Lecture 2

1 Dilute Absorbing Medium

- In order to obtain light-matter interaction for transparent media (such as fiber), we can dope this material with absorbing material (such as Erbium).
- $\chi = \chi_{host} + \chi_{transition}$
- We can ignore χ''_{host} since we are operating at a frequency away from its resonance frequency.

$$\begin{aligned}\epsilon_r &= 1 + \chi_{host} + \chi_{transition} \\ &= \epsilon_{r_{\text{host}}} \left(1 + \frac{\chi_{transition}}{\epsilon_{r_{\text{host}}}}\right)\end{aligned}$$

Square root to get the refractive index and n_b is the refractive index of the host material:

$$n_{eff} = n_b \left(1 + \frac{\chi_{transition}}{n_b^2}\right)^{1/2}$$

Assume $\chi_{transition} << n_b^2$ so that we apply Taylor expansion:

$$\begin{aligned}n_{eff} &= n_b \left(1 + \frac{\chi_{transition}}{2n_b^2}\right) \\ &= n_b + \frac{\chi_{transition}}{2n_b} \\ &= n_b + \frac{\chi'_{transition}}{2n_b} + j\frac{\chi''_{transition}}{2n_b}\end{aligned}$$

For a propagating wave:

$$e^{-jn_{eff}k_0z} = e^{-j\left(n_b + \frac{\chi'_{transition}}{2n_b}\right)k_0z} e^{\frac{\chi''_{transition}}{2n_b}k_0z}$$

Remarks:

- The first term represents the phase change and the second term represents the amplitude change.
- $e^{\frac{k_0\chi''z}{2n_b}} = e^{-\frac{\alpha}{2}z}$ (amplitude change)
- The field's amplitude decays by $\frac{\alpha}{2}$, and the power decays by α.

Using the derived expression for χ'':

$$\alpha = -\frac{k_0\chi''}{n_b} = \frac{N_a q^2}{m\eta\omega\epsilon_0 n_b} \frac{k_0}{1 + \left(\frac{2(\omega-\omega_0)}{\eta}\right)^2}$$

Multiply by $\frac{\eta^2}{\eta^2}$:

$$-\frac{k_0\chi''}{n_b} = \frac{N_a q^2}{m\eta\omega\epsilon_0 n_b}\frac{k_0\eta^2}{\eta^2 + 4(\omega-\omega_0)^2}$$

From $\omega = 2\pi\nu$ and $k_0 = \frac{\omega}{c_0}$:

$$\begin{aligned}\alpha &= \frac{N_a q^2}{m\eta\cancel{\omega}\epsilon_0 n_b}\frac{\cancel{\omega}}{c_0}\frac{\eta^2}{\eta^2 + 4(2\pi(\nu-\nu_0))^2}\\ &= \frac{N_a q^2}{m\eta\epsilon_0 n_b c_0}\frac{\eta^2}{\eta^2 + 4(4\pi^2)(\nu-\nu_0)^2}\end{aligned}$$

$4(4\pi^2) = (4\pi)^2$, so:

$$\begin{aligned}\alpha &= \frac{N_a q^2}{m\eta\epsilon_0 n_b c_0}\frac{\left(\frac{\eta}{4\pi}\right)^2}{\left(\frac{\eta}{4\pi}\right)^2 + (\nu-\nu_0)^2}\\ &= \frac{N_a q^2}{m\cancel{\eta}\epsilon_0 n_b c_0}\frac{\eta^{\cancel{2}}}{4(2\pi)(2\pi)}\frac{1}{\left(\frac{\eta}{4\pi}\right)^2 + (\nu-\nu_0)^2}\\ &= \frac{N_a q^2}{m\epsilon_0 n_b c_0}\frac{\frac{1}{2\pi}\frac{\eta}{2\pi}}{4}\frac{1}{\left(\frac{\eta}{4\pi}\right)^2 + (\nu-\nu_0)^2}\end{aligned}$$

Let $\Delta\nu = \frac{1}{2\pi\tau}$, so $\eta = \frac{1}{\tau} = 2\pi\Delta\nu$, so $\Delta\nu = \frac{\eta}{2\pi}$:

$$\alpha = \frac{N_a q^2}{4m\epsilon_0 n_b c_0}\frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu-\nu_0)^2}$$

Given that $\tau_r = \frac{6\pi\epsilon_0 m c_0^3}{q^2\omega_0^2 n_b}$, multiply by $\frac{\tau_r}{\tau_r}$:

$$\begin{aligned}\alpha &= \frac{1}{\tau_r}\frac{N_a \cancel{q^2}}{4\cancel{m\epsilon_0}n_b\cancel{c_0}}\frac{6\pi\cancel{\epsilon_0 m}c_0^2}{\cancel{q^2}\omega_0^2 n_b}\frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu-\nu_0)^2}\\ &= \frac{3\pi N_a}{2\tau_r}\left(\frac{c_0}{2\pi\nu_0 n_b}\right)^2\frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu-\nu_0)^2}\end{aligned}$$

Using $c = \nu\lambda$:

$$\begin{aligned}\alpha &= \frac{3\pi N_a}{2(4\pi^2)\tau_r}\lambda^2\frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu-\nu_0)^2}\\ &= \frac{3N_a}{8\pi\tau_r}\lambda^2\frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu-\nu_0)^2}\end{aligned}$$

Notes:

- λ is the wavelength inside the material ($\lambda = \frac{\lambda_0}{n}$)
- $f_1(\nu) = \frac{3N_a}{8\pi\tau_r}\lambda^2$ and $f_2(\nu) = \frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2+(\nu-\nu_0)^2}$
- From this derivation:

$$\begin{aligned}\nu &= \frac{c}{\lambda}\\ \frac{d\nu}{d\lambda} &= -\frac{c}{\lambda^2}\\ \Delta\nu &= -\frac{c}{\lambda^2}\Delta\lambda\\ \Delta\lambda &= -\frac{\lambda^2}{c}\Delta\nu\end{aligned}$$

Since λ^2 is in 10^{-12} range, and $\frac{1}{c}$ is in 10^{-8} range, so any change in frequency ($\Delta\nu$) will be multiplied by 10^{-20}. We can conclude that $f_1(\nu)$ is a very slowly varying function of ν and can be considered as a constant.

- $f_2(\nu)$ has a Lorentzian shape and is called the line shape function

$$g_{\nu_0}(\nu) = \frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu - \nu_0)^2}$$

  - Maximum at $\nu = \nu_0$

$$g_{\nu_0}(\nu_0) = \frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu_0 - \nu_0)^2} = \frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2} = \frac{2}{\pi\Delta\nu}$$

  - FWHM is at $\nu = \nu_0 \pm \frac{\Delta\nu}{2}$

$$g_{\nu_0}(\nu_0 \pm \frac{\Delta\nu}{2}) = \frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\frac{\Delta\nu}{2})^2} = \frac{\frac{\Delta\nu}{2\pi}}{2(\frac{\Delta\nu}{2})^2} = \frac{1}{\pi\Delta\nu}$$

- Note that the line shape function is normalized:

$$\int_{-\infty}^{\infty} g_{\nu_0}(\nu) d\nu = 1$$

- Classically, α is always positive meaning that only attenuation is possible (no amplification).

2 Absorption Cross Section

$$\alpha = \frac{3N_a\lambda^2}{8\pi\tau_r} g_{\nu_0}(\nu) = N_a\sigma(\nu_0)$$

where $\sigma(\nu_0) = \frac{3\lambda^2}{8\pi\tau_r} g_{\nu_0}(\nu)$ is the absorption cross section.

$$\begin{aligned} I &= I_0 e^{-\alpha z} = I_0 e^{-N_a\sigma z} \\ \frac{dI}{dz} &= -N_a\sigma I_0 e^{-N_a\sigma z} = -N_a\sigma I \\ \frac{-dI}{I} &= N_a\sigma dz = \frac{\#\text{ atoms}}{A\cancel{dz}}\sigma\cancel{dz} \end{aligned}$$

where:

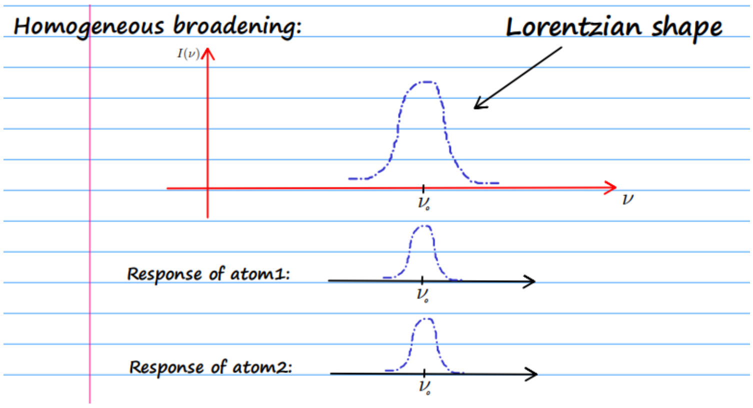
- $\frac{-dI}{I}$ is the fractional change in intensity (decrease in intensity)
- N_a is the atomic density (number of atoms per unit volume)
- σ is the absorption cross section (represents an opaque area in the material)

3 Broadening of Spectral Lines

3.1 Homogeneous Broadening

All atoms have the same resonance frequency and Lorentzian shape. The total summation of all the Lorentzian functions is a Lorentzian function.

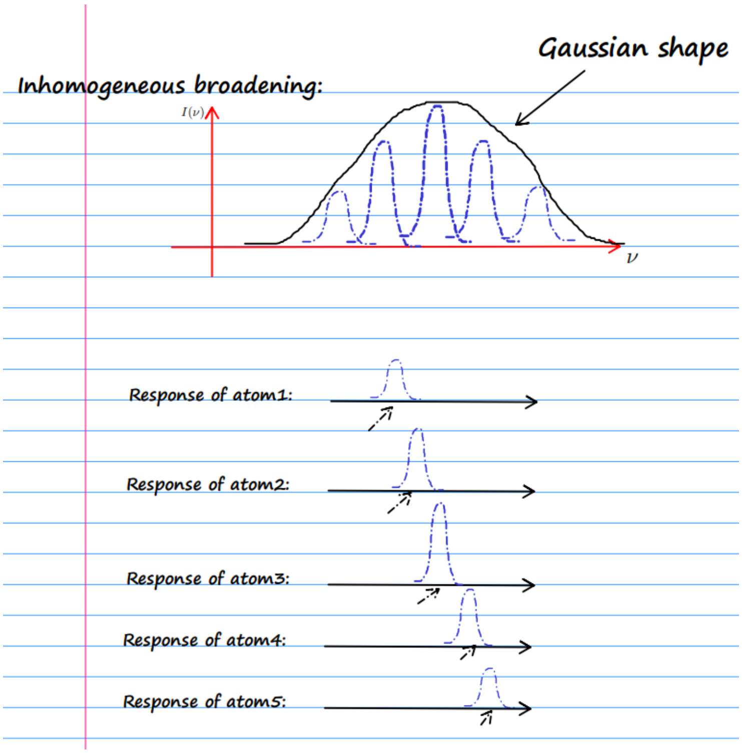
- Natural / Lifetime Broadening
- Collision Broadening



3.2 Inhomogeneous Broadening

Each group of atoms has a different resonance frequency and line shape function. The total summation of all the Lorentzian functions is a Gaussian function.

- Doppler Broadening



3.2.1 Doppler Broadening

- Due to the randomness of the atoms' velocities and directions of motion, each atom will experience the light at a different frequency.
- This usually happens in gases where the atoms are free to move.
- Examples:
  - Atom is moving towards the light

$$\nu_{atom} = \nu_{source}\left(1 + \frac{v}{c}\right)$$

  - Atom is moving away from the light

$$\nu_{atom} = \nu_{source}\left(1 - \frac{v}{c}\right)$$

– Atom is moving in the speed of light

$$\nu_{atom} = \nu_{source}\left(1 - \frac{c}{c}\right) = 0$$

We want to get the source frequency ($\nu_{source} = \nu_0'$) equivalent to the atom's resonance frequency ($\nu_{atom} = \nu_0$)

$$\nu_0' = \frac{\nu_0}{(1 - \frac{v}{c})} \approx \nu_0\left(1 + \frac{v}{c}\right)$$

Modify the line shape function:

$$\begin{aligned} g_{\nu_0}(\nu) &= \frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu - \nu_0')^2} \\ &= \frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu - \nu_0(1 + \frac{v}{c}))^2} \end{aligned}$$

There is multiple line shape functions depending on the velocities of the atoms, so we need to average over all the velocities:

$$\overline{g_{\nu_0}} = \int_{-\infty}^{\infty} g(\nu) f(v) dv$$

where $f(v)$ is the probability density function (PDF) of the velocities. From statistical thermodynamics, the PDF of the velocities is a Gaussian function with zero mean (no external force applied) and variance σ_v^2:

$$f(v) = \frac{1}{\sqrt{2\pi}\sigma_v} e^{-\frac{v^2}{2\sigma_v^2}}$$

The variance is given by:

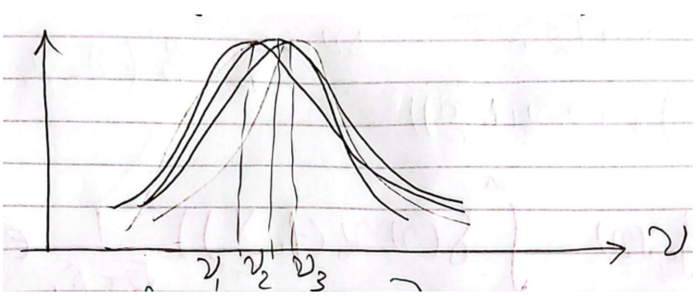
$$\sigma_v^2 = E[v^2] - \cancel{E[v]^2} = \frac{k_B T}{m}$$

So:

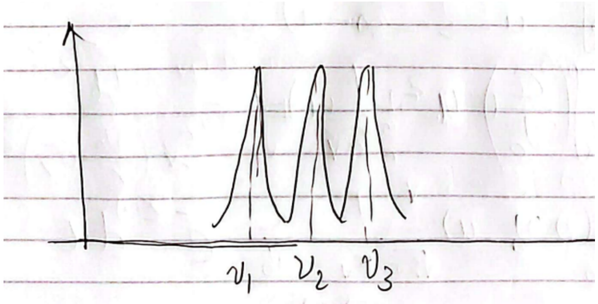
$$\overline{g_{\nu_0}(\nu)} = \int_{-\infty}^{\infty} \frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu - \nu_0(1 + \frac{v}{c}))^2} \frac{1}{\sqrt{2\pi}\sigma_v} e^{-\frac{v^2}{2\sigma_v^2}} dv$$

This is a very difficult integral to solve, so we can solve at 2 limits:

- **$\sigma_v \to 0$:** This means that the variance in atoms' velocities is very small, so the shift in resonance frequency (ν_0) is very small, so the overall line shape function is a Lorentzian function (Homogeneous Broadening).



- **$\Delta\nu \to 0$:** This means that the line width is very small that for each velocity the line shape function can be considered as a delta function (Inhomogeneous Broadening).



For $\Delta\nu \to 0$:

$$\frac{\frac{\Delta\nu}{2\pi}}{(\frac{\Delta\nu}{2})^2 + (\nu - \nu_0(1+\frac{v}{c}))^2} = \delta\left(\nu - \nu_0\left(1 + \frac{v}{c}\right)\right)$$

So:

$$\overline{g_{\nu_0}(\nu)} = \int_{-\infty}^{\infty} \delta\left(\nu - \nu_0\left(1 + \frac{v}{c}\right)\right) \frac{1}{\sqrt{2\pi}\sigma_v} e^{-\frac{v^2}{2\sigma_v^2}} dv$$

To utilitize $\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$, let:

$$u = \nu_0 \frac{v}{c} \Rightarrow v = \frac{c}{\nu_0} u \Rightarrow dv = \frac{c}{\nu_0} du$$
$$v \to -\infty \Rightarrow u \to -\infty$$
$$v \to \infty \Rightarrow u \to \infty$$

Delta function will be:

$$\delta\left(\nu - \nu_0\left(1 + \frac{v}{c}\right)\right) = \delta(\nu - \nu_0 - u)$$

The Gaussian function will be:

$$\frac{1}{\sqrt{2\pi}\sigma_v} e^{-\frac{v^2}{2\sigma_v^2}} dv = \frac{1}{\sqrt{2\pi}\sigma_v} e^{-\left(\frac{c}{\nu_0\sigma_v}\right)^2 \frac{u^2}{2}} \frac{c}{\nu_0} du$$

Let $\sigma_d = \frac{\nu_0 \sigma_v}{c}$, where σ_d is the standard deviation of the Doppler Broadening (depends on temperature):

$$\overline{g_{\nu_0}(\nu)} = \int_{-\infty}^{\infty} \delta(\nu - \nu_0 - u) \frac{1}{\sqrt{2\pi}\sigma_d} e^{\frac{u^2}{2\sigma_d^2}} du$$

Since the delta function is even, we can multiply by -1 inside the bracket:

$$\overline{g_{\nu_0}(\nu)} = \int_{-\infty}^{\infty} \delta(u - (\nu - \nu_0)) \frac{1}{\sqrt{2\pi}\sigma_d} e^{-\frac{u^2}{2\sigma_d^2}} du$$
$$= \frac{1}{\sqrt{2\pi}\sigma_d} e^{-\frac{(\nu-\nu_0)^2}{2\sigma_d^2}}$$

The final value is the Gaussian envelope of the Doppler Broadening.