac{ed}{1 + e \cos \theta} = \frac{a(1 - e^2)}{1 + e \cos \theta} = \frac{b^2/a}{1 + e \cos \theta} = \frac{b^2/a}{1 + e \left( \frac{a(\cos E - e)}{r} \right)} \\ \implies 1 &= \frac{b^2/a}{r + e(a(\cos E - e))} \\ \implies r &= \frac{b^2}{a} - e(a(\cos E - e)) = \frac{b^2}{a} + ae^2 - ea \cos E = a(1 - e \cos E) \end{aligned}$$

$$(166) \quad \boxed{r = a(1 - e \cos E)}$$

Compare this to Eq. (4.6-9) on pp. 215 of Sec. 4.6 in "The Kepler Problem" of Bate, Mueller, White (1971) [36].

For the coordinates with the origin at the *focus* F, it is given by

$$(167) \quad \begin{aligned} x &= a(\cos E - e) \\ y &= b \sin E = a\sqrt{1 - e^2} \sin E \end{aligned}$$

Compare this to Eqns. (4.6-7), (4.6-8) on pp. 215 of Sec. 4.6 in "The Kepler Problem" of Bate, Mueller, White (1971) [36].

Indeed, for  $E = 0$ ,  $x = a - ae$  (the distance from the focus F to the vertex is  $a - ae$ ),  $E = \pi/2$ ,  $x = -ae$  (the distance from the focus F and back to the center C) and for  $E = \pi$ ,  $x = -a - ae$  (the distance to the other vertex).

22.5.1. *Path in terms of eccentric anomaly.* Take the time derivative of Eq. 167:

$$(168) \quad \begin{aligned} \dot{x} &= -a \sin E \dot{E} \\ \dot{y} &= a\sqrt{1 - e^2} \cos E \dot{E} \end{aligned}$$

To determine  $\dot{E}$ , consider both expressions for the radial distance  $r$  for Eq. 166 and Eq. 126:

$$\begin{aligned} r &= \frac{a(1 - e^2)}{1 + e \cos \theta} = \frac{b}{1 + e \cos \theta} \\ \implies \dot{r} &= \frac{-b}{(1 + e \cos \theta)^2} (-e \sin \theta) \dot{\theta} = \frac{be \sin \theta \dot{\theta}}{(1 + e \cos \theta)^2} = \\ \dot{r} &= \frac{r^2 e}{b} \sin \theta \dot{\theta} \\ r &= a(1 - e \cos E) \\ \implies \dot{r} &= +ae \sin E \dot{E} \\ \implies ae \sin E \dot{E} &= \frac{r^2 e}{b} \sin \theta \dot{\theta} \text{ or } a \sin E \dot{E} = \frac{r^2}{b} \sin \theta \dot{\theta} = \frac{\hat{L}_0 \sin \theta}{b} = \frac{\hat{L}_0}{b} \frac{y}{r} \\ \implies \dot{E} &= \frac{\hat{L}_0}{ab} \frac{y}{r \sin E} = \frac{\hat{L}_0}{ar} = \frac{b}{a^{3/2} r} \end{aligned}$$

Compare the expression for  $\dot{E}$  with Eq. (4.6-11) on pp. 216 of Bate, Mueller, White (1971) [36].

Armed with an expression for  $\dot{E}$ , plug this back into the expression for  $\dot{r}$ :

$$(169) \quad \dot{r} = ae \sin E \left( \frac{\hat{L}_0}{ar} \right) = \frac{e \sin E \hat{L}_0}{r}$$

Compare this to the equation on pp. 216, "Position and Velocity - a function of time" in Ch. 4 of Bate, Mueller, White (1971) [36].

Use the expressions above in Eqns. 163. For  $\mathbf{r}, \mathbf{r}_0$  centered at focus F:

$$\begin{aligned} xv_{0,y} - yv_{0,x} &= a(\cos E - e)b \cos E_0 \dot{E}_0 - b \sin E (-a \sin E_0 \dot{E}_0) = \\ &= ab \dot{E}_0 [(\cos E - e) \cos E_0 + \sin E \sin E_0] = \frac{b^2}{\sqrt{ar}} [\cos(E - E_0) - e \cos E_0] = \frac{\hat{L}^2 \left( \frac{-m}{2E} \right)}{\sqrt{GM} \sqrt{\frac{m}{-2E} r}} [\cos(E - E_0) - e \cos E_0] = \\ &= \frac{\hat{L}_0 b}{r \sqrt{GM}} [\cos(E - E_0) - e \cos E_0] \end{aligned}$$

Comparing this against the "right-hand side (RHS)" with  $f(\mathbf{r}_0 \times \mathbf{v}_0)$ , the RHS has a magnitude of  $f \hat{L}_0$  in the same direction as  $(\mathbf{r} \times \mathbf{v}_0)$ . Thus

$$(170) \quad f = \frac{b}{r \sqrt{GM}} [\cos(E - E_0) - e \cos E_0]$$

For the expression involving  $g$  in Eqns. 163,

$$\mathbf{r} \times \mathbf{r}_0 = xy_0 - yx_0 = a(\cos E - e)b \sin E_0 - b \sin E a(\cos E_0 - e) = -ab [(\cos E_0) \sin E - (\cos E - e) \sin E_0]$$

with

$$ab = a \hat{L}_0 \sqrt{\frac{-m}{2E}} = a \hat{L}_0 \left( \frac{a}{GM} \right)^{1/2} = \left( \frac{a^3}{GM} \right)^{1/2} \hat{L}_0$$

For the RHS with  $g\mathbf{v}_0 \times \mathbf{r}_0$ ,  $\mathbf{v}_0 \times \mathbf{r}_0$ , by the right-hand rule or the right-hand orientation, it is in the *opposite* direction as  $\mathbf{r} \times \mathbf{r}_0$ . Thus, apply a negative side on the RHS.



(171) 
$$\implies g = \left(\frac{a^3}{GM}\right)^{1/2} [(\cos E_0 - e) \sin E - (\cos E - e) \sin E_0]$$

Part 15. General Relativity

Part 16. WE Heraeus International Winter School on Gravity and Light

INTRODUCTION (FROM EY)

The International Winter School on Gravity and Light held *central lectures* given by Dr. Frederic P. Schuller. These lectures on General Relativity and Gravity are unequivocally and undeniably, the best and most lucid and well-constructed lecture series on General Relativity and Gravity. The mathematical foundation from topology and differential geometry from which General Relativity arises from is solid, well-selected in rigor. The lectures themselves are well-thought out and clearly explained.

Even more so, the International Winter School provided accompanying Tutorial Sessions for each of the lectures. I had given up hopes in seeing this component of the learning process ever be put online so that anyone and everyone in the world could learn through the Tutorial process as well. I was afraid that nobody would understand how the Tutorial or “Office Hours” session was important for students to digest and comprehend and work out-doing exercises-the material presented in the lectures. This International Winter School gets it and shows how online education has to be done, to do it in an excellent manner, moving forward.

For anyone who is serious about learning General Relativity and Gravity, I would simply point to these video lectures and tutorials.

What I want to do is to build upon the material presented in this International Winter School. Why it’s important to me, and to the students and practicing researchers out there, is that the material presented takes the student from an introduction to the research frontier. That is the stated goal of the International Winter School. I want to dig into and help contribute to the cutting edge in research and this entire program with lectures and tutorials appears to be the most direct and sensible route directly to being able to do research in General Relativity and Gravity. -EY 20150323

20200321 update: Someone collected all the lecture videos and tutorials:

<http://mathswithphysics.blogspot.com/2016/07/the-we-heraeus-international-winter.html>

23. LECTURE 1: TOPOLOGY

23.1. Lecture 1: Topological Spaces.

**Definition 55.** *Let  $M$  be a set.*

*A **topology**  $\mathcal{O}$  is a subset  $\mathcal{O} \subseteq \mathcal{P}(M)$ ,  $\mathcal{P}(M)$  power set of  $M$ : set of all subsets of  $M$ . satisfying*

- (i)  $\emptyset \in \mathcal{O}, M \in \mathcal{O}$
- (ii)  $U \in \mathcal{O}, V \in \mathcal{O} \implies U \bigcap V \in \mathcal{O}$
- (iii)  $U_\alpha \in \mathcal{O}, \alpha \in \mathcal{A} \implies (\bigcup_{\alpha \in \mathcal{A}} U_\alpha) \in \mathcal{O}$

$\mathcal{O}$ } utterly useless

**Definition 56.**  $\mathcal{O}_{\text{standard}} \subseteq \mathcal{P}(\mathbb{R}^d)$

EY : 20150524  
I’ll fill in the proof that  $\mathcal{O}_{\text{standard}}$  is a topology.

*Proof.*  $\emptyset \in \mathcal{O}_{\text{standard}}$   
since  $\forall p \in \emptyset, \exists r \in \mathbb{R}^+ : \mathcal{B}_r(p) \subseteq \emptyset$  (i.e. satisfied “vacuously”)

Suppose  $U, V \in \mathcal{O}_{\text{standard}}$ .

Let  $p \in U \bigcap V$ . Then  $\exists r_1, r_2 \in \mathbb{R}^+$  s.t.  $\mathcal{B}_{r_1}(p) \subseteq U$   
 $\mathcal{B}_{r_2}(p) \subseteq V$

Let  $r = \min \{r_1, r_2\}$ .  
Clearly  $\mathcal{B}_r(p) \subseteq U$  and  $\mathcal{B}_r(p) \subseteq V$ . Then  $\mathcal{B}_r(p) \subseteq U \bigcap V$ . So  $U \bigcap V \in \mathcal{O}_{\text{standard}}$ .

Suppose,  $U_\alpha \in \mathcal{O}_{\text{standard}}, \forall \alpha \in \mathcal{A}$ .  
Let  $p \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ . Then  $p \in U_\alpha$  for at least 1  $\alpha \in \mathcal{A}$ .  
 $\exists r_\alpha \in \mathbb{R}^+$  s.t.  $\mathcal{B}_{r_\alpha}(p) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ . So  $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{O}_{\text{standard}}$  □

23.2. **2. Continuous maps.**

23.3. **3. Composition of continuous maps.**

23.4. **4. Inheriting a topology.** EY : 20150524  
I’ll fill in the proof that given  $f$  continuous (cont.), then the restriction of  $f$  onto a subspace  $S$  is cont. If you want a reference, check out Klaus Jänich [?, pp. 13, Ch. 1 Fundamental Concepts, Sec. Continuous Maps]  
If cont.  $f : M \rightarrow N, S \subseteq M$ , then  $f|_S$  cont.

*Proof.* Let open  $V \subseteq N$ , i.e.  $V \in \mathcal{O}_N$  i.e.  $V$  in the topology  $\mathcal{O}_N$  of  $N$ .

$$f|_S^{-1}(V) = \{m \in M \mid f|_S(m) \in V\}$$

Now  $f^{-1}(V) = \{m \in M \mid f(m) \in V\}$ .

So  $f^{-1}(V) \bigcap S = f|_S^{-1}(V)$   
Now  $f$  cont. So  $f^{-1}(V) \in \mathcal{O}_N$ .  
and recall  $\mathcal{O}_S := \{U \bigcap S \mid U \in \mathcal{O}_M\}$ .  
so  $f^{-1}(V) \bigcap S = f|_S^{-1}(V) \in \mathcal{O}_S$  i.e.  $f|_S^{-1}(V)$  open.  
 $\implies f|_S$  cont. □

TOPOLOGY TUTORIAL SHEET

filename : `main.pdf`

The WE-Heraeus International Winter School on Gravity and Light: Topology

EY : 20150524

What I won’t do here is retype up the solutions presented in the Tutorial (cf. [https://youtu.be/\\_XkhZQ-hNLs](https://youtu.be/_XkhZQ-hNLs)): the presenter did a very good job. If someone wants to type up the solutions and copy and paste it onto this LaTeX file, in the spirit of open-source collaboration, I would encourage this effort.

Instead, what I want to encourage is the use of as much CAS (Computer Algebra System) and symbolic and numerical computation because, first, we’re in the 21st century, second, to set the stage for further applications in research. I use Python and Sage Math alot, mostly because they are open-source software (OSS) and fun to use. Also note that the structure of Sage Math modules matches closely to Category Theory.

In checking whether a set is a topology, I found it strange that there wasn’t already a function in Sage Math to check each of the axioms. So I wrote my own; see my code snippet, which you can copy, paste, edit freely in the spirit of OSS here, titled `topology.sage`:

[gist github ernestyalumni topology.sage](#)

[Download topology.sage](#)

Loading `topology.sage`, after changing into (with the usual Linux terminal commands, `cd`, `ls`) by

`sage: load('‘topology.sage’')`



Exercise 2: Topologies on a simple set.

Question Does  $\mathcal{O}_1 := \dots$  constitute a topology  $\dots$ ?

Solution: Yes, since we check by typing in the following commands in Sage Math:

```
emptyset in O_1
Axiom2check(O_1) # True
Axiom3check(O_1) # True
```

Question What about  $\mathcal{O}_2 \dots$ ?

Solution: No since the 3rd. axiom fails, as can be checked by typing in the following commands in Sage Math:

```
emptyset in O_2
Axiom2check(O_2) # True
Axiom3check(O_2) # False
```

24. LECTURE 2: TOPOLOGICAL MANIFOLDS

Lecture 2: Manifolds. Topological spaces:  $\exists$  so man that mathematicians cannot even classify them.

For spacetime physics, we may focus on topological spaces  $(M, \mathcal{O})$  that can be charted, analogously to how the surface of the earth is charted in an atlas.

24.1. Topological manifolds.

Definition 57. A topological space  $(M, \mathcal{O})$  is called a ***d-dimensional topological method*** if  $\forall p \in M : \exists U \in \mathcal{O}, U \ni p : \exists x : U \subseteq M \rightarrow x(U) \subseteq \mathbb{R}^d \quad (M, \mathcal{O}), (\mathbb{R}^d, \mathcal{O}_{std})$

(i)  $x$  ***invertible*** :

$$x^{-1} : x(U) \rightarrow U$$

(ii)  $x$  ***continuous***

(iii)  $x^{-1}$  ***continuous***

24.2. Terminology.

24.3. **3. Chart transition maps.** Imagine 2 charts  $(U, x)$  and  $(V, y)$  with overlapping regions.

24.4. **4. Manifold philosophy.** Often it is desirable (or indeed the way) to define properties (“continuity”) of real-world object  $(\mathbb{R} \xrightarrow{\gamma} M)$  by judging suitable coordinates not on the “real-world” object itself, but on a chart-representation of that real world object.

EY’s add-ons. This lecture gives me a good excuse to review Topology and Topological Manifolds from a mathematician’s point of view. I find John M. Lee’s **Introduction to Topological Manifolds** book good because it’s elementary and thorough and it’s fairly recent (2010) so it’s up to date [?]. See my notes and solutions for the book; it’s a file titled `LeeJM_IntroTopManifolds_sol.pdf` of which I’ll try to keep the pdf and LaTeX file available for download on my ernestyalumni Google Drive (so try to search for it on Google).

filename: Sheet\_1.2.pdf

Exercise 4: Before the invention of the wheel.

Another one-dimensional topological manifold. Another one?

Consider set  $F^1 := \{(m, n) \in \mathbb{R}^2 | m^4 + n^4 = 1\}$ , equipped with subset topology  $\mathcal{O}_{std}|_{F^1}$

Question  $x : F^1 \rightarrow \mathbb{R}$  is what?.

Solution . EY : 20150525 The tutorial video [https://youtu.be/ghfEQ3u\\_B6g](https://youtu.be/ghfEQ3u_B6g) is really good and this solution is how I’d write it, but it’s really the same (I needed the practice).

$$\begin{aligned} x : F^1 &\rightarrow \mathbb{R} \\ (m, n) &\mapsto m \end{aligned}$$

If  $m = 0, n^4 = 1$  so  $n = \pm 1$  so it’s not injective.

Let the closed  $n$ -dim. upper half-space  $\mathbb{H}^n \subseteq \mathbb{R}^1$ . Then

$$\begin{aligned} \mathbb{H}^n &= \{(x_1 \dots x_n) \in \mathbb{R}^n | x_n \geq 0\} \\ \text{int}\mathbb{H}^n &= \{(x_1 \dots x_n) \in \mathbb{R}^n | x_n > 0\} \\ -\mathbb{H}^n &= \{(x_1 \dots x_n) \in \mathbb{R}^n | x_n \leq 0\} \\ -\text{int}\mathbb{H}^n &= \{(x_1 \dots x_n) \in \mathbb{R}^n | x_n < 0\} \end{aligned}$$

Question This map  $x$  may be made injective by restricting its domain to either of 2 maximal open subsets of  $F^1$ . Which ones?.

Solution .

Let

$$\begin{aligned} U_+ &= F^1 \cap \text{int}\mathbb{H}^2 \\ U_- &= F^1 \cap -\text{int}\mathbb{H}^2 \end{aligned}$$

Look at

$$\begin{aligned} x^4 &= 1 - n^4 \\ \implies x &= \pm(1 - n^4)^{1/4} \end{aligned}$$

Then for

$$\begin{aligned} x_+^{-1} : (-1, 1) &\subseteq \mathbb{R} \rightarrow U_+ \\ m &\mapsto (m, (1 - m^4)^{1/4}) \\ x_-^{-1} : (-1, 1) &\subseteq \mathbb{R} \rightarrow U_- \\ m &\mapsto (m, -(1 - m^4)^{1/4}) \end{aligned}$$

$x_+, x_-$  injective (since left inverse exists).

Question Construct injective  $y$ .

Solution .

Let

$$\begin{aligned} V_+ &= F^1 \cap \text{int}\mathbb{H}^1 \\ V_- &= F^1 \cap -\text{int}\mathbb{H}^1 \end{aligned}$$

Then

$$\begin{aligned} y_+ : V_+ &\rightarrow (-1, 1) \subseteq \mathbb{R} \\ (m, n) &\mapsto n \\ y_- : V_- &\rightarrow (-1, 1) \subseteq \mathbb{R} \\ (m, n) &\mapsto n \end{aligned}$$

**Question** Construct inverse  $y^{-1}$ . **Solution** .

For

$$\begin{aligned} y_+^{-1} : (-1, 1) &\subseteq \mathbb{R} \rightarrow V_+ \\ n &\mapsto ((1 - n^4)^{1/4}, n) \\ y_-^{-1} : (-1, 1) &\subseteq \mathbb{R} \rightarrow V_- \\ n &\mapsto -(1 - n^4)^{1/4}, n) \end{aligned}$$

$y_+, y_-$  injective (since left inverse exists).

Note  $(-1, 0) \notin U_+, U_-$   
 $(1, 0) \notin U_+, U_-$   
and

$(0, 1) \notin V_+, V_-$   
 $(0, -1) \notin V_+, V_-$

**Question** construct *transition map*  $x \circ y^{-1}$ .

**Solution** .

$$\begin{aligned} x_+ y_+^{-1} : (0, 1) &\subseteq \mathbb{R} \rightarrow (0, 1) \subseteq \mathbb{R} \\ n &\mapsto (1 - n^4)^{1/4} \\ x_- y_+^{-1} : (-1, 0) &\subseteq \mathbb{R} \rightarrow (0, 1) \subseteq \mathbb{R} \\ n &\xrightarrow{y_+^{-1}} ((1 - n^4)^{1/4}, n) \xrightarrow{x_-} (1 - n^4)^{1/4} \\ x_+ y_-^{-1} : (0, 1) &\subseteq \mathbb{R} \rightarrow (-1, 0) \subseteq \mathbb{R} \\ n &\mapsto -(1 - n^4)^{1/4} \\ x_- y_-^{-1} : (-1, 0) &\subseteq \mathbb{R} \rightarrow (-1, 0) \subseteq \mathbb{R} \\ n &\mapsto -(1 - n^4)^{1/4} \end{aligned}$$

**Question** ... Does the collection of these domains and maps form an atlas of  $F^1$ ?

Yes, with atlas

$$\mathcal{A} = \left\{ \begin{pmatrix} U_+, x_+ \\ U_-, x_- \end{pmatrix}, \begin{pmatrix} V_+, y_+ \\ V_-, y_- \end{pmatrix} \right\}$$

Clearly

$$\begin{aligned} U_+ \cup U_- \cup V_+ \cup V_- &= (F^1 \cap \text{int} \mathbb{H}^2) \cup (F^1 \cap -\text{int} \mathbb{H}^2) \cup (F^1 \cap \text{int} \mathbb{H}^1) \cup (F^1 \cap -\text{int} \mathbb{H}^1) = \\ &= F^1 \cap \mathbb{R}^2 \setminus \{(0, 0)\} = F^1 \end{aligned}$$

and (the point is that)  $x_{\pm}, y_{\pm}$  are homeomorphisms of open sets of  $F^1$  onto open sets of 1 dim.  $\mathbb{R}^1$  (namely  $(-1, 1) \subseteq \mathbb{R}^1$ ), and so  $\mathcal{A}$  is an atlas of  $F^1$ .

25. LECTURE 3: MULTILINEAR ALGEBRA

Lecture 3: Multilinear Algebra (International Winter School on Gravity and Light 2015)

We will **not** equip space(time) with a vector space structure. Do you know where

$$\begin{aligned} 5 \cdot \text{Paris} &=? \\ \text{Paris} + \text{Vienna} &=? \\ \text{lie ?} \end{aligned}$$

Moreover, the tangent spaces  $T_p M$  (lecture 5) smooth manifolds (Lecture 4)  
Beneficial to first study vector spaces abstractly for two reason

- (i) for construction of  $T_p M$  one needs an intermediate vector space  $C^\infty(M)$
- (ii) tensor technique are most easily understood in an abstract setting.

25.1. Vector spaces.

**Definition 58.** A vector space  $(V, +, -)$  is

- (i) a set  $V$
- (ii)  $+: V \times V \rightarrow V$  “addition”
- (iii)  $\cdot: \mathbb{R} \times V \rightarrow V$  “s-multiplication” EY : 20160317 s for “scalar”

satisfying:

$$CANIADDU$$

$$\begin{aligned} C^+ : \quad & v + w = w + v \\ A^+ : \quad & (u + v) + w = u + (v + w) \\ N^+ : \quad & \exists 0 \in V : \forall v \in V : v + 0 = v \\ I^+ : \quad & \forall v \in V : \exists (-v) \in V : v + (-v) = 0 \\ A : \quad & \lambda \cdot (\mu + v) = (\lambda \cdot \mu) \cdot v \quad (\forall \lambda, \mu \in \mathbb{R}) \\ D : \quad & (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v \\ D : \quad & \lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w) \\ U : \quad & 1 \cdot v = v \end{aligned}$$

*Terminology.* An element of a vector space is often referred to, informally as a vector.

*Example.* def. **set** of polynomials (fixed) degree  $\mathcal{P} := \{p : (-1, +1) \rightarrow \mathbb{R} | p(x) = \sum_{n=0}^N p_n \cdot x^n\}$

Thought bubble: is  $\square$  a vector?  
 $\square(x) = x^2$

No  $\square \in \mathcal{P}$ .  
 $+: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$

$$(p, q) \mapsto p + q$$

where  $(p + q)(x) = p(x) +_{\mathbb{R}} q(x)$

$$\cdot: \mathbb{R} \times \mathcal{P} \rightarrow \mathcal{P}$$

$$(\lambda, p) \mapsto \lambda \cdot p$$

where  $(\lambda \cdot p)(x) := \lambda \cdot_{\mathbb{R}} p(x)$

Thought bubble:  $\square$  a vector?      Yes, but who cares?  
 $(\mathcal{P}, +, \cdot)$  is a vector space.  
 $\square \in \mathcal{P}$

25.2. **Linear maps.** These are the structure-respecting maps between vector spaces.  
EY : 20160316 out of tradition, they’re called “linear” maps

**Definition 59.**  $(V, +_V, \cdot_V)$  and  $(W, +_W, \cdot_W)$  vector spaces  
Then a map

$$\varphi : V \rightarrow W$$

is called **linear** if

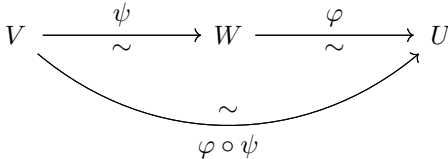
- (i)  $\varphi(v +_V \tilde{v}) = \varphi(v) +_W \varphi(\tilde{v})$
- (ii)  $\varphi(\lambda \cdot_V v) = \lambda \cdot_W \varphi(v)$

Example. :  $\delta : \mathcal{P} \rightarrow \mathcal{P}$   
 $p \mapsto \delta(p) := p'$

linear:

- (i)  $\delta(p + q) = (p +_P q)'$   $\underbrace{=}_{\text{sum rule}} p' +_P q' = \delta(p) +_P \delta(q)$
- (ii)  $\delta(\lambda p) = (\lambda p)' = \lambda \cdot p' \stackrel{\text{sum rule}}{=} \lambda \cdot \delta(p)$

Notation:  $\varphi : V \rightarrow W$  linear  $\iff \varphi : V \xrightarrow{\sim} W$



25.2.1. Example\*.  $\delta \circ \delta : \mathcal{P} \xrightarrow{\sim} \mathcal{P}$

25.3. **Vector space of Homomorphisms.** fun fact:  $(V, +, \cdot)$   $(W, +, \cdot)$  vector spaces  
def.  $\text{Hom}(V, W) := \{\varphi : V \xrightarrow{\sim} W\}$  set.  
We can make this into a vector spaces.

$$\oplus : \text{Hom}(V, W) \times \text{Hom}(V, W) \rightarrow \text{Hom}(V, W)$$
$$(\varphi, \psi) \mapsto \varphi \oplus \psi$$

where  $(\varphi \otimes \psi)(v) := \varphi(v) +_W \psi(v)$   
 $\otimes : \dots$  similarly.  
 $(\text{Hom}(V, W), \oplus, \otimes)$  **is** a vector space.

25.3.1. Example\*.  $\text{Hom}(\mathcal{P}, \mathcal{P})$  is a vector space.  
 $\delta \in \text{Hom}(\mathcal{P}, \mathcal{P})$   
 $\delta \circ \delta \in \text{Hom}(\mathcal{P}, \mathcal{P})$

$\vdots$   
 $\underbrace{\delta \circ \dots \circ \delta}_M \in \text{Hom}(\mathcal{P}, \mathcal{P})$

$$\implies 5 \circ \delta \oplus_{\text{Hom}(\mathcal{P}, \mathcal{P})} \delta \circ \delta \in \text{Hom}(\mathcal{P}, \mathcal{P})$$

25.4. **Dual vector space.** heavily used special case:  
 $(V, +, \cdot)$  vector space:

**Definition 60.**

$$V^* := \{\varphi : V \xrightarrow{\sim} \mathbb{R}\} = \text{Hom}(V, \mathbb{R})$$

$$\underbrace{(V^*, \oplus, \otimes)}_{\text{dual vector space (to } V\text{)}} \text{ is a vector space}$$

Terminology:  $\varphi \in V^*$  is called, informally, a covector.

Example.  $I : \mathcal{P} \xrightarrow{\sim} \mathbb{R}$

i.e.  $I \in \mathcal{P}^*$

def.  $I(p) := \int_0^1 dx p(x)$

linear: 
$$I(p + q) = \int_0^1 dx \underbrace{(p + q)(x)}_{p(x) + q(x)} = \dots = I(q) + I(p)$$

$$I(\lambda p) = \lambda \cdot I(p)$$

i.e.  $I = \int_0^1 dx$

25.5. **Tensors.**

**Definition 61.** Let  $(V, +, \cdot)$  be a vector space.

An  $(r, s)$ -tensor  $T$  over  $V$   $r, s \in \mathbb{N}_0$   
is a multi-linear map

$$T : \underbrace{V^* \times \dots \times V^*}_r \times \underbrace{V \times \dots \times V}_s \xrightarrow[\sim]{\underbrace{\qquad}_{r+s}} \mathbb{R}$$

25.5.1. Example. :  $T$  (1,1)-tensor

$$\begin{array}{ll} T(\varphi + \psi, v) = T(\varphi, v) + T(\psi, v) & T(\varphi, v + w) = T(\varphi, v) + T(\varphi, w) \\ T(\lambda \varphi, v) = \lambda \cdot T(\varphi, v) & T(\varphi, \lambda \cdot v) = \lambda T(\varphi, v) \end{array}$$

$$\begin{array}{l} T(\varphi + \psi, v + w) = \\ = t(\varphi, v) + T(\varphi, w) + T(\psi, v) + t(\psi, w) \end{array}$$

Excursion: Given  $T : V^* \times V \xrightarrow{\sim} \mathbb{R}$   
 $\phi_T : V \xrightarrow{\sim} (V^*)^* \underbrace{=}_{\dim V < \infty} V$

Define  $v \mapsto \underbrace{T(\cdot, v)}_{V^* \xrightarrow{\sim} \mathbb{R}}$

Given  $\phi : V \xrightarrow{\sim} V$

Construct  $T_\phi : V^* \times V \xrightarrow{\sim} \mathbb{R}$

$$(\varphi, v) \mapsto \varphi(\phi(v))$$

$\implies$  given  $T : T = T_{\varphi_T}$

given  $\phi : \phi = \phi_{T_\phi}$

*Example.*  $g : P \times P \xrightarrow{\sim} \mathbb{R}$   
 $(p, q) \mapsto \int_{-1}^1 dx p(x) q(x)$   
is a  $(0, 2)$ -tensor over  $P$ .  
Info: If  $T \in \text{Hom}(V, W)$

25.6. **Vectors and covectors as tensors.**

**Theorem 24.** *(including proof)*  
“covector”  $\varphi \in V^* \iff \varphi : V \xrightarrow{\sim} \mathbb{R} \iff \varphi(0, 1)\text{-tensor}.$

**Theorem 25.**  $v \in V \underset{\dim V < \infty}{=} (V^*)^* \iff v : V^* \xrightarrow{\sim} \mathbb{R} \iff v \text{ is } (1, 0)\text{-tensor}.$

25.7. **Bases.**

**Definition 62.**  $(V, +, \cdot)$  vector space.  
A subset  $B \subset V$  is called  
a basis if

Thought bubble: Hamel (L.A.) EY : 20160316 Hamel basis, Linear Algebra

$$\forall v \in V \quad \exists \text{ \underline{finite} } \underbrace{F}_{\{f_1, \dots, f_n\}} \subset B : \exists ! \underbrace{v^1, v^2, \dots, v^n}_{\in \mathbb{R}}, \quad v = v^1 f_1 + \dots + v^n f_n$$

**Definition 63.** If  $\exists$  basis  $\mathcal{B}$  with finitely many elements, say  $d$  many, then we call  $d =: \dim V$   
This is well-defined.

Remark:  $(V, +, \cdot)$  be a finite-dim. vector space.  
Having chosen a basis  $e_1, \dots, e_n$  of  $(V, +, \cdot)$  we may uniquely associate  
(Thought bubble: this requires a chosen basis)

$$v \mapsto (v^1, \dots, v^n) \text{ called the components of } v \text{ w.r.t. chosen basis}$$

where:  $v^1 e_1 + \dots + v^n e_n + n = v$

25.8. **Basis for the dual space.** choose Basis  $e_1, \dots, e_n$  for  $V$   
can choose Basis  $\epsilon^1, \dots, \epsilon^n$  for  $V^*$   
However, more economical to require  
once  $e_1, \dots, e_n$  on  $V$  has been chosen, that

$$\epsilon^a(e_b) = \delta_b^a$$

This uniquely determines choice of  $\epsilon^1, \dots, \epsilon^n$  from choice of  $e_1, \dots, e_n$

**Definition 64.** If a basis  $\epsilon^1, \dots, \epsilon^n$  of  $V^*$  satisfies this, it is called the **dual basis** (of the dual space)

Example:  $P$  ( $N = 3$ )  
 $e_0(x) = 1$   
 $e_1(x) = x$   
 $e_0, e_1, e_2, e_3$  basis if  $e_2(x) = x^2 \{ e_a(x) := x^a$   
 $e_3(x) = x^3$   
 $\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3$  dual basis  $\epsilon^a := \frac{1}{a!} \partial^a|_{x=0}$

*Proof.*  $\epsilon^a(e_b) = \delta_b^a$

25.9. **Components of tensors.** Let  $T$  be an  $(r, s)$ -tensor on a finite-dim. vs.  $V$ . Then define the  $(r + s)^{\dim V}$  many real numbers.

$$\underbrace{T^{i_1 \dots i_r}_{j_1 \dots j_s}}_{\in \mathbb{R}} := T(\epsilon^{i_1}, \dots, \epsilon^{i_r}, e_{j_1}, e_{j_2}, \dots, e_{j_s})$$

$i_1 \dots i_r, j_1 \dots j_s \in \{1, \dots, \dim V\}$   
Thought bubble:  $\underbrace{T^{i_1 \dots i_r}_{j_1 \dots j_s}}_{\in \mathbb{R}}$  are the components of the tensor w.r.t. chosen basis

Useful: Knowing components (and basis) one can reconstruct the entire tensor.

*Example.*  $T$   $(1, 1)$ - tensor

$$T^i_j := T(\epsilon^i, e_j)$$

reconstruct

$$\begin{aligned} T(\varphi, v) &= T\big(\sum_{i=1}^{\dim V} \varphi_i \epsilon^i, \sum_{j=1}^{\dim V} v^j e_j\big) \quad \varphi_i \in \mathbb{R}, v^j \in \mathbb{R} \\ &= \sum_{i=1}^{\dim V} \sum_{j=1}^{\dim V} \varphi_i v^j \underbrace{T(\epsilon^i, e_j)}_{T^i_j} \\ &=: \varphi_i v^j T^i_j \end{aligned}$$

26. LECTURE 4: DIFFERENTIABLE MANIFOLDS

so far: top. mfd.  $(M, \mathcal{O})$   
 $\dim M = d$   
we wish to define a notion of differentiable  
curves  $\mathbb{R} \rightarrow M$   
function  $M \rightarrow \mathbb{R}$   
maps  $M \rightarrow N$

26.1. **1. Strategy.** choose a chart  $(U, x)$   
 $\gamma : \mathbb{R} \rightarrow M$  portion of curve in chart domain

$$\begin{array}{ccc} \gamma : \mathbb{R} & \xrightarrow{\quad} & U \\ & \searrow x \circ \gamma & \downarrow x \\ & & x(U) \subseteq \mathbb{R}^d \end{array}$$

idea. try to “lift” the undergraduate notion of differentiability of a curve on  $\mathbb{R}^d$  to a notion of differentiability of a curve on  $M$

□ Problem Can this be well-defined under change of chart?

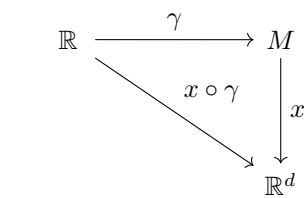
EY : 20151109 Schuller says:  $C^k$  is easy to work with because you can judge  $k$ -times cont. differentiability from existence of all partial derivatives **and** their continuity. There are examples of maps that partial derivatives exist but are not  $D^k$ ,  $k$ -times differentiable.

**Theorem 26** (Whitney). *Any  $C^{k \geq 1}$ -atlas,  $\mathcal{A}_{C^{k \geq 1}}$  of a topological manifold contains a  $C^\infty$ -atlas.*

Thus we may w.l.o.g. always consider  $C^\infty$ -manifolds, “smooth manifolds”, unless we wish to define Taylor expandibility/-complex differentiability ...

EY : 20151109 Hassler Whitney <sup>7</sup>

**Definition 68.** A smooth manifold (  $\underbrace{M, \mathcal{O}}_{top. \text{ mfd.}}$  ,  $\underbrace{\mathcal{A}}_{C^\infty\text{-atlas}}$  )



EY: 20151109 Schuller was explaining that the trajectory is real in  $M$ ; the coordinate maps to obtain

coordinates is  $x \circ \gamma$

26.3. **4. Diffeomorphisms.**  $M \xrightarrow{\phi} N$

If  $M, N$  are naked sets, the structure preserving maps are the bijections (invertible maps).  
e.g.  $\{1, 2, 3\} \rightarrow \{a, b\}$

**Definition 69.**  $M \cong_{set} N$  (set-theoretically) isomorphic if  $\exists$  bijection  $\phi : M \rightarrow N$

Examples.  $\mathbb{N} \cong_{set} \mathbb{Z}$   
 $\mathbb{N} \cong_{set} \mathbb{Q}$  (EY: 20151109 Schuller says from diagonal counting)  
 ~~$\mathbb{N} \cong_{set} \mathbb{R}$~~

Now  $(M, \mathcal{O}_M) \cong_{top} (N, \mathcal{O}_N)$  (topl.) isomorphic = “homeomorphic”  $\exists$  bijection  $\phi : M \rightarrow N$   
 $\phi, \phi^{-1}$  are continuous.

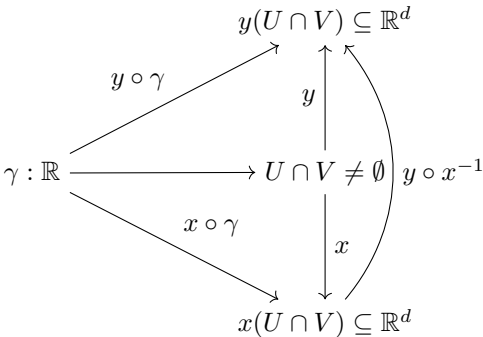
$(V, +, \cdot) \cong_{vec} (W, +_w, \cdot_w)$  (EY: 20151109 vector space isomorphism) if  
 $\exists$  bijection  $\phi : V \rightarrow W$  linearly  
finally

**Definition 70.** Two  $C^\infty$ -manifolds  
 $(M, \mathcal{O}_M, \mathcal{A}_M)$  and  $(N, \mathcal{O}_N, \mathcal{A}_N)$  are said to be **diffeomorphic** if  $\exists$  bijection  $\phi : M \rightarrow N$  s.t.

$$\phi : M \rightarrow N$$

$$\phi^{-1} : N \rightarrow M$$

are both  $C^\infty$ -maps



$x \circ \gamma$  undergraduate differentiable (“as a map  $\mathbb{R} \rightarrow \mathbb{R}^d$ ”)

$$\underbrace{y \circ \gamma}_{\text{maybe only continuous, but not undergraduate differentiable}} = \underbrace{(y \circ x^{-1})}_{\text{continuous}} \circ \underbrace{(x \circ \gamma)}_{\text{undergrad differentiable}} = y \circ (x^{-1} \circ x) \circ \gamma$$

At first sight, strategy does not work out.

26.2. **2. Compatible charts.** In section 1, we used any imaginable charts on the top. mfd.  $(M, \mathcal{O})$ .

To emphasize this, we may say that we took  $U$  and  $V$  from the *maximal atlas*  $\mathcal{A}$  of  $(M, \mathcal{O})$ .

**Definition 65.** Two charts  $(U, x)$  and  $(V, y)$  of a top. mfd. are called  $\clubsuit$ -compatible if either

- (a)  $U \cap V = \emptyset$  or
- (b)  $U \cap V \neq \emptyset$

chart transition maps have undergraduate  $\clubsuit$  property.

EY : 20151109 e.g. since  $\mathbb{R}^d \rightarrow \mathbb{R}^d$ , can use undergradate  $\clubsuit$  property such as continuity or differentiability.

$$y \circ x^{-1} : x(U \cap V) \subseteq \mathbb{R}^d \rightarrow y(U \cap V) \subseteq \mathbb{R}^d$$
$$x \circ y^{-1} : y(U \cap V) \subseteq \mathbb{R}^d \rightarrow x(U \cap V) \subseteq \mathbb{R}^d$$

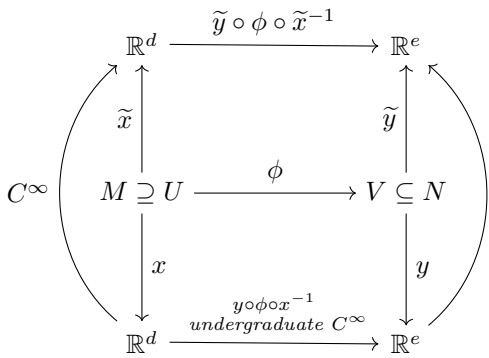
Philosophy:

**Definition 66.** An atlas  $\mathcal{A}_\clubsuit$  is a  $\clubsuit$ -compatible atlas if any two charts in  $\mathcal{A}_\clubsuit$  are  $\clubsuit$ -compatible.

**Definition 67.** A  $\clubsuit$ -manifold is a triple (  $\underbrace{M, \mathcal{O}}_{top. \text{ mfd.}}$  ,  $\mathcal{A}_\clubsuit$  )  $\mathcal{A}_\clubsuit \subseteq \mathcal{A}_{maximal}$

$\clubsuit$	undergraduate $\clubsuit$	
$C^0$	$C^0(\mathbb{R}^d \rightarrow \mathbb{R}^d) =$	continuous maps w.r.t. $\mathcal{O}$
$C^1$	$C^1(\mathbb{R}^d \rightarrow \mathbb{R}^d) =$	differentiable (once) and is continuous
$C^k$		$k$ -times continuously differentiable
$D^k$		$k$ -times differentiable
$\vdots$		
$C^\infty$	$C^\infty(\mathbb{R}^d \rightarrow \mathbb{R}^d)$	
$\cup$		
$C^\omega$	$\exists$ multi-dim. Taylor exp.	
$\mathbb{C}^\infty$	satisfy Cauchy-Riemann equations, pair-wise	

<sup>7</sup><http://mathoverflow.net/questions/8789/can-every-manifold-be-given-an-analytic-structure>



**Theorem 27.**  $\#$  = number of  $C^\infty$ -manifolds one can make out of a given  $C^0$ -manifolds (if any) - up to diffeomorphisms.

$\dim M$	$\#$	
1	1	Morse-Radon theorems
2	1	Morse-Radon theorems
3	1	Morse-Radon theorems
4	uncountably infinitely many	
5	finite	surgery theory
6	finite	surgery theory
$\vdots$	finite	surgery theory

EY : 20151109 cf. <http://math.stackexchange.com/questions/833766/closed-4-manifolds-with-uncountably-many-differentiable-structures>  
The wild world of 4-manifolds

TUTORIAL 4 DIFFERENTIABLE MANIFOLDS

EY : 20151109 The [gravity-and-light.org](http://gravity-and-light.org) website, where you can download the tutorial sheets *and* the full length videos for the tutorials and lectures, are no longer there. = (  
Hopefully, the YouTube video will remain: [https://youtu.be/FXPdKxOq1KA?list=PLFeEvEPtX\\_ORQ1ys-7VIsKlBWz7RX-FaL](https://youtu.be/FXPdKxOq1KA?list=PLFeEvEPtX_ORQ1ys-7VIsKlBWz7RX-FaL)

**Exercise 1: True or false?.** *These basic questions are designed to spark discussion and as a self-test.*

Tick the correct statements, but not the incorrect ones!

- (a) The function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , ...
- - 
  - ..., defined by  $f(x) = |x^3|$ , lies in  $C^3(\mathbb{R} \rightarrow \mathbb{R})$ .

EY : 20151109 **Solution 1a3.** For  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = |x^3| = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$

$$f'(x) = \begin{cases} 3x^2 & \text{if } x \geq 0 \\ -3x^2 & \text{if } x < 0 \end{cases}$$
$$f''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

Thus,

$f(x) = |x^3| \in C^1(\mathbb{R})$  but  $f(x) \notin C^2(\mathbb{R}) \subseteq C^3(\mathbb{R})$

- 
- 

- (b)  
(c)

**Short Exercise 4: Undergraduate multi-dimensional analysis .**

*A good notation and basic results for partial differentiation.*

For a map  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  we denote by the map  $\partial_i f : \mathbb{R}^d \rightarrow \mathbb{R}$  the partial derivative with respect to the  $i$ -th entry.

**Question .:** Given a function

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}; (\alpha, \beta, \delta) \mapsto f(\alpha, \beta, \delta) := \alpha^3 \beta^2 + \beta^2 \delta + \delta$$

calculate the values of the following derivatives:

**Solution .:**

- $(\partial_2 f)(x, y, z) =$
- $(\partial_1 f)(\square, \circ, *) =$
- $(\partial_1 \partial_2 f)(a, b, c) =$
- $(\partial_3^2 f)(299, 1222, 0) =$

EY: 20151110

For  $f(\alpha, \beta, \delta) := \alpha^3 \beta^2 + \beta^2 \delta + \delta$ , or  $f(x, y, z) = x^3 y^2 + y^2 z + z$ ,

$$(\partial_2 f) = 2(x^3 y + y z)$$

$$(\partial_1 f) = 3x^2 y^2$$

$$(\partial_1 \partial_2 f) = 6x^2 y$$

$$(\partial_3^2 f) = 0$$

and so

- $(\partial_2 f)(x, y, z) = 2(x^3 y + y z)$
- $(\partial_1 f)(\square, \circ, *) = 3\square^2 \circ^2$
- $(\partial_1 \partial_2 f)(a, b, c) = 6a^2 b$
- $(\partial_3^2 f)(299, 1222, 0) = 0$

**Exercise 5: Differentiability on a manifold.**

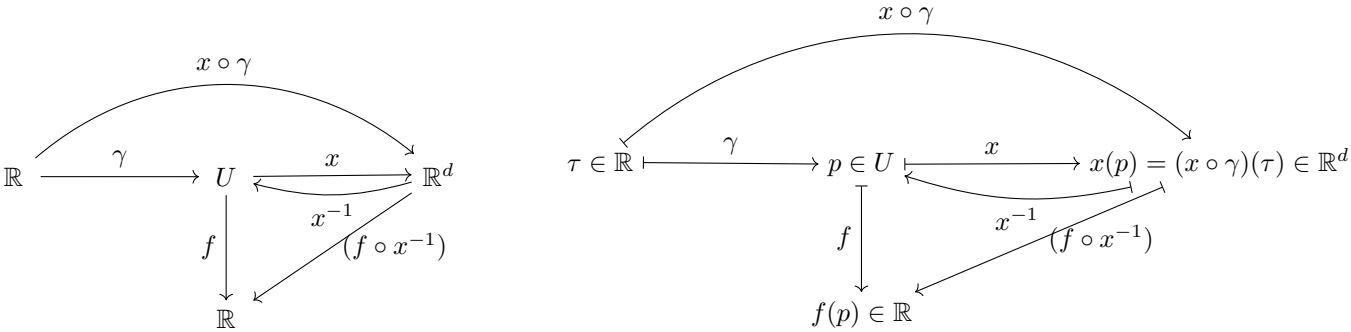
*How to deal with functions and curves in a chart*

Let  $(M, \mathcal{O}, \mathcal{A})$  be a smooth  $d$ -dimensional manifold. Consider a chart  $(U, x)$  of the atlas  $\mathcal{A}$  together with a smooth curve  $\gamma : \mathbb{R} \rightarrow U$  and a smooth function  $f : U \rightarrow \mathbb{R}$  on the domain  $U$  of the chart.

**Question .:** Draw a commutative diagram containing the chart domain, chart map, function, curve and the respective represen-

tatives of the function and the curve in the chart.

**Solution .:**





**Question ::** Consider, for  $d = 2$ ,

$$(x \circ \gamma)(\lambda) := (\cos(\lambda), \sin(\lambda)) \text{ and } (f \circ x^{-1})((x, y)) := x^2 + y^2$$

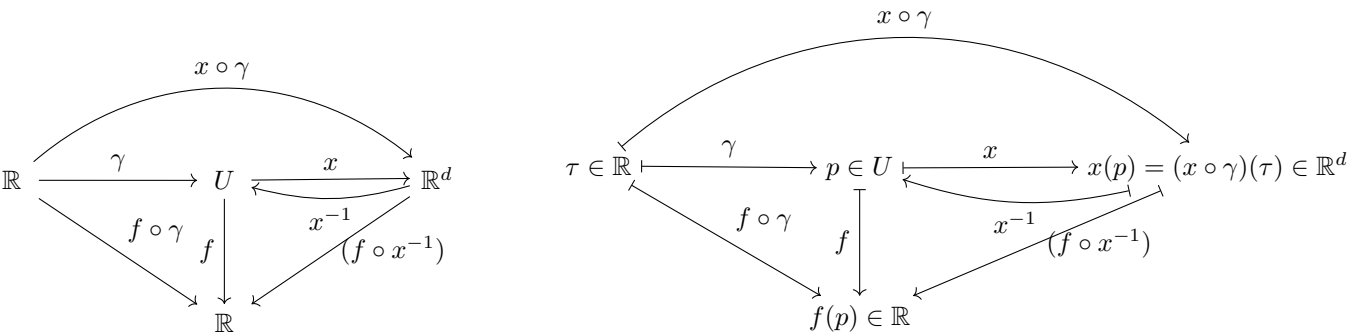
Using the chain rule, calculate

$$(f \circ \gamma)'(\lambda)$$

explicitly.

**Solution ::**

EY : 20151109 Indeed, the domains and codomains of this  $f\gamma$  mapping makes sense, from  $\mathbb{R} \rightarrow \mathbb{R}$  for



$$(f \circ \gamma)'(\lambda) = (Df) \cdot \dot{\gamma}(\lambda) = \frac{\partial f}{\partial x^j} \dot{\gamma}^j(\lambda) = 2x(-\sin \lambda) + 2y \cos \lambda = 2(-\cos \lambda \sin \lambda + \sin \lambda \cos \lambda) = 0$$

## 27. LECTURE 5: TANGENT SPACES

lead question: “what is the velocity of a curve  $\gamma$  point  $p$ ?

### 27.1. Velocities.

**Definition 71.**  $(M, \mathcal{O}, \mathcal{A})$  smooth mfd.

curve  $\gamma : \mathbb{R} \rightarrow M$  at least  $C^1$ .

Suppose  $\gamma(\lambda_0) = p$

The **velocity** of  $\gamma$  at  $p$  is the linear map

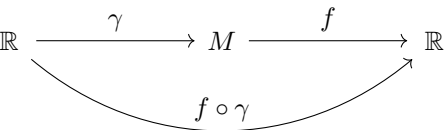
$$v_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R}$$

$$C^\infty(M) := \{f : M \rightarrow \mathbb{R} | f \text{ smooth function} \} \text{ equipped with } (f \oplus g)(p) := f(p) + g(p)$$

$$(\lambda \otimes g)(p) := \lambda \cdot g(p)$$

$\sim$  denotes linear map on top of  $\rightarrow$ .

$$f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0)$$



intuition

Schuller says: children run around the world. Temperature function as temperature contour lines. You feel the temperature.

You observe the rate of change of temperature as you run around.  $f$  is temperature.

past: “  $\underbrace{v^i}_{\text{vector}}(\partial_i f) = (\underbrace{v^i \partial_i}_{\text{vector}})f$

### 27.2. Tangent vector space.

**Definition 72.** For each point  $p \in M$

def the **set** “tangent space  $\neq_0 M$  at  $p$  “

$$T_p M := \{v_{\gamma,p} | \gamma \text{ smooth curves} \}$$

picture:

rather  $\overline{M}$  than (embedded)  $p \in T_p M$  EY : 20151109 see [https://youtu.be/pepU\\_7NJSGM?t=12m38s](https://youtu.be/pepU_7NJSGM?t=12m38s) for the picture

Observation:  $T_p M$  can be made into a vector space.

$$\begin{aligned} \oplus : T_p M \times T_p M &\rightarrow \\ (v_{\gamma,p} \oplus v_{\delta,p})(\underbrace{f}_{\in C^\infty(M)}) &:= v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f) \\ \odot : \mathbb{R} \times T_p M &\rightarrow \text{Hom}(C^\infty(\mathbb{R}), \mathbb{R}) \\ (\alpha \odot v_{\gamma,p})(f) &:= \alpha \cdot_{\mathbb{R}} v_{\gamma,p}(f) \end{aligned}$$

Remains to be shown that

- (i)  $\exists \sigma$  curve :  $v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$
- (ii)  $\exists \tau$  curve :  $\alpha \odot v_{\gamma,p} = v_{\tau,p}$

Claim:  $\tau : \mathbb{R} \rightarrow M$

where  $\mu_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ , does the trick.

$$\mapsto \tau(\lambda) := \gamma(\alpha\lambda + \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \quad r \mapsto \alpha \cdot r + \lambda_0$$

$$\tau(0) = \gamma(\lambda_0) = p$$

$$\begin{aligned} v_{\tau,p} &:= (f \circ \tau)'(0) = (f \circ \gamma \circ \mu_\alpha)'(0) \\ &= (f \circ \gamma)'(\lambda_0) \cdot \alpha = \\ &= \alpha \cdot v_{\gamma,p} \end{aligned}$$

Now for the sum:

$$v_{\gamma,p} \oplus v_{\delta,p} \stackrel{?}{=} v_{\sigma,p}$$

make a choice of chart  $(\underbrace{U}_{\ni p}, x)$  In cloud: ill definition alarm bells.

and define:

Claim:

$$\sigma : \mathbb{R} \rightarrow M$$

$$\sigma(\lambda) := x^{-1}(\underbrace{(x \circ \gamma)(\lambda_0 + \lambda) + (x \circ \delta)(\lambda_1 + \lambda) - (x \circ \gamma)(\lambda_0)}_{\mathbb{R} \rightarrow \mathbb{R}^d})$$

does the trick.

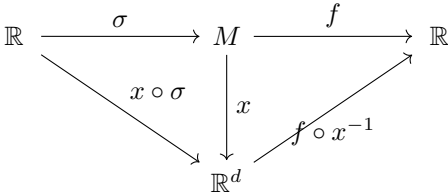
*Proof.* Since:

$$\begin{aligned} \sigma_x(0) &= x^{-1}((x \circ \gamma)(\lambda_0) + (x \circ \delta)(\lambda_1) - (x \circ \gamma)(\lambda_0)) \\ &= \delta(\lambda_1) = p \end{aligned}$$

Now:

$$\begin{aligned} v_{\sigma_x,p}(f) &:= (f \circ \sigma_x)'(0) = \\ &= (\underbrace{(f \circ x^{-1})}_{\mathbb{R}^d \rightarrow \mathbb{R}} \circ \underbrace{(x \circ \sigma_x)}_{\mathbb{R} \rightarrow \mathbb{R}^d})'(\gamma) = \underbrace{(x \circ \sigma_x)'(0)}_{(x \circ \gamma)'(\lambda_0) + (x \circ \delta)'(\lambda_1)} \cdot (\partial_i(f \circ x^{-1}))(x(\underbrace{\sigma(0)}_p)) = \\ &= (x \circ \gamma)'(\lambda_0)(\partial_i(f \circ x^{-1}))(x(p)) + (x \circ \delta)(\lambda_1)(\partial_i(f \circ x^{-1}))(x(p)) \\ &= (f \circ \gamma)'(\lambda_0) + (f \circ \delta)'(\lambda_1) = \\ &= v_{\gamma,p}(f) + v_{\delta,p}(f) \quad \forall f \in C^\infty(M) \end{aligned}$$

$$v_{\gamma,p} \oplus v_{\delta,p} = v_{\sigma,p}$$



picture: (cf. [https://youtu.be/pepU\\_7NJSGM?t=39m5s](https://youtu.be/pepU_7NJSGM?t=39m5s))

$$\begin{aligned}\gamma &: \mathbb{R} \rightarrow M \\ \delta &: \mathbb{R} \rightarrow M\end{aligned}$$

$(\gamma \oplus)(\lambda) := \gamma(\lambda) + \delta(\lambda)$   
EY : 20151109 Schuller says adding trajectories is chart dependent, bad. Adding velocities is good.

27.3. **Components of a vector wrt a chart.**

**Definition 73.** Let  $(U, x) \in \mathcal{A}_{smooth}$ .

$\gamma : \mathbb{R} \rightarrow U$   
Let  $\gamma(0) = p$  .  
Calculate

$$\begin{aligned}v_{\gamma,p}(f) &:= (f \circ \gamma)'(0) = \underbrace{((f \circ x^{-1}) \circ (x \circ \gamma))'}_{\mathbb{R}^d \rightarrow \mathbb{R}}(0) \\ &= \underbrace{(x \circ \gamma)^{i'}}_{\dot{\gamma}_x^i(0)} \cdot \underbrace{(\partial_i(f \circ x^{-1}))(x(p))}_{=:(\frac{\partial f}{\partial x^i})_p}\end{aligned}$$

think cloud  $f : M \rightarrow \mathbb{R}$

$$= \boxed{\dot{\gamma}_x^i(0) \cdot \left(\frac{\partial}{\partial x^i}\right)_p} f \quad \forall f \in C^\infty(M)$$

$\therefore$  as a map.

$$v_{\gamma,p} \underbrace{=}_{\text{use of chart}} \underbrace{\dot{\gamma}_x^i(0)}_{\text{“components of the velocity } v_{\gamma,p}\text{”}}$$

$$\underbrace{\left(\frac{\partial}{\partial x^i}\right)}_{\text{basis elements of the } T_p M \text{ wrt which the components need to be understood. “chart induced basis of } T_p M\text{”}}$$

Picture: [https://youtu.be/pepU\\_7NJSGM?t=1h16s](https://youtu.be/pepU_7NJSGM?t=1h16s)

27.4. **4. Chart-induced basis.**

**Definition 74.**  $(U, x) \in \mathcal{A}_{smooth}$   
the  $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_p U \subseteq T_p M$   
constitute a **basis** of  $T_p U$

*Proof.* remains: linearly independent

□

$$\begin{aligned}\lambda^i \left(\frac{\partial}{\partial x^i}\right)_p &\stackrel{!}{=} 0 \\ \implies \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p (x^j) &= \lambda^i \partial_i (\underbrace{x^j \circ x^{-1}}_p)(x(p)) = x^j \circ x^{-1} : \mathbb{R}^d \rightarrow \mathbb{R} \\ &= \lambda^i \delta_i^j = \lambda^j \quad j = 1, \dots, d\end{aligned}\qquad \begin{aligned}(\alpha^1, \dots, \alpha^d) &\mapsto \alpha^j\end{aligned}$$

in cloud:  $x^j : U \rightarrow \mathbb{R}$  differentiable

□

**Corollary 1.**  $\dim T_p M = d = \dim M$

Terminology:  $X \in T_p M \rightarrow \exists \gamma : \mathbb{R} \rightarrow M : X = v_{\gamma,p}$  and  
 $\exists \underbrace{X_1^1, \dots, X^d}_{\in \mathbb{R}} : X = X^i \left(\frac{\partial}{\partial x^i}\right)_p$

27.5. **5. Change of vector components under a change of chart.** **✖** vector does **not** change under change of chart.  
Let  $(U, x)$  and  $(V, y)$  be overlapping charts and  $p \in U \cap V$ .  
Let  $X \in T_p M$

$$X_{(y)}^i \cdot \left(\frac{\partial}{\partial y^i}\right)_p \underbrace{=}_{(V,y)} X \underbrace{=}_{(U,x)} X_x^i \left(\frac{\partial}{\partial x^i}\right)_p$$

to study change of components formula:

$$\begin{aligned}\left(\frac{\partial}{\partial x^i}\right)_p f &= \partial_i(f \circ x^{-1})(x(p)) = \\ &= \partial_i(\underbrace{(f \circ y^{-1})}_{\mathbb{R}^d \rightarrow \mathbb{R}} \circ \underbrace{(y \circ x^{-1})}_{\mathbb{R}^d \rightarrow \mathbb{R}^d})(x(p)) \\ &= (\partial_i(y^i \circ x^{-1}))(x(p)) \cdot (\partial_j(f \circ y^{-1}))(y(p)) = \\ &= \boxed{\left(\frac{\partial y^p}{\partial x^i}\right)_p \cdot \left(\frac{\partial f}{\partial y^j}\right)_p} f \\ \implies X_{(x)}^i \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_p &= X_{(y)}^j \left(\frac{\partial}{\partial y^j}\right)_p \\ \implies \boxed{X_{(y)}^j} &= \left(\frac{\partial y^j}{\partial x^i}\right)_p X_{(x)}^i\end{aligned}$$

27.6. **6. Cotangent spaces.**  $T_p M = V$   
trivial  $(T_p M)^* := \{\varphi : T_p M \xrightarrow{\sim} \mathbb{R}\}$   
Example:  $f \in C^\infty(M)$

$$\begin{aligned}(df)_p &: T_p M \xrightarrow{\sim} \mathbb{R} \\ X &\mapsto (df)_p(X)\end{aligned}$$

i.e.  $\boxed{(df)_p \in T_p M^*}$

$(df)_p$  called the gradient of  $f$   $p \in M$ .  
Calculate components of gradient w.r.t. chart-induced basis  $(U, x)$

**Definition 77.** A *section*  $\sigma$  of a bundle

$$\begin{array}{c} E \\ \pi \downarrow \nearrow \sigma \\ M \end{array}$$

require  $\pi \circ \sigma = id_M$

Schuller says: in quantum mechanics, Aside:  $\psi : M \rightarrow \mathbb{C}$

27.9. **Tangent bundle of smooth manifold.**  $(M, \mathcal{O}, \mathcal{A})$  smooth manifold

(a) as a **set**  $TM := \dot{\bigcup}_{p \in M} T_p M$

(b) surjective  $\pi : TM \rightarrow M$  the *unique* point  $p \in M$ ,  $X \in T_p M$

$$\begin{array}{ccccc} & X \mapsto p & & & \\ \text{situation: } & \underbrace{TM}_{\text{set}} & \xrightarrow{\pi} & \underbrace{M}_{\text{smooth manifold}} & \\ & \text{surjective map} & & & \end{array}$$

(c) Construct topology on  $TM$  that is the coarsest topology such that  $\pi$  (just) continuous. (“initial topology with respect to  $\pi$ ”).

$$\mathcal{O}_{TM} := \{\text{preim}_{\pi}(U) | \mathcal{U} \in \mathcal{O}\}$$

Show: Tutorial  $\mathcal{O}_{TM}$  Schuller says this is shown in the tutorial  $(TM, \mathcal{O}_{TM})$

Construction of a  $C^\infty$ -atlas on  $TM$  from the  $C^\infty$ -atlas  $\mathcal{A}$  on  $M$ .

$$\mathcal{A}_{TM} := \{(T\mathcal{U}, \xi_x) | (U, x) \in \mathcal{A}\}$$

where

$$\begin{array}{l} \xi_x : T\mathcal{U} \rightarrow \mathbb{R}^{2 \cdot \dim M} \\ X \mapsto \underbrace{((x^1 \circ \pi)(X), \dots, (x^d \circ \pi)(X), (dx^1)_{\pi(X)}(X), \dots, (dx^d)_{\pi(X)}(X))}_{(U, x) - \text{ coords of } \pi(X) \text{ (} d \text{ many )}} \end{array}$$

where  $X \in T_{\pi(X)} M$   
 $X = X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)}$

$$\begin{aligned} (dx^j)_{\pi(X)}(X) &= (dx^j)_{\pi(X)} \left( X^i_{(x)} \left( \frac{\partial}{\partial x^i} \right)_{\pi(X)} \right) = \\ &= X^i_{(x)} \delta^j_i = X^j_{(x)} \end{aligned}$$

Write  $\xi_x^{-1} : \xi_x(TU) \subseteq \mathbb{R}^{2\dim M} \rightarrow TU$

$$(\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) := \beta^i \left( \frac{\partial}{\partial x^i} \right) \underbrace{x^{-1}(\alpha^1, \dots, \alpha^d)}_{\pi(X)}$$

$$\begin{aligned} ((df)_p)_j &:= (df)_p \left( \left( \frac{\partial}{\partial x^j} \right)_p \right) \\ &= \left( \frac{\partial f}{\partial x^j} \right)_p = \partial_j (f \circ x^{-1})(x(p)) \end{aligned}$$

**Theorem 28.** Consider chart  $(U, x) \implies x^i : U \rightarrow \mathbb{R}$

Claim:  $(dx^1)_p, (dx^2)_p, \dots, (dx^d)_p$  basis of  $T_p^* M$   
 $\implies$  In fact: dual basis:

$$(dx^a)_p \left( \left( \frac{\partial}{\partial x^b} \right)_p \right) = \left( \frac{\partial x^a}{\partial x^b} \right)_p = \dots = \delta^a_b$$

27.7. **7. Change of components of a covector under a change of chart:**

$$\underbrace{T_p^* M}_{\ni \omega} \text{ with } \omega_{(y)}(dy^j)_p = \omega = \omega_{(x)i}(dx^i)_p$$

$$\implies \boxed{\omega_{(y)i} = \frac{\partial x^j}{\partial y^i} \omega_{(x)j}}$$

LECTURE 6: FIELDS

cf. **Lecture 6: Fields (International Winter School on Gravity and Light 2015)**

So far:

$$\begin{array}{c} T_p M \\ \vdots \downarrow \\ T_p^* M \\ \vdots \downarrow \\ \vdots \end{array},$$

now  
in Thought Cloud: theory of bundles

27.8. **Bundles.**

**Definition 75.** A *bundle* is a triple

$$E \xrightarrow{\pi} M$$

$E$  smooth manifold “**total space**”  
 $\pi$  smooth map (surjective) “*projection map*”  
 $M$  smooth manifold “*base space*”

Example  $E$  = cylinder  $M$  = circle

**Definition 76.**

$$E \xrightarrow{T} M$$

bundle.  
 $p \in M$   
define **fibre over**  $p$   
 $:= \text{preim}_{\pi}(\{p\})$

Check:

$$\begin{aligned} &(\xi_y \circ \xi_x^{-1})(\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) = \\ &= \xi_y \left( \beta^i \left( \frac{\partial}{\partial x^i} \right)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \right) \\ &= \left( \dots, (y^i \circ \pi)(\beta^m \cdot \left( \frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)}), \dots, \dots (dy^i)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \left( \beta^m \left( \frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)} \right), \dots \right) = \\ &= (\dots, (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d), \dots, \dots, \underbrace{\beta^m \left( \frac{\partial y}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)}}_{\beta^m \left( \frac{\partial y}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)}} \left( \left( \frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)} \right)) \end{aligned}$$

Check transition map:  $(U, x), (V, y), \quad U \cap V \neq \emptyset$   
 $\left( \frac{\partial y}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)} = \partial_m(y^i \circ x^{-1})(x \circ (x^{-1}(\alpha^1 \dots \alpha^d))) = \partial_m(y^i \circ x^{-1})(\alpha^1 \dots \alpha^d)$  smooth.

upshot

$$\underbrace{TM}_{\text{smooth manifold}} \xrightarrow{\pi} \underbrace{M}_{\text{smooth manifold}}$$

bundle, called the tangent bundle.

3. Vector fields.

**Definition 78.** A *smooth vector field*  $\chi$  is a smooth map, (where)

$$\begin{array}{c} TM \\ \pi \downarrow \curvearrowright \\ M \end{array} \chi$$

Example:

$$\begin{array}{c} TS^1 \\ \downarrow \pi \\ S^1 \end{array}$$

4. The  $C^\infty(M)$ -module  $\Gamma(TM)$ .  
 $C^\infty(M)$ -module  $\leftarrow (C^\infty(M), +, \cdot)$  (satisfies)  $C^+, A^+, N^+, I^+, C^+, A^+, N^+, D^+$ . Not a field. A ring.

$$\text{set } \Gamma(TM) = \{ \chi \mid M \rightarrow TM \mid \text{smooth section} \}$$
$$(\chi \oplus \tilde{\chi})(f) := (\chi f) \underbrace{+}_{C^\infty(M)} \tilde{\chi}(f)$$

$$(\underbrace{g}_{C^\infty(M)} \odot \xi)(f) := \underbrace{g}_{C^\infty(M)} \cdot \chi(f)$$

$$\begin{aligned} \chi : M &\rightarrow TM \\ p &\mapsto \chi(p) \\ \chi f : M &\rightarrow \mathbb{R} \\ p &\mapsto \chi(p)f \end{aligned}$$

$(\Gamma(TM), \oplus, \odot)$   $C^\infty(M)$  - **module**

upshot: set of all smooth vector fields can be made into a  $C^\infty(M)$ -module.

Fact:

- (1)  $\text{ZFC} \implies$  every vector space has a basis. (You have to have C - axiom of choice in set theory)
- (2) no such result exists for modules.

This is a shame, because otherwise, we could have chosen (for any manifolds) vector fields,

$$\chi_{(1)}, \dots, \chi_{(d)} \in \Gamma(TM)$$

and would be able to write every vector field  $\Xi$

$$\chi = \underbrace{f^i}_{\text{component functions}} \cdot \chi_{(i)}$$

Simple counterexample

Schuller says: Take a sphere, Morse Theorem, every smooth vector field must vanish at 2 pts. “mustn’t choose a global basis”

However:  $\frac{\partial}{\partial x^i} : U \xrightarrow{\text{smooth}} TU$

$$p \mapsto \left( \frac{\partial}{\partial x^i} \right)_p$$

27.10. **Tensor fields.** so far  
 $\Gamma(M)$  =”set of vector fields”  $C^\infty(M)$ -module  
 $\Gamma(T^*M)$  = “covector fields”  $C^\infty(M)$ -module

**Definition 79.** An  $(r, s)$ -tensor field  $T$  is a multi-linear map

$$T : \underbrace{\Gamma(T^*M) \times \dots \times \Gamma(T^*M)}_r \times \Gamma(TM) \times \dots \times \Gamma(TM) \xrightarrow{\sim} C^\infty(M)$$

Example:  $f \in C^\infty(M)$

$$\begin{aligned} df : \Gamma(TM) &\xrightarrow{\sim} C^\infty(M) \\ \chi &\mapsto df(\chi) := \chi[f] \end{aligned}$$

$df$  (0, 1)-T.F. (tensor field)  
where  $(\chi f) \left( \underbrace{p}_{\in M} \right) := \underbrace{\chi(p)}_{\in T_p M} f$   
can check:  $df$  is  $C^\infty$ –linear

LECTURE 7: CONNECTIONS

cf. [Lecture 7: Connections \(International Winter School on Gravity and Light 2015\)](#)

So far: saw that a vector field  $X$  can be used to provide a directional derivative

$$\nabla_X f := Xf$$

of a function  $f \in C^\infty(M)$ .

Remark: from now on: consider mostly vector fields.  
Notational overkill?

$$\nabla_X f = Xf = (df)(X)$$

In Thought Bubble:  $\nabla_X(f \cdot g) = X(fg) = (Xf) \cdot g + fX(g)$  Product rule, because it’s a derivative.  
not quite:

$$\begin{aligned} X &: C^\infty(M) \rightarrow C^\infty(M) \\ df &: \Gamma(TM) \rightarrow C^\infty(M) \\ \nabla_X &: C^\infty(M) \rightarrow C^\infty(M) \end{aligned}$$

$$\begin{array}{ccc} \nabla_X : C^\infty(M) & \longrightarrow & C^\infty(M) \\ \vdots \downarrow & & \vdots \downarrow \\ \nabla_X : TM^p \otimes T^*M^q \text{ i.e.} & \longrightarrow & TM^p \otimes T^*M^q \text{ i.e.} \\ \left( \begin{smallmatrix} p \\ q \end{smallmatrix} \right) \text{ tensor field} & & \left( \begin{smallmatrix} p \\ q \end{smallmatrix} \right) \text{ tensor field} \end{array}$$

**1. Directional derivatives of tensor fields.** We formulate a wish list of properties which the  $\nabla_X$  acting on a tensor field should have.

In Thought Bubble: Any remaining freedom in choosing  $\nabla$ , will need to be provided as additional structure beyond  $(M, \mathcal{O}, \mathcal{A})$

**Definition 80** (connection). *In Thought Bubble: linear connection, covariant derivative, affine connection*

A **connection**  $\nabla$  on a smooth manifold  $(M, \mathcal{O}, \mathcal{A})$  is a map that takes a pair consisting of a vector (field)  $X$  and a  $(p, q)$ -tensor field  $T$  and sends them to a  $(p, q)$ -tensor (field)  $\nabla_X T$  satisfying

- (i)  $\nabla_X f = Xf \quad \forall f \in C^\infty(M)$
- (ii)  $\nabla_X(T + S) = \nabla_X T + \nabla_X S$
- (iii)  $\nabla_X \underbrace{(T(\omega, Y))}_{\in C^\infty(M)} = (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y)$

*In Thought Bubble: for  $(1, 1)$ -TF  $T$ , but analogously for any  $(p, q)$  - TF*

*"Leibnitz" rule.*

- (iv)  $\nabla_{fX+Z} T = f\nabla_X T + \nabla_Z T$   
 $f \in C^\infty(M)$

A manifold with connection is quadruple  $(M, \mathcal{O}, \mathcal{A}, \nabla)$

Remark:  $\nabla_X$  is the extension of  $X$ .

$\nabla$  — " ———- of  $d$

**2. New structure on  $(M, \mathcal{O}, \mathcal{A})$  required to fix  $\nabla$ .** Q: How much freedom do we have in choosing such a structure.

Consider  $X, Y$  vector fields

$$\begin{aligned} \nabla_X Y & \underbrace{=}_{\text{In Thought Bubble: } (U, x)} \nabla_{X^i \frac{\partial}{\partial x^i}} \left( Y^m \frac{\partial}{\partial x^m} \right) \\ & \underbrace{=}_{\text{(iii)}} X^i \left( \nabla_{\frac{\partial}{\partial x^i}} Y^m \right) \frac{\partial}{\partial x^m} + X^i Y^m \underbrace{\nabla_{\frac{\partial}{\partial x^i}} \left( \frac{\partial}{\partial x^m} \right)}_{\text{(iii)}} \\ & \underbrace{=}_{\text{(i)}} X^i \left( \frac{\partial}{\partial x^i} Y^m \right) \frac{\partial}{\partial x^m} + X^i Y^m \underbrace{\Gamma_{mi}^q(x)}_{\text{connection coefficient functions (on } M) \text{ of } \nabla \text{ wrt } (U, x)} \frac{\partial}{\partial x^q} \end{aligned}$$

$$\underbrace{\left( \underbrace{T}_{(p,q)} \otimes \underbrace{S}_{(r,s)} \right)}_{C^\infty(M)} (\omega, \dots, X, \dots) := T(\omega, \dots, X, \dots) \underbrace{\cdot}_{C^\infty(M)} S(\dots, \dots)$$

$$\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S)$$

**Definition 81** (Connection coefficient functions).  $(M, \mathcal{O}, \mathcal{A}, \nabla), (\mathcal{U}, x) \in \mathcal{A}$ .

Then the **connection coefficient functions** ("Γ"s) with respect to (wrt)  $(U, x)$  on the  $(\dim(M))^3$  many functions

$$\begin{aligned} \Gamma_{jk}^i : \mathcal{U} &\rightarrow \mathbb{R} \\ p &\mapsto \left( dx^i \left( \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) \right) (p) \end{aligned}$$

Thus:

$$(\nabla_X Y)^i = X^m \left( \frac{\partial}{\partial x^m} Y^i \right) + \Gamma_{nm}^i \underbrace{\cdot}_{C^\infty(M)} Y^n X^m$$

*Remark:* On a chart domain  $U$ , choice of the  $(\dim M)^3$  functions  $\Gamma_{jk}^i$  suffices to fix the action of  $\nabla$  on a *vector* field.

Fortunately, the same  $(\dim M)^3$  functions fix the action of  $\nabla$  on any tensor field.

*key observation:*

$$\nabla_{\frac{\partial}{\partial x^m}} (dx^i) = \sum_{jm}^i dx^j$$

but now:

$$\begin{aligned} \underbrace{\nabla_{\frac{\partial}{\partial x^m}} \underbrace{\left( dx^i \left( \frac{\partial}{\partial x^j} \right) \right)}_{\delta_j^i}}_{\parallel \text{ (iii)}} &= \frac{\partial}{\partial x^m} (\delta_j^i) = 0 \\ &= \left( \nabla_{\frac{\partial}{\partial x^m}} dx^i \right) \left( \frac{\partial}{\partial x^j} \right) + dx^i \underbrace{\left( \nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^j} \right)}_{\Gamma_{jm}^q \frac{\partial}{\partial x^q}} = 0 \\ \implies \left( \nabla_{\frac{\partial}{\partial x^m}} dx^i \right) \left( \frac{\partial}{\partial x^j} \right) &= -\Gamma_{jm}^i \end{aligned}$$

Summary so far:

$$\begin{aligned} (\nabla_X Y)^i &= X(Y^i) \underbrace{+}_{\text{act on } \mathbf{vector} \text{ field}} \Gamma_{jm}^i Y^j X^m \\ (\nabla_X \omega)_i &= X(\omega_i) + -\Gamma_{im}^j \omega_j X^m \end{aligned}$$

Note that for the immediately above expression for  $(\nabla_X Y)^i$ , in the second term on the right hand side,  $\Gamma_{jm}^i$  has the last entry at the bottom,  $m$  going in the direction of  $X$ , so that it matches up with  $X^m$ . This is a good mnemonic to memorize the index positions of  $\Gamma$ .

similarly, by further application of Leibnitz

$T$  a  $(1, 2)$ -TF (tensor field)

$$(\nabla_X T)_{jk}^i = X(T_{jk}^i) + \Gamma_{sm}^i T_{jk}^s X^m - \Gamma_{jm}^s T_{sk}^i X^m - \Gamma_{km}^s T_{js}^i X^m$$



Question: If in a Euclidean space, the  $\Gamma$ 's all vanish in a (then existing) global chart.

Answer: Yes, but: What is a Euclidean space:

$(M = \mathbb{R}^n, \mathcal{O}_{\text{st}}, \mathcal{A})$  smooth manifold.

Assume  $(\mathbb{R}^n, \text{id}_{\mathbb{R}^n}) \in \mathcal{A}$  and

$$(\Gamma_{(x)}^i)_{jk} = dx^i \left( (\nabla_{\underline{\mathbf{E}}})_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) \stackrel{!}{=} 0$$

*Intuition:*

$\mathbb{R}^2$ :  $\nabla_{\text{Euclidean}}$

$\mathbb{R}^2$ :  $\nabla_{\text{Hyperbolic}}$

**Definition 82.**  $X$  vector field on  $(M, \mathcal{O}, \mathcal{A}, \nabla)$ .

Then divergence of  $X$  is the function:

$$\text{div}(X) := \left( \nabla_{\frac{\partial}{\partial x^i}} X \right)^i$$

*Claim:* chart-independent.

### 3. Change of $\Gamma$ 's under change of chart.

$(U, x), (V, y) \in \mathcal{A}$  and  $U \cap V \neq \emptyset$

$$\Gamma_{jk}^i(y) := dy^i \left( \nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} \right) = \frac{\partial y^i}{\partial x^q} dx^q \left( \nabla_{\frac{\partial x^p}{\partial y^k} \frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j} \frac{\partial}{\partial x^s} \right)$$

Note  $\nabla_{fX}$  is  $C^\infty$ -linear for  $fX$

covector  $dy^i$  is  $C^\infty$ -linear in its argument

$$\begin{aligned} \implies \Gamma_{jk}^i(y) &= \frac{\partial y^i}{\partial x^q} dx^q \left( \frac{\partial x^p}{\partial y^k} \left[ \left( \nabla_{\frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j} \right) \frac{\partial}{\partial x^s} + \frac{\partial x^s}{\partial y^j} \left( \nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^s} \right) \right] \right) = \\ &= \frac{\partial y^i}{\partial x^q} \underbrace{\frac{\partial x^p}{\partial y^k} \frac{\partial}{\partial x^p}}_{\frac{\partial}{\partial y^k}} \frac{\partial x^s}{\partial y^j} \delta_s^q + \frac{\partial y^i}{\partial x^q} \frac{\partial x^p}{\partial y^k} \frac{\partial x^s}{\partial y^j} \Gamma_{sp}^q(x) \end{aligned}$$

in summary:

$$(172) \quad \Gamma_{jk}^i(y) = \frac{\partial y^i}{\partial x^q} \frac{\partial^2 x^q}{\partial y^j \partial y^k} + \frac{\partial y^i}{\partial x^q} \frac{\partial x^s}{\partial y^j} \frac{\partial x^p}{\partial y^k} \Gamma_{sp}^q(x)$$

Eq. (172) is the change of connection coefficient function under the change of chart  $(U \cap V, x) \rightarrow (U \cap V, y)$

#### 4. Normal Coordinates.

Let  $p \in M$  of  $(M, \mathcal{O}, \mathcal{A}, \nabla)$

Then one can construct a chart  $(U, x)$  with  $p \in U$  such that

$$\Gamma(x)^i_{(jk)}(p) = 0$$

at the point  $p$ . **Not** necessarily in any neighborhood.

*Proof.* Let  $(V, y)$  be any chart (with)  $p \in V$ .

Thus, in general:  $\Gamma(y)^i_{jk} \neq 0$

Then consider a new chart  $(U, x)$  to which one transits by virtue of

$$(x \circ y^{-1})^i(\alpha^1, \dots, \alpha^d) := \alpha^i - \frac{1}{2} \Gamma(y)^i_{(jk)}(p) \alpha^j \alpha^k$$

$$p = x^{-1}(\alpha^1, \dots, \alpha^d)$$

$$\left( \frac{\partial x^i}{\partial y^j} \right)_p = \partial_j(x^i \circ y^{-1}) = \delta_j^i - \Gamma(y)^i_{mj}(p) \alpha^m|_{\alpha=0} = \delta_j^i$$

$$\frac{\partial x^i}{\partial y^k \partial y^j}(p) = -\Gamma(y)^i_{kj}(p)$$

$$\implies \Gamma(x)^i_{jk}(p) = \Gamma(y)^i_{jk}(p) - \Gamma(y)^i_{kj}(p) = 0$$

$$= \Gamma(y)^i_{[jk]}(p) = T(y)^i_{jk}$$

Terminology:  $(U, x)$  is called a **normal coordinate chart** of  $\nabla$  at  $\mathbf{p} \in \mathbf{M}$ .

□

#### Tutorial 7 Connections. Exercise 1. : True or false?

- (a)
- $\nabla_{fX}Y = f\nabla_XY$  by definition so  $\nabla_{fX} = f\nabla_X$  i.e.  $\nabla_X$  is  $C^\infty(M)$ -linear in  $X$
  - $f \in C^\infty(M)$  is a  $(0, 0)$ -tensor field.  $\nabla_X f = Xf \equiv X(f)$  by definition.
  - If the manifold is flat, I'm assuming that means that the manifold is globally a Euclidean space, and by definition,  $\Gamma = 0$ .

$$\nabla_X Y = X^j \frac{\partial}{\partial x^j} (Y^i) \frac{\partial}{\partial x^i} + \Gamma_{jk}^i Y^k X^j \frac{\partial}{\partial x^i} = X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x^i} + 0$$

and similarly for any  $(p, q)$ -tensor field, i.e.

$$\nabla_X T = X^j \frac{\partial T^{i_1 \dots i_p}_{j_1 \dots j_q}}{\partial x^j}$$

•

$$\nabla_X f = X^j \frac{\partial f}{\partial x^j} = X \cdot \text{grad}(f)$$

•

- $\forall (U, x) \in \mathcal{A}$ , locally (after working out the first few cases, and doing induction, one can look up the expression for the local form; I found it in Nakahara's **Geometry, Topology and Physics**, Eq. 7.26, and it needs to be modified for the convention of order of bottom indices for  $\Gamma$ :

$$\nabla_\nu t_{\mu_1 \dots \mu_q}^{\lambda_1 \dots \lambda_p} = \partial_\nu t_{\mu_1 \dots \mu_q}^{\lambda_1 \dots \lambda_p} + \Gamma_{\kappa\nu}^{\lambda_1} t_{\mu_1 \dots \mu_q}^{\kappa \lambda_2 \dots \lambda_p} + \dots + \Gamma_{\kappa\nu}^{\lambda_p} t_{\mu_1 \dots \mu_q}^{\lambda_1 \dots \lambda_{p-1} \kappa} - \Gamma_{\mu_1 \nu}^{\kappa} t_{\kappa \mu_2 \dots \mu_q}^{\lambda_1 \dots \lambda_p} - \dots - \Gamma_{\mu_q \nu}^{\kappa} t_{\mu_1 \dots \mu_{q-1} \kappa}^{\lambda_1 \dots \lambda_p}$$

Clearly,  $\nabla_X$  is uniquely fixed  $\forall p \in M$  by choosing each of the  $(\dim M)^3$  many connection coefficient functions  $\Gamma$ .

- (b)
- $\nabla : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$
  - $\nabla : (p, q)$ -tensor field  $\mapsto (p, q)$ -tensor field
  - By definition,  $\nabla$  satisfies the Leibniz rule.

•

•

•

**Exercise 2. : Practical rules for how  $\nabla$  acts** Torsion-free covariant derivative boils down to a connection coefficient function  $\Gamma$  that is symmetric in the bottom indices.

•

$$\nabla_X f = X(f) = X^i \frac{\partial f}{\partial x^i}$$

•

$$(\nabla_X Y)^a = X^i \frac{\partial Y^a}{\partial x^i} + \Gamma_{jk}^a Y^j X^k$$

•

$$(\nabla_X \omega)_a = X^i \frac{\partial \omega_a}{\partial x^j} - \Gamma_{ak}^i \omega_i X^k$$

•

$$(\nabla_m T)^a_{bc} = \frac{\partial}{\partial x^m}(T^a_{bc}) + \Gamma^a_{im} T^i_{bc} - \Gamma^i_{bm} T^a_{ic} - \Gamma^j_{cm} T^a_{bj}$$

•

$$(\nabla_{[m} A)_{n]} = (\nabla_m A)_n - (\nabla_n A)_m = \frac{\partial A_n}{\partial x^m} - \Gamma^i_{nm} A_i - \left( \frac{\partial A_m}{\partial x^n} - \Gamma^i_{mn} A_i \right) = \frac{\partial A_m}{\partial x^m} - \frac{\partial A_m}{\partial x^n}$$

•

$$(\nabla_m \omega)_{nr} = \frac{\partial \omega_{nr}}{\partial x^m} - \Gamma^i_{nm} \omega_{ir} - \Gamma^i_{rm} \omega_{ni}$$

Exercise 3. : Connection coefficients

Question .

The connection coefficient functions  $\Gamma$  in chart  $(U \cap V, y)$  is given, in terms of chart  $(U \cap V, x)$  as follows:  
Recall Eq. (172)

$$\Gamma^i_{jk}(y) = \frac{\partial y^i}{\partial x^q} \frac{\partial^2 x^q}{\partial y^j \partial y^k} + \frac{\partial y^i}{\partial x^q} \frac{\partial x^s}{\partial y^j} \frac{\partial x^p}{\partial y^k} \Gamma^q_{sp}(x)$$

LECTURE 8: PARALLEL TRANSPORT & CURVATURE (INTERNATIONAL WINTER SCHOOL ON GRAVITY AND LIGHT 2015)

**Experiment.**  $\nabla_{v_\gamma} Y \stackrel{!}{=} 0$   
(please move in this fashion)

round sphere  
 $(S^2, \mathcal{O}, \mathcal{A}, \nabla)$

ellipsoid  
 $(S^2, \mathcal{O}, \mathcal{A}, \tilde{\nabla})$

ellisoid embedded in  $\mathbb{R}^3$  with  $\mathbb{R}^3$  Euclidean connection  
 $(\mathbb{R}^3, \nabla_E)$

1. Parallelity of vector fields.  $(M, \mathcal{O}, \mathcal{A}, \nabla)$  manifold with connection.

**Definition 83** (parallelly transported, parallel). (1) *A vector field  $X$  on  $M$  is said to be **parallelly transported** along a smooth curve  $\gamma : \mathbb{R} \rightarrow M$  if*

$$\nabla_{v_\gamma} X = 0$$

*i.e.*

$$(\nabla_{v_{\gamma, \gamma(\lambda)}} X)_{\gamma(\lambda)} = 0$$

(2) *A slightly weaker condition than "parallelly transported" is "parallel":*

$$(\nabla_{v_{\gamma, \gamma(\lambda)}} X)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)}$$

for  $\mu : \mathbb{R} \rightarrow \mathbb{R}$

Example: Euclidean plane  $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_E)$ ,  
*parallelly transported  $X$  along  $\gamma$ , parallel  $X$  along  $\gamma$ , not parallel, not parallelly transported.*

2. Autoparallely transported curves. Thought bubble: self- is "Auto"

Child: "How do I get to the toy store." Adult: "Just follow your nose."

(Move in direction of nose means move along curve whose tangent vector is your nose. Means move in the straightest line. We are not talking about shortest line because we do not have a notion of distance.)

**Definition 84.** *A (smooth) curve  $\gamma : \mathbb{R} \rightarrow M$  is called **autoparallely transported** if*

$$(173) \qquad \qquad \qquad \nabla_{v_\gamma} v_\gamma = 0$$

$$\iff (\nabla_{v_{\gamma, \gamma(\lambda)}} v_\gamma)_{\gamma(\lambda)} = 0$$

*Remark:* Define weaker notion of autoparallel curve  
Thought bubble: !

$$\nabla_{v_\gamma} v_\gamma = \mu v_\gamma$$

Example. Euclidean plane  $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_E)$   
autoparallely transported (uniform straight motion, Newton's First law), autoparallel (straight motion) but not autoparallely transported,

3. Autoparallel equation.

Autoparallely transported curve  $\gamma$

Consider that portion of the curve that lies in  $U$ , where  $(U, x) \in \mathcal{A}$ .

$$\nabla_{v_\gamma} v_\gamma = 0$$

Express  $\nabla_{v_\gamma} v_\gamma = 0$  in terms of chart representation.

$$(\nabla_{v_\gamma} v_\gamma) = \left( \nabla_{\dot{\gamma}^m(x) (\frac{\partial}{\partial x^m})_{\gamma(\lambda)}} \dot{\gamma}^n(x) \frac{\partial}{\partial x^n} \right)$$

where

$$\begin{aligned} v_{\gamma, \gamma(\lambda)} &= \dot{\gamma}^m(x) \cdot \left( \frac{\partial}{\partial x^m} \right) \\ \gamma^m(x) &:= x^m \circ \gamma \\ \implies (\nabla_{v_\gamma} v_\gamma) &= \\ &= \underbrace{\dot{\gamma}^m \frac{\partial \dot{\gamma}^q}{\partial x^m}}_{\ddot{\gamma}^q(x)^m} \cdot \frac{\partial}{\partial x^q} + \dot{\gamma}^m \dot{\gamma}^n \Gamma^q_{nm} \frac{\partial}{\partial x^q} \end{aligned}$$

20200306 EY: For further clarification,

$$\begin{aligned} \dot{\gamma}(t) &= v(\gamma(t)) \in T_{\gamma(t)} U \\ \dot{\gamma}(t) &= v^i(\gamma^1(t), \dots, \gamma^n(t)) \\ \frac{d}{dt} \dot{\gamma}^i(t) &= \frac{d}{dt} v^i(\gamma^1(t), \dots, \gamma^n(t)) = \frac{d}{dt} v^i(x^1, \dots, x^n) \\ &= \frac{\partial v^i}{\partial x^j}(x) \dot{x}^j = \dot{\gamma}^j \frac{\partial \dot{\gamma}^i}{\partial x^j} \end{aligned}$$

in summary:

$$(174) \qquad \qquad \qquad \ddot{\gamma}^m_{(x)}(\lambda) + (\Gamma^m_{(x)})_{ab}(\gamma(\lambda)) \dot{\gamma}^a_{(x)}(\lambda) \dot{\gamma}^b_{(x)}(\lambda) = 0$$

chart expression of the condition that  $\gamma$  be autoparallely transported.

Examples.

- (a) Euclidean plane,  $U = \mathbb{R}^2$ ,  $x = \text{id}_{\mathbb{R}}$ ,  $\Gamma(x)^i_{jk} = 0$   
 $\implies \ddot{\gamma}(x)^m = 0 \implies \gamma(x)^m(\lambda) = a^m \lambda + b^m a, b \in \mathbb{R}^d$

- (b) round sphere  $(S^2, \mathcal{O}, \mathcal{A}, \nabla_{\text{round}})$   
Consider a chart:

$$\begin{aligned}x(p) &= (\theta, \varphi) \\ \theta &\in (0, \pi) \\ \varphi &\in (0, 2\pi)\end{aligned}$$

all other  $(\Gamma^1_{11} = 0)$  vanish.

$$\Gamma(x)^1_{22}(x^{-1}(\theta, p)) := -\sin \theta \cos \theta \quad \Gamma(x)^2_{21} = \Gamma(x)^2_{12} := \cot \theta$$

Sloppy notation (classical mechanics)

$$\begin{aligned}x^1(p) &= \theta(p) \\ x^2(p) &= \varphi(p)\end{aligned}$$

autoparallel equation

$$\begin{aligned}\ddot{\theta} + \Gamma^1_{22}\dot{\varphi}\dot{\varphi} &= 0 \\ \ddot{\varphi} + 2\Gamma^2_{12}\dot{\theta}\dot{\varphi} &= 0\end{aligned}$$

$$\iff \begin{cases} \ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\varphi}\dot{\varphi} = 0 \\ \ddot{\varphi} + 2\cot(\theta)\dot{\theta}\dot{\varphi} = 0 \end{cases}$$

Solve: e.g.  $\theta(\lambda) = \frac{\pi}{2}$     $\varphi(\lambda) = \underline{\omega} \cdot \lambda + \varphi_0$

Similarly,  
 $\nabla_{\text{potato}}, (S^2, \mathcal{O}, \mathcal{A})$

**3. Torsion.** Q: Can one use  $\nabla$  to define tensors on  $(M, \mathcal{O}, \mathcal{A}, \nabla)$ ?

**Definition 85.** The *torsion* of a connection  $\nabla$  is the  $(1, 2)$ -**tensor field**

(175)

$$T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y])$$

(Inside a cloud)

$[X, Y]$  vector field defined by

$$[X, Y]f := X(Yf) - Y(Xf)$$

*Proof.* check  $T$  is  $C^\infty$ -linear in each entry

$$\begin{aligned}T(f \cdot \omega, X, Y) &= f \cdot \omega(\dots) = fT(\omega, X, Y) \\ T(\omega + \psi, X, Y) &= \dots = T(\omega, X, Y) + T(\psi, X, Y) \\ T(\omega, fX, Y) &= \omega(\nabla_{fX} Y - \nabla_Y(fX) - [fX, Y]) = \\ &= \omega(f\nabla_X Y - (Yf)X - f\nabla_Y X - f[X, Y] + (Yf)X) = fT(\omega, X, Y)\end{aligned}$$

since

$$\begin{aligned}[fX, Y]g &= fX(Yg) - Y(fXg) = \\ &= fX(Yg) - (Yf)(Xg) - fY(Xg) = \\ &= f \cdot [X, Y]g - (Yf)Xg = \\ &= (f \cdot [X, Y] - (Yf)X)g \\ T(\omega, X, Y) &= -T(\omega, Y, X) \quad \checkmark\end{aligned}$$

□

**Definition 86.** A  $(M, \mathcal{O}, \mathcal{A}, \nabla)$  is called *torsion-free* if  $T = 0$

In a chart

$$\begin{aligned}T^i_{ab} &:= T\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = dx^i(\dots) \\ &= \Gamma^i_{ab} - \Gamma^i_{ba} = 2\Gamma^i_{[ab]}\end{aligned}$$

From now on, in these lectures, we only use torsion-free connections.

**4. Curvature.**

**Definition 87.** The *Riemann curvature* of a connection  $\nabla$  is the  $(1, 3)$ -tensor field

(176)

$$Riem(\omega, Z, X, Y) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z)$$

*Proof.* do it:  $C^\infty$ -linear in each slot.

□

Tutorials  $Riem^i_{jab} = \dots$

*Algebraic relevance of Riem.*

$$(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) = R(\cdot, Z, X, Y) + \nabla_{[X, Y]}Z$$

In one chart  $(U, x)$

$$\underbrace{(\nabla_a)}_{=\nabla_{\frac{\partial}{\partial x^a}}} \nabla_b Z - \nabla_b \nabla_a Z)^m = Riem^m_{\phantom{m} nab} Z^n + \nabla \left[ \underbrace{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}}_{=0} \right] Z$$

$$(\nabla_a \nabla_b Z)^m - (\nabla_b \nabla_a Z)^m = Riem^m_{\phantom{m} nab} Z^n$$

*Geometric significance.*

Idea:

$$[X, Y] = 0$$

$$\text{If } [X, Y] \neq 0$$

$$(\delta Z)^m = \dots = R^m_{\phantom{m} nab} X^a Y^b Z^n \delta s \delta t + \mathcal{O}(\delta s^2 \delta t, \delta t^2 \delta s)$$

Holonomy

## TUTORIAL 8 PARALLEL TRANSPORT &amp; CURVATURE

**Exercise 1.**

**Exercise 2. : Where connection coefficients appear**

It was suggested in the tutorial sheets and hinted in the lecture that the following should be committed to memory.

**Question : Recall the autoparallel equation for a curve  $\gamma$ .**

(a)

$$\nabla_{v_\gamma} v_\gamma = 0$$

(b)

$$\begin{aligned} \nabla_{v_\gamma} v_\gamma &= \nabla_{\dot{\gamma} \frac{\partial}{\partial x^\mu}} v_\gamma = \dot{\gamma}^\nu \nabla_{\partial_\nu} v_\gamma = \dot{\gamma}^\nu \left[ \frac{\partial v_\gamma^\mu}{\partial x^\nu} + \Gamma_{\mu\nu}^\rho v_\gamma^\mu \right] \frac{\partial}{\partial x^\rho} = \dot{\gamma}^\nu \left[ \frac{\partial \dot{\gamma}^\rho}{\partial x^\nu} + \Gamma_{\mu\nu}^\rho \dot{\gamma}^\mu \right] \frac{\partial}{\partial x^\rho} = 0 \\ \implies &\boxed{\dot{\gamma}^\rho + \Gamma_{\mu\nu}^\rho \dot{\gamma}^\mu \dot{\gamma}^\nu} \end{aligned}$$

as, for example, for  $F(x(t))$ ,

$$\frac{dF(x(t))}{dt} = \dot{x} \frac{\partial F}{\partial x} = \frac{d}{dt} F$$

so that

$$\dot{\gamma}^\nu \frac{\partial v_\gamma^\mu}{\partial x^\nu} = \frac{d}{d\lambda} v_\gamma^\mu = \frac{d^2}{d\lambda^2} \gamma^\mu$$

**Question : Determine the coefficients of the Riemann tensor with respect to a chart  $(U, x)$ .**

Recall this manifestly covariant definition

$$\text{Riem}(\omega, Z, X, Y) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z)$$

We want  $R_{jab}^i$ .  
now

$$\nabla_X \nabla_Y Z = \nabla_X \left( (Y^\mu \frac{\partial}{\partial x^\mu} Z^\rho + \Gamma_{\mu\nu}^\rho Z^\mu Y^\nu) \frac{\partial}{\partial x^\rho} \right) = (X^\alpha \frac{\partial}{\partial x^\alpha} (Y^\mu \frac{\partial}{\partial x^\mu} Z^\rho + \Gamma_{\mu\nu}^\rho Z^\mu Y^\nu) + \Gamma_{\alpha\beta}^\rho (Y^\mu \frac{\partial}{\partial x^\mu} Z^\alpha + \Gamma_{\mu\nu}^\alpha Z^\mu Y^\nu) X^\beta) \frac{\partial}{\partial x^\rho}$$

For  $X = \partial_a$ ,  $Y = \partial_b$ ,  $Z = \partial_j$ , then the partial derivatives of the coefficients of the input vectors become zero.

$$\implies \nabla_{\partial_a} \nabla_{\partial_b} \partial_j = \frac{\partial}{\partial x^a} (\Gamma_{jb}^i) + \Gamma_{\alpha a}^i \Gamma_{jb}^\alpha$$

Now

$$[X, Y]^i = X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial X^i}{\partial x^j}$$

For coordinate vectors,  $[\partial_i, \partial_j] = 0 \ \forall i, j = 0, 1 \dots d$ .

Thus

$$\boxed{R_{jab}^i = \frac{\partial}{\partial x^a} \Gamma_{jb}^i - \frac{\partial}{\partial x^b} \Gamma_{ja}^i + \Gamma_{\alpha a}^i \Gamma_{jb}^\alpha - \Gamma_{\alpha b}^i \Gamma_{ja}^\alpha}$$

**Question :  $\text{Ric}(X, Y) := \text{Riem}_{amb}^m X^a Y^b$  define (0, 2)-tensor?.**

Yes, transforms as such:

**EY developments.** I roughly follow the spirit in Theodore Frankel's **The Geometry of Physics: An Introduction** Second Ed. 2003, Chapter 9 Covariant Differentiation and Curvature, Section 9.3b. The Covariant Differential of a Vector Field. P.S. EY : 20150320 I would like a copy of the Third Edition but I don't have the funds right now to purchase the third edition: go to my tilt crowdfunding campaign, <http://ernestyalumni.tilt.com>, and help with your financial support if you can or send me a message on my various channels and ernestyalumni gmail email address if you could help me get a hold of a digital or hard copy as a pro bono gift from the publisher or author.

The spirit of the development is the following:

“How can we express connections and curvatures in terms of forms?” -Theodore Frankel.

From Lecture 7, connection  $\nabla$  on vector field  $Y$ , in the “direction”  $X$ ,

$$\nabla_{\frac{\partial}{\partial x^k}} Y = \left( \frac{\partial Y^i}{\partial x^k} + \Gamma_{jk}^i Y^j \right) \frac{\partial}{\partial x^i}$$

Make the ansatz (approche, impostazione) that the connection  $\nabla$  acts on  $Y$ , the vector field, first:

$$\nabla Y(X) = \left( X^k \frac{\partial Y^i}{\partial x^k} + \Gamma_{jk}^i Y^j X^k \right) \frac{\partial}{\partial x^i} = X^k \left( \nabla_{\frac{\partial}{\partial x^k}} Y \right)^i \frac{\partial}{\partial x^i} = (\nabla_X Y)^i \frac{\partial}{\partial x^i} = \nabla_X Y$$

Now from Lecture 7, Definition for  $\Gamma$ ,

$$dx^i \left( \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) = \Gamma_{jk}^i$$

Make this ansatz (approche, impostazine)

$$\nabla_{\frac{\partial}{\partial x^j}} = (\Gamma_{jk}^i dx^k) \otimes \frac{\partial}{\partial x^i} \in \Omega^1(M, TM) = T^*M \otimes TM$$

where  $\Omega^1(M, TM) = T^*M \otimes TM$  is the set of all  $TM$  or vector-valued 1-forms on  $M$ , with the 1-form being the following:

$$\begin{aligned} \Gamma_{jk}^i dx^k &= \Gamma_j^i \in \Omega^1(M) & i &= 1 \dots \dim(M) \\ & & j &= 1 \dots \dim(M) \end{aligned}$$

So  $\Gamma_j^i$  is a  $\dim M \times \dim M$  matrix of 1-forms (EY !!!).

Thus

$$\nabla Y = (d(Y^i) + \Gamma_j^i Y^j) \otimes \frac{\partial}{\partial x^i}$$

So the connection is a (smooth) map from  $TM$  to the set of all vector-valued 1-forms on  $M$ ,  $\Omega^1(M, TM)$ , and then, after “eating” a vector  $Y$ , yields the “covariant derivative”:

$$\nabla : TM \rightarrow \Omega^1(M, TM) = T^*M \otimes TM$$

$$\nabla : Y \mapsto \nabla Y$$

$$\nabla Y : TM \rightarrow TM$$

$$\nabla Y(X) \mapsto \nabla Y(X) = \nabla_X(Y)$$

Now

$$\left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] f = \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial x^j} \right) - \frac{\partial}{\partial x^j} \left( \frac{\partial}{\partial x^i} \right) = 0$$

(this is okay as on  $p \in (U, x)$ ;  $x$ -coordinates on same chart  $(U, x)$ )

EY : 20150320 My question is when is this nontrivial or nonvanishing (i.e. not equal to 0).

$$[e_a, e_b] = ?$$

for a frame  $(e_c)$  and would this be the difference between a tangent bundle  $TM$  vs. a (general) vector bundle?

Wikipedia helps here. cf. wikipedia, “Connection (vector bundle)”

$$\nabla : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E) = \Omega^1(M, E)$$

$$\nabla e_a = \omega_{ab}^c f^b \otimes e_c$$

$f^b \in T^*M$  (this is the dual basis for  $TM$  and, note, this is for the manifold,  $M$

$$\nabla_{f_b} e_a = \omega_{ab}^c e_c \in E$$

$$\omega_a^c = \omega_{ab}^c f^b \in \Omega^1(M)$$

is the connection 1-form, with  $a, c = 1 \dots \dim V$ . EY : 20150320 This  $V$  is a vector space living on each of the fibers of  $E$ . I know that  $\Gamma(T^*M \otimes E)$  looks like it should take values in  $E$ , but it’s meaning that it takes vector values of  $V$ . Correct me if I’m wrong: ernestyalumni at gmail and various social media.

Let  $\sigma \in \Gamma(E)$ ,  $\sigma = \sigma^a e_a$

$$\nabla \sigma = (d\sigma^c + \omega_{ab}^c \sigma^a f^b) \otimes e_c \text{ with}$$

$$d\sigma^c = \frac{\partial \sigma^c}{\partial x^b} f^b$$

$$\implies \nabla_X \sigma = \left( X^b \frac{\partial \sigma^c}{\partial x^b} + \omega_{ab}^c \sigma^a X^b \right) e_c = X^b \left( \frac{\partial \sigma^c}{\partial x^b} + \omega_{ab}^c \sigma^a \right) e_c$$

LECTURE 9: NEWTONIAN SPACETIME IS CURVED!

**Axiom 1** (Newton I:). *A body on which no force acts moves uniformly along a straight line*

**Axiom 2** (Newton II:). *Deviation of a body’s motion from such uniform straight motion is effected by a force, reduced by a factor of the body’s reciprocal mass.*

Remark:

- (1) 1st axiom - in order to be relevant - must be read as a measurement prescription for the geometry of space ...
- (2) Since gravity universally acts on every particle, in a universe with at least two particles, gravity must not be considered a force if Newton I is supposed to remain applicable.

27.11. **Laplace’s questions.** Laplace \*1749

†1827

Q: “Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of (no other) force we postulated to move along straight lines in this curved space?”

Answer: No!

*Proof.* gravity is a force point of view

$$m\ddot{x}^\alpha(t) = F^\alpha(x(t))$$

$$m\ddot{x}^\alpha(t) = \underbrace{mf^\alpha}_{F^\alpha}(x(t))$$

$$-\partial_\alpha f^\alpha = 4\pi G\rho \text{ (Poisson)}$$

$\rho$  mass density of matter

$$m\ddot{x}^\alpha(t) = \underbrace{mf^\alpha}_{F^\alpha}(x(t))$$

True?

(EY : 20150330) You know this,  $F = Gm_1m_2/r^2$

Yes

weak equivalence principle

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0$$

Laplace asks: Is this ( $\ddot{x}(t)$ ) of the form

$$\ddot{x}^\alpha(t) + \Gamma_{\beta\gamma}^\alpha(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0$$

Conclusion: One cannot find  $\Gamma$  s such that Newton’s equation takes the form of an autoparallel.

□

Question (from audience) We can evaluate the autoparallel equation pointwise?! But at each point, we can set the Gammas to zero?!

Then, one should be able to write Newton’s second law in the usual form?

Prof. Schuller: you (observer) fall with the mass (i.e. accelerated reference frame) and so you transform  $\Gamma$ ’s to be 0. The problem with this is if you do the same experiment in the North pole and fall with the body. If someone else at the South Pole does the same experiment at the same time, with that same transformation (of reference frames), the effect of gravity cannot be transformed out.

In a homogeneous gravitational field, you can possibly transform away gravity,  $\Gamma = 0$ . But in an inhomogeneous gravitational field, no.

**2. The full wisdom of Newton I.** use also the information from Newton’s first law that particles (no force) move uniformly introduce the appropriate setting to talk about the difference easily

insight: in spacetime uniform & straight motion in space is simply **straight** motion

So let’s try in spacetime:

let  $x : \mathbb{R} \rightarrow \mathbb{R}^3$

$X : \mathbb{R} \rightarrow \mathbb{R}^4$   
be a particle’s trajectory in space  $\longleftrightarrow$  worldline (history) of the particle  
 $t \mapsto (t, x^1(t), x^2(t), x^3(t)) :=$   
 $:= (X^0(t), X^1(t), X^2(t), X^3(t))$

That’s all it takes:

Trivial rewritings:

$$\dot{X}^0 = 1$$
$$\implies \begin{matrix} \ddot{X}^0 & = & 0 \\ \ddot{X}^\alpha - f^\alpha(X(t)) \cdot \dot{X}^0 \cdot \dot{X}^0 & = & 0 \end{matrix} \quad (\alpha = 1, 2, 3) \implies \begin{matrix} a = 0, 1, 2, 3 \\ \boxed{\ddot{X}^a + \Gamma_{bc}^a \dot{X}^b \dot{X}^c = 0} \\ \text{antoparallel eqn in } \underline{\text{spacetime}} \end{matrix}$$

Yes, choosing  $\Gamma_{ab}^0 = 0$

$$\Gamma_{\beta\gamma}^\alpha = 0 = \Gamma_{0\beta}^\alpha = \Gamma_{\beta 0}^\alpha$$

only:  $\Gamma_{00}^\alpha \stackrel{!}{=} -f^\alpha$

Question: Is this a coordinate-choice artifact?

No, since  $R_{0\beta 0}^\alpha = -\frac{\partial}{\partial x^\beta} f^\alpha$  (only non-vanishing components) (tidal force tensor, – the Hessian of the force component)

Ricci tensor  $\implies R_{00} = R_{0m0}^m = -\partial_\alpha f^\alpha = 4\pi G\rho$

Poisson:  $-\partial_\alpha f^\alpha = 4\pi G \cdot \rho$

writing:  $T_{00} = \frac{1}{2}\rho$

$$\implies \boxed{R_{00} = 8\pi GT_{00}}$$

Einstein in 1912  $\boxed{R_{ab} \Rightarrow 8\pi G T_{ab}}$

Conclusion: Laplace’s idea works in spacetime

Remark

$$\Gamma^\alpha_{00} = -f^\alpha \\ R^\alpha_{\beta\gamma\delta} = 0 \qquad \alpha, \beta, \gamma, \delta = 1, 2, 3$$

$R_{00} = 4\pi G \rho$

Q: What about transformation behavior of LHS of

$$\underbrace{\ddot{x}^a + \Gamma^a_{bc} \dot{X}^b \dot{X}^c}_{\underbrace{(\nabla_{v_X} v_X)^a}_{:= a^a \text{ "acceleration vector"}}} = 0$$

3. The foundations of the geometric formulation of Newton’s axiom. new start

**Definition 88.** A *Newtonian spacetime* is a quintuple

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t)$$

where  $(M, \mathcal{O}, \mathcal{A})$  4-dim. smooth manifold

$$t : M \rightarrow \mathbb{R} \text{ smooth function}$$

(i) “There is an absolute space”

$$(dt)_p \neq 0 \qquad \forall p \in M$$

(ii) “absolute time flows uniformly”

$$\nabla dt \underbrace{=}_\text{space of (0,2)-tensor fields} 0 \qquad \text{everywhere}$$

$\nabla dt$  is a (0,2)-tensor field

(iii) add to axioms of Newtonian spacetime  $\nabla = 0$  torsion free

**Definition 89.** absolute space at time  $\tau$

$$S_\tau := \{p \in M | t(p) = \tau\} \\ \xrightarrow{dt \neq 0} M = \coprod S_\tau$$

**Definition 90.** A vector  $X \in T_p M$  is called

(a) future-directed if

$$dt(X) > 0$$

(b) spatial if

$$dt(X) = 0$$

(c) past-directed if

$$dt(X) < 0$$

picture

Newton I: The worldline of a particle under the influence of no force (gravity isn’t one, anyway) is a future-directed autoparallel i.e.

$$\nabla_{v_X} v_X = 0 \\ dt(v_X) > 0$$

and (iii)  $\nabla$  is torsion-free.

Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m} \iff m \cdot \mathfrak{a} = F$$

where  $F$  is a spatial vector field:

$$dt(F) = 0$$

**Convention:** restrict attention to atlases  $\mathcal{A}_{\text{stratified}}$  whose charts  $(\mathcal{U}, x)$  have the property

$$\begin{array}{ccc} x^0 : \mathcal{U} \rightarrow \mathbb{R} \\ x^1 : \mathcal{U} \rightarrow \mathbb{R} \\ \vdots \\ x^3 \end{array} \qquad x^0 = t|_{\mathcal{U}} \qquad \implies \begin{array}{c} 0 \stackrel{\text{“absolute time flows uniformly”}}{=} \nabla dt \\ 0 = \nabla_{\frac{\partial}{\partial x^a}} dx^0 = -\Gamma^0_{ba} \qquad a = 0, 1, 2, 3 \end{array}$$

Let’s evaluate in a chart  $(\mathcal{U}, x)$  of a stratified atlas  $\mathcal{A}_{\text{sheet}}$ : Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m}$$

in a chart.

$$(X^0)'' + \Gamma^0_{cd} \cancel{(X^a)'} \cancel{(X^b)'} \text{stratified atlas} = 0$$

$$(X^\alpha)'' + \Gamma^\alpha_{\gamma\delta} X^{\gamma'} X^{\delta'} + \Gamma^\alpha_{00} X^{0'} X^{0'} + 2\Gamma^\alpha_{\gamma 0} X^{\gamma'} X^{0'} = \frac{F^\alpha}{m} \qquad \alpha = 1, 2, 3$$

EY: 20160623: where the factor of 2 comes from torsion free  $\nabla$

$$\implies (X^0)''(\lambda) = 0 \implies X^0(\lambda) = a\lambda + b \quad \text{constants } a, b \quad \text{with}$$

$$X^0(\lambda) = (x^0 \circ X)(\lambda) \stackrel{\text{stratified}}{=} (t \circ X)(\lambda)$$

convention parametrize worldline by absolute time

$$\frac{d}{d\lambda} = a \frac{d}{dt}$$

$$\begin{aligned} a^2 \ddot{X}^\alpha + a^2 \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + a^2 \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + 2a^2 \Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0 &= \frac{F^\alpha}{m} \\ \implies \underbrace{\ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta} \dot{X}^\gamma \dot{X}^\delta + \Gamma^\alpha_{00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\gamma 0} \dot{X}^\gamma \dot{X}^0}_{\mathfrak{a}^\alpha} &= \frac{1}{a^2} \frac{F^\alpha}{m} \end{aligned}$$

Student question: How can you get the Lagrange equations in this formalism?

Schuller says: In *all* systems they take the form like this:

$$L = \underbrace{T}_{\frac{m}{2} g(v_x, v_x)} + V$$

where  $v_x$  is velocity to the spatial curves  $x$ .

Schuller says:  $\frac{m}{2} \sqrt{g(v_x, v_x)}$ , parameter invariance, you get straight curves; if you take  $\sqrt{\phantom{x}}$  away,  $\frac{m}{2} g(v_x, v_x)$  you get straight, *uniform* curves.

Schuller says: Only in inertial systems, which is defined as:

$\nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} = 0$

inertial frame, non-accelerating, non-rotating.

Schuller says: Apart from  $g$ , a 2-tensor, you need to know  $\omega$ .

Schuller references 2nd. year Mechanik in German: <https://www.video.uni-erlangen.de/course/id/272>



cf. [Lecture 10: Metric Manifolds \(International Winter School on Gravity and Light 2015\)](#)

We establish a structure on a smooth manifold that allows one to assign vectors in each tangent space a length (and an angle between vectors in the same tangent space).

From this structure, one can then define a notion of length of a curve.

Then we can look at shortest curves.

Requiring then that the shortest curves coincide with the straightest curves (wrt  $\nabla$ ) will result in  $\nabla$  being determined by the metric structure.

$$g \overset{\text{straight=short}}{\overset{T=0}{\rightsquigarrow}} \nabla \rightsquigarrow \text{Riem}$$

28.1. Metrics.

**Definition 91.** A metric  $g$  on a smooth manifold  $(M, \mathcal{O}, \mathcal{A})$  is a  $(0, 2)$ -tensor field satisfying

- (i) symmetry  $g(X, Y) = g(Y, X) \quad \forall X, Y \text{ vector fields}$
- (ii) non-degeneracy: the musical map

$$\begin{aligned} \text{“flat” } \flat : \Gamma(TM) &\rightarrow \Gamma(T^*M) \\ X &\mapsto \flat(X) \end{aligned}$$

where  $\flat(X)(Y) := g(X, Y)$   
 $\flat(X) \in \Gamma(T^*M)$

In thought bubble:  $\flat(X) = g(X, \cdot)$   
... is a  $C^\infty$ -isomorphism in other words, it is invertible.

Remark:  $(\flat(X))_a$  or  $X_a$

$(\flat(X))_a := g_{am} X^m$   
Thought bubble:  $\flat^{-1} = \sharp$   
 $\flat^{-1}(\omega)^a := g^{am} \omega_m$

$\flat^{-1}(\omega)^a := (g^{\text{“-1”}})^{am} \omega_m \implies$  not needed. (all of this is not needed)

**Definition 92.** The  $(2, 0)$ -tensor field  $g^{\text{“-1”}}$  with respect to a metric  $g$  is the symmetric

$$\begin{aligned} g^{\text{“-1”}} : \Gamma(T^*M) \times \Gamma(T^*M) &\rightarrow C^\infty(M) \\ (\omega, \sigma) &\mapsto \omega(\flat^{-1}(\sigma)) \quad \flat^{-1}(\sigma) \in \Gamma(TM) \end{aligned}$$

chart:  $g_{ab} = g_{ba}$   
 $(g^{-1})^{am} g_{mb} = \delta_b^a$

Example:  $(S^2, \mathcal{O}, \mathcal{A})$   
 $\text{chart } (\mathcal{U}, x)$   
 $\varphi \in (0, 2\pi)$   
 $\theta \in (0, \pi)$   
define the metric

$$g_{ij}(x^{-1}(\theta, \varphi)) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{bmatrix}_{ij}$$

$R \in \mathbb{R}^+$   
“the metric of the round sphere of radius  $R$ ”

$A^a{}_m v^m = \lambda v^a$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 & \\ & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}$$

28.2. **Signature.** Linear algebra:

$$g_{am} v^m = \lambda \cdot v^a? \rightsquigarrow$$

$(1, 1)$  tensor has eigenvalues  
 $(0, 2)$  has signature  $(p, q)$  (well-defined)

$\begin{pmatrix} (+ + +) \\ (+ + -) \\ (+ - -) \\ (- - -) \end{pmatrix}$

$\left. \vphantom{\begin{pmatrix} (+ + +) \\ (+ + -) \\ (+ - -) \\ (- - -) \end{pmatrix}} \right\} d + 1 \text{ if } p + q = \dim V$

**Definition 93.** A metric is called **Riemannian** if its signature is  $(+ + \cdots +)$   
**Lorentzian** if  $(+ - \cdots -)$

28.3. **Length of a curve.** Let  $\gamma$  be a smooth curve.  
Then we know its veloctiy  $v_{\gamma, \gamma(\lambda)}$  at each  $\gamma(\lambda) \in M$ .

**Definition 94.** On a Riemannian metric manifold  $M, \mathcal{O}, \mathcal{A}, g)$ , the **speed** of a curve at  $\gamma(\lambda)$  is the number

$$(\sqrt{g(v_\gamma, v_\gamma)})_{\gamma(\lambda)} = s(\lambda)$$

F. Schuller: “I feel the need for speed.” -Top Gun.  
(I feel the need for speed, then I feel the need for a metric)  
Aside:  $[v^a] = \frac{1}{T}$   
 $[g_{ab}] = L^2$   
 $[\sqrt{g_{ab} v^a v^b}] = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}$

**Definition 95.** Let  $\gamma : (0, 1) \rightarrow M$  a smooth curve.  
Then the **length of  $\gamma$**  is the number

$$\mathbb{R} \ni L[\gamma] := \int_0^1 d\lambda s(\lambda) = \int_0^1 d\lambda \sqrt{(g(v_\gamma, v_\gamma))_{\gamma(\lambda)}}$$

F. Schuller: “velocity is more fundamental than speed, speed is more fundamental than length”  
Example: reconsider the round sphere of radius  $R$   
Consider its equator:  
 $\theta(\lambda) := (x^1 \circ \gamma)(\lambda) = \frac{\pi}{2}$   
 $\varphi(\lambda) := (x^2 \circ \gamma)(\lambda) = 2\pi\lambda^3$   
 $\theta'(\lambda) = 0$   
 $\varphi'(\lambda) = 6\pi\lambda^2$

on the same chart  $g_{ij} = \begin{bmatrix} R^2 & \\ & R^2 \sin^2 \theta \end{bmatrix}$

F.Schuller: do everything in this chart

$$\begin{aligned} L[\gamma] &= \int_0^1 d\lambda \sqrt{g_{ij}(x^{-1}(\theta(\lambda), \varphi(\lambda)))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)} = \int_0^1 d\lambda \sqrt{R^2 \cdot 0 + R^2 \sin^2(\theta(\lambda))36\pi^2\lambda^4} = \\ &= 6\pi R \int_0^1 d\lambda \lambda^2 = 6\pi R [\frac{1}{3}\lambda^3]_0^1 = 2\pi R \end{aligned}$$

**Theorem 29.**  $\gamma : (0,1) \rightarrow M$  and  $\sigma : (0,1) \rightarrow (0,1)$  *smooth bijective and increasing “reparametrization”*

$$L[\gamma] = L[\gamma \circ \sigma]$$

*Proof.*  $\implies$  Tutorials

28.4. **Geodesics.**

**Definition 96.** A curve  $\gamma : (0,1) \rightarrow M$  is called a **geodesic** on a Riemannian manifold  $(M, \mathcal{O}, \mathcal{A}, g)$  if its a stationary curve with respect to a length functional  $L$ .

Thought bubble: in classical mechanics, deform the curve a little,  $\epsilon$  times this deformation, to first order, it agrees with  $L[\gamma]$

**Theorem 30.**  $\gamma$  geodesic iff it satisfies the Euler-Lagrange equations for the Lagrangian

$$\begin{aligned} \mathcal{L} : TM &\rightarrow \mathbb{R} \\ X &\mapsto \sqrt{g(X, X)} \end{aligned}$$

In a chart, the Euler Lagrange equations take the form:

$$\left( \frac{\partial \mathcal{L}}{\partial \dot{x}^m} \right)' - \frac{\partial \mathcal{L}}{\partial x^m} = 0$$

F.Schuller: this is a chart dependent formulation [here](#):

$$\mathcal{L}(\gamma^i, \dot{\gamma}^i) = \sqrt{g_{ij}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda)}$$

Euler-Lagrange equations:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} &= \frac{1}{\sqrt{\dots}} g_{mj}(\gamma(\lambda)) \dot{\gamma}^j(\lambda) \\ \left( \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} \right)' &= \left( \frac{1}{\sqrt{\dots}} \right)' g_{mj}(\gamma(\lambda)) \cdot \dot{\gamma}^j(\lambda) + \frac{1}{\sqrt{\dots}} (g_{mj}(\gamma(\lambda)) \ddot{\gamma}^j(\lambda) + \dot{\gamma}^s (\partial_s g_{mj}) \dot{\gamma}^j(\lambda)) \end{aligned}$$

Thought bubble: reparametrize  $g(\dot{\gamma}, \dot{\gamma}) = 1$  (it’s a condition on my reparametrization)

By a clever choice of reparametrization  $(\frac{1}{\sqrt{\dots}})' = 0$

$$\frac{\partial \mathcal{L}}{\partial \gamma^m} = \frac{1}{2\sqrt{\dots}} \partial_m g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)$$

putting this together as Euler-Lagrange equations:

$$g_{mj} \ddot{\gamma}^j + \partial_s g_{mj} \dot{\gamma}^s \dot{\gamma}^j - \frac{1}{2} \partial_m g_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$$

Multiply on both sides  $(g^{-1})^{qm}$

$$\ddot{\gamma}^q + (g^{-1})^{qm} (\partial_i g_{mj} - \frac{1}{2} \partial_m g_{ij}) \dot{\gamma}^i \dot{\gamma}^j = 0$$

$$\ddot{\gamma}^q + (g^{-1})^{qm} \frac{1}{2} (\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) \dot{\gamma}^i \dot{\gamma}^j = 0$$

geodesic equation for  $\gamma$  in a chart.

$$(g^{-1})^{qm} \frac{1}{2} (\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) =: \Gamma^q_{ij}(\gamma(\lambda))$$

Thought bubble:  $\left( \frac{\partial \mathcal{L}}{\partial \xi_x^{a+\dim M}} \right)'_{\sigma(x)} - \left( \frac{\partial \mathcal{L}}{\partial x^i_x} \right)_{\sigma(x)} = 0$

□ **Definition 97.** “Christoffel symbol”  ${}^{L.C.}\Gamma$  are the connection coefficient functions of the so-called Levi-Civita connection  ${}^{L.C.}\nabla$

We usually make this choice of  $\nabla$  if  $g$  is given.

$$(M, \mathcal{O}, \mathcal{A}, g) \rightarrow (M, \mathcal{O}, \mathcal{A}, g, {}^{L.C.}\nabla)$$

abstract way:  $\nabla g = 0$  and  $T = 0$  (torsion)  
 $\implies \nabla = {}^{L.C.}\nabla$

**Definition 98.** (a) The Riemann-Christoffel curvature is defined by

$$R_{abcd} := g_{am} R^m_{bcd}$$

(b) Ricci:  $R_{ab} = R^m_{amb}$   
Thought bubble: with a metric,  ${}^{L.C.}\nabla$

(c) (Ricci) scalar curvature:

$$R = g^{ab} R_{ab}$$

Thought bubble:  ${}^{L.C.}\nabla$

**Definition 99.** Einstein curvature  $(M, \mathcal{O}, \mathcal{A}, g)$

$$G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R$$

Convention:  $g^{ab} := (g^{“-1”})^{ab}$

F. Schuller: these indices are not being pulled up, because what would you pull them up with (student) Question: Does the Einstein curvature yield new information?

Answer:

$$g^{ab} G_{ab} = R_{ab} g^{ab} - \frac{1}{2} g_{ab} g^{ab} R = R - \delta_a^a R = R - \frac{1}{2} \dim M R = (1 - \frac{d}{2}) R$$

**Tutorial 9: Metric manifolds. Exercise 3: Levi-Civita Connection.** Suppose torsion-free  $T = 0$  and metric-compatible connection  $\nabla g = 0$

**Question Recall**  $T = 0$  on a chart.

$$\Gamma^c_{ba} = \frac{1}{2} (g^{-1})^{cm} \left( \frac{\partial g_{bm}}{\partial x^a} + \frac{\partial g_{ma}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^m} \right)$$

or

$$\Gamma^a_{bc} = \frac{1}{2} (g^{-1})^{am} \left( \frac{\partial g_{bm}}{\partial x^c} + \frac{\partial g_{mc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^m} \right)$$

29. SYMMETRY

EY : 20150321 This lecture tremendously and lucidly clarified, for me at least, what a symmetry of the Lie algebra is, and in comparing structures  $(M, \mathcal{O}, \mathcal{A})$  vs.  $(M, \mathcal{O}, \mathcal{A}, \nabla)$ , clarified differences, and asking about differences is a good way to learn, the difference between  $\mathcal{L}$  and  $\nabla$ , respectively.

Feeling that the round sphere

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}})$$

has rotational symmetry, while the potato

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}})$$

does not.

29.1.

29.2. Important

29.3. **Flow of a complete vector field.** Let  $(M, \mathcal{O}, \mathcal{A})$  smooth  $X$  vector field on  $M$

**Definition 100.** A curve  $\gamma : I \subseteq \mathbb{R} \rightarrow M$  is called an integral curve of  $X$  if

$$v_{\gamma, \gamma(\lambda)} = X_{\gamma(\lambda)}$$

**Definition 101.** A vector filed  $X$  is **complete** if all integral curves have  $I = \mathbb{R}$  EY: 20150321 (i.e. domain is all of  $\mathbb{R}$ )

Ex. minute 48:30 EY : reall good explanation by F.P.Schuller; take a pt. out for an incomplete vector field.

**Theorem 31.** compactly supported smooth vector field is complete.

**Definition 102.** The flow of a complete vector field  $X$  is a 1-parameter family

$$h^X = \mathbb{R} \times M \rightarrow M$$

where  $\gamma_p : \mathbb{R} \rightarrow M$  is the integral curve of  $X$  with

$$\begin{aligned} \gamma(0) &= p \\ \textit{Then for fixed } \lambda \in \mathbb{R} \end{aligned}$$

$$h^X_\lambda : M \rightarrow M \text{ smooth}$$

picture  $h^X_\lambda(S) \neq S$  ( if  $X \neq 0$ )

29.4. **Lie subalgebras of the Lie algebra  $(\Gamma(TM), [\cdot, \cdot])$  of vector fields.**

(a)  $\Gamma(TM) = \{ \text{ set of all vector fields } \}$   $C^\infty(M)$ -module =  $\mathbb{R}$ -vector space

$$\implies [X, Y] \in \Gamma(TM) \qquad [X, Y]f := X(Yf) - Y(Xf)$$

- (i)  $[X, Y] = -[Y, X]$
- (ii)  $[\lambda X + Z, Y] = \lambda[X, Y] + [Z, Y]$
- (iii)  $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$   
 $(\Gamma(TM), [\cdot, \cdot])$  Lie algebra

(b) Let  $X_1 \dots X_s$  for  $s$  (many) vector fields on  $M$ , such that

**Tutorial 11 Symmetry. Exercise 1. : True or false?**

(a) 

- 
- $\phi^* : T^*N \rightarrow T^*M$  i.e.  $\phi^*\nu(X) = \nu(\phi_*X)$  for smooth  $\phi : M \rightarrow N$ , so the pullback of a covector  $\nu \in T^*N$  maps to a covector in  $T^*M$ .

- 
- 
- 
- 

(b)

(c)

**Exercise 2. :** Pull-back and push-forward

**Question .** Let's check this locally

$$\begin{aligned} \phi^*(df)(X) &= (df)(\phi_*X) = (df)(X^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}) = X^i \frac{\partial y^j}{\partial x^i} \frac{\partial f}{\partial y^j} \text{ where } \phi_*X = X^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \\ d(\phi^*f)(X) &= d(f(\phi))(X) = \frac{\partial f}{\partial y^j} \frac{\partial y^j}{\partial x^i} dx^i(X) = X^i \frac{\partial y^j}{\partial x^i} \frac{\partial f}{\partial y^j} \end{aligned}$$

So

$\phi^*(df) = d(\phi^*f)$

 $\forall p \in M, \forall X \in \mathfrak{X}(M)$

The big idea is that this is a showing of the **naturality** of the pullback  $\phi^*$  with  $d$ , i.e. that this commutes:

$$\begin{array}{ccc} \Omega^1(M) & \xleftarrow{\phi^*} & \Omega^1(N) \\ d \uparrow & & d \uparrow \\ C^\infty(M) & \xleftarrow{\phi^*} & C^\infty(N) \end{array}$$

**Question .**

$$\begin{aligned} (\phi_*)^a_b &:= (dy^a)(\phi_*(\frac{\partial}{\partial x^b})) \\ \text{Let } g &\in C^\infty(N) \\ \phi_*\left(\frac{\partial}{\partial x^b}\right)g &= \frac{\partial x^b}{g}\phi(p) = \frac{\partial}{\partial x^b}g\phi x^{-1}x(p) = \frac{\partial}{\partial x^b}(gyy^{-1}\phi x^{-1})(x) = \\ &= \frac{\partial}{\partial x^b}(gy^{-1}(y\phi x^{-1}(x(p)))) = \frac{\partial g^b}{\partial y} \bigg|_y \frac{\partial y^a}{\partial x^b} \bigg|_x = \frac{\partial y^a}{\partial x^b} \frac{\partial g}{\partial y^a} \end{aligned}$$

Then

$$\phi_*\left(\frac{\partial}{\partial x^b}\right) = \frac{\partial y^a}{\partial x^b} \frac{\partial}{\partial y^a}$$

and so

$$(\phi_*)^a_b = \frac{\partial y^a}{\partial x^b}$$

**Question .**

Exercise 3. :Lie derivative-the pedestrian way

Question . While it is true that  $\forall p \in S^2$ , for  $x(p) = (\theta, \varphi)$ , and  $(yix^{-1})(\theta, \varphi) = (y^1, y^2, y^3) \in \mathbb{R}^3$  and that, at this point

$p$ ,  $(y^1)^2/a^2 + (y^2)^2/b^2 + (y^3)^2/c^2 = 1$ , this doesn't imply (EY: 20150321 I think) that, globally, it's an ellipsoid (yet). In the familiar charts given, spherical chart  $(U, x) \in \mathcal{A}$  and  $(\mathbb{R}^3, y = \text{id}_{\mathbb{R}^3}) \in \mathcal{B}$  it looks like an ellipsoid, but change to another choice of charts, and it could look something very different.

Question .

Equip  $(\mathbb{R}^3, \mathcal{O}_{\text{st}}, \mathcal{B})$  with the Euclidean metric  $g$ , and pullback  $g$ .  
Note that the pullback of the inclusion from  $\mathbb{R}^3$  onto  $S^2$  for the Euclidean metric is the following:

$$i^*g\left(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j}\right) = g\left(i_*\frac{\partial}{\partial\theta^i}, i_*\frac{\partial}{\partial\theta^j}\right) = g\left(\frac{\partial x^a}{\partial\theta^i}\frac{\partial}{\partial x^a}, \frac{\partial x^b}{\partial\theta^j}\frac{\partial}{\partial x^b}\right) = g_{ab}\frac{\partial x^a}{\partial\theta^i}\frac{\partial x^b}{\partial\theta^j}$$

With  $g_{ab} = \delta_{ab}$ , the usual Euclidean metric, this becomes the following:

$$g_{ij}^{\text{ellipsoid}} = \frac{\partial x^a}{\partial\theta^i}\frac{\partial x^a}{\partial\theta^j}$$

At this point, one should get smart (we are in the 21st century) and use some sort of CAS (Computer Algebra System). I like Sage Math (version 6.4 as of 20150322). I also like the Sage Manifolds package for Sage Math.  
I like Sage Math for the following reasons:

- Open source, so it's open and freely available to anyone, which fits into my principle of making online education open and freely available to anyone, anytime
- Sage Math structures everything in terms of Category Theory and Categories and Morphisms naturally correspond to Classes and Class methods or functions in Object-Oriented Programming in Python and they've written it that way

and I like Sage Manifolds for roughly the same reasons, as manifolds are fit into a category theory framework that's written into the Python code. e.g.

```
sage: S2 = Manifold(2, 'S^2', r'\mathbb{S}^2', start_index=1) ; print S2
sage: print S2
2-dimensional manifold 'S^2'
sage: type(S2)
<class 'sage.geometry.manifolds.manifold.Manifold_with_category'>
```

With code (I've provided for convenience; you can make your own as I wrote it based upon to example of  $S^2$  on the sage-manifolds documentation website page), load it and do the following:

cf. <https://github.com/ernestyalumni/diffgeo-by-sagemnfd/blob/master/S2.sage>  
<http://sagemanifolds.obspm.fr/examples.html>

```
sage: load("S2.sage")
sage: U_ep = S2.open_subset('U_{ep}')
sage: eps.<the,phi> = U_ep.chart()
sage: a = var("a")
sage: b = var("b")
sage: c = var("c")
sage: inclus = S2.diff_mapping(R3, {(eps, cart): [ a*cos(phi)*sin(the), b*sin(phi)*sin(the),c*cos(the) ]} , name="inc", latex_name=r'\mathcal{i}')
sage: inclus.pullback(h).display()
inc_*(h) = (c^2*sin(the)^2 + (a^2*cos(phi)^2 + b^2*sin(phi)^2)*cos(the)^2) dthe*dthe - (a^2 - b^2)*cos(phi)*cos(the)*sin(phi)*sin(the) dthe*dphi - (a^2 - b^2)*cos(phi)*cos(the)*sin(phi)*sin(the) dphi*dthe + (b^2*cos(phi)^2 + a^2*sin(phi)^2)*sin(the)^2 dphi*dphi
sage: inclus.pullback(h)[2,2].expr()
(b^2*cos(phi)^2 + a^2*sin(phi)^2)*sin(the)^2
```

A new open subset  $U_{\text{ep}}$  was declared in  $S^2$ , a new chart  $(U_{\text{ep}}, (\theta, \phi))$  was declared, the constants,  $a, b, c$ , were declared, and the inclusion map given in the problem

$$y \circ i \circ x^{-1} : (\theta, \phi) \mapsto (a \cos \phi \sin \theta, b \sin \phi \sin \theta, c \cos \theta)$$

Then the pullback of the inclusion map  $\rangle$  was done on the Euclidean metric  $h$ , defined earlier in the file

S2.sage

. Then one can access the components of this metric and do, for example,

simplify\_full(),full\_simplify(), reduce\_trig()

on the expression.

In Python, I could easily do this, and give an answer quick in LaTeX:

```
sage: for i in range(1,3):
....:     for j in range(1,3):
....:         print inclus.pullback(h)[i,j].expr()
....:         latex(inclus.pullback(h)[i,j].expr() )
....:
c^2*sin(the)^2 + (a^2*cos(phi)^2 + b^2*sin(phi)^2)*cos(the)^2
```

(EY: I'll suppress the LaTeX output but this sage math function gives you LaTeX code) and so

$$i^*g = c^2 \sin (the)^2 + \left(a^2 \cos (\phi)^2 + b^2 \sin (\phi)^2\right) \cos (the)^2 d\theta \otimes d\theta + \\ -2\left(a^2 - b^2\right) \cos (\phi) \cos (the) \sin (\phi) \sin (the) d\theta \otimes d\phi + \\ +\left(b^2 \cos (\phi)^2 + a^2 \sin (\phi)^2\right) \sin (the)^2 d\phi \otimes d\phi$$

Question .

```
sage: polar_vees = eps.frame()
sage: X_1 = - sin(phi) * polar_vees[1] - cot( the ) * cos(phi) * polar_vees[2]
sage: X_2 = cos( phi ) * polar_vees[1] - cot( the ) * sin( phi ) * polar_vees[2]
sage: X_3 = polar_vees[2]
sage: X_2.lie_der(X_1).display()
(cos(the)^2 - 1)/sin(the)^2 d/dphi
sage: X_3.lie_der(X_1).display()
cos(phi) d/dthe - cos(the)*sin(phi)/sin(the) d/dphi
sage: X_3.lie_der(X_2).display()
sin(phi) d/dthe + cos(phi)*cos(the)/sin(the) d/dphi
```

Indeed, one can check on a scalar field  $f_{\text{eps}} \in C^\infty(S^2)$ :

```
sage: f_eps = S2.scalar_field({eps: function('f', the, phi ) }, name='f' )
sage: (X_1( X_2(f_eps)) - X_2(X_1(f_eps) ) ).display()
U_{ep} --> R
(the, phi) |--> -D[1](f)(the, phi)
sage: X_2.lie_der(X_1) == -X_3
True
sage: X_3.lie_der(X_1) == X_2
True
sage: X_3.lie_der(X_2) == -X_1
True
```

$$\implies \boxed{[X_i, X_j] = -\epsilon_{ijk} X_k}$$

So  $\text{span}_{\mathbb{R}}\{X_1, X_2, X_3\}$  equipped with  $[\ ,\ ]$  constitute a Lie subalgebra on  $S^2$  (It's closed under  $[\ ,\ ]$ )

30. INTEGRATION

30.1.

30.2.

30.3. Volume forms.

**Definition 103.** On a smooth manifold  $(M, \mathcal{O}, \mathcal{A})$

a  $(0, \dim M)$ -tensor field  $\Omega$  is called a volume form if

- (a)  $\Omega$  vanishes nowhere (i.e.  $\Omega \neq 0 \ \forall p \in M$ )
- (b) totally antisymmetric

$$\Omega(\dots, \underbrace{X}_{ith}, \dots, \underbrace{Y}_{jth}, \dots) = -\Omega(\dots, \underbrace{Y}_{ith}, \dots, \underbrace{X}_{jth}, \dots)$$

In a chart:

$$\Omega_{i_1 \dots i_d} = \Omega_{[i_1 \dots i_d]}$$

Example  $(M, \mathcal{O}, \mathcal{A}, g)$  metric manifold  
construct volume form  $\Omega$  from  $g$   
In any chart:  $(U, x)$

$$\Omega_{i_1 \dots i_d} := \sqrt{\det(g_{ij}(x))} \epsilon_{i_1 \dots i_d}$$

where **Levi-Civita symbol**  $\epsilon_{i_1 \dots i_d}$  is defined as  $\epsilon_{123 \dots d} = +1$   
 $\epsilon_{1 \dots d} = \epsilon_{[i_1 \dots i_d]}$

*Proof.* (well-defined) Check: What happens under a change of charts

$$\begin{aligned} \Omega(y)_{i_1 \dots i_d} &= \sqrt{\det(g(y)_{ij})} \epsilon_{i_1 \dots i_d} = \\ &= \sqrt{\det(g_{mn}(x)) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j}} \frac{\partial y^{m_1}}{\partial x^{i_1}} \dots \frac{\partial y^{m_d}}{\partial x^{i_d}} \epsilon_{[m_1 \dots m_d]} = \\ &= \sqrt{|\det g_{ij}(x)|} \left| \det \left( \frac{\partial x}{\partial y} \right) \right| \det \left( \frac{\partial y}{\partial x} \right) \epsilon_{i_1 \dots i_d} = \sqrt{\det g_{ij}(x) \epsilon_{i_1 \dots i_d}} \text{sgn} \left( \det \left( \frac{\partial x}{\partial y} \right) \right) \end{aligned}$$

Consider the following:

$$\begin{aligned} \Omega(y)(Y_{(1)} \dots Y_{(d)}) &= \Omega(y)_{i_1 \dots i_d} Y_{(1)}^{i_1} \dots Y_{(d)}^{i_d} = \\ &= \sqrt{\det(g_{ij}(y))} \epsilon_{i_1 \dots i_d} Y_{(1)}^{i_1} \dots Y_{(d)}^{i_d} = \\ &= \sqrt{\det(g_{mn}(x)) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j}} \epsilon_{i_1 \dots i_d} \frac{\partial y^{i_1}}{\partial x^{m_1}} \dots \frac{\partial y^{i_d}}{\partial x^{m_d}} X^{m_1} \dots X^{m_d} = \\ &= \sqrt{\det(g_{mn}(x)) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j}} \det \left( \frac{\partial y}{\partial x} \right) \epsilon_{m_1 \dots m_d} X^{m_1} \dots X^{m_d} = \\ &= \sqrt{\det(g_{mn}(x))} \left| \det \left( \frac{\partial x}{\partial y} \right) \right| \det \left( \frac{\partial y}{\partial x} \right) \epsilon_{m_1 \dots m_d} X^{m_1} \dots X^{m_d} = \\ &= \sqrt{\det(g_{mn}(x))} \epsilon_{m_1 \dots m_d} \text{sgn} \left( \det \left( \frac{\partial x}{\partial y} \right) \right) X^{m_1} \dots X^{m_d} = \text{sgn}(\det \left( \frac{\partial x}{\partial y} \right)) \Omega_{m_1 \dots m_d}(x) X^{m_1} \dots X^{m_d} \end{aligned}$$

If  $\det \left( \frac{\partial y}{\partial x} \right) > 0$ ,

$$\Omega(y)(Y_{(1)} \dots Y_{(d)}) = \Omega(x)(X_{(1)} \dots X_{(d)})$$

This works also if Levi-Civita symbol  $\epsilon_{i_1 \dots i_d}$  doesn't change at all under a change of charts. (around 42:43 <https://youtu.be/2XpnbvPy-Zg>)

Alright, let's require,  
restrict the smooth atlas  $\mathcal{A}$   
to a subatlas ( $\mathcal{A}^\uparrow$  still an atlas)

$$\mathcal{A}^\uparrow \subseteq \mathcal{A}$$

s.t.  $\forall (U, x), (V, y)$  have chart transition maps  $y \circ x^{-1}$   
 $x \circ y^{-1}$

s.t.  $\det \left( \frac{\partial y}{\partial x} \right) > 0$   
such  $\mathcal{A}^\uparrow$  called an **oriented** atlas

$$(M, \mathcal{O}, \mathcal{A}, g) \implies (M, \mathcal{O}, \mathcal{A}^\uparrow, g)$$

Note: associated bundles.  
Note also:  $\det \left( \frac{\partial y^b}{\partial x^a} \right) = \det(\partial_a(y^b x^{-1}))$   $\frac{\partial y^b}{\partial x^a}$  is an endomorphism on vector space  $V$ .

$\varphi : V \rightarrow V$   
 $\det \varphi$  independent of choice of basis  
 $g$  is a  $(0, 2)$  tensor field, not endomorphism (not independent of choice of basis)  $\sqrt{|\det(g_{ij}(y))|}$

**Definition 104.**  $\Omega$  be a volume form on  $(M, \mathcal{O}, \mathcal{A}^\uparrow)$  and consider chart  $(U, x)$

□ **Definition 105.**  $\omega_{(X)} := \Omega_{i_1 \dots i_d} \epsilon^{i_1 \dots i_d}$  same way  $\epsilon^{12 \dots d} = +1$   
 $\epsilon^{[\dots]}$

one can show

$$\omega_{(y)} = \det \left( \frac{\partial x}{\partial y} \right) \omega_{(x)}$$

scalar density

30.4. Integration on one chart domain  $U$ .

**Definition 106.**

(177)

$$\int_U f \stackrel{(U,y)}{=} \int_{y(U)} d^d \beta \omega_{(y)}(y^{-1}(\beta)) f_{(y)}(\beta)$$

*Proof.* : Check that it’s (well-defined), how it changes under change of charts

$$\begin{aligned} \int_U f \stackrel{(U,y)}{=} \int_{y(U)} d^d \beta \omega_{(y)}(y^{-1}(\beta)) f_{(y)}(\beta) &= \stackrel{(U,y)}{=} \int_{x(U)} \int d^d \alpha \left| \det \left( \frac{\partial y}{\partial x} \right) \right| f_{(x)}(\alpha) \omega_{(x)}(x^{-1}(\alpha)) \det \left( \frac{\partial x}{\partial y} \right) = \\ &= \int_{x(U)} d^d \alpha \omega_{(x)}(x^{-1}(x)) f_{(x)}(\alpha) \end{aligned}$$

On an oriented metric manifold  $(M, \mathcal{O}, \mathcal{A}^\uparrow, g)$

$$\int_U f := \int_{x(U)} d^d \alpha \underbrace{\sqrt{\det(g_{ij}(x))(x^{-1}(\alpha))}}_{\sqrt{g}} f_{(x)}(\alpha)$$

30.5. Integration on the entire manifold.

31. LECTURE 13: RELATIVISTIC SPACETIME

Recall, from Lecture 9, the definition of Newtonian spacetime

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t)$$

$\nabla$  torsion free

$t \in C^\infty(M)$

$dt \neq 0$

$\nabla dt = 0$      (uniform time)

and the definition of relativistic spacetime (before Lecture )

$$(M, \mathcal{O}, \mathcal{A}^\uparrow, \nabla, g, T)$$

$\nabla$  torsion-free

$g$  Lorentzian metric(+ − − −)

$T$  time-orientation

31.1. Time orientation.

**Definition 107.**  $(M, \mathcal{O}, \mathcal{A}^\uparrow, g)$  a Lorentzian manifold. Then a time-orientation is given by a vector field  $T$  that

- (i) does **not** vanish anywhere

(ii)  $g(T, T) > 0$

Newtonian vs. relativistic  
Newtonian  
 $X$  was called future-directed if

$$dt(X) > 0$$

$\forall p \in M$ , take half plane, half space of  $T_p M$

also stratified atlas so make planes of constant  $t$  straight  
relativistic

half cone  $\forall p, q \in M$ , half-cone  $\subseteq T_p M$

This definition of spacetime  
Question

I see how the cone structure arises from the new metric. I don’t understand however, how the  $T$ , the time orientation, comes in

Answer  
 $(M, \mathcal{O}, \mathcal{A}, g)$   $g \stackrel{(\quad)}{\leftarrow} + - - -$   
requiring  $g(X, X) > 0$ , select cones  
 $T$  chooses which cone

This definition of spacetime has been made to enable the following physical postulates:

(P1) The worldline  $\gamma$  of a massive particle satisfies

- (i)  $g_{\gamma(\lambda)}(v_{\gamma, \gamma(lambda)}, v_{\gamma, \gamma(\lambda)}) > 0$

(ii)  $g_{\gamma(\lambda)}(T, v_{\gamma, \gamma(\lambda)}) > 0$

□ (P2) Worldlines of massless particles satisfy

- (i)  $g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) = 0$

(ii)  $g_{\gamma(\lambda)}(T, v_{\gamma, \gamma(\lambda)}) > 0$

picture: spacetime:

Answer (to a question)  $T$  is a smooth vector field,  $T$  determines future vs. past, “general relativity: we have such a time orientation; smoothness makes it less arbitrary than it seems” -FSchuller,

Claim: 9/10 of a metric are determined by the cone

spacetime determined by distribution, only one-tenth error

31.2. **Observers.**  $(M, \mathcal{O}, \mathcal{A}^\uparrow, \nabla, g, T)$

**Definition 108.** An observer is a worldline  $\gamma$  with

$$g(v_\gamma, v_\gamma) > 0$$

$$g(T, v_\gamma) > 0$$

together with a choice of basis

$$v_{\gamma, \gamma(\lambda)} \equiv e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$$

of each  $T_{\gamma(\lambda)} M$  where the observer worldline passes, if  $g(e_a(\lambda), e_b(\lambda)) = \eta_{ab} =$

$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{ab}$$

precise: observer = smooth curve in the frame bundle  $LM$  over  $M$

31.2.1. Two physical postulates.

(P3) A **clock** carried by a specific observer  $(\gamma, e)$  will measure a **time**

$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)})}$$

between the two “events”

$$\gamma(\lambda_0)$$

“start the clock”

and

$$\gamma(\lambda_1)$$

“stop the clock”

Compare with Newtonian spacetime:

$$t(p) = 7$$

Thought bubble: proper time/eigentime  $\tau$



$M = \mathbb{R}^4$

$\mathcal{O} = \mathcal{O}_{\text{st}}$

Application/Example.  
 $\mathcal{A} \ni (\mathbb{R}^4, \text{id}_{\mathbb{R}^4})$

$g : g_{(x)ij} = \eta_{ij} \quad ; \quad T^i_{(x)} = (1, 0, 0, 0)^i$

$\implies \Gamma^i_{(x) \ jk} = 0 \text{ everywhere}$

$\implies (M, \mathcal{O}, \mathcal{A}^\uparrow, g, T, \nabla) \quad \text{Riem} = 0$

$\implies \text{spacetime is flat}$

This situation is called special relativity.

Consider two observers:

$\gamma : (0, 1) \rightarrow M$

$\gamma^i_{(x)} = (\lambda, 0, 0, 0)^i$

$\delta : (0, 1) \rightarrow M$

$\alpha \in (0, 1) : \delta^i_{(x)} = \begin{cases} (\lambda, \alpha \lambda, 0, 0)^i & \lambda \leq \frac{1}{2} \\ (\lambda, (1 - \lambda)\alpha, 0, 0)^i & \lambda > \frac{1}{2} \end{cases}$

let’s calculate:

$\tau_\gamma := \int_0^1 \sqrt{g_{(x)ij} \dot{\gamma}^i_{(x)} \dot{\gamma}^j_{(x)}} = \int_0^1 d\lambda 1 = 1$

$\tau_\delta := \int_0^{1/2} d\lambda \sqrt{1 - \alpha^2} + \int_{1/2}^1 \sqrt{1^2 - (-\alpha)^2} = \int_0^1 \sqrt{1 - \alpha^2} = \sqrt{1 - \alpha^2}$

Note: piecewise integration

Taking the clock postulate (P3) seriously, one better come up with a realistic clock design that supports the postulate.  
idea.

2 little mirrors

(P4) Postulate

Let  $(\gamma, e)$  be an observer, and

$\delta$  be a *massive* particle worldline that is parametrized s.t.  $g(v_\gamma, v_\gamma) = 1$  (for parametrization/normalization convenience)

Suppose the observer and the particle *meet* somewhere (in spacetime)

$\delta(\tau_2) = p = \gamma(\tau_1)$

*This* observer measures the 3-velocity (spatial velocity) of this particle as

(178)

$v_\delta : \epsilon^\alpha(v_{\delta, \delta(\tau_2)})e_\alpha \quad \alpha = 1, 2, 3$

where  $\epsilon^0, \boxed{\epsilon^1, \epsilon^2, \epsilon^3}$  is the unique dual basis of  $e_0, \boxed{e_1, e_2, e_3}$

EY:20150407

There might be a major correction to Eq. (178) from the Tutorial 14 : Relativistic spacetime, matter, and Gravitation, see the second exercise, Exercise 2, third question:

(179)

$v := \frac{\epsilon^\alpha(v_\delta)}{\epsilon^0(v_\delta)}e_\alpha$

Consequence: An observer  $(\gamma, e)$  will extract quantities measurable in his laboratory from objective spacetime quantities always like that.

Ex:  $F$  Faraday  $(0, 2)$ -tensor of electromagnetism:

$F(e_a, e_b) = F_{ab} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}$

observer frame  $e_a, e_b$

$E_\alpha := F(e_0, e_\alpha)$

$B^\gamma := F(e_\alpha, e_\rho)\epsilon^{\alpha\beta\gamma}$  where  $\epsilon^{123} = +1$  totally antisymmetric

**31.3. Role of the Lorentz transformations.** Lorentz transformations emerge as follows:

Let  $(\gamma, e)$  and  $(\tilde{\gamma}, \tilde{e})$  be observers with  $\gamma(\tau_1) = \tilde{\gamma}(\tau_2)$

(for simplicity  $\gamma(0) = \tilde{\gamma}(0)$ )

Now

$e_0, \dots, e_1 \quad \text{at } \tau = 0$

$\text{and } \tilde{e}_0, \dots, \tilde{e}_1 \quad \text{at } \tau = 0$

both bases for the same  $T_{\gamma(0)}M$

Thus:  $\tilde{e}_a = \Lambda^b_a e_b \quad \Lambda \in GL(4)$

Now:

$\eta_{ab} = g(\tilde{e}_a, \tilde{e}_b) = g(\Lambda^m_a e_m, \Lambda^n_b e_n) =$   
 $= \Lambda^m_a \Lambda^n_b \underbrace{g(e_m, e_n)}_{\eta_{mn}}$

i.e.  $\Lambda \in O(1, 3)$

Result: Lorentz transformations relate the *frames* of *any two observers* at the same point.

“ $\tilde{x}^\mu - \Lambda^\mu_\nu x^\nu$ ” is utter nonsense

**Tutorial.** I didn’t see a tutorial video for this lecture, but I saw that the Tutorial sheet number 14 had the relevant topics. Go there.

32. LECTURE 14: MATTER

two types of matter

point matter

field matter

point matter

massive point particle

more of a phenomenological importance

field matter

electromagnetic field

more fundamental from the GR point of view

both classical matter types

**32.1. Point matter.** Our postulates (P1) and (P2) already constrain the possible particle worldlines.

But what is their precise law of motion, possibly in the presence of “forces”,

(a) without external forces

$S_{\text{massive}}[\gamma] := m \int d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)})}$

with:

$g_{\gamma(\lambda)}(T_{\gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) > 0$

dynamical law Euler-Lagrange equation

similarly

$$\begin{aligned} S_{\text{massless}}[\gamma, \mu] &= \int d\lambda \mu g(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) \\ \delta_\mu \qquad g(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) &= 0 \\ \delta_\gamma \qquad \text{e.o.m.} \end{aligned}$$

Reason for describing equations of motion by actions is that composite systems have an action that is the sum of the actions of the parts of that system, possibly including “interaction terms.”

Example.

$$S[\gamma] + S[\delta] + S_{\text{int}}[\gamma, \delta]$$

- (b) presence of external forces  
or rather presence of fields to which a particle “couples”

Example

$$S[\gamma; A] = \int d\lambda m \sqrt{g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)})} + qA(v_{\gamma, \gamma(\lambda)})$$

where  $A$  is a **covector field** on  $M$ .  $A$  fixed (e.g. the electromagnetic potential)

Consider Euler-Lagrange eqns.  $L_{\text{int}} = qA_{(x)}\dot{\gamma}_{(x)}^m$

$$m(\nabla_{v_\gamma} v_\gamma)_a + \underbrace{\left(\frac{\partial L_{\text{int}}}{\partial \dot{\gamma}_{(x)}^m}\right)}_{*} - \frac{\partial L_{\text{int}}}{\partial \gamma_{(x)}^m} = 0 \implies \boxed{\begin{aligned} m(\nabla_{v_\gamma} v_\gamma)^a &= \underbrace{-qF_m^a \dot{\gamma}^m}_{\text{Lorentz force on a charged particle in an electromagnetic field}} \end{aligned}}$$

$$\frac{\partial L}{\partial \dot{\gamma}^a} = qA_{(x)a}, \qquad \left(\frac{\partial L}{\partial \dot{\gamma}^m}\right) = q \cdot \frac{\partial}{\partial x^m}(A_{(x)m}) \cdot \dot{\gamma}_{(x)}^m$$

$$\frac{\partial L}{\partial \gamma^a} = q \cdot \frac{\partial}{\partial x^a}(A_{(x)m}) \dot{\gamma}^m$$

$$* = q \left( \frac{\partial A_a}{\partial x^m} - \frac{\partial A_m}{\partial x^a} \right) \dot{\gamma}_{(x)}^m \quad = q \cdot F_{(x)am} \dot{\gamma}_{(x)}^m$$

$F \leftarrow$  Faraday

$$S[\gamma] = \int (m\sqrt{g(v_\gamma, v_\gamma)} + qA(v_\gamma))d\lambda$$

32.2. **Field matter.**

**Definition 109.** *Classical (non-quantum) field matter is any tensor field on spacetime where equations of motion derive from an action.*

Example:

$$S_{\text{Maxwell}}[A] = \frac{1}{4} \int_M d^4x \sqrt{-g} F_{ab} F_{cd} g^{ac} g^{bd}$$

$A$  (0,1)-tensor field

= thought cloud: for simplicity one chart covers all of  $M$

– for  $\sqrt{-g}$  (+ – – –)

$$F_{ab} := 2\partial_{[a}A_{b]} = 2(\nabla_{[a}A)_{b]}$$

Euler-Lagrange equations for fields

$$0 = \frac{\partial \mathcal{L}}{\partial A_m} - \frac{\partial}{\partial x^s} \left( \frac{\partial \mathcal{L}}{\partial \partial_s A_m} \right) + \frac{\partial}{\partial x^s} \frac{\partial}{\partial x^t} \frac{\partial^2 \mathcal{L}}{\partial \partial_t \partial_s A_m}$$

Example ...

**inhomogeneous** Maxwell

thought bubble  $j = qv_\gamma$

$$(\nabla_{\frac{\partial}{\partial x^m}} F)^{ma} = j^a$$

$$\partial_{[a}F_{b]} - ()$$

homogeneous Maxwell

Other example well-liked by textbooks

$$S_{\text{Klein-Gordon}}[\phi] := \int_M d^4x \sqrt{-g} [g^{ab}(\partial_a \phi)(\partial_b \phi) - m^2 \phi^2]$$

$\phi$  (0,0)-tensor field

**32.3. Energy-momentum tensor of matter fields.** At some point, we want to write down an action for the metric tensor field itself.

But then, this action  $S_{\text{grav}}[g]$  will be added to any  $S_{\text{matter}}[A, \phi, \dots]$  in order to describe the total system.

$$S_{\text{total}}[g, A] = S_{\text{grav}}[g] + S_{\text{Maxwell}}[A, g]$$

$$\delta A \qquad \implies \text{Maxwell's equations}$$

$$\delta g_{ab} \quad : \quad \boxed{\frac{1}{16\pi G} G^{ab}} + (-2T^{ab}) = 0$$

$G$  Newton’s constant

$$G^{ab} = 8\pi G_N T^{ab}$$

**Definition 110.**  $S_{\text{matter}}[\Phi, g]$  is a matter action, the ***so-called energy-momentum tensor*** is

$$T^{ab} := \frac{-2}{\sqrt{-g}} \left( \frac{\partial \mathcal{L}_{\text{matter}}}{\partial g_{ab}} - \partial_s \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \partial_s g_{ab}} + \dots \right)$$

– of  $\frac{-2}{\sqrt{g}}$  is Schrödinger minus (EY : 20150408 F.Schuller’s joke? but wise)

choose all sign conventions s.t.

$$T(\epsilon^0, \epsilon^0) > 0$$

Example: For  $S_{\text{Maxwell}}$ :

$$T_{ab} = F_{am} F_{bn} g^{mn} - \frac{1}{4} F_{mn} F^{mn} g_{ab}$$

$$T_{ab} \equiv T_{\text{Maxwell}ab}$$

$$T(e_0, e_0) = \underline{E}^2 + \underline{B}^2$$

$$T(e_0, e_\alpha) = (E \times B)_\alpha$$

Fact: One often does not specify the fundamental action for some matter, but one is rather satisfied to assume certain properties / forms of

$$T_{ab}$$

Example Cosmology: (homogeneous & isotropic)  
perfect fluid

of pressure  $p$  and density  $\rho$  modelled by

$$T^{ab} = (\rho + p)u^a u^b - pg^{ab}$$

radiative fluid

What is a fluid of photons:

$T_{\text{Maxwell}}^{ab}g_{ab} = 0$

observe:  $T_{\text{p.f.}}^{ab}g_{ab} \stackrel{!}{=} 0$

$= (\rho + p)u^au^bg_{ab} - p \underbrace{g^{ab}g_{ab}}_4$

$\leftrightarrow \rho_p 0 4 p = 0$

$\rho = 3 p$

$p = \frac{1}{3}\rho$

Reconvene at 3 pm? (EY : 20150409 I sent a Facebook (FB) message to the International Winter School on Gravity and Light: there was no missing video; it continues on Lecture 15 immediately)

**Tutorial 14: Relativistic Spacetime, Matter and Gravitation. Exercise 2: Lorentz force law.**

**Question electromagnetic potential.**

33. LECTURE 15: EINSTEIN GRAVITY

Recall that in Newtonian spacetime, we were able to reformulate the Poisson law  $\Delta \phi = 4\pi G_N \rho$  in terms of the Newtonian spacetime curvature as

$R_{00} = 4\pi G_N \rho$

$R_{00}$  with respect to  $\nabla_{\text{Newton}}$

$G_N =$  Newtonian gravitational constant

This prompted Einstein to postulate < 1915 that the relativistic field equations for the Lorentzian metric  $g$  of (relativistic) spacetime

$R_{ab} = 8\pi G_N T_{ab}$ ✓

However, this equation suffers from a problem

LHS:  $(\nabla_a R)^{ab} \neq 0$

generically

RHS:

$(\nabla_a T)^{ab} = 0$

thought bubble: = formulated from an action

Einstein tried to argue this problem away.

Nevertheless, the equations cannot be upheld.

33.1. **Hilbert.** Hilbert was a specialist for variational principles.

To find the appropriate left hand side of the gravitational field equations, Hibert suggested to start from an action

$S_{\text{Hilbert}}[g] = \int_M \sqrt{-g} R_{ab} g^{ab}$

thought bubble = “simplest action”

aim: varying this w.r.t. metric  $g_{ab}$  will result in some tensor

$G^{ab} = 0$

33.2. **Variation of  $S_{\text{Hilbert}}$ .**

$0 \stackrel{!}{=} \underbrace{\delta}_{gi} S_{\text{Hilbert}}[g] = \int_M [\underbrace{\delta \sqrt{-g} g^{ab} R_{ab}}_1 + \underbrace{\sqrt{-g} \delta g^{ab} R_{ab}}_2 + \underbrace{\sqrt{-g} g^{ab} \delta R_{ab}}_3]$

and  $1 : \delta \sqrt{-g} = \frac{-(\det g) g^{mn} \delta g_{mn}}{2 \sqrt{-g}} = \frac{1}{2} \sqrt{-g} g^{mn} \delta g_{mn}$

thought bubble

$\delta \det(g) = \det(g) g^{mn} \delta g_{mn}$

e.g. from

$\det(g) = \exp \text{tr} \ln g$

ad 2:  $g^{ab} g_{bc} = \delta_c^a$

$\implies (\delta g^{ab}) g_{bc} + g^{ab} (\delta g_{bc}) = 0$

$\implies \delta g^{ab} = -g^{am} g^{bn} \delta g_{mn}$

ad 3:

$\Delta R_{ab} \underbrace{=}_{\text{normal coords at point}} \delta \partial_b \Gamma_{am}^m - \delta \partial_m \Gamma_{ab}^m + \Gamma \Gamma - \Gamma \Gamma =$

$= \partial_b \delta \Gamma_{am}^m - \partial_m \delta \Gamma_{ab}^m =$

$= \nabla_b (\delta \Gamma)_{am}^m - \nabla_m (\delta \Gamma)_{ab}^m$

$\implies \sqrt{-g} g^{ab} \delta R_{ab} = \sqrt{-g}$

“if you formulate the variation properly, you’ll see the variation  $\delta$  commute with  $\partial_b$ ” EY : 20150408 I think one uses the integration at the bounds, integration by parts trick

$\Gamma_{(x)jk}^i - \tilde{\Gamma}_{(x)jk}^i$  are the components of a (1, 2)-tensor.

Notation:  $(\nabla_b A)^i{}_g =: A^i{}_{j;b}$

$\implies \sqrt{-g} g^{ab} \delta R_{ab}$

$\underbrace{=}_{\nabla g=0} \sqrt{-g} (g^{ab} \delta \Gamma_{am}^m)_{;b} - \sqrt{-g} (g^{ab} \delta \Gamma_{ab}^m)_{;m} = \sqrt{-g} A^b{}_{;b} - \sqrt{-g} B^m{}_{;m}$

Question: Why is the difference of coefficients a tensor?

Answer:

$\Gamma_{(y)jk}^i = \frac{\partial y^i}{\partial x^m} \frac{\partial x^m}{\partial y^j} \frac{\partial x^q}{\partial y^k} \Gamma_{(x)~ ,nq}^m + \frac{\partial y^i}{\partial x^m} \frac{\partial^2 x^m}{\partial y^j \partial y^k}$

Collecting terms, one obtains

$0 \stackrel{!}{=} \delta S_{\text{Hilbert}} = \int_M [\frac{1}{2} \sqrt{-g} g^{mn} \delta g_{mn} g^{ab} R_{ab} - \sqrt{-g} g^{am} g^{bn} \delta g_{mn} R_{ab} + \underbrace{(\sqrt{-g} A^a)_{,a}}_{\text{surface}} - \underbrace{(\sqrt{-g} B^b)_{,b}}_{\text{surface term}}]$

$= \int_M \sqrt{-g} \delta \underbrace{g_{mn}}_{\text{arbitrary variation}} [\frac{1}{2} g^{mn} R - R^{mn}] \implies G^{mn} = R^{mn} - \frac{1}{2} g^{mn} R$

Hence Hilbert, from this “mathematical” argument, concluded that one may take

$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G_N T_{ab}$

Einstein equations

$S_{E-H}[g] = \int_M \sqrt{-g} R$

33.3. **3. Solution of the  $\nabla_a T^{ab} = 0$  issue.** One can show ( $\rightarrow$  Tutorials) that the Einstein curvature

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$$

satisfy the so-called contracted differential Bianchi identity

$$(\nabla_a G)^{ab} = 0$$

33.4. **Variants of the field equations.**

(a) a simple rewriting:

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab} = T_{ab}$$

$$G_N = \frac{1}{8\pi}$$

Contract on both sides  $g^{ab}$

$$R_{ab} - \frac{1}{2}g_{ab}R = T_{ab}||g^{ab}$$

$$R - 2R = T := T_{ab}g^{ab} \\ \implies R = -T$$

$$\implies R_{ab} + \frac{1}{2}g_{ab}T = T_{ab}$$

$$\iff R_{ab} = (T_{ab} - \frac{1}{2}Tg_{ab}) =: \hat{T}_{ab}$$

$R_{ab} = \hat{T}_{ab}$

(b)

$$S_{E-H}[g] := \int_M \sqrt{-g}(R + 2\Lambda)$$

thought bubble:  $\Lambda$  cosmological constant

History:

1915:  $\Lambda < 0$  (Einstein) in order to get a non-expanding universe

>1915:  $\Lambda = 0$  Hubble

today  $\Lambda > 0$  to account for an accelerated expansion

$\Lambda \neq 0$  can be interpreted as a contribution

$-\frac{1}{2}\Lambda g$  to the energy-momentum

“dark energy”

Question: surface terms scalar?

Answer: for a careful treatment of the surface terms which we discarded, see, e.g. E. Poisson, “A relativist’s toolkit”

C.U.P. “excellent book”

Question: What is a constant on a manifold?

Answer:  $\int \sqrt{-g}\Lambda = \Lambda \int \sqrt{-g}1$

[back to dark energy]

[Weinberg, QCD, calculated]

idea: 1 could arise as the vacuum energy of the standard model fields

$\Lambda_{\text{calculated}} = 10^{120} \times \Lambda_{\text{obs}}$

“worst prediction of physics”

Tutorials: check that

- Schwarzschild metric (1916)
- FRW metric
- pp-wave metric
- Reisner-Nordstrom

$\implies$  are solutions to Einstein’s equations

in high school

$$m\ddot{x} + m\omega^2 x^2 = 0$$

$$x(t) = \cos(\omega t)$$

ET: [elementary tutorials]

study motion of particles & observers in Schwarzschild S.T.

Satellite: Marcus C. Werner

Gravitational lensing

odd number of pictures Morse theory (EY:20150408 Morse Theory !!!)

Domenico Giulini

Hamiltonian form Canonical Formulations

Key to Quantum Gravity

TUTORIAL 13 SCHWARZSCHILD SPACETIME

EY : 20150408 I’m not sure which tutorial follows which lecture at this point.

The tutorial video is excellent itself. Here, I want to encourage the use of CAS to do calculations. There are many out there. Again, I’m partial to the Sage Manifolds package for Sage Math which are both open-source and based on Python. I’ll use that here.

Exercise 1. Geodesics in a Schwarzschild spacetime

Question Write down the Lagrangian.

Load “Schwarzschild.sage” in Sage Math, which will always be available freely here <https://github.com/ernestyalumni/diffgeo-by-sagemnfd/blob/master/Schwarzschild.sage>:

```
sage: load("Schwarzschild.sage")
4-dimensional manifold 'M'
open subset 'U_sph' of the 4-dimensional manifold 'M'
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
```

and so on.

Look at the code and I had defined the Lagrangian to be

L

. To get out the coefficients of  $L$  of the components of the tangent vectors to the curve, i.e.  $t', r', \theta', \phi'$ , denoted

tp,rp,thp,php

in my .sage file, do the following:

```
sage: L.expr().coefficients(tp)[1][0].factor().full_simplify()
(2*G_N*M_0 - r)/r
sage: L.expr().coefficients(rp)[1][0].factor().full_simplify()
-r/(2*G_N*M_0 - r)
sage: L.expr().coefficients(php)[1][0].factor().full_simplify()
r^2
sage: L.expr().coefficients(thp)[1][0].factor().full_simplify()
r^2*sin(th)^2
```

Question There are 4 Euler-Lagrange equations for this Lagrangian. Derive the one with respect to the function  $t(\lambda)!$ .

```
sage: L.expr().diff(t)
0
```

This confirms that  $\frac{\partial L}{\partial t} = 0$

For  $\frac{d}{d\lambda} \frac{\partial L}{\partial t'}$ , then one needs to consider this particular workaround for Sage Math (computer technicality). One takes derivatives with respect to declared variables (declared with var) and then substitute in functions that are dependent upon  $\lambda$ , and then take the derivative with respect to the parameter  $\lambda$ . This does that:

```
sage: L.expr().diff( thp ).factor().subs( r == gamma1 ).subs( thp == gamma3.diff( tau ) ).subs( th == gamma3 ).diff(tau)\
....: .factor()
2*(2*cos(gamma3(tau))*gamma1(tau)*D[0](gamma3)(tau)^2 + 2*sin(gamma3(tau))*D[0](gamma1)(tau)*D[0](gamma3)(tau)
+ gamma1(tau)*sin(gamma3(tau))*D[0, 0](gamma3)(tau))*gamma1(tau)*sin(gamma3(tau))
```

**Question** Show that the Lie derivative of  $g$  with respect to the vector fields  $K_t := \frac{\partial}{\partial t}$ .

The first line defines the vector field by accessing the frame defined on a chart with spherical coordinates and getting the time vector. The second line is the Lie derivative of  $g$  with respect to this vector field.

```
sage: K_t = espher[0]
sage: g.lie_der(K_t).display() # 0, as desired
0
```

EY : 20150410 My question is this:  $\forall X \in \Gamma(TM)$  i.e.  $X$  is a vector field on  $M$ , or, specifically, a section of the tangent bundle, then does

$$\mathcal{L}_X g = 0$$

instantly mean that  $X$  is a symmetry for  $(M, g)$ ?  $\mathcal{L}_X g$  is interpreted geometrically as how  $g$  changes along the flow generated by  $X$ , and if it equals 0, then  $g$  doesn’t change.

34.

35.

36. CANONICAL FORMULATION OF GR I

Dynamical and Hailtonian formulation of General Relativity.

Purpose

- (1) formulate and solve initial-value problems
- (2) integrate Einstein’s Equations by numerical codes
- (3) characterize degrees of freedom
- (4) characterize isolated systems, associated symmetry groups and conserved quantities, like Energy/Mass, Momenta (linear and angular), Poincaré charges
- (5) starting point for “canonical quantization” program.

How. We will rewrite Einstein’s Eq. in form of a *constrained Hamiltonian system*.



$$\underbrace{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R}_{G_{\mu\nu}} + \underbrace{\Lambda}_{\text{kosm. const.}} g_{\mu\nu} = \underbrace{k}_{\frac{8\pi G}{c^4}} T_{\mu\nu}$$

(− + ++)

$$T^{\mu\nu} = \begin{pmatrix} W & \frac{1}{c} S^m \\ cg^m & \mathbf{t}^{mn} \end{pmatrix}$$

- $W$  = Energy density (1 component)
- $g^m$  = Momentum density, (3 components)
- $S^m$  = Energy current-density (3 components)
- $\mathbf{t}^{mn}$  = Momentum current-density (6 components)

$$T^{\mu\nu} = T^{\nu\mu} \implies S^m = c^2 g^m$$

10 independent komp. (components)

Phys. dim.  $[T^{\mu\nu}] = \frac{J}{m^3}$   
 $[G^{\mu\nu}] = \frac{1}{m^2}$

$$k = \frac{\text{curvature}}{\text{Energy} \cdot \text{density}}$$

$$[k] = \frac{1}{m^2} / \frac{J}{m^3}, \quad {}^2k = \frac{\text{Curvature}}{\text{mass density}} = \left( \frac{1}{1.5 \text{ AU}} \right)^2 / \text{Density of water}$$
$$= \left( \frac{1}{10 \text{ km}} \right)^2 / \text{Nuclear density in core of neutron star} \simeq 5 \cdot 10^{17} \text{ kg}/m^3$$

If “Ein” for Einstein Tensor,  $G_{\mu\nu} = \text{Ein} \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right)$

$$\text{Ein}(v, w) = \frac{1}{4} [\text{Ein}(v + w, v + w) - \text{Ein}(v - w, v - w)]$$

$$\text{Ein}(w, w) = -g(w, w) \sum_{\perp w} \text{Sec}$$

where  $\perp w$  is take the sum over any triple of mutually perp. 2-planes in  $\perp w$

$$\text{Sec}(\text{Span}\{v, w\}) = \frac{\text{Riem}(v, w, v, w)}{[g(v, w)]^2 - g(v, v)g(w, w)}$$

“sectional curvature”

Identity:  $\nabla_\mu G^{\mu\nu} = 0$  (follows from twice-contracted II. Bianchi Identity

$$\sum_{\lambda\mu\nu \text{ cycl}} \nabla_\lambda R_{\alpha\beta\mu\nu} = 0 \text{ )}$$

$$\underbrace{\partial_0 G^{0\nu}}_{\text{contains at most 1st time der.}} + \underbrace{\partial_k G^{k\nu} + \Gamma G + \Gamma G}_{\text{contains at most 2nd. time derivatives}} \equiv 0$$

$\implies$  4 out of 10 Einstein Eq. do not evolve the fields but rather constrain the initial data. The space-space components (6 Eqns.) are the evolution Eqns.

- 10 Einstein Eq. - 4 constraints (underdetermined elliptic type)
- \ - 6 evolution equations (undetermined hyperbolic type)

37.

38.

39.

40. LECTURE 22: BLACK HOLES

Only depends on Lectures 1-15, so does lecture on “Wednesday”  
Schwarzschild solution also vacuum solution (from tutorial EY : oh no, must do tutorial)  
Study the Schwarzschild as a vacuum solution of the Einstein equation:  
 $m = G_N M$  where  $M$  is the “mass”

$$g = \left( 1 - \frac{2m}{r} \right) dt \otimes dt - \frac{1}{1 - \frac{2m}{r}} dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi)$$

in the so-called Schwarzschild coordinates

$$\begin{matrix} t & r & \theta & \varphi \\ (-\infty, \infty) & (0, \infty) & (0, \pi) & (0, 2\pi) \end{matrix}$$

What staring at this metric for a while, two questions naturally pose themselves:

(i) What exactly happens  $r = 2m$ ?

$t$  $r$

$(-\infty, \infty)$  $(0, 2m) \cup (2m, \infty)$

$\theta$  $\varphi$

$(0, \pi)$  $(0, 2\pi)$

(ii) Is there anything (in the real world) beyond  $t \rightarrow -\infty$ ?

$t \rightarrow +\infty$

idea: Map of Linz, blown up  
Insight into these two issues is afforded by stopping to stare.  
Look at *geodesic* of  $g$ , instead.

40.1. **Radial null geodesics.** null -  $g(v_\gamma, v_\gamma) = 0$   
Consider null geodesic in “Schd”

$$S[\gamma] = \int d\lambda \left[ \left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) \right]$$

with  $[\dots] = 0$   
and one has, in particular, the  $t$ -eqn. of motion:

$$\left( \left(1 - \frac{2m}{r}\right) \dot{t} \right)' = 0$$

$\implies$

$\left(1 - \frac{2m}{r}\right) \dot{t} = k$

= const.

Consider radial null geodesics  
 $\theta \stackrel{!}{=} \text{const.}$   $\varphi = \text{const.}$   
From  $\square$  and  $\square$

$$\implies \dot{r}^2 = k^2 \leftrightarrow \dot{r} = \pm k$$

$$\implies r(\lambda) = \pm k \cdot \lambda$$

Hence, we may consider

Case A:  $\oplus$

$$\tilde{t}(r) := t(\pm k \lambda)$$

$$\frac{d\tilde{t}}{dr} = \frac{\dot{\tilde{t}}}{\dot{r}} = \frac{k}{\left(1 - \frac{2m}{r}\right) k} = \frac{r}{r - 2m}$$

$$\implies \tilde{t}_+(r) = r + 2m \ln |r - 2m|$$

(**outgoing** null geodesics)  
Case b.  $\pm$  (Circle around  $-$ , consider  $-$ ):

$$\tilde{t}_-(r) = -r - 2m \ln |r - 2m|$$

(**ingoing** null geodesics)  
Picture

40.2. **Eddington-Finkelstein.** Brilliantly simple idea:  
change (on the domain of the Schwarzschild coordinates) to different coordinates, s.t.  
in those new coordinates,  
*ingoing* null geodesics appear as straight lines, of slope  $-1$   
This is achieved by

$$\bar{t}(t, r, \theta, \varphi) := t + 2m \ln |r - 2m|$$

Recall: ingoing null geodesic has

$$\tilde{t}(r) = -(r + 2m \ln |r - 2m|) \quad (\textit{Schdcoords})$$

$$\iff \bar{t} - 2m \ln |r - 2m| = -r - 2m \ln |r - 2m| + \text{const.}$$
$$\therefore \bar{t} = -r + \text{const.}$$

(Picture)  
*outgoing* null geodesics

$$\bar{t} = r + 4m \ln |r - 2m| + \text{const.}$$

Consider the new chart  $(V, g)$  while  $(U, x)$  was the Schd chart.

$$\bigcup_{\text{Schd}} \bigcup \{ \text{horizon} \} = V$$

“chart image of the horizon”  
Now calculate the *Schd metric*  $g$  w.r.t. Eddington-Finkelstein coords.

$$\bar{t}(t, r, \theta, \varphi) = t + 2m \ln |r - 2m|$$

$$\bar{r}(t, r, \theta, \varphi) = r$$

$$\bar{\theta}(t, r, \theta, \varphi) = \theta$$

$$\bar{\varphi}(t, r, \theta, \varphi) = \varphi$$

EY : 20150422 I would suggest that after seeing this, one would calculate the metric by your favorite CAS. I like the Sage  
Manifolds package for Sage Math.

[Schwarzschild\\_BH.sage on github](#)  
[Schwarzschild\\_BH.sage on Patreon](#)  
[Schwarzschild\\_BH.sage on Google Drive](#)

```
sage: load('Schwarzschild_BH.sage')
4-dimensional manifold 'M'
expr = expr.simplify_radical()
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
Launched png viewer for Graphics object consisting of 4 graphics primitives
```

Then calculate the Schwarzschild metric  $g$  but in Eddington-Finkelstein coordinates. Keep in mind to calculate the set of  
coordinates that uses  $\bar{t}$ , not  $\tilde{t}$ :

```
sage: gI.display()
gI = (2*m - r)/r dt*dt - r/(2*m - r) dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: gI.display( X_EF_I_null.frame())
gI = (2*m - r)/r dtbar*dtbar + 2*m/r dtbar*dr + 2*m/r dr*dtbar + (2*m + r)/r dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
```



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