The Magic of Charles Sanders Peirce

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Abstract

C. S. Peirce was an amazing intellectual who spearheaded American philosophy and contributed to mathematics, statistics, physics, and geology. He also invented highly original and unperformable mathematical card tricks. We analyze Peirce's work through his card tricks.

1 Introduction

Charles Sanders Peirce (1839–1914) was an impossibilist: impossible to understand and impossible to ignore. One of the founders of the American school of philosophy, pragmatism (with Oliver Wendell Holmes, William James, and John Dewey), Peirce is revered as the father of semiotics. He made serious contributions to mathematical logic. He spent years as a working physicist and geologist. He was an early contributor to statistics, using kernel smoothers and robust methods 100 years before they became standard fare. A spectacular appreciation of Peirce's contributions is in Louis Menand's 2001 Pulitzer Prize-winning book, *The Metaphysical Club: A Story of Ideas in America*. Indeed, if you get nothing more from our article than a pointer to this one item, you will thank us (!).

At the same time, Peirce is very difficult to parse. He wrote in a convoluted, self-referential style, went off on wild tangents, and took very strong positions. The breadth of his interests makes Peirce hard to summarize; we find the following effort of Menand [39, p. 199] useful.

> What does it mean to say we "know" something in a world in which things happen higgledy-piggledy? Virtually all of Charles Peirce's work —an enormous

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> body of writing on logic, semiotics, mathematics, astronomy, metrology, physics, psychology, and philosophy, large portions of it unpublished or unfinished — was devoted to this question. His answer had many parts, and fitting them all together — in a form consistent with his belief in the existence of a personal God — became the burden of his life. But one part of his answer was that in a universe in which events are uncertain and perception is fallible, knowing cannot be a matter of an individual mind "mirroring" reality. Each mind reflects differently — even the same mind reflects differently at different moments — and in any case reality doesn't stand still long enough to be accurately mirrored. Peirce's conclusion was that knowledge must therefore be social. It was his most important contribution to American thought, and when he recalled, late in life, how he came to formulate it, he described it — fittingly — as the product of a group. This was the conversation society he formed with William James, Oliver Wendell Holmes, Jr., and a few others in Cambridge in 1872, the group known as the Metaphysical Club.

Should a modern reader take Peirce seriously? For example, in Menand's summary above, Peirce comes across as staunchly anti-Bayesian in his approach to uncertainty. At least one of your authors is a Bayesian [12]. Is it worth figuring out Peirce's views?

It turns out that Peirce had a lifelong fascination with card tricks. At the end of his life he wrote a 77-page paper detailing some of his magical inventions. We thought, "This we can evaluate!" Are the tricks amazing and performable? No. Are they original and interesting? Yes.

Peirce's magical manuscript, "Some amazing mazes" [35], is no more readable than his semiotics. Its centerpiece is a deadly 15-minute effect with cards repeatedly dealt into piles and picked up in odd ways. The conclusion is hard to appreciate. He had an awareness of this, writing "Please deal the cards carefully, for few would want to see it again." Despite all this, Peirce's methods contain four completely original ideas. Each of these can be developed into a charming, performable trick. The ideas have some mathematical depth; along the way Peirce gives card trick proofs of Fermat's little theorem and the existence of primitive roots.

It is our task to bring Peirce's ideas into focus. In what follows the reader will find Peirce's

- cyclic exploitation principle
- dyslexia effect
- packet pickup
- primitive arrangement principle

We will try to make both the mathematical ideas and magical applications transparent. Along the way, new tricks and math problems surface.

Peirce's trick is explained, more or less as he did it, in the following section. We then break it into pieces in Sections 3–6 and 8. In each, we have tried to take Peirce's idea and make a good trick. We also develop the mathematical underpinnings. If this juxtaposition between magic and mathematics seems strange to you, take a look at Section 7 before proceeding.

This presents one of our best tricks, "Together again". The reader mostly interested in magic should look at "Concentration", our development of Peirce's dyslexic principle in Section 4. The "Tarot trick", an application of Peirce's cyclic principle in Section 3 is also solid entertainment. If you are interested in magic, please read Section 5, our attempt to make a one-row version of the dyslexic principle. It needs help yet to make it a good trick.

The mathematics involved goes from low to high, from the simplest facts of number theory and permutations through probabilistic combinatorics and the Riemann hypothesis. We have tried to entwine it with the magic and make it accessible but, if it's not your thing, just skip over.

The final section comes back to the larger picture of evaluating Peirce's work through the lens we have chosen. One of Peirce's gurus was the American polymath Chauncey Wright. They corresponded about card tricks. In an appendix we unpack some of their tricks, discussed 50 years before "Some amazing mazes".

There is an enormous Peirce literature, in addition to his 13 volumes of collected work [34, 20]. Surely the best popular introduction is [39]. This may be be supplemented by [41]. For Peirce's card trick (and a friendly look at his weirder side), see Martin Gardner's *Fractal Music, Hypercards and More... Mathematical Recreations from SCIENTIFIC AMERICAN Magazine* [17, Chap. 4]. Gardner's *The Whys of a Philosophical Scrivener* [18, Chap. 2] — "Truth: Why I Am Not a Pragmatist" — is a useful review of the ups and downs of pragmatism. *His Logic Machines and Diagrams* [19] gives an extensive review of Peirce's many contributions to logic. [21, 22] gives a number theoretic analysis of Peirce's card trick; Alex Elmsley's "Peirce arrow" and "Through darkest Peirce" [23, 24] take the magic further. For Peirce and statistics, see [30], [45], and [11].

The Essential Peirce [32, 33] is a careful selection of Peirce's key writings. The Wikipedia page for Peirce points to numerous further paths of investigation: the website of The Charles S. Peirce Society and their quarterly *Transactions*, the website of Commens: Digital Companion to C.S. Peirce, and the website of the Peirce Edition Project being rewarding destinations.

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2 Peirce's "Some amazing mazes"

In this section we first describe what the trick looks like and then give an explanation of why it works.

2.1 What the trick looks like

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The routine that Peirce performed "at the end of some evening's card play" looked like this: the performer removes the Ace through Queen of Hearts and the Ace through Queen of Spades from a normal deck of cards, arranging each packet in order A, 2, 3, ... The 13 Hearts are put in a face-down pile on the table, off to one side. The 12 Spades are held face down as if about to deal in a card game. They are dealt face up into two piles, say left, right, left, right, ..., so the piles look like:

$$\begin{array}{cc} \vdots & \vdots \\ 5 & 6 \\ 3 & 4 \\ A & 2 \end{array}$$

The performer stops before placing the last card (the black Queen) and discards it face up off to one side, replacing it with the top card of the heart pile (the Ace). Now the two face-up piles of six cards are gathered together, the right pile on the left, so the face-up Ace of Hearts winds up in the center of the stack.

This whole procedure is repeated 12 times (!!!). Each time the final dealt card will be black. It is placed onto the growing discard pile, face up, and replaced by the current top card of the heart pile. At first, it is not surprising that the last card is black, since all or most of the dealt pile is, as well. But as more and more red cards are switched in, it becomes more surprising. On the 12^{th} deal, all the blacks will have been replaced by reds. This is the ending of the first part of the trick. Perhaps it is wise for the performer to point this out, saying something like "You win if the last card is red–nope, not this time; well, let's just add another red winner for you and try again." After this, the performer places the red King with the rest of the Hearts on the face of the Heart packet and turns the whole packet face down.

The second part of the trick begins now. The black packet that accumulated as cards were discarded is taken up, turned face down and dealt, left to right, in a long row on the table. The red packet is turned face down and "mixed" by dealing it into a number of hands suggested by a spectator (e.g., seven hands). These cards are then picked up to form a single pile via "Peirce's pickup" (explained in Section 6). This may be repeated a time or two and the red packet may even be cut. However, at the end, the performer openly cuts the red packet to bring the King of Hearts to the bottom and then places this packet face down.

The denouement of the second part of the trick has begun. There is a row of black cards on the table and a pile of red cards. All cards are face down. The performer announces she will make but one simple adjustment and move a certain number of black cards cyclicly, from the left side of the black row to the right. (Say three cards are moved; the actual number depends on the spectator's choices via a rule explained later.) The trick is concluded with what we will call Peirce's dyslexic effect. The performer claims that the black card will reveal the location of any red card. "What red card do you want to find?" Suppose the spectator says "Six." The performer counts over six from the left end of the black row and turns up the card in that position: let's say it's a nine. After announcing "the cards say your card is nine down in the red packet" and counting down without disturbing the order, the ninth card is turned up and it is indeed the requested six. This may be repeated several times, each time the row of black cards "knows" where any requested red card is located. During the repetitions, if desired, the packet of reds can be dealt into a number of piles finishing with a Peirce pickup.

This is the conclusion of the second part and indeed of the complete routine. Let us try to explain how strange an ending it is. In a standard, modern card trick, when a row of cards is revealed it often has a strikingly simple order. For example, a classical trick has the deck cut into four piles and the top card of each pile is turned up to show an Ace in each location. Or, a row of seven might be turned up to reveal 1-31-1945, the spectator's day of birth. More elegantly, four piles are cut off the deck and the top four cards are turned up, appearing as a random failure, say, A, Q, 9, 7. With a wink, the performer turns the piles over to reveal their bottom cards: they are the A, Q, 9, and 7 of matching colors. These are easy-to-understand endings.

Consider Peirce's ending. There are two packets of cards and the card at position j in one packet tells the position of the card of value j in the other. It is hard to parse but completely original. People have been doing such revelations for over 500 years and we have never seen *anything* like this dyslexic effect. In Sections 5 and 6 we will shine some mathematical and magical light in this corner. For the moment, let us review.

Peirce's trick involves an interminable amount of dealing for two fairly subtle results: never a red card shows, and the dyslexic revelation. His contributions need unpacking; we turn to this in sections 3-8. To conclude this section we give a much more "hands on" description of Peirce's trick together with a stand alone explanation of why it works. This is not needed to follow the rest of the paper and the reader may want to return to it in working through Peirce's trick with cards in hand.

2.2 Why the trick works.

Peirce worked "mod 13" but all can be done with any prime number p of cards such that 2 is a primitive root of p (see Section 9 for more on primitive roots). Thus 5, 11, 13, 19 or 29 cards can be used, for example.

To explain the mathematical underpinnings of Peirce's card trick, we will first consider in some detail what happens with cards Ace through 10.

Analysis of a two pile shuffle. Let us start with 10 cards, say the Ace through 10 of Spades. We will think the Ace as having the value 1. The cards arranged face-down in the order $1, 2, \ldots 10$ from top to bottom (in hand, so to speak). We now deal the cards alternately into two face-up piles from left to right, with the left pile getting the first card 1 and the right pile getting the next card 2. After all 10 cards have been dealt, the face-up piles look like:

9	10
7	8
5	6
3	4
1	2
left	*right*

We then take the left pile and place it on top of the right pile, and then *turn the combined pile over*, so that now the cards are all face-down. The new pile now looks like:

Position	*Value*
1	2
2	4
3	6
4	8
5	10
6	1
7	3
8	5
9	7
10	9

Notice that the card in position i has the value $2i \pmod{11}$. (How nice!) In general, if this operation is repeated k times then the card in position i will have the value $2^k i \pmod{11}$. In particular, if $k = 10$ then the cards will come back to their original order (since $2^{10} \equiv 1 \pmod{11}$).

Now for Peirce's trick, we start with *two* sets of 11 cards, say Ace through Jack of Spades (the *black* cards) and Ace through Jack of Hearts (the *red* cards). To begin with we discard the Jack of Spades since it will never be used (actually it can be used at the very end). So at the beginning, we have the black cards face-down in the order $1, 2, \ldots, 10$ from top to bottom (in hand so to speak) and the red cards face-down on the table in order $1, 2, \ldots, 10$, Jack from top to bottom.

We now perform the "shuffle" described above. That is, we begin dealing the 10 black cards face-up into two alternating piles on the table, starting with the left pile first, etc. However, when we reach card 10, instead of placing it on top of the right pile, we instead place it face-up on the table starting a new black pile, and we take the top card of the red pile (which is an Ace or 1) an place it face-up on top the the right pile. So the piles now look like:

9	1
7	8
5	6
3	4
1	2

We use i to indicate that card i is red. As before, we now combine the left and right piles into one pile by placing the left pile on top of the right pile, and turning combined pile upside down, so that the cards are all face-down. Here is the situation after this first step.

Position	*Value*
1	2
2	4
3	6
4	8
5	①
6	1
7	3
8	5
9	7
10	9

The (slightly depleted) red pile now has the face-down cards in order $2, 3, \ldots, 10$ from top to bottom and the (new) black face-up pile has the single black card 10.

Now perform this step again on the combined pile. That is, we deal the cards face-up into two piles alternatively, left to right with the left coming first, except when we come to the last card (which is a 9), we place it face-up on the new black pile, and put the next card from the red pile (which is a ②) as the last card on the right pile. Now the left and right piles look like:

7	②
3	5
①	1
6	8
2	4
left	*right*

We then combine the two piles into one by placing the left pile on top of the right pile and turning the combined pile upside-down so that all the cards are face-down. What we now have after this step is:

Position	*Value*
1	4
2	8
3	1
4	5
5	②
6	2
7	6
8	①
9	3
10	7

Let us now perform this "Peirce shuffle" altogether 10 times. We tabulate the combined face-down piles after each step below (the step is indicated on top in bold).

Position	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**8**	**10**
1	2	4	8	5	1	2	3	4	5	6
2	4	8	5	1	2	3	4	5	6	7
3	6	1	2	4	8	5	1	2	3	4
4	8	5	1	2	3	4	5	6	7	8
5	1	2	3	4	5	6	7	8	9	10
6	1	2	4	8	5	1	2	3	4	5
7	3	6	1	2	4	8	5	1	2	3
8	5	1	2	3	4	5	6	7	8	9
9	7	3	6	1	2	4	8	5	1	2
10	9	7	3	6	1	2	4	8	5	1

In the meanwhile, the new black face-up pile has been building up. Turning it over so that all the cards are face-down, it now looks like:

Position	*Value*
1	10
2	9
3	7
4	3
5	6
6	1
7	2
8	4
9	8
10	5

The first interesting claim is that the final red pile and the final black pile are **inverse permutations** of each other. (Check!). (See Section 4 for background on inverses). Let us see why.

First consider the black pile. Its entries are formed by always taking the card in position 10 in the current list and placing on the (new) black pile. But the card in the $10th$ position goes through the values $10, 2 \cdot 10, 2^2 \cdot 10, \ldots, 2^9 \cdot 10 \pmod{11} = 10, 9, 7, 3, 6, 1, 2, 4, 8, 5$ as the piles are created. So this is the arrangement of the black pile. Each entry is twice the value (mod 11) of the value before it. So what is the rule that links the position i to the value? It is just this:
Rule 1: The card at position i in the black pile has the value $2^{4+i} \pmod{11}$.
The 4 in the exponent was needed to adjust the cycle so that the card in position 6 has value $1\ (= 2^{10} \pmod{11})$.

Now let's look at the red pile. The trick here is represent the position as a power of 2 (mod 11). Thus, We could express the above table as:

Position	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$1 \equiv 2^{10}$	2	4	8	5	①	②	③	④	⑤	⑥
$2 \equiv 2^1$	4	8	5	①	②	③	④	⑤	⑥	⑦
$3 \equiv 2^8$	6	1	2	4	8	5	①	②	③	④
$4 \equiv 2^2$	8	5	①	②	③	④	⑤	⑥	⑦	⑧
$5 \equiv 2^4$	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩
$6 \equiv 2^9$	1	2	4	8	5	①	②	③	④	⑤
$7 \equiv 2^7$	3	6	1	2	4	8	5	①	②	③
$8 \equiv 2^3$	5	①	②	③	④	⑤	⑥	⑦	⑧	⑨
$9 \equiv 2^6$	7	3	6	1	2	4	8	5	①	②
$10 \equiv 2^5$	9	7	3	6	1	2	4	8	5	①

We saw that on the very first step, the card in position 10 (the black 10) was replaced by the first red card ①. At step **2**, the card in position 10 (the black 9) was replaced by the next red card ②. In general, once a card reaches position 10, it gets replaced at the next step by next unused red card which will be in position 5. This is why the successive cards in position 5 are ①, ②, ③, etc. Now consider the card in position 8. It started out with value $8 \equiv 2^3 \pmod{11}$. It would have hit the value 10 at step **2** but 10 was already replaced by ①. That is why the card in position 8 at step **3** is ①. The next card in position 8 in step **3** should have been $2 \cdot 10 \equiv 9 \pmod{11}$ but 9 has already been replaced by ②. That's why the card in position 8 at step **3** is ②. In general, once a ① appears in a position in some step, then the subsequent cards in that position are ②, ③, ... to the end. So the question is, what is the final value in position 2^j at the end? Well, the value in that position at step **k** is $2^{j+k} \pmod{11}$. Thus, at step $\mathbf{4-j} \pmod{\mathbf{10}}$, the value is 5. The next value at that position should have been 10 but 10 was replaced by ①. So how many more steps does it take us to get to the end from here? It takes just $10-(4-j) = 6+j \pmod{10}$ more steps to get to step **10**. Thus, we have

Rule 2: The card in position 2^j in the red pile has the value $6+j \pmod{11}$.

We now need to show that these rules are reciprocal. To do this, we just bump up the index in Rule 1 by 6 to get:

Rule 1'. The card at position $i+6$ in the black pile has the value $2^i \pmod{11}$ since $2^{i+10} \equiv 2^i \pmod{11}$.

This shows that the two permutations are inverses of each other.

The Peirce Pickup–preliminary analysis for the general case. Here, we begin with a pile of n face-down cards with the card in position i from the top having value i. We choose an arbitrary integer k *relatively prime* to n satisfying $1 \leq k < n$. We now deal the cards into k piles, starting from the left and turning the cards face-up as we go. Define $r = n \pmod{k}$. Thus, $n = kt - r$ for some integer t. We now choose an arbitrary pile, say pile s and place it on top of pile $s+r$. Here the pile numbers start with 1 on the left and are always reduced modulo k. We then take this larger pile and place it on top of pile $s+2r$. We have to be careful to count the positions of the missing piles when we are doing this pickup. We continue this process of picking up piles until all the piles have been combined into one pile of n face-up cards. We then turn the pile over so that we now have a pile of n face-down cards. We then make a final cut so as to bring the card n to the bottom of the face-down stack. As usual, the question now is: What is the value of the card in position i from the top? The answer is this:

Rule 3. The card at position i has the value $ki \pmod{n}$.

As an example, take $n = 13,\ k = 5$ so that $r = 3$. After the piles are dealt, we have the situation:

11	12	13		
6	7	8	9	10
1	2	3	4	5

Let us choose $s = 2$ for the starting pile to pick up. It is placed on top of pile 5 and the combined pile is placed on pile $5 + 3 \equiv 3 \pmod{5}$, etc. In the combined pile, we then cut the deck so that the 13 is on the bottom. The final pile after being turned over to become face-down is:

Position	*Value*
1	5
2	10
3	2
4	7
5	12
6	4
7	9
8	1
9	6
10	11
11	3
12	8
13	13

As advertised, the card in position i has value $5i \pmod{13}$. Now why does this work in general. Well, consider the i^{th} pile before the piles have been collected. It has all the cards with value congruent to i modulo k arranged in increasing order from bottom to top. So the top card of this pile is either $kt + i$ or $kt + i - n$ depending on whether $i \le r$ or not. Hence, if we add k to the value of this top card, then in either case we get the value $k(t+1) + i - n = i - r$. But this is just value of the bottom card of the $(i-r)^{th}$ pile which is the pile we are putting on top of pile i. Thus, when go from the top card of pile i to the bottom card of pile $i - r$, we simply add $k \pmod{n}$ to the value of the top care of pile i. Since this holds for any pile-to-pile connection, then in the inverted combined list, the value of the card in position $s + 1$ is obtained from the value of the card at position s by adding $k \pmod{n}$. This implies that the value of the card at position i must have the form $ki + c$ for some c. Now, the final cut which brings the card with value n to the bottom is just a cyclic shift of the values. In particular, it forces the value of card n (on the bottom) to have the value n. Thus, this cyclic shift forces c to be zero! In other words, for the final stack, we have

Rule 3. In the final stack of the Peirce Pickup using k piles, the card in position i from the top has value $ki \pmod{n}$.

The same arguments show that the following more general rule holds for the Peirce Pickup.

Rule 3'. In the final stack of the Peirce Pickup using k piles with an arbitrary initial list of values, the value of the card in position i is the value of card in position $ki \pmod{n}$ in the original list.

Finishing the analysis. For the final part of the trick we begin by placing the red card 11 (= Jack of Hearts) at the bottom of the red stack. We next choose (actually, the spectator chooses) an arbitrarily number k with $1 \leq k < 11$. We define $r = 11 \pmod{k}$. Thus, for some t, $11 = kt + r$. We then perform the "Peirce Pickup" on this red stack, normalizing at the end so that the card with value 11 is on the bottom. For example, suppose we choose $k = 4$. Thus, in this case starting with pile 2, for example, we go from

Position	*Value*
1	6
2	7
3	4
4	8
5	10
6	5
7	3
8	9
9	2
10	1
11	11

by stacking to

2	1	11	
10	5	3	9
6	7	4	8

by recombining starting with pile 2 to

Position	*Value*
1	1
2	5
3	7
4	2
5	10
6	6
7	9
8	8
9	11
10	3
11	4

by inverting to

Position	*Value*
1	4
2	3
3	11
4	8
5	9
6	6
7	10
8	2
9	7
10	5
11	1

by cutting the 11 to the bottom to

Position	*Value*
1	8
2	9
3	6
4	10
5	2
6	7
7	5
8	1
9	4
10	3
11	11

Now what is the rule for obtaining the value of the card at position i? For the starting red stack, the card at position 2^j had the value $6+j$. By **Rule 3'**, we know that after performing the Peirce Pickup with $k = 4$, the card at position i now has the value originally held by the card at position $4i \pmod{11}$. But by **Rule 2**, the card at position $4 \cdot 2^i = 2^{j+2}$ in the red stack has the value $6 + j + 2 = 8 + j \pmod{11}$. This is just the value of the card in position j after completing the Peirce Pickup. Replacing the index $j + 2$ by j, this says that the card in position 2^j in the final red list has the value $6 + j \pmod{11}$ (If we had chosen $k = 3$ for example, then we would have used $3 \cdot 2^j = 2^{j+8}$ since $2^8 \equiv 3 \pmod{11}$).

Now for the final cut of the (long ignored) black deck. Recall, it was (face-down).

Position	*Value*
1	10
2	9
3	7
4	3
5	6
6	1
7	2
8	4
9	8
10	5

and the card in position i has the value 2^{4+i}. Since the card in position 1 in the final red list has the value 8, we want to cut the black deck so that in it, the card in position 8 will have the value 1. This is necessary if the permutation are to be inverses. To do this we simply cut the bottom two cards to the top, thus forming

Position	*Value*
1	8
2	5
3	10
4	9
5	7
6	3
7	6
8	1
9	2
10	4
(11)	(11)

If desired, the discarded black card 11 (the Jack of Spades) can be appended to this list with no harm. In this final black deck, the card in position i now has the value 2^{i+6} since the indices were shifted by 2. Thus, making sure that some pair (here, 1 and 8) are "reciprocal" guarantees that all pairs are reciprocal. Of course, this is the inverse of the rule governing the arrangement in the final red list.

It should be clear how the arguments will go in the general case that the original number of cards n is prime (or a prime power) which has 2 as a primitive root. For this case, the rules for the red and black lists must be modified accordingly. In particular, they are:

Rule 1 (general):The card in position i in the black pile has the value $2^{i+\frac{n-3}{2}}$ (mod n).

Rule 2 (general): The card in position 2^j in the red pile has the value $j + \frac{n+1}{2}$ $\pmod{n}$.

3 Peirce's cyclic principle

The first part of Peirce's trick has cards repeatedly dealt into two piles, the last card switched for one of another color and the two piles combined so the new card goes into the center; the old card is set off to one side. Working with 12 cards it turns out that on each repetition, the last card dealt is always one of the old cards, so all 12 cards are eventually replaced. As there are more and more new cards, it becomes more and more surprising that you don't hit one at the end of the deal.

This section abstracts Peirce's idea and shows how to make a standalone trick from it. Begin with the abstraction. You can replace "deal into two piles" with any repetitive procedure for any number of cards n as long as your procedure first repeats after n repetitions. Let us explain by example.

Example 1 (Down and under). *Take five cards, in order* $1, 2, 3, 4, 5$ *from top down. Hold them face down as if dealing in a card game. Deal the top card onto the table, take off the next card and put it under the current four in hand (put it under). Deal the next card face down onto the card on the table, put the next under, then down, then under, then down, and put the last one down as well. The cards will be in order* $2, 4, 5, 3, 1$. *Instead of switching the 2 for a different color, just turn it face up. You will find this recycles after exactly five repetitions. Further, the last card is always face down (before being turned over).*

Example 2 (Two-pile deal). *Consider next dealing* $2n$ *cards repeatedly into two piles. We will follow Peirce and start with the cards face down, turning them up as they are dealt, say left/right, left/right,.... So the cards start out:*

$$\begin{array}{cc} \vdots & \vdots \\ 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{array}$$

At the end, put the last card on the right pile, but leave it face down. Put the left pile on the right pile and turn all the cards back over so they are all face down (with one face up in the middle). This is one deal. How many deals does it take to recycle? The answer depends on n *in an unknowable way. Indeed, Peirce's two-pile deal is what magicians call an inverse perfect shuffle. Since the original bottom card winds up inside, it is an inverse in-shuffle. The number of repeats, call it* r, *to recycle satisfies* $2^r \equiv 1 \pmod{2n+1}$. *For example, when* $2n = 12$ *as for Peirce, we must find the smallest power of 2 so that* $2^r \equiv 1 \pmod{13}$. *By brute force, the successive powers of* $2 \pmod{13}$ *are* $2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1$. *The first repeat comes after 12 doublings. We say "2 is a primitive root (mod 12)". It is natural to ask if there are other values of* n *so that 2 is a primitive root (mod* $2n+1$*). It is unknown if this happens for infinitely many* n. *It does if the Generalized Riemann Hypothesis holds. The Riemann Hypothesis is perhaps the most famous problem in mathematics (with a* $\$1,000,000$ *prize for its solution). Our simple card trick leads into deep waters.*

Some mathematics

To learn more about perfect shuffles (and down/under shuffles) the reader may consult [13, Chap. 7]. There are many other ways of mixing that repeat after exactly n. These are called "n-cycles" and it is not hard to see that there are $(n-1)!$ of them. However, if you pick a shuffle at random, it's not likely to be an n-cycle. Call the shuffle σ and suppose it first repeats after $r(\sigma)$ iterations. [25] showed that r is about $e^{(\log n)^2/2}$. More formally, for large n,

$$P\left\{\frac{\log r(\sigma) - \frac{(\log n)^2}{2}}{\sqrt{(\log n)^3/3}} \leq x\right\} \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

To say this in English: the log of the repetition number $r(\sigma)$ fluctuates about $(\log n)^2/2$ at scale $\sqrt{(\log n)^3/3}$ with fluctuations following the bell-shaped curve.

Permutations such as σ can easily have $r(\sigma)$ much larger than n. For example, when $n = 52$, the largest order is $180,180$. If you look up permutations in an elementary group theory book, you will find that a permutation can be decomposed into cycles and that $r(\sigma)$ is the least common multiple of the different cycle lengths. This makes it easy to determine $r(\sigma)$. If $g(n)$ is the largest possible order of a permutation of n — $g(n)$ is called Landau's function, with its own Wikipedia page — Landau showed that

$$\lim_{n\to\infty} \frac{\log g(n)}{\sqrt{n \log n}} = 1.$$

Thus, roughly, the largest order is $e^{\sqrt{n \log n}}$ which is much, much larger than $n = e^{\log n}$.

Some magic

That's enough about the mathematics of Peirce's cyclic principle. How can a reasonable trick be formed? Here is a simple illustration, a variation of a trick called "turning the tarot" by Dave Arch, Syzygy **3**, (1996), p. 245.

Example 3 (A tarot trick)**.** *The performer shows seven tarot cards, drawn from the Major Arcana. (If you want to follow along, you can work with any seven face-down playing cards.) Explaining that some people believe that tarot cards can predict the future, the performer asks a spectator to select a lucky number between 1 and 7; say 4 is named. The performer asks the spectator to mix the seven cards and then place cards singly from top to bottom of the packet, turning up the fourth. At this stage, a brief interpretation of the meaning of this card is given: a bit of homework is required to present this effect. The deal is repeated, the spectator putting cards singly from top to bottom and turning up the fourth. The performer continues*

the interpretation and adds, "If, as we go along, we have a repeat of an earlier card, the reading is off. If, by luck or fate, we get all the way through without a repeat, the reading has meaning." As continued, a fresh card is turned up each time. With a little practice, the successive interpretations can be linked together to cohere.

We think it is clear that this simple cycle has been amplified into a solid piece of entertainment. The trick always works because 7 is a prime and any number between 1 and 7 has no common factors with 7 and so will first repeat after seven repetitions.

If you want to do the trick with n cards — for example, the full Major Arcana of the tarot deck has $n = 22$ — the spectator must name a number having no common factor with n. The number of these has a name, $\phi(n)$ or *Euler's phi function*. For example, $\phi(22) = \phi(2) \times \phi(11) = 10$. The chance of success is $\phi(n)/(n-1)$. On average, for a typical n, this is about $6/\pi^2 \doteq 0.61$. In math language we are saying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\phi(n)}{n} = \frac{6}{\pi^2}.$$

To improve your chances, if a spectator names a number having a common factor with the deck size, you can say: "Deal your number of cards singly from top to bottom and turn up the *next* card." The chance that j or $j+1$ have no common factors with n is approximately

$$2 \cdot \frac{6}{\pi^2} - \prod_p \left(1 - \frac{2}{p^2}\right) \doteq 0.87$$

(the product is over all primes p). We are just having fun mixing magic and mathematics. For practical performance, use a prime number for n.

Peirce amplified the cyclic principle, merging it with what we call his dyslexic principle. We explain and abstract this next.

4 Peirce's dyslexic principle

This section unpacks the strange ending of Peirce's card trick. Remember the setting: there are two rows of cards on the table. The *value* of the card in position j in the first row is the *position* of value j in the second row. Here is an example when $n = 5$:

First row	3	5	1	2	4
Second row	3	4	1	5	2

To find the position of value 5 in the second row, look in the fifth position of the first row. The 4 there tells you there is a 5 in position 4 of the second row. This works for any value. It also works with the rows reversed: the 2 in position 5 of the second row tells us that the value of the card in position 2 of the first row is a 5. Switching rows switches *value* and *position*. What's going on?

While Peirce doesn't spell it out, his principle amounts to a classical statement: the two permutations involved are *inverses* of each other. To explain, think of a row of cards as a rule σ for assigning a value $\sigma(i)$ to position i. For the first row in the previous example, this is represented:

i	1	2	3	4	5
$\sigma(i)$	3	5	1	2	4

Here, 1 is assigned $\sigma(1) = 3$ and 5 is assigned $\sigma(5) = 4$. These assignments are called permutations. Now, permutations can be multiplied. If σ and π are permutations, $\pi\sigma(i) = \pi(\sigma(i))$; first do σ and then do π. With σ as given previously, use π defined:

face-down i	1	2	3	4	5
$\pi(i)$	5	4	3	2	1

Then $\pi(\sigma(1)) = 3$ because σ takes 1 to 3 and π fixes 3.

i	1	2	3	4	5
$\pi\sigma$	3	1	5	4	2

Note that this is different than $\sigma\pi$. The *identity* permutation ι fixes everything:

i	1	2	3	4	5
$\iota(i)$	1	2	3	4	5

Two permutations π and σ are *inverses* if $\pi\sigma = \iota$. The reader can check that the two permutations used in our $n = 5$ example are inverses:

σ	3	5	1	2	4
π	3	4	1	5	2

We usually write σ^{-1} for the inverse of σ.

Our finding

Peirce's dyslexic principle is equivalent to the two rows of cards being inverse to one another. Multiplication and inverse are the backbone of modern algebra and group theory. Peirce has let them in the back door as a card trick. We don't know if he did this backward, i.e., "Hmm, let's see, how can I make a card trick out of inverses?" As far as we know, no one before or since has had such a crazy, original idea.

Peirce did much more: he found a completely non-obvious way of arriving at two inverse permutations. We hope that the reader finds the following as surprising as we did.

Consider Peirce's trick. He started with two sets of 12 cards in order A, 2, ..., Q (the red King is not really part of the picture). He repeatedly dealt the first set into two piles, discarding the last and switching it for the current top card of the second set. The discards from the first set were placed successively in a face-up pile on the table. After going through the two piles, he dealt them into two rows on the table. Amazingly,

$$\boxed{\text{the two rows are inverses.}}$$

Peirce's discovery suggests two questions. One mathematical, the other magical. How else can two inverse permutations be formed? How can this inverse relationship be exploited to make a good trick?

There is a charming, simple, surprising way to get inverses. Start with two rows of n cards, in order $1, 2, \ldots, n$ on the table. Suppose the top row is all red and the bottom row is all black. It doesn't matter if all cards are face up or all cards are face down.

INVERSES:
Repeatedly take any red card in the top row, any black card in the bottom row, and switch their positions. Continue until all the red cards are in the bottom row. The two permutations are inverse.

$$\begin{matrix}1&2&3&4&5\\1&2&3&4&5\end{matrix}\longrightarrow\begin{matrix}1&4&3&4&5\\1&2&3&2&5\end{matrix}\longrightarrow\begin{matrix}1&4&1&4&5\\3&2&3&2&5\end{matrix}\longrightarrow\begin{matrix}2&4&1&4&5\\3&1&3&2&5\end{matrix}\longrightarrow\begin{matrix}2&4&1&5&5\\3&1&3&2&4\end{matrix}\longrightarrow\begin{matrix}2&4&1&5&3\\3&1&5&2&4\end{matrix}$$

Please check that the last two permutations are inverses. The spectators can be allowed to choose these random transpositions or the performer can choose them, seemingly haphazardly, to ensure a given permutation in the first row. At least one of your authors still finds it viscerally surprising that any old way of switching the two rows will work.

Marty Isaacs has suggested another way to get two inverse rows: start with two rows of cards, the top row in order $1, 2, ..., n$ face up. The bottom row in the same order, face down. Have the spectator switch two face up cards, say 2 and 7 in the face up row. You now switch the cards at POSITIONS 2 and 7 in the bottom row. this may be continued for as many switches as you like. The two rows will always be inverse permutations. If both rows are face up, at any stage values may be switched in either row with positions switched in the other. Finally, If our original method of producing inverses is used, as above, a few further such switches will keep the rows inverse.

Peirce's original procedure can be abstracted. Here is our main finding.

Our rule

Consider *any* n-cycle σ, e.g., a repetitive procedure on n cards which first recycles after n repetitions. Take two packets of cards, the red packet and the black packet, originally in the same order, $1, 2, 3, \ldots, n$. Repeatedly perform σ to the red packet *but* at a fixed step k, switch the current red card for the current top card of the black packet, setting the red cards face up into a pile at one side. After n steps, the two packets are in inverse orders.

We leave the proofs of both rules to our readers. The "cards in hand" method should prove completely convincing.

Turn next to the magical problem. How can inverses be exploited to make a good trick? Peirce just baldly revealed things by example. To see the problem, consider a possible slam-bang ending: with two piles of cards in inverse permutation order, deal one of the piles in a row face down on the table and the second pile in a row face up on the table. Suppose $n = 10$ and the two rows are as follows:

Face down	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$	$\boxed{5}$	$\boxed{6}$	$\boxed{4}$	$\boxed{9}$	$\boxed{7}$	$\boxed{10}$	$\boxed{8}$
Face up	3	1	2	6	4	5	8	10	7	9

We want to put the face-up cards, starting with the leftmost 3, one at a time, onto their face-down mates in the top row. The two rows determine one another, so it should be easy (or at least possible). Okay, where should the 3 go? Well, the position of the $\boxed{3}$ in the top row is the value of the third card in the bottom row, a 2, so put the face-up 3 on the second card in the first row. Next to be placed is the 1: the position of the 1 in the top row is the value of the first card in the bottom row. Oops, this has already been placed, but we remember it was a 3. So place the 1 in the third place of the top row. A similar problem occurs for placing the 2, the 4, and the 5. Your memory may be better than ours but trying to flawlessly match the cards during a live performance seems foolish. Trouble mounts when the bottom permutation, call it σ, has a drop $\sigma(i) < i$, for then the information has been used up.

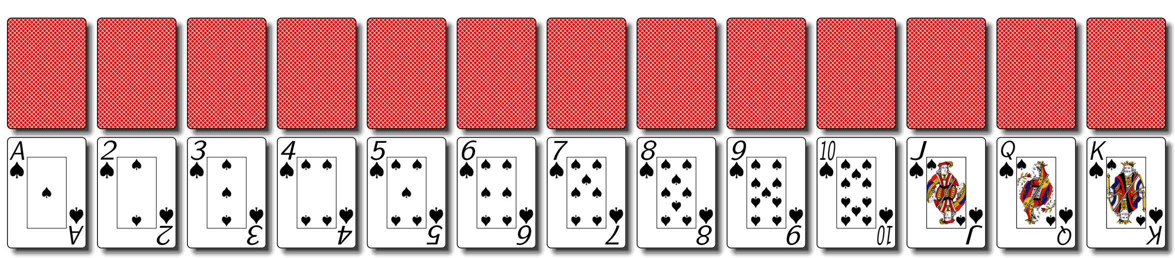
Okay, we can't use it for magic, what about math? Pick a permutation at random: how many drops does it have? This is a well studied problem; indeed, one of us has written several papers on drops [2, 4, 5, 6] because they occur in studying the mathematics of juggling. Permutations with a given number of drops are in bijection with permutations with a given number of descents. (A permutation σ has a descent at i if $\sigma(i+1) < \sigma(i)$, so *3* 1 5 *4* 2 has three descents.) Euler studied descents, and your second author wrote about them [7]. The bottom line is, a typical permutation has about $n/2 \pm \sqrt{n/12}$ drops and the fluctuations follow the bell-shaped curve. This is too many for practical work.

Peirce himself offered an even more convoluted way to reveal the inverse relation. His manuscript contains "A second curiosity", sections 643–645 (or [36]), which is an amazing two-dimensional extension requiring the audience to understand a novel numbering system. It remains to be unpacked.

We offer in what follows our best effort to make a solid performable trick from Peirce's dyslexic principle.

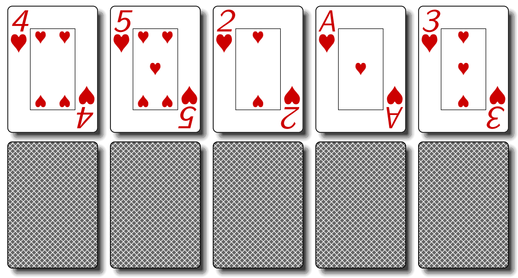
Example 4 (Concentration, without memory)**.** *Concentration — also known as pairs, match-up, pexeso, or shinkei suijaku — is a widely played family card game in which a deck of cards is laid out on the table. Two cards are turned over in each round and the object is to turn up matching pairs, i.e., the two red Aces or the two black Jacks. If the pair turned up matches, you remove them and get a point. If not, they are turned face down again and all players try to remember their positions for future rounds. The following magic trick version makes for good entertainment.*

Effect *Show, say, the Ace through King of Hearts and the Ace through King of Spades, in order. Lay the Hearts face down in a row on the table and the Spades face up in a row below them.*

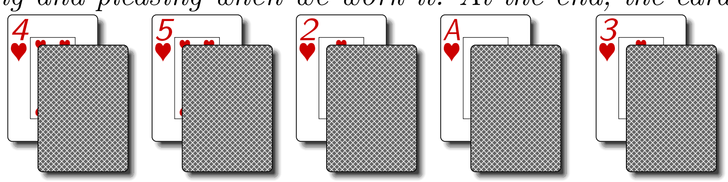


Turn your back on the proceedings and instruct the spectator to take any face-down card in the top row and switch it with any face-up card in the bottom row. This is repeated until the bottom row is all face down and the top row is all face up. In practice, this should be done slowly, repeating the instructions, perhaps with commentary about trying to remember where the cards go. To finish, rapidly pick up the face-down cards, one by one, and place them on *a face-up card. At the end, thirteen perfect matches are displayed–an Ace on an Ace, and so on.*

Method *The trick (sort of) works itself. Of course, it will take a bit of practice to perform it in a rapid, efficient way. Here is how it goes: suppose you are doing it with five cards and when you turn around you see:*



As you go along, you will leave the face-up cards in place and put their matching face-up cards alongside. Where is the face-down 4? Look at position 4 in the top row; it's the Ace. The face-down 4 is in position 1(!). Similarly, the face-down 5 is in position 3, the face-down 2 is in position 5, the face-down ace is in position 4, and the face-down 3 is in position 2. We find it surprising and pleasing when we work it. At the end, the cards appear as:



The face-down cards half-cover their face-up mates; conclude by revealing the perfect matches.

How it works *At the conclusion of the initial swapping phase — switching face-up and face-down pairs — the top and bottom rows make up an inverse pair of permutations. By Peirce's dyslexic principle, the position of the face-down value j is determined by the value of the face-up card at position j. That's it.*

Performance details *Any number n of cards can be used. Perhaps a complete suit is a good balance between difficulty and boredom. Two matching sets of alphabet cards could be used, or pictures of Disney characters, movie stars, abstract symbols, or ESP cards.*

It is best to involve the audience's "concentration" along the way. After the initial switches, turn back to face the audience and ask, "Does anyone know where the Ace is?" Or, pointing to a middle card, "Anyone remember what this card is?"

You don't have to match the cards left to right as described. You can have a spectator point to any face-up card and find its mate as you are questioning him.

We hope that Peirce would approve of our efforts to make a performable trick from his dyslexic principle. We give more developments in the following section.

5 More dyslexia

A "one row" version of Peirce's dyslexic principle can be developed by considering involutions. Work with an even number of cards, say $2n$. Put them face up in order left to right, for example, with $2n = 10$:

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$$

Now pick any pair of distinct cards. Turn them face down and switch their positions. For example, if 3 and 7 are chosen the row appears as:

$$1 \quad 2 \quad \blacksquare \quad 4 \quad 5 \quad 6 \quad \blacksquare \quad 8 \quad 9 \quad 10$$

With the face-down cards being 7, 3 (left to right). Continue, each time picking a random pair of cards, turning them face-down and switching their positions. After n such moves, all the cards are face down and in a seemingly random arrangement. If you perform this as a trick, turn your back and have the spectator make the switches.

You may now turn around and have the cards "magically" sort themselves: here is how it goes. As a presentation gambit, ask "Who knows where the 7 is?" or "Who can remember the position of any card now?" Ask the spectator to turn over the desired card. Say it is the 6 in position 4. Switch the 6 with the face-up card in position 6 and move that face-down card back to position 4; say "We'll just put the 6 in its proper place." Keep going. Have the spectator choose another face-down card (other than the 4th), turn it up and switch the two as before. After n such moves, half the cards are face up and in their correct positions. End the trick by turning over all the remaining face-down cards, revealing that the row is sorted correctly.

This is close enough to the surface that you may just see through it: do you? We are reminded of the old line, "It's not much of a trick but it makes you think–makes you think it's not much of a trick."

The trick doesn't have to be performed with cards. Indeed, what do logarithms, migrated, spheroidal, and multipronged have in common? All are familiar words spelled with all letters distinct. Here is a little trick based on any one of these.

One-handed concentration The performer removes eight letter tiles from a small bag and lays them out on the table. They spell

"I'll turn my back and ask you to make a mess by switching pairs: pick up any two tiles, switch their positions, and turn them face down like this." Say N and I are switched. The row becomes

$$\boxed{A} \quad \blacksquare \quad \boxed{O} \quad \boxed{R} \quad \boxed{E} \quad \boxed{X} \quad \blacksquare \quad \boxed{C}$$

The switched tiles are placed slightly below their original positions. The performer continues: "Jane, please choose two tiles, switch them and put them where they belong, just below. How about you, Patricia?" Keep going until all tiles are face down.

"Was anybody watching? Who remembers what wound up in the first position?" The performer points to the face-down tile in the original A position and has the spectators guess before turning it up. "Let's put it back where it started." Say the E tile is in the first position; switch it with the face-down tile at the original E position. At this stage the row appears as

$$\blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare \quad \boxed{E} \quad \blacksquare \quad \blacksquare \quad \blacksquare$$

The performer moves to the second position, originally the N: "Does anyone remember what's here?" Whatever it is, switch it with the face-down tile now at that letter's original position.

$$\blacksquare \quad \blacksquare \quad \blacksquare \quad \blacksquare \quad \boxed{E} \quad \boxed{X} \quad \blacksquare \quad \blacksquare$$

One more iteration — "Does anyone think they know any tile for sure?" — Someone points out a face-down tile not in the first two positions; say the C shows up. Switch it with the face-down tile at the original C position. At this point, there is one final pair which hasn't been touched. Without making a big deal of it, the performer casually switches these two face-down tiles: "That was hard work. I prefer magic! What was our original word?" The performer snaps his fingers and turns over the tiles in order to spell "A-N-O-R-E-X-I-C" once more.

It isn't a great trick but has a certain charm. You can experiment with the length of the word used; it's more surprising to get more letters right but takes additional "clowning around time" to complete. An ambitious performer might be able to use

T H E Q U I C K B R O W N F O X J U M P E D O V E R T H E L A Z Y D O G.

Of course, you could do it with a row of cards, originally in order A 1 2 3 4 5 6 7 8 9 10, having four cards turned up in the original memory test and switching the last two. With cards, an opportunity to do a little manipulation presents itself. After four have been placed face up during the memory phase, gather the remaining six, managing to switch the two that need to be switched as you do so. Give the face-down packet of six a "false shuffle" and deal them back face down into the missing places. Finish as before. Does this help the mystery? Maybe; you have to decide. Instead of a false shuffle, you can mix the six by dealing them under and down and then replacing them in the correct order (a bit of study and practice being required for this).

As ever, there are further possibilities: there *can* be repeated letters. Anagrams such as "calipers/replica" or "discounter/introduces/reductions" suggest a trick which starts out with one word and ends up with another. To go back to the start of this section, you could use "dyslexic".

As advertised, it is a one-row version of Peirce's dyslexic principle. Call the final permutation of $2n$ cards, when all are face down after first switching n pairs, π. So $\pi(1)$ is the card at position 1 and so on. The procedure forces $\pi\pi =$ identity; applying π twice winds up leaving every card in its original place. We write $\pi = \pi^{-1}$ or have $\pi(i) = j$ if and only if $\pi(j) = i$. These are the *fixed point free involutions*. We don't want to keep writing all that, so let's call them "big involutions" for this section.

How many big involutions are there? The answer is neat,

$$(2n-1)(2n-3)\dots 5 \cdot 3 \cdot 1;$$

we write $(2n-1)!!$ for this "taking every other term" (the so-called *skip factorial*). When $2n = 4$, $(2n-1)!! = 3$. The three involutions are:

2 1 4 3   3 4 1 2   4 3 2 1

When $2n = 6$, $(2n-1)!! = 5 \cdot 3 = 15$. It is instructive to write these all out (at least once in one's life).

Whenever we meet a new object we ask, "What does a typical one look like?" Okay, what does a typical big involution "look like"? This innocent question leads to complex destinations. To keep things civil, we will not explore this subject in detail, but just get things started.

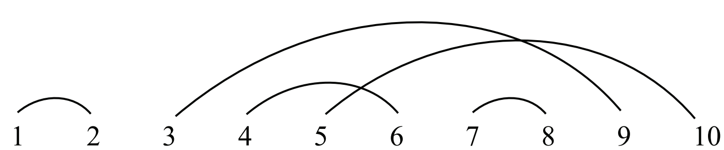
Descents

One obvious feature of a permutation is its up/down pattern. A permutation π has a descent at i if $\pi(i+1) < \pi(i)$. Thus, for $2n = 10$ the involution *2* 1 *9* 6 *10* 4 *8* *7* *3* 5 has six descents as indicated. Let $D(\sigma)$ be the number of descents. A big involution must have at least one descent, and this can occur, e.g., 4 5 6 1 2 3 has $D(\sigma) = 1$. As many as $2n-1$ can occur, e.g., 6 5 4 3 2 1 has $D(\sigma) = 5$. We may now ask: "Pick a big involution at random; how many descents does it typically have, and how are they distributed?" Does a typical σ have more or less descents on average than a typical permutation without any restrictions? A random permutation has $(2n-1)/2$ descents on average. It is easy to see that for involutions the average is larger,

$$\frac{2n-1}{2} + \frac{2n-1}{2n-2}$$

(so only about 1 larger). We won't dig in and do the work here but it can be shown that a random big involution has about n descents with fluctuations of order $\sqrt{n}$ and, normalized by its mean and standard deviation, the fluctuations follow the bell-shaped curve. This is almost exactly the same as what happens for a random permutation. A careful proof and extension to the distribution of descents in other conjugacy classes will be found in the (unfinished) thesis of USC's Gene Kim.

The reader may discover that there is an extensive enumerative literature on descents in permutations, going back to Euler. Entry to this body of work can be found in [44, 8, 43, 47]. There is some parallel development for random big involutions in [37]. To open this door, think of a big involution as a "perfect matching" of $2n$ things. These can be diagrammed by drawing an arc between each switched pair. Thus, $\pi = 2\ 1\ 9\ 6\ 10\ 4\ 8\ 7\ 3\ 5$ appears as:



This representation suggests a host of questions, such as, how many arcs cross? The crossing number in the previous example is $c(\pi) = 2$. The answers are sometimes lovely. For example, the number of noncrossing matchings, $c(\pi) = 0$, is the Catalan number $\binom{2n}{n}/(n+1)$. For a look at what is known, and pointers to an extensive literature, see [3, 14, 15] and [27]. More probabilistic results are in [28].

Perhaps the deepest result about random matchings is the Baik–Rains theorem [1] to determine the limiting distribution of the length of the longest increasing subsequence, $l(\pi)$. Returning to the previous sequence $l(\mathit{2}\ 1\ 9\ \mathit{6}\ 10\ 4\ \mathit{8}\ 7\ 3\ 5) = 3$. They show that $l(\pi)$ is about $2\sqrt{2n}$ with fluctuations of size $(2n)^{1/6}$ having remarkable Tracy–Widom distributions. See [44] for a detailed overview of these notable results.

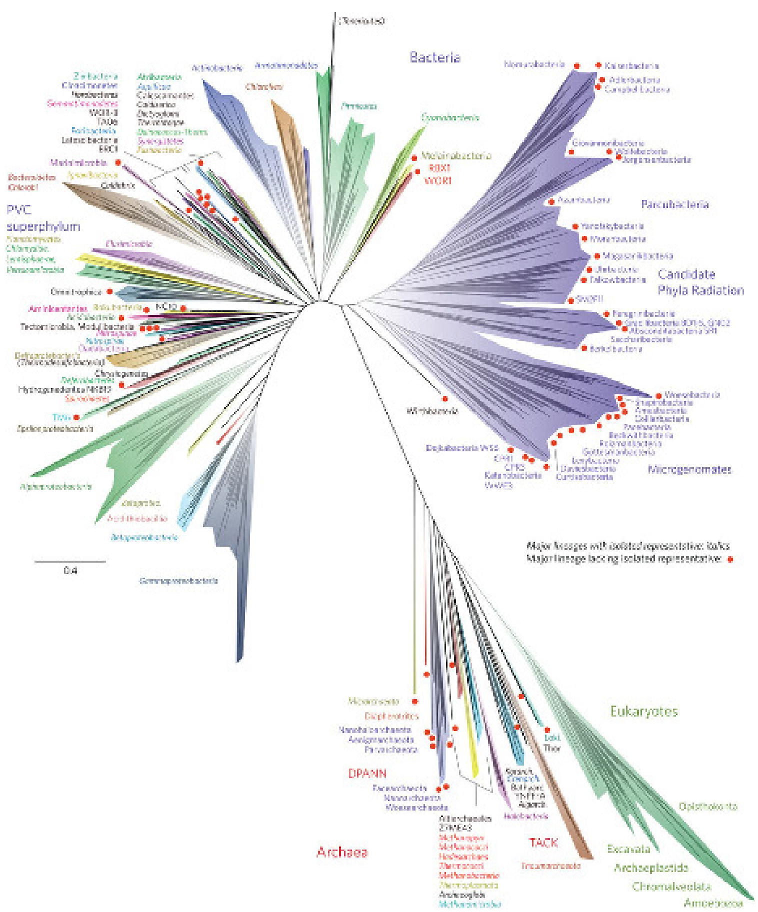


Figure 1: The new Tree of Life, depicted at "Understanding Evolution", a web site of the University of California Museum of Paleontology (`evolution.berkeley.edu/evolibrary/news/160505_treeoflife`), 29 September 2017.

To conclude, there is another coding of big involutions widely used in biology and linguistics. Figure 1 depicts one entry in the world of phylogenetic trees [26]. These are leaf-labeled "family trees". The three involutions on four objects correspond as

$$(1\ 2)(3\ 4) \qquad (1\ 3)(2\ 4) \qquad (1\ 4)(2\ 3)$$

There are $(2n-1)!!$ such trees with n leaves. The bijection with matchings is useful computationally [9, 10]. One can also ask questions about the shape of the tree associated with π [40]. Our bottom line for this section is that both the math and magic are interesting.

6 Peirce's packet pickup

Magic tricks devoted to dealing cards into piles and assembling the piles in various orders have been performed for at least 400 years. Charles Peirce introduced a truly new principle in this domain. He used his principle both as part of card tricks and as a way of teaching children basic arithmetic.

All of Peirce's dealing was done with the cards face down in the hand to start (say in order $1, 2, 3, \ldots, n$ from the top down). The cards are dealt *face up* into k piles. The piles are picked up, in a careful order described in a moment, and then turned back face down. For example, ten cards dealt into five piles look like:

$$\begin{array}{ccccc} 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

Suppose the piles are picked up left to right and all are turned back face down. The final order is 5, 10, 4, 9, 3, 8, 2, 7, 1, 6. If ten cards are dealt into two piles and assembled left to right, the order becomes 2, 4, 6, 8, 10, 1, 3, 5, 7, 9. To see how this can be used, consider 25 cards dealt into five piles and assembled in this way. If this action is repeated, you will find the cards back in their original order.

Here comes Peirce's contribution. In all of the previous examples, the total number of piles is a divisor of the total number of cards, so things worked out neatly. Peirce figured out a way to have things work out neatly with any number n of cards and any number k of piles where k is relatively prime to n. As an illustration, suppose 13 cards are dealt into five piles. They appear as:

$$\begin{array}{ccccc} 11 & 12 & 13 & & \\ 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

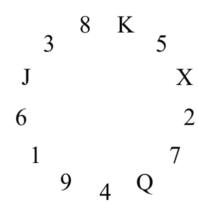
Peirce's rule says: pick up the pile on which the last card is dealt. Say this is j from the left. Drop it on the pile j to the right (going around the corner if needed); $j = 3$ in the arrangement above. Drop these two piles on the pile j to the right, and continue until there is only one big pile. It is important to count empty spaces as you go around. Thus, if the piles in the example are in places 1, 2, 3, 4, 5, you drop pile 3 on pile 1, both on pile 4, this packet on pile 2, and finally all on pile 5.

What's neat about it? Peirce's pickup results in the card in position i being of value ki (mod n). In the illustration above, $n = 13$, $k = 5$, and the final arrangement is 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8, 13. That is, the top card (position 1) is 5 ($= 5 \times 1$). The next card is 10 ($= 5 \times 2$). The next is 2 because $= 5 \times 3 = 15 = 2 \pmod{13}$ and so on. There are all kinds of bells and whistles that can be added to this: starting the pickup with any pile, using the minimum of j and $k - j$, and others. Some of this may be found in [17, Chap. 4] or [21, 22] but it is also fun to try your own variations. We have recorded many of ours in Section 7.

It is hard to know the exact history. Peirce said he developed the trick around 1860 and he certainly used his pickup in an early unpublished manuscript, "Familiar letters about the art of reasoning" (1890), reprinted in [?]. This was part of a many-sided effort that Peirce made to write a popular school arithmetic book. It was one of dozens of failed schemes to earn a living. He did receive a number of advances and produced a large number of partial chapters. Peirce used cyclic arithmetic as a way of teaching addition (mod n) and introduced his dealing procedure as a way of teaching multiplication (mod n). In "Some amazing mazes" he analyzed repeated deals into different numbers of piles and gave a way of doing one final deal to return the deck to the original order. He also combined his deals with primitive roots to make a magic trick.

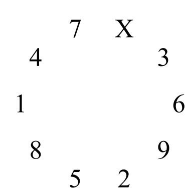
We give here a simpler magic trick, introduced in his "Familiar letters", which has not been presented before.

Example 5 (A simple Peirce trick)**.** *For illustration, work (mod 13) with cards labeled A, 2, 3, ..., 10, J, Q, K. Deal into either five or eight piles and perform a Peirce pickup. Then deal the cards face down around a circle as:*



Ask a spectator, "What card would you like me to find?" Suppose she says, "The Jack (= 11)*." Starting from the King, count 11 clockwise to wind up at 3; turn it up. Then move counterclockwise three positions starting from the King and show the requested Jack. In general, to find a count forward a from the King, turn up b, then count b counterclockwise from the King and turn up a. Briefly, if* $\pi(i)$ *is the label at position* i*, with the King* $K = 0$*, then* $\pi(i)+\pi(-i) = 0 \pmod{13}$*. Hence,* $5+8 = 3+10 = J+2 = 6+7 = 1+Q = 9+4 = 13$.

Peirce gives variations; ten cards dealt into three piles and properly picked up twice results in:



This is reminiscent of Peirce's dyslexic principle, with a single pile of cards. We have given our best tricks with the principle in Sections 4 and 5. Peirce used card tricks many times in his efforts to liven up the teaching of arithmetic. It is surely worth looking further at his work.

In the next section we present an original development of Peirce's pickup, showing how a quite nice magic trick can be based on his idea.

7 A performable Peirce trick

Peirce discovered a remarkable principle whereby after any number of shuffles, one further shuffle brings the deck back to its original order. His shuffles involve dealing a deck of cards into any number of piles and picking up the piles. He discovered that special pickup sequences permit analysis; this section presents an analysis of his shuffle. We begin with an example that makes a fine performable trick, and then explain how the general principle works. As usual, the example is designed so that the reader can follow along with cards in hand. We urge you to go get a deck.

Together again

The performer removes the 13 Spades from a regular deck of cards and arranges them in order from ace at the top through King at the bottom. “This is a story about the trials of couples in modern times.” The performer removes the King and Queen, turns them face up and places them onto the rest of the face-down packet.

“The King and Queen represent an ordinary couple, trying to get through life. Of course, life has its ups and downs. Would you cut the cards and complete the cut?” The packet of cards is spread face down. The King and Queen are face up together. “Even though no one knew where they would wind up, at least they were together. Now here comes trouble. Would you, sir, give me any woman’s first name?”

Whatever name is suggested, the cards are dealt into piles one at a time, from left to right as in a card game, into the number of piles equal to the number of letters in the name. For the example “Sue”, three piles are dealt. Say the letters out loud as you deal, S-U-E, S-U-E, etc. As you deal, the face-up King and Queen will be dealt into separate piles. “When Sue came into their lives, they separated.”

Next, the piles must be picked up in a special order. The general rule will be given later. If you are following along — and you should be! — the piles are called 1, 2, 3 from left to right and pile 3 should be placed on pile 1, with pile 2 on top of these.

Now point to a woman and ask her to provide a man’s name: say it’s “Barry”. Deal the cards into five piles, one for each letter of the name. Pick up by placing pile 3 on pile 1, pile 5 on these, 2 on these, and 4 on top of all, These deals can be continued. You can ask how many children a couple has and deal that number of piles, or simply ask for a small number. To get to the finish, we will stop here.

To finish, a deal must be made which brings the King and Queen together. The number of piles will depend on the configuration but is easy to find. Spread the cards and count how many are between the King and Queen. In our example it will be five or six depending on the initial cut. Add 1 to this number and deal that many piles. If you like, you can name an appropriate word out loud, e.g., “divorce” for a deal into seven piles. In this case, your final deal will be into six or seven piles. If six, pick up pile 1, place it on 2, the whole on 3, then on 4, 5, and 6 in turn. If you deal seven piles, pick up pile 7, place it on 6, the whole on 5, then 4, then 3, then 2 and 1.

In either case, your final pickup and deal will have brought the King pair together. Spread the cards and cut the face-up pair to the top. The final patter goes as follows: “These trials and tribulations have brought the couple back together. What about the rest of their world, how did they fare through all this chaos?" Here, the rest of the packet is spread face up and shown to be in perfect order, from Ace through Jack.

Some analysis

To explain how the trick works, and how it can be adapted to other deck sizes, an excursion into cyclic arithmetic is necessary. We begin by explaining a general rule for picking up piles which makes further analysis simple. This is our version of Peirce's rule (he dealt cards face up). Our development benefits from contributions of Bob Page, a California card man.

Peirce's pickup Let n cards be dealt into p piles. The deal is left to right as in a card game. Hence, if 13 cards numbered from 0 to 12 are dealt into five piles they would wind up as:

$$\begin{array}{ccccc} 10 & 11 & 12 & & \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 & 4 \end{array}$$

Suppose that the number of cards is not evenly divisible by the number of piles. Then there will be some piles that are "short". Call the number of short piles s. In the example here, $s = 2$. The piles are gathered by choosing a base pile, and consecutively picking up piles at distance s, moving clockwise around a circle, and placing on the base pile.

Suppose 13 cards are dealt into five piles. If the five piles are numbered 1 through 5 from left to right and the leftmost pile is chosen as base, the pickup proceeds by placing pile 3 on pile 1 — remember there are $s = 2$ short piles — then pile 5 on these, pile 2 next, and finally pile 4 on top. The deck will be in final arrangement:

$$\begin{array}{ccccccccccccc} 8 & 3 & 11 & 6 & 1 & 9 & 4 & 12 & 7 & 2 & 10 & 5 & 0 \\ \text{top} & & & & & & & & & & & & \text{bottom} \end{array}$$

Starting with a different base pile merely results in a cyclic shift. Note that the arrangement has consecutive cards differing by 5 (mod 13). The next theorem shows how this works in general.

Theorem 1. *If n cards are dealt into p piles, with n and p having no common factors, and the piles are picked up by putting every sth pile on the first pile where s is the remainder when n is divided by p, then the card originally at position j winds up at position*

$$\frac{-j}{p} - 1 \pmod{n}.$$

To explain the notation, consider what $-1/p \pmod{n}$ means. This is a number such that

$$p \cdot \left(\frac{-1}{p}\right) \equiv -1.$$

It is a basic fact of elementary number theory that any p larger than 0 and less than n has such an inverse. For example, if n is 13 and p is 5, $-1/p = 5$ because $5 \times 5 = 25 = 26 - 1 \equiv -1$

Table 1: Values of $-1/p \pmod{13}$ when $n = 13$.

p	1	2	3	4	5	6	7	8	9	10	11	12
$-1/p$	12	6	4	3	5	2	11	8	10	9	7	1

(mod 13). The inverses of all possible values of p when $n = 13$ are given in 1. There is no simple formula for $-1/p$. It can be found efficiently by using the Euclidean algorithm, but for small n a bit of trial and error usually suffices.

Going back to the example: 13 cards numbered from 0 to 12 are dealt into five piles and assembled via the Peirce pickup into one pile with card 0 at the bottom. The cards are in order

$$8 \;\; 3 \;\; 11 \;\; 6 \;\; 1 \;\; 9 \;\; 4 \;\; 12 \;\; 7 \;\; 2 \;\; 10 \;\; 5 \;\; 0\,.$$

The theorem says card j goes to $5j - 1 \pmod{13}$ because $-1/5 = 5$. Thus card in position 0 winds up at $5 \times 0 - 1 = -1 = 12 \pmod{13}$. The card at position 1 winds up in position $4 = 5 \times 1 - 1$ (remember the top card is in position 0). The card in position 8 winds up in $5 \times 8 - 1 = 39 = 0 \pmod{13}$.

The theorem has corollaries that yield card tricks. The first explains the trick, showing how many piles the final deal must be to bring back the original order.

Corollary 2. *If n cards are dealt into p_1, then $p_2, \ldots$, then p_k piles, the final position of a card at position j is*

$$-1 + \frac{1}{p_k} - \frac{1}{p_{k-1}p_k} + \cdots + \frac{(-1)^k}{p_2 \cdots p_{k-1}p_k} + \frac{(-1)^k j}{p_1 p_2 \cdots p_{k-1}p_k} \pmod{n}.$$

If one final deal into

$$\frac{(-1)^{k+1}}{p_1 p_2 \cdots p_k} \pmod{n}$$

piles is made, the deck will return to its original order.

Example 6. *In the trick used to illustrate this section, deals into three and then five piles were made. Now $3 \times 5 = 15 = 2 \pmod{13}$, and $-1/2 = 6$ from 1.*

It is perfectly possible to do the calculation required to bring the trick to a successful conclusion in your head. This allows the trick to be worked with all cards face down. The face-up cards allow the determination without calculation. Some further ideas are contained in the practical suggestions at the end of this section.

The next corollary abstracts Peirce's original discovery. It is followed by two tricks using the structure captured in the corollary.

Corollary 3. *Suppose that $n = a \cdot b$. If n and p have no common factors and p piles are dealt and picked up according to Peirce's pickup, then cards at position $j, j + a, j + 2a, \ldots$ are permuted among each other, for each $j = 0, 1, 2, \ldots, b - 1$.*

To explain, consider a 12-card deck made up of 4 clubs, 4 Hearts, and 4 Spades. Suppose the original order is

$$\clubsuit \, \heartsuit \, \spadesuit \, \clubsuit \, \heartsuit \, \spadesuit \, \clubsuit \, \heartsuit \, \spadesuit \, \clubsuit \, \heartsuit \, \spadesuit \, .$$

If the cards are dealt into five (or seven) piles and picked up properly, every fourth card will be of the same suit. Twelve can also be represented as 2×6, and this order would be preserved as well. Thus, if the cards are arranged in suit order as above, or alternate as picture-spot-picture-spot, etc., the order is still preserved.

One way to capitalize on Peirce's discovery is to use numbers like 25 or 49, squares of primes, which can be dealt into any number of piles other than the primes themselves and still preserve the order. In this form, the principle was discovered by the English card man, Roy Walton. His trick was published in the 1967 volume of the privately published magic journal, *Pallbearer's Review*. The Walton trick, along with our own variation, is a fine example of how the ideas of this section can be made into magic.

Draw poker plus

In this trick a deck of 25 cards is repeatedly mixed by dealing into piles of varying sizes. At the finish, five hands of poker are dealt and it is discovered that every hand is a pat hand, e.g., a straight, flush, full house, four-of-a-kind, or straight flush.

The working is more or less automatic. Remove any five pat poker hands and arrange them five apart: a 25-card deck is stacked 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, . . . , where 1 represents any card from the first hand, etc. The deck can be freely cut. Walton allowed the pack to be dealt into a spectator's choice of two, three, four, or six piles. The piles are assembled by picking up the one at the leftmost position and placing it on top of the next pile to the left, placing these two packets on the next pile to the left, and so on until one final packet has been formed. Walton's discovery is that every fifth card will still be from the same poker hand. The pack can be cut and dealt repeatedly; whenever you wish, deal five hands. Each will contain a pat poker hand.

To connect with Corollary 3, observe that when 25 cards are dealt into two, three, four, or six piles, the first pile has an extra card and the remaining piles are short. The Peirce pickup agrees with Walton's pickup so Corollary 3 is in force. Of course, we now know how to preserve things if seven, eight, nine, . . . piles are used.

Over the years, we have developed a different use of Walton's idea. This is a nice example where less is more. Let's all do the trick that follows next.

Draw poker minus

In effect, a packet of cards is dealt into piles to mix them. Finally, they are dealt into five hands. The performer explains that of course one of the hands must be best, but which one? He lets the spectator freely choose any of the hands. The others are shown and while some fairly good poker hands turn up, inevitably the hand chosen beats them all.

This trick begins as in *draw poker plus* explained previously. Five pat poker hands are removed and stacked every fifth. Have the spectator cut. Spread the cards face up, casually asking that a subliminal impression be formed. Ask for a small number of piles: if two, three, four, or six, deal and pick up, asking the spectator to observe carefully to ensure proper cardtable procedure is used. Have the cards cut. Ask for another number of piles, deal, and pick up. Say that you hope that the spectator is subconsciously following along. Continue until the number of piles requested is five, and deal into those five piles.

Point out that one of the hands wins (most probably). Ask the spectator to place his hand on one. This is a good time to build up the tension. Ask if he feels sure about his choice or wants to change his mind. Finally, the unchosen four hands are formed into a pile with patter like: "Do you want to trade for this hand? No? How about this one?", etc. The 20 remaining cards are in a pile on the table. Pick them up by removing the top card and using it as a scoop to lift the others. This shifts one card from the top to the bottom. This should be done surreptitiously, with a distracting remark such as, "Have I come near the cards you chose, or interfered with your selection in any way?"

Now fan off the top five cards of the 20-card packet. This will no longer be a pat hand, though it may be a pretty good hand. Continue to fan off packets of five cards to find the best hand among the four in the pile. Remind the spectator that the cards were cut freely and mixed at his discretion. Ask that the selected hand be turned up one card at a time and make the most of it.

Some practical details The hands chosen may be flushes, straights, or full houses. Avoid four-of-a-kinds since there is a small chance (1 in 5) that transferring the card from top to bottom won't break the hand.

Instead of openly transferring the top card, those who are adept can use slight-of-hand. For example, if the first card dealt off the 20-card packet is dealt from the bottom, all will still go well.

There are many variants possible. Craig Snyder has suggested using 25 tarot cards. These can be repeatedly cut and dealt, and this can be made to look like part of a fortune-telling ritual. A final deal into five piles is made and a hand selected. This will be one of the five pat hands originally set. With a bit of thought it is easy to decide which hand — if all else fails you can try marking the cards — and you can have five prepared "readings" (preferably inscribed on parchment) put into five different pockets. Remove the appropriate reading and go to town drawing out the drama, turn up the five chosen cards and reveal their interpretation.

Another potential use involves an ESP deck. This consists of 25 cards depicting symbols such as $\bigcirc$, $+$, $\int\!\!\int\!\!\int$, $\square$, $*$, each repeated five times. These can be arranged so that every fifth card is the same design. Then, when five hands are dealt and the spectator selects one, it consists of just one symbol throughout. Again, with a bit of preparation you can determine which symbol the hand contains. Turn your back, have one card freely selected from the chosen hand and concentrated upon. Reveal it in a dramatic manner.

Alternatively, have several spectators select a hand and then a card within that hand, and read their minds all at once. It is also possible to arrange the cards initially so that every fifth card is different. Then the final chosen hand of five cards has no repeated values; this seems less easy to turn into a miracle.

Our next section returns to Peirce's last novel magical principle.

8 Peirce's primitive principle

Suppose that the number of cards p is a prime, like 11 or 13. A basic fact of number theory is that there are some numbers a so that $1, a, a^2, \ldots, a^{p-1}$ are all distinct (mod p). For example, when $p = 11$, $a = 2$ is a primitive root:

$$\begin{array}{c|cccccccccc} j & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ a^j & 1 & 2 & 4 & 8 & 5 & 10 & 9 & 7 & 3 & 6 \end{array}$$

A classical theorem of Fermat (Fermat's little theorem) says for any nonzero a, $a^{p-1} \equiv 1$ (mod p), so this is as far as it goes. When $p = 11$, $a = 3$ is *not* a primitive root: the successive powers of 3 mod 11 are 1, 3, 9, 5, 4. There are $\phi(p-1)$ distinct primitive roots mod p, where ϕ is Euler's ϕ function, the number of a relatively prime to $p-1$ (so $\phi(10) = 5$), but there is no "formula" for finding one. Indeed, it is unknown if 2 is a primitive root for infinitely many primes; see the Wikipedia entry for the Artin conjecture. Peirce gave his own proof of the existence of primitive roots as part of his "Some amazing mazes" card routine. He also gave a card-dealing proof of Fermat's little theorem.

If a is a primitive root mod p then p cards dealt into a piles, and picked up à la Peirce, will recycle after exactly $p-1$ repetitions. This occurs at the start of Peirce's trick with $p = 13$, $a = 2$. However, Peirce used primitive roots in a much more subtle way. Suppose the $p-1$ values $1, 2 \ldots, p-1$ are arranged with a^j in position j ($j = 0$ being the first position) through $p-2$. When $p = 13$ and $a = 2$, the values are in order

$$1 \;\; 2 \;\; 4 \;\; 8 \;\; 3 \;\; 6 \;\; \mathrm{Q} \;\; \mathrm{J} \;\; 9 \;\; 5 \;\; 10 \;\; 7\,.$$

Now, deal into b piles and pick up à la Peirce. This takes the card originally at position j to position bj. Suppose that $b = a^k$ and the cards are in order $a^0, a^1, a^2, \ldots$. After the deal, they are in order $a^k, a^{k+1}, a^{k+2}, \ldots, a^{k+p-1}$. The *exponents* behave in a simple way, all shifting by k. Dealing again, say into c piles with $c = a^l$, results in an additional shift by l. If after a series of deals the total shift is by m, one additional deal into d piles, with $d = a^{-m}$, will bring the cards back to order.

Peirce used this in an ingenious way. He worked with two piles, the Hearts in order $1, a, a^2, \ldots,$ and the Spades in *inverse* order. If after any number of deals of the Hearts, the final arrangement consists of a shift by m (in the exponent), merely cutting m cards from the bottom of the Spades pile to the top results in the two orderings being inverses. We call this use of primitive roots, in connection with inverses, *Peirce's primitive principle.*

The procedure of going between $b = a^j$ and j back to b is a number theoretic version of logarithms. Turning multiplication into addition has well known benefits. In recent years, it forms the basis of a raft of cryptographic schemes [42]. It is easy, given a and p, to find a^j. It seems difficult to go backward: given a^j (mod p), find j. This is the problem of finding logarithms in finite fields and extensive, very sophisticated theoretical work has been developed to replace simple trial and error. The best algorithms still require $2^{p^{1/3}}$ steps, insuring the security of the associated crypto schemes.

The use of primitive roots to perform computations (and prove theorems) in number theory is classical, going back to Euler. Lists of primitive roots for small primes were compiled and tables for going from a^j to j were published. We think that Peirce wanted to make a card trick that would illustrate these ideas, going well beyond the mechanism of almost all other magic tricks. The primitive element theorem underlies our tricks with de Bruijn sequences, and we recommend [13, Chap. 2–4] for further developments.

9 Summing up

Peirce's magical contribution has been broken into pieces:

- the cyclic principle
- the dealing principle
- the dyslexic principle
- the primitive principle

Each of these has depth and absolute originality. We have tried to show how these can be broken off, modified, extended, and adapted to make solid, entertaining magic tricks. Peirce *combined* them. This section tries to show that the whole is more than (and less than) the sum of its parts.

Let's first dispose of "less than". The bottom line is that Peirce's "Some amazing mazes" is a poor, essentially unperformable trick. Martin Gardner called it "...surely the most complicated and fantastic card trick ever invented."

Now for "more than". Peirce managed to *combine* his principles into one seamless whole. He begins with two packets in standard order. After repeated dealing — his cyclic principle, which has some mild entertainment built in — the two packets are in inverse order, his dyslexic principle. But they are not in just any order: Peirce has arranged that they are in "primitive root order". This is because 2 is a primitive root mod 13 and the last card dealt has double the value of its position; this is his primitive principle. Next, Peirce mixes the cards by repeatedly dealing into spectator-chosen numbers of piles. He picks up carefully, his dealing principle. Finally, by making a single simple cut in the other packet — his primitive principle fully applied — he again has the two packets in inverse order. The finale is a display of the dyslexic principle. It is astounding to layer these concepts in such a fluid fashion.

Let us go back to a much larger picture: the impossible task of showing how his magic illuminates his huge body of work. Of course, this can be dangerous business. Consider evaluating Linus Pauling, who won two Nobel Prizes, on the basis of his strange, late-in-life fixation with vitamin C. We knew Pauling then and many of his peers regarded him as a crackpot.

We feel Peirce's magic work is enlightening. At a distance, and even after a first, second, and third look, his card trick is an unperformable nightmare: tedious dealing that goes on and on coupled with a weak, ungraspable effect that is sure to leave the audience confused. But Gardner also expressed its underlying value: "I cannot recommend it for entertaining friends unless they have a passion for number theory, but for a teacher who wants to 'motivate' student interest in congruence arithmetic, it is superb."

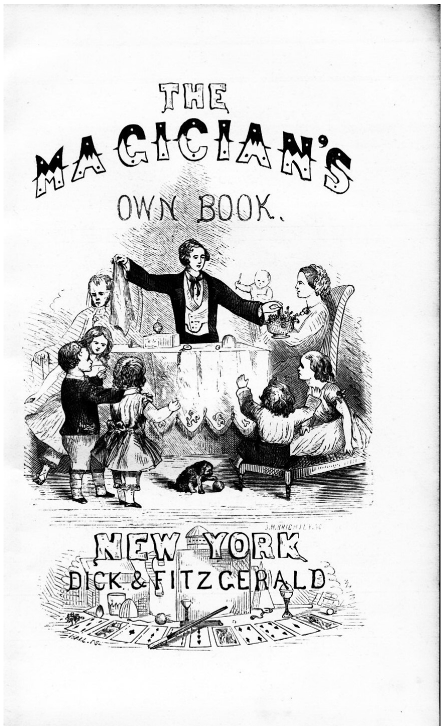
Writing now in 2017, we have been trying to unpack Peirce over a 60-year period, having first come across his work in 1957. Long years of study have revealed fascinating new ideas there, completely original and with every possibility of generating marvelous new magic tricks. It is at least plausible that Peirce's work in *any* of the myriad subjects on which he wrote has similar complexity and uniqueness, and innumerable scholars have taken up the challenge of this research. In our survey of the Peirce literature, we find it has developed its own language and themes. We suggest a return to the original Peirce corpus; take an essay and try to follow along and make your own sense of it.

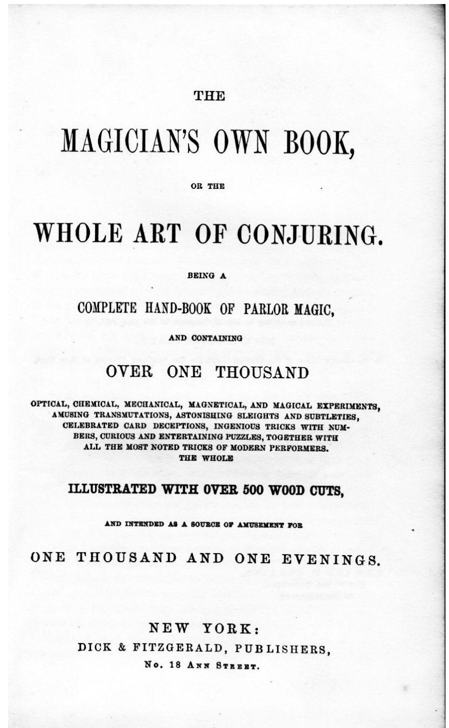
A A Peirce magic letter

We don't know much about Peirce's interactions with magic. Introducing "Some amazing mazes" he wrote:

> About 1860 I cooked up a melange of effects of most of the elementary principles of cyclic arithmetic; and ever since at the end of some evening's card play, I have occasionally exhibited it in the form of a "trick" (though there is really no trick about the phenomenon) with the uniform result of interesting and surprising all the company, albeit their mathematical powers have ranged from a bare sufficiency for an altruistic tolerance of cards up to those of some of the mightiest mathematicians of the age, who assuredly with a little reflection could have unraveled the marvel.

Peirce (1839–1914) was 21 in 1860. At about this time there were regular magic performances in Boston theaters, attracting performers such as John Henry Anderson, Signor Blitz, and Andrew MacAllister. (Much more detail can be found in [29].) One could find magic books in libraries and printshops. These were often compendia of sports and games with some magic thrown in. Titles such as *The Boy's Own Book of Sports and Games* (1859), *The Modern Cabinet of Arts* (1846, 1856) by Thornton, *Wyman's Book of Magic* (1851), and *The Whole Art of Legerdemain* (1830, 1852) were common.




THE

MAGICIAN'S OWN BOOK,

OR THE

WHOLE ART OF CONJURING.

BEING A

COMPLETE HAND-BOOK OF PARLOR MAGIC,

AND CONTAINING

OVER ONE THOUSAND

OPTICAL, CHEMICAL, MECHANICAL, MAGNETICAL, AND MAGICAL EXPERIMENTS, AMUSING TRANSMUTATIONS, ASTONISHING SLEIGHTS AND SUBTLETIES, CELEBRATED CARD DECEPTIONS, INGENIOUS TRICKS WITH NUMBERS, CURIOUS AND ENTERTAINING PUZZLES, TOGETHER WITH ALL THE MOST NOTED TRICKS OF MODERN PERFORMERS. THE WHOLE

ILLUSTRATED WITH OVER 500 WOOD CUTS,

AND INTENDED AS A SOURCE OF AMUSEMENT FOR

ONE THOUSAND AND ONE EVENINGS.

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One of the better books, *The Magician's Own Book* (1857), incorporated recent American tricks. There are some self-working tricks of the type Peirce favored. These include "The pairs repaired" (*Mutus nomen dedit cocis* or the Latin card trick), "The 21 card trick", "The clock trick", and tricks with a pre-arranged pack. All of these are pretty tame compared to what Peirce was cooking up. Many mathematical card tricks go back hundreds of years — see [16] for a wonderful history — but Peirce's inventions are much, much deeper.

By remarkable good luck, a long letter from Peirce to his friend Chauncey Wright in September of 1865 in which he described card tricks has recently turned up [31]. Wright was one of Peirce's gurus, a central figure in the birth of Cantabrigian-American philosophy alongside Peirce, William James, and Oliver Wendell Holmes. Fascinating details are in [39], including their shared interest in card tricks.

After apologizing for not returning a borrowed book, Peirce commences:

> I have invented a little trick at cards. Take a pack containing a multiple of four cards, an equal number of each suit. Arrange them in regular order in their suits. Milk the pack three times. That is, take alternate cards from the face and back of the pack and put them in a pile on the table, back up, until the cards are exhausted. Do this three times. Then, holding the cards back up, count out four into the other hand so as to reverse their order. Then count four *underneath* these so as *not* to reverse their order. Then four *above*, so as to reverse their order again and so on till the pack is exhausted. Then turn the pack over and deal out in four hands.

This combination of milk and over/under shuffles is original. The careful study by Monge ("Reflexions sur un tour de cartes", 1773) concerning over/under shuffles is here being extended to encompass dealing off packets and reversing some of the cards along the way.

Peirce begins some mathematical analysis for milk shuffles: where does card j end up, and how many repeats to recycle.

In a postscript, Peirce describes a few extensions and variations. Once of them is a little "story trick". In brief, remove the Ace of Hearts (the priest) and two couples, the Queen and Jack of Hearts and the Queen and Jack of Spades. Arrange them at the top of a face-up pack as A, Q, J, Q, J; the Ace is five cards down. "The couples went to see a priest to arrange a joint wedding, but then the complexities of life intervened." Place different cards, here denoted X, between the couples as

$$\begin{gathered} \text{A, X, Q, X, J, X, X,} \\ \text{Q, X, X, X, J} \end{gathered}$$

and then cut six cards from the top to the bottom of the pack. "The storms of life continued." Turn the deck face down and proceed as follows: deal the top card on the table, the next under the pack, the next on the table (onto the first card dealt there), then under, then down, ... At the finish — "when life settled down the hymeneal altar was found intact" — spread the deck face up to show the couples and priest together.

Peirce called this story "a three-volume novel" and went on to an even more elaborate variation. Unknowingly, we applied some of Peirce's principles in a similar story; see "Together again" in Section 7. We may hope for further adventures of Peirce in our shared magicians' land and perhaps some supporting material from Chauncey Wright.

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