The Teridentity and Peircean Algebraic Logic

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Abstract. A main source of inspiration for the work on Conceptual Graphs by John Sowa and on Contextual Logic by Rudolf Wille has been the Philosophy Charles S. Peirce and his logic system of Existential Graphs invented at the end of the 19th century. While Peirce has described the system in much detail, there is no formal definition which suits the requirements of contemporary mathematics.

In his book *A Peircean Reduction Thesis: The Foundations of topological Logic*, Robert Burch has presented the Peircean Algebraic Logic (PAL) which aims to reconstruct in an algebraic precise manner Peirce's logic system.

Using a restriction on the allowed constructions, he is able to prove the Peircean Reduction Thesis, that in PAL all relations can be constructed from ternary relations, but not from unary and binary relations alone. This is a mathematical version of Peirce's central claim that the category of thirdness cannot be decomposed into the categories of firstness and secondness.

Removing Burch's restriction from PAL makes the system very similar to the system of Existential Graphs, but the proof of the Reduction Thesis becomes extremely complicated. In this paper, we prove that the teridentity relation is – as also elaborated by Burch – irreducible, but we prove this without the additional restriction on PAL. This leads to a proof of the Reduction Thesis in PAL without the restriction.

Introduction

The influence of Peirce's philosophy on the development of the theory of conceptual structures is visible in many areas. Both conceptual graphs (see [Sow84], [Sow92], [Sow00]) and the developments in contextual logic (see [Arn01], [Wil00], [Wil00b], [DaK05]) are influenced by his ideas in general and his system of existential graphs in particular.

Philosphical ideas and Peirce's work on formalizing logic converge on the Reduction Thesis. "The triad is the lowest form of relative from which all others can be derived." (MS 482 from [PR67]). This expresses both his philosophical believe that the categories of firstness, secondness and thirdness suffice and no category of fourthness etc. is needed. Also it is to be understood that all relatives (these correspond to relations in nowadays mathematical terminology) can be generated from triads (ternary relations) but not from unary and binary relations alone. Peirce was conviced that at least on the mathematical level this thesis can be proven. According to Herzberger in [Her81] Peirce mentioned he found a proof, but no corresponding publication has been found.

In his article [Her81], Herzberger summarizes Peirce's understanding on the thesis and provides a first approach for an algebraic proof. In [Bur91], Burch gives a more extended and elaborated framework. He shows that his framework, the Peircean Algebraic Logic is able to represent the same relations as the existential graphs. However, to prove the Reduction Thesis, he imposes a restriction on the constructions in PAL. The juxtaposition of graphs (this corresponds to the product Def. 1(1)) is only allowed as last or before last operation. Removing this restriction make PAL a little simpler (our version of PAL needs only one join-operator as opposed to two in [Bur91]) and probably more alike to the system of existential graphs. The proof of the reduction thesis in contrast becomes exceedingly difficult.

Many attempts have failed for non-obvious reasons. In fact, often the parts that seemed to be obvious turned out to be wrong afterwards. For this reason we present the complete mathematical proof of the difficult part of the reduction thesis. Due to space restrictions, we will not show the part that any relation can be constructed (in PAL) from ternary relations. For this, we refer to [Her81], [Bur91] or [HCP04].

Organization of this Paper

In the following section we present the various tools needed to describe the relations that can be generated from unary and binary relations. Each subsection will be commented about the purpose of the following definitions. Then the representation theorem for the relations generatable without ternary relations will be presented. The paper concludes with a short final section consisting of only the reduction thesis.

Mathematical Notations

To avoid disambiguities, we define some abbreviations used in this paper. The set of all m-ary relations over some set A is denoted by $\mathrm{Rel}^{(m)}(A) := \{\varrho \subseteq A^m\}$ (and relations will be denoted by greek letters). Please note, that also empty relations have arities, that is for $n \neq m$ the empty relations $\emptyset^n \subseteq A^n$ and $\emptyset^m \subseteq A^m$ are considered to be different. Often we will talk about the places of a relation. If m is the arity of a relation, we will write $\underline{m}$ instead of $\{1, \ldots, m\}$.

A tuple $(a_1, \ldots, a_n)$ wil be shortened to the notation $\underline{a}$ if the arity of the relation the tuple belongs to can be derived from the context. If not otherwise noted, A denotes an arbitrary set.

1 Peircean Algebraic Logic (PAL)

The operations of the *Peircean Algebraic Logic (PAL)* are closely related to the existential graphs that Peirce developed in the late 1890s. They have been identified Burch in [Bur91] as the fundamental operations in Peirce's understanding of the manipulation of relations. For a detailed discussion of these operations we refer to [Bur91], for this paper we adopt Burch's operations.

Definition 1. Let A be an arbitrary set, let $\varrho \in \mathrm{Rel}(A)$ be an m-ary and $\sigma \in \mathrm{Rel}(A)$ an n-ary relation. We define the following operations:

(PAL1) The *product* of relations:

$$\varrho \times \sigma := \{(a_1, \ldots, a_m, b_1, \ldots, b_n) \in A^{m+n} \mid \underline{a} \in \varrho, \underline{b} \in \sigma\},$$

(PAL2) for $1 \leq i < j \leq m$ the *join* of i and j of a relation is defined by

$$\delta^{i,j}(\varrho) :=$$
$$\{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m) \in A^{m-2} \mid \exists\, \underline{a} \in \varrho, a_i = a_j\}$$

(PAL3) $\neg\varrho := \{\underline{a} \in A^m \mid \underline{a} \notin \varrho\}$ (the *complement* of ϱ),
(PAL4) if α is a permutation on $\underline{m}$, then
$\pi_\alpha(\varrho) := \{(a_1, \dots, a_m) \mid (a_{\alpha(1)}, \dots, a_{\alpha(m)}) \in \varrho\}$.

Remark 1. Let ϱ be an m-ary relation, let $1 \leq i < j \leq m$ and let α be the permutation on $\underline{m}$ with

$$\alpha(k) := \begin{cases} k & \text{if } 1 \leq k < i \\ m-1 & \text{if } k = i \\ k-1 & \text{if } i < k < j \\ m & \text{if } k = j \\ m-2 & \text{if } j < k \leq m \end{cases}$$

such that π_α moves the i-th and j-th place to the $m-1$-th and m-th place, preserving the order of the other columns. Then we have $\delta^{i,j}(\varrho) = \delta^{m-1,m}(\pi_\alpha(\varrho))$. For this reason we will often only investigate the specific case $\delta^{m-1,m}(\varrho)$ as the general case can be derived together with the permutation operation.

Definition 2. Let A be an arbitrary set and $\Sigma \subseteq \mathrm{Rel}(A)$ a set of relations over A. Let

$$\Sigma_0 := \Sigma \cup \{\mathsf{id}_3\}, \tag{1}$$

where $\mathsf{id}_3 := \{(a, a, a) \mid a \in A\}$, and let

$$\begin{aligned} \Sigma_{i+1} := \quad & \Sigma_i \\ & \cup \{\varrho \times \sigma \mid \varrho, \sigma \in \Sigma_i\} \\ & \cup \{\delta^{i,j}(\varrho) \mid \varrho \in \Sigma_i \text{ and } 1 \leq i < j \leq \mathrm{ar}(\varrho)\} \\ & \cup \{\neg\varrho \mid \varrho \in \Sigma_i\} \\ & \cup \{\pi_\alpha(\varrho) \mid \varrho \in \Sigma_i \text{ and } \alpha \text{ is permutation of } \underline{\mathrm{ar}(\varrho)}\} \end{aligned}$$

We set $\langle\Sigma\rangle_{\mathsf{PAL}} := \bigcup_{i\in\mathbb{N}} \Sigma_i$ and call this the set of the *relations generated from Σ by PAL*.

Analogously we define $\langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ as the set of *relations generated from Σ without Teridentity* using the same construction, but replacing Eq. (1) by $\Sigma_0 := \Sigma$.

Analogous to the above construction we can syntactically define the terms describing a relation constructed with PAL.

Definition 3. Let Σ be a set and $\mathrm{ar} : \Sigma \to \mathbb{N}$ be a mapping. Every element of Σ is called an *atomic term*. If we use full PAL (as opposed to PAL without teridentity) then id_3 is also a term with $\mathrm{ar}(\mathsf{id}_3) = 3$. If s and t are terms, then $(t \times s)$ is a term with $\mathrm{ar}(t \times s) = \mathrm{ar}(t) + \mathrm{ar}(s)$. If $1 \leq i < j \leq \mathrm{ar}(t)$, then $\delta^{i,j}(t)$ is a term with $\mathrm{ar}(\delta^{i,j}(t)) = \mathrm{ar}(t) - 2$. Also $\neg t$ is a term with $\mathrm{ar}(\neg t) = \mathrm{ar}(t)$. If α is a permutation on $\overline{\mathrm{ar}(t)}$, then $\pi_\alpha(t)$ is a term with $\mathrm{ar}(\pi_\alpha(t)) = \mathrm{ar}(t)$. Analogous to the set of generated relations, the set of all terms is denoted by $\langle \Sigma \rangle_{\mathsf{PAL}}$ if all operations from PAL are allows, and $\langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}}$ if teridentity is not included.

If $\Sigma \subseteq \mathrm{Rel}(A)$ and ar corresponds to the usual arity-mapping, we define the *standard interpretation* $[\![\cdot]\!] : \langle \Sigma \rangle_{\mathsf{PAL}} \to \mathrm{Rel}(A)$ by $[\![t]\!] := \sigma$ for $t = \sigma \in \Sigma$, $[\![\mathsf{id}_3]\!] := \{(a, a, a) \mid a \in A\}$, $[\![(t \times s)]\!] := [\![t]\!] \times [\![s]\!]$, $[\![\neg t]\!] := \neg [\![t]\!]$ and $[\![\pi_\alpha(t)]\!] := \pi_\alpha([\![t]\!])$.

From the last two definitions one easily derives the following corollary:

Corollary 1. *Let* $\Sigma \subseteq \mathrm{Rel}(A)$ *and let* ar *correspond to the usual arity-mapping. Then for terms* $t \in \langle \Sigma \rangle_{\mathsf{PAL}}$ *and* $s \in \langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}}$ *one has* $\mathrm{ar}(t) = \mathrm{ar}([\![t]\!])$ *and* $\mathrm{ar}(s) = \mathrm{ar}([\![s]\!])$*, and also* $[\![t]\!] \in \langle \Sigma \rangle_{\mathsf{PAL}}$ *and* $[\![s]\!] \in \langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}}$*. Conversely for every relation* $\varrho \in \langle \Sigma \rangle_{\mathsf{PAL}}$ *and* $\sigma \in \langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}}$ *exist terms* $t \in \langle \Sigma \rangle_{\mathsf{PAL}}$ *and* $s \in \langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}}$ *with* $[\![t]\!] = \varrho$ *and* $[\![s]\!] = \sigma$*.*

Remark 2. Different terms may be interpreted as the same relation. For instance, for $\varrho, \sigma, \tau \in \Sigma \subseteq \mathrm{Rel}(A)$ the following identity of interpretations $[\![((\varrho \times \sigma) \times \tau)]\!] = [\![(\varrho \times (\sigma \times \tau))]\!]$ is easy to see, but formally the terms $((\varrho \times \sigma) \times \tau)$ and $(\varrho \times (\sigma \times \tau))$ are different.

1.1 Connected Places

Associated with PAL is a graphical notation, as presented in [Bur91] and [HCP04]. In the graphical representation it is obvious how places (called hooks in [Bur91]) are connected with each other. As we need the notion of connectedness but will not introduce the graphical representation, we define connectedness formally following the constructions by PAL-terms.

Definition 4. Let $\Sigma \subseteq \mathrm{Rel}(A)$ and $t \in \langle \Sigma \rangle_{\mathsf{PAL}}$ be a term. The places $k, l \in \underline{\mathrm{ar}(t)}$ are said to be t-connected if

(i) $t \in \Sigma \cup \{\mathsf{id}_3\}$ or
(ii) $t = u \times v$ and k, l are u-connected or if $(k - m), (l - m)$ are v-connected or
(iii) $t = \delta^{m-1,m}(u)$ and k, l are u-connected or
(iii') $t = \delta^{m-1,m}(u)$ and $k, m-1$ and l, m are u-connected or
(iii") $t = \delta^{m-1,m}(u)$ and $l, m-1$ and k, m are u-connected or
(iv) $t = \neg u$ and k, l are u-connected or
(v) $t = \pi_\alpha(u)$ and $\alpha^{-1}(k), \alpha^{-1}(l)$ are u-connected.

A set $P \subseteq \underline{\mathrm{ar}(t)}$ is said to be *t-connected* if the elements of P are pairwise t-connected.

For the reduction thesis the relations generated by PAL without teridentity are very important. The following lemma is a first indication on a special property of these relations.

Lemma 1. *Let $\Sigma := \mathrm{Rel}^{(1)}(A) \cup \mathrm{Rel}^{(2)}(A)$, let t be a $\langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}}$-term and let $X \subseteq \underline{\mathrm{ar}(t)}$ be t-connected. Then $|X| \leq 2$.*

Proof. For atomic terms the assertion trivially holds. The case $t = \mathsf{id}_3$ is not possible because only PAL without teridentity is considered. If $t = (u \times v)$, it is easy to verify that two places can only be t-connected if they are both less or equal to $\mathrm{ar}(u)$ or both strictly greater. This means that either $\max(X) \leq \mathrm{ar}(u)$ or $\min(X) > \mathrm{ar}(u)$, and consequently X is u-connected or $\{x - \mathrm{ar}(u) \mid x \in X\}$ is v-connected. By the induction hypothesis one concludes $|X| = |\{x - \mathrm{ar}(u) \mid x \in X\}| \leq 2$. Now let us consider the case $t = \delta^{m-1,m}(u)$ where $m := \mathrm{ar}(u)$. If there are $x, y \in X$ with $x \neq y$ such that x and y are u-connected, one conludes from the induction hypothesis that x and y cannot be u-connected to $m-1$ or m, therefore the cases Def. 4(iii') and (iii") cannot apply for x and y and there can be no third element t-connected to x or y. If all $x, y \in X$ with $x \neq y$ are not u-connected then they must be (to be t-connected) be u-connected to $m-1$ or m. Therefore in this case $X \subseteq (\{k \in \underline{m} \mid k, m-1\ u\text{-connected}\} \cup \{k \in \underline{m} \mid k, m\ u\text{-connected}\}) \setminus \{m-1, m\}$ and therefore $|X| \leq 2 + 2 - 2 = 2$. For $t = \neg u$ the set X is t-connected $\iff$ X is

u-connected, therefore the assertion holds. For $t = \pi_\alpha(u)$ the assertion can easily be seen because α is a bijection and one can therefore apply the inverse mapping: X is t-connected $\iff \{\alpha^{-1}(x) \mid x \in X\}$ is u-connected. $\square$

1.2 Essential Places

Later we will introduce representations of relations as unions of intersections of special relations. Formally, these special relations have to have the same arity as the relation represented. However, they are essentially unary or binary relations. To make formally clear what "essentially" means, we introduce the notion of "essential places".

Definition 5. Let $\varrho \in \mathrm{Rel}(A)$ be an m-ary relation and $i \in \underline{\mathrm{ar}(\varrho)}$ a place of the relation. A place i is called a *fictitious place of ϱ* if

$$\underline{a} \in \varrho, b \in A \implies (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_m) \in \varrho$$

A non-fictitious place is called *essential place of ϱ*. The set of essential places of ϱ is denoted by $E(\varrho)$.

Essential places are the places of the relation, where one cannot arbitrarily exchange elements in the tuple.

Lemma 2. *For any m-ary relation $\varrho \in \mathrm{Rel}^{(m)}(A)$ holds*

$$\varrho \in \{\emptyset^m, A^m\} \iff E(\varrho) = \emptyset.$$

Proof. $\implies$ is easy to see. For $\impliedby$ let $\varrho \in \mathrm{Rel}^{(m)}(A) \setminus \{\emptyset^m\}$. Let $\underline{a} \in \varrho$ and $\underline{b} \in A^m$. Every $i \in \underline{m}$ is a fictitious place of ϱ, therefore a_i can be replaced by b_i and one gets $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_m) \in \varrho$. Consecutively applying this exchange for all places shows $\underline{b} \in \varrho$ and consequently $\varrho = A^m$. $\square$

The following lemmata are useful to show in the representation Theorem 1 that the special relations of the representations are essentially at most binary.

Lemma 3. *Let A be a set with at least two elements, let ϱ be an m-ary relation and let σ be another relation over A. Then*

(i) $E(\varrho) \subseteq \underline{m}$,
(ii) $E(\mathsf{id}_3) = \underline{3}$,
(iii) $E(\varrho \times \sigma) = E(\varrho) \cup \{m+i \mid i \in E(\sigma)\}$,
(iv) $E(\delta^{m-1,m}(\varrho)) \subseteq E(\varrho) \setminus \{m-1, m\}$
(v) $E(\neg\varrho) = E(\varrho)$
(vi) $E(\pi_\alpha(\varrho)) = \{\alpha^{-1}(i) \mid i \in E(\varrho)\}$.

Proof. (i) and (ii) are trivial. For (iii) let $\varrho \in \mathrm{Rel}^{(m)}(A)$ and $\sigma \in \mathrm{Rel}^{(n)}(A)$. For $1 \le i \le m$ and $i \notin E(\varrho)$ one has the following equivalencies: $(a_1, \dots, a_m, b_1, \dots, b_n) \in \varrho \times \sigma, c \in A \iff \underline{a} \in \varrho, \underline{b} \in \sigma, c \in A \iff a_i \in A, (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_m) \in \varrho, \underline{b} \in \sigma \iff a_i \in A, (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_m, b_1, \dots, b_n) \in \varrho \times \sigma$ and similarly for $1 \le i \le n$ $(a_1, \dots, a_m, b_1, \dots, b_n) \in \varrho \times \sigma, c \in A \iff \underline{a} \in \varrho, \underline{b} \in \sigma, c \in A \iff b_i \in A, \underline{a} \in \varrho, (b_1, \dots, b_{i-1}, c, b_{i+1}, \dots, b_n) \in \sigma \iff b_i \in A, (a_1, \dots, a_m, b_1, \dots, b_{i-1}, c, b_{i+1}, \dots, b_n) \in \varrho \times \sigma$. Therefore $\neg E(\varrho \times \sigma) = (\underline{m} \setminus E(\varrho)) \cup \{m+i \mid i \in \underline{n} \setminus E(\sigma)\} \iff E(\varrho \times \sigma) = E(\varrho) \cup \{m+i \mid i \in E(\sigma)\}$. (iv) Let $i \in (\underline{m-2} \setminus E(\varrho))$, $c \in A$ and $(a_1, \dots, a_{m-2}) \in \delta^{m-1,m}(\varrho)$, then there exists $b \in A$ with $(a_1, \dots, a_{m-2}, b, b) \in \varrho \implies (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_{m-2}, b, b) \in \varrho \implies (a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_{m-2}) \in \delta^{m-1,m}(\varrho)$. Therefore $\underline{m-2} \setminus E(\varrho) \subseteq \neg E(\delta^{m-1,m}(\varrho)) \iff E(\delta^{m-1,m}(\varrho)) \subseteq E(\varrho) \setminus \{m-1, m\}$. (v) If $\varrho = \emptyset^m$ then this follows from Lem. 2. Otherwise let $\underline{a} \in A^m \setminus \varrho$ and $i \in \underline{m} \setminus E(\varrho)$. Let us assume that $i \in E(\neg\varrho)$. Then there is some $c \in A$ such that $(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_m) \in \varrho$. But because $i \notin E(\varrho)$ and $a_i \in A$ this implies $\underline{a} \in \varrho$, contradiction. Therefore $\underline{m} \setminus E(\varrho) = \underline{m} \setminus E(\neg\varrho) \iff E(\neg\varrho) = E(\varrho)$. (vi) is easy to verify. □

Lemma 4. *Let $S \subseteq \mathrm{Rel}^{(m)}(A)$ for some $m \in \mathbb{N}$. Then*

$$E\left(\bigcap S\right) \subseteq \bigcup_{\sigma \in S} E(\sigma).$$

Proof. Let $i \in \underline{m} \setminus (\bigcup_{\sigma \in S} E(\sigma))$, $\underline{a} \in \bigcap S$ and $c \in A$. Then for all $\sigma \in S$ holds $i \notin E(\sigma)$ and therefore $(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_m) \in \sigma$ and consequently $(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_m) \in \bigcap S$, therefore $\underline{m} \setminus (\bigcup_{\sigma \in S} E(\sigma)) \subseteq \neg E(\bigcap S) \iff E(\bigcap S) \subseteq \bigcup_{\sigma \in S} E(\sigma)$. □

Lemma 5. *Let $S \subseteq \mathrm{Rel}^{(m)}(A)$ for some $m \in \mathbb{N}$. Then*

$$\delta^{m-1,m}(\bigcap S) = \left(\bigcap_{\substack{\sigma \in S \\ \{m-1,m\} \cap E(\sigma) = \emptyset}} \delta^{i,j}(\sigma) \right) \cap \delta^{m-1,m} \left(\bigcap_{\substack{\sigma \in S \\ \{m-1,m\} \cap E(\sigma) \neq \emptyset}} \sigma \right).$$

Proof. $\subseteq$: Let $\underline{a} \in \delta^{i,j}(\bigcap S)$. Then there exists some $c \in A$ such that $\underline{b} := (a_1 \ldots, a_{m-2}, c, c) \in \bigcap S$, therefore for all $\sigma \in S$ also $\underline{b} \in \sigma \implies \underline{a} \in \delta^{m-1,m}(\sigma)$ and also $\underline{a} \in \delta^{m-1,m}\left(\bigcap\{\sigma \in S \mid \{m-1,m\} \cap E(\sigma) \neq \emptyset\}\right)$.

$\supseteq$: Let $\underline{a} \in \left(\bigcap\{\delta^{i,j}(\sigma) \mid \sigma \in S, \{m-1,m\} \cap E(\sigma) = \emptyset\}\right) \cap \delta^{m-1,m}\left(\bigcap\{\sigma \in S \mid \{m-1,m\} \cap E(\sigma) \neq \emptyset\}\right)$. Then there exists some $c \in A$ such that $(a_1, \ldots, a_{m-2}, c, c) \in \bigcap\{\sigma \in S \mid \{m-1,m\} \cap E(\sigma) \neq \emptyset\}$, that is $(a_1, \ldots, a_{m-2}, c, c) \in \sigma$ for all σ with $\{m-1,m\} \cap E(\sigma) \neq \emptyset$. For every $\sigma \in S$ with $\{m-1,m\} \cap E(\sigma) = \emptyset$ there is some $d_\sigma \in A$ such that $(a_1, \ldots, a_{m-2}, d_\sigma, d_\sigma) \in \sigma$. Because of $m-1, m \notin E(\sigma)$ one can replace the d_σ by c and gets $(a_1, \ldots, a_{m-2}, c, c) \in \sigma$. As this tuple is in each $\sigma \in S$ one concludes $\underline{a} \in \delta^{i,j}(\bigcap S)$. $\square$

1.3 Core and Comparable

The final proof was so difficult because the join between relations does not conserve inclusions. They do in many cases but not in all. To cover the exceptions we found the notions of "core" and "comparable".

Definition 6. Let $\varrho \in \mathrm{Rel}^{(2)}(A)$ be a binary relation. Then the *core of ϱ* is defined as the set

$$K(\varrho) := \{c \in A \mid (a,b) \in \varrho \implies (a,c) \in \varrho\}.$$

Corollary 2.

$$\neg K(\neg\varrho) = \{c \in A \mid \exists a, b \in A : (a,b) \notin \varrho \text{ and } (a,c) \in \varrho\}.$$

Definition 7. Let $\sigma, \tau \in \mathrm{Rel}^{(1)}(A) \cup \mathrm{Rel}^{(2)}(A)$. We define

$$\sigma \preceq \tau :\iff \begin{cases} \sigma \subseteq \tau & \text{if } \mathrm{ar}(\sigma) = \mathrm{ar}(\tau) = 1 \\ \sigma = \tau & \text{if } \mathrm{ar}(\sigma) = \mathrm{ar}(\tau) = 2 \\ \sigma \subseteq K(\tau) & \text{if } \mathrm{ar}(\sigma) = 1,\ \mathrm{ar}(\tau) = 2 \\ \neg K(\neg\sigma) \subseteq \tau & \text{if } \mathrm{ar}(\sigma) = 2,\ \mathrm{ar}(\tau) = 1 \end{cases}$$

To simplify notation we set $\varrho^{-1} := \varrho$ for any unary relation $\varrho \in \mathrm{Rel}^{(1)}(A)$. We say the relations σ and τ are *comparable* if

$$\sigma \preceq \tau \;\text{ or }\; \tau \preceq \sigma$$

and we say σ and τ are *inverted comparable* if σ^{-1} and τ^{-1} are comparable.

The following lemma shows that the comparability is conserved under some PAL-operations. The proof also shows the points where the simple inclusion posed problems.

Lemma 6. *Let $\varrho_1, \sigma_1 \in \mathrm{Rel}^{(1)}(A)$ and let $\varrho_2 \in \mathrm{Rel}^{(2)}(A)$ such that ϱ_1 and ϱ_2 are comparable and ϱ_1 and σ_1 are comparable.*

- (i) *$\neg\varrho_1$ and $\neg\varrho_2$ are comparable.*
- (ii) *ϱ_1 and $\delta^{1,2}(\tau \times \varrho_2)$ are comparable for any $\tau \in \mathrm{Rel}^{(1)}(A)$.*
- (ii′) *ϱ_1 and $\delta^{2,3}(\tau \times \varrho_2)$ are comparable for any $\tau \in \mathrm{Rel}^{(2)}(A)$.*

Proof. (i) follows trivially from Def. 7. (ii) Let $\sigma := \delta^{1,2}(\tau \times \varrho_2) = \{c \in A \mid \exists a \in \tau : (a,c) \in \varrho_2\}$. If $\sigma \in \{\emptyset^1, A^1\}$ the assertion holds. Otherwise, we have two possibilities for ϱ_1 and ϱ_2 to be comparable. If $\varrho_1 \preceq \varrho_2$, then for $t \in \varrho_1 \subseteq K(\varrho_2)$ we have that from $s \in \sigma \neq \emptyset^1$ follows that there is some $a \in A$ such that $a \in \tau, (a,s) \in \varrho_2 \xRightarrow{\text{Def. 6}} a \in \tau, (a,t) \in \varrho_2 \implies t \in \sigma$, that is $\varrho_1 \subseteq \sigma \implies \varrho_1 \preceq \delta^{1,2}(\tau \times \sigma)$. If $\varrho_2 \preceq \varrho_1$ let $s \in \sigma$, then there exists some $a \in \tau$ with $(a,s) \in \varrho_2$. From $\sigma \neq A^1$ we deduce that there is some $b \in A$ with $(a,b) \notin \varrho_2$. Therefore by Cor. 2 $s \in \neg K(\neg\varrho) \subseteq \varrho_1$, therefore $\sigma \subseteq \varrho_1 \implies \delta^{1,2}(\tau \times \varrho_2) \preceq \varrho_1$.

The proof for (ii’) is similar. Let $\sigma := \delta^{2,3}(\tau \times \varrho_2)$. If for σ holds $\forall a \in A : ((\{a\} \times A \subseteq \sigma)$ or $(\{a\} \times A \subseteq \neg\sigma))$, then $K(\sigma) = A^1 \implies \varrho_1 \preceq \delta^{2,3}(\tau \times \varrho_2)$. Otherwise we consider the two cases. If $\varrho_1 \preceq \varrho_2$ then let $t \in \varrho_1$ and $(a,b) \in \sigma$. Then there exists $c \in A$ such that $(a,c) \in \tau, (c,b) \in \varrho_2 \xRightarrow{t \in \varrho_1 \subseteq K(\varrho_2)} (a,c) \in \tau, (c,t) \in \varrho_2 \implies (a,t) \in \sigma$ therefore $t \in K(\sigma)$ and consequently $\varrho_1 \subseteq K(\sigma) \implies \varrho_1 \preceq \delta^{2,3}(\tau \times \varrho_2)$. If $\varrho_2 \preceq \varrho_1$ then there are for every $c \in \neg K(\neg\sigma)$ elements $a, b \in A$ with $(a,b) \in \sigma$ and $(a,c) \notin \sigma$. From $(a,b) \in \sigma$ one deduces the existence of $d \in A$ with $(a,d) \in \tau$ and $(d,a) \in \varrho_2$. Let us assume $(d,c) \in \varrho_2$ then one has together with $(a,d) \in \tau$ that $(a,c) \in \sigma$, contradiction. Therefore $(d,c) \notin \varrho_2$ and together with $(d,a) \in \varrho_2$ one gets by Cor. 2 that $c \in \neg K(\neg\varrho)$, that is $\neg K(\neg\sigma) \subseteq \neg K(\neg\varrho) \subseteq \varrho_1 \implies \delta^{2,3}(\tau \times \varrho_2) \preceq \varrho_1$. $\square$

The following lemma was the second key to the final proof. In a certain sense it is counter-intuitive. While we attempt to prove that places that are not connected are (in a certan way) independent, this lemma allows us to represent a connected graph (in the graphical representation the element c connects the four relations ϱ_1, ϱ_2, σ_1 and σ_2) by the intersection of four other graphs (each element $c_1, \dots, c_4$ is connecting only two operations). Of course, this is not possible in general, but only for comparable relations.

Lemma 7 (crux). *Let $\varrho_1, \sigma_1 \in \mathrm{Rel}^{(1)}(A)$ and $\varrho_2, \sigma_2 \in \mathrm{Rel}^{(2)}(A)$ such that ϱ_1 and ϱ_2 are comparable and σ_1 and σ_2 are comparable. Then for any $a, b \in A$*

$$\begin{aligned}
\exists c \in A : c &\in \varrho_1 \cap \sigma_1, && \text{Ⓐ}\\
(a,c) &\in \varrho_2, && \text{Ⓑ}\\
(b,c) &\in \sigma_2 && \text{Ⓒ}\\
&\Longleftrightarrow \\
\exists c_1, c_2, c_3, c_4 \in A : c_1 &\in \varrho_1 \cap \sigma_1, && \text{ⓐ}\\
(a,c_2) &\in \varrho_2,\ c_2 \in \sigma_1, && \text{ⓑ}\\
c_3 &\in \varrho_1,\ (b,c_3) \in \sigma_2, && \text{ⓒ}\\
(a,c_4) &\in \varrho_2,\ (b,c_4) \in \sigma_2 && \text{ⓓ}
\end{aligned}$$

Proof. " $\Longrightarrow$ " is obvious. For "$\Leftarrow$" we have to consider several cases.
I) $\varrho_1 \preceq \varrho_2$ $(*)$ and $\sigma_1 \preceq \sigma_2$ $(**)$: Then we can set $c := c_1$. Condition Ⓐ is then the same as ⓐ, in particular it follows that $c_1 \in \varrho_1 \underset{(*)}{\subseteq} K(\varrho_2)$. Together with $(a,c_2) \in \varrho_2$ from ⓑ we get by the definition of the core that $(a,c_1) \in \varrho_2$, that is Ⓑ. Analogously we conclude from ⓐ that $c_1 \in \sigma_1$ and by the equations $(**)$ and ⓑ that $(b,c_1) \in \sigma_2$, that is Ⓒ.
II) $\varrho_1 \preceq \varrho_2$ $(*)$ and $\sigma_2 \preceq \sigma_1$ $(**)$. There are two subcases: i) $\forall c' \in A : (b,c') \in \sigma_2$ $(\square)$ and ii) $\exists c' \in A : (b,c') \notin \sigma_2$ $(\square\square)$. For II.i), we chose $c = c_1$, as in I) we conclude, that c_1 fulfills Ⓐ and Ⓑ. From

($\Box$) we obtain Ⓒ. For II.ii), we can set $c = c_3$. We have by ⓒ that $c_3 \in \varrho_1$. From ($\Box\Box$) and $(b, c_3) \in \sigma_2$ (from ⓒ) we deduce by Cor. 2 that $c_3 \in \neg K(\neg\sigma_2) \underset{(**)}{\subseteq} \sigma_1$, and therefore $c_3 \in \varrho_1 \cap \sigma_1$, that is Ⓐ.
III) The case $\varrho_2 \preceq \varrho_1$ and $\sigma_1 \preceq \sigma_2$ is handled analogously to II). If $\forall c' \in A : (a, c') \in \varrho_2$, we can set $c = c_1$, if $\exists c' \in A : (a, c') \notin \varrho_2$, we chose $c = c_2$.
IV) Finally, we consider $\varrho_2 \preceq \varrho_1$ $(*)$ and $\sigma_2 \preceq \sigma_1$ $(**)$. Now we have four subcases:
IV.i) $\forall c' \in A : (a, c') \in \varrho_2$ $(\triangle)$ and $\forall d' \in A : (b, d') \in \sigma_2$ $(\triangledown)$. For $c = c_1$ we get Ⓐ from ⓐ, Ⓑ from $(\triangle)$, and Ⓒ from $(\triangledown)$.
IV.ii) $\exists c' \in A : (a, c') \notin \varrho_2$ $(\triangle)$ and $\forall d' \in A : (b, d') \in \sigma_2$ $(\triangledown)$. We show that $c = c_2$ is a possible choice. With $(\triangle)$ and $(a, c_2) \in \varrho_2$ we obtain by Cor. 2 that $c_2 \in \neg K(\neg\varrho_2) \underset{(*)}{\subseteq} \varrho_1$. Also from ⓑ we know $c_2 \in \sigma_1$ and therefore Ⓐ. Condition Ⓑ follows directly from ⓑ, while Ⓒ follows from $(\triangledown)$.
IV.iii) $\forall c' \in A : (a, c') \in \varrho_2$ and $\exists d' \in A : (b, d') \notin \sigma_2$. This case is analogous to IV.ii), we can set $c = c_3$.
IV.iv) $\exists c' \in A : (a, c') \notin \varrho_2$ $(\triangle)$ and $\exists d' \in A : (b, d') \notin \sigma_2$ $(\triangledown)$. From $(\triangle)$ and $(a, c_4) \in \varrho_2$ (from ⓓ) we deduce (by Cor. 2) that $c_4 \in \neg K(\neg\varrho_2) \underset{(*)}{\subseteq} \varrho_1$, analogously from $(\triangledown)$ and $(b, c_4) \in \sigma_2$ (ⓓ again), that $c_4 \in \neg K(\neg\sigma_2) \underset{(**)}{\subseteq} \sigma_1$, therefore $c_4 \in \varrho_1 \cap \sigma_1$, that is Ⓐ. Conditions Ⓑ and Ⓒ follow from ⓓ. $\Box$

1.4 $\bigcup\bigcap$-representations of Relations

In this section we introduce the notion of $\bigcup\bigcap$-representations. They correspond to disjunctive-conjunctive (normal) forms of first-order predicate logic formulas. This was our oldest approach to the problem but failed because we needed the crux lemma to assure that the special relations are essentialy generatable without teridentity (see the property (iii) in the representation Theorem 1).

Definition 8. Let $\Sigma \subseteq A$ and let $\varrho \in \mathrm{Rel}^{(n)}(A)$. Then we say the set $\mathcal{S} \subseteq \mathfrak{P}(\mathrm{Rel}^{(n)}(A))$ is a $\bigcup\bigcap$-*representation* of ϱ if

(i) $\varrho = \bigcup\{\bigcap S \mid S \in \mathcal{S}\}$ and
(ii) all $S \in \mathcal{S}$ are finite.

Let t be a $\langle\Sigma\rangle_{\mathsf{PAL}}$-term. Then $\mathcal{S} \subseteq \mathfrak{P}(\mathrm{Rel}^{(n)}(A))$ is said to be a $\bigcup\bigcap$*-representation consistent with* t if $\mathcal{S}$ is a $\bigcup\bigcap$-representation of $\varrho := [\![t]\!]$ and we have for all $\sigma \in \bigcup \mathcal{S}$:

(iii) $E(\sigma)$ is t-connected and
(iv) $\sigma\restriction_{E(\sigma)} \in \langle\Sigma\rangle_{\mathsf{PAL}}$.

The following lemmata show how $\bigcup\bigcap$-representations have to be transformed to provide a $\bigcup\bigcap$-representation of the result of PAL-operations.

Lemma 8. *Let* $\varrho_1, \varrho_2 \in \mathrm{Rel}(A)$ *and let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be* $\bigcup\bigcap$*-representations of* ϱ_1 *and* ϱ_2 *respectively. Then*

$$\mathcal{S} := \{\{\sigma_1 \times A^{\mathrm{ar}(\varrho_2)} \mid \sigma_1 \in S_1\} \cup \{A^{\mathrm{ar}(\varrho_1)} \times \sigma_2 \mid \sigma_2 \in S_2\} \mid S_1 \in \mathcal{S}_1,\ S_2 \in \mathcal{S}_2\}$$

is a $\bigcup\bigcap$*-representation of* $\varrho_1 \times \varrho_2$.

Proof. Let $m := \mathrm{ar}(\varrho_1)$ and $n := \mathrm{ar}(\varrho_2)$, let $\underline{a} \in A^m$ and $\underline{b} \in A^n$. It is easy to see that for any relation $\tau_1 \in \mathrm{Rel}^{(m)}(A)$ it is equivalent $\underline{a} \in \tau_1 \iff (a_1, \dots, a_m, b_1, \dots, b_n) \in \tau_1 \times A^n$ $(*)$ and analogously for any $\tau_2 \in \mathrm{Rel}^{(n)}(A)$ it is equivalent $\underline{b} \in \tau_2 \iff (a_1, \dots, a_m, b_1, \dots, b_n) \in A^m \times \tau_2$ $(*')$. Consequently

$$\begin{aligned}
& (a_1, \dots, a_m, b_1, \dots, b_n) \in \varrho_1 \times \varrho_2 \\
\overset{\text{Def. 1(1)}}{\iff} & \underline{a} \in \varrho_1,\ \underline{b} \in \varrho_2 \\
\overset{\text{Def. 8}}{\iff} & \exists S_1 \in \mathcal{S}_1, S \in \mathcal{S}_2 :\ \underline{a} \in \bigcap S_1 \text{ and } \underline{b} \in \bigcap S_2 \\
\overset{(*),(*')}{\iff} & \exists S_1 \in \mathcal{S}_1, S \in \mathcal{S}_2 : \\
& \qquad (a_1, \dots, a_m, b_1, \dots, b_n) \in ((\bigcap S_1) \times A^n) \cap (A^m \times (\bigcap S_2)) \\
\overset{\text{Def of } \mathcal{S}}{\iff} & \exists S \in \mathcal{S} :\ (a_1, \dots, a_m, b_1, \dots, b_n) \in \bigcap S \\
\overset{\text{Def. 8}}{\iff} & (a_1, \dots, a_m, b_1, \dots, b_n) \in \bigcup\{\bigcap S \mid S \in \mathcal{S}\}
\end{aligned}$$

□

Lemma 9. *Let* $\mathcal{S}_1$ *be a* $\bigcup\bigcap$*-representation of* $\varrho \in \mathrm{Rel}(A)$*. Then*

$$\mathcal{S} := \{\{\neg\tau(S_1) \mid S_1 \in \mathcal{S}_1\} \mid \tau : \mathcal{S}_1 \to \bigcup \mathcal{S}_1,\ \tau(S) \in S \text{ for all } S \in \mathcal{S}_1\}$$

is a $\bigcup\bigcap$*-representation of* $\neg\varrho$.

Proof. Basically we use de Morgan's law and the distributivity of $\cap$ and $\cup$, although in a generalized version. We show first, that every tuple not in ϱ is an element of the relation describe by $\mathcal{S}$. Let $m := \mathrm{ar}(\varrho)$ and $\underline{a} \in \neg\varrho$. Because $\mathcal{S}_1$ is a $\bigcup\bigcap$-representation of ϱ and by Def. 8 one can conclude that for every $S \in \mathcal{S}_1$ there is some relation $\sigma_S \in S$ such that $\underline{a} \notin S$ (otherwise $\underline{a} \in \bigcap S \subseteq \varrho$, contradiction). The mapping $\tau_{\underline{a}} : \mathcal{S}_1 \to \bigcup \mathcal{S}_1$ with $\tau_{\underline{a}}(S) := \sigma_S$ is obviously a choice function as used in the $\bigcup\bigcap$-representation $\mathcal{S}$ and $\underline{a} \notin \tau_{\underline{a}}(S_1)(\iff \underline{a} \in \neg\tau_{\underline{a}}(S_1))$ for all $S_1 \in \mathcal{S}_1$ and consequently $\underline{a} \in \bigcap\{\neg\tau_{\underline{a}}(S)_1) \mid S_1 \in \mathcal{S}_1\} \subseteq \bigcup\{\bigcap S \mid S \in \mathcal{S}\}$.

After having shown that every tuple in $\neg\varrho$ is described by $\mathcal{S}$, one can similarly show that every element not in $\neg\varrho$, that is every $\underline{a} \in \varrho$ is not described by $\mathcal{S}$. By Def. 8 we see that there is some $S_{\underline{a}} \in \mathcal{S}_1$ such that $\underline{a} \in \bigcap S_{\underline{a}}$. Then for any choice function $\tau : \mathcal{S}_1 \to \bigcup \mathcal{S}_1$ one has $\underline{a} \in \tau(S_{\underline{a}})(\iff \underline{a} \notin \neg\tau(S_{\underline{a}}) \supseteq \bigcap\{\neg\tau(S_1) \mid S_1 \in \mathcal{S}\}$, consequently $\underline{a} \notin \bigcup\{\bigcap S \mid S \in \mathcal{S}\}$. $\square$

Lemma 10. *Let $\mathcal{S}$ be a $\bigcup\bigcap$-representation consistent with some $\langle\Sigma\rangle_{\mathsf{PAL}}$-term t. Then there exists a $\bigcup\bigcap$-representation $\mathcal{S}'$ consistent with t with*

(i) $\forall \sigma_1, \sigma_2 \in S \in \mathcal{S}' : \sigma_1 \subseteq \sigma_2 \implies \sigma_1 = \sigma_2$,
(ii) $\forall \sigma \in \bigcup \mathcal{S}' : \sigma \neq \emptyset^{\mathrm{ar}(t)}$,
(iii) $\forall \sigma \in \bigcup \mathcal{S}' : \sigma \neq A^{\mathrm{ar}(t)}$,
(iv) $S \neq \emptyset$ *for all* $S \in \mathcal{S}'$ *if* $|\mathcal{S}'| > 1$ *and*
(v) $\bigcup \mathcal{S}' \subseteq \bigcup \mathcal{S}$.

Proof. (i) Because of ii every set $S \in \mathcal{S}$ is finite. For that reason there are minimal relations. Let $\widetilde{S} := \min_{\subseteq} S$, then $\bigcap \widetilde{S} = \bigcap S$ and $\mathcal{S}_{(i)} := \{\widetilde{S} \mid S \in \mathcal{S}\}$ fulfills condition (i) and $\bigcup \mathcal{S}_{(i)} \subseteq \bigcup \mathcal{S}$ ($\square$). Part (ii): The empty relation $\emptyset^{\mathrm{ar}(\varrho)}$ absorbes all other relations, that is $\bigcap S = \emptyset^{\mathrm{ar}(\varrho)}$ if $\emptyset^{\mathrm{ar}(\varrho)} \in S$ and consequently the $\bigcup\bigcap$-representation $\mathcal{S}_{(ii)} := \{S \mid S \in \mathcal{S}_{(i)}, \emptyset^{\mathrm{ar}(\varrho)} \notin S\}$ fulfills conditions (i) and (ii) and by ($\square$) also $\bigcup \mathcal{S}_{(ii)} \subseteq \bigcup \mathcal{S}$ ($\square\square$). Part (iii): The full relation $A^{\mathrm{ar}(\varrho)}$ has no influence on the intersection of relations, that is for all $S \in \mathcal{S}$ holds $\bigcap S = \bigcap(S \setminus \{A^{\mathrm{ar}(\varrho)}\}$ and for that reason $\mathcal{S}_{(iii)} := \{S \setminus \{A^{\mathrm{ar}(\varrho)} \mid S \in \mathcal{S}_{(ii)}\}$ fulfills conditions (i)–(iii) and by ($\square\square$) also $\bigcup \mathcal{S}_{(iii)} \subseteq \bigcup \mathcal{S}$ ($\square\square\square$). Finally, because of $\bigcap \emptyset = A^{\mathrm{ar}(\varrho)}$ it is either $\varrho = A^{\mathrm{ar}(\varrho)}$ and $\mathcal{S}' := \{\emptyset\}$ fulfills conditions (i)–(v) or $\emptyset \notin \mathcal{S}_{(iii)}$ and $\mathcal{S}' := \mathcal{S}_{(iii)}$ fulfills conditions (i)–(v). $\square$

Definition 9. A $\bigcup\bigcap$-representation fulfilling the conditions (i), (ii), (iii) and (iv) of Lem. 10 is said to be *normalized*.

Lemma 11. *Let $\varrho_1 \in \mathrm{Rel}^{(1)}(A)$ and $\varrho_2 \cup \mathrm{Rel}^{(2)}(A)$ be arbitrary relations. Then ϱ_1 and ϱ_2 are comparable $\emptyset^1$ and A^1, and ϱ_1 is comparable with $\emptyset^2$ and A^2 too.*

Proof. For $\emptyset^1$ and A^1 this is trivial. For $\emptyset^2$ and A^2 it follows from $K(\emptyset^2) = K(A^2) = A^1$. $\square$

2 The Representation Theorem

After the mathematical tools have been prepared, we can now prove the first central result, the representation theorem for the relations generated from unary and binary relations in PAL without the teridentity. Many parts of the proof are rather technical because many subcases have to be distinguished. The most difficult case is the join operation, there the crux lemma is applied to conserve the property (iii) that the special relations are essentialy unary or binary.

Theorem 1. *Let $\Sigma : \mathrm{Rel}^{(1)}(A) \cup \mathrm{Rel}^{(2)}(A)$. Then there is for every $\langle\Sigma\rangle_{\mathsf{PAL}}$-term t a $\bigcup\bigcap$-representation $\mathcal{S}$ consistent with t such that*

(i) $|E(\sigma)| \leq 2$ *for all* $\sigma \in \bigcup\mathcal{S}$,
(ii) $\forall\sigma_1, \sigma_2 \in \bigcup\mathcal{S} : E(\sigma_1) \cap E(\sigma_2) \neq \emptyset \implies \sigma_1{\restriction}_{E(\sigma_1)}$ *and* $\sigma_2{\restriction}_{E(\sigma_2)}$ *are comparable (if* $\max(E(\sigma_1) \cup E(\sigma_2)) \in E(\sigma_1) \cap E(\sigma_2)$*) or inverted comparable (otherwise) and*
(iii) $\{\sigma{\restriction}_{E(\sigma)} \mid \sigma \in \bigcup\mathcal{S}\} \subseteq \langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$.

Proof. The proof works by induction over the possible constructions of a $\langle\Sigma\rangle_{\mathsf{PAL}}$-term. When checking the consistency of a $\bigcup\bigcap$-representation with a term, we do not have to consider the condition Def. 8(iv) because condition (iii) of the theorem is stronger. Condition (i) of the theorem holds by Lem. 1 for all $t \in \langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$. This condition makes clear that condition (ii) is well-defined.

I) If t is atomic then $t = \sigma \in \Sigma$. We can therefore simply set $\mathcal{S} := \{\{\sigma\}\}$. Conditions (ii) and (iii) hold trivially. Obviously, $\mathcal{S}$ is a $\bigcup\bigcap$-representation of $\sigma = [\![t]\!]$. Due to $E(\sigma) \subseteq \underline{\mathrm{ar}(\sigma)}$ the representation is consistent with t by Def. 4(i).

II) The case $t = \mathsf{id}_3$ is not possible because $\mathsf{id}_3 \notin \Sigma = \Sigma_0$.

III) If $t = t_1 \times t_2$ then there exist by the induction hypothesis $\bigcup\bigcap$-representations $\mathcal{S}_1$ and $\mathcal{S}_2$ consistent with t_1 and t_2 respectively. Let $n_1 := \mathrm{ar}(t_1)$ and $n_2 := \mathrm{ar}(t_2$. We set $\mathcal{S}$ as in Lem. 8. By this lemma we see that $\mathcal{S}$ is a $\bigcup\bigcap$-representation of $[\![t_1]\!] \times [\![t_2]\!] = [\![t]\!]$.

It is easy to see that $E(\sigma_1 \times A^{n_2}) = E(\sigma_1)$ and $E(A^{n_1} \times \sigma_2) = \{n_1 + i \mid i \in E(\sigma_2)\}$ for all $\sigma_1, \sigma_2 \in \mathrm{Rel}(A)$.Using the induction hypothesis we can deduce $\sigma^1 \times A^{n_2}\restriction_{E(\sigma_1 \times A^{n_2})} = \sigma_1\restriction_{E(\sigma_1)} \in \langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ ($\triangle$) and $A^{n_1} \times \sigma_2\restriction_{E(A^{n_1} \times \sigma_2)} = \sigma_2\restriction_{E(\sigma_2)}$ and therefore condition (iii) of the theorem holds.

The consistency of $\mathcal{S}$ with t follows by Def. 4(ii). Now let $\tau_1, \tau_2 \in \bigcup\mathcal{S}$. If τ_1 is of the form $\sigma_1 \times A^{n_2}$ and τ_2 of the form $A^{n_1} \times \sigma_2$ (or vice-versa) for some $\sigma_1 \in \bigcup\mathcal{S}_1$ and $\sigma_2 \in \mathcal{S}_2$ then $E(\tau_1) \subseteq \{1, \ldots, n_1\}$ and $E(\tau_2) \in \{n_1 + 1, \ldots, n_1 + n_2\}$, that is $E(\tau_1) \cap E(\tau_2) = \emptyset$ and they do not fulfill the premise of condition (ii). If $\tau_1 = \sigma_1 \times A^{n_2}$ and $\tau_2 = \sigma_2 \times A^{n_2}$ for $\sigma_1, \sigma_2 \in \bigcup\mathcal{S}_1$ then $\tau_1\restriction_{E(\tau_1)}$ and $\tau_2\restriction_{E(\tau_2)}$ are comparable by ($\triangle$) and the induction hypothesis. Analogously we can show that they are comparable if they are both of the form $A^{n_1} \times \sigma$ for some $\sigma \in \bigcup\mathcal{S}_2$. This proves condition (ii) of the theorem.

IV) The most complicated case is $t = \delta^{i,j}(t_1)$. Let $\mathcal{S}_1$ be a $\bigcup\bigcap$-representation consistent with t_1 and $m := \mathrm{ar}(t_1)$. There are three subcases: IV.i) if every place $p \in \underline{m}$ is t_1-connected to i or j then by Def. 4(iii-iii") we have that $\underline{m}$ is t-connected. Then $\{\{[\![t]\!]\}\}$ is trivially a $\bigcup\bigcap$-representation consistent with t. As for the atomic case we see that conditions (ii) and (iii) hold.

Otherwise (IV.ii) and IV.iii)), there is some place in $\overline{\mathrm{ar}(t_1)}$ which is not t_1-connected to i or j. Let $m := \mathrm{ar}(t_1)$. Without loss of generality we assume $i = m$ and $j = m - 1$ and that 1 is not t_1-connected to these two places. Let $\mathcal{S}_1$ be a normalized $\bigcup\bigcap$-representation consistent with t_1 with the properties given in the theorem (which exists by the induction hypothesis and Lem. 10) and let $S \in \mathcal{S}_1$.

IV.ii) If $m - 1$ and m are t_1-connected, then we have by Lem. 4 and condition (i) that for $\tau := \bigcap\{\sigma \mid \sigma \in S,\ E(\sigma) \cap \{m-1, m\} \neq \emptyset\}$ we have $E(\tau) \subseteq \{m - 1, m\}$. By Lem. 3(iv) and Lem. 2 we see that $\delta^{m-1,m}(\tau) \in \{\emptyset^{m-2}, A^{m-2}\}$.

By Lem. 5 we see that $\delta^{m-1,m}(\bigcap\{\sigma \in S\}) = \bigcap\{\delta^{i,j}(\sigma) \mid \sigma \in S \text{ and } m - 1, m \notin E(\sigma)\} \cap \delta^{m-1,m}(\tau)$. If $\delta^{m-1,m}(\tau) = \emptyset^{m-2}$ we set $S' := \{\emptyset\}$, otherwise $S' := \{\delta^{m-1,m}(\sigma) \mid \sigma \in S \text{ and } m-1, m \notin E(\sigma)\}$

and have in both cases $\delta^{m-1,m}(\bigcap\{\sigma \in S\}) = \bigcap S'$. We also have $\{\sigma'\restriction_{E(\sigma')} \mid \sigma' \in S'\} \subseteq \{\sigma\restriction_{E(\sigma)} \mid \sigma \in S\} \cup \{\emptyset^{m-2}\}$ and therefore conditions (i) and (iii) hold.

We set $\mathcal{S} := \{S' \mid S \in \mathcal{S}_1\}$, applying the same construction to all elements of $\mathcal{S}_1$. It is easy to see that this is a $\bigcup\bigcap$-representation of $[\![t]\!]$. We have $\bigcup\mathcal{S} \subseteq \bigcup\mathcal{S}_1 \cup \{\emptyset^{m-2}\}$ and by the induction hypothesis (condition ii) and Lem. 11 we see that condition (ii) holds and that $\mathcal{S}$ is consistent with t.

IV.iii) Finally we consider the case, that 1 is not t_1-connected to $m-1$ or m, but some place other than these three is connected to $m-1$ or m.

If there is some relation $\widetilde{\varrho}_1 \in S$ with $E(\widetilde{\varrho}_1) = \{m\}$ we set $\varrho_1 := \widetilde{\varrho}_1\restriction_{\{m\}}$ (because $\mathcal{S}_1$ is normalized and by condition (ii) there can be at most one such relation), otherwise we set $\widetilde{\varrho}_1 := A^m$ and $\varrho_1 := A^1$ (and therefore $\varrho_1 = \widetilde{\varrho}_1\restriction_{\{m\}})$). In the second case we have $E(\widetilde{\varrho}_1) = \emptyset \subseteq \{m\}$. Analogously, if there is some relation $\widetilde{\sigma}_1 \in S$ with $E(\widetilde{\sigma}_1) = \{m-1\}$ we set $\sigma_1 := \widetilde{\sigma}_1\restriction_{\{m-1\}}$, and otherwise $\widetilde{\sigma}_1 := A^m$ and $\sigma_1 := A^1$.

If there is some place $k \in \underline{m} \setminus \{m-1, m\}$ which is t_1-connected to m and some relation $\widetilde{\varrho}_2 \in S$ with $E(\widetilde{\varrho}_2) = \{k, m\}$ (we will refer to this condition by $(\lhd)$) we set $\varrho_2 := \widetilde{\varrho}_2\restriction_{\{k,m\}}$. By condition (i) there can be at most one such place, and by condition (ii) and Lem. 10(i) there can be at most one such relation. If there is no such relation or no such place $(\not\lhd)$ we set $k := 1$, $\widetilde{\varrho}_2 := A^m$ and $\varrho_2 := A^2$. In all these cases we have $\widetilde{\varrho}_2\restriction_{\{k,m\}} = \varrho_2$ and that ϱ_1 and ϱ_2 are comparable (the latter by condition (ii) or Lem. 11).

Similarly, if there is some place $l \in \underline{m} \setminus \{m-1, m\}$ which is t_1-connected with $m-1$ and some relation $\widetilde{\sigma}_2 \in S$ with $E(\widetilde{\sigma}_2) = \{l, m-1\}$ $(\rhd)$ we set $\sigma_2 := \widetilde{\sigma}_2\restriction_{\{l,m-1\}}$. If no such place or no such relation exists $(\not\rhd)$ we set $l := 1$, $\widetilde{\sigma}_2 := A^m$ and $\sigma_2 := A^2$. We have $E(\widetilde{\sigma}_2) \subseteq \{l, m-1\}$ and $\widetilde{\sigma}_2\restriction_{\{l,m-1\}} = \sigma_2$ in all cases and that σ_1 and σ_2 are comparable.

Because $\mathcal{S}_1$ is normalized we know that $\emptyset^m \notin \{\widetilde{\varrho}_1, \widetilde{\varrho}_2, \widetilde{\sigma}_1, \widetilde{\sigma}_2\}$ and that $m-1, m \notin E(\widetilde{\varphi})$ for all $\widetilde{\varphi} \in S \setminus \{\widetilde{\varrho}_1, \widetilde{\varrho}_2, \widetilde{\sigma}_1, \widetilde{\sigma}_2\}$.

Let $\tau_{11} := \varrho_1 \times \sigma_1$ and $\widetilde{\tau}_{11} := A^{m-2} \times \tau_{11}$. Then we have $\widetilde{\tau}_{11}\restriction_{E(\widetilde{\tau}_{11})} = \tau_{11} \in \langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (if $\widetilde{\varrho}_1, \widetilde{\sigma}_1 \in S$) or $\widetilde{\tau}_{11}\restriction_{E(\widetilde{\tau}_{11})} = \varrho_1 \in \langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (if $\widetilde{\varrho}_1 \in S, \widetilde{\sigma}_1 = A^m$) or $\widetilde{\tau}_{11}\restriction_{E(\widetilde{\tau}_{11})} = \sigma_1 \in \langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (if $\widetilde{\varrho}_1 = A^m, \widetilde{\sigma}_1 \in S$) or $\widetilde{\tau}_{11}\restriction_{E(\widetilde{\tau}_{11})} = A^0 \in \langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (if $\widetilde{\varrho}_1 = \widetilde{\sigma}_1 = A^m$), that is in general $\widetilde{\tau}_{11}\restriction_{E(\widetilde{\tau}_{11})} \in \langle\Sigma\rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ $(\Box)$.

We set $\widetilde{\tau}_{21} := \{\underline{a} \in A^m \mid (a_k, a_m) \in \varrho_2,\ a_{m-1} \in \sigma_1\}$ and $\tau_{21} := \varrho_2 \times \sigma_1$. We get $\widetilde{\tau}_{21}\restriction_{\{k,m-1,m\}} = \pi_{(23)}(\tau_{21})$ in general and $\widetilde{\tau}_{21}\restriction_{E(\widetilde{\tau}_{21})} = \pi_{(23)}(\tau_{21}) \in \langle \Sigma \rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (if $(\triangleleft)$ and $\widetilde{\sigma}_1 \in S$), $\widetilde{\tau}_{21}\restriction_{E(\widetilde{\tau}_{21})} = \varrho_2 \in \langle \Sigma \rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (if $(\triangleleft)$ and $\widetilde{\sigma}_1 = A^m$), $\widetilde{\tau}_{21}\restriction_{E(\widetilde{\tau}_{21})} =\in \sigma_1 \langle \Sigma \rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (if $(\ntriangleleft)$ and $\widetilde{\sigma}_1 \in S$) and $\widetilde{\tau}_{21}\restriction_{E(\widetilde{\tau}_{21})} = A^0 \in \langle \Sigma \rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (if $(\ntriangleleft)$ and $\widetilde{\sigma}_1 = A^m$), that is in general $\widetilde{\tau}_{21}\restriction_{E(\widetilde{\tau}_{21})} \in \langle \Sigma \rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (□□).

Analogously we set $\widetilde{\tau}_{12} := \{\underline{a} \in A^m \mid a_m \in \varrho_1,\ (a_l, a_{m-1}) \in \sigma_2\}$ and get $\widetilde{\tau}_{12}\restriction_{\{l,m-1,m\}} = \pi_{(132)}(\varrho_1 \times \sigma_2)$ and $\widetilde{\tau}_{12}\restriction_{E(\widetilde{\tau}_{12})} \in \langle \Sigma \rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (□□□).

In this subcase we know that some place in $\underline{m} \setminus \{m-1, m\}$ is t_1-connected to $m-1$ or m, therefore we can deduce $k \neq l$.

We set $\tau := \varrho_2 \times \sigma_2$ and $\widetilde{\tau}_{22} := \{\underline{a} \in A^m \mid (a_k, a_m) \in \varrho_2,\ (a_l, a_{m-1}) \in \sigma_2\}$. We get $\widetilde{\tau}_{22}\restriction_{\{k,l,m-1,m\}} = \pi_{(243)}(\tau_{22})$ if $k < l$, $\widetilde{\tau}_{22}\restriction_{\{k,l,m-1,m\}} = \pi_{(1243)}(\tau_{22})$ if $l < k$, for the restriction on the essential places we have:

	$(\triangleleft)$			$(\ntriangleleft)$	
	$(\triangleright)$		$(\ntriangleright)$	$(\triangleright)$	$(\ntriangleright)$
	$k < l$	$l < k$			
$\widetilde{\tau}_{22}\restriction_{E(\widetilde{\tau}_{22})}$	$\pi_{(243)}(\tau_{22})$	$\pi_{(1243)}(\tau_{22})$	ϱ_2	σ_2	$\emptyset^2$

and in all cases $\tau_{22}\restriction_{E(\tau_{22})} \in \langle \Sigma \rangle_{\mathsf{PAL}\setminus\{\mathsf{id}_3\}}$ (□□□□).

As noted before, ϱ_1 and ϱ_2 are comparable as are σ_1 and σ_2. Therefore we can apply the crux lemma 7 and get

$$
\begin{aligned}
& \exists a_1,\dots,a_{m-2}:\ (a_1,\dots,a_{m-2}) \in \delta^{m-1,m}\big(\bigcap\{\widetilde{\varrho}_1,\widetilde{\varrho}_2,\widetilde{\sigma}_1,\widetilde{\sigma}_2\}\big)\\
\iff & \exists a_1,\dots,a_{m-2},c \in A:\ (a_1,\dots,a_{m-2},c,c) \in \bigcap\{\widetilde{\varrho}_1,\widetilde{\varrho}_2,\widetilde{\sigma}_1,\widetilde{\sigma}_2\}\\
\xLeftrightarrow{1,\dots,m-2\notin\begin{cases}E(\widetilde{\varrho}_1)\\E(\widetilde{\varrho}_2)\\E(\widetilde{\sigma}_1)\\E(\widetilde{\sigma}_2)\end{cases}} & \exists a_k,a_l,c \in A:\ c \in \varrho_1\cap\sigma_1,\ (a_k,c)\in\varrho_2,\ (a_l,c)\in\sigma_2\\
\xLeftrightarrow{\text{crux lemma}} & \exists a_k,a_l,c_1,c_2,c_3,c_4 \in A: c_1 \in \varrho_1\cap\sigma_1,\\
& (a_k,c_2)\in\varrho_2, c_2\in\sigma_1,\ c_3\in\varrho_1,(a_l,c_3)\in\sigma_2,\\
& (a_k,c_4)\in\varrho_2,(a_l,c_4)\in\sigma_2\\
\Leftrightarrow & \exists a_k,a_l,c_1,c_2,c_3,c_4 \in A:\ (c_1,c_1)\in\tau_{11},\\
& (a_k,c_2,c_2)\in\tau_{21},\ (c_3,a_l,c_3)\in\tau_{12},\ (a_k,c_4,a_l,c_4)\in\sigma_2\\
\xLeftrightarrow{1,\dots,m-2\notin\begin{cases}E(\widetilde{\varrho}_1)\\E(\widetilde{\varrho}_2)\\E(\widetilde{\sigma}_1)\\E(\widetilde{\sigma}_2)\end{cases}} & \exists a_1,\dots,a_{m-2},c_1,\dots,c_4 \in A:\ (a_1,\dots,a_{m-2},c_1,c_1)\in\widetilde{\tau}_{11},\\
& (a_1,\dots,a_{m-2},c_2,c_2)\in\widetilde{\tau}_{21},\ (a_1,\dots,a_{m-2},c_3,c_3)\in\widetilde{\tau}_{12},\\
& (a_1,\dots,a_{m-2},c_4,c_4)\in\widetilde{\tau}_{22}\\
\Leftrightarrow & \exists a_1,\dots,a_{m-2}\in A:\ (a_1,\dots,a_{m-2})\in\bigcap\{\delta^{m-1,m}(\widetilde{\tau}_{xy})\mid x,y\in\{1,2\}\}
\end{aligned}
$$

Now, let $S' := \{\delta^{m-1.m}(\sigma) \mid \sigma\in S,\ m-1,m\notin E(\sigma)\}\cup\{\delta^{m-1,m}(\widetilde{\tau}_{x,y}) \mid x,y\in\{1,2\}\}$. By Lem. 5 we get $\delta^{m-1,m}(\bigcap S) = \bigcap S'$. For $\sigma\in S,\ m-1,m\notin E(\sigma)$ we have $\delta^{m-1,m}(\sigma)\restriction_{E(\delta^{m-1,m}(\sigma))} = \sigma\restriction_{E(\sigma)}$ and together with (□), (□□), (□□□) and (□□□□) we see that conditions (i) and (iii) hold for all relations in S'.

Now we set $\mathcal{S} := \{S' \mid S\in\mathcal{S}_1\}$, applying the same construction to all elements in $\mathcal{S}_1$. As before it is easy to see that conditions (i) and (iii) hold for all relations in $\bigcup\mathcal{S}$ and that $\mathcal{S}$ is a $\bigcup\bigcap$-representation of $[\![t]\!]$, which is due to Def. iii consistent with t.

To show that condition (ii) holds requires again to consider several cases. Let $\widetilde{\varphi}_1,\widetilde{\varphi}_2\in\bigcup\mathcal{S}$ such that no place in $E(\widetilde{\varphi}_1)\cap E(\widetilde{\varphi}_2)$ is t_1-connected to $m-1$ or m. By Def. 4(iii-iii”) and the foregoing construction we see that there are relations $\widetilde{\psi}_1,\widetilde{\psi}_2\in\bigcup\mathcal{S}$ such that

$\widetilde{\varphi}_1 = \delta^{m-1,m}(\widetilde{\psi}_1)$, $\widetilde{\varphi}_2 = \delta^{m-1,m}(\widetilde{\psi}_2)$ and $m-1, m \notin E(\widetilde{\psi}_1) \cup E(\widetilde{\psi}_2)$. Consequently, $\widetilde{\varphi}_1\restriction_{E(\widetilde{\varphi}_1)} = \widetilde{\psi}_1\restriction_{E(\widetilde{\psi}_1)}$ and $\widetilde{\varphi}_2\restriction_{E(\widetilde{\varphi}_2)} = \widetilde{\psi}_2\restriction_{E(\widetilde{\psi}_2)}$ are comparable according to the induction hyposthesis.

If for $\widetilde{\varphi}_1, \widetilde{\varphi}_2 \in \bigcup \mathcal{S}$ there is some place in $E(\widetilde{\varphi}_1) \cap E(\widetilde{\varphi}_2)$ which is t_1-connected to $m-1$ or m, then we have due to (i) $E(\widetilde{\varphi}_1), E(\widetilde{\varphi}_2) \subseteq \{k, l\}$. There are several possibilities. IV.iii.a) $\widetilde{\varphi}_1 = \delta^{m-1,m}(\widetilde{\psi}_1)$ and $\widetilde{\varphi}_2 = \delta^{m-1,m}(\widetilde{\psi}_2)$ with $E(\widetilde{\varphi}_1) = E(\widetilde{\varphi}_2) \in \{\{k\}, \{l\}\}$. Then $\widetilde{\psi}_i\restriction_{E(\widetilde{\psi}_i)} = \widetilde{\varphi}_i\restriction_{E(\widetilde{\varphi}_i)}$ for $i \in \{1, 2\}$ which are comparable due to the induction hypothesis. IV.iii.b) $\widetilde{\varphi}_1 = \delta^{m-1,m}(\widetilde{\psi}_1)$ with $E(\widetilde{\psi}_1) = E(\widetilde{\varphi}_2) = \{k\}$, but $\widetilde{\varphi}_2$ is of the form $\widetilde{\varphi}_2 = \delta^{m-1,m}(\widetilde{\tau}_{21})$ or $\widetilde{\varphi}_2 = \delta^{m-1,m}(\widetilde{\tau}_{22})$. From $k \in E(\widetilde{\varphi}_2) \underset{L.\ 3(\mathrm{iv})}{\subseteq} E(\widetilde{\psi}_2)$ we deduce $\widetilde{\tau}_{21}\restriction_{E(\widetilde{\tau}_{21})} \in \{\pi_{(23)}(\tau_{21}), \varrho_2, \pi_\alpha(\tau_{22})\}$ with $\alpha = (243)$ if $k < l$ and $\alpha = (1243)$ otherwise. Consequently, $\widetilde{\psi}_2\restriction_{E(\widetilde{\psi}_2)} \in \{\delta^{2,3}(\varrho_2 \times \sigma_1), \delta^{2,3}(\varrho_2 \times A^1), \delta^{2,3}(\varrho_2 \times \sigma_2), \delta^{2,3}(\sigma_2^{-1} \times \varrho_2^{-1}\}$ (if $k < l$ we have $\delta^{2,3}(\tau_{22})$ and if $l < k$ it is the join of the inverted relations). By the induction hypothesis are $\varphi_2\restriction_{E(\varphi_2)} = \widetilde{\psi}_2\restriction_{E(\widetilde{\psi}_2)}$ and $\widetilde{\varrho}_2\restriction_{E(\widetilde{\varrho}_2)} = \varrho_2$ inverted comparable, by applying Lemma 6(ii) we see that $\widetilde{\varphi}_1\restriction_{E(\widetilde{\varphi}_1)}$ and $\widetilde{\varphi}_2\restriction_{E(\widetilde{\varphi}_2)}$ are inverted comparable (or comparable if $l < k$).

IV.iii.c) The case $\widetilde{\varphi}_1 = \delta^{m-1,m}(\widetilde{\psi}_1)$ with $E(\widetilde{\psi}_1) = \{l\}$ and $\widetilde{\varphi}_2 = \delta^{m-1,m}(\widetilde{\tau}_{12})$ or $\widetilde{\varphi}_2 = \delta^{m-1,m}(\widetilde{\tau}_{22})$ is handled analogously to case IV.iii.b).

IV.iii.d) Finally, if $E(\widetilde{\varphi}_1) = E(\widetilde{\varphi}_2) = \{k, l\}$ then both must be of the form $\widetilde{\varphi}_1 = \delta^{m-1,m}(\widetilde{\tau}_{22})$ and $\widetilde{\varphi}_2 = \delta^{m-1,m}(\widetilde{\tau}'_{22})$ with $\widetilde{\tau}_{22}\restriction_{E(\widetilde{\tau}_{22})} = \varrho_2 \times \sigma_2$ and $\widetilde{\tau}'_{22}\restriction_{E(\widetilde{\tau}'_{22})} = \varrho'_2 \times \sigma'_2$ (they may come from different sets $S_1, S_2 \in \mathcal{S}_1$). However, by induction hypothesis ϱ_2 and ϱ'_2 are comparable and therefore $\varrho_2 = \varrho'_2$ and for the same reason $\sigma_2 = \sigma'_2$ and consequently $\widetilde{\tau}_{22} = \widetilde{\tau}'_{22}$ and therefore $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ are equal and comparable (and inverted comparable).

Therefore, condition (ii) also holds for $\mathcal{S}$ in the case IV.iii).

V) If $t = \neg t_1$ then there exists by the induction hypothesis $\bigcup\bigcap$-representations $\mathcal{S}_1$ consistent with t_1. By Lem. 9 the set $\{\{\neg\tau(S_1) \mid S_1 \in \mathcal{S}_1\} \mid \tau : \mathcal{S}_1 \to \bigcup \mathcal{S}_1,\ \tau(S) \in S \text{ for all } S \in \mathcal{S}_1\}$ is a $\bigcup\bigcap$-representation of $\neg[\![t_1]\!] = [\![t]\!]$ which is consitent with t by Def. 4(iv) and Lem. 3(v). Condition (i) follows from Lem. 3(v), condition (ii) follows from Lem. 6(i).

VI) If $t = \pi_\alpha(t_1)$ for some permutation α on $\underline{\mathrm{ar}\, t}$, then there exists by the induction hypothesis $\bigcup\bigcap$-representations $\mathcal{S}_1$ consistent with t_1. Oviously $\mathcal{S} := \{\{\pi_\alpha(\sigma) \mid \sigma \in S\} \mid S \in \mathcal{S}_1\}$ is a $\bigcup\bigcap$-representation consistent with $[\![t]\!]$ (by Lem. 3(vi))). It is easy to see that conditions (i) and (iii) hold. For relations $\pi_\alpha(\sigma), \pi_\alpha(\tau) \in \mathcal{S}$ with $\alpha(i) \in E(\pi_\alpha(\sigma)) \cap E(\pi_\alpha(\tau))$. If $|E(\sigma) \cup E(\tau)| = 1$ this trivially holds, if $|E(\sigma)| = |E(\tau)| = 2$ we can deduce using Lem. 1 $E(\sigma) = E(\tau)$ and because σ and τ are comparable by the induction hypothesis even $\sigma = \tau$ and therefore $\pi_\alpha(\sigma)$ and $\pi)\alpha(\tau)$ are equal, comparable and inverted comparable. Otherwise, we can assume without loss of generality that $|E(\sigma)| = 2$ and $|E(\tau)| = 1$. Let $E(\sigma) = \{i, j\}$ and $i \in E(\tau)$. By the induction hypothesis we know that $\sigma\restriction_{E(\sigma)}$ and $\tau\restriction_{E(\tau)}$ are comparable if $i < j$ and inverted comparable if $j < i$. If α inverts the order of i and j we get that $\pi_\alpha(\sigma)\restriction_{E(\pi_\alpha(\sigma))} = (\sigma\restriction_{E(\sigma)})^{-1}$. We see that then eihter $\pi_\alpha(\tau)\restriction_{E(\pi_\alpha(\tau))}$ and $\pi_\alpha(\sigma)\restriction_{E(\pi_\alpha(\sigma))}$ are comparable (if $\alpha(i) < \alpha(j)$) or inverted comparable (otherwise). Therefore condition (ii) holds in all cases. $\square$

3 A Peircean Reduction Thesis

The representation theorem presented in the last section allows to prove the difficult part of the Peircean Reduction Thesis. It is easy to see that all relations can be generated from the unary and binary relations – if we the teridentity can be used. Consequently it must be impossible to generate the teridentity itself from unary and binary relations. Using the properties of the $\bigcup\bigcap$-representation provided by Theorem 1 we can prove that the teridentity cannot be generated, therefore at least one ternary relation is needed besides unary and binary relations to generate all relations.

Theorem 2. *Let* $|A| \geq 2$ *and* $\Sigma := \mathrm{Rel}^{(1)}(A) \cup \mathrm{Rel}^{(2)}(A)$. *Then*

$$\langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}} \subsetneq \langle \Sigma \rangle_{\mathsf{PAL}}.$$

Proof. By definition we have $\mathsf{id}_3 \in \langle \Sigma \rangle_{\mathsf{PAL}}$. Let $\varrho \in \langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}}$ be a ternary relation. Let us assume $\mathsf{id}_3 = \varrho$. Let $t \in \langle \Sigma \rangle_{\mathsf{PAL} \setminus \{\mathsf{id}_3\}}$ be a term describing ϱ. Then there exists by Thm. 1 and Lem. 10 a normalized $\bigcup\bigcap$-representation $\mathcal{S}$ consistent with t. From Lem. 1 we can deduce that there is some place in $\underline{3}$ which is not t-connected to the other two places. Without loss of generality we assume that 1 is not t-connected to 2 or 3. Let $S \in \mathcal{S}$. Because $\mathcal{S}$ is normalized, there is some tuple $(x,x,x) \in S$. Let $y \in A$ with $x \neq y$. For any relation $\sigma \in S$ with $1 \notin E(\sigma)$ we have $\sigma = A^1 \times \sigma\restriction_{\{2,3\}}$ and therefore $(y,x,x) \in \sigma$. If for no $\sigma \in S$ we have $1 \in E(\sigma)$ we have therefore $(y,x,x) \in \bigcap S \subseteq \bigcup\{\bigcap S \mid S \in \mathcal{S}\} = \varrho$ but $(y,x,x) \notin \mathsf{id}_3$, a contradiction.

Therefore, for every $S \in \mathcal{S}$ there has to be a relation ϱ_S with $1 \in E(\varrho_S)$, and because 1 is not t-connected to any other place we have $E(\varrho_S) = \{1\}$. Let $a, b \in A$ be two distinct elements. Let S_a be an element of $\mathcal{S}$ with $(a,a,a) \in \bigcap S_a$ (and therefore $a \in \varrho_{S_a}$), and analogously S_b such that $(b,b,b) \in \bigcap S_b$ and therefore $b \in \varrho_{S_b}$. By condition (ii) of Thm. 1 we know that ϱ_{S_a} and ϱ_{S_b} are comparable and therefore $a \in \varrho_{S_a} \subseteq \varrho_{S_b}$ or $b \in \varrho_{S_b} \subseteq \varrho_{S_a}$. Because $\mathcal{S}$ is normalized and by condition (ii) of Thm. 1 we know that ϱ_{S_a} is the only relation $\sigma \in S_a$ with $1 \in E(\sigma)$ (and likewise for ϱ_{S_b}), therefore we can deduce either $(a,b,b) \in \bigcap S_b \subseteq \varrho$ or $(b,a,a) \in \bigcap S_a \subseteq \varrho$ in both cases a contradiction to $\varrho = \mathsf{id}_3$. Therefore the teridentity is not representable in PAL without teridentity. $\square$

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