Gaussian Distribution (Multi-variable), ~~De~~

$$\mathcal{N}(x|\mu,\Sigma) = \frac{1}{(2\pi|\Sigma|)^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\}$$

log-likelihood

$$\ln p(x|x,\Sigma) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)$$

take derivative and let it equal to zero.

$$\frac{\partial \ln p(x|\mu,\Sigma)}{\partial \mu} = 0 \quad \Rightarrow \quad \frac{\partial -\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}{\partial \mu} \Rightarrow \bar{\mu} = \frac{1}{N}\sum_{n=1}^{N} x_n$$

$$\frac{\partial \ln p(x|\mu,\Sigma)}{\partial \Sigma} = 0 \quad \Rightarrow \quad \Sigma = \frac{1}{N}\sum_{n=1}^{N}(x_n-\bar{\mu})(x_n-\bar{\mu})^T$$

and $p(x) = \sum_{k=1}^{K} \pi_k \mathcal{N}(x|\mu_k,\Sigma_k)$

for the log-likelihood:

$$\ln p(X|\mu,\Sigma,\pi) = \sum_{n=1}^{N}\ln p(x_n) = \sum_{n=1}^{N}\ln\left\{\sum_{k=1}^{K}\pi_k \mathcal{N}(x_n|\mu_k,\Sigma_k)\right\}$$

for $\gamma_k = p(k|x) = \frac{p(k)p(x|k)}{p(x)}$ (Bayenes Rule)

$$\Rightarrow \gamma_k = \frac{\pi_k \mathcal{N}(x|\mu_k,\Sigma_k)}{\sum_{j=1}^{K}\pi_j \mathcal{N}(x|\mu_j,\Sigma_j)}$$

After deriviation, we can obtain a E-M method for the GMM models.

EM methods for GMM

1. Initialization

initialize μ, Σ, π

2. E-step:

$$\gamma_j(x) = \frac{\pi N(x|\mu, \Sigma)}{\sum_{j=1}^{K} \pi_j N(x|\mu_j, \Sigma_j)}$$

3. M-step:

$$\mu_j = \frac{\sum_{n=1}^{N} \gamma_j(x_n) x_n}{\sum_{n=1}^{N} \gamma_j(x_n)}$$

$$\Sigma_j = \frac{\sum_{n=1}^{N} \gamma_j(x_n)(x_n - \mu_j)(x_n - \mu_j)^T}{\sum_{n=1}^{N} \gamma_j(x_n)}$$

$$\pi_j = \frac{1}{N} \sum_{n=1}^{N} \gamma_j(x_n)$$

4. log-likelihood:

$$\ln p(x|\mu, \Sigma, \pi) = \sum_{n=1}^{N} \ln\left(\sum_{k=1}^{K} \pi_k N(x_n|\mu_k, \Sigma_k)\right)$$

5. check convegence, if not, repeat step 2 3 4