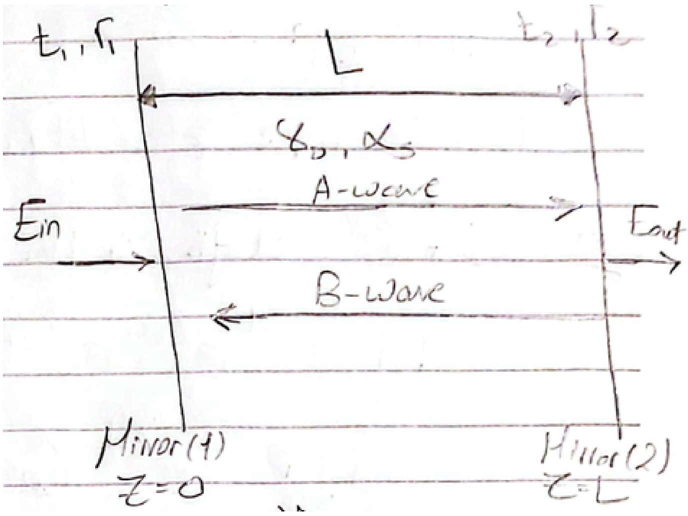
# Lecture 7

## 1 Fabry Perot Amplifier

It is similar to FP LASER, but we adjust the mirrors' reflectivity to prevent oscillations, so we can consider it an amplifier with a positive feedback loop but $A\beta < 1$.




$$A(L) = A(0)e^{-j\beta L}e^{-\frac{\alpha_s}{2}L}e^{\frac{\gamma_o}{2}L} \tag{1}$$

$$B(0) = B(L)e^{-j\beta L}e^{-\frac{\alpha_s}{2}L}e^{\frac{\gamma_o}{2}L} \tag{2}$$

$$A(0) = t_1 E_{in} + r_1 B(0) \tag{3}$$

$$B(L) = r_2 A(L) \tag{4}$$

From (3):

$$E_{in} = \frac{A(0) - r_1 B(0)}{t_1} \tag{5}$$

From (1):

$$E_{out} = t_2 A(L) = t_2 A(0)e^{-j\beta L}e^{-\frac{\alpha_s}{2}L}e^{\frac{\gamma_o}{2}L} \tag{6}$$

Substitute (4) in (2):

$$B(0) = r_2 A(0)e^{-2j\beta L}e^{-\alpha_s L}e^{\gamma_o L} \tag{7}$$

From (5) and (6):

$$\frac{E_{out}}{E_{in}} = \frac{t_1 t_2 A(0)e^{-j\beta L}e^{-\frac{\alpha_s}{2}L}e^{\frac{\gamma_o}{2}L}}{A(0) - r_1 B(0)}$$

From (7):

$$\frac{E_{out}}{E_{in}} = \frac{t_1 t_2 e^{-j\beta L}e^{-\frac{\alpha_s}{2}L}e^{\frac{\gamma_o}{2}L}}{1 - r_1 r_2 e^{-2j\beta L}e^{-\alpha_s L}e^{\gamma_o L}}$$

$$
\begin{aligned}
\text{Gain} &= \frac{P_{out}}{P_{in}} = \frac{|E_{out}|^2}{|E_{in}|^2} = \frac{E_{out}E_{out}^*}{E_{in}E_{in}^*} \\
&= \frac{t_1 t_2 e^{-j\beta L} e^{-\frac{\alpha_s}{2}L} e^{\frac{\gamma_o}{2}L}}{1 - r_1 r_2 e^{-2j\beta L} e^{-\alpha_s L} e^{\gamma_o L}} \frac{t_1^* t_2^* e^{+j\beta L} e^{-\frac{\alpha_s}{2}L} e^{\frac{\gamma_o}{2}L}}{1 - r_1^* r_2^* e^{+2j\beta L} e^{-\alpha_s L} e^{\gamma_o L}} \\
&= \frac{|t_1|^2 |t_2|^2 e^{-\alpha_s L} e^{\gamma_o L}}{1 - 2\Re\{r_1 r_2 e^{-\alpha_s L} e^{\gamma_o L} e^{-2j\beta L}\} + |r_1 r_2|^2 e^{-2\alpha_s L} e^{2\gamma_o L}}
\end{aligned}
$$

Note that $z + z^* = a + jb + a - jb = 2a = 2\Re\{z\}$

$$
\begin{aligned}
&\Re\{r_1 r_2 e^{-\alpha_s L} e^{\gamma_o L} e^{-2j\beta L}\} \\
&= r_1 r_2 e^{-\alpha_s L} e^{\gamma_o L} \cos(2\beta L) \\
&= r_1 r_2 e^{-\alpha_s L} e^{\gamma_o L} - 2 r_1 r_2 e^{-\alpha_s L} e^{\gamma_o L} \sin^2(\beta L)
\end{aligned}
$$

Simplifing the denumerator:

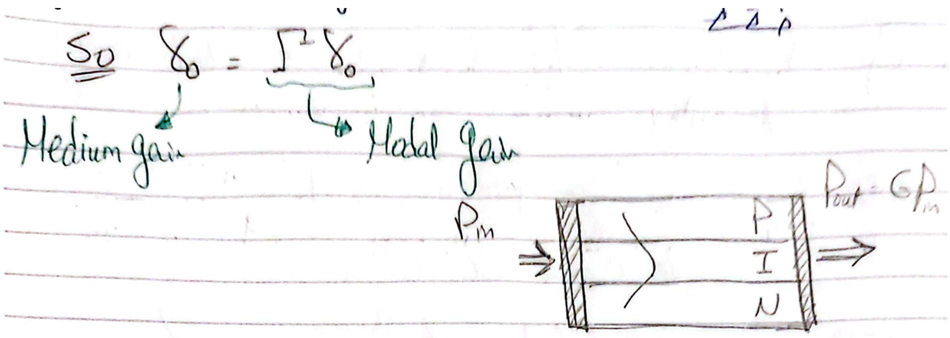
$$
\begin{aligned}
&1 - 2\Re\{r_1 r_2 e^{-\alpha_s L} e^{\gamma_o L} e^{-2j\beta L}\} + |r_1 r_2|^2 e^{-2\alpha_s L} e^{2\gamma_o L} \\
&= 1 - 2 r_1 r_2 e^{-\alpha_s L} e^{\gamma_o L} + |r_1|^2 |r_2|^2 e^{-2\alpha_s L} e^{2\gamma_o L} + 4 r_1 r_2 e^{-\alpha_s L} e^{\gamma_o L} \sin^2(\beta L)
\end{aligned}
$$

$r_1 = \sqrt{R_1}$, $r_2 = \sqrt{R_2}$ and let $G_o = e^{-\alpha_s L} e^{\gamma_o L}$:

$$
\begin{aligned}
\text{Gain} &= \frac{(1 - R_1)(1 - R_2) G_o}{1 - 2\sqrt{R_1 R_2} G_o + R_1 R_2 G_o^2 + 4\sqrt{R_1 R_2} G_o \sin^2(\beta L)} \\
&= \frac{(1 - R_1)(1 - R_2) G_o}{(1 - \sqrt{R_1 R_2} G_o)^2 + 4\sqrt{R_1 R_2} G_o \sin^2(\beta L)}
\end{aligned}
$$

**Notes:**

- If $R_1 = R_2 = 0$ (AR coating) then Gain $= G_o =$ TWA gain
- If $\gamma_o = 0 \Rightarrow$ passive cavity, but if $\gamma_o \neq 0 \Rightarrow$ active cavity
- Note that FPA has a high gain because the waves stay in the gain medium from the mirrors reflectivity unlike the TWA where the waves leave the gain medium after the first pass.
- For a certain gain FPA length is less than TWA length.
- $G_o$ is the single path gain and more accurately is given by: $G_o = e^{-\alpha_s L} e^{\Gamma \gamma_o L}$ where $\Gamma$ is the confinement factor because the gain is only in the core (I) in the PIN structure.




Maximum gain is when $\sin^2(\beta L) = 0$:

$$
\beta L = q\pi \rightarrow \frac{2\pi \nu n}{c_o} L = q\pi \Rightarrow \nu_{q_{max}} = \frac{q c_o}{2nL} = q\nu_F
$$

where $\nu_{q_{max}}$ are the frequencies that give maximum gain and $\nu_F$ is the free spectral range ($\Delta f$ between 2 longitudinal modes in the cavity).

$$
Gain_{max} = \frac{(1 - R_1)(1 - R_2) G_o}{(1 - \sqrt{R_1 R_2} G_o)^2}
$$

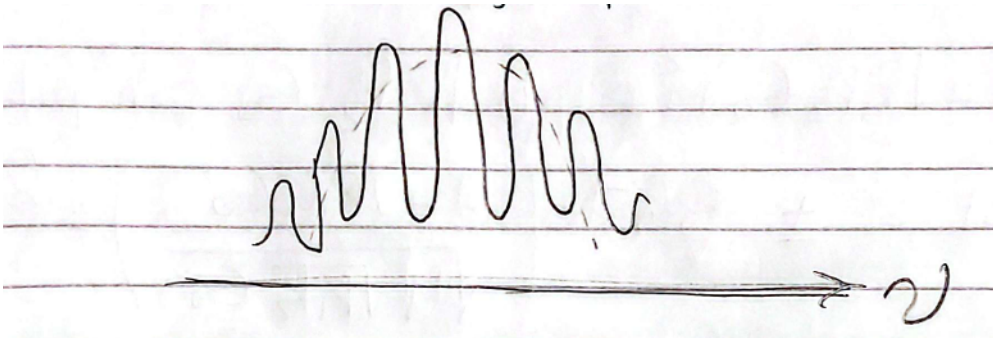
Minimum gain is when $\sin^2(\beta L) = 1 \to \sin(\beta L) = \pm 1$:

$$\beta L = (2q+1)\frac{\pi}{2} \to \frac{2\pi\nu n}{c_o}L = (2q+1)\frac{\pi}{2} \Rightarrow \nu_{q_{min}} = \frac{(2q+1)\,c_o}{4nL}$$

where $\nu_{q_{min}}$ are the frequencies that give minimum gain.

$$\begin{aligned} Gain_{min} &= \frac{(1-R_1)(1-R_2)G_o}{(1+\sqrt{R_1R_2}G_o)^2 + 4\sqrt{R_1R_2}G_o} \\ &= \frac{(1-R_1)(1-R_2)G_o}{1+R_1R_2G_o^2 - 2\sqrt{R_1R_2}G_o + 4\sqrt{R_1R_2}G_o} \\ &= \frac{(1-R_1)(1-R_2)G_o}{1+R_1R_2G_o^2 + 2\sqrt{R_1R_2}G_o} \\ &= \frac{(1-R_1)(1-R_2)G_o}{(1+\sqrt{R_1R_2}G_o)^2} \end{aligned}$$

Note that $G_o$ is frequency dependent because $\gamma_o$ is frequency dependent, so the gain is frequency dependent.




We will define a new parameter called the ripples $\rho = \frac{G_{max}}{G_{min}}$. This parameter puts a spec on the AR coating reflectivity when designing a TWA.

$$\rho_{dB} = 10\log\left(\frac{G_{max}}{G_{min}}\right) = 20\log\left(\frac{1+\sqrt{R_1R_2G_o}}{1-\sqrt{R_1R_2G_o}}\right)$$

To get FWHM:

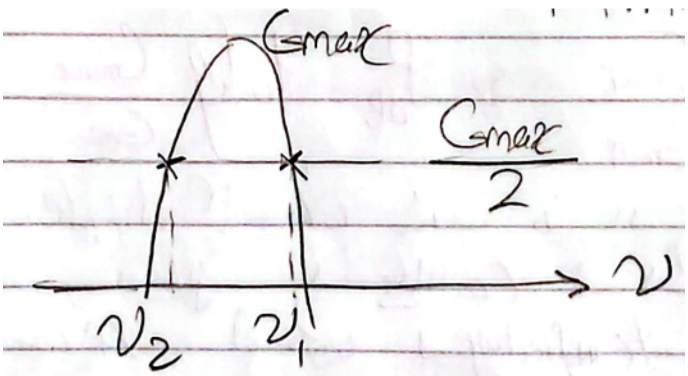
$$G\left(\nu + \frac{\Delta\nu}{2}\right) = \frac{G_{max}}{2} = \frac{(1-R_1)(1-R_2)G_o}{2(1-\sqrt{R_1R_2}G_o)^2} = \frac{(1-R_1)(1-R_2)G_o}{(1-\sqrt{R_1R_2}G_o)^2 + 4\sqrt{R_1R_2}G_o\sin^2(\beta L)}$$

So:

$$\begin{aligned} (1-\sqrt{R_1R_2}G_o)^2 &= 4\sqrt{R_1R_2}G_o\sin^2(\beta L) \\ \beta L = \pm\sin^{-1}\left(\frac{1-\sqrt{R_1R_2}G_o}{\sqrt{4\sqrt{R_1R_2}G_o}}\right) &= \frac{2\pi\nu_{1,2}n}{c_o}L \end{aligned}$$

So:

$$\Delta\nu_{FWHM} = \nu_1 - \nu_2 = 2\frac{c_o}{2\pi nL}\sin^{-1}\left(\frac{1-\sqrt{R_1R_2}G_o}{\sqrt{4\sqrt{R_1R_2}G_o}}\right)$$



$\frac{c_o}{\pi n L} = \frac{c_o}{2nL}\frac{2}{\pi} = \nu_F \frac{2}{\pi}$, so:

$$\Delta\nu_{FWHM} = \frac{2}{\pi}\nu_F \sin^{-1}\left(\frac{1-\sqrt{R_1R_2}G_o}{\sqrt{4\sqrt{R_1R_2}G_o}}\right)$$

$\sin^{-1}(x) \approx x$ if $x << 1$, so if $G_o >> 1$ and $R = \sqrt{R_1R_2}$:

$$\Delta\nu_{FWHM} \approx \frac{2\nu_F}{\pi}\frac{1-\sqrt{R_1R_2}G_o}{\sqrt{4\sqrt{R_1R_2}G_o}} = \frac{\nu_F}{\pi}\frac{1-RG_o}{\sqrt{R}G_o}$$

We will define a new parameter Finesse (F), which is similar to the Quality factor ($Q = \frac{f}{\Delta f}$). Q is used when having a single resonnance frequency, but F is used when having multiple resonnance frequencies. Higher F means better selectivity.

$$F = \frac{\nu_F}{\Delta\nu} = \frac{\pi\sqrt{R}G_o}{1-RG_o}$$

Finesse is usually defined for a passive cavity (such as an ideal filter), where there is no loss or gain ($\gamma_o = \alpha_s = 0$, so $G_o = 1$):

$$F = \frac{\pi\sqrt{R}}{1-R}$$

In case of a practical filter, $\gamma_o = 0$ and $\alpha_s \neq 0$, so $G_o = e^{-\alpha_s L}$:

$$F = \frac{\pi\sqrt{R}e^{-\alpha_s L}}{1-Re^{-\alpha_s L}}$$

Another equation for practical filter Finesse is:

$$\alpha_r = \alpha_s + \frac{1}{2L}ln\left(\frac{1}{R_1R_2}\right) = \alpha_s + \frac{1}{L}ln\left(\frac{1}{\sqrt{R_1R_2}}\right) = \alpha_s + \frac{1}{L}ln\left(\frac{1}{R}\right)$$
$$e^{-\alpha_r L} = e^{-\alpha_s L}e^{-ln\left(\frac{1}{R}\right)} = e^{-\alpha_s L}e^{ln(R)} = Re^{-\alpha_s L}$$
$$F = \frac{\pi\sqrt{R}e^{-\frac{\alpha_s}{2}L}}{1-Re^{-\alpha_s L}} = \frac{\pi e^{-\frac{\alpha_r}{2}L}}{1-e^{-\alpha_r L}}$$

If $\alpha_r L << 1$ then $1 - e^{-\alpha_r L} \approx \alpha_r L$, so:

$$F = \frac{\pi(1-e^{-\frac{\alpha_r}{2}L})}{\alpha_r L} = \frac{\pi}{\alpha_r L}$$

FPA is FP LASER if gain tends to infinity, so $G_{max} \to \infty$:

$$1 - \sqrt{R_1R_2}G_o = 0 \Rightarrow G_o = \frac{1}{\sqrt{R_1R_2}}$$

Using $G_o = e^{-\alpha_s L}e^{\gamma_o L}$:

$$e^{-\alpha_s L}e^{\gamma_o L} = \frac{1}{\sqrt{R_1R_2}} \Rightarrow (\gamma_o - \alpha_s)L = \frac{1}{2}ln\left(\frac{1}{R_1R_2}\right)$$
$$\Rightarrow \gamma_o = \alpha_s + \frac{1}{2L}ln\left(\frac{1}{R_1R_2}\right)$$

Photon lifetime is the time a photon stays in the cavity before leaving it, which controls the bandwidth of the cavity. The photon lifetime is given by:

$$\tau_{ph} = \frac{1}{\alpha_r v_g} \approx \frac{n}{\alpha_r c_o}$$

Proof:

$$I(z) = I(0)e^{-\alpha_r z}$$
$$\frac{dI(z)}{dz} = -\alpha_r I(0)e^{-\alpha_r z} = -\alpha_r I(z)$$

So:

$$\frac{dI}{dt} = \frac{dI}{dz}\frac{dz}{dt} = -\alpha_r I v_g = -\alpha_r I \frac{c_o}{n}$$

So:

$$I(t) = I(0)e^{-\alpha_r \frac{c_o}{n} t} \Rightarrow \tau_{ph} = \frac{1}{\alpha_r c}$$

If losses decrease, the photon lifetime increases meaning it will stay longer in the cavity.