RELATIONS AND PEIRCE'S GRAPHICAL LOGIC

by

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Abstract

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This thesis is focused on a formal logic system invented by C.S. Peirce, called Existential Graphs. While this system has been proved to contain all the same theorems as standard first-order systems, it has an unusual metalogical property, called Peirce's Reduction Thesis. In this paper, I explain how this metalogical property results from two semantic restrictions on how graphs or diagrams may be constructed. I then defend these two restrictions. The defense of these requires a modified view of how to understand the extension of relations.

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Abbreviations

I will use the following abbreviations for references to Peirce's primary source material

CP 3.328
Collected Papers of Charles Sanders Peirce, Charles Hartshorne, Paul Weiss, and A. Burks, eds. (Cambridge, MA: Harvard University Press, 1935, 1958).

"CP" refers to "Collected Papers" followed by Volume and Paragraph number. Thus 3.328 is Volume 3, paragraph 328.

EP 2.328
The Essential Peirce: Selected Philosophical Writings, Vol.1 (1867-1893), Nathan Houser and Christian Kloesel, eds. (Indianapolis, IN: Indiana University Press, 1992)

The Essential Peirce: Selected Philosophical Writings, Vol. 2 (1893-1913), The Peirce Edition Project, ed. (Indianapolis, IN: Indiana University Press, 1998)

"EP" refers to "Essential Peirce" followed by Volume and Paragraph number. Thus 2.328 is Volume 2, paragraph 328.

W1.328
Writings of Charles S. Peirce: A Chronological Edition, Max Fisch *et al*., eds.
Vol. 1—1857-1866, Max Fisch, gen. ed. (1982)
Vol. 2—1867-1871, Edward C. Moore, gen. ed. (1984)
Vol. 3—1872-1878, Christian J.W. Kloesel. gen. ed. (1986)
Vol. 4—1879-1884, Christian J.W. Kloesel. gen. ed. (1986)
Vol. 5—1884-1886, Christian J.W. Kloesel. gen. ed. (1993)

"W" refers to "Writings" followed by Volume and Paragraph number.

NEM 4.328
New Elements of Mathematics, Carolyn Eisele, ed. (Mouton Publishers, The Hague, 1976)

"NEM" refers to "New Elements of Mathematics" followed by Volume and Paragraph number.

Introduction

I.

Late in an impressive career of logical achievement, Peirce invented a logical system called the Existential Graphs (hereafter EG). It is a peculiar system, consisting of logical "graphs", or diagrams,[1] rather than linear formulae. In addition to being conceptually peculiar, it also has peculiar properties as a system, which we will discuss. Being graphical, it follows in the tradition of Euler and Venn, who each constructed a system of logical diagrams that could be used as a tool to determine if certain inferences are valid.

Peirce's system of existential graphs could similarly be used as a tool or aid in drawing inferences. But he insisted that this was not their primary function. Their primary function was to serve as a vehicle of discovery.

"[The] purpose and end [of a system of logical symbols] is simply and solely the investigation of the theory of logic, and not at all the construction of a calculus to aid the drawing of inferences. These two purposes are incompatible, for the reason that the system devised for the investigation of logic should be as analytical as possible, breaking up inferences into the greatest possible number of steps, and exhibiting them under the most general categories possible; while a calculus would aim, on the contrary, to reduce the number of processes as much as possible, and to specialize the symbols so as to adapt them to special kinds of inferences" (CP 4.373)

This distinction in purpose has been a stumbling block for logicians considering Peirce's logic, particularly the graphs. But it is a vital distinction, as is emphasized by scholars of EG—including, for example Sun-Joo Shin who devotes a chapter of her book on EG to the distinction.

In the passage above, Peirce speaks of "the most general categories possible". What is meant by this idea? Peirce claimed to have learned philosophy from intensely

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[1] I follow Peirce in using the terms "graph" and "diagram" interchangeably. Peirce uses the term 'graph' for a specific kind of diagram made up of spots and lines. He follows his friends and colleagues, J.J. Sylvester and W.K. Clifford who introduced the term in their mathematical study of chemical diagrams. See CP 3.470 and CP 4.535 for discussion of Clifford and Sylvester.

[2] One would think that this should say that monads cannot be constructed *from dyads and triads*; dyads cannot be constructed from monads *and triads*; and triads cannot be constructed from monads and dyads. In fact, Peirce shows that all addicities can be constructed from triads (EP 2.346). This may seem to suggest that triads are the only primitive type of ingredient, but in fact Peirce thinks that triads effectively presuppose

reading the *Critique of Pure Reason* to the point that he almost knew it by heart (EP 2.424). Obviously this is hyperbole, but it appropriately suggests his debt to Kant. He followed Kant particularly in the following ways.

First, Kant teaches that the task of metaphysics is to ascertain what are the elements of reality. Second, in order to do this, metaphysics must look to logic. We will explain this idea in a bit more detail in Chapter Two.

Through an analysis of formal logic, Kant arrives at a list of twelve categories, under four headings (A80/B106). For Kant, these categories are the fundamental concepts that the Understanding contains within itself and brings to reality (ibid). Since, with Kant's "Copernican turn", objects conform to knowledge (Bxvi), these categories are the logical building blocks of thought and the metaphysical building blocks of reality (insofar as reality is accessible to thought).

Peirce's criticism of Kant (from the 30,000 ft view) is straightforward. He finds that Kant's list is not correct since Kant's analysis of formal logic was insufficient. So Peirce's task is thus to find *within the discipline of logic* what are the "indecomposable elements" (EP 2.362), and then build up metaphysics on that foundation. He finds that there are three logical elements. These logical elements are three forms of predication—embodied logically as three primitive classes of logical predicates: those that apply to one subject, two subjects, and three subjects respectively (EP 2.425). Peirce calls these monads, dyads and triads. We may think of them as relations of addicity one, two and three, respectively.

But this will sound peculiar to anyone not familiar with Peirce. If these logical elements are divided merely according to addicity, why stop at three? The answer to this question is the central topic of this paper.

II.

Peirce's claim that monads, dyads, and triads are primitive is a claim that has come to be known as the Reduction Thesis. I will discuss it in more detail in the last chapter of this paper, but I will introduce it here. It is a two-part claim.

The **first** part asserts that from three distinct types of logical predicates—namely monads, dyads and triads—all predicates of higher addicities are formable. This is to say that these three types of predicates are *sufficient* to form predicates of all addicities.

The **second** part asserts that these three logical types are irreducible. That is to say that dyads cannot be constructed exclusively from monads, and triads cannot be constructed exclusively from monads and dyads.[2] This is to say that all three of these types of predicates are *necessary*.

But it is a bit unclear how we should understand the Reduction Thesis. One major question is, does the thesis hold in *any* logical system? Peirce often seems to talk as if he thinks it does. But, at least in the way the thesis is stated above, that answer has proved to be No. As I will discuss later in the paper, Quine (and Löwenheim) showed that any relation with addicity >2 can be "reduced" to a dyadic predicate (though explanation of what this means will have to be left until then). Quine's proof seems to defeat the second prong of the thesis, by showing that triads are translatable into dyads, and therefore not irreducible.

Peirce seemed to think that the Reduction Thesis was most clearly defensible when considered within EG. This raises the question: is the Reduction Thesis to be understood as a metalogical property of *that* (rather peculiar) system? In one sense, the answer is Yes. And this has largely been the focus of the technical literature on the Reduction Thesis (as I will discuss in Chapter Five). For, while EG[3] "has the same theorems as first-order predicate calculus with identity…theorem isomorphism is a weak condition and the structures embedded in the graphs make this a very different system" (Brunning, 257).

[2] One would think that this should say that monads cannot be constructed *from dyads and triads*; dyads cannot be constructed from monads *and triads*; and triads cannot be constructed from monads and dyads. In fact, Peirce shows that all addicities can be constructed from triads (EP 2.346). This may seem to suggest that triads are the only primitive type of ingredient, but in fact Peirce thinks that triads effectively presuppose monads and dyads. But explaining these subtleties would take us off course. The battleground for the Reduction Thesis is thus the question whether triads can be constructed from dyads, which is the question I will focus on in this paper.

[3] Specifically one part of EG, called Beta, with which I am most concerned in this paper.

The Reduction Thesis holds in EG because EG makes two seemingly peculiar restrictions on conceptual combination. These restrictions will make little or no sense until we introduce the graphs in the next chapter. But, here we can introduce them in very broad strokes in the following way. To explain these restrictions we must first note that, in EG, individuals (quantified variables) are represented by lines. The restrictions I mention are: (i) when these lines are joined, they can only be joined one to another (rather than three or more joining at a node), and (ii) when a line "branches", it must branch in threes. In my own reading of the secondary literature on the Reduction Thesis (particularly the technical literature), most authors devote their energy to showing *that* the thesis holds in a system that makes these sorts of restrictions. While I do not at all mean to belittle this technical work, it seems clear enough to me (through a few simple examples) that, given these two restrictions, the Reduction Thesis holds in a system like EG. But, admittedly, EG is a strange system, so *why* should we care if an obscure system has some obscure property? Furthermore, Peirce invented EG in such a way that it has these restrictions. To an extent, he crafted it so that the Reduction Thesis would hold.[4] So, I believe the interesting question is *why* a logical system *should* make these restrictions in the first place. This is the question I aim to consider.

Of course, one could decide from prior metaphysical commitments (or perhaps, more accurately, prior metaphysical *preferences*) that certain properties of relations should be captured in a logical system. For example, one might hold that relations (as opposed to objects) have some sort of ontological priority. Then one could endeavor to create a logical system in which relations are primitive terms and objects derivative. And this would be highly interesting. That system may reveal certain metalogical properties; and if it possessed certain elegance, it may even persuade some metaphysicians to subscribe to those metaphysical commitments that prompted its construction.

But I follow Peirce in maintaining that such a project would get the order wrong. Logic must teach us how to do metaphysics, not vice versa.[5] Peirce thought that his most

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[4] See Brunning, 253

[5] Peirce holds that, "most of the metaphysical conceptions…are nothing but logical conceptions applied to real objects, and can only attain elucidation in logical study" (EP 2.376); Similarly, "This is the spirit of the Kantian doctrine that metaphysical concepts are logical concepts applied somewhat differently from their logical application" (EP,

significant philosophical discoveries arose out of a close and careful study of logic, and he seemed to think that these discoveries were most defensible from the bulwark of a well-crafted logical system.[6]

Obviously, if Peirce crafted EG in such a way that the Reduction Thesis holds in that system (but not in others), then he cannot argue that *this fact* (that the Reduction Thesis holds in EG) reveals anything substantial about logic generally. But if he discovered *within the discipline of logic itself* that certain forms of conceptual combination are primitive, then the story is a bit different. I am convinced that Peirce *discovered,* through his logical investigations, that there is something primitive in these two features of conceptual combination, and this is part of what I aim to defend.

Thus, in this paper I aim:

First, to defend how Peirce came to believe that these restrictions on conceptual combination were primitive

Second to argue that, if these forms of conceptual combination are primitive, then a logical system (for purposes of discovery; not necessarily for purposes of aiding inference) should reflect these. I think a graphical system like EG does that best.

Finally, I suggest a new way to think about the Reduction Thesis. Peirce believed that a flaw in his own earlier logical system was that "the very triadic relations which it does not recognize it does itself employ" (1904 letter to Victoria Welby, cited in Burch, APRT, xv). I follow Robert Burch and Jacqueline Brunning in thinking that this is a clue to how we should think about the Reduction Thesis. In that vein, I argue that we can think of the Reduction Thesis as a claim about metalogical proofs that reduce triads to dyads, and I give an example of how we might make that idea precise. I will leave the question open whether the thesis is a *logical* claim about metalogical proofs (in the way we would normally think of metalogical theorems), or a *semiotic* claim about such metalogical proofs. This distinction will hopefully be clarified in the coming pages.

2.393). See also EP 2.424, and EP 2.257, passages that similarly emphasize that metaphysics rests on logic, not the other way around.

[6] See Peirce's 1905 letter: "[Royce] attacks my one-two-three doctrine in the very field where it is most obviously defensible, that of formal logic" (cited in Herzberger, 57).

In the course of this paper I will distinguish three different types of first-order logical systems. I will follow the secondary literature on Peirce's logic in referring to these as Algebraic, Quantificational, and Graphical logic systems.

Algebraic systems do not contain quantifiers, but have other ways of dealing with quantification (Merrill, 158). This sort of a system takes relations as primitive terms and defines permissible operations on these relations that preserve validity.[7] Peirce's logical systems of 1870 and 1880 are algebraic systems, and Burch's system called PAL (Peircean Algebraic Logic) is a modern version that aims to incorporate insights from EG in an algebraic way. Contemporary mathematicians working on Peircean logic, including Frithjof Dau, Joachim Hereth Correia and Reinhard Pöschel, use PAL. Burch argues that Peirce preferred algebraic logic over quantificational logic "until well into the 1890s" (Burch, APRT,1), at which point Peirce pivoted towards graphical logic.

Quantificational systems will be most familiar to contemporary scholars in logic. These sorts of systems employ quantifiers, and define semantics of the system set-theoretically. While one quantifier can be defined in terms of the other, at least one quantifier is a primitive term, and it quantifies over a domain of individuals, which is also primitive. "[Quantification] seems to require the existence of the objects over which it takes place" (Guinness, 36). Thus, in quantificational systems, generally relations are understood extensionally—that is to say, they are understood *to be* sets of n-tuples. Thus, relations are defined in terms of individuals instead of the other way around.

The only graphical system that I consider is EG, though perhaps there are others. Peirce's system of Entitative Graphs (a precursor to EG) is certainly another such system, which is very similar to EG but importantly different in certain respects (and in Peirce's view inferior). If algebraic systems take relations as primitive, and quantificational systems take individuals (and at least one quantifier term) as primitive, how should we understand a graphical system? Syntactically, EG takes relations as primitive in the same way that an algebraic system would. And Burch's contemporary system PAL is intended to parallel EG, as Burch emphasizes. The key difference, then, between a system like EG

[7] "…the terms of algebraic logic may naturally be understood to stand for *relations*…it follows…that reasoning is primarily, most elementarily, reasoning *about relations*" (Burch, APRT, 2)

and one like PAL is in its syntactic presentation. Graphical systems require a "graphical syntax…[which is] two-dimensional…requiring a surface rather than a line for its inscription" (Burch, APRT, 2). Peirce, however, believed that EG had certain theoretical virtues over any algebraic system. These virtues are best explained using semiotics, which we will elaborate in Chapter Two. Peircean scholars including Burch and Kenneth Ketner note that the study of graphical logic shades into topology in addition to logic as traditionally understood.

III.

Above, I noted that the Reduction Thesis could be understood as either being a *logical* claim about metalogical arguments or a *semiotic* one. This, of course, raises a question about the relationship between logic and semiotics. But first, perhaps we should consider what semiotics is in the first place, and why we should care about the discipline at all.

The Peircean scholar, Max Fisch, notes that semeiotic[8] was important even for the logical positivists, and points out that the first section of the planned Encyclopedia on Unified Science was to be devoted to semiotics (Fisch, 346). Semiotics is the study of signs. I will discuss signs in much more detail in Chapter Two. But here we may consider a sign very generally as anything that indicates or represents anything else. As such, anything meaningful—a word, a gesture, a scientific instrument, a percept interpreted as being at all informative, a noise—is a sign. Fisch notes that, "Peirce's general theory of signs is so general as to entail that, whatever else anything may be, it is also a sign" (ibid, 357). In this sense, any logical or mathematical argument consists of signs—be they diagrams, words, formulae, variables, etc. And thus, it would make sense for an account of unified science to begin with a theory that aims to give a general account of how anything at all can be meaningful.[9]

[8] Fisch, for reasons he explains, follows Peirce in using this spelling. I will generally use the more common spelling—semiotic.

[9] This characterization of semiotics seems to cover both semantics and metasemantics, as they would be defined in the philosophy of language. This is no mistake. Peirce is partly concerned with a general taxonomy of signs into different types, which is an empirical exercise (even if those signs be mathematical signs). But he is also concerned with giving

Here one might worry, however, that we are seeking to build logic on top of semiotics, which would seem to risk making logic psychological. But Peirce did not believe logic rested on semiotics (whatever that might mean). However, his conception of the relationship between logic, semiotics, and mathematics is complex and there is a rich secondary literature on the topic.[10] According to Peirce, "mathematical logic is formal logic. Formal logic, however developed, is mathematics" (CP 4.240). Formal logic, like mathematics, draws necessary conclusions from certain initial conditions—e.g. by defining formal operations or semantic rules in such and such a way, such and such results *must* hold. But Peirce disagreed with the logical positivists who wanted to build mathematics on logic, claiming instead that mathematics is pre-logical (Kerr-Lawson, 77; CP 4.239). For Peirce, mathematics *draws* necessary conclusions (CP 3.558), but these conclusions are purely hypothetical and do not refer to anything real—"the truths of mathematics are truths about ideas merely" (NEM 4.xv). *Deductive* logic, in contrast, is "the science of *drawing* necessary conclusions" (CP 4.239)—that is to say, deductive logic is concerned with studying *the reasoning* involved in mathematical arguments.

But, furthermore, deductive logic is not the only branch of logic. Logic, more generally, is concerned with studying reasoning in all sorts of arguments—not just mathematical ones. We will consider this idea in Chapter Two. Thus Peirce claims, "Formal logic, however, is by no means the whole of logic, or even its principal part" (CP 4.240). Peirce held that it was also a task of logic (in the general sense) to define "metalogical"[11] principles that embody *good* reasoning. He was acutely aware that logic in this sense must be a normative science. To make this idea a bit more clear, we might

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an account of the general structure of all signs. He also intends this project to involve *all* signs, not just linguistic ones. As Dau notes, "For Peirce, semiotics is not a mere metatheory of linguistics" (Dau, 26). In this regard, the following passage is helpful: "Logic is itself a study of signs…There are three ways in which signs can be studied, first as to the general conditions of their having any meaning, which is the *Grammatica Speculativa* of Duns Scotus [Bellucci, 2018 focuses on this], second as to the conditions of their truth, which is logic, and thirdly, as to the conditions of their transferring their meaning to other signs" (NEM 4.331). Here we see that the line between semiotics and logic is blurry.

[10] See particularly Guiness, 1997; Van Evra, 1997; Bellucci, 2018; and Kerr-Lawson, 1997

[11] To my knowledge, he does not use the term. But, as I will argue, some of his ideas are best understood as being metalogical.

consider a more familiar system—Gentzen's Natural Deduction. If we define the deductive rules in a certain way, certain necessary consequences follow, which would be mathematical in Peirce's sense. These might be consequences within the system—like the consequence that a certain sequent is provable from another, given the deductive rules; or the consequences may be metalogical—like the consequence that the system as a whole is sound and complete in respect to the semantics. But we may also think that a deductive system like Gentzen's is useful because it embodies deductive principles that are "natural", and because it is sound and complete with respect to a semantics that embodies "good" reasoning. This latter concern would be a question for philosophy of logic, rather than, say, metalogic, which is not concerned with normative questions.

This being said, this paper is devoted primarily to formal logic, not logic in this normative sense. However, if there is some discoverable *semiotic* property of *any* metalogical proof of a certain type (e.g. any proof that reduces triads to dyads), I believe that this would be significant. Admittedly, it may be *more* significant if this property is strictly logical rather than semiotic—then it would be a metalogical claim. Whether it is logical depends on how we can precisely formalize the idea, which I will discuss in Chapter Five. But even if it is a semiotic claim, this does not mean it is a normative claim, though it may have implications for the normative study of logic. In this paper I will argue, first, that certain modes of conceptual combination are primitive. This is not a normative question; it is a question of logic and semiotics. But if this is true, then I maintain that a logical system whose purpose is to reveal the way that we reason (not necessarily those with other purposes) *should* incorporate that idea. This is a normative claim, but it rests on the rather stable normative premise that any logical system whose purpose is to reveal the essence of how we reason should incorporate insights about how we reason. Thus, my efforts are primarily devoted to arguing that these restrictions are indeed primitive. That is not a normative question.

Fisch notes, "…one of the strengths [Peirce] came to prize in his existential graphs is that, without loss of rigor and even with some gain, they lend themselves to a pragmatic understanding of logic itself" (Fisch, 370). Semiotics is relevant since it attempts to give a general account of how a logical system is to be applied to reality (particularly in the context of science). And if it can do so without loss of rigor, then, on Peirce's view as well as my own, this should be seen as a virtue of the system.

As noted above, Peirce's views on the relationship between logic and mathematics situate him in a different camp from the logical positivists. As Guinness says, "Mathematical logic, especially in the logicist version of Russell, was held to *contain* 'all' mathematics; algebraic logic [the tradition of Peirce] maintained some *relationship* with it" (Guinness, 29). While Guinness elaborates several mathematical themes that differentiate Peirce's tradition from that of the logical positivists, in my view, the primary difference comes down to how they each see logic connecting to reality. The positivists attempt to make that connection at the level of formal semantics, by defining truth in a particular way. But Peirce has a more complex picture. For him the whole formal system is hypothetical and does not itself refer to anything real. It must be *mapped* to Reality, and he wants to give a general account of how that mapping takes place.

On this topic, it is worth keeping in mind a distinction between two traditions in formal logic, which is emphasized in the secondary literature. These two traditions are occasionally given different names, but the one, usually called the tradition of "algebraic logic", includes Boole, De Morgan, Jevons, Peirce and Schröder, among others. After Peirce and Schröder, it includes Löwenheim, Skolem, Tarski and perhaps others.[12] The second tradition, occasionally called the tradition of "mathematical logic", includes Weierstrass, Cantor, Frege, Dedekind, Hilbert, Peano and Russell, among others. The term "algebraic" may recall my definition above of an algebraic logical system. But the distinction here has less to do with how each tradition conceives of relations, and more to do with how each tradition understands the task of logic.[13] In that regard, the respective attitudes towards the relationship between mathematics and logic are quite central. However, the distinction between these two traditions is still loosely related to the distinction between algebraic and quantificational logic discussed above. Guinness argues

[12] See Guinness, 24-31; Shields, 50; Anellis, 271; Hawkins 119; Brady, 191
[13] Hintikka associates Peirce with the model-theoretic tradition, and associates the "mathematical" tradition with the proof-theoretic tradition (Hintikka, 116). Iliff similarly claims, "the tradition of Frege, Peano, Whitehead, and Russell is essentially deduction-theoretic; by contrast, the tradition of Boole, De Morgan, Peirce, Schröder, Löwenheim, and Skolem is essentially model-theoretic" (Illiff, 204).

that, just as quantificational logic seems to require the existence of individuals in a domain, the tradition of mathematical logic seems to require existence of sets and sets of sets, which "constitutes a considerable problem for logicism" (Guinness, 36-37). In this vein Guinness praises "the caution of the algebraic logicians" (ibid, 29). Peirce's contributions to formal logic are rooted in the same mathematical tradition as his father, Benjamin Peirce, who was a major figure in 19th century mathematics, generalizing the study of algebra beyond concern only with quantity with his work *Linear Associative Algebra*, 1870.[14]

As mentioned above, these traditions had different attitudes about the relationship between logic, mathematics and semiotics. For example, Guinness tells us that,

"both traditions were much concerned with names; but whereas the mathematical logicians laid stress on the difference between names and definite descriptions (partly for motivations in mathematics itself, and for Russell also because of the paradoxes), Peirce set off toward semiotics (which contains a theory of proper names)…" (Guinness, 31-32).

It is interesting, on this topic, to consider that the former tradition, to which Peirce belongs, never had a real problem with the paradoxes.[15] Perhaps this suggests that there is more hope in a tradition that is self-consciously aware of semiotic subtleties.

IV.

My paper is primarily devoted to logic, and aims to defend why there are three *logical* elements (not less and not more). But it also (at least implicitly) looks forward to

[14] See Quine, WRML, 3; Guinness, 35; Van Evra, 152.

[15] Brady cites Schröder, Löwenheim and Skolem on this topic, who build from Peirce's logic (through Schröder): "Schröder acknowledges the influence of Peirce in many places, in particular 1895:4: 'Peirce has given us one such foundation [i.e., an independent foundation for the algebra of relatives], which takes its departure from the consideration of 'elements' or 'individuals'; the comparison of the thereby created, totally peculiar basis of the entire logic with other foundations can only be instructive. We therefore follow this course.' Löwenheim (1940:1), in turn, states that he worked entirely within Schröder's system: 'When endeavoring to analyze mathematics logically, the paradoxes discovered by Russell and others have always appeared as the most formidable obstacle…I myself have never encountered such difficulties, because to analyze logically has always meant to adapt to Schröder's relative calculus.' Finally, Skolem (1920: 254) acknowledges that he is improving on Löwenheim's results…" (Brady, 191).

the metaphysical application of this logical doctrine, so it is worth making some brief comments about what that application looks like.

The two (related) metaphysical views that grow out of Peirce's logical discoveries are his Pragmatism (or Pragmat*ici*sm)[16] and his scientific realism.

I have often heard repeated that pragmatism is a doctrine that upholds the maxim that a proposition is true if it "works". This idea is at best a thorny distraction from Peirce's philosophy and my project here, and at worst an utter misconstrual of what Peirce was up to. In 1907 Peirce somewhat modestly claims that, "pragmatism is, in itself, no doctrine of metaphysics, no attempt to determine any truth of things. It is merely a method of ascertaining the meanings of hard words and of abstract concepts" (EP 2.400). The hard words and abstract concepts that Peirce primarily had in mind were those in science and philosophy, though he, no doubt, saw the pragmatic method as having import beyond that. He states (his own version of) the pragmatic maxim as follows:

"the rational purport of a word or other expression, lies exclusively in its conceivable bearing upon the conduct of life"; therefore "since obviously nothing that might not result from experiment can have any direct bearing upon conduct, if one can define accurately all the conceivable experimental phenomena which the affirmation or denial of a concept could imply, one will have therein a complete definition of the concept, and *there is absolutely nothing more in it*" (EP, 2.332).

Another way of stating the maxim is the following:

"Pragmatism is the principle that every theoretical judgment expressible in a sentence in the indicative mood is a confused form of thought whose only meaning, if it has any, lies in its tendency to enforce a corresponding practical maxim expressible as a conditional sentence having its apodosis in the imperative mood" (EP 2.134-5, 1903).

These passages align with Peirce's interpretation of the quantifiers, as we will elaborate in Chapter Three. As I will discuss towards the end of Chapter Four, this notion is also relevant to Peirce's conception of the extension of relations. Peirce's rather novel scientific and metaphysical views grow out of logical considerations—particularly about how to understand relations. And I believe these ideas have implications for how we can

[16] Pragmat*ici*sm is Peirce's term for his own version of pragmatism, coined in 1905 in order to differentiate it from that of others including James, Royce and Schiller. He hoped it would be a term "which is ugly enough to be safe from kidnappers" (EP 2.335).

think about essences in metaphysics and in the philosophy of science—particularly in regard to Natural Kinds.

This brings us to his realism. Peirce was deeply concerned about the question whether universals and laws are real, or are merely a product of the mind. And he held that the division of philosophers into realists and nominalists on this question "cuts across more familiar contrasts between rationalists and empiricists, naturalists and transcendentalists, and so affords a novel perspective on the philosophical tradition" (Forster, 1). Peirce came to be a committed realist on this question, and rather vehemently criticized his earlier nominalist tendencies—particularly as embodied in his two papers "The Fixation of Belief" (1877) and "How to Make our Ideas Clear" (1878).[17] It is ironic (and unfortunate) that today these are probably Peirce's two most often read papers. Peirce does acknowledge, however, with a nod to Ockham, that nominalism is the simpler of the two theories. Thus, he claims, "Everybody ought to be a nominalist at first, and to continue in that opinion until he is driven out of it by the *force majeure* of irreconcilable facts" (CP 4.1).

Claudine Tiercelin characterizes the later Peirce's metaphysical view as "dispositional realism…within a scientific realistic metaphysics" (Tiercelin, 125). As she clarifies, it is a version of Scholastic (specifically Scotistic) Realism, but with a pragmatist twist. What does this mean? As we have suggested above, Peirce maintains that there are three fundamental elements in logic. This leads him to make the Kantian step of building metaphysical analogs of these three concepts—the three metaphysical categories. His criticism of Scotus (and of the scholastics generally) is simply to say that they failed to build the third metaphysical "category" into their account of essences. Peirce's "pragmatist twist" is thus a correction of Scotus' position, which had incorrectly characterized essence as "behavior of a thing." Peirce's pragmatic realism redefines essence as the "*habit of behavior*" of a thing (ibid, 133), drawing a subtle distinction and thereby shedding new light on the old problem of the reality of universals. As Tiercelin argues, this has much relevance for contemporary debates about Natural Kinds.

The error in the debate about universals, says Peirce, lies in the assumption that they must exist in the mode of being of the second metaphysical category, when it would

[17] See, for example, EP 2.455-7 for his criticism; also Fisch, 368

be more proper to say that they exist in the mode of being of the third. We will discuss this idea in a bit more detail at the end of Chapter Four after giving an account of how Peirce conceives of the referents of relations. Unfortunately, until that time, this discussion will likely be unclear—particularly since we have not yet defined what Peirce's three metaphysical categories are. But it is important to recognize that Peirce's (professed) correction of Scholastic metaphysics is made possible by discoveries that fall squarely within the discipline of logic.

Tiercelin agrees with Randall Dipert in maintaining that, " 'logic, especially the logic of relations played a central role in the development of Peirce's philosophy' (2004:287) and that his logic of relations had a decisive impact on the right realistic metaphysics one should adopt" (Tiercelin, 125). She goes on to quote Peirce's statement of his plan to defeat nominalism:

"My plan for defeating nominalism is not simple nor direct; but it seems to me sure to be decisive, and to afford no difficulties except the mathematical toil that it requires. For as soon as you have once mounted the vantage-ground of the logic of relatives, which is related to ordinary logic precisely as the geometry of three dimensions is to the geometry of points on a line, as soon as you have scaled this height, I say, you find that you command the whole citadel of nominalism, which must thereupon fall almost without another blow" (CP 4.1).

I believe that the crux of his metaphysical argument in favor of scientific realism lies within the second pillar of the Reduction Thesis. If triads really are irreducible and elemental in the way Peirce thinks they are, then the citadel of nominalism falls. If they are not, then it stands.

As noted above, Peirce believed that the Existential Graphs "lend themselves to a pragmatic understanding of logic itself" (Fisch, 370), and they do so in a way that maintains logical rigor. As Peirce says, "It is one of the chief advantages of Existential Graphs, as a guide to Pragmaticism, that it holds up thought to our contemplation with the wrong side out, as it were" (CP 4.7). This allows us to observe hidden relations that we did not see before. He also claims that the graphs "put before us moving pictures of thought, [and by that] I mean thought in its essence free from physiological and other accidents" (CP 4.8).

If the two forms of conceptual combination that I discuss are really primitive in a logical sense, then these should be incorporated into a logical system whose purpose is to exhibit thought in this way. EG is just such a system, and if this system displays the conclusion of the Reduction Thesis, then I believe there is strong reason to take that thesis seriously.

V.

This paper will proceed in the following way.

The first chapter will give a technical introduction to EG as a system. Towards the end of this chapter, we will give an account of the graphical embodiment of the two forms of conceptual combination I have been discussing—as embodied in the rules of transformation for Beta graphs (roughly speaking these are deductive rules for the part of EG that is theorem isomorphic to first-order predicate logic with identity).

In Chapter Two, I will introduce four trichotomies that are central to Peirce's semiotics and to his logic. In the course of this explanation, I will also explain Peirce's early discovery (in 1866) of a flaw in Kant's logical reasoning. This discovery, explained in Peirce's paper "Memoranda on the Aristotelian Syllogism" was a vital stepping-stone for Peirce's thinking about the logical "elements", and, I believe, lends support for my understanding of the Reduction Thesis.

Chapter Three is devoted to clarifying how Peirce understands the referents of relations. I give context to Peirce's logical work with some discussion of Boole and De Morgan, and I identify three important Peircean themes that are germinal in De Morgan's work. I argue that these ideas emerged through a consideration of how to incorporate relations into a logical system. They are germinal in De Morgan because De Morgan was the first logician to take relations seriously in formal logic. I argue that these themes prompt important questions about the referents of relations.

During the course of this discussion, I distinguish (i) relational facts from (ii) representational relations—i.e. relations in a representational system, like a language or logical system. And I distinguish the latter into (ii.a) "unsaturated" relations, or relational predicates, and (ii.b) "saturated" relations. I argue that, on Peirce's view, the referents of saturated relations (ii.b) are relational facts (i). But this leaves the difficult question as to what is the referent of an unsaturated relation. I attempt to answer that question in Chapter Three and Chapter Four. I argue that Peirce has two conceptions in mind for the referents of unsaturated relations. One is a "God's-eye-view", which is similar to the typical contemporary view, that a relation is a class of n-tuples. But I argue that Peirce also has a second conception in mind that involves intension, and which better tracks the way we actually acquire new knowledge.

I end Chapter Three by explaining Peirce's conception of diagrams. I follow him in taking a quite broad notion of what constitutes a diagram. This discussion is intended to make the case that we should scrutinize the diagrammatic features within any logical system—including an algebraic or quantificational system—and should ask which system best captures the features of relations that I discuss.

Chapter Four considers one particular operation in Peirce's algebraic systems that he came to view as privileged. It is called the relative product operation. In this chapter I give context as to how this operation led to the discovery of quantification, and defend *why* it is privileged. The answer to this question is related to the discussion of the referents of relations. After clarifying why Relative Product is privileged, I state the more nuanced view as to how we should understand relations in light of this form of conceptual combination. As mentioned, I maintain that Peirce is working with two views in regard to the referent of a relation—one is the God's-eye-view, and one is the more nuanced view. If we take the God's-eye-view, it is impossible to see how Relative Product is *determinative*—that is, how it determines reference and gives new information. But, by considering the more nuanced view—which, I believe, actually tracks how concepts develop and how we acquire new information—we can see how this operation is privileged in permitting concepts to mutually determine each other through combination. This sort of combination is embodied in the relative product operation. When Peirce moves to EG, this mode of conceptual combination is embodied as one of the restrictions that we have discussed. This is one of the two important restrictions that the Reduction Thesis depends on.

Chapter Five explicitly addresses the Reduction Thesis. Having defended the restrictions on conceptual combination and the features of relations that ought to be captured, I propose that we should understand the Reduction Thesis as a claim about any metalogical argument that purports to reduce triads to dyads. According to this view, the Reduction Thesis claims that any metalogical argument of this sort *must itself employ* triadicity. I then consider how to make that idea logically precise, and identify a clue through the work of Jakko Hintikka, though I leave it open-ended whether this is a logical claim about metalogical proofs of this sort or a semiotic claim about such metalogical proofs.

During the course of this paper, I will occasionally give historical details. While my purpose is not historical exegesis, I believe some of these details give context to Peirce's discoveries and to his conceptual development about these topics, showing that they were genuinely *discoveries* that emerged from a careful study of logic.

Chapter One
Technical Explanation of EG

Background on EG

Peirce took up the study of logic at age twelve (~1851), when he read his brother's copy of Whately's *Elements of Logic* (Bellucci, 28; Iliff, 193). His first published works on logic were in 1866, in conjunction with a series of lectures he gave at Harvard on philosophy (Bellucci, 15).

In 1870 he published his first major paper on logic—a lengthy and ambitious paper on the logic of relations entitled "Description of a Notation of the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole's Calculus of Logic" (hereafter **DNLR**). From 1879-1884 he taught logic at Johns Hopkins University, his only academic post in his life (most of his career working as an experimental scientist and applied mathematician). During the tenure at Johns Hopkins, in 1880, he expanded on DNLR with another major paper called "On the Algebra of Logic" (hereafter **AL**). Three years later, in 1883, his student O.H. Mitchell discovered quantification (likely with Peirce's assistance), and in the journal where Mitchell published his discovery (edited by Peirce), Peirce incorporated Mitchell's distinction into his own system, also adding an improvement.[18] In his 1885 paper "On the Algebra of Logic: A Contribution to the Philosophy of Notation" (hereafter **ALPT**), Peirce brought quantificational logic fully to fruition in a complete system. Schröder then took up work within that system, and systematized and expanded on Peirce's work after 1885.

In 1886, Peirce's friend and colleague Alfred Bray Kempe published his "Memoir on the Theory of Mathematical Form," and sent a copy to Peirce (Fisch, 333). Kempe's treatise "treats of the representation of relationships by 'Graphs,' which is Clifford's name for a diagram, consisting of spots and lines, in imitation of the chemical

[18] By all accounts in the secondary literature, Peirce had no knowledge of Frege's work (Frege independently discovering quantification in *Begriffsschrift*, 1879), nor did Frege have any awareness of Peirce's in 1879. See Hawkins, 134-137 for an account of historical details regarding Frege and Peirce's cognizance of one another's work (or, more appropriately, lack thereof).

diagrams showing the constitution of compounds" (CP 3.468).[19] Kempe's "proto-topological ideas" excited Peirce "who saw in the work a means of developing the logic of relations (which Kempe himself had not done)" (Guinness, 36). Peirce was inspired by this work, which also posed a "formidable objection" to Peirce's own views (CP 3.423). Kempe's paper even caused Peirce to reconsider his commitment to the metaphysical categories (see CP 3.423-424), an historical detail we will discuss briefly in Chapter Five.

In 1897, Peirce wrote a logical paper called "The Logic of Relatives" where he reviewed Schröder's *Algebra und Logik der Relative* (1895) (the third volume of Schröder's massive *Vorlesungen über die Algebra der Logik*), and presented a new graphical logical system called the Entitative Graphs. While that paper was on its way to press (in September 1896), Peirce wrote to Paul Carus, the editor of *The Monist*, that he would like to make a modification to that system (the letter is now published in R482). This modification was the earliest version of the Existential Graphs—what Peirce later called "the strongest of all my logical papers" (Bellucci, 174). His modification effectively turned the Entitative Graphs "inside out" (Roberts, 28 (Peirce, MS 485)), reversing conventions for the 'and' and 'or' connectives, and having the blank "sheet of assertion" represent 'the true' rather than 'the false' (CP 4.434). And thus the Existential Graphs were born.[20] Carus rejected the request and published the original, thinking Peirce would go on to make further changes (Atkin, 190). But he did not.

Peirce's work on logic after 1897 (until his death in 1914) was largely concentrated on the system of Existential Graphs (EG). I will discuss this system throughout this paper, but it will be useful to give a very informal introduction to the system here first. My purpose here is not to introduce the system in a rigorous way, but is rather merely to give an idea for what this system looks like in order to give some context to my future arguments about graphical logic.

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[19] Kempe's work is related to that of two other mathematicians and early graph-theorists, who were also colleagues of Peirce's—W.K. Clifford and J.J. Sylvester. Sylvester also worked at John's Hopkins at the same time as Peirce. See Atkin 188-9. Also see CP 3.470 and CP 4.535 for discussion of Clifford and Sylvester.

[20] " 'Existential' first occurs in the *Logic Notebook* on June 9, 1898 (p. 102r) and frequently thereafter" (Roberts, 30). It is so named because "the fundamental symbol…'expresses the relation of existence' (Ms 485, p. 1).

The system has three parts—Alpha, Beta, and Gamma.[21] The Alpha system is isomorphic to languages of Sentential Logic, and the Beta system is isomorphic to First-Order Predicate Logic with Identity. Both have been proven to be sound and complete deductive systems—soundness by Jay Zeman in 1964 and Completeness by Don Roberts in 1973.[22] The Gamma system is a modal system with interesting logical and philosophical features, but it is beyond my scope. All parts involve scripture of graphs on a blank two-dimensional "sheet of assertion" (hereafter **SA**) (CP 4.396)—in practice, a piece of paper, blackboard, computer screen, etc. As such, the graphs involve a two-dimensional syntax.

Typically, presentation of each part (Alpha and Beta, for our purposes) involves two stages: First are formation rules, which account for how to scribe and read permissible graphs (analogous to a syntax and semantics of well-formed formulae). Second are rules of transformation, which are the rules for transforming graphs into other logically deducible graphs. These can be understood as deductive rules.

Alpha—Introduction

In Alpha, there are three primitive types of symbols—the sheet of assertion itself, the cut, and sentential symbols.[23] "Sentential symbols A_1, A_2,…, represent propositions, and a cut represents negation" (Shin, 38). As Roberts notes, the SA can be thought to represent the " 'universe of discourse', that is, the domain of objects to be talked about" (Roberts, 31). But one feature that is unusual in EG is that this domain is not explicitly specified until propositions are asserted on the SA. The blank SA can therefore be thought to semantically represent "the true", but in a manner that is entirely vague. Thus,

[21] Peirce also added an additional variation called the Tinctured Existential Graphs, which are aimed at formalizing non-assertoric statements as well as assertoric ones. They involve twelve possible tinctures for the sheet of assertion (the tinctures are inspired by the use of tinctures in heraldry), which indicate mode of being and grammatical mood (CP 4.554). The tinctures are beyond my scope.
[22] See Shin, 10, 59; Roberts, 139-151
[23] Roberts, 31; Shin speaks of "two kinds of primitive symbols" (Shin, 37), informally omitting the SA as a primitive symbol, but her formal account of EG includes the SA as itself a graph.

"whatever we write upon it can be thought of as making the representation of the universe more determinate (Ms 455, pp.2-3)" (Roberts, 31). Roberts also helpfully asks,

"What does SA express? Whatever is taken for granted at the outset to be true of the universe of discourse. The sheet thus functions as a kind of all-purpose axiom, and we will understand the first convention [that the SA itself is a graph] to be a statement of this axiom" (ibid, 32).

The rules for what constitutes a "sentential symbol" are extremely open-ended. A sentential symbol might be a full sentence in English or any other language taken for granted as meaningful between Graphist—"Quasi-utterer" (CP 4.551)—and Interpreter—"Quasi-interpreter" (ibid).[24] Or a sentential symbol may be purely formal—e.g. P, Q, etc. So the following would both be permissible (tokens of) Alpha graphs[25]:

(1) (2)

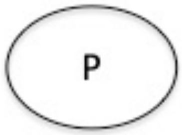
P The Prime Minister is on vacation

A cut, which is a contiguous self-connecting line (typically an oval, though any shape is permissible), asserts the negation of whatever is contained in the area within it. "A cut drawn upon the sheet of assertion severs the surface it encloses, called the **area** of the cut, from the sheet of assertion" (CP 4.414). So if graph (2) above asserts that 'The Prime Minster is on vacation', then graph (4) below asserts that 'it is not the case that the Prime Minster is on vacation'. Similarly, graph (3) asserts ¬P.

[24] Peirce explains, "these conventions are supposed to be mutual understandings between two persons: a *Graphist*, who expresses propositions according to the system of expression called that of *Existential Graphs*, and an *Interpreter*, who interprets those propositions and accepts them without dispute" (CP 4.395). Since Peirce conceives of thought as dialogue, these need not be separate individual persons though; thus he attaches the prefix "quasi" elsewhere. See also CP 4.431-2; Ochs, 207.

[25] Although my presentation here is extremely informal, it is worth drawing the distinction between graphs (as types) and their tokens. Frithjof Dau makes some helpful criticisms of Shin in his book *Mathematical Logic with Diagrams*, based on EG. One of his central points is that we must conceive of graphs as abstract mathematical (specifically topological) entities (types), whose replicas (tokens) we employ in any particular context (Dau, 44-45).

(3) (4)



No order of sentential symbols within the same *area* is acknowledged in EG. Thus, an additional convention is that two or more sentential symbols that are juxtaposed are to be interpreted as being conjunctively asserted. Thus graph (5) asserts that: 'the Prime Minster is on vacation and the Prime Minister is not in his office'; *or* similarly, that 'The Prime Minister is not in his office and the Prime Minster is on vacation'. Since these two propositions are logically equivalent, they are not recognized as distinct in EG.

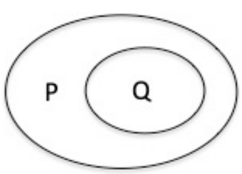
(5)

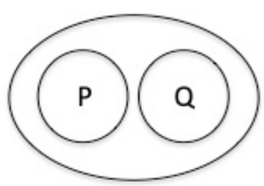
The Prime Minister is on vacation

The Prime Minister is not in his office

Since a sentential language with negation and conjunction is truth-functionally complete, we may build more complex Alpha graphs using these conventions. "[Peirce's] conventions correspond to an informal semantics of the system" (Shin, 72). For example, graph (6) asserts: $\neg(P\wedge\neg Q)$, or equivalently, $P\rightarrow Q$; and (7) asserts: $\neg(\neg P\wedge\neg Q)$, or equivalently $P\vee Q$.[26]

(6) (7)



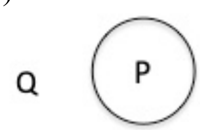


[26] For ease of presentation, we are implicitly taking these two conventions—the juxtaposition convention for conjunction and the cut convention for negation—as basic. But for philosophical reasons, Peirce himself actually took the conditional form as basic—what he referred to as the "scroll." In regard to the scroll, Peirce says, "the consequent of a conditional proposition asserts what is true, not throughout the whole universe of possibilities considered, but in a subordinate universe marked off by the antecedent" (CP 4.435); See also Roberts, 35; Shin, 72.

Alpha—Reading Methods

The translations of graphs (6) and (7) above employ a method called **endoporeutic** (Roberts, 39; CP 4.561), in which one proceeds inwardly in reading the graphs. In explaining this method, Shin and Roberts importantly emphasize (e.g. Shin, 62; Roberts, 39) that graph (6) is not to be interpreted as asserting that "'Q is true and P is false', even though Q is evenly enclosed and P is oddly enclosed" (Roberts, 39). This is because when we proceed inwardly, we first encounter a cut, which contains the rest of the graph within it. This enclosure represents the scope of the negation. Informally continuing the reading of graph (6), then, we would interpret any juxtaposed subgraphs within the area as being asserted conjunctively.[27] In the case of graph (6), there are two subgraphs: (i) P, and (ii) the cut that contains Q. This latter is interpreted as $\neg$Q. So we end up with the sentence $P \wedge \neg Q$ contained within the area of a cut—i.e. within the scope of the negation. This translates to $\neg(P \wedge \neg Q)$. The graph that asserts the sentence mentioned above—'Q is true and P is false'—then, would be the following:

(8)



Endoporeutic is the method proposed by Peirce and taken up by Zeman, and is the most straightforward way of translating Alpha graphs. Shin, however, identifies certain disadvantages in this method—perhaps most notably that translating graphs in this way obscures visual features of the graphs that make them more intuitive. As such she offers a reading algorithm for Alpha graphs that combines multiple reading methods. I will not comment on her reading algorithm, however.[28]

[27] Dau offers a helpful criticism of Shin regarding the undefined term subgraph, and he discusses several borderline cases (Dau, 42) that reveal the problem with a vague definition. For our informal purposes, however, we will not discuss these questions in any detail. But we can understand 'subgraph' roughly to mean any token of a graph, A, within the area of another graph, B, which could be identified through an endoporeutic reading of B.

[28] Shin also offers a new reading algorithm for Beta, which I will also not address in detail.

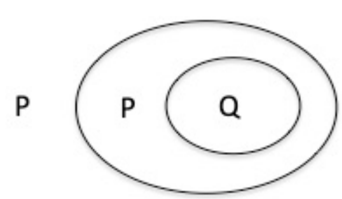
Alpha—Transformation Rules

As noted above, in addition to the rules for construction and interpretation of graphs, there are rules of transformation—analogous to deductive rules. Shin's reformulations of the Alpha rules are easiest to understand. Peirce stated five rules of transformation (and Zeman and Roberts follow him here). The fifth is a rule for adding or removing a double cut (corresponding to double negation in a typical deductive system). The other four rules are symmetrical—R1 and R2 being rules of erasure and insertion, respectively; and R3 and R4 being rules of "iteration" and "deiteration", respectively (Shin, 81; CP 4.505). Shin keeps the rule for double cuts, but reframes and simplifies the other four rules. She preserves the symmetry but simplifies these by framing them as consisting of erasure or insertion in either evenly enclosed or oddly enclosed areas. Her rules are stated in the following chart (Shin, 85):

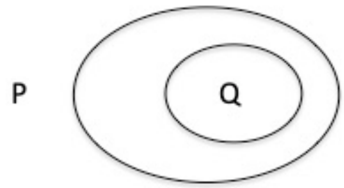
	E-area	O-area
Erase	X	X if there is another X either in the same area or in the next-outer area
Draw	X if there is another X either in the same area or in the next-outer area	X

Here, E-areas are areas enclosed in an even number of cuts (a graph enclosed in zero cuts is stipulated to be in an E-area), and O-areas are areas enclosed in an odd number of cuts. A few examples of simple valid arguments will show how the rules work.

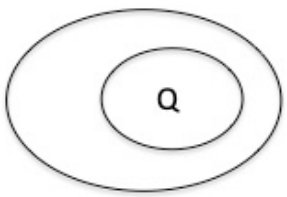
<u>**Modus Ponens:**</u> We begin with the premises (i) P, and (ii) P→Q, and aim to deduce Q. The premises represented graphically are thus:



P (a subgraph) occurs twice—once in an O-area and once in an E-area, which is the next-outer area. Thus we may employ the rule of erasure for a graph in an O-area. This gives:



From here, we are permitted to erase the token of P, since we can erase any graph in an E-area. This gives:

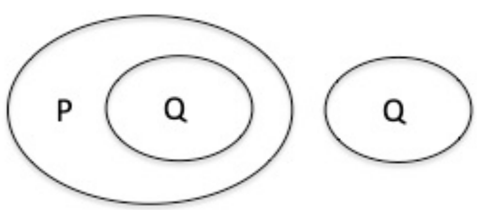


Finally, using the fifth rule (which allows us to add or erase a double cut around any graph, we may erase the double cut around Q, leaving:

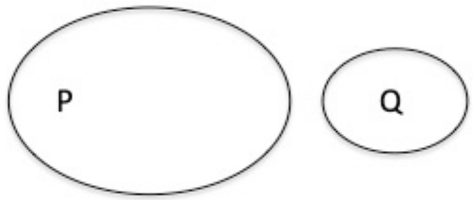
Q

And our proof is done.

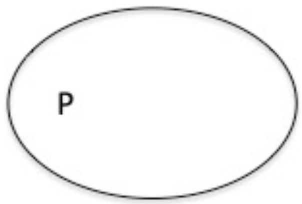
Modus Tollens: We begin with the premises (i) P→Q and (ii) ¬Q, and aim to deduce ¬P. The premises represented graphically are thus:



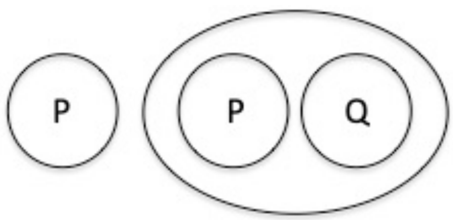
In a similar manner to the proof above, we notice that the subgraph "cut Q", which would be translated as ¬Q, occurs twice—once in an O-area and once in an E-area, which is the next outer area. Thus, by a similar process to what we used above, we may employ the rule of erasure for a token of a graph in an O-area to get:



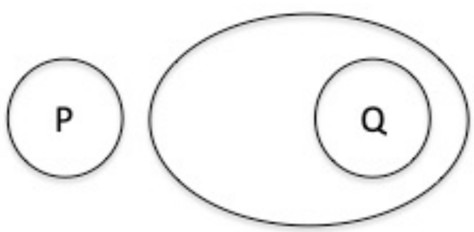
Since a token of any subgraph may be erased in an E-area, we may erase the second token of “cut Q”, which leaves the following graph, which would be translated as ¬P. Thus our proof is done.



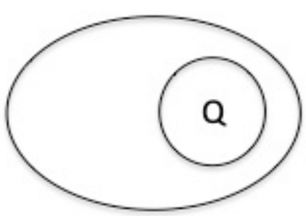
Disjunctive Syllogism: We begin with the premises (i) P∨Q, and (ii) ¬P, and aim to deduce Q. The premises would be represented graphically as:



Again, we see that “cut P”—the graph, which would be translated as ¬P occurs twice. We may erase the oddly-enclosed token since there is another token in the next outer area. This yields:



Then, in a similar manner to the proofs above, we can erase the token of “cut P” since it is in an E-area to get



Which by erasure of a double cut (the fifth rule), yields

Q

and so our proof is done.

Beta—Brief Introduction

The Beta system is an extension of the Alpha system, in the same way that a first-order language is an extension of a sentential language. Beta is "analogous to a pure symbolic first-order language with an equality symbol, but without a constant symbol. This system has three kinds of primitive vocabulary: predicate symbols, cut, and line" (Shin, 39).

The cut in Beta, as with Alpha, represents negation of whatever is contained within it. The significant additions are thus the predicate symbols and the lines (called 'lines of identity'). The need for lines of identity can be seen to arise from consideration of rhemata. '**Rhema**' (also called 'rheme'; plural rhemata) is a technical term of Peirce's, which is effectively equivalent to a predicate. He claims, "A blank form of proposition produced by such erasures as can be filled, each with a proper name, to make a proposition again, is called a *rhema*" (CP 4.438). So in the case of the monadic predicate 'man' the rhema would be '__is a man', where the empty slot could be filled by an indexical word like 'this' or a proper name like 'Socrates', etc. Similarly, we might have a dyadic rhema like '__is the father of__', or a triadic rhema like '__gives__to__', etc.

In the context of EG, Peirce introduces a related term, '**spot**': "In this system, the unanalyzed expression of a rhema shall be called a spot" (CP 4.441). We can think of a spot as a token of a rhema (which would be the type). Each spot is an n-adic relation of addicity ≥1. As such, we can imagine each spot as having a certain number of '**hooks**', which number corresponds to the addicity of the relation (ibid).

Peirce gives an example, where we have two partial graphs:

(Fig. 9)

A is greater than___

___is greater than B (CP 4.442).

These would indicate that 'A is greater than something' and that 'something is greater than B'. But suppose we want to assert that the two 'somethings' are the same individual.

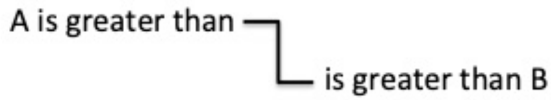
Peirce points out that this cannot be accomplished by simply adding a dyadic rhema in the following way:

(Fig. 10)

A is greater than___
___is greater than B
___is greater than___ (ibid)

This is because "the 'somethings,' being indesignate, cannot be described in general terms. It is necessary that the signs of them should be connected in fact. No way of doing this can be more perfectly iconic than…" (CP 4.442) (figure below also from CP 4.442)

(Fig. 11)



The single 'line of identity' signifies that it is the same individual (i) than which A is greater and (ii) that is greater than B. And thus, in general, "a heavy line, called a **line of identity**, shall be a graph asserting the numerical identity of the individuals denoted by its two extremities" (CP 4.444).[29]

In the example above, the predicates are partially "saturated." In a typical contemporary logical system we could signify the partially saturated predicate "A is greater than___" by a formula such as $\exists x(Gax)$. Here we have one relate of *G* being signified by a quantified variable (namely x), and another relate being indicated by an individual constant. EG makes no use of individual constants, so it treats constant names as monadic predicates. In the example above, the second predicate "___is greater than B" could be signified by the formula $\exists y(Gyb)$. If these two formulae are asserted together, as in Figure 9, nothing tells us that the variables x and y pick out the same individual object. But when we connect the lines of identity, as in Figure 11, then we have a graph equivalent to any of the following first-order formulae (which are all logically equivalent):

(i) $\exists x(Gax \wedge Gxb)$

[29] The convention of having lines of identity be "heavy" lines is simply to distinguish them from the lines used as cuts.

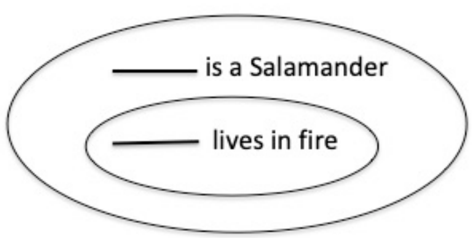
(ii) $\exists x \exists y((Gax \wedge Gyb) \wedge (x=y))$
(iii) $\exists y \exists x((Gyb \wedge Gax) \wedge (x=y))$

While these are logically equivalent, however, this example raises questions about scope of quantifiers. Peirce has ways of dealing with this issue, but they require no other syntactic conventions other than lines of identity (LOIs), cuts, and predicate symbols. However, any two sentential or first-order formulae that are logically equivalent will have exactly the same graph in EG. Peirce saw this as a virtue of the system.

A related concern, however, is how we might signify sentences with universal quantifiers. Obviously a sentence with a universal quantifier—$\forall v\ (\phi v)$—is logically equivalent to a sentence with only existential quantifiers—$\neg\exists v \neg(\phi v)$. Using clues from the previous discussion, we might realize that the negation symbol is the main connective in this latter formula, so we might (correctly) imagine that we would want the subgraph (translated as $\exists v \neg(\phi v)$) to be inscribed within a cut. But then we still have the problem of having a negation within the scope of an existential quantifier. How should this be handled?

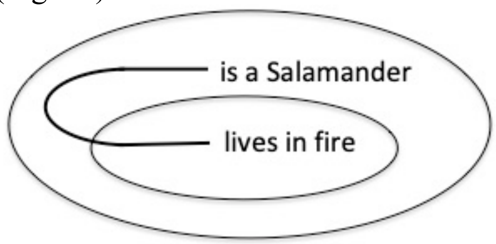
Peirce discusses this case with the following example: "How shall we express the proposition 'Every salamander lives in fire,' or 'If it be true that something is a salamander, then it will always be true that *that something* lives in fire'?" (CP 4.449). He first considers the simpler case where we do not assert the identity of "the [two] somethings" (ibid). This would be graphed in the following way (ibid):

(Fig. 12)



Now, if we want to assert that these two "somethings" are identical, then it would make sense to connect them, producing the following graph:

(Fig. 13)



But Peirce is keenly aware of the difficulties here, where a line of identity crosses a cut. As he points out, it might be natural to read Figure 13 as saying, 'There is something which, if it be a salamander, lives in fire'—that is, as asserting $\exists x(Sx \rightarrow Fx)$ rather than $\forall x(Sx \rightarrow Fx)$, which is the interpretation we want. He thus adds the following conventions to ensure the right interpretation: first, that "this part of the line of identity [that crosses a cut] is not a graph; for a graph must be either outside or inside of each [cut]" (CP 4.449). This convention is important for considerations of how to handle subgraphs in the rules of transformation. Secondly, he stipulates that points on a cut must be interpreted to lie outside the cut (Convention No. 8) (CP 4.450); and finally, that "the junction by a line of identity of a point on a [cut] to a point within the close of the [cut] shall assert of such individual as is denoted by the point on the [cut], according to the position of that point by Convention No. 8, a hypothetical conditional identity…" (CP 4.451).

I will not go into further detail on more complex Beta graphs. Zeman has a method for translating Beta graphs into first-order sentences with quantifiers, which proceeds by steps that turn Beta graphs into "quasi-Alpha graphs" by assigning variables and erasing lines of identity (Shin, 101). I will not discuss this reading algorithm in detail, but I will make a few brief points about it.

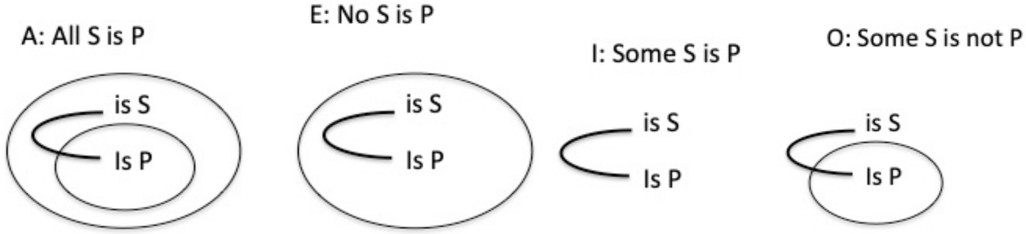
Zeman's method adds new quantified variables each time that a line of identity crosses a cut.[30] Like the endoporeutic reading method for Alpha graphs, Zeman's method is systematic and comprehensive but is wanting in its intuitive appeal, and obfuscates helpful visual features of the graphs that differentiate existential from universal quantification (Shin, 100). For example, the translation of the graph above would require two quantified variables and an assertion of equality of the variables. According to this reading algorithm, Fig. 13 would be translated in the following way (where *S* is the

[30] Technically, there is another term for a "line of identity" that crosses a cut, which we will discuss briefly. But for now, we can just think of it as a line of identity.

monadic predicate 'is a Salamander' and L is the monadic predicate 'lives in fire'): $\neg\exists x(Sx \wedge \exists y(x{=}y \wedge \neg Ly))$. This is logically equivalent to: $\forall x(Sx \rightarrow Lx)$, but of course it is a rather cumbersome formula. The problem only gets compounded with more complex graphs.

Shin offers an improvement to Zeman's reading algorithm, which is helpful in making the graphs more intuitive. But I will not address her reading algorithm here either, since it is not directly relevant to my purposes. I will, however, give the following examples of simple first-order formulae—those of traditional categorical syllogism:

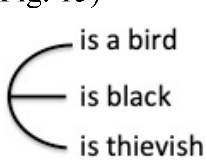
(Fig. 14)



Here, we can see clearly that A and O are contradictories, since A is formable from simply adding a cut around the whole subgraph O. E and I are contradictories for the same reason. Intuitively, we can see that the graph for I asserts that some single individual (indicated by the line of identity) is both S and P. Its negation, E, asserts (reading endoporeutically) that it is not the case that something is both S and P. As noted above, the graph for O is more complex since a line of identity crosses a cut. But (very informally), if read endoporeutically, it could be imagined as saying 'there is some individual which is S, and which is identical with some individual that is not P'—i.e. $\exists x(Sx \wedge \exists y(x{=}y \wedge \neg Py))$, which is logically equivalent to $\exists x(Sx \wedge \neg Px)$. The graph for A negates this, and thus asserts that $\neg\exists x(Sx \wedge \neg Px)$, which is equivalent to $\forall x(Sx \rightarrow Px)$.

Peirce also, importantly, permits branching of lines of identity. The following graph (Peirce's example) asserts that some black bird is thievish (CP 4.445):

(Fig. 15)

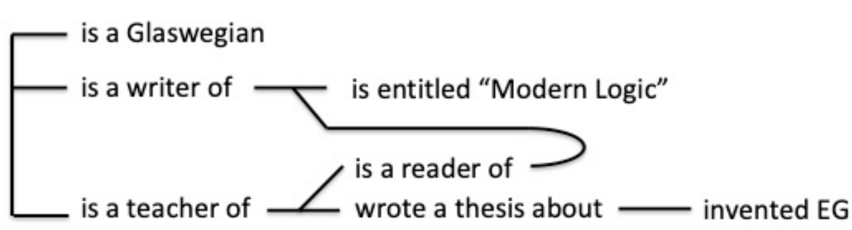


After introducing this example, he explains, “now it is plain that no number of mere bi-terminal bonds, each terminal occupying a spot’s hook, can ever assert the identity of three things, although when we once have a three-way branch, any higher number of terminals can be produced from it” (CP 4.445). And thus he adds the convention that, “a branching line of identity shall express a triad rhema signifying the identity of the three individuals, whose designations are represented as filling the blanks of the rhema by coincidence with the three terminals of the line” (4.446). So branching must happen from a point of **teridentity**—that is a point where *three* branches meet.

This is one of the two peculiar rules, related to conceptual combination, which I alluded to in the introduction. This paper will largely be occupied with considering the logical and philosophical motivation for this convention as well as another—the convention that only two lines of identity can be joined to each other—which we will discuss shortly.

Since a line of identity can branch, Peirce introduces another key term: “The totality of all the lines of identity that join one another is termed a **ligature**. A ligature is not generally a graph, since it may be part in one area and part in another” (CP 4.416).[31] The following graph seems complex, but is rather straightforward to interpret since no ligature crosses a cut. It contains four ligatures.

(Fig. 16)

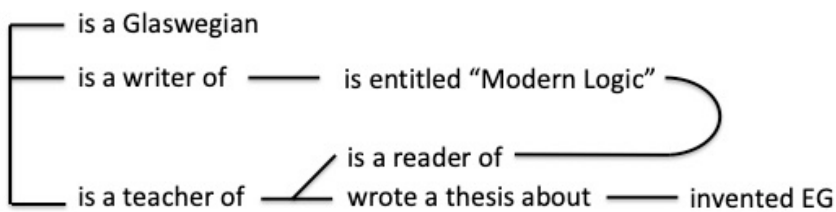


[31] Confusingly, Shin puts this term to different use. She calls the branches of a line of identity “ligatures” (Shin, 39), and uses the term “LI network” for a “connected web of branches” (Shin, 113, 119-120)—i.e. what Peirce means by a ligature. She also does not discuss Peirce’s subtle but important point that a ligature (in Peirce’s sense) is not technically a graph (CP 4.416, 4.449, 4.499). This idea is related to Frithjof Dau’s criticism of Shin that she does not draw a sharp distinction between graphs and their tokens (Dau, 57). This is also related to a flaw in her reading algorithm, which I believe results from her imprecise definition of ‘branching’ (Shin, 128; Dau, 57).

The ligatures are visually distinguished as each being a network of one or more connected lines. The proposition expressed by this graph would be very convoluted in English, but would be something like the following (though the names of the individuals signified by the ligatures are merely a heuristic and not part of the translation of the graph): There is something (A) that is a Glaswegian, and that is a writer of something (B) that is entitled "Modern Logic"; and A is a teacher of something (C); and C is a reader of B; and C wrote a thesis about something (D) that invented the Existential Graphs.

One might wonder, though, why we couldn't symbolize the same proposition by the following graph.

(Fig. 17)

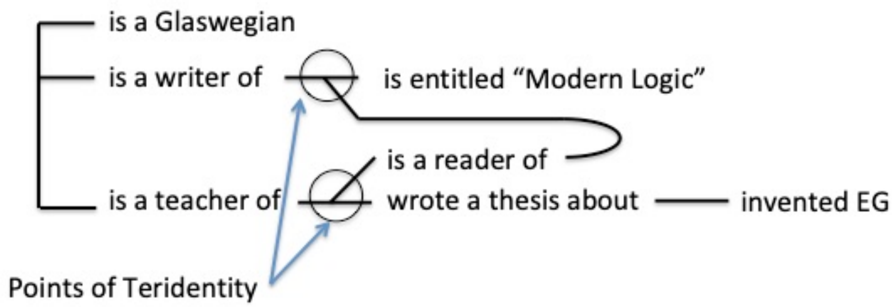


The only difference between this graph and Fig. 16 concerns the line passing from the predicate "is a reader of" to the predicate "is entitled 'Modern Logic'." The first reason that we could not construct this graph is that the predicate "is entitled 'Modern Logic'" is a *monadic* predicate with only one hook. It could be expressed in the form of a rhema as "__is entitled 'Modern Logic'", which can only be filled by *one* indexical word or proper name to produce a complete proposition such as "This book is entitled 'Modern Logic'." The graph in Fig. 17 above incorrectly treats that predicate as a dyadic predicate, and so is not permissible in EG.

The second point to make is that Fig. 17 would consist of five ligatures, instead of four, which was the intended interpretation. We want to assert that it is the same individual (the book) that is written by the Glaswegian and read by the thesis writer. Fig. 17 does not assert this either.

When we do graph this proposition in the correct way, as in Fig. 16, there are two points of teridentity—that is, points where three lines meet.

(Fig. 18)



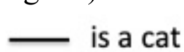
Why not imagine these as points where two lines meet, though? After all, if we removed the predicate '__is a reader of__' with its two corresponding lines of identity, then each of the lines to which it attaches here would seem to be one line rather than two. Peirce's answer is rooted in the idea that lines of identity can only be bonded one to another. But, if LOIs can only bond one at a time, then how is the graph above formable at all? It seems like one line of identity simply binds to the middle of another. In fact, this is not what happens, but to explain why, we must briefly consider the transformation rules for Beta.

Beta—Transformation Rules

I will not present the rules of transformation for Beta in any real detail since they are more complex than the rules for Alpha, and mostly not necessary to my project. But, as I have noted, two components of the rules are important: First is the restriction that I mentioned above—that lines of identity must branch in threes. Second is the restriction that only one line of identity can bind to another.

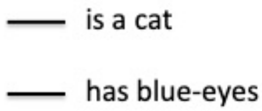
To see these, let us consider a very simple graph that asserts 'something is a cat'.

(Fig. 19)



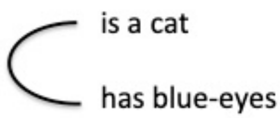
Now imagine that we want to graph 'something is a cat' and 'something has blue-eyes', without making any assertion that these two 'somethings' are the same individual. This would be shown by the following graph.

(Fig. 20)



Now suppose we *do* want to assert that it is the same individual, which would make the proposition 'some cat has blue-eyes'. In this case, we would combine the lines of identity to form the following graph.

(Fig. 21)



This joining of lines of identity asserts new information. But since

$\exists x Cx \wedge \exists x Bx \nvDash \exists x(Cx \wedge Bx)$,

Fig. 21 is not a permissible transformation from Fig. 20. But if we know that

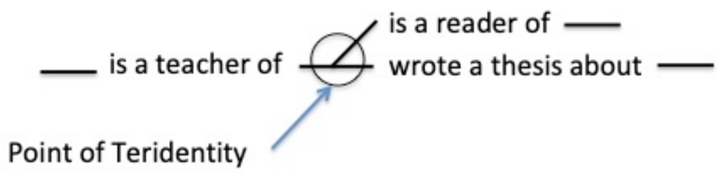
$(\exists x Cx \wedge \exists y By) \wedge x{=}y$

then we should be able to deduce that $\exists x(Cx \wedge Bx)$. And in EG, if we knew that these two 'somethings' were identical, then this sort of joining of lines of identity would be possible.

But it is vital to note that only two loose ends can bind to each other. In this paper, particularly in Chapters Three and Four, I will argue why Peirce thinks that combination of loose ends should be restricted in this way.

With the idea in mind that only two loose ends can bind, let us return to the example from Fig. 16. In that graph, it seems like the lines of identity coming from the predicate 'is a reader of' simply join the middle of other lines of identity. As mentioned, this is not actually the case, but some explanation is required to understand why. Let us examine one part of that graph—particularly one point of teridentity:

(Fig. 22)

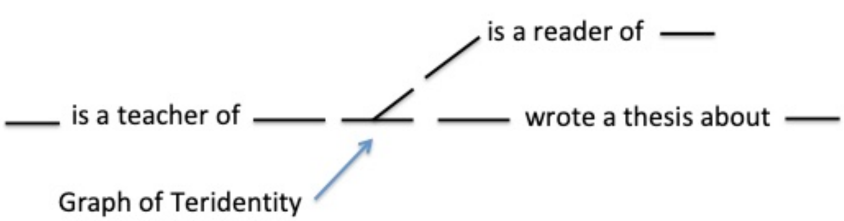


We can understand why this is a point of teridentity through some analysis. The predicate '__is a teacher of__' has two loose ends (i.e. unconnected lines of identity attached to hooks), as does the predicate '__wrote a thesis about__'. So imagine that we disconnect the joined line of identity that asserts that it is the same individual who occupies the second position of the 'is a teacher of' predicate and the first position of the 'wrote a thesis about' predicate. That graph would be the following:

(Fig. 23)

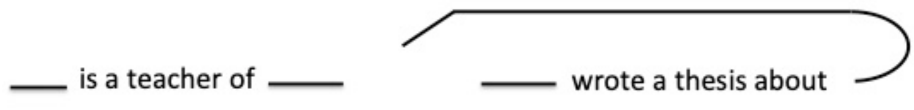
Since the '__is a reader of__' predicate also has two loose ends, we can position it in the same way. Then what we must do in order to assert that this same individual also occupies the first position of the 'reader' predicate is to insert a graph of teridentity—that is a graph with three loose ends, which asserts that three individuals are all identical.

(Fig. 24)



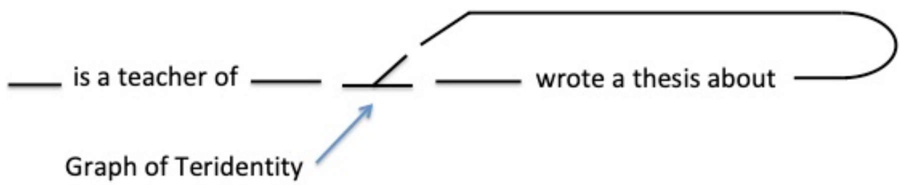
Then we would just need to connect the loose ends (one to another) in order to get the graph we want. Through connecting multiple graphs of teridentity, we can form graphs with any higher number of loose ends. We can also form graphs that assert that the same individual occupies multiple slots of a predicate. Suppose, for example, that we wanted to say that this individual wrote an autobiographical thesis. Thus we could arrange the lines of identity in the following way:

(Fig. 25)



But, as we have explained, the assertion that these lines refer to the same individual requires a graph of teridentity, which connects these three lines.

(Fig. 26)



On Peirce's view, this method is supremely analytical. It dissects reasoning into the greatest number of distinct steps. But it also does not overanalyze, drawing superficial distinctions—representing logically equivalent formulae, like $\neg\exists x(Sx \wedge \neg Px)$ and $\forall x(Sx \rightarrow Px)$, by one and the same graph. In the coming chapters, I will be attempting to give an account of *why* the two restrictions on conceptual combination outlined above (that lines must branch in threes and that loose ends must be joined one to another) are primitive. If they are justified in some robust logical sense, then the result is that the Reduction Thesis holds. And that claim has significant implications for logic and for metaphysics.

Chapter Two

Logic and Semiotics: The Trichotomies

Over the course of this paper, it will be necessary to discuss ideas that come out of Peirce's semiotics. In order to understand the relationship between Peirce's logic and his semiotics, we must have an understanding of several trichotomies that are central to his overall philosophical project.

Peirce confesses to having "a leaning to the number three in philosophy" (EP 1.247). He claims that any division into threes must rest upon the conceptions of First, Second, and Third—"ideas so broad that they may be looked upon rather as moods or tones of thought, than as definite notions" (EP 1.247). Viewed as numerals, for the purpose of making enumerations (ibid), it would be futile to consider their signification themselves. But since Peirce intends to use these "conceptions" philosophically he endeavors to give an (albeit extremely general) account of their meaning. "The First is that whose being is simply in itself…[it] must therefore be present and immediate…it cannot articulately be thought…every description of it must be false to it" (ibid, 248). "The Second is that which is what it is by force of something to which it is second…it meets us in such facts as Another, Relation, Compulsion, Effect, Dependence, Independence, Negation…" (ibid, 248). "The Third is that which is what it is owing to things between which it mediates and which it brings into relation to each other" (ibid, 248). As stated, however, these quasi-concepts already use language of being, immediacy, force, mediation, etc. As stated, they already carry at least metaphysical connotation, if not being already full-blown metaphysical ideas. And, indeed, Peirce speaks of the "universal categories" of Firstness, Secondness, and Thirdness (another trichotomy we will consider) in a metaphysical way throughout his philosophical career. But the conviction that there are three universal metaphysical categories is simply a corollary from two ideas: (i) the conviction that metaphysics must rest on logic, and (ii) the conviction that there are three logical categories—that is, three irreducible logical forms of predication. We will discuss these three logical categories more in the coming pages.

In this chapter, I will discuss the following trichotomies:

First: I will discuss Peirce's account of the structure of all signs. This is explained by the trichotomy Sign, Object, Interpretant. His triadic account of the sign is novel, and is deeply connected to other logical discoveries, as we will see.

Second, Peirce divides signs in general into three primary types. These are icons, indices, and symbols. Each of these types of sign indicates its object in a different fundamental way. While Peirce's rich semiotic theory elaborates many different types of signs, his division of types of signs always begins with this trichotomy.

Third, we will consider Peirce's "categories" Firstness, Secondness, and Thirdness. We have alluded to these a bit in the introduction as well as above. Following Kant's project, Peirce finds that there are three primitive elements in logic; and he thus makes the Kantian step to concluding that there are three metaphysical categories in Nature. I believe it is a legitimate worry that these might be phenomenological or metaphysical categories from the beginning, but I aim to defend Peirce in arguing that they are built on the logical categories, which were *discovered within the discipline of logic.*

During the course of discussion of the categories, we will take a detour through the fourth trichotomy, which led to their discovery. This is the trichotomy that differentiates three forms of reasoning: Abduction, Induction, and Deduction. Peirce's discovery of these came early in his career—about 1866. In this section, I will discuss one of his earliest discoveries—a logical error in Kant's reasoning. This finding led to the discovery of three independent forms of reasoning; but it also led to the discovery that a "metalogical" argument may employ certain principles that are not explicit in the object language, but are nonetheless required to justify the metalogical argument.[32] I believe that this is the crucial stepping-stone in how to understand the Reduction Thesis.

Peirce's account of Induction is also important for my argument in Chapter Four. There I will argue that Peirce's understanding of the referents of relations is similar to a key feature of Inductive reasoning. Finally, we will end this chapter by returning to a consideration of Firstness, Secondness, and Thirdness.

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[32] Again, Peirce does not explicitly use these terms, but I argue that his ideas are roughly the same. He does explicitly distinguish between signs of illation (the implication involved in claiming 'A, therefore B'), and the conditional (the implication involved in claiming 'if A, then B'). We will also explain this distinction in more detail below.

Sign, Object, Interpretant

Perhaps the most important of Peirce's trichotomies is the distinction between sign, object, and interpretant. According to Peirce's own account, it was early in his career (at least by 1868 or so) that he became convinced that all thinking is performed in Signs (EP 2.447). He also says, similarly, that "all reasoning is an interpretation of signs" (EP 2.4). This discovery was reinforced later in his life by his rereading of Plato's *Theaetetus.*[33] So what is a sign?

Perhaps the following definition (from Peirce's 1895 essay "Of Reasoning In General") is a good place to start:

> "A sign is a thing which serves to convey knowledge of some other thing, which it is said to *stand for* or *represent*. This thing is called the *object* of the sign; the idea in the mind that the sign excites, which is a mental sign of the same object, is called an *interpretant* of the sign" (EP, 2.13).

This definition sounds decidedly anthropomorphic. And yet, as Peirce claims elsewhere, "Anthropomorphic is what pretty much all conceptions are at bottom" (EP 2.151-2). In this regard he is a follower of Kant, maintaining that our human faculties are deeply and intimately involved in what we call 'reality' in a way that is inescapable. This idea is put well by another philosopher who came to see Kant's relevance for contemporary philosophy, namely Hilary Putnam. He writes, echoing Peirce's sentiment, that,

> "the elements of what we call 'language' or 'mind' penetrate so deeply into what we call 'reality' that the very project of representing ourselves as being 'mappers' of something 'language-independent' is fatally compromised from the very start" (Putnam, RHF, 28).

If our faculties (for Kant, faculties of Intuition and Understanding) are intimately involved in reality itself, as Kant teaches, then the search for objective knowledge must begin with an examination of the fundamental categories of thought. This is obviously

[33] See O'Hara, SPF, 10 and O'Hara Appendix, which contains Peirce's letter to Victoria Welby about the *Theaetetus*. See also Max Fisch "The Range of Peirce's Relevance", 31 where Fisch cites Peirce's annotated copy of Lutoslawski's book on Plato: "On page 376, where Lutoslawski quotes Plato's Socrates as defining thought at *Theaetetus* 189E as 'a conversation of the soul with itself,' Peirce writes in the margin: 'This is, I think, Plato's greatest contribution to thought" (Fisch, RPR, 30-31).

Kant's project in the Transcendental Analytic. Peirce affirms this overall project, but he aims to generalize:

> "The first question, and it was a question of supreme importance…was whether or not the fundamental categories of thought really have that sort of dependence upon formal logic that Kant asserted. I became thoroughly convinced that such a relation really did and must exist. After a series of inquiries, I came to see that Kant ought not to have confined himself to divisions of propositions, or 'judgments,' as the Germans confuse the subject by calling them, but ought to have taken account of all elementary and significant differences of form among signs of all sorts, and that above all, he ought not to have left out of account fundamental forms of reasoning" (EP, 2.424).

According to Peirce, Kant's overall project was headed in the right direction. It aimed to determine the structures of reality by considering the fundamental structures of thought; and this would come primarily by an investigation into formal logic. The principal lacuna in that project was simply an insufficient account of formal logic, particularly Kant's "neglect of the logic of relations" (EP 2.219). Peirce would delve deeply into this topic, as we will see. We will also consider "the fundamental forms of reasoning" to which Peirce alludes in the passage above. But for now, our central aim is to clarify Peirce's basic trichotomy sign-object-interpretant. As noted above, Peirce was convinced that all thinking takes place in signs. In that case, an understanding of the basic structure of signs in general would offer a clue to finding the fundamental categories of thought, which would in turn provide a clue about the metaphysical categories of reality.

Peirce was convinced that his sign-object-interpretant trichotomy was extremely general, and non-psychological (perhaps even anti-psychological)[34]. In his later work and

[34] There is some debate in the secondary literature as to whether or not Peirce truly advocated anti-psychologism in regard to logic. Susan Haack argues that he is a proponent of "weak psychologism" where logic is *prescriptive* of mental processes—that is, it prescribes how we *should* think. According to Haack, this is contrasted with the "Strong psychologism" of Kant who holds that logic is *descriptive* of mental processes, and is also contrasted with "anti-psychologism" of, say, Frege, where logic has nothing to do with mental processes (Haack, PL, 238). However, other thinkers—e.g. Francesco Bellucci—identify Peirce as an anti-psychologist, citing for example Peirce, W 1:164 "Logic has nothing to do with the operations of the understanding, acts of the mind or facts of the intellect." Part of the complexity here is that Peirce often spoke of formal logic *as* mathematics, but conceived of logic generally as being a broader study than we would normally consider today. Peirce, in one of his most complex and significant papers, "Prolegomena to an Apology of Pragmaticism", 1906, claims that, "I must be understood as talking not psychology, but the logic of mental operations" (CP 4.539). We

correspondence he speaks of the sign-object-interpretant distinction using language of determination and causality, which goes from object to sign to interpretant (and not vice versa).[35] As late as 1908 he also laments that he cannot find the language to make his broader, more general notion intelligible:

"I define a Sign as anything which is so determined by something else, called its Object, and so determines an effect upon a person which effect I call its Interpretant, that the latter is thereby mediately determined by the former. My insertion of 'upon a person' is a sop to Cerberus, because I despair of making my own broader conception understood…" (EP, 2.478 (1908)).[36]

Part of the exegetical challenge in understanding Peirce's "anthropomorphism" may be aided by recognizing that he does not see Thought as something exclusively human.

"Thought is not necessarily connected with a brain. It appears in the work of bees, of crystals, and throughout the purely physical world; and one can no more deny that it is really there, than that the colors, the shapes, etc. of objects are really there. Consistently adhere to that unwarrantable denial, and you will be driven to some form of idealistic nominalism akin to Fichte's. Not only is thought in the organic world, but it develops there" (CP 4.551).

Peirce's technical term *interpretant* is likely the cornerstone of his entire semiotic theory, and possibly of his entire system of philosophy. It differentiates his semiotic theory from the theories of other semioticians, like Augustine or Saussure, whose theories of the sign are dyadic rather than triadic. It is also a concept that has deep implications for Peirce's formal logic. The interpretant is so named "because it fulfills the office of an interpreter" (EP 1.5). Peirce's discovery that any meaningful sign requires an interpretant is perhaps made clearer with an example that he gives. Imagine looking in a French-

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can see, though, that the relationship between logic, mathematics and semiotics is complex.

[35] See Peirce's 1909 letter to James (EP 2.497) where he uses the German word *bestimmen* to characterize the sign relation, and Writings 2.155 for explanation of that term: "fixed to be thus". He explains that a sign is something, "which being *bestimmt* by…its Object…in its turn *bestimmt* the mind of an interpreter…" (ibid). In the 1909 letter he notes, "you will probably object that *bestimmt*, to your mind, means 'causes' or 'caused.' Very well, so it does to mine…" (EP 2.497). But then he goes on to explain that we should understand 'cause' not just in terms of efficient cause, but also in terms of Aristotelian final cause. This passage is also related to his discussion of cause on EP 2.315, which we will discuss in Chapter Three.

[36] See Fisch, 342-344 for a discussion of the sop to Cerberus.

English dictionary and finding the word *homme*; "…we shall find opposite to it the word *man*, which, so placed represents *homme* as representing the same two-legged creature which *man* itself represents" (EP 1.5). In this case, the word *homme* is a sign[37]; similarly the word *man* is a sign. But these two signs are brought into a particular relation by a *third* sign—i.e. the convention where the two words are juxtaposed, which is common in translational dictionaries. This third sign represents the two signs (the relata in the relation) as being signs of the same object. And this third sign is *required* in order to convey this information—that these are signs of the same object. Peirce's concept of the *interpretant* attempts to generalize this idea; he argues that the interpretant is required in order for the sign to represent *its* object. The interpretant represents the sign and object as being in relation. To make this a bit clearer, consider another example. Suppose a child is in bed at Christmas time. This child lives nowhere near train tracks, but he is awoken to hear hissing steam and squeaking metal outside his window. This sound (for the sake of argument we will consider it as *one* sound) is a sign. That sign conjures up (*bestimmt*) the mental image of a train. In this case, the Polar Express itself is the object, the sound is the sign, and the mental image is the interpretant. But if the child never realized the mental image—i.e. if the sign never had an interpretant—then the sign (the noise) would not properly be a sign of *its* object (the train) at all.

In Peirce's late defense of pragmatism (1905), he takes up this notion of translation again. He claims, "The rational meaning of every proposition lies in the future. How so? The meaning of a proposition is itself a proposition. Indeed, it is no other than the very proposition of which it is the meaning: it is a translation of it" (EP 2.340). But naturally there are many possible ways to translate a proposition. So we might be driven to ask, what sort of translation is the *essential* one? The answer, according to the pragmaticist is, "that form in which the proposition becomes applicable to human conduct" (ibid), specifically human conduct in a controlled, lawlike, habitual manner.

It is worth noting the close affinity between the concept of interpretant and the concept of mind. We may even go as far as Robert Burch does in arguing that, "a full

[37] More precisely, the particular collection of graphemes constitutes a token of the type, which is the word in the more general sense. See CP 4.537 (1906). Russell makes the same point in IMT, 23, 54-55 (1940).

account of what Peirce meant by a relation would have to involve the conception of a mind" (Burch, 1991, ix). In Burch's own logical system—called Peircean Algebraic Logic, or PAL, which is inspired by EG but rendered algebraic—Burch symbolizes the concept of 'mind' formally as an interpretation function. There is much to provoke thought here, but we are not yet in a position to discuss Burch. In regard to EG, however, Don Roberts, another pioneering scholar of EG (giving its completeness proof in 1973) notes that the entire system of EG is "a rough and generalized diagram of the mind" (Roberts, 113). As noted above, it is perhaps significant that as late as 1908 (Peirce died in 1914), Peirce was still using the notion of mind to explain the concept of an interpretant. But viewed another way, this is not so surprising. Consider a natural sign such as smoke from a fire, which indicates its object—namely the fire. We might be tempted to think that there is nothing human involved here. But, if the sign is to have meaning at all, and *indicate* its object, then there must be something to which it indicates meaning—and that something would seem to be mindlike. Peirce is sure to emphasize that this does not mean a *human* mind, though. In addition to speaking of Thought existing in the work of bees, crystals, etc., in later years, criticizing his earlier nominalism, Peirce also builds *possibility* into his account of semeiosis. A diamond is not hard because it *does* or *will* produce certain experimental results; it is hard because the conditional is true that *if it were* subjected to certain experiments, *then it would* produce certain results (Haack, Pragmatism, 645).

Icon, Index, Symbol

According to Peirce all signs have the triadic structure sign-object-interpretant. For Peirce, signs are divisible into three principal kinds—icons, indices and symbols.[38] From this trichotomy, Peirce overlays others and his taxonomy becomes more complex,

[38] In certain earlier works (e.g. ALPT, 3.360, (1885)), Peirce uses the term 'token' instead of 'symbol'. Later he went on to change the meaning of this technical term to the meaning that is more familiar now—i.e. distinguishing token from type. In fact, there is a third technical term in that trichotomy tone-token-type (EP 2.488), but it is little discussed. I suspect that he used the term token in these works because he saw the word as the paradigmatic example of a symbol. Each individual instance of a word is a token of a type (the word in the general sense).

but he consistently maintains this trichotomy, and defines the other sub-division of signs in terms of it.

"Firstly, there are *likenesses*, or icons; which serve to convey ideas of the things they represent simply by imitating them" (EP 2.5). Icons are signs, which are interpreted as representing their objects by means of resemblance. Some examples might include a portrait painting, which indicates the person painted by virtue of the fact that it (the painting) resembles the person. Another example would be a drawing of a triangle on paper, whose object is the triangle understood as a type (EP 2.306). In our discussion of the Existential Graphs, we will see in more detail the value Peirce placed on icons in logic. In this regard he praised the work of Euler who created a system of logical diagrams that employs icons. Euler uses circles to represent classes. In his system, the geometric relation of enclosure (one circle being enclosed within another) *resembles* the abstract relation of set-membership (one set being a subset of another). Because the one relation resembles the other, it is suited to represent the other as an icon.

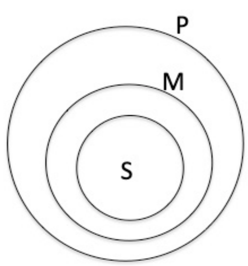
Here, perhaps we ought to pause though, since at present we are using the word 'resemblance' vaguely. If one relation resembles another, does that mean that *every* property of the one—e.g. transitivity, reflexivity, etc.—is a property of the other? Or does it mean only that *some* of the properties of one are the same? In our later discussion of Abduction, this question will be made clearer. But we can say now that Peirce understands resemblance to be intentionally vague. We might say that for a sign to (be interpreted to) resemble its object, every property *we are aware of* must be the same between the sign and the object. If we notice that there is some property of the one that is different from the other, there is no longer resemblance, but the reasoning that decides this is a different form of reasoning, which is more complex. Furthermore, the sign would then become more complex, on Peirce's view. So we should understand "resemblance" simply as that feature by which the sign produces an interpretant. For example, if a poorly drawn triangle is still sufficient to produce the idea of a triangle, then it is an icon. If it is drawn so poorly that the idea is not produced, then either it is not properly a sign, or it must be stipulated to represent its object (the triangle as type). As we will explain briefly, in this case, it would become a symbol.

Though there is some vagueness to the notion of resemblance, icons are extremely useful—particularly in mathematics. A drawing of a triangle is mathematically useful because it resembles the type, which is inaccessible except through tokens. Furthermore, these sorts of icons in mathematics seem to be quite essential to mathematical reasoning. Max Fish quotes Peirce in saying, "the icon is very perfect in respect to signification, bringing its interpreter face to face with the very character signified. For this reason it is the mathematical sign *par excellence*' (NEM, 4:242)" (Fisch, 335-6). In Euler's case, the resemblance allows us to "experiment" upon the diagram, and observe new facts about the geometrical relation, which can then be applied to the abstract subset relation. An example will hopefully make this idea a bit more clear.

Peirce asks us to consider the following syllogism (4.350):

> All men are passionate,
> All Saints are men;
> Therefore, All saints are passionate

In Euler's diagrams this would be diagrammatically represented thus:



The major premise tells us that the class of men is a subset of the class of those who are passionate. Again, in the diagram, classes are represented by circles. But the sign that represents one circle as inscribed within another is an iconic sign of the abstract relation (which is the object of that sign). The diagram is useful because it *displays* the transitivity of the geometrical relation (of inclusion). We are then permitted to apply this property (transitivity) to the abstract subset relation, and *see* the conclusion of the inference. For this reason, as we will discuss in more detail, Peirce is very interested in diagrams and their application in logic. We have already seen how EG goes far beyond Euler's diagrams; but in creating EG, Peirce was vigilant in trying to maintain the element of iconicity.[39]

In his 1904 essay "New Elements", Peirce describes icons and indices as "degenerate" signs. The term "degenerate" is borrowed from geometry—"degenerate in the sense in which two coplanar lines form a degenerate conic" (EP 2.306). Peirce notes that "a pure icon is independent of any purpose…it asserts nothing" (ibid). As soon as it is interpreted as asserting anything, on Peirce's view, it is technically a symbol, though it may possess iconic features. It is in this sense that the pure icon or index, which asserts nothing, is degenerate. We will discuss this more below, but it is also worth noting that Peirce believed that the most perfect logical system would embody all three semiotic elements. By doing so, the "moving pictures of thought" (CP 4.8) of diagrams in the system would most closely resemble thought itself.

Peirce's second major kind of sign is the index. "Secondly, there are *indications,* or indices; which show something about things, on account of their being physically connected with them" (EP 2.5). Peirce goes on to give a few examples:

"such is a guidepost, which points down the road to be taken, or a relative pronoun, which is placed just after the name of the thing intended to be denoted, or a vocative exclamation, as 'Hi! There,' which acts upon the nerves of the person addressed and forces his attention" (ibid).

It is clear that Peirce intends these divisions to cross-cut the Augustinian distinction between natural and conventional signs. In fact, his examples of indices are often from the natural world. For example, the vocative exclamation above is not chiefly a sign because of any meaning signified; rather the index is a mere sound, which through direct physical connection (i.e. sound waves) acts upon the nerves of the person addressed. In other places, he gives the example of smoke being an index of a fire, and of a weathervane being an index of wind direction since it is moved by physical contact with its object (namely the wind) (EP 2.406).[40]

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[39] See Ketner, 1987: "I propose to use the term *logical graph* to designate any diagram which iconizes logical relations by means of geometrical relations {MS 483:02, 1898-9}" (Ketner, 550, quoting Peirce).

[40] In 1866 Peirce realized that logical operations could, in theory, be performed by electrical switches, anticipating Claude Shannon's groundbreaking discovery by over 50 years (Peirce, W5.xliv, W5.421-3). This idea depended on another anticipated

And yet, a number of these examples reveal ways that the divisions are not always entirely clear-cut. Consider the weathervane. Like the non-verbal vocative exclamation, it is an index. But unlike the mere exclamation, it conveys information, namely the direction of the wind. In this sense, it possesses qualities of an index but is more properly a symbol. Similarly, if the vocative exclamation were verbal—for example "Watch out!"—it would signify indexically in the same way that the mere exclamation does, but it also conveys information and is thereby a symbol. Another example might be a mirror that reflects an image. It is iconic insofar as it resembles the image that produced it. But the image in the mirror (here considered as sign) is produced by direct physical connection with its object, so it is both iconic and indexical. Peirce's semiotic taxonomy is quite exhaustive in this regard, and he goes to great lengths to accommodate these sorts of distinctions (see particularly EP 2.289-299). But the subdivision is always triadic and always begins with the division into icon, index, and symbol.

An important corollary of Peirce's definition of an index is that for a sign to be an index at all, its object must exist (e.g. EP 2.278). Traditionally speaking, this means that it must exist in the actual mode. But Peirce underwent significant development in his thinking about possibility—moving from a nominalist position early on (about universals) to a metaphysics that embraced Aristotelian "real possibility."[41] He attempted

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discovery—namely his anticipation of Sheffer that only one truth-functional connective ('nand') was necessary for expressive completeness (CP 4.12 (1880); Guinness, 31). The case of a logic gate in an electrical circuit is an interesting semiotic example. Considered in itself, the logic gate would be an index; insofar as it is in direct contact with an electrical current, it is a sign (an index) of that current. The current acts on the sign (the logic gate) by physical connection, as in the case of sound waves acting on an eardrum, and the logic gate either stops that current or allows it to pass. In either case, the immediate interpretant (a division of the concept of interpretant that we will not address) is something non-human, and non-mind-like. This immediate interpretant can be understood to be either the flow of electricity to the next logic gate, or the arrest of that flow. But the logic gate also is designed to convey information, and does so by virtue of intrinsic features that were designed in a particular way. Thus the final interpretant of the logic gate would be something more complex—for example the display of a number on a screen, which is the conclusion of some arithmetical operation that the logic gate played some part in performing. In this sense, the logic gate should be understood as a symbol. Obviously the complete semiotic account here would get extremely complex.

[41] See, for example, EP 2.180 where Peirce discusses Aristotle's doctrine of 'real possibility': "The doctrine of Aristotle is distinguished from substantially all modern

to integrate notions of possibility and necessity into the Gamma part of the Existential Graphs, but that subject is unfortunately beyond the scope of this paper. The Beta graphs (the part that is theorem isomorphic to first-order logic with identity) consider a state of the universe *de inesse*—that is, at a single instant. "A proposition *de inesse* relates to a single state of the universe, like the present instant. Such a proposition is altogether true or altogether false" (CP 4.376). Peirce was acutely aware of the logical and metaphysical challenges to truth-functionality that come by incorporating time into logic. And indeed, late in his life, he anticipated later developments in modal logic, trivalent logical systems and other ideas in non-standard logic that attempt to deal with these sorts of challenges.[42]

Finally, thirdly, "there are *symbols*, or general signs, which have become associated with their meanings by usage. Such are most words, and phrases, and speeches, and books, and libraries" (EP, 2.5). Symbols represent their objects merely because they are interpreted as such. That is to say, they are interpreted to represent their objects by virtue of convention. With symbols—more so than with icons and indices—the need for interpretation is most explicit. For example, in considering the case above that distinguishes the geometrical relation of inclusion from the abstract relation of subsethood, we may be liable to overlook the fact that these are separate relations at all. But we can see immediately that unfamiliar symbols are meaningless. The symbol '⌫' will not have any meaning to someone who has never used a computer, but those who have will recognize it as representing the 'delete' button. Because the symbol makes explicit its need for an interpreter, Peirce occasionally calls it the most genuine form of sign.[43]

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philosophy by its recognition of at least two grades of being. That is, besides *actual reactive existence*, Aristotle recognizes a germinal being, an *esse in potentia* or I like to call it an *esse in futuro.* In places Aristotle has glimpses of a distinction between ενεργεια and εντελεχεια."

[42] See Fisch, 171 on Peirce's trivalent logic; See also Hawkins: "Peirce's work is a vista on modern formalism, intuitionism, model theory and nonstandard analysis" (Hawkins, 121).

[43] See EP 2.164. This also relates to the technical distinction he makes between a genuine sign and a degenerate sign, as was mentioned above (EP 2.306; EP 2.171). This distinction is actually quite relevant for the Reduction Thesis, as Burch and Brunning discuss. But it would take us too far into semiotics to consider it in detail. However, we will note that a genuine index *involves* an icon, and a genuine symbol *involves* an index

Symbols also “both denote and connote” (Bellucci, 40). While an index denotes without any connotation, and an icon connotes without any denotation, a symbol does both, and thus, “only symbols have information” (ibid). Some of the examples given above—for example a weathervane—express information, e.g. the direction of the wind. Insofar as it gives information, the weathervane is properly a symbol, though it possesses indexical elements (again Peirce goes to great lengths to give a proper taxonomy).

Symbols are also especially important for Peirce, since the very notion of “convention”, which characterizes the symbol, suggests a habitual or lawlike *pattern* of interpretation. “[Symbols] alone express laws” (EP 2.308). In his discussion of logical diagrams that prefaces the presentation of the Existential Graphs, Peirce notes:

> “There must be an interpreter, since the graph, like every sign founded on convention, only has the sort of being that it has if it is interpreted; for a conventional sign is neither a mass of ink on a piece of paper or any other individual existence, nor is it an image present to consciousness, but is a special habit or rule of interpretation and consists precisely in the fact that certain sorts of ink spots—which I call its replicas—will have certain effects on the conduct, mental and bodily, of the interpreter” (CP 4.431).

Here another important concept comes to light—the concept of a replica. Peirce notes, “Now it is of the essential nature of a symbol that it determines an interpretant, which is itself a symbol. A symbol, therefore, produces an endless series of interpretants” (EP, 323).[44] This feature of symbols is what allows them to express laws and habits.

Peirce made much of his related distinction between type and token, and obviously this important distinction is still germane to contemporary analytic philosophy.[45] Individual replicas are tokens of the same symbol (the type). Max Fisch notes, “The most striking features of these later writings [1903-1911] are the high frequency of focus on pragmaticism and the development of a semeiotic realism out of

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and an icon (EP 2.310). This is part of the theoretical virtue of EG—it involves all three types of signs.

[44] See also EP 2.310 and EP 2.317: “A sign has its being in its adaption to fulfill a function…It is, therefore, what it is understood to be. Hence if two symbols are used, without regard to any differences between them, they are replicas of the same symbol.” He goes on to give the example of the words ‘he’ and ‘him’.

[45] The first place I have found where Peirce uses this distinction is in 1902 (EP 2.125). It is explained, perhaps, most clearly in CP 4.537 (1906). Additional explanation can be found in EP 2.488 (1908), and EP 2.483 where he characterizes “famisigns”, which is an equivalent name for ‘type’.

the type-token distinction" (Fisch, 338). The distinction between type and token, for Peirce, is a semiotic distinction, and can help us understand the nature of symbols in general. As Fisch says elsewhere, "Tokens are signs in the first place of their types, and only thereby are they signs of anything else" (ibid, 357). Furthermore, Peirce sees the manner in which a token signifies its type to be an instance of something like law:

"…the word [in his example, 'man] is not a thing. What is its nature? It consists in the really working general rule that three such patches [the letters of the word 'man'] seen by a person who knows English will effect his conduct and thoughts according to a rule" (CP 4.447).

This distinction also helps to clarify Peirce's professed anti-psychologism: "thinking is a matter for psychology, thought for logic. Thought is type; thinking is token" (Fisch, 360).

Symbols are of three primary kinds, all of which are very relevant to logic. "The *Symbol*, or relatively genuine form of [Sign], divides by trichotomy into the Term, the Proposition, and the Argument" (EP 2.164). In the essay "New Elements" (1904), he equates his technical words 'rhema' and 'term' suggesting that a 'term' is to be understood as an n-adic predicate: "If from a propositional symbol we erase one or more of the parts which separately denote its objects, the remainder is what is called a *rhema*; but I shall take the liberty of calling it a *term*" (EP 2.308). We will discuss the concept of a rhema (also called rheme) in some detail in this paper. That concept undergoes quite a bit of conceptual development in Peirce's logic, as we will see in Chapter Four. In his mature view, the number of blanks to be filled (after we erase the parts of the proposition denoting its objects) represents the addicity of the rhema. As we will see, Peirce thinks that rhemata are of three primitive types—those with addicity of 1, 2 and 3 respectively. This claim that there are three and only three primitive types of rhemata (i.e. relations) is essentially what is claimed in the Reduction Thesis.

Firstness, Secondness, Thirdness

This trichotomy also grows out of Kant, and is indeed the foundation upon which Peirce's entire metaphysics rests.

"Kant taught that our fundamental conceptions are merely the ineluctable ideas of a system of logical forms; nor is any occult transcendentalism requisite to show that this is so, and must be so. Nature only appears intelligible so far as it appears rational, that is, so far as its processes are seen to be like processes of thought…That there are three elementary forms of categories is the conclusion of Kant, to which Hegel subscribes[46]; and Kant seeks to establish this from the analysis of formal logic" (CP 3.422).

He goes on here and elsewhere to strongly criticize Kant—primarily Kant's insufficient analysis of formal logic, but again, he sees himself as continuing and refining Kant's project, rather than departing from it.

I have mentioned that the categories arise out of logic. As Peirce himself describes, they are the result of making "the Kantian step of transferring the conceptions of logic to metaphysics" (NEM 4.331). We alluded to them at the beginning of this chapter. Firstness is

"the mode in which anything would be for itself, irrespective of anything else…Now this mode of being can only be apprehended as a mode of feeling. For there is no other mode of being which we can conceive as having no relation to the possibility of anything else" (NEM 4.332).

Elsewhere (e.g. EP 2.150) he gives Firstness the name Quality.

Secondness is a mode of being which involves one thing reacting with another. In characterizing Secondness, he often speaks of force or effort, which always involves an equal and opposite force—two sides of the same dyadic reaction between two things. "There could not be effort without equal resistance any more than there could be a resistance without an equal effort that it resists" (EP 2.150). He also characterizes Secondness as "Pairedness" (NEM 4.332), recalling its logical basis.

Thirdness is often characterized by the synonyms mediation, representation, generality, law, and habit. He claims, "Every law or general rule, expresses a thirdness" (NEM 4.333).

"Thirdness is found wherever one thing brings about a Secondness between two things. In all such cases, it will be found that Thought plays a part. By thought is meant something like the meaning of a word, which may be 'embodied in,' that is, may govern,

[46] Peirce interprets Hegel's "three stages of thought" as his "universal categories", in contrast to his long list of particular categories. He notes, "I regard the fact that I reached the same result as he did by a process as unlike his as possible, at a time when my attitude toward him was rather one of contempt than of awe, and without being influenced by him in any discernible way however slightly, as being a not inconsiderable argument in favor of the correctness of the list" (EP 2.148). Peirce criticizes Hegelianism from a metaphysical perspective, however. He claims that it attempts to reduce everything to Thirdness, unlike Pragmatism, which respects the elementarity of all three of the categories (EP 2.345).

this or that, but is not confined to any existent…Thought is of the nature of a habit, which determines the suchness of that which may come into existence, when it does come into existence…If, however, there be any regularity that never will be and never would be broken, that has a mode of being consisting in this destiny or determination of the nature of things that the endless future shall conform to it, that is what we call a *law*" (EP, 2.269).

It is important to note that, for Peirce, this trichotomy informs all other divisions. Peirce *discovered* the categories by making the Kantian step from his discovery about logical concepts; but once discovered, the categories map onto all other divisions in Peirce's metaphysics. They even are mapped onto the fundamental Sign relation. "The Sign, in general, is the third member of a triad; first a thing as thing, second a thing as reacting with another thing [its object]; and third a thing as representing another to a third [the interpretant]" (NEM 4.331). This talk of categories may seem like loose talk. And indeed, it must be left for some other paper to defend how these phenomenological or metaphysical categories are derived from their logical analogs. My focus here is on the logical categories—the three primitive types of logical predicates. But I believe some understanding of the metaphysical categories will be helpful for our later discussion.

At this point, we must take a detour through the final trichotomy that we will consider, for it was Peirce's discovery (in his twenties) of a flaw in Kant's reasoning which led to his discovery of three forms of reasoning, and then to the categories—Firstness, Secondness, and Thirdness. Throughout his life he maintained that the categories could be defended most strongly from the domain of logic.[47] But his discussion of the categories often shades into phenomenology, as well as metaphysics. And while he resolutely maintained that metaphysics is built on logic and not vice versa,[48] his concept of phenomenology is a bit hard to pin down. So it is, admittedly, a bit unclear whether these categories are phenomenological, which are then applied to logic, and then to metaphysics, or whether they are discovered in logic and then applied to

[47] See, for example, Peirce's 1905 letter to William James: "[Royce] attacks my one-two-three doctrine in the very field where it is most obviously defensible, that of formal logic" (cited in Herzberger, 57).
[48] Again, see for example: EP 2.376 "most of the metaphysical conceptions…are nothing but logical conceptions applied to real objects"; also EP, 2.393; EP 2.424; EP 2.257

phenomenology and metaphysics.[49] As I have discussed, the picture is also complicated by his views about the relationship between logic, semiotics, and mathematics. Moreover, Peirce notes that it took him twenty-five years of intense logical study (until roughly 1892) to reach a "provisionally final result" (EP, 2.425) regarding the fundamental logical modes of predication (i.e. the foundation of the categories). But his work during the years 1867-1892 is heavily influenced by the idea of the categories. So this raises legitimate questions. However, here we must draw the important distinction between, on the one hand, biographical questions about how Peirce himself first made this discovery, and on the other hand questions about how Peirce's logical system is most defensible. The latter enquiry may look to the former for clues, but that must be the extent of it. In that vein, my task here is not chiefly biographical, though I take clues from the biographical account of Peirce's discoveries.

A Detour: Deduction, Induction, Abduction and Memoranda Concerning the Aristotelian Syllogism

In a 1905 letter to Mario Calderoni, Peirce claims that his elusive proof of Pragmatism would begin with an account of the "three elementary kinds of reasoning" (Ochs, 198): Abduction, Induction, and Deduction.[50] His discovery that there are three distinct forms of reasoning grows out of his consideration of a fallacy that he found in Kant. In the following section I will explain Peirce's discovery of this fallacy. I believe his discovery is chiefly interesting to us in that it clarifies how we ought to understand the Reduction Thesis. This discovery was also the key stepping stone for Peirce's discovery of the logical elements, though it took twenty-five years or so to remove the haze. That discovery, in very broad strokes, was that there may be certain structures in an argument that are ineliminable. That is to say, even though an object-level argument may be "reduced" in the object language, there may be structures from the reduced argument that smuggle their way into the meta-level argument and which are *required* there in order to perform the object-level reduction. This idea will hopefully be made clearer in the following section. I believe that this discovery is the linchpin for how we should

[49] Bellucci discusses Peirce's later engagement with phenomenology (Bellucci, 51).
[50] Also see Max Fisch, "Proof of Pragmatism", 362

understand the Reduction Thesis. As I will argue later, I believe the Reduction Thesis is making a claim about any metalogical argument that reduces triads to dyads—namely that the metalogical argument itself must employ triads in order to perform the reduction. After explaining Peirce's discovery of the germ of this idea, we will return to a consideration of the three forms of reasoning—Abduction, Induction, and Deduction. Bellucci explains:

"As explained by Murray Murphey (1961, 57-63), around 1865 the study of syllogism had convinced Peirce that the three syllogistic figures are essentially different. The next step was to seek for a similar difference among inferences *in general.* This step, Murphey says, was related to another important point: the discovery that in an argument the premises are a *sign* of the conclusion." (Bellucci, 16-17).

We will consider these discoveries presently.

In the paper *Die falsche Spitzfindigkeit der vier syllogistischen Figuren* ("On the False Subtlety of the Four Syllogistic Figures"), Kant was aiming to show that the second, third, and fourth syllogistic figures could be "reduced" to Barbara, proving that necessary reasoning exhibited in Barbara was logically primitive in a way that the reasoning in the other syllogistic forms is not.

Peirce, however, discovered that, "the very reasoning by which [Kant] reduces the indirect moods to Barbara…itself introduce[s] an additional logical principle" (CP 4.2). Peirce's proof is published in "Memoranda Concerning the Aristotelian Syllogism" (privately published in 1866). As Peirce claims in that paper, "it appears that no syllogism of the second or third figure can be reduced to the first without taking for granted an inference which can only be expressed syllogistically in that figure from which it has been reduced" (Peirce, Memoranda, 8 {CP 2.807}).

In Memoranda, Peirce's argument operates on two different "levels" (to speak vaguely). On one level he is aiming to give a formal account of the steps whereby an argument (syllogism) of one form is transformed (i.e. reduced) into the form of another. But, then, on a second "meta-level" he examines the argument that implicitly justifies those steps of transformation—i.e. the argument justifying the steps employed in the reduction (considered in the form of a syllogism). His discovery is that, when the "meta-level" argument is expressed in a syllogistic form, it has the same form as the syllogism from which the reduction took place, proving that this form of argument cannot be eliminated. This is a subtle point, and it makes *Memoranda* rather confusing to read, but I believe it is an extremely significant discovery, which has major implications for Peirce's Reduction Thesis.

This was one of Peirce's earliest discoveries, but he consistently emphasized it in his later work on logic—beginning his 1880 "On the Algebra of Logic" (AL) with an account of this idea, and as late as 1909 writing to William James,

" 'I find myself bound, in a way which I discovered in the sixties, to recognize that there are concepts which, however we may attempt to analyze them, will always be found to enter intact into one or the other or both of the components into which we may fancy that we have analyzed them' (NEM 3:851)"[51]

Francesco Bellucci provides a wonderfully clear and detailed account of Peirce's discovery in his book *Peirce's Speculative Grammar—Logic as Semiotic* (2018). In what follows, I will draw from Bellucci to explain Peirce's discovery in more detail and its relation to the three forms of inference, to the categories (Firstness, Secondness, Thirdness), and to the fundamental sign relation (sign-object-interpretant). Again, I wish to argue that this discovery is important because it sheds light on how we can understand the Reduction Thesis.

Peirce's argument against Kant turns on the notion of "leading principles", which is an idea that goes back to the medieval logicians. Consider the following syllogism in Barbara (the first figure):

(B) *Rule:* All men are mortal[52]
Case: Napoleon III is a man
Result: Napoleon III is mortal.

Here the major premise asserts a rule; the minor premise asserts a case; and the result makes the valid inference that the case falls under the rule. But what permits us to call this inference valid? Peirce's answer is that there must be some (purely formal) "logical leading principle" (Bellucci, 24) that asserts that the conclusion follows from the

[51] Cited in Bellucci, 32

[52] Bellucci gives this premise in the more common way: "All men are mortal"; however, Peirce, in Memoranda (and elsewhere), frames universal statements in the form "*Any* man is mortal" (CP 2.794). This phrasing is more appropriate given Peirce's interpretation of the quantifiers, which we will discuss in Chapter Three.

premises by virtue of form. This leading principle does not have to do with men or mortals or anything material, but, being purely formal, could be stated in the following way (ibid, 25):

(b) If a rule and a case falling under it are true, then their result is true.

But since (b) is a rule governing (B), suppose we construct an argument using the syllogistic structure of Rule, Case, Result, where (b) is a *premise*. We would get the following (ibid, 25-26):

(B′) *Rule*: If a rule and a case falling under it are true, then their result is true [i.e. b]
Case: That all men are mortal and that Napoleon III is a man are rule and case, whose result is that Napoleon III is mortal [i.e. B]
Result: If it is true that all men are mortal and that Napoleon III is a man, it is true that Napoleon III is mortal.

At this point we might similarly ask what permits us to assert the validity of (B′), and thereby seek to identify *its* leading principle. "But, evidently enough, the leading principle of (B′) is (b) itself. Therefore, analysis must have stopped at the preceding step" (ibid, 26). Bellucci later goes on to say,

"Peirce's point is that the very setting in motion of the regress is the symptom that we have reached the end of logical analysis: the principle L of an argument A is *logical* when, if it is used as the rule of an argument A′ in which A appears as the result, the leading principle of A′ is L itself" (ibid, 26).

While Bellucci, in this section, follows Peirce in keeping the explanation in terms of Aristotelian logic, I think it is helpful to consider this in terms of modern logic (though keep in mind, this discovery was published in 1866 and Peirce's full account of quantification was not published until 1885). Argument (B) expressed in modern logic would be the following:

(B_M) *Rule:* $\forall x\,(Mx \rightarrow Dx)$
Case: M*n*
Result: D*n*

Deductively—for example, in Gentzen's *Naturliche Kalkul* ($\vdash_{NK}$)—the validity of (B_M) would depend on deductive steps, which are truth-preserving—i.e. (i) universal instantiation/elimination and (ii) Modus ponens, or conditional elimination. In Natural Deduction, the deductive steps are proved valid by showing, through mathematical induction (what Peirce called Fermatian inference (NEM 4.78)) that the deductive inferences ($\vdash_{NK}$) preserve the corresponding semantic interpretation ($\vDash$) of the well-formed formulae.

Peirce is obviously taking up a different strategy. But, the important point to note here is that, even in the case of modern logic, the meta-logical argument for soundness and completeness of a first-order deductive system—in the case of Natural Deduction, the argument that:

$$(\phi \vdash_{NK} \psi) \Leftrightarrow (\phi \vDash \psi)$$

—relies on a concept of implication (expressed by the connective $\Leftrightarrow$) which is second-order, and therefore not expressible *within* that deductive system itself.

I believe Peirce is touching on a similar idea here, though he has a different method for defining when the inference is "logical"—i.e. he does not rely on an account of the semantics of the logical connectives considered as truth-functions. He does this for deliberate reasons. In AL (1880), Peirce characterizes leading principles as habits, which are *good* insofar as they lead to truth; and his account of logic begins with a very human, very physical account of inference. It seems that he is not merely content to consider logical connectives as truth-functions, leaving aside the question of whether the inferences permissible in that system align with our intuitions about what constitutes good reasoning. He wants to build a logical system that aligns with our habits of good inference (those that lead to truth), explicitly acknowledged in the formal system as leading principles. He is also concerned, in a similar way, with "mapping" logical systems to Nature.[53] As he explains in 1892,

"Describe and describe and describe, and you never can describe a date, a position, or any homaloidal quantity. You may object that a map is a diagram showing localities; undoubtedly, but not until the law of the projection is understood, nor even then unless at least two points on the map are somehow previously identified with points in nature" (CP 4.419).

Though Peirce's "metalogical" arguments about leading principles do not appeal to first-order semantics in the way that contemporary metalogical soundness and

[53] Peirce's insights about mapping are, no doubt, partly thanks to his longtime employment with in the US Coast Survey doing work in coastal mapping.

completeness arguments would, he does, importantly, distinguish between the sign of illation (P ∴ Q)—expressed verbally as ‘P, therefore Q’—and the sign of the conditional (P—< Q) expressed verbally as ‘If P, then Q’ (CP 3.165).[54] Yet, using the notion of leading principles, he explains the truth-functional identity of the two relations, and goes on to say that, “from the identity of the relation expressed by the copula with that of illation, springs an algebra” (CP 3.182).

This “identity” can be demonstrated by one of Peirce’s examples from 1880 (CP 3.171):

Peirce claims that the following argument

P Q R S T
∴ C

can be broken up into two, namely, first:

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[54] As we will discuss in Chapter Four, part of the challenge in reading Peirce’s papers on algebraic logic is the fact that “the dyadic relation or binary operator (—<)…could be variously interpreted as material implication, a partial ordering, class inclusion, and set elementhood” (Anellis, 279). In 1870 he describes it as the symbol for “*inclusion in* or *being as small as*”, which is a transitive relation (CP 3.47). In 1880 he defines it as “the copula [which, in the proposition P_i —< C_i] signifies primarily that every state of things in which a proposition of the class P_i is true is a state of things in which the corresponding propositions of the class C_i are true” (CP 3.165). He also (prior to 1885), occasionally talks of this relation as if it were quantified—e.g. following DeMorgan, he expresses the universal affirmative statement ‘Every *a* is *b*’ symbolically as *a*—< *b* (CP 3.177). Then, in 1885, he alludes to a truth-functional interpretation of this connective: “the proposition *a*—< *b* is true if *a* is false or if *b* is true, but is false if *a* is true while *b* is false” (CP 3.375).

As we will discuss in Chapter Three, the challenge in interpreting the symbol —< is related to the idea that, in the 1870-1885 papers, Peirce does not draw a clear distinction between members of a set and the singleton sets formable from members of that set. As such the set-theoretic distinction between the relations expressed by the symbols $\in$ and $\subseteq$ is obfuscated. See Quine: “The notion of unit classes seems to have first become explicit in [Peano’s] ‘Démonstration de l’integrabilité des equations différentielles ordinaires’ in 1890” (Quine, PaL, 268). Though, Quine also notes that Frege clearly distinguished the cases differently in *Begriffsschrift* of 1879 (Quine, PaL, 268b). That being said, in later papers, Peirce draws a clear distinction between these two concepts, and has some very interesting comments on the distinction between individuals and unit classes, which we will address later in this paper (NEM 3:371; Hawkins, 133 and 130). On this point, Hawkins notes that Russell confuses ‘if p, then q’ with ‘p, therefore q’—an error related to use-mention distinction (Hawkins, 127)

P Q R S
∴ T—< C

and, second:

T—< C
T
∴ C

Then Peirce says, "by repeating this process, any argument may be broken up into arguments of two premises each" (ibid).

As Bellucci explains, the relationship between illation and the copula (i.e. conditional) goes back to the medieval logicians. Indeed "Peirce himself acknowledges in 1898, he had learnt from the medieval doctors, who 'always called the minor premise the antecedent and the conclusion the consequent' (R 441 CSP 15)" (Bellucci, 27-28). This idea that one can pass from illation to the conditional is the central idea of the medieval Deduction Theorem. Bellucci cites Peter King: the " 'medieval deduction theorem permits the logician to pass between conditional and inferential formulations of the same claim, without any logical baggage getting lost in the transfer, as it were' (King 2001, 133)" (ibid, 28).

Peirce sees illative transformation (P, therefore Q), which is "the only transformation, relating solely to truth, that a system of symbols can undergo" (CP 4.375) as the "passage from a symbol [sign] to an *interpretant* [my emphasis], generally a partial interpretant" (ibid). He goes on to describe the conditional (If P then Q) as a sign that signifies that illative transformation i*s possible*. In the conditional, the sign indicates its *object* (ibid).[55]

The next question for Peirce was how we know we have reached the end of analysis of leading principles. We have already alluded to the answer above, but a further example from Bellucci (summarized from Peirce, 1869b) will be helpful (Bellucci, 27).

[55] It is perhaps worth mentioning here the difference in wording. The illative transformation involves *passage* from sign to interpretant, but he does not say that conditional involves passage from sign to object, since the direction of semeiosis is always asymmetrical—the object determines the sign to determine an interpretant; or, rather, the object determines the interpretant mediately through the sign.

Consider any general argument, whose premises are denoted by *P*, whose conclusion is denoted by *C*, and whose leading principle is denoted by *L*. "Then if the whole of the leading principle be expressed as a premise, the argument will become

L and *P*
$\therefore$ *C*

"But this new argument must also have its leading principle, which may be denoted by L'." (ibid). But Peirce's point is that L' must already be "contained in" *L*, since "were it not, then we would need to lay it down as a premise along with *L* and *P*" (ibid) As Bellucci explains later, "In Peirce's terms, the incomplete (material) argument 'P, therefore C' is "completed" (i.e. reduced to a formal argument) by laying down its leading principle—'If P, then C'—as a further premise" (ibid, 29). Peirce's "proof of the logical nature of a leading principle" (ibid) is obtained by comparing *L* and L' as we did when comparing the leading principle of (B) and (B′) above. In that example, the leading principle of (B′)—namely (b)—was already expressed (purely formally) as one of the premises of (B′). According to Peirce's argument, this reveals that the leading principle is a *logical* leading principle, which is ineliminable from the argument. As Bellucci says, "This was a momentous discovery, and a striking example of the kind of 'elementarity' that logical leading principles have for Peirce: they cannot be analyzed or, which is the same, can be analyzed only by or through themselves" (Bellucci, 33).

Here we return to Peirce's criticism of Kant. The flaw in Kant's reasoning was to fail to examine the structure of the inferential steps performed (perhaps implicitly) in the reduction itself. "[Kant's] reduction of a syllogism in the second figure to a syllogism in the first figure depends upon the equivalence between 'No X is Y' and 'No Y is X'" (Bellucci, 31). Kant characterizes this as an immediate inference that requires no middle term, but Peirce shows that it is capable of analysis by rendering it into syllogistic form as:

No X is Y
All Y is Y
No Y is X

But this (implicit) syllogism is *itself* a syllogism of the second figure. So a syllogism of the second figure is required in the argument that "reduces" a syllogism of the second figure. Thus, Peirce showed that Kant's proofs of reduction (of the second and third syllogistic figures to the first) possess implicit structure that is ineliminable in analysis. Peirce found that the fourth figure can be reduced using a combination of the second and third, showing that it is not primitive in the same way. This discovery also led Peirce to the belief that "all processes of thought must be viewed as processes of substitution…" (Thompson, 14). The criterion, then, for when one sign can be substituted for another, is when and only when the one sign has the same object as the other. We will discuss this idea in more detail in Chapter Three.

Having made this discovery (that the syllogistic structures of the second and third figures are ineliminable in analysis), Peirce then claims (in a later account of the discovery) that he was led to consider whether there might be "forms of probable reasoning analogous to the second and third figures…" (CP 4.2). He found a clue in Aristotle who (in the Prior Analytics II.23) considers induction as a probable syllogism in the third figure (ibid). Peirce thus associated Induction with the third figure, but realized that there is an additional primitive form of inference, whose structure mirrors that of a syllogism in the second figure—to which he gave the name Abduction (also called Hypothesis, Retroduction, and occasionally 'conjecture'—e.g. EP 2.443).

From here, he reconsidered Kant's categories. With this view of leading principles and substitution, he realized that "every premiss, whether stated conditionally or categorically, is for syllogistic purposes a rule of substitution" (Thompson, 17). In that case Kant's third division in the table of judgments into Categorical, Hypothetical and Disjunctive (A70/B95) is wrong. These are one and the same. So his table of categories must similarly be revised. Peirce notes, "This led me to see that the relation between subject and predicate, or antecedent and consequent, is essentially the same as that between premiss and conclusion" (CP 4.3). This related discovery was also extremely significant. If the relation between antecedent and consequent is essentially the same as the relation between premise and conclusion, then propositions and arguments have the same essential structure. In fact, as we noted in discussing symbols, Peirce argues (somewhat surprisingly) that even simple *terms* possess this structure, and are therefore not *essentially* different from propositions or whole arguments. Terms have structure because (insofar as they are meaningful) they are symbols, through which an object mediately determines an interpretant.

Thus, through the concept of leading principles, and the discovery of three distinct forms of reasoning, Peirce realized that there is a structural identity underlying each of the three fundamental types of symbols used in logic—terms, propositions, and arguments. This structure comes from the fact that they are all signs and thus governed by the sign-object-interpretant trichotomy. Peirce's logic was, of course, far from complete at this point; he was yet to explore the logic of relations. But these (related) discoveries stayed with him through his later logical work, and are important for how we should understand the Reduction Thesis.

Deduction, Induction, Abduction, continued

What are these three primitive forms of reasoning—Abduction, Induction, and Deduction? Peirce's account in the 1867 essay "On a New List of Categories" helps to clarify. He claims, "In an argument, the premises form a representation of the conclusion, because they indicate the interpretant of the argument, or representation representing it to represent its object" (EP 1.9). This idea will guide the classification of argument forms. If, in an argument, the premises are themselves signs of the conclusion, then it makes sense to ask *what sort* of signs they are, according to the triadic division of signs that we have already laid out. In short:

1) In an abductive argument, the premises *iconically* represent the conclusion
2) In an inductive argument, the premises *indexically* represent the conclusion
3) In a deductive argument, the premises *symbolically* represent the conclusion

In this discussion, we are primarily interested in Induction, since I believe this form of argument provides a clue for how we can understand the referents of relations. But let us first briefly consider **Abduction**. According to Peirce, an abductive argument takes the following form (EP 1.9):

M is, for instance, P', P'', P''', and P^{iv}
S is P', P'', P''', and P^{iv}
S is M

The conclusion of this argument is a hypothesis. In scientific investigation, one would next *deduce* necessary (risky) consequences that must be true if this hypothesis is true.

Then one would perform experimental tests, and gather evidence. If the hypothesis stands up to this testing, then through *inductive* reasoning (not deductive reasoning), which examines that evidence, one draws a conclusion, affirming or rejecting the hypothesis.[56] Clearly, the argument form above is not deductively valid, but deductive validity is not relevant to an abductive argument. In scientific investigation, hypotheses are not proposed willy-nilly. They are proposed because scientists have some sort of provisional belief that they might be true. But neither is the reasoning through which hypotheses are developed strictly deductive. For in that case, there would be no need for experimentation. As Peirce claims in the essay "Pragmatism as the Logic of Abduction",

"the hypothesis cannot be admitted, even as a hypothesis, unless it be supposed that it would account for the facts or some of them. The form of inference therefore is this: The surprising fact, C, is observed; But if A were true, C would be a matter of course. Hence, there is reason to suspect that A is true" (EP, 2.231).

Peirce claims that, in an abductive argument, "the premises are or represent a *likeness* [my emphasis] of the conclusion" (ibid).[57] The argument above has the structure of a syllogism of the second figure where the conjunctive term 'P', P'', P''', and P^{iv}' is the middle term.[58]

As Bellucci also clarifies, Peirce saw abductive inferences (as well as inductive inferences) as inversions of deductively valid syllogisms.[59] The schematic above could be seen as a sort of inversion of Barbara, where (i) the *Result* from Barbara is made into the

[56] At this point, the hypothesis is not true in some objective sense. To think that would be to mistake induction for deduction. The hypothesis under consideration would, rather, be affirmed in a fallible way. The degree of scientist's belief in the truth of the hypothesis may, perhaps, be measured by examining conduct, where the scientific community takes on habits that regard the hypothesis as true until some new information causes us to doubt it. On this topic, one can perhaps see how Karl Popper's fallibilism is influenced by Peirce, and how Peirce's philosophy of science may even be able to shed new light on the question of falsification. See Eugene Freeman's essay "C.S. Peirce and Objectivity in Philosophy", and Popper's own response to the essay "Freeman on Peirce's Anticipation of Popper". Both appear in *The Relevance of Charles Peirce*, 1983.

[57] There is some equivocation in the schema above whether S *resembles* M or whether the two premises taken together *resemble* the conclusion. In later explanations of Abduction, Peirce characterizes it more like inferring the antecedent (e.g. EP 2.231), though his explanation there also importantly involves the concept of resemblance.

[58] See Bellucci, 44 for a more in depth account.

[59] Ochs also discusses this point (118-121).

major premise (i.e. *Rule*) of the new syllogism, (ii) the *Rule* from Barbara is made into the minor premise (i.e. *Case*), and (iii) the *Case* from Barbara is made into the new conclusion (i.e. *Result*). For example (taken from Bellucci, 44):

> *Result*: S is M_1, M_2, M_3, M_4
> *Rule*: P is M_1, M_2, M_3, M_4
> *Case*: S is P

Bellucci also gives a non-schematic example of an inference like this. From the two premises (P1) 'This fruit is spherical, bright, fragrant, juicy', and (P2) 'oranges are spherical, bright, fragrant, juicy', one hypothesizes (C) therefore, this fruit is an orange (ibid).

Here it is quite important to note that, while *classification* of this and the other forms of argument might properly be a task of logic (as Peirce attempts to show in *Memoranda*), abductive reasoning *itself* is not properly logical. Ochs explains this idea well, noting that, "perceptual judgments function just like abductions", though they are "pre-logical" (Ochs, 115). Thus, abductive inferences involve both habits and perceptual judgments about surprising facts. Ochs schematizes an abductive inference in a similar way to what we have above, but with a bit more verbal detail:

> A well recognized kind of object, M, has for its ordinary predicates P′, P′′, P′′′, etc., indistinctly recognized.
> The suggesting Object, S, has these same predicates P′, P′′, P′′′, etc.
> Hence, S is of the kind M (8.64) (Ochs, 115).

Ochs, citing Peirce, goes on to explain, "Here the major premise 'is not actually thought, though it is in the mind habitually' (8.65); the minor premise represents the surprising fact; the conclusion 'has the peculiarity of not being abstractly thought, but actually seen' (ibid)" (Ochs, 115). At this point we may recall the question above (from our discussion of icons) whether resemblance between two things means that they possess all the same properties or only some? I believe we should understand resemblance to vaguely mean 'all', but only until one comes up that is different. If we find one that is different, then we rely on a *deductive* principle that $(a=b) \rightarrow (\phi a \leftrightarrow \phi b)$, which would be used to prove that a and b are not identical, but this "proof" would rely on deductive reasoning in addition to abductive reasoning.

An **inductive** argument takes this form (EP 1.9):

S', S'', S''', and S^{iv} are taken as samples of the collection M
S', S'', S''', and S^{iv} are P
All M is P

This form of argument could be seen as a syllogism of the third figure, where the conjunctive term 'S', S'', S''', and S^{iv}' is the middle term. Also, in the same way that we characterized an abductive inference as a sort of inversion of a deductively valid syllogism, we can see the schema above for Induction as the result of swapping *Rule* and *Result* in Barbara.[60]

Result: S', S'', S''', and S^{iv} are M
Case: S', S'', S''', and S^{iv} are P
Rule: All P is M

A non-schematic example of an inductive inference like this takes the two premises (P1) Neat, swine, sheep, and deer are herbivore, (P2) Neat, swine, sheep and deer are cloven-hoofed, and concludes (C) therefore, cloven-hoofed animals are herbivore (Bellucci, 43).

In this case, according to Peirce, "the premises are an index of the conclusion" (EP 1.9). The collection { S', S'', S''', and S^{iv}} is a sign of the class of which it is a subcollection, namely M. That is to say, it *stands for* M.[61] In Bellucci's words, "the sample is an *index*…of the whole" (Bellucci, 43). In the actual operation of science this seems quite plausible. We never have the entire class of any natural kind ostensively available to us. Yet we still speak scientifically in terms of universally quantified

[60] The following schematic is taken from Bellucci, 43, but I have substituted Peirce's letters for the letters that Bellucci uses, for sake of consistency. One may notice as well that Bellucci's schematic concludes 'All P is M', while Peirce's concludes 'All M is P'. Of course, deductively, these are not equivalent. Understood inductively, however, I believe the difference can be explained by distinguishing—in Ochs' language—the "suggesting object" from the "well recognized kind of object" (Ochs, 115). The "suggesting object" is to be subsumed under the class of the "well recognized kind of object".

[61] This discussion recalls a passage where Peirce credits Kant for effectively raising the philosophical question of how we are justified in making universal claims about experience. "Kant declares that the question of his great work is 'How are synthetical judgments *a priori* possible?' By *a priori* he means universal; by synthetical, experiental (i.e. relating to experience, not necessarily derived wholly from experience). The true question for him should have been, 'How are universal propositions relating to experience to be justified?' But let me not be understood to speak with anything less than profound and almost unparalleled admiration for that wonderful achievement, that indispensible stepping-stone of philosophy" (CP 4.92).

propositions about those kinds, at least until we find some quality that M possesses, but that P does not, or vice versa. But, again, in that case, the inference that 'M is not P' would then also involve deduction.[62]

Later in this paper, in Chapter Four on the relative product operation, I will argue that this idea of a sample *standing for* a class can help us to understand how Peirce conceives of the referents of relations. I will argue (and that argument will require much more explanation than I can give here) that we must distinguish between the referent of a relation in two different senses: we might call these the complete extensional sense (or God's-eye-view sense), and the more nuanced particular sense. But, *further*, I will argue that the referent of a relation in the nuanced particular sense is an *index* of the referent of that relation in the God's-eye-view sense.

The crux of the issue is that Peirce wants to give a *semiotic* account of how these two concepts (of the referents of relations) *are related*. I argue that the latter is an index of the former in the same way that a particular sample of cats is an index of the full class of cats. This makes Peirce's task immensely difficult. For, now we have (i) the concept of (the referent of) a relation in the latter (particular) sense, (ii) the concept of (the referent of) a relation in the former (complete, extensional 'God's eye view') sense. But we also now have a new (iii) higher-order relation *between these two relations*.

The final primitive form of reasoning is **Deduction**. "Deduction is the only necessary reasoning. It is the reasoning of mathematics. It starts from a hypothesis, the truth or falsity of which has nothing to do with the reasoning; and of course its conclusions are equally ideal" (EP 2.205). Charles Peirce follows his father Benjamin Peirce who defined mathematics as the science that *draws* necessary conclusions, thus

[62] As with Abduction above, there is some equivocation here whether Peirce takes the two premises to be an index of the conclusion, or whether he takes the middle term—i.e. the set { S', S'', S''', and S^{iv}}, which is a subset of M and of P—to be an index of M and/or of P. In the latter case, it must be an index of both, but this does not determine whether M is a subset of P, or if P is a subset of M. I believe the answer to that question is best understood, as in Abduction, by taking either M or P as a "suggesting" class or a "well recognized" class, which determines the order. Based on other passages from Peirce, which we will discuss, I take his key point here to be the latter one—that { S', S'', S''', and S^{iv}}, a subset of a larger class, is an *index* of that class, which represents the whole class in the manner of an index.

generalizing the domain of mathematics beyond concern only with numbers.[63] B. Peirce's research culminated in his major work *Linear Associative Algebra* (1870), which was "explicitly a comparative study of certain uninterpreted algebras" (Quine, WRML, 3). Peirce's paper DNLR, published that same year, is a work that follows in the tradition of B. Peirce's LAA.[64] However, while mathematics *draws* necessary conclusions, deductive logic concerns itself with studying the *drawing* of necessary conclusions (CP 4.239). In *New List* Peirce does not give a schematic example of the form of deductive reasoning, which is no accident, since obviously deductive reasoning takes many schematic forms.[65]

Bellucci gives the following helpful clarification of the semiotic element involved in Deduction: "in deduction a symbol is symbolized (substituted with a symbol) on the principle that a symbol of a symbol is a symbol of the same object" (Bellucci, 44). This is contrasted with Induction, which involves substituting a symbol for an index—e.g. substituting the symbol M (representing a whole class) for the conjunctive term 'S′, S″, S‴, and S^{iv}', which was an index of that class. In the case of Abduction, a symbol is substituted with an icon (Bellucci, 45). Thus, "the leading principle of inference in general is that 'the sign of a sign of an object is itself a sign of the object" (Bellucci, 48). This leading principle effectively fulfills the office of an interpretant, which asserts that a certain relation holds between a sign and an object; in asserting that that relation holds, the interpretant is itself a sign of that same object.

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[63] "The philosophical mathematician, Dr. Richard Dedekind, holds mathematics to be a branch of logic. This would not result from my father's definition, which runs, not that mathematics is the science of *drawing* necessary conclusions—which would be deductive logic—but that it is the science which *draws* necessary conclusions. It is evident, and I know as a fact, that he had this distinction in view" (CP 4.239). Also see Guinness: "Peirce's philosophy of necessary conclusions was an extension of his father's position, which was oriented around algebra, to a general assertion of mathematics as matterless" (Guinness, 36). Also see Hawkins: "these and other of Peirce's views on mathematics anticipate modern formalism and intuitionism" (Hawkins, 119).

[64] See Van Evra: "There is also evidence that the two works are related at a deeper level. The relationship which Charles established between logic and mathematics in DNLR appears to reflect ideas which his father set forth in LAA concerning the foundations of mathematics. What Charles Peirce *does*, that is, is similar to what Benjamin Peirce *says*" (Van Evra, 152).

[65] Peirce also draws a distinction between two overall forms of Deduction—Corollarial and Theorematic deduction. We will discuss these in Chapter Five as they relate to the Reduction Thesis.

Firstness, Secondness, Thirdness, continued

We return to the categories. Peirce's three "categories" are first presented in his 1867 essay "On a New List of Categories" (EP 1.1-10). There he calls them Quality, Relation, and Representation (EP 1.6), though later he claims that Quality, Reaction, and Mediation would do better (CP 4.4). "But for scientific terms, Firstness, Secondness, and Thirdness, are to be preferred as being entirely new words without any false associations whatever" (ibid). Interestingly, he explains in a different essay that when he gave the name "relation" to the second category, he "was not aware that there are relations which cannot be analyzed into relations between pairs of objects" (NEM 4.331). These sorts of relations are the irreducible triads to which we have alluded, and which we will discuss in more detail in Chapter Five.

As noted earlier, these are "excessively general ideas, so very uncommonly general that it is far from easy to get any but a vague apprehension of their meaning…" (CP 4.3). Again, as we noted earlier, we should be wary. For any attempt to describe these categories almost always shades into metaphysics. The very names that Peirce uses—Quality, Reaction, Mediation—have echoes of metaphysics and phenomenology.

However, I wish to argue (as I believe Peirce wishes to argue) that the claim that there are three fundamental categories in Nature is, in effect, just a metaphysical corollary of the Reduction Thesis. Put another way, the idea that there are three fundamental categories in Nature is simply the natural conclusion that one draws if one holds (i) that the Reduction Thesis is true in the realm of logic, and (ii) that logical truths reveal something true about Nature. In this paper, I aim to consider the logical categories (i.e. relations of addicity 1, 2, and 3), not the phenomenological or metaphysical categories. But, if the Reduction Thesis holds in the more general way that I suggest, then I believe the natural consequence is that there are three fundamental (metaphysical) categories in Nature, though perhaps the "essence" of these requires defense in some other paper. But if there are three universal categories in nature, then this has implications for metaphysics and the philosophy of science, as I suggested in the introduction, though spelling those out in detail must also be the task for some other project. However, if the Reduction Thesis does hold in this more general way, then there are also implications for formal logic. A system of formal logic (for the purpose of revealing how reasoning takes place) ought to (attempt to) give an accurate account of the primitive elements in reasoning; but it should make logical facts explicit wherever that is possible (like the fact—if it is a logical fact!—that one cannot reduce triads to dyads in a metalogical proof without employing triads). If this is a logical fact (or if we are content with it being a semiotic fact), then I believe we have good reason to take EG seriously—not necessarily as a calculus to aid inference (though perhaps it has virtue in that respect as well), but because it brings these metatheoretical ideas to light, which can help guide us towards a better metaphysics.

Chapter Three

Facts, Relations, and Peirce's Logical Inheritance

"This subject [of relations], although always recognized as an integral part of logic, has been left untouched on account of its intricacy. It is as though a geographer, finding the whole United States, its topography, its population, its industries, etc. too vast for convenient treatment, were to content himself with a description of Nantucket" (CP 3.416)

Peirce's Logical Inheritance

The two logicians that most influenced Peirce early in his career were Boole and De Morgan. He saw great promise in Boole's project of "expressing logical concepts in mathematical terms and then analyzing them with the available mathematical technologies" (Brady, 174). But he saw two distinct and significant problems in Boole's system, which he would end up winding together during the course of his own extension of Boole's work.

The first was the problem with Boole's treatment of particular propositions. The second was the fact that Boole's calculus was incapable of dealing with relations. What do we mean by saying that Peirce would wind these issues together in his later work? As we will see, they are brought together by the concept of Relative Product—the primitive operation of "application of a relation" (CP 3.68), which implicitly achieves existential quantification. Chapter Four will be devoted to an account of this operation.

While I will not discuss Boole's work in detail, it is worth giving a brief example of the first problem I mention. Boole attempts to signify 'some' by using an "indefinite symbol *v*", and expresses the proposition 'Some y's are not x's' by the equation: $vy=v(1-x)$ (Boole, 61). Peirce notices, however, that by Boole's rules, this equation can be transformed into $vx=v(1-y)$, which would be translated as 'some x's are not y's'. But it is invalid to conclude 'some x's are not y's' from 'some y's are not x's'. So something has gone wrong in Boole's system.[66] Peirce argues that the problem arises because "[Boole's] symbol *v* does not denote any particular subset and, therefore, as in the above argument, can be used to denote two different things resulting in a contradiction if they are equated"

[66] This example is found in DNLR (1870) (CP 3.138). Brady also gives some helpful clarification (Brady, 175). In Boole's equation, *x* and *y* represent classes; '1' represents the universal set; and 1- *x* represents the class of everything except *x*'s.

(Brady, 175). The challenge here will be to come up with a way to indefinitely identify a member or subset of a set and yet ensure that reference to *that* member or subset is maintained where necessary. In the next chapter we will consider how Peirce addresses this in DNLR (1870), and how his solution grows out of considerations of conceptual combination.[67]

As mentioned above, the second problem that Peirce saw in Boole's logical algebra was that it was incapable of dealing with relational propositions and inferences. Tiercelin notes that it was in the early 1860s that Peirce began "to see the incompleteness of traditional syllogistic and of Boole's algebra of classes, and the necessity of taking relations into account" (Tiercelin, 126). But Peirce was not the first logician to explore the logic of relations in a mathematical way. That title goes to Augustus De Morgan. And, while Peirce saw more promise in the technical workings of Boole's system, he was more deeply influenced philosophically by De Morgan.

According to Peirce's own account (EP, 2.424), it was 1866 when, as a 26 or 27 year-old, DeMorgan sent him a copy of his "On the Logic of Relations" (published 1859). [68] No doubt, part of the reason that such an eminent mathematician as De Morgan would take note of a 26-year-old youngster, was due to the fact that Charles was the son of Benjamin Peirce. Peirce notes that he fell on De Morgan's treatise, and,

"before many weeks had come to see in it, as De Morgan had already seen, a brilliant and astonishing illumination of every corner and every vista of logic…[He] stood indeed like Aladdin (or whoever it was), gazing upon the overwhelming riches of Ali Baba's cave, scarce capable of making a rough inventory of them…" (EP 2.425)

[67] Merrill writes, "We are surprised to read Peirce's claim that Boole's problems with hypothetical and particular propositions were what 'first led me to seek for the present extension of Boole's logical notation' (3.138) [1870]…let me note that this is a striking claim about the connection between the logic of relatives and issues of quantification. It is natural to suggest that problems in expressing relational propositions were the main source for modern quantification theory; but Peirce's claim reverses the order of influence" (Merrill, 164). I believe that Peirce was actually considering both issues—the problem with particular propositions, and the need to account for relations—since he was talking about the need to take relations into account quite early on (see Tiercelin, 126 below). But what seems certain is that he was *not* aware at this early stage how these two issues were connected.
[68] See Iliff, 195: Peirce claimed to have first seen the paper in 1866, but based on historical scholarship, "it is now accepted that he first saw the paper in late 1868" (Iliff, 195).

In retrospect, we certainly do not have to look far for valid relational inferences that are incapable of being captured by Aristotelian logic. Peirce gives an example in one of his earliest works: "A is half of B; therefore A is less than B" (EP 1.9). This valid inference is not expressible in Boole's calculus or in the form of a syllogism. Certainly anyone can see that the conclusion follows from the premise; but a formal account of why this is so would require examining the 'half of' relation as well as the 'less than' relation and determining whether the latter is deducible from the former. Traditional (Aristotelian) syllogistic reduces arguments to a formal structure where the only formal relations expressed are those of subsethood (or set membership). Thus the argument above would have to be rendered into the form: 'All As are Half-Bs', and 'All Half-Bs are Less-than-Bs', therefore 'All As are Less-than-Bs'. The minor premise here is effectively the inference in question, so the syllogism does not account for why the inference is valid; it merely assumes it as a premise.[69]

Peirce remarks, however, that after his initial reading of De Morgan's work in 1866, it took him about twenty-five years to approach "a provisionally final result" (EP 2.425). This result was the discovery,

"that indecomposable predicates are of three classes; first, those which, like neuter verbs, apply but to a single subject; secondly those which like simple transitive verbs have two subjects each, called in the traditional nomenclature of grammar (generally less philosophical than that of logic) the 'subject nominative' and the 'object accusative', although the perfect equivalence of meaning between 'A affects B' and 'B is affected by A' plainly shows that the two things they denote are equally referred to in the assertion[70]; and thirdly, those predicates which have three subjects, or correlates. These last…never express mere brute fact, but always some relation of an intellectual nature, being either constituted by action of a mental kind or implying some general law" (EP, 2.425).

How Peirce arrived at this idea becomes clearer through a study of his earlier logical writings, where he was working out ideas including quantification, combination of relatives and class terms, combination of multiple relatives, and degrees of relatives

[69] De Morgan also gives the following example of an inference inexpressible in syllogism: 'every man is an animal' therefore 'every head of a man is the head of an animal' (De Morgan, 216); this example is referenced by Peirce in W 2.90
[70] Peirce already suggested this idea of the equivalence between 'A affects B' and 'B is affected by A' in 1867 in his "On a New List of Categories". However the idea does not originate with him; in that same essay he cites Abelard (EP 1.4).

(i.e. addicity[71]), to name a few. Part of the difficulty, as we will discuss, comes from the fact that Peirce initially viewed relative terms like 'father' or 'lover' as nouns or nominalized verbs, rather than as verbs or predicates. While his later writings reveal more clarity, his earlier logical papers show that these ideas are evolving. He alternates between different notations, and equivocates about certain distinctions that he became clearer about later.

A few important Peircean themes are already germinal in De Morgan's work. In what follows, I will specify three such themes—though this is by no means intended as an exhaustive list. As Peirce suggests, De Morgan was the first logician to take relations seriously. Part of what led De Morgan to discover the richness of relations was a careful attention to the meaning of *facts*, which De Morgan saw to be *themselves relational*. (At this stage we are using the term 'relation' vaguely; we will clarify the relevant meanings later in this chapter). But, further, in endeavoring to construct a logical system that accurately captures relational inferences, certain properties of the relations (as abstract entities in a logical system), and properties of their combination were revealed. Part of what I aim to defend in outlining these three themes is that Peirce's *discoveries* about relations came out of a careful study of logic itself. These ideas are germinal in De Morgan because De Morgan was the first to attempt to give a formal account of relations. But these discoveries were by no means obvious. As Peirce says, it took him twenty-five years to reach a provisionally final result. I maintain that Peirce went a great deal beyond De Morgan, logically and philosophically, in clarifying these discoveries, but I wish to argue that they were just that—discoveries—which emerged out of a thorough study of this complex topic.[72]

Relations More than Mere Connection

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[71] Peirce uses the term addinity, which is effectively the same as our current notion of addicity (see CP 3.465).

[72] Peirce consistently emphasized that it was study of logic as well as the study of the structure of mathematical reasoning (which, for Peirce, is a part of logic), which led him to his philosophical convictions—particularly his discovery of the categories, and his Pragmatic Realism. For example, "…I aver that weighty reason have moved me to the adoption of my opinion; and I am also anxious that it should be understood that those reasons have not been psychological at all, but are purely logical" (CP 4.540 (1906)).

The **first** germinal theme in De Morgan that I wish to discuss is the idea that a relation is more than merely connection between two things, but is rather logically (though not yet metaphysically) a third thing wherein the relata are unified. In consideration of this idea, we should be vigilant to ensure that it is a logical point and not an extra-logical one—e.g. a metaphysical or phenomenological point. I aim to defend that it is neither of these, but I leave it an open question as to whether it may be a semiotic point. This gets into the difficult question about the relationship between logic, mathematics and semiotics. As explained earlier, Peirce's views here are a bit complex, and we will not discuss them in much detail. But he certainly saw semiotics as a vital discipline.

DeMorgan claims that,

"any two objects of thought brought together by the mind, and thought together in one act of thought, are *in relation*. Should any one deny this by producing two notions of which he defies me to state the relation, I tell him that he has stated it himself: he has made me think the notions in the relation of alleged *impossibility of relation*; and has made his own objection commit suicide. Two thoughts cannot be brought together in thought except by a thought: which last thought contains their *relation*" (DeMorgan, 218).

But what De Morgan precisely means here by "relation" may not be entirely precise. On the one hand, it seems that this "relation" is something metaphysical, existing in Nature. And yet, De Morgan (like Peirce) is concerned with how to logically express that fact, and thus arises the consideration of *Relations* as formal logical entities. We must draw some subtle distinctions.

Relations and Facts—Some Terminological Clarification

Peirce writes in 1892 that, "a relation is a fact about a number of things." He goes on, "Not only is every fact really a relation, but your thought of the fact *implicitly* represents it as such" (CP 3.416). Thus we already have a distinction between (i) relations qua facts, and (ii) relations qua representations (your thought of the fact, which *represents* the fact as a relation).

Throughout his writings, Peirce emphasizes the difference between a grammatical approach to the structure of propositions and a logical approach. A grammatical approach takes clues from grammar, perhaps first drawing a distinction between subject and object;

then, perhaps, further drawing the distinction between direct object and indirect object, etc. This sort of analysis has largely been the default in Aristotelian logic. But, as Peirce emphasizes, and as De Morgan already saw, this distinction already smuggles in metaphysical baggage.

De Morgan uses the example of when the two notions ‘white’ and ‘ball’ are combined into a compound notion, as in the phrase ‘white ball’ (De Morgan, 219). He argues, “The metaphysical distinction of the ball being a substance, of which the whiteness is an inherent accident, is extralogical” (ibid). Logic can just as easily speak of ‘rotundity of the white’ as it can of ‘whiteness of the ball’. And, further, since logic *can* analyze away this metaphysical notion of substance, it *ought* to. Metaphysics should rest on logic, not vice versa.

Peirce writes in 1903 that, “in a triadic fact, say, for example ‘A gives B to C’, we make no distinction in the ordinary logic of relations between the subject nominative, the direct object, and the indirect object” (EP 2.170). He claims, rather, that from the logical perspective, this proposition has three logical subjects. It is a “mere affair of English grammar” that we can express the same fact in different ways—for example, by saying that ‘C receives B from A’ (ibid). In this example, the latter sentence seems to assert something different, but this superficial difference is merely grammatical; logically, it expresses the same fact.[73] Later, using the example of the proposition “Napoleon ceded Louisiana to the United States”, Peirce admits that, “there is a sense in which we can continue to say that [this proposition] has but one Subject…the ordered triplet ‘Napoleon—Louisiana—the United States” (CP 4.543). Peirce suggests that we may view logical subjects in either way, though he notes that “the view that there are three subjects is…preferable for most purposes, in view of its being so much more analytical” (CP 4.543). He also emphasizes that logical subjects are different from metaphysical

[73] As we will see, Peirce permits us to distinguish logically equivalent propositions in his algebraic systems. For example, he describes the proposition that ‘A is a servant of B’, and notes that we may take the converse of this proposition and thus form a new proposition ‘B is the master of A’ (CP 3.147). In the algebraic systems, these would still be *separate* propositions whose referent is the same fact. But, in EG, they would have the *same* graph. Peirce saw this as a virtue of the latter system.

subjects, or 'substances' since the former "may be composed of characters, of elementary facts, etc" (CP 4.546).[74]

We have said that, for Peirce, facts are relational. But what precisely does this mean? As mentioned above, we certainly want to (at least) draw a distinction between (i) facts that exist in Nature, and which are therefore elements of Reality, and (ii) relations which are representational—that is to say, relations which are elements of an abstract system, whether that be a natural language, a logical system or anything else.

But here we must note two subtle points. First, Peirce holds that facts have the same *structure* as representational relations—it is only by virtue of this sameness of structure that the latter can represent the former at all. We cannot refer to facts themselves without employing representations. So what does it mean to say that facts are relational? It means that we can only represent them by using relational representations. (We will discuss this idea below, after discussing a second point).

Second, the concept of (what we have been calling) 'representational relations' actually includes two distinct concepts.

One (ii.a) is the concept of a relational predicate, or "unsaturated" relation. For these, Peirce (in his later logical work) invented the technical term rhema (or rheme). "A blank form of proposition produced by such erasures as can be filled, each with a proper name, to make a proposition again, is called a *rhema*, or, relatively to the proposition of which it is conceived to be a part, the *predicate* of that proposition" (CP 4.438). He also argues that, "the only difference between my rhema and the 'term' of other logicians is that the latter contains no explicit recognition of its own fragmentary nature" (EP 2.310). In the context of quantificational (extensional) logic, we may view this as an n-place predicate—e.g. the two-place predicate *Rxy*—without any specification as to the extension of *R* except its addicity.[75] We should also note that any 'partially saturated

[74] See also EP 2.408; EP 2.494

[75] This way of talking, of course, raises the question whether *Rxy* is bound by quantifiers or is an open formula. In EG, open formulae with free variables are not expressible (Roberts, 49). Thus, in that system, the scripture of a rhema on the SA with unconnected lines of identity would assert an existentially quantified predicate. But, in DNLR and AL, Peirce is, of course, not yet thinking this way because he has yet to discover quantification. We may take my term "unsaturated relation", or Peirce's equivalent term "rhema", as existentially quantified relations, but it will be important to note that they do

relation' is a rhema. For example '__is the father of__' would be a rhema, but so would '__is the father of Sasha'.

The second distinct concept (ii.b) is of a "saturated" relation. We may view this as the logical representation of a relational fact—e.g. the assertion of *Rab*, which involves the determinate assertion that *a* and *b* are in the relation *R*. Here, a passage from Peirce is helpful. He writes in 1906:

"A *state of things* is an abstract constituent part of reality, of such a nature that a proposition is needed to represent it. There is but one *individual*, or completely determinate, state of things, namely the all of reality. A *fact* is so highly a precissively abstract state of things, that it can be wholly represented in a simple proposition" (EP, 2.378).

This definition of *fact* is helpful. A fact is an element of reality, but an element of reality that is abstracted so as to be distinct from everything else in reality.[76] Furthermore, a fact is something that can be expressed by a proposition. Propositions can be true or false. As such, the predicates and logical subjects of that proposition must be *determinate*. The individuals involved in the "universe of discourse" must be "object[s] existing in the universe in a well-understood category; that is, having such a mode of being as to be determinate in reference to every character as wholly possessing it or else wholly wanting it" (CP 4.461). Prior to being determinate, they are vague and therefore if a predicate were applied to them, the proposition it expresses would not strictly be true or false. The

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not yet involve any assertion as to the extension of the relation in question. So the only thing we know about that relation is its addicity.

[76] This idea also has implications for how we should understand causality. See EP 2.315 for a more detailed discussion where Peirce criticizes Mill's "extraordinary misconception of the word 'cause'" (EP 2.315). Peirce notes that Mill mistakenly speaks of the cause of a "singular event", rather than a " 'fact,' which is an element of the event" (ibid)—i.e. an abstraction. The consequence is that Mill's conception of cause must include "the totality of all circumstances attending the event…[which must be] the Universe of being in its totality" (ibid). On Peirce's own view, "the cause is another 'fact'" (ibid). In his more detailed explanation that follows, he argues that we must take account of further causal distinctions—e.g. internal v. external cause, formal v. material cause, and perhaps most importantly, final cause in addition to efficient cause (ibid). This latter distinction motivates Peirce to make the rather bold claim that, "every sufficiently complete symbol is a final cause of, and 'influences,' real events, in precisely the same sense in which my desire to have the window open, that is, the symbol in my mind of the agreeability of it, influences the physical facts of my rising from my chair, going to the window, and opening it" (EP 3.317).

law of excluded middle would not apply. Peirce has much to say about the concept of 'individuals' and characterizes them using the mathematical concept of limits.[77] We will not discuss these ideas in detail except to note that, " 'Individual' is usually and well defined as that which is absolutely determinate" (EP 2.408). Thus, as facts are abstractions, so are the individuals involved in them. In the context of the Beta graphs (and first-order logic generally) we take propositions to be *de inesse*, which is a technical term of Peirce's, meaning "relating to a single state of the universe" (CP 4.376). *De inesse* propositions are contrasted with *modal* propositions, which Peirce considers in the gamma graphs.[78] The gamma graphs are beyond my scope in this paper.

This does not mean, however, that we can only make claims about "saturated" relations. We can also, of course, assert proposition that have indefinite subjects, which we do whenever we assert universally and existentially quantified propositions. But, for Peirce, these too assert the existence *de inesse* of their subjects. We will give a more detailed account of how Peirce interprets the quantifiers later in this chapter; but for now we will note that, for Peirce, to say, for example, that " 'Some woman is adored by every Catholic'…means that a well-disposed person with sufficient means could find an index whose object should be a woman such that allowing an ill-disposed person to select an index whose object should be a Catholic, that Catholic *would adore* [my emphasis] that woman" (EP 2.168). The future conditional tense is important here. The *fact* involved in this proposition is the adoring of the woman by the Catholic, but the (representational) relation of which this fact is the referent is effectively the consequent of a conditional, whose antecedent involves "filling the gaps" of the proposition with particulars—i.e. any particular Catholic you please, and *one* particular woman, the Holy Mother. But the proposition itself is either true or false *de inesse*. That is to say, *now, in this state of the universe*, it is either possible or not to identify such particulars, which, once filling the blanks of the proposition, that proposition would be either true or false. We will say more about this interpretation of the quantifiers later.

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[77] CP 3.93-96 (DNLR); CP 3.216-217 (AL)

[78] "a proposition is either *de inesse* (the phrase used in the *Summulae* [*Summulae Logicales* of Petrus Hispanus, taught in the schools from about 1200 A.D.], the highest medieval authority for terms of logic) or *modal* (Latin *modalis*, Abelard)" (EP 2.283).

This is all made more complex (though perhaps also more clear) by noting the first subtle point mentioned above—namely, that according to Peirce, facts themselves have semiotic structure: “What we call a ‘fact’ is something having the structure of a proposition, but supposed to be an element of the very universe itself” (EP 2.304). It is only by virtue of this sameness of structure that propositions are fit to represent facts. Peirce goes on to say, “the purpose of every sign is to express ‘fact,’ and by being joined with other signs, to approach as nearly as possible to determining an interpretant which would be the *perfect Truth*, the absolute Truth…”(EP 2.304).

Therefore, importantly, the only way we can analyze the structure of *facts* is by analyzing the structure of *propositions*. “Thus, the question whether a fact is to be regarded as referring to a single thing or to more is a question of the form of proposition under which it suits our purpose to state the fact” (CP 3.418). As such, we must look to logic to determine what are the unanalyzable or indecomposable elements. As we have noted, Peirce finds that there are three, and aims to defend this notion with the Reduction Thesis.

Here some additional definitions from Peirce are, perhaps, helpful:

Rheme: “…[in an assertion] let a number of the proper designations of individual subjects be omitted, so that the assertion becomes a mere blank form for an assertion which can be reconverted into an assertion by filling all the blanks with proper names. I term such a blank form a rheme.” (CP 4.354)

As we noted above, a rheme (or rhema) can be understood either as an unsaturated predicate or a partially saturated predicate (CP 4.438). But it must have at least one blank. For our purposes, a rheme can be understood as an unsaturated (or partially saturated) relation—(ii.a) above.

Correlate: “the objects whose designations fill the blanks of a complete relative are called the *correlates*” (CP 3.466).

A “complete relative”, for our purposes, can just be understood as a rhema whose blanks are all filled. Those blanks are filled by correlates. These are essentially the same as logical subjects. Correlates are representational entities (they exist in a logical system). They may refer to individuals, characters, elementary facts, etc. (CP 4.546).

Relative rheme: a rheme where "the number of blanks exceeds one" (CP 4.354); "A relative rheme signifies its corresponding relation" (ibid).

For example, '__is the father of__' would be a (dyadic) relative rheme. Similarly, '__sells__to__for the price of__' would be a tetradic relative rheme.

Relation: "A Relation is a substance whose being and identity precisely consist in this; its being, in *the possibility* [my emphasis] of a fact which could be precisely asserted by filling the blanks of a corresponding relative rheme with proper names; its identity, in its being in all cases so expressible by the same relative rheme" (CP 4.354).

Here we note a few important features of this definition. First, recall from the previous definition that, "a relative rheme signifies its corresponding relation" (ibid). So, in Peirce's terminology, while a relative rheme is a representational entity—that is, a term in a logical system—the relation is the real thing it signifies. So, relation, as used here, means the referent of an unsaturated relation, in the language I have used above.

Second, Peirce points out that the identity of a relation consists in it being expressible by the same relative rheme. This will be most clear with an example. Suppose we have a fact (in Nature) that Barack is the father of Sasha. We represent this "saturated relation" logically by the formula *Rbs*. The terms *b* and *s* are correlates; and the term *R* is, in Peirce's language, a rheme, specifically a relative rheme, with two blanks. Now suppose we also have the fact (in Nature) that Phillip is the father of Alexander. We represent this "saturated relation" by the formula *Rpa*. Again, *p* and *a* are correlates, and *R* is a relative rheme (an "unsaturated relation"). But it is *the very same* relative rheme (i.e. "unsaturated relation") that we got by considering *Rbs*. This, of course, raises the question: how can it be the same if it is abstracted from a different fact? The answer given by quantificational logic is that the relation (*R*) *is* a set of ordered pairs, which includes both $\langle b,s \rangle$ and $\langle p,a \rangle$. Peirce's answer is more nuanced, as we will see.

The third point to note about this definition is that the being of a relation consists in the *possibility* that a fact could be precisely asserted by filling the blanks of the relative rheme by which it is signified. It is significant that Peirce speaks of possibility rather than actuality. He deliberately does *not* say that a relation exists iff the relative rheme (which signifies it) whose blanks are filled *actually* signifies a fact. That would be to say that *the fact* exists *de inesse*—i.e. now, in actuality. But Peirce realizes that it does not exist now.

The fact may exist in the future, *but its possibility exists now*. In other words, it is a fact now that this future fact is either possible or not. Herein lies Peirce's realism, and his criticism of his own earlier nominalism. As Bellucci says, "Aristotle supposes that a general term is equal to a sum of singulars, which is a doctrine that Peirce cannot admit, for 'the extension of a universal term consists in the total of possible things to which it is applicable and not merely to those that are found to occur (W 1:263)" (Bellucci, 34).

In the next chapter, we will often use the term 'relative', which is Peirce's preferred term in much of his logical writing. He notes that, "A **relative** is…an icon, or image, without attachments to experience, without 'a local habitation and a name,' but with indications of the need of such attachments" (CP 3.459). As noted above, in Peirce's language, a relative is a relative term, which signifies a corresponding relation. However, I will follow modern usage and occasionally still use the word *relation* to mean a representational entity, but I will try to be clear when I am talking about the representational entity, and when I am talking about its referent. I believe the definitions above are helpful in bringing out how Peirce is thinking about relatives/relations and their referents. But in the discussions that follow, there are two key distinctions we should keep in mind: First is the distinction between

(i) Facts, which have a relational structure, and

(ii) Relations as representational entities (in a logical system or language).

Second is the distinction between

(ii.a) Unsaturated relations in a logical system (e.g. *R*), and

(ii.b) Saturated relations in a logical system (e.g. *Rab*).

The referent of a saturated relation (ii.b) is a fact (i). But the question as to the referent of an unsaturated relation (ii.a) is much more difficult to answer. Again, I will argue that Peirce has two different answers in mind—one being the 'God's eye view' and one being the more nuanced, particular view, which I will attempt to explain in this chapter and the next.

Referents of "Unsaturated Relations" (Rhemata)

How should we define truth and falsity in light of the distinctions we have drawn? Peirce's clearest answer concerns the concepts of *logical depth* and *logical breadth*—a distinction that is intimately related to some other important concept pairs, which we have discussed and will discuss: logical subject(s) and logical predicate/rhema; extension and intension/comprehension; object and interpretant; and even antecedent and consequent (though this last will likely not be clear yet).[79]

Logical Depth: Peirce claims that, "the totality of the predicates of a sign, and also the totality of the characters it signifies, are indifferently each called its logical depth" (EP 2.305). If we imagine, as noted above, that propositions consist of the union of predicates and logical subjects, then we can understand logical depth (of a logical subject) to be all of the predicates that "fall under" that given logical subject. Peirce goes on to explain that some synonyms of this concept are "the *comprehension* of the Port-Royalists, the *content* (*Inhalt*) of the Germans, the *force* of De Morgan, [and] the *connotation* of J.S. Mill (this last is objectionable)" (ibid). Peirce also associates logical depth with the concept of an interpretant: e.g. "[a sign's] *Interpretant* is the *Signification* of the concept, its *Inhalt*, its 'connotation' (to use a bad term)" (letter to James, EP 2.497). This concept is also effectively the same as intension, in the sense that the full intension of a general noun like 'man' (considered as logical subject), or of a proper name like 'Socrates', would be the totality of all of the predicates that can truly be applied to that concept (considered as logical subject).

Logical Breadth: Again, making use of the distinction between logical subject and predicate, we can consider logical breadth as the totality (or set) of all logical subjects that "fall under" a predicate. In Peirce's words, "the totality of the subjects, and also, indifferently, the totality of the real objects of a sign, is called the logical *breadth*" (EP 2.305). He again gives synonyms, which include "*extension* of the Port-Royalists…the *sphere* (*Umfang*) of translators from the German, the *scope* of De Morgan, the *denotation* of J.S. Mill" (ibid).

Here, it is perhaps worth flagging a point that cannot be made clear until later in this paper. Peirce maintains that all signs are indefinite either in breadth or in depth, and he aims to give an account of how they gain definiteness or determination. Thus, as I have mentioned, I believe that, on Peirce's view, we must distinguish between complete logical breadth, or extension in the 'God's-eye-view' sense, and denotation in actual

[79] See EP 2.305

practice (particularly in science). Peirce notes that, “denotation essentially takes a part for its whole” (EP 2.322). Since “universes”—i.e. receptacles or classes of logical subjects (CP 4.545)—and “categories”—i.e. classes of predicates (ibid)—are “enormously large, very promiscuous, and known but in small part” (CP 4.544), they must be “denoted by Indices” (ibid). I will attempt to give a proper account of this idea in the next chapter. In that chapter I will argue that we can take a clue from Peirce’s account of Induction, where a sample (i.e. subset) of a class represents the complete class by being an index of it.

Where logical depth was associated with the concept of the *interpretant* (of a sign), logical breadth is associated with the concept of an *object* of a sign (CP 4.542). Peirce asks, “What, for example, is the object of ‘runs? Answer: it is something, a runner. What is the object of ‘kills’? Answer: it is a pair of indesignate individuals, the one a killer, the other killed by him. So giver has for its object a triplet of related indesignate singulars…” (EP 2.408). Here, the sign under consideration is a predicate, or rheme. The individual n-tuple of “indesignate singulars” is the object of the sign. But the sign also has its interpretant, which should be understood intensionally. When object and interpretant are unified in a proposition, the blanks of the rheme are filled and we get a saturated relation, whose referent is a fact—e.g. the fact of this individual killing that individual. Logical breadth is effectively the same concept as extension (in a “God’s-eye-view” sense), in that the complete extension of a concept like ‘man’ (considered now as the rheme or predicate ‘__is a man’) would be the complete set of logical subjects, each of which could fill the blank and in so doing make a true proposition.

The example above (‘__is a man’) is obviously a monadic predicate. In the case of relational predicates (i.e. predicates with addicity >1), logical breadth (as a “God’s-eye-view” concept) is essentially the same as the monadic case, except that the logical subjects falling under the relational predicate would be considered as ordered n-tuples. To give an account of the logical breadth of various predicates, then, we must divide these predicates according to their addicity, in order to determine what sorts of logical subjects fall under them—e.g. pairs, triples, etc.

Further, as I have intimated, Peirce’s understanding of the referents (the logical breadth) of “unsaturated” relations (concept (ii.a) from above) is rather complex, and involves a semiotic component, which I will attempt to bring out in the next chapter. The referent of a "saturated" relation (concept (ii.b) above)—e.g. *Rab*—is easier to account for; its referent is a particular *fact* (concept (i) above). In this case, we consider a (relational) fact in Nature, which we symbolize in a logical system by a relational predicate that is "saturated" with indices representing its logical subjects (in this example, the logical subjects are indicated by the individual constants *a* and *b*).

But, of course we would also like to know, what is the referent of *R*? This is a trickier question. In one sense, it is the purely formal skeletal thing that is abstracted from *Rab*. But this abstracted skeletal thing (*R*) is the *very same thing* that we get when we analyze *Rcd*, a representational relation that represents a *different* particular fact in Nature.

Obviously, contemporary logic would define *R to be* the set of ordered n-tuples, but I believe that Peirce would not make that leap. For one, Peirce never claims that relations *themselves are* sets of n-tuples. This is also further supported by the fact that in EG there is no semantic feature according to which we may consider the extension of unsaturated relations (the *R'*s) as sets of ordered n-tuples. I believe that one reason why Peirce would criticize the interpretation of unsaturated relations as sets of n-tuples is based on the distinction between matter and form. When we take a relational proposition—e.g. *Rab*—it is only through a process of abstraction, by removing the material elements, *a* and *b*, that we get the skeletal thing *R*. To say that *R is itself a set of n-tuples* seems to be smuggling matter into an entity that was supposed to be purely formal. But, obviously, if we take *R* to be merely formal, then it seems to just be the blank form of a two-place relation. In that case, it seems we would have no way to distinguish it from a different relation, say *S*. And further, if we consider it as purely formal it seems hard to explain how it is the very same thing abstracted from two different facts.

We began this digression by asking, how shall we define truth and falsity, in light of the distinction between (i) relations qua facts, (ii.a) unsaturated representational relations, and (ii.b) saturated representational relations. It is propositions that are true or false. And a proposition is formed from the combination of a predicate (or predicates) and logical subjects that fill its blanks. So we can say (loosely) that what makes a proposition true is alignment of logical depth and logical breadth. A little more precisely, we can say that a proposition is true *iff* the predicate is included in the logical depth of its subjects, and the logical subject(s) are included in the logical breadth of the predicate. This definition (though I do not intend it to be a definition in a very strict way) involves both extension and intension (to use more familiar terminology). It also aligns with Bellucci, who claims,

> "Logic is interested in truth, and 'truth' may in a general sense be defined as 'the concurrence of the extension and comprehension of a representation which has extension and comprehension independent of one another' (W 2:4). Only symbols, then, can be true, because only symbols both denote and connote" (Bellucci, 38).

Bellucci explains elsewhere that icons connote but do not denote, and indices denote but do not connote. Symbols do both—they "denote by connoting" (W 1:272)—and so only symbols can strictly speaking be true. We could illustrate this with an example.

Consider the street-cry from our earlier discussion of indices. If the street-cry is just a noise, then it denotes by drawing the attention of the interpreter; but it asserts nothing, and so cannot be true or false. However, if a street-cry, in addition to being a noise, merely drawing attention of the interpreter, were to also give information—e.g. if the crier yelled "Watch out!"—then the cry ceases to be a pure index, and is actually a symbol with indexical features. "Watch out!" is really an abbreviation of a proposition, such as, perhaps, "you should watch out!" If it turns out that there was no danger, then that proposition would be false, since it was not the case that you should watch out.

The same holds in the case of icons. Suppose one is in a house of mirrors and sees a shrunken reflection of oneself. This image is certainly an iconic sign, but could only be seen as "false" insofar as one interprets it as a sign that is giving information—that is, insofar as one interprets it as a symbol. In this case, the proposition that the iconic symbol (implicitly) asserts would amount to something like: "the object of this reflection has the proportions displayed by this image." Since the object—i.e. the person—has very different proportions than the shrunken image, we could say that the symbol (i.e.

proposition) was false. But we can only say it is false insofar as it is a symbol, giving information.[80]

But, it is vital in our account of logical truth, to note that the account above is only how truth would be defined in some God's-eye-view kind of way.

Peirce points out that, "no cognition and no Sign is absolutely precise, not even a Percept; and indefiniteness is of two kinds, indefiniteness as to what is the Object of the Sign, and indefiniteness as to its Interpretant, or indefiniteness in Breadth and in Depth" (CP 4.543). If we had the complete logical depth of a subject and complete logical breadth of a predicate accessible to us, then truth would be defined in the way we noted above. But this is not how things work in the case of actual reasoning.

Bellucci explains that in Peirce's day, "the law that *the greater the extension, the less the comprehension* was daily bread of logicians; so we find it in Hamilton, Jevons, and many others" (Bellucci, 39). For example, suppose the logical subject 'horse' has an extension—namely the class of horses. But when the concept is precisified to be "black horse", this increases comprehension/intension of the term, but decreases its extension, since the new extension cuts out all non-black horses. "However, Peirce argues, this is true only in a perfectly immobile state of knowledge" (ibid). Bellucci explains that we may attempt to divide the class of men into man-risible and man-non-risible. But we *discover* that the latter class is empty. Thus, the comprehension (intension) of the term 'man' increases without a corresponding decrease in extension. We have new information. According to Peirce (explained by Bellucci), this is how synthetic propositions give new information in actual practice: "when we discover that 'bodies are heavy,' the comprehension [i.e. intension] of 'body' is increased by the character 'heavy,' while the extension of 'heavy thing' is increased by the addition of bodies" (Bellucci, 38). This foreshadows the mode of combination of relatives that we will see in the relative product operation. The point of the next chapter is to show how Peirce's

[80] On this topic, recall that Peirce divides symbols into terms, propositions, and arguments. Right now we are only talking of propositions, but since arguments and even terms have the same structure, the same principles apply to them. With terms, this is the most difficult to see. But the central point, in that case, is that object and interpretant are related in a similar way that logical subject and logical predicate are related in a proposition.

discovery of quantification grew out of his attempt to give a (logical) account of how logical depth and breadth are made more determinate.

A worry here is that we are sliding into epistemological talk. But I believe Peirce would characterize it instead as an investigation into the logic of discovery, or the logic of reasoning in actual practice. In Chapter Two, we explained that, while we do not *employ* abductive and inductive reasoning in logic, it is yet a task of logic to *classify* these. Similarly, if actual practice of reasoning about relations (particularly in science) implicitly or explicitly involves a more nuanced account of the referents of those relations, then it is a task of logic to examine that reasoning. Peirce, of course, understands that the complete and precise 'God's eye view' account of the referents of relations works one way. And this account is, of course, valuable. But I follow Peirce in thinking that we also need an account of how this theory of reference maps to reality. I believe that in logic and metaphysics today we have a problem where we *either* assume there is some simple, direct mapping of our logical theories to reality, which results in an overreaching and hubristic metaphysical view; *or* we ignore the question of mapping to reality, and end up metaphysical relativists. I maintain that Peirce's views provide some hope for navigating through the Scylla and Charybdis in this dilemma.

What is the point of all this? Why go into all this detail on different conceptions of the meaning of relations? The answer comes back to indefiniteness and determination. The way that we acquire knowledge is through the use of unsaturated relations as intensional entities, not by examining the extension of a relation in some God's-eye-view way. The latter way assumes that we already know everything about the extension of the relation, and it implies that intension is irrelevant. In the next chapter we will discuss how considerations about relations led to the discovery of quantification—which is achieved (in a representational system) through conceptual combination (specifically the relative product operation). This conceptual combination happens in a *particular way*—through identifying that different indices (i.e. variables) refer to the same object. And I will argue that Peirce *discovered* that it happens this way. This sort of conceptual combination takes a particular form in a formal system. So an ideal logical system (for purposes of discovery) should only permit that sort of conceptual combination. I maintain that this is why Peirce only permits two loose ends to bind to each other in EG, which is one of the key pillars of the Reduction Thesis.

Relations More than Mere Connection—Continued

Let us consider some examples to illustrate the idea that a relation is more than a mere connection between two things. Peirce notes, "when you think 'this is blue,' the demonstrative 'this' shows you are thinking of something just brought up to your notice; while the adjective shows that you recognize a familiar idea as applicable to it" (CP 3.417). We noted above that facts have the same structure as propositions. This is to say that they include a verb. We might imagine that someone closes their eyes in an aquarium and opens them to see a tank of water and simply thinks the idea "blueness" without any verbal copula. But if this thought is anything more than mere feeling, then "blueness" is really just shorthand for the assertion "blueness is here," which is a proposition capable of being true or false. If we followed the traditional approach to propositions (with its myopic fidelity to *grammatical* structure), we might imagine that in the assertion 'this is blue', we have three elements—namely subject, copula, and predicate.[81] Peirce's mature view, however, is that the predicate includes the verb—thus we have (at least), a logical subject 'this', and a predicate '__is blue', which includes the verb. We might consider another simple proposition with only two words: 'Spot runs'. This proposition, similarly, consists of a logical subject—the proper name 'Spot', and the predicate '__runs'.

However, in a proposition involving a monadic predicate (like the two examples above), we are not dealing with two things but three. As we noted, we have (i) this, and (ii) '__is blue'; but these two things alone do not constitute a proposition that is capable of being true or false. We must also have the unity of 'this' and '__is blue', which unity is a *third thing*. This is importantly different from the third thing being the copula. To see this, consider the second example where we have a proposition 'Spot runs'. Here there

[81] See EP 2.308-9. Peirce admits that Aristotle (in *De Interpretatione* 17a, 19b) "dissects a proposition into subject, predicate, and verb, yet as long as Greek was the language which logicians had in view, no importance was attached to the substantive verb 'is,' because the Greek permits it to be omitted. It was not until the time of Abelard, when Greek was forgotten, and logicians had Latin in mind, that the copula was recognized as a constituent part of the logical proposition" (EP 2.308-9).

are only two words—the logical subject 'Spot' and the predicate '__runs'—but we still need a third thing that signifies that these are unified—i.e. something to assert that *Spot* is the one that is running (i.e. that the subject Spot belongs to the logical breadth of the predicate '__runs') and that *running* is what Spot is doing (i.e. that '__runs' is a predicate that belongs to the logical depth of the subject Spot). As Peirce says using a different example: "it is important not to forget that no more do 'Socrates' and 'is wise' make a proposition unless there is something to indicate that they are to be taken as signs of the same object" (EP 2.310). What is this object? It is the *fact* in Nature, which is the referent of a saturated relation—in this case, the referent of the proposition Wa ('Socrates is wise').

We can illustrate this point (that monadic propositions require three things) from the perspective of extensional quantificational logic. Simply from the existence of a—the proper name of the object brought to our attention—and the assertion that 'something is blue', that is, $\exists xBx$, it does not follow that 'a is blue'—i.e. Ba. To get that proposition, we need an additional piece of information that asserts that $a \in \mathrm{Ext}[B]$.

Similarly, in the case of a dyadic predicate we have four things—each of the two relata (correlates), the dyadic rhema, and the union of these. But, again, simply from the existence of a and the existence of b—or even from the existence of the ordered pair $\langle a,b \rangle$—and the assertion that $\exists x \exists y\ Rxy$, we cannot conclude Rab without the additional assertion that $\langle a,b \rangle \in \mathrm{Ext}[R]$. In the case of a triadic predicate, similarly, we have five things. Tiercelin emphasizes this point as well, quoting Peirce: "the very idea of a compound supposes two parts, at least, and a whole, or three objects, at least, in all…" (CP 7.537) (Tiercelin, 130).[82]

Here one might object, however, by pointing out that not all propositions have a definite logical subject. From the perspective of extensional quantificational logic, this is to say that not all propositions involve individual constants. Certainly Ba is a proposition, but so is $\exists xBx$. In this case we do not have a 'this' in the same way that we do in the

[82] In the case of a dyadic predicate, we could view $\langle a,b \rangle$ as a single entity, in which case we have three things. Similarly, we could take any n-tuple as a single thing, in which case any higher-addicity predicate would consist of three things. But the point is that we always have a minimum of three things.

previous example. So does an existentially quantified monadic proposition like $\exists x Bx$ still consist of three things? For Peirce, it does because of his interpretation of the quantifier. He argues,

"So in 'some cat is blue-eyed' the subject is not 'some cat' but 'something,' the predicate being '__is a blue-eyed cat'. 'Something' means that sufficient knowledge would enable us to replace the 'something' by a monstrative index and still keep the proposition true. 'Anything' means that the interpreter of the proposition is free to replace the 'anything' by such a monstrative index as he will, and still the proposition will be true" (EP 2.173).

The predicate in this example is a conjunction of two monadic predicates ('is a cat' and 'is blue eyed') but this makes little difference for the broader point. The point is that Peirce conceives of existentially quantified propositions as asserting the truth of a conditional—namely, that *if* one replaced the 'something' (i.e. the variable bound by the existential quantifier) with the right sort of monstrative index, *then* the proposition *would be* true. This conditional is either true or false *de inesse*-–i.e. in this present state of the universe. But the structure of the (in this case monadic) proposition, at the point when it becomes saturated, would still consist of three things—the monadic predicate, the logical subject (which has now been identified), and the union of these.[83]

What is the fact to which these sorts of quantified propositions refer? It is a conditional fact, but as the later Peirce emphasized (in criticizing his own earlier

[83] See Peirce's 1909 letter to James: "The Object of every Sign is an Individual, usually an Individual Collection of Individuals. Its *Subjects*, i.e. the Parts of the Sign that denote the Partial objects, are either *directions for finding the Objects* or are *Cyroids*, i.e. signs of single Objects…Such for example are all *abstract* nouns, which are names of single characters, the personal pronouns, and the demonstrative and relative pronouns, etc." (EP 2.494). This idea of a subject being a set of "directions for finding the Objects" also recalls Peirce's discussion of scientific definitions: "If you look into a textbook of chemistry for a definition of *lithium*, you may be told that it is that element whose atomic weight is 7 very nearly. But if the author has a more logical mind he will tell you that if you search among minerals that are vitreous, transluscent, grey or white, very hard, brittle, and insoluble, for one which imparts a crimson tinge to an unluminous flame, this mineral being triturated with lime…can be partly dissolved in muriatic acid; and if this solution be evaporated, and the residue be extracted with sulphuric acid…will yield a pinkish silvery metal that will float on gasoline; and the material of *that* is a specimen of lithium" (CP 2.286). I have left out some of the scientific details, but the point is that this "definition" gives a recipe for how one can go about getting ostensive acquaintance with the thing defined. See CP 4.548 (1906), and EP 2.408 (1907) for further discussion of the existential quantifier.

nominalism), “a true [conditional] ‘would be’ is as real as an actuality” (EP 2.456). Peirce explains that these “would-be’s” are copious in mathematics, and generally “[take] the following form: proceed according to such and such a general rule. Then, if such and such a concept is applicable to such and such an object, the operation will have such and such a general result; and conversely” (EP 2.410-11).

This can also be conceptualized more easily in EG. As we have noted, in that system there are no proper names; rather proper names function like monadic predicates. Thus, in EG, loose ends (unattached lines of identity emerging from hooks) of predicates are indices that indefinitely represent ‘something’. In this sense, any “unsaturated predicate” with loose ends, is already existentially quantified (from the perspective of quantificational logic). But what happens with conceptual combination is that the number of quantifiers in a proposition effectively decreases by one, since it is affirmed that two of the indices (from the perspective of quantificational logic these indices are the bound variables) represent the same object.

We have been speaking of the idea that a relation is more than mere connection between two things from the perspective of quantificational logic, but there is also a semiotic component. We noted above that ‘Socrates’ (a logical subject) and ‘__is wise’ (a rhema) do not alone make a proposition, unless there is something to indicate that they are signs of the same object (EP 2.310). As we noted above, this object should be understood as a fact. This recalls Bellucci’s point that we highlighted in Chapter Two, namely that, “the leading principle of inference in general is that ‘the sign of a sign of an object is itself a sign of the object” (Bellucci, 48). We have been considering symbols—specifically propositions, and have claimed that the object of a saturated relational proposition is a fact in Nature. For example, take the proposition ‘Bob is a servant of Alice’. We might represent this fact in quantificational logic with the sentence *Sba*. But in an argument, perhaps we know that a different symbol—e.g. *Mab*, representing ‘Alice is a master of Bob’—represents *this very same* fact. It has the same object. In this case the symbol (i.e. proposition) *Mab* can be substituted for *Sba* in the argument, without making the argument invalid. Through the substitution, *Mab* is now also a symbol of *Sba*. And it is permitted to be a symbol of *Sba* because it has the same object. Thus, we see that the leading principle of deduction is that the symbol of a symbol of an object is itself a symbol of that object. In mathematical arguments, this sort of substitution is perhaps most explicit.

The idea that a relation is more than merely a connection between two things is also related to the Reduction Thesis. Hawkins cites a 1904 letter to Victoria Welby where Peirce objects to Russell, claiming, “the point is that triads evidently cannot be so reduced since the very relation of a whole to two parts is a triadic relation” (Hawkins, 134). But, in all this, Peirce also says (NEM 3:824), “it is not pretended, and is not true…that a dyadic relation is a triadic relation” (Anellis, 292). So what are we to think?

The key idea in regard to the Reduction Thesis hinges on a distinction between the object-language argument and the metalogical argument. As noted in Chapter Two, Peirce made this important discovery in *Memoranda*, and it became central for his thinking about the Reduction Thesis. We will discuss this idea in more detail with the third germinal theme. But the point is one about metalogical arguments. Even when speaking of monadic relations in a metalogical argument, we must speak of three things. In the course of that argument, then, one must employ a triadic relation to describe (in metalogical terms) the relation that holds between logical subject and predicate. So, in that sense, the metalogical argument requires *employing* triads (even though we are not talking *about* a triadic relation. In the relation *Rab*, for example, Peirce is not saying that *a* and *b* are secretly in a triadic relation; they are in a dyadic relation. But when we talk *about* that relation, we need to talk about *a*, *b*, the unsaturated relation *R*, and union of these—i.e. the idea that *a* and *b* are in the relation *R*. And all four of these are required. As mentioned above, we may instead take $\langle a,b\rangle$ to be one thing (rather than two), but in that case, we are still talking about three things at minimum. So we see that, at the metalevel, triadicity is basic.[84]

[84] For Peirce, this idea is also tied to how we should understand the identity relation. Identity, as defined in DNLR and elsewhere, is not primitive for Peirce, since the proposition x=y is capable of analysis into x—< y and y—< x (CP 3.48). The meaning of the connective —< is ambiguous, and will be explained in the next chapter; but it can be taken here to mean (ambiguously) ‘implies’ and/or ‘includes’. Peirce makes a similar point in his argument against Royce and Kempe that unsymmetrical relations are primitive since “the combination of two unsymmetrical relations will yield a symmetrical relation, whereas the combination of two symmetrical relations will never yield an

Relations as Primitive

The **second** germinal Peircean theme in De Morgan is the idea that relations themselves could be ontologically primitive. As De Morgan points out, "language hesitates at *realizing* notions which are not objectively called *things*" (DeMorgan, 219). The distinction between 'whiteness of the ball' and 'rotundity of the white' is a metaphysical one that depends on the metaphysical notion of substance being applied to one of the terms of the compound 'white ball'. If we are capable of analyzing away this distinction, then at least logically speaking, substance appears not to be a primitive notion. However, we are incapable of analyzing away relations, so perhaps relations *are* primitive.

Burch claims, "Peirce was certainly some sort of realist with regard to relations simpliciter…in his view, relations as such were fundamental, whereas individual entities were derivative by means of (hypostatic) abstraction from them" (Burch, APRT, ix). Hypostatic abstraction is effectively the process where we consider something referenced by an adjective as a noun—e.g. converting the concept 'red' to the noun 'redness'.[85] Part of the reason for this view comes from ideas that we have discussed above. As we noted, Peirce conceives of facts as things that are abstracted. In giving an account of truth and falsity of the propositions that refer to those facts, we consider the logical universe *de inesse*. In this account, the individuals involved in the facts are really abstractions as well. It would take us too far afield to talk in detail about Peirce's conception of individuals, but it would be more appropriate to think of the logical subject of the proposition 'Spot runs' as 'Spot-here-now' rather than Spot the enduring metaphysical subject. In all of this, it seems more natural, then, to consider the facts as basic.

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unsymmetrical relation" (Anellis, 291). I believe 'unsymmetrical' here is meant, from the perspective of quantificational logic, to mean antisymmetric—i.e. $\forall x \forall y\ ((Rxy \wedge Ryx) \rightarrow x=y)$. Anellis gives a more detailed account on 291-294, where he also explains how Peirce shows that, "while a dyadic successor relation is enough to order the integers…Peirce has shown that ordering among rationals is a triadic relation" (ibid, 293).

[85] "hypostatic abstraction—which gives mathematics half its power" (EP 2.352). See also footnote on EP 2.350-1 for discussion of hypostatic abstraction as it relates to the distinction between the distributively general and the collectively general.

Furthermore, if semiotics is as general as Peirce thinks it is, then we could properly say that everything is a sign. In this case, signs are primitive. But if signs contain the sort of structure that Peirce thinks they do—i.e. the sign-object-interpretant structure—then we could also say that, that from a semiotic perspective, the sign-object-interpretant *relation* is primitive.

Relation vs. Judgment

A **third** theme that is germinal in De Morgan is the distinction between a proposition itself and the assertion or denial of that proposition. De Morgan makes the criticism that, that "in the ordinary books on logic, the relation before the mind is confusedly mixed up with the judgment, the assertion or denial of the relation" (DeMorgan, 215). Peirce similarly criticizes "those logicians who follow the lead of Germans, [and] instead of treating of propositions speak of 'judgments' (*Urtheile*). They regard a proposition as merely an expression in speech or writing of a judgment" (Peirce, EP 2.311). Peirce makes the point that judgment involves something *additional*—namely acceptance of the proposition as true.

This distinction also has important implications in regard to the paradoxes. Peirce maintains that a false paradox arises when we confuse a proposition itself with the acceptance of the proposition as true. To illustrate this, he considers the two propositions 'It rains', and 'I assert that it rains' (EP 2.167). He points out that their negations are different—'It does not rain' v. 'I do not assert that it rains' (ibid). So Peirce's conclusion is that, "the proposition 'It rains' does not *itself* [my emphasis] assert that I assert it rains; but when I utter the proposition 'It rains' I afford you the evidence of your senses that I assert it rains" (EP 2.167).[86]

The central point in regard to this germinal theme is this: Peirce, following De Morgan, delved deeply into issues about how to represent relations, leading to all of the distinctions we cited above; and in the course of that investigation, he realized that we need to distinguish between propositions themselves and the assertion or denial of those propositions. This brought to light an immensely important distinction that he had already

[86] See Hawkins, 127 for explanation of Peirce's response to Russell regarding "Lewis Carrol's so-called 'logical paradox'."

clued into in Memoranda—that we must take account of the form or structure of our reasoning in the meta-level argument. This notion will be absolutely key in how he thinks about the Reduction Thesis, as we will see. This distinction permits us, for example, to draw the distinction between the monadic relation that we are reasoning *about*, and the triadic relation that we *must employ* in reasoning *about* that monadic relation.

This is also related to Peirce's definition of validity itself. Ordinarily, we understand validity to be an assertion about the truth of a conditional property of an argument, namely that: if the premises are true, then the conclusion is true. As Bellucci explains, this amounts to saying that an argument is valid iff "the object of the premises is also an object of the conclusion" (Bellucci, 16). We saw this idea in the example above involving substitution of equivalent symbols. Logical inferences are expressed formally by substitution of symbols that are equivalent, and therefore, "in an argument…a symbol is substituted for another that has *the same object* (i.e. which is true when the former is true" (ibid, 16).

Hawkins makes a similar point, but clarifies the role that Thirdness plays in the concept of validity itself. He claims,

"Peirce considered an argument, for example, within the province of logic in the restricted sense, to be 'a sign whose interpretant represents its object as being an ulterior sign through a law, namely, the law that the passage from all such premises to such conclusions tends to truth' (2.263)" (Hawkins, 117).

Hawkins claims that this anticipates Tarski's conception of logical consequence, and he makes the additional observation that Peirce anticipated many conceptions of model theory (ibid). A longer passage from Hawkins is worth quoting to clarify this notion:

"The point is that in the context of Peirce's semiotic, if A_n is a deductive argument, there is a relation K_n on A_n iff K_n is a relation on the illative transformation of a sign S_i into the *finis operis* S_{i+m}, so that K_n is 'requisite besides the premises to determine the necessary…truth of the conclusion (2.465). The effect of K_n is a *habit*: 'The truth is that an inference is 'logical,' if, and only if, it is governed by a habit that would in the long run lead to the truth' (L 463 [December 1908]/*Lieb ed. 1953:30*)."[87]

[87] Interestingly, Hilary Putnam criticizes (the metaphysical analog of) this idea in Peirce on the basis of Hawking Radiation. He says in a footnote of "Pragmatism and Realism": "In Peirce's case, however, the story is more complex because Peirce was willing to make *empirical* assumptions (ones that are not compatible with today's physics) to guarantee that his theory of truth would not be incompatible with his realism. Given his fallibilism, Peirce might even be happy that his theory has been refuted by a scientific

This passage recalls our earlier discussion of leading principles. The key ideas are (i) that leading principles are required over and above the deductive steps of inference within a logical system (i.e. they are metalogical principles) and (ii) that leading principles *are* (and not merely express) habits. In one sense, certainly, the formal statement of a leading principle *expresses* that leading principle. But, as Peirce says later (1907), "The real and living logical conclusion *is* that habit; the verbal formulation merely expresses it" (EP 2.418). Here we see a key feature of Peirce's pragmatism. The final interpretant[88] of an argument is not an abstract statement, but a *habit* that accepts that statement as true. This second idea is strongly emphasized in Peirce's 1880 "On the Algebra of Logic" (hereafter AL). There, Peirce begins with a rather psychological account of logic, claiming,

> "A habit of inference may be formulated in a proposition which shall state that every proposition *c,* related in a given general way to any true proposition *p*, is true. Such a proposition is called the *leading principle* of the class of inferences whose validity it implies. When the inference is first drawn, the leading principle is not present to the mind, but the habit it formulates is active in such a way that, upon contemplating the believed premiss, by a sort of perception the conclusion is judged to be true" (CP 3.164).

AL is one of the most psychologistic of Peirce's logical essays (if not *the* most). It begins with a physiological account of how logic itself arises (CP 3.154), talking of nerve stimulation and cerebral habits. That being said, it is no doubt a technical paper on mathematical logic, but it is clear that Peirce is aiming at constructing a system where rules of inference track habitual leading principles of what we would call *good* reasoning. In this sense, perhaps, Peirce's task is not so different from Gentzen's in aiming at deductive rules that are "natural", though Peirce may conceive of logic in this broader sense as more prescriptive than descriptive.[89] Again, the *evaluation* of those rules as good

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discovery. The discovery in question—by Steven Hawking—is that there is such a thing as the *irretrievable destruction of information*. This refutes Peirce's claim that we are entitled to believe that scientific investigation could discover the answer to *any* factual question if sufficiently long continued, and that claim is necessary to Peirce's defense of the realistic character of his notion of truth" (Putnam, "Pragmatism and Realism", 51). Addressing this concern, however, would take us too far afield from logic.

[88] a technical subdivision of the concept of interpretant

[89] See Susan Haack, "Philosophy of Logics" where she argues that Peirce is not anti-psychologistic, like Frege, but rather thinks it is part of the task of logic to define what constitutes good reasoning.

or bad will be a normative task for the philosophy of logic or some other discipline. But the *identification* of those leading principles as leading principles is a task for metalogic.

We must be careful, though. The claim that 'habitual leading principles lead, in the long run, to the truth' is (as Putnam suggests) an empirical claim, which seems to be based on observation of past success in reasoning. There is therefore a legitimate worry that Peirce is basing logic on empirical observation. But this gets at the challenge of defining the task of logic. In one sense, formal logic is simply mathematics, but in another broader sense, logic is concerned with laying the groundwork for how we should inquire into the truth of things. In this sense, Peirce is deeply concerned about linking abstract systems to reality, and he seems to think of logic in the very broad sense as prescriptive of what constitutes good reasoning. Of course, this raises the question 'good' in what sense? We might, at least provisionally answer, good in the sense that our inquiry leads to the truth in the long run. And after all, what else would we want?

Diagrammatization

As soon as we acknowledge the distinction between (i) relations qua facts and (ii) relations qua representations, a question arises as to how to account for relations in a representational system. In addressing that question, it is, perhaps, most natural to begin by examining the structures implicit in our natural languages (and this has indeed been the unfortunate strategy in much of the history of logic). But Peirce had seen early on the limits of that sort of analysis, and followed Boole and De Morgan in pursuing a more exact mathematical approach.

Any logical system that purports to represent relations will lay out rules through which arguments can be represented and their validity evaluated. Peirce argues that to show the validity of any argument (whose validity we could conceivably doubt), one would have to employ a diagram (CP 3.418). Obviously the term "diagram" here is meant in a very general way. It does not even have to be visual—"such a diagram has got to be either auditory or visual, the parts being separated in the one case in time, in the other in space" (ibid). In the final chapter of this paper, we will note another of Peirce's discoveries in regard to diagrams—he came to realize that a necessary semiotic ingredient of any diagram or argument is the very space upon which the diagram is written. This insight led him to revise the Entitative Graphs, where the blank sheet of assertion (SA) was understood as 'the false' (or an absurdity) instead of 'the true', as it is in EG. Peirce saw the advantage of EG over the earlier system to be primarily semiotic—in that it represents inference far more naturally and iconically.

Since a diagram exhibits "the form of a relation" (CP 4.530), on Peirce's view, we may then experiment upon these diagrams in a way that is not essentially different from the experiments that one performs upon specimens in the natural sciences. In the case of chemistry, for example, we might perform experiments upon a particular sample of some chemical compound, but we are hardly concerned about the properties of *that* specific compound—i.e. the token. We only care about the experimental results insofar as they reveal something about the molecular structure in general—i.e. the type (ibid).

In this more general sense, any mathematical equation is a diagram. Peirce offers an example in Prolegomena:

"let f_1 and f_2 be the two distances of the two foci of a lens from the lens. Then,

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f_o}.$$

This equation is a diagram of the form of the relation between the two focal distances and the principal focal distance; and the conventions of algebra (and all diagrams, nay all pictures, depend on conventions) in conjunction with the writing of the equation, establish a relation between the very *letters* f_1, f_2, f_o regardless of their significance, the form of which relation is the *Very Same* as the form of the relation between the three focal distances that these letters denote" (ibid).

If we understand diagrams in this broader sense, then we can make sense of Peirce's claim that, "all deductive reasoning, even simple syllogism, involves an element of observation" (CP 3.363). In his later work, Peirce would speak of experimentation upon diagrams, which is not essentially different than experimentation in the physical sciences. According to Peirce,

"a method of forming a diagram is called *algebra*. All speech is but such an algebra, the repeated signs being the words, which have relations by virtue of the meanings associated with them. What is commonly called *logical algebra* differs from other formal logic only in using the same formal method with greater freedom" (CP 3.418) (1892).

He then goes on to suggest the virtues of a new graphical logical system he has been working on. It is "a far more powerful method of diagrammatisation than algebra…being an extension at once of algebra and of Clifford's method of graphs" (ibid) (1892). This was the system that would evolve into EG.

Once one has a formalized algebra—that is, a system of formal logic—the work is not done, however. Since that system affords diagrammatization, it may reveal previously hidden relations in new ways. These might be hidden properties of the relations *within* that system, or they might be properties of the system as a whole. Peirce was keenly attentive to the fact that he was pursuing this sort of a project, as we see in his statement of three distinct objectives in ALPT, 1885:

"I shall thus attain three objects. The first is the extension of the power of logical algebra over the whole of its proper realm. The second is the illustration of the principles which underlie all algebraic notation. The third is the enumeration of the essentially different kinds of necessary inference; for when the notation which suffices for exhibiting one inference is found inadequate for explaining another, it is clear that the latter involves an inferential element not present to the former. Accordingly, the procedure contemplated should result in a list of categories of reasoning, the interest of which is not dependent upon the algebraic way of considering the subject" (CP 3.364).

Peirce's first objective, here, is the most obvious—he is attempting to construct a formal system that is powerful enough to deal with relations. But, secondly, he wants to give a general account of leading principles (those which "underlie *all* algebraic notation"), which are required to justify inferential steps in any argument. And finally, he is concerned with giving an account of the different kinds of necessary inference, which should result in a list of categories of reasoning. We should note that, in this paper, he is only concerned with necessary inference, so he is not considering Induction or Abduction. But, as I will argue in the next chapter, he is keenly attentive (in this essay and others) to examine how reasoning takes place, and how concepts are combined and made more determinate. One of the principal results of this (which took time to work out) is, I argue, a distinction between two independent conceptions of the referents of unsaturated relations.

Chapter Four

Relative Product

Introduction to Chapter

In an algebraic logical system, as we have noted, relations are taken as primitive terms. But in algebraic logic, in addition to the relations themselves, we also need to define permissible *operations* on those relations. These operations account for how relations are employed in inferences. One such operation—namely relative product—is particularly important. As Herzberger notes, "So prominent did it [the relative product operation] become in Peirce's thought that some of his statements treat it as the very paradigm of conceptual combination" (Herzberger, 44).[90] For our purposes, Relative Product is chiefly interesting in this regard. That operation is something of a metonym for Peirce's logic as a whole. It helps us see how conceptual combination is determinative, and what forms of conceptual combination are primitive. Those primitive forms should be the kinds that are permissible in a logical system for purposes of discovery. Certainly, one could come up with arbitrary semantic interpretations for truth-functional connectives and quantifiers, and construct a deductive system that is sound and complete with respect to these, but that system would not cut Nature at its joints. Peirce came to believe that relative product exhibits the primitive mode of conceptual combination, and he created EG in a way that displays this. The result of this restriction on permissible combination of relations is that the Reduction Thesis holds.

Of course, one could claim here that we are arguing in a circle—constructing a logical system with certain restrictions and then triumphantly pointing out a property of that system, which is a direct result of this restriction. And yes, Peirce intentionally constructed EG to have this restriction. But I wish to argue that this restriction is non-arbitrary in the same way that any good deductive leading principle is non-arbitrary. I believe Peirce *discovered* that conceptual combination and determination genuinely happen through a process that is captured by the relative product operation.

[90] Brunning, Burch, and Brady, and others similarly emphasize this point—e.g. "relative product was the 'privileged mode' of combination" (Brunning, 255).

In that regard, another key point that I aim to bring out in this chapter is that this form of conceptual combination (embodied in the relative product operation) was a *genuine discovery* that came out of dealing rigorously with questions in logic itself. By examining the evolution of the relative product operation in Peirce's logic, we can begin to see how this privileged mode of conceptual combination was not predetermined. As noted in the last chapter, Peirce saw two major problems with the logic of his day. It was incapable of expressing particular propositions, and it was incapable (with the exception of De Morgan's system, which had its own flaws and limitations) of accounting for inferences that involve relations.

It was relative product that led to Peirce's discovery of quantification, and to his conception of relations as predicates that include the copula. His earlier systems (in 1870 and 1880), which still employed relative product, did not yet view relations in this way. They were still bound to the traditional view that propositions are made up of three parts—subject, copula, and predicate. It was a breakthrough when Peirce realized that relations should be seen as n-adic *predicates*, which include the copula—rather than merely as class names. This development will hopefully be made clearer in the coming pages.

In what follows, my purpose is not to give a full account of Peirce's systems as presented in any of the logical papers I discuss. Rather, I aim to sketch the evolution of this one important operation, to show how it can help us understand conceptual determination that occurs through combining relational terms. Explaining relative product, however, will require a fair bit of background. As such, I will first give some context for how to understand DNLR and AL. Then I will discuss the relative product operation and how it leads to existential quantification. In the last part of this chapter I will explain how this is related to the view I have been developing about the referents of unsaturated relations.

In this chapter, I will employ the terms 'relative' and 'relation'. I take 'relative' to mean a relative term. In this sense, there is no substantive difference between a 'relative' and a representational relation (i.e. a relational term in a formal system), as we discussed in the previous chapter. But I do intend to maintain the distinction between saturated and unsaturated (representational) relations. As we discussed in the last chapter, the key question then becomes, what is the referent of an unsaturated relation?

Description for a Notation of the Logic of Relatives

A typical presentation of a logical system begins with syntax. And Peirce's work in logic, considered on a large scale, is no different. His first major logical paper is published in 1870, entitled "Description of a Notation of the Logic of Relatives, Resulting from an Amplification of the Conceptions of Boole's Calculus of Logic,"[91] (DNLR), and it is primarily concerned with a proper syntax for a logic of relatives,[92] which will combine insights from Boole and De Morgan.[93] That being said, DNLR includes material beyond syntax, as we will see. If facts are primitive in nature, and if we are aiming to construct a logical system to represent those facts, we must consider the primitive logical elements, their means of combination into well-formed aggregates—i.e. well-formed formulae, well-formed complex graphs, etc.—and the operations upon them.

In a typical first-order quantificational system, the primitive elements are usually understood to be the objects in the domain. In the first-order context, n-adic predicates and relations are defined *as being* sets of ordered n-tuples of these objects. The relations are constituted by these objects—that is to say, they (the n-adic predicates or relations) are syntactically more complex than these objects. Furthermore, quantifiers quantify over the domain of these objects.

During the years 1870-1885, Peirce was working towards a theory of first-order quantification, which achieves full expression in his 1885 paper "On the Algebra of

[91] He had published works in formal logic a few years earlier (1866, 1867, 1868) but they were nowhere near as ambitious.

[92] See, for example Burch, PARR, 207: "[commentators on DNLR] have undervalued the fact that at its outset Peirce erects a very clear distinction between (what logicians currently call) *syntax* and (what logicians currently call) *semantics*…the work is an endeavor to create a *notation*, that is to say, a *syntactical* structure. In particular the syntax is to be a syntax for the logic of relatives". See also Burch, PARR, 208-209.

[93] Peirce did not see his 1870 paper as marking a break with Boole or De Morgan; rather he saw it "as an 'enriched', 'beautified' and 'completed' *generalization* of it (4.5)" (Tiercelin, 128).

Logic: A Contribution to the Philosophy of Notation" (ALPT).[94] Daniel D. Merrill observes that, "the relative operations of 1870, and even 1883," (which we shall discuss in this chapter), "disappeared entirely by 1885" (Merrill, 166). Based on this, Merrill argues that, "Peirce clearly believed that the QLR [Quantificational Logic of Relations] was superior to the ALR [Algebraic Logic of Relations]" (ibid). However, after 1885, when Schröder was systematizing and expanding upon Peirce's work on quantification (culminating in Schröder's *Vorlesungen uber die Algebra der Logik*, 1890, 1895), Peirce himself moved away from quantificational logic. This move is peculiar to many logicians studying Peirce's work. Why pivot right after making the discovery of quantification, which is largely seen as the breakthrough that brought on modern logic? The mainstream view of Peirce's logic after 1885 is perhaps best embodied in the dismissive words of Quine:

"The logical enterprise to which Peirce devoted most attention and attached most importance in later years was his evolving system of 'entitative' or 'existential' graphs (1897, 1903) [Quine, whether intentionally or not, lumps these together]. It is a complex and cumbersome apparatus, and seems anachronistic at so late a date, when Peano's transparent and efficient logical notation was already inspiring Whitehead and Russell to embark on *Principia Mathematica* and Peirce's equally efficient notation of 'Σ' and 'Π' had long since inspired Schröder" (Quine, PL, 264).

Hilary Putnam, in his essay "Peirce the Logician" reflects autobiographically on his own (surprising) discovery of Peirce's role in the history of logic, recounting his own previous

[94] The contributions of Peirce's student at John's Hopkins, O.H. Mitchell should also not be neglected. Peirce credits Mitchell with effectively introducing the quantifier into logic with Mitchell's 1883 paper, published in "Johns Hopkins Studies in Logic", a publication of which Peirce was the editor. Peirce was also a contributor to this particular publication, with his now well-known paper "Note B" (published in CP 3.328-358, which takes Mitchell's insight distinguishing the quantifier and the "Boolean part" of an expression, and incorporates it into his (Peirce's) own system. Peirce notes in 1885, "All attempts to introduce this distinction [of some and all] into the Boolian algebra were more or less complete failures until Mr. Mitchell showed how it was to be effected" (CP 3.393). However, Mitchell's versions of the quantifiers were limited in power. They only dealt with binary connectives, which Peirce then generalized to any n-adic relation (Brady, 187, 189); Mitchell's quantifiers also "enjoy no flexibility of scope" (Quine, 262).

bias in neglecting Peirce's work, having been brought up in a logical tradition where this neglect was rather systematic.[95]

Peirce, no doubt, did see the advantages of quantificational logic over algebraic logic—in fact as late as 1897 he praises the former in the very essay where he has introduced the Entitative Graphs:

"Besides the algebra just described, I have invented another which seems to me much more valuable. It expresses with the utmost facility everything which can be expressed by a graph, and frequently much more clearly than the unabridged graphs above" (CP 3.499).[96]

But the view of Peirce's logical work post-1885 as mere anachronistic grumblings is far from the mark. He would not have abandoned a perfectly good logical system without reason. Furthermore, he went to great lengths to articulate these reasons, and went so far as to call the Existential Graphs his *chef d'ouvre* (CP 4.346-7). Part of the purpose of this chapter is to demonstrate why Peirce found QLR wanting.

But before understanding Peirce's reasons for moving away from quantificational logic (post 1885), I believe we must first consider his work that led to the discovery of quantification. This investigation begins with DNLR (1870). In that work, many of the

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[95] John Sowa notes, "The primary reason for the focus on Frege at the expense of Peirce was not their logic, but their philosophy. Frege addressed narrow questions that could be expressed in logic; instead of broadening the scope of logic, many of his followers dismissed, attacked or ridiculed attempts to address broader issues" (Sowa, Peirce's Contribution to 21st cent). See also Shields, 50 on the neglect of Peirce and the two traditions in formal logic. Benjamin Hawkins' essay "Peirce and Russell: The History of a Neglected Controversy" also provides historical and philosophical details on Peirce's neglect in the history of logic.

[96] One major blow to Peirce's earlier algebraic system (which perhaps contributed to the mainstream favoring of quantificational logic) was Korselt's counterexample, cited by Brady, which "[shows] that this elimination of quantifiers in favor of relative product and the other operations of the algebra of relations does not always work" (Brady, 190). Korselt discovered that "the statement that the domain has at most four elements" is a first-order statement that could not be expressed in "the condensed form of the calculus of relatives (i.e. the fragment restricted to relative product, converse and the Boolean operations" (ibid). As Brady explains, this discovery became a springboard for Löwenheim and Skolem to prove their celebrated theorems. But Brady elsewhere cites Löwenheim as saying that he worked "entirely within Schröder's system (ibid, 191), which was built upon Peirce's work in 1870-1885. A contemporary algebraic system like Burch's PAL does not have this limitation, however—nor does EG.

key ideas from later works, including quantification, are germinal.[97] Robert Burch even goes so far as to claim that "by 1870 Peirce had invented and elaborated a logic of relations at least as powerful in expressiveness as the standard contemporary logic of today" (Burch, PARR, 232). Burch also argues that, "in Peirce's concept of the application of a relation [i.e. the relative product operation], the embryo of his Reduction Thesis is contained" (Burch, PARR, 206). By 1885, Relative Product will evolve into existential quantification, in a manner that I will attempt to make clear. As we noted above, Peirce abandons quantificational logic in his later work, *but* in his graphical logic (EG), he retains a concept strikingly similar to the relative product operation—in the form of his rules of transformation for lines of identity. As I have noted, I believe this fact is significant, and provides a clue as to why graphical logic better displays Peirce's logical discoveries.

Peirce begins DNLR by crediting De Morgan for pioneering the study of relative terms, but claims that his algebraic notation for relatives is wanting. Boole's logical algebra, on the other hand, is advantageous in that it welcomes convenient manipulation. Thus, Peirce asks,

"whether it [Boole's logical algebra] cannot be extended over the whole realm of formal logic, instead of being restricted to that simplest and least useful part of the subject, the logic of absolute terms, which, when he [Boole] wrote, was the only formal logic known. The object of this paper is to show that an affirmative answer can be given to this question" (CP 3.45).

While I do not intend to undertake a full explanation and analysis of DNLR (or the other papers that I will consider in this chapter), it will be useful to introduce a few symbols and definitions that arise in DNLR, as well as in the other logical papers we will consider.

Symbols and Definitions in DNLR and AL

[97] See, for example, Quine: "what is especially interesting in this monograph [DNLR] is Peirce's atomistic approach to relations as sums or aggregates of what he called simple relatives—what we have think of as ordered pairs (or triples or quadruples as the case may be). It was this feature that was destined to evolve into quantification through Peirce's writings of ten to thirteen years later" (Quine, PL, 259).

First: Peirce employs the sign '—<' to indicate the relation of "inclusion in" or "being as small as". As we will discuss, there is ambiguity here. Peirce claims to use the symbol '—<' in place of '≤', for reasons of convenience but also for philosophical reasons. The latter symbol "seems to represent the relation it expresses as being compounded of two others which in reality are complications of this" (CP 3.48). Elsewhere, Peirce similarly defends the idea that '—<' is more simple/primitive than the relation of equality.[98] He also argues (as is made explicit in Burch's 1991 proof of the Reduction Thesis) that the later-developed relation of teridentity is more simple/primitive than equality. Teridentity is expressed in contemporary quantificational logic as 'x=y=z' (Burch, SEP), but in EG as the meeting point of three branches of a ligature. As Peirce says, "it is identity *and* identity, but this 'and' is a distinct concept, and is precisely that of teridentity" (CP 4.561). But we are not yet in a place to address those ideas. Yet we will note that, in DNLR, Peirce defines equality in terms of inclusion: "to say that *x*=*y* is to say that *x*—< *y* and *y*—< *x*" (CP 3.48).

As we noted above, there is equivocation here between elements of a set and singleton sets composed of those members (i.e. between the relations expressed by the contemporary symbols '∈' and '⊆').[99] In DNLR, Peirce explains the meaning of the symbol '<' in the logical context: "The sign 'less than' is to be so taken that 'f<m' means every Frenchman is a man, but there are men besides Frenchmen" (CP 3.66). In this example, Peirce clearly seems to have in mind the concept of a subset (rather than the concept of an element), and we can thus take the '<' symbol here as indicating *proper* subsethood—i.e. '⊂'. In contrast, "f—< m…means 'every Frenchman is a man,' without saying whether there are any other men or not" (ibid), which is to say that Peirce is taking the '—<' symbol to mean general subsethood—i.e. '⊆'. Obviously some of the confusion here also comes from the fact that here Peirce is implicitly incorporating universal quantification into this symbol without acknowledging it. That problem is closely related to the lack of distinction between elementhood and subsethood.[100]

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[98] E.g. NEM 3.821; See Anellis, 291-294.

[99] This point is also made by Iliff, 197; Quine, "Peano as Logician", 268; Dipert, 59

[100] However, in Peirce's later work (at least by 1903), as we will see, he is quite clear on the distinction, and criticizes Russell's treatment of this topic. As Hawkins explains,

In Peirce's account of equality, there is also implicit but unrecognized universal quantification, which causes similar problems. Peirce notes that he follows Boole "in taking the sign of equality to signify identity" (CP 3.66).

An additional notion to keep in mind in DNLR is that Peirce follows Boole in being guided by "certain principles of analogy" (CP 3.46) with ordinary algebra and arithmetic. Quine, perhaps fairly, disparages the "procrustean forcing of analogies with themes and theorems of classical mathematics" (Quine, PL, 259). In regard to this tendency, Peirce later criticized himself. Van Evra quotes Peirce in 1879 (nine years after the publication of DNLR), admitting that, "there has been on the part of Boole and also of myself a straining after analogies of this kind with a neglect of the differences between the two algebras…" (MS 575) (Van Evra, 156). Yet Van Evra also notes, in regard to mathematical analogy, that Peirce went a great deal beyond Boole, who limited analogy "by rigidly confining it to the area of elementary algebraic operations and their inverses" (ibid, 149). Peirce, on the other hand, "[employs] devices such as logarithms, derivatives, complex numbers, quaternions, transcendental equations, non-Euclidean geometry, and more, all within a strictly logical context and without regard to the limits of interpretability" (ibid, 149-150). Obviously, this makes for exhausting (and sometimes unproductive) reading, but DNLR is certainly not lacking in ambition. Van Evra, humorously, comments that, "Peirce's full work on the logic of relatives is an extension of Boole's logic in roughly the same way that space travel is an extension of walking" (ibid, 148). For the sake of brevity (as well as my own sanity and that of my readers!), I will not discuss any of Peirce's more exotic mathematical analogies. But it is worth noting that Peirce, following Boole, is partly concerned with using the mathematical properties of these concepts in a way that can parallel the logical properties.[101] In the context of relative product, for example, Peirce is thinking chiefly in terms of matrix

depending on which of two distinct concepts of a set one is using (we will explain these at the end of this chapter), singleton sets are either distinguished or not (Hawkins, 133). On this topic, an interesting historical detail is that Peirce "appears to have discovered [the power-set construction] independently of Cantor" (Guinness, 30). Guinness cites a number of references on this topic.

[101] See CP 3.61.1 for his motives in using these mathematical signs

multiplication, as Iliff explains in detail. I believe that this idea helps to explain Peirce's detour through an extensional understanding of relations.[102]

Logical Terms

Now let us consider the "logical terms" Peirce uses. In DNLR, he distinguishes

(1) Absolute terms
(2) Simple relative terms
(3) Conjugative terms (CP 3.63).

Burch and Merrill consider these through the lens of Peirce's mature view (which also aligns with our current logical notions). As such, they claim that we may think of these as one-place, two-place, and three-place *predicates*, respectively (Merrill, 159; Burch, PARR, 209). However, as I will discuss, (and as Merrill and Burch no doubt realize) Peirce is not yet thinking this way in his early logical work (DNLR and AL). It is only through a long process of development that he comes to see the fundamental logical terms as one-place, two-place, and three-place *predicates*, and the crucial step in that insight is realizing that the copula—with its verbal, i.e. existential element—is a part of the predicate, and not a separate logical entity. As I will explain below, in DNLR and AL, he views Absolute terms, Simple relative terms, and Conjugative terms as having addicities of 0, 1, and 2 respectively—i.e. *one less* than we would normally think. To give a brief example, in DNLR, he would view 'man' as an absolute term with addicity of 0, instead of viewing it as the predicate '__is a man' with addicity 1. Similarly, he would view 'father of__' as a simple relative term with addicity 1, instead of viewing it as the predicate '__is a father of__' with addicity 2 (as he would later).

The exegesis of Peirce's logical papers is also complicated by the fact that, by 1880 in AL, he has changed his terminology and put old terms to new uses. There he distinguishes "non-relative" terms and the operations upon them—e.g. "non-relative addition and multiplication" (CP 3.499)—from relative terms and relative operations. These "non-relative" terms—also called "terms of singular reference" (CP 3.219)—are effectively the "absolute terms" of 1870—that is, one-place predicates.

[102] Even in ordinary arithmetical multiplication, Peirce speaks of multiplication involving "one factor's being taken relatively to the other (as we write 3×2 for a triplet *of* pairs…)" (CP 3.61). Obviously we can see an analogy here with an extensional view of relations.

But, so far, his presentation roughly parallels that of a typical contemporary system, in the sense that monadic predicate logic is introduced before the logic of relations with higher addicities and identity (his presentation takes the same approach in ALPT, 1885). "Simple relatives" from 1870 are, in 1880, called "dual relatives" (CP 3.220), and relatives of greater addicities are called "plural relatives" (CP 3.219).[103] Thus, in AL (1880), we have:

(1) Non-relative terms / Terms of singular reference
(2) Dual relatives
(3) Plural relatives

As I mentioned at the beginning of this chapter, there is some debate as to what precisely Peirce means by 'relatives' and by 'relations'. By a 'relative' Peirce just means a 'relative term' that has blanks in it (CP 4.354). But of course this raises the question as to what is the referent of such a term. In the previous chapter we clarified that a 'relative rheme' is an unsaturated relation in a representational system, which includes the copula/verb—e.g. the *R* that is abstracted from *Rab* (which is the saturated relation). This is Peirce's mature view. But as he is building logical systems, he is wrestling with the question as to what is the referent of relative terms. Tracing his answer to that question is difficult because what he means by a relative term changes.

He begins by considering relative terms as predicate nouns that do not include the copula/verb, with addicity one less than we would expect. On this first view, the referent of a relative term is conceived in one way. But then, along with Mitchell, he discovers quantification, which changes how he conceives of relative terms (they contain the copula, and are verb-like). This change in his conception of relative terms is wrapped up with a change in how he conceives of the referents of those relative terms. With the discovery of quantification, Peirce is closest to the extensional view of reference. In fact, in my own reading of Peirce's work and of the secondary literature, it seems to me that it

[103] It is, perhaps, worth noting that in AL, 1880 Peirce "confusingly…now puts his old term 'simple relative' to an opposite use: the complement of an individual relative. The pair of Abraham and Isaac is an individual relative; the sum or class of all pairs except that one he now calls a simple relative" (Quine, PL, 260). I will not be making use of this concept, though.

was his consideration of the extensional view of reference that partly led to the discovery of quantification (this point is emphasized by Quine, Merrill and Iliff in particular).

But my purpose in this chapter is to show that this is not the whole story. The ancestor of quantification is relative product, but the thing that quantification evolved into (for Peirce) actually looks more like relative product than quantification. I argue that this is because he hit on something fundamental with relative product—namely the idea that determination comes through combination of concepts. And if that is *really* how we acquire new information and new knowledge, then this idea should be reflected in logic, at least in a logical system for purposes of discovery (as it is explicitly in EG, where only two loose ends can be joined). Furthermore, there is a very important metalogical result that comes when we incorporate that idea into a logical system. The result is that the Reduction Thesis holds.[104]

Later in this chapter we will consider possible interpretations of the referents of relations and I will offer my own position, which incorporates a semiotic component, and which is, I believe, a novel account that has not been considered in the secondary literature. The account I will offer is related to this restriction on conceptual combination.

Peirce came to view relative product as privileged because, in it, indeterminate relational concepts (rhemes) mutually determine each other, which he believed was generally the way that we gain new information through our inferences. But this fact only became clear after Peirce saw that relative product accomplished existential quantification. This idea was tied to the insight that the verb (or copula) is included in relational predicates in an essential way—an insight Peirce did not see in DNLR or AL, when we was first making use of the relative product operation. He (likely) was motivated to take up relative product as a relevant operation for combining concepts simply by considering the way that we combine relational expressions in ordinary language. But he *discovered* that there is important *structure* within this sort of operation,

[104] As mentioned earlier, the Reduction Thesis results from *two* important restrictions on the transformation of lines of identity. One is that only two loose ends can bind (which is the focus in this chapter). The other—equally important—is that lines of identity must branch from points of teridentity—i.e. must branch in threes. This is also a restriction that arises out of logical considerations—particularly considerations about the identity relation.

which implicitly accomplishes existential quantification. In considering how this was a discovery, we must consider how Peirce thought of the three sorts of terms that can be combined through relative product.

First, we must note that, in the 1870-1880 essays, all three of the logical terms outlined above are conceived as *nouns*—i.e. "nominalized forms of relation terms…" (Merrill, 161)—and not as verbs or predicates, as Peirce will conceive of them later.[105] As Burch notes, this tendency in the early work to regard relatives as nouns rather than verbs or verb phrases (which is obviously the typical way in contemporary first-order logic), "might perhaps be explained by the fact that in his early logical works, Peirce is still tied to the idea that all propositions are basically subject-predicate in form" (Burch, PARR, 211). In logic now, we would represent a 1-place predicate like 'man' by a predicate symbol that implicitly includes the copula, such as *M*, to which an individual constant or quantified variable can be attached. Similarly, a 2-place relation like lover would be represented by a predicate/relation symbol, such as *L*, to which *two* individual constants or quantified variables can be attached.

In DNLR and AL, Peirce is not yet thinking this way. He is bound to class logic. As such, he thinks of the denotation/extension of an "absolute term"—i.e. a non-relative term—like man, as simply the collection (or class) of men.[106]

In the case of relative terms, things become more complex. In later logical work (1883 and after), Peirce will speak of relatives as predicates, which include the copula. For these he introduces the term "rheme", or "rhema" (CP 4.354). As we have already discussed, a relative rheme "differs from a relative *term* only in retaining the 'copula,' or signal of assertion" (CP 3.420 (1892). The term '__is the father of__' is a rheme, which

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[105] Recall that, by Peirce's own account, it was not until roughly 1892 that he had come to a "near approach toward a provisionally final result"—that result being "that indecomposable *predicates* [my emphasis] are of three classes": i.e. those that apply to one subject, two subjects, and three subjects (EP 2.425).

[106] Dipert notes that Peirce translated Cantor's *Menge* as collection, not set (53). Guinness also notes that collection is umbrella term to cover sets, classes, assemblies (Guinness, 29). I will use these terms interchangeably. However, as we will see later in this chapter, Peirce had two distinct conceptions of a class defined by some property ϕ, which he called a *sam* and a *gath*. After we introduce those terms, I will follow Peirce's usage, but in the meantime, I draw no distinction between the terms 'class', 'collection', and 'set'.

has an addicity of 2, since there are two blanks to be filled. In the 1870 notation, however, the term 'father of' is viewed as a 'simple relative' (in 1880 'dual relative') with addicity of 1, rather than addicity of 2 (in both earlier essays).

In DNLR, Peirce explains that each of his three different types of logical terms constitutes a class. A simple relative, like 'father of', has a "logical form [that] involves the conception of relation, and which require[s] the addition of another term to complete the denotation" (CP 3.63). Keeping with the 'father of' example, this other term that is required is the absolute term that saturates the second place of the relation. In Peirce's view (circa 1870), 'father of' does not denote anything. But once it is combined (through an operation) with an additional absolute term, such as 'Malia', we get a new relative—i.e. 'father of Malia'. For Peirce (circa 1870), this new term (of addicity 0) names a class—the class of all things that are a 'father of Malia'. The class has only one member, namely Barack.

This passage above is telling, though, since it brings to light a key idea in the background of Peirce's thinking—namely that the relative term cannot properly denote anything until its blank(s) is (/are) filled in by absolute terms. This idea gets retained once Peirce conceives of relatives as rhemata, which contain the copula. In that case, similarly, '__is a man', or '__is the father of__', or '__gives__to__' are not strictly denotative. They do not pick out particular objects. Now, with an extensional interpretation in view, we would say that they pick out a set of ordered n-tuples. But even after Peirce makes the shift to viewing relative terms as rhemata, his interpretation of the quantifiers is still different from our more common contemporary view. As we discussed in the last chapter, he sees quantifiers as essentially asserting a conditional claim.

"Thus, a proposition whose subject is distributively universal…such as 'Any man will die,' allows the interpreter, after collateral observation has disclosed what single universe is meant, to take any individual of that universe as the Object of the proposition, giving, in the example above, the equivalent 'If you take any individual you please of the universe of existent things, and if that individual is a man, it will die'" (EP 2.408) (1907).

The key point in discussing the previous idea of addicity *n*-1 and then later addicity *n* has to do with what is happening with determination. When blanks are removed, we are performing analysis. What is left is not a relative of addicity *n*-1, but a relative of addicity *n*, which can be filled with logical subjects. Obviously, when a noun or proper name fills in one of those blanks, the proposition gains determination. It becomes "saturated". We symbolize this in quantificational logic by attaching individual constants to a predicate term. But when blanks are combined through the relative product operation, greater determination is also achieved through existential quantification. The new information is that the object that some index (coefficient) x stands for is the same object that some other index (coefficient) y stands for.

Thus, from the get-go, Peirce has combination in mind—that is, (i) combination of relatives and non-relatives (as in the example above), and (ii) combination of relatives with other relatives. This brings us to the relative product operation.

Operations

In presenting a logical syntax, in addition to the primitive elements in the system, we must consider the operations that are permissible. Peirce's algebraic logic in DNLR (and in AL) "contains no individual variables and no quantifiers" (Merrill, 159). Rather, it contains, beyond the primitive terms and connectives that we have outlined above, the following operations on relations (briefly and informally stated)[107]:

(1) Converse: If the servant relation indicates (again, we are still speaking vaguely about denotation of relations) that Bob is a servant of Alice, then the Converse of this relation indicates that Alice is a *master* of Bob.
(2) Relative Product: If we have two relations—'lover of' and 'servant of'—then their relative product is 'lover of a servant of'. To use a "saturated" example: if Tom is a lover of Dick, and Dick is a servant of Harry, then Tom is a lover of a servant of Harry.
(3) Involution[108]: "the involution of lover and servant (l^s) is *lover of every servant*" (Merrill, 159)
(4) Backwards involution: "the backwards involution of lover and servant (l_s) is *lover of only servants*" (ibid).

Peirce also introduces the operation of relative addition, but for our purposes we will not address this. Furthermore, these four operations are not logically independent of one another. As Merrill notes, "the two involutions can be defined using complementation

[107] These are explained well by Merrill, 159
[108] Also called 'Exponential'—e.g. Brady, 176.

and relative product" (ibid).[109] But Peirce saw relative product as special, considering it "the most fundamental of all modes of conceptual combination" (Burch, APRT, 1-2).

Peirce notes,

"I shall adopt for the conception of multiplication *the application of a relation*, in such a way that, for example, *lw* shall denote whatever is lover of a woman" (3.68). Peirce goes on to say that, "this notation is the same as that used by Mr. De Morgan, although he appears not to have had multiplication in mind" (ibid).[110]

As Brady argues, "Relative product is the cornerstone of Peirce's algebra, and his first tool for representing the quantifiers…" (Brady, 177). How so?

Let us follow Peirce in, first, taking the case of application of a relative (a relative term of addicity 1 or higher) to a class (a non-relative term).[111] Peirce gives the example where *lw* denotes "whatever is a lover of a woman" (CP 3.68). Here we have the relative term lover (conceived as 'lover of__', with an addicity of 1) and the non-relative term 'woman' with an addicity of zero. It is easy to imagine the relation being "saturated"—in Peirce's terms, "completed" (CP 3.466)—which results in a new determinate predicate 'lover of a woman' (Brady, 177 explains this in a similar way). We might then view *lw* as a sort of new non-relative term, "standing for the class of all lovers of women" (Martin, 26). The generalized cases where a relation is applied to a relation (or to a relation *and* a class term) would be similar. For example, if *l* denotes the 'lover of' relation, and *s*

[109] Brady also explains this idea: "Drawing upon De Morgan's findings, Peirce observes that relative product and involution are connected via complementation (Brady, 178-9).
[110] The idea of application of a relation to a relation is already germinal in De Morgan, who employs the term relative product (see DNLR 3.68; Brady, 178; De Morgan, 241); Herzberger also points out: "Originally De Morgan's, this operation [relative product] was taken over and extended by Peirce to apply to relations of any degree" (Herzberger, 44). It is also, perhaps, instructive to consider why Peirce chooses multiplication to be the parallel algebraic concept for application of a relation. He is thinking primarily of matrix multiplication (Iliff, 198d and 200b)—and is concerned with ensuring that the operation is associative (e.g. CP 3.53). The result of a multiplication (with numbers as arguments) would be a matrix. This idea of multiplication of matrices results in him seeing relations as extensional. In this respect, again, Charles is influenced by his father, Benjamin Peirce (Van Evra, 152). The operations in DNLR parallel those of algebra (Burch, PARR, 208).
[111] As Martin notes, this "notion of the application of a relation to a class [rather than to another relation] here is something new" (Martin, 25) different from what De Morgan was doing. Brady also notes that Peirce, in 1867, was "experimenting…in mixing Boole's algebra of classes with binary relations" (Brady, 176). This leads to his exponential notation in DNLR (which we will not discuss).

denotes the ‘servant of’ relation, and *w* denotes the non-relative term ‘woman’, then we might think that *ls* stands for the *class of* ‘lovers of servants’, or that *lsw* stands for the class of ‘lovers of servants of women’. By the same token, we may imagine *l* and *s* as standing for classes—the class of lovers and the class of servants

But there is a major problem with this sort of view. As Merrill points out:

> “More basically, *l* and *s* cannot stand solely for classes of individuals, since then the results of using the relative operations would not make sense. For instance, *ls* would have to stand for the class of lovers of servants; yet this is not a function of the class of lovers and the class of servants. It must, instead, be explained in terms of the *relations* expressed by *l* and *s*” (Merrill, 162).[112]

If we simply have two classes—lovers and servants—each of which is conceived simply as the extension of a non-relative term, then there is no way to express the sort of combination that is going on with the relative product operation, since the relative product operation takes into account the *order* of the arguments in the relation. An example where the relations are “saturated” should hopefully make this clearer.

Suppose Alice is a lover of Bob, and Bob is a servant of Tom. In this case, so long as we are considering lovers and servants as *classes* of non-relative terms, Alice would be a member of the class of lovers, and Bob would be a member of the class of servants. Or, similarly, Alice would be a member of the class of ‘lovers of Bob’, and Bob would be a member of the class of ‘servants of Tom’. But neither of these gets us what we need. What we need in order to get the relative product—that Alice is a lover of a servant of Tom—is a specification that Bob is *loved by* Alice and a *servant of* Tom. This cannot be achieved by considering relatives as non-relative classes, because in that case, any notion of order is obliterated.

Here Peirce stumbles a bit, but he attempts to address the concern by doing two things. First, he restricts relative product to only operate on relations (not 0-addicity non-relative terms), and second he introduces a notation (the “comma notation”) to increase addicity by one.[113] To clarify this, Peirce explains the equivalence between the “absolute term” ‘man’ and the “relative term” ‘man that is___’. In a similar way, he notes, “not

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[112] Ochs makes a similar observation here in his discussion of Brunning (Brunning, “A Brief Account of Peirce’s Development of the Algebra of Relations”, 1983). See Ochs, 224.

[113] See Brady, 178 for some explanation

only may any absolute term be thus regarded as a relative term, but any relative term may in the same way be regarded as a relative with one correlate more" (CP 3.73). The important difference between what he is doing here vs. what he is doing in his later logical work is that, later, he incorporates the copula into the relative term, thus fully making it a verb or predicate (in his terminology, a rhema). He does not do that with this "comma operation".

For this reason, the comma operation is roundabout and largely unhelpful, but it shows that he is wrestling with how to give an account of extension of newly combined relatives. As we have suggested, his later explanations are much clearer, viewing the "non-relative terms" from 1880 ("absolute terms" from 1870) as monadic relatives (that include the copula) from the beginning. For example, in 1897 he clarifies that, in the majority of languages,

> "Distinctive common nouns either do not exist or are exceptional formations…If a language has a verb meaning 'is a man', a noun 'man' becomes a superfluity…The best treatment of the logic of relatives, as I contend, will dispense altogether with class names and only use such verbs" (CP 3.459).

With this in mind, let us return to the example above, but view lovers and servants as dyadic relations with addicity of 2 (rather than 1). The relative product of these two dyadic relations—l and s—is expressed symbolically as $ls(i,j)$. But what this operation asserts is that there is something, k, such that i is a lover of k and k is a servant of j. That is to say that $ls(i,j)$ means $\exists k\ (l(i,k) \wedge s(k,j))$. And thus we see that "contained within [the relative product operation], there is an implicit existential quantifier" (Brady, 178). As Peirce claims in 1883, while characterizing relative product (with the example of the lover and benefactor relations), this is called "particular combination, because it implies the *existence* of something *loved by* its relate and a *benefactor of* its correlate" (CP 3.332). For this reason, relative product is historically interesting, since it is the evolutionary linchpin in giving an historical account of how quantification emerged in modern logic.[114] And, indeed, some scholars (e.g. Merrill) seem to see its significance as ending there. Like an extinct evolutionary ancestor, it is strange yet familiar and, no

[114] Though Frege discovered quantification independently in 1879.

doubt, interesting, but is far less effective than the thing that it evolved into—namely existential quantification.

Some passages from Peirce also suggest that he saw the operation (and indeed the whole algebraic system) in a similar light. For example, we read as late as 1897, in the very paper where he presents the Entitative Graphs (the precursor to the Existential Graphs),

"Besides the algebra just described [his algebraic system], I have invented another [the quantificational system] which seems to me much more valuable. It expresses with the utmost facility everything which can be expressed by a graph, and frequently much more clearly than the unabridged graphs described above" (CP 3.499).

This passage is undoubtedly a point in favor of Merrill's view (that Peirce preferred QLR), and a strike against mine. But I believe this statement can be accounted for if we recall the two different views of the task of logic, which Peirce frequently emphasized in his discussion of EG. In his presentation of that system (as we mentioned in the introduction), he notes, "That purpose and end [of a system of logical symbols] is simply and solely the investigation of the theory of logic, and not at all the construction of a calculus to aid the drawing of inferences. These two purposes are incompatible…" (CP 4.373).[115] Quantificational logic may indeed be preferable for many (even most!) applications in aiding inference. But Peirce came to be convinced that EG best demonstrated the fundamental elements of reasoning, and the fundamental modes of conceptual combination (expressed in that system by the rules of transformation).

That idea of conceptual combination begins with relative product. This operation takes two relatives (relations) whose correlates (arguments) are indeterminate to some degree. In their combination, however, the existence of a third object is determined. We have new information.

Could one push back here? Do we really have new information? In my own study of relative product, I have stumbled a good deal trying to understand exactly what new information the relative product operation achieves. So I think it is worth giving some further explanation. Suppose we want to take the relative product of two dyadic relations—the lover relation and the benefactor relation. Note that, here we are

[115] Also see Shin, chapter 6

considering these as 2-place predicates (dyadic rhemes) that include the copula. We might represent the '__is the lover of__' relation by the symbolic expression l_{ij} (or the modern equivalent L*xy*). But this raises a question whether this relation is already existentially quantified or not. In most logical systems (at least those with which I am familiar), the formula L*xy* would not be well formed. It would either have to be saturated with individual constants—and thus become, for example, *Lab*; or it would have to *already* be existentially quantified. But if this formula is already existentially quantified, then it seems that relative product does not "achieve" anything at all.

To see what it achieves, let us imagine that the 'lover of' relation and 'benefactor of' relation are both existentially quantified. Thus we have (i) $\exists w \exists x\ Lwx$ and (ii) $\exists y \exists z\ Byz$. In this example, the relative product of these two relations asserts that there is at least one case in which $x=y$. Relative product adds more information in that it asserts the identity of the object in the second slot of the first relation and the first slot of the second relation. Thus, it gives greater determination and new information. It is this feature whereby *determination* is accomplished through conceptual combination that is deeply central to Peirce's overall logical and philosophical project.

Quantification and the Extensional view of the Referent of a Relative

At this stage, we have been mainly speaking extensionally about the referents of unsaturated relations (that is, speaking of their referents as sets of ordered n-tuples). As I have suggested, I believe that in order to understand Peirce, we must take a detour through the extensional view, (as Peirce himself does). But I believe Peirce's mature view about the referents of relations is more nuanced than that, as I will attempt to show. However, the secondary literature is mixed. In the following section, I will attempt to explain how Peirce's considerations about the referents of relatives led to the discovery of quantification. An important step in this process was viewing the referents of relatives extensionally. But, as I have noted, I do not believe this is Peirce's final view. So after explaining the extensional view, I will try to explain my own position.

For similar reasons to those we have outlined above, Merrill interprets Peirce as having an extensional view of relations. And there is good reason to think that Peirce has this extensional view in mind. Later in DNLR, Peirce introduces the term "elementary relatives" which seem to be effectively individual n-tuples (see CP 3.121). Then he claims that, "every relative may be conceived of as a logical sum [set-theoretic union] of elementary relatives…" (ibid). Obviously, this sounds extremely close to our contemporary extensional view of relations. Martin gives a helpful commentary on this section, noting that, "now, for the first time in Peirce's published logic papers, ordered pairs, triples, and so on, are explicitly admitted" (Martin, 19). But, as Martin points out, this is an "enormous ontological extension" of the system (ibid). After all, ordered n-tuples are different logical entities than the primitive terms that were introduced at the beginning of the paper.

In AL, ten years later, Peirce comes a step closer to an extensional view of relations. There, he considers an infinite block of dual relatives (i.e. a two-dimensional matrix), in the following way (CP 3.220).[116]

A:A	A:B	A:C	A:D	A:E	etc.
B:A	B:B	B:C	B:D	B:E	etc.
C:A	C:B	C:C	C:D	C:E	etc.
D:A	D:B	D:C	D:D	D:E	etc.
E:A	E:B	E:C	E:D	E:E	etc.
etc.	etc.	etc.	etc.	etc.	etc.

This block does not represent the extension of any specific relation—like 'lover of'; rather it represents the block of all possible ordered pairs formable from the objects in the domain (the relevant universe). For example, if the domain is a domain of people, then this is the set of all possible ordered pairs of people. Each of the pairs—e.g. A:B—are "individual relatives" (ibid). Peirce claims that "the logical sum [i.e. union] of all the relatives in this infinite block will be the relative universe, ∞, where

x —< ∞

whatever dual relative x may be" (CP 3.220). So x is to be understood as one of these "individual relatives"—i.e. one particular n-tuple. And by "x —< ∞", Peirce means to say that this individual is a member of the infinite block. Here recall our earlier discussion about the symbol '—<'.

[116] He also speaks of generalizing to higher dimensions—"triple individual relatives may be arranged in a cube, and so forth" (CP 3.220)

The symbol '∞' has nearly the same meaning as the symbol '1' that Peirce uses elsewhere—it represents "unity", or the universal set, and is contrasted with '0,' which represents the empty set (CP 3.61). The only difference between '1' and '∞' is that '∞' symbolizes a set of *relatives* (i.e. n-tuples), rather than a set of individual objects.[117]

After conceiving of the "relative universe" in this way in 1880, Peirce goes on in 'Note B' (1883) to effectively define a semantics of relations that ranges over it. The same matrix appears at the beginning of 'Note B' (CP 3.329).

Imagine that we take all the objects n in the logical universe (the logical universe is just the domain). If we are considering a 2-place relation, then, the "*relative* universe", ∞, will be the $n \times n$ matrix that includes all possible ordered pairs formable from this domain. (In the case of a 3-place relation, the relative universe would be the $n \times n \times n$ matrix, etc.). So the "relative universe" is not the extension of any particular relation, but is the extension of all possible relations.

Now we must contrast "individual relatives", which are particular (we may think of these as an individual n-tuples) with relatives in a more general way. Peirce notes that, "every relative, like every term of singular reference, is general; its definition describes a system in general terms; and, as general, it may be conceived…as a logical sum of simple relatives" (CP 3.220).[118] A general relative might be something like the 'father of' relation. If A is a father of B, then the *individual relative* pair A:B would be included in the general relation—i.e. in the logical sum of such individual relatives. I believe that it is significant that, in the passage above, Peirce speaks of the "definition" of the general relative, which "describes a system in general terms"—suggesting that he has *intension* in mind, as well as *extension*—but for the moment let us continue to just think of these general relatives (like 'father of' and 'lover of') extensionally.

[117] In a similar manner to our criticisms above, we should also note that there is some set-theoretic ambiguity as to the denotation of the symbol '0'. R.M. Martin, for example, cites Peirce's equation from CP 3.67: "[0]=0" which is "supposed to [indicate] that the cardinal number of the null class is zero." But, as Martin notes, that equation does not accomplish this since, "the '0' is being used ambiguously for the null class and for the number zero" (Martin, 15). See also Guinness, 29-30.
[118] Peirce actually describes two ways that we may conceive of a general relative; but for sake of simplicity, I am only describing one.

As Iliff notes, "In Note B [1883] Peirce used truth values [0 and 1], instead of field elements [i.e. objects], in the matrices…[which] are added and multiplied in the usual way, except that 1+1=1 (3.331/W4:454-455)" (Iliff, 199). Thus, if we want to consider a general relative, like 'lover of', we can imagine that the infinite block is filled in with 1s and 0s, depending on whether the relation is true of the pair. For example, the cell containing the individual relative C:D would be filled with a 1 just in case C is a lover of D. At that point, we can perform summation (and multiplication) on the block, or on rows and columns in the block.[119]

Peirce uses the 'Σ' symbol for 'some', "suggesting a sum" and the 'Π' symbol for 'all "suggesting a product" (CP 3.393), and he employs variable coefficients as indices, which can be "filled" by whatever n-tuple from the block we please. He claims, in 'Note B' (1883) that,

"any proposition whatever is equivalent to saying that some complexus of aggregates and products of such numerical coefficients is greater than zero [i.e. the empty set]. Thus

$\Sigma_i \Sigma_j l_{ij} > 0$

means that something is a lover of something; and

$\Pi_i \Sigma_j l_{ij} > 0$

means that everything is a lover of something" (CP 3.351).

Quine's discussion of 'Note B' is helpful:

"Here we may think of all individuals as arbitrarily numbered and we may think of l_{ij} as 1 or 0 according as person number i does or does not love person j. Then $\Sigma_j l_{ij}$ will be a number, a strictly arithmetical sum of ones and zeroes: a one for each person who person number i loves. So $\Sigma_j l_{ij}$ is how many people person number i loves. So $\Sigma_j l_{ij} > 0$ if and only if person number i loves some people. Quantification dawns" (Quine, PL, 261).

Similarly, in the case of multiplication, if there is even one cell (in the relevant row or column) that is filled with a 0, then this is sufficient to reduce the product of the

[119] Iliff emphasizes that Peirce's discovery here was possible because of his knowledge of matrix multiplication in linear algebra: "To an extent heretofore unrecognized, Peirce's discovery of the quantifiers was based on his expertise in sophisticated techniques of abstract algebra; it was not merely a simple generalization of the Boolean sum and product" (Iliff, 201). One can also see that in DNLR, Peirce was already thinking this way about matrix multiplication (e.g. CP 3.127).

whole row or column to 0. Thus $\Pi_j l_{ij} > 0$ iff every object in the domain loves *j*; or, in other words, $\Pi_j l_{ij} > 0$ iff every cell in the *j* column is filled with a 1.

This sort of operation is made possible by distinguishing between the two parts of a proposition—(i) the "pure Boolian expression referring to an individual" and (ii) the "quantifying part saying what individual this is" (CP 3.393). Peirce credits Mitchell for discovering the distinction: "All attempts to introduce this distinction [of *some* and *all*] into the Boolian algebra were more or less complete failures until Mr. Mitchell showed how it was to be effected" (ibid).[120]

With this notation, we can make sense of Peirce's 1883 equation for relative product—in that example using the relations of lover (*l*) and benefactor (*b*):

$(lb)_{ij} = \Sigma_x (l)_{ix} (b)_{xj}$ (CP 3.333)[121]

Similarly, in this notation, we would express the proposition 'something is a lover of something' by the formula $\Sigma_i \Sigma_j\, l_{ij} > 0$. Obviously this notation is suggestive of our contemporary notation: $\exists i\ \exists j\ (Lij)$, which is no accident. Peirce points out that we may drop the inequality symbol, leaving $\Sigma_i \Sigma_j l_{ij}$ to mean that 'something is a lover of something' (3.351).

Iliff notes that in ALPT (1885), "Peirce reinterpreted the letters given as subscripts as propositions instead of as truth values (3.392). In today's terms these subscripts can be regarded as atomic formulae. He also reinterpreted Σ and Π as denoting the logical notions of 'for some' and 'for all'" (Iliff, 200-201)—dropping the detour through arithmetic. After that point, it is just a notational change rather than a conceptual change to express existentially and universally quantified propositions in contemporary form.

The Referents of Unsaturated Relations—My Own View Explained

As we have noted earlier, "a relative is a *relative term*" (Merrill, 161). Then, of course, the important question is: what is the referent of a relative term? Merrill gives

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[120] See Brady, 180-184 for an account of Mitchell's symbolism, which is quite different from Peirce's.

[121] Since this formula is understood mathematically in the way I have described, it is allowed to contain free variables. Once Peirce reinterprets the subscripts as propositions rather than truth-values, variables must be bound by quantifiers.

“three candidates [which] have been explored in the secondary literature: a relative term is a term which stands for:

1) “the domain of a relation, such as the class of lovers”
2) “the result of compounding a relation and a class, such as the class of lovers of men”
3) “a relation, as expressed by ‘*x* loves *y*’ or ‘*x* is a lover of *y*’” (Merrill, 161).

Merrill (I believe successfully) refutes the first two interpretations, for reasons we have already discussed. The class of lovers of servants is not a function of the class of lovers and the class of servants. Relative Product requires that we take into account the order of the relata. Viewing the extension of relative terms simply as non-relative classes obfuscates that order.

But, there are really still two interpretations left over, not only one. Even in ‘Note B’ and ALPT, where Peirce’s view sounds most extensional, he never says that relations *themselves are* sets of n-tuples, or logical sums (union) of individual relatives. Furthermore, he speaks of general relatives (i.e. unsaturated relations) as having *definitions that describe in general terms* (implying an intensional understanding). For example: “A relative is a term whose definition describes what sort of a system of objects that is whose first member…” (CP 3.218); and “Every relative, like every term of singular reference, is general; its definition describes a system in general terms…” (3.220). This shows that, in actual applications of determining reference, denotation only comes through conceptualizing the intension.

Brunning argues, “Peirce had two distinct notions of relation. One was the standard notion of a relation as a set of n-tuples. Peirce often referred to these as merely formal relations. The other was a relation as a relative concept” (Brunning, 225).[122]

I believe that Peirce has both concepts of relation in mind—in fact, he explicitly says as much, although, as I have attempted to show, he equivocates in his earlier work,

[122] In fact, Brunning’s notion of a “relation as a relative concept” is a bit different from mine. She seems to think that, if we take the second meaning of relation, there are just three relative concepts—monads, dyads and triads. So she is not thinking about relations as having an intensional essence—e.g. the ‘fatherhood’ relation being a relative (intensional) concept, and the ‘to-the-left-of’ relation being another different relative intensional concept, as I am arguing. Burch also gives an intensional account, but he formalizes the idea with possible world semantics.

since he is working out these discoveries as he goes. Hawkins, in his fascinating essay on the "neglected controversy" between Peirce and Russell, points out that,

"Peirce, unlike Russell, was not committed to extensionality…[he] conceded that the *extension of a property* ϕ is the class defined by 'ϕ', and he casually noticed in 1902 that 'A class, of course, is the total of whatever objects there may be in the universe which are of a certain [qualitative] description' (1.204)" (Hawkins, 129).

Peirce's own words in 1903 (quoted by Hawkins) are extremely instructive here, where he advances two distinct meanings of a set/class/collection ϕ:

"The one I shall call a *gath* which is simply the word 'gather' with the last syllable dropped. The other I call a *sam* which is the word 'same' with the last letter dropped. I also like this word because it is so much like the word *sum*, in the phrase *sum total*. It also recalls the German *Samlung*. A collection, in the sense of a *gath*, is a subject which is a pure Secondness without Firstness, and whose only mode of being is whatever existence it may have; and this consists in the existence of certain other existents, or pure Seconds, called its *members*…A *sam* is an *ens rationis* whose essence is the being of a definite quality (imputed to the *sam*) and whose existence is the existence of whatever subject there may be possessing that quality…No doubt the easiest way to conceive of the *sam* is to imagine that you have a common noun, without specifying what noun it is, and to think that the noun signifies some quality which is possessed by anything to which it applies, but is not possessed by anything to which it does not apply. Now you are to imagine a single thing which is composed of parts. Nothing is done to these parts to put them into their places in the whole: their mere existence locates them in the whole. Now think of this *rule* [my emphasis] as describing the whole. If any individual object can properly have that common noun predicated of it, it is a part of the single object called a *sam*; if not, it is not. That gives you the idea of the *sam*. Now to get the idea of a *gath*, you are to consider that those individual objects might change their qualities without losing their individual identity; so that limiting ourselves to any instant any individual object which at that instant forms a part of the *sam* forever forms a part of an object of which no object not at that instant a part of the sam is a part, and this individual composite whole which has nothing to do with the qualities of its members is a *gath*…So that for every gath there is a corresponding *sam*. But it is not true that for every *sam* there is a corresponding *gath*…What has been said of qualities is equally true of relations, which may be regarded as the qualities of sets of individuals. That is to say, if any form of relation is logically possible between the members of two *gaths*, a relation of that form actually exists between them. (MS 469: 8^p-36^p passim)" (quoted in Hawkins 129-30).

This passage is deeply interesting and reveals that (at least by 1903) Peirce was crystal clear about these two different ways of defining the extension of relations. This discussion also foreshadows later developments of possible world semantics.[123] One part of the passage above that is a bit confusing is where he says that "for every *gath* there is a corresponding *sam*…but it is not true that for every *sam* there is a corresponding *gath*." I think that the best way to understand this is as follows: if a gath is a collection whose extension is merely stipulated (pure Secondness without Firstness), then merely by virtue of forming the collection, we have a new qualitative property—namely the property of being a member of this collection. Thus, "for every *gath*, there is a corresponding *sam.*" However, we may have an idea of a definite quality without there being anything that possesses that quality. As such, the *sam* has no members, and thereby does not have a corresponding *gath*.

The long passage quoted above is also interesting in the way that it distinguishes *essence* of a *sam* from *existence* of that *sam*. The existence seems simply to be the set of n-tuples, considered as seconds. But the essence is the embodiment of a "definite quality" (i.e. an intensional quality). In logic today, typically we would formalize this notion by an appeal to some modal semantics of possible worlds. In that case, we might say, for example, that, while the extension of the fatherhood relation in the actual world is *this* set of pairs, in other possible worlds, the fatherhood relation may have a different extension[124]. *But*, (if I may take a detour into metaphysics) Peirce helps to show that perhaps this view is still wedded to the idea that essence must be defined by existence—or, rather, that essence must only be characterized in the form of Secondness. Peirce's Pragmatic Realism is largely an appeal to take account of Thirdness (habit of behavior) in defining essence of a thing. Here, it's clearly also Firstness that we are missing.

The appeal to take account of Thirdness is Peirce's "Pragmatist twist" on Scotistic Realism. The problem with the contemporary debate (for Peirce, and, I believe, still for us now) about universals is that universals are exclusively understood in the mode of Secondness. Peirce's idea offers a new insight for how to think about the extension of relations, which has an application in metaphysics generally as well as in the philosophy

[123] The distinction is also, of course, relevant for thinking about issues in the metaphysics of Time. As mentioned, Peirce attempted to incorporate these sorts of distinctions into the gamma graphs, which are beyond my scope.
[124] This is Burch's strategy—e.g. Burch, APRT, 39-40. Though he claims that, "this work's reference to possible worlds is heuristic only" (ibid, 39).

of science for thinking about Natural Kinds—which, of course, is a related issue. It is (so easily, but according to Peirce, incorrectly) assumed that for something to be *real*, it must exist in the manner, or mode of being, of a Second. But, in fact, we also have to consider *both* Firstness—in considering relations as bearing an intensional property ϕ—as well as Thirdness (in defining membership of relations not merely by behavior (the Scotistic view) but by *habit of behavior* (the Pragmatic Realist view).

As Hawkins goes on to say,

"According to Peirce, the generality of ϕ being 'peculiar to the category of quality,' is merely potential' (1.427). In fact, by virtue of Peirce's professed 'extreme form of realism' (NEM 4:345), a relation $\varnothing$ is *universale inter res*, and 'it does not contain any individuals at all. It only contains general conditions, which *permit* the determination of individuals' (6.185)" (Hawkins, 130).

Hawkins, citing Martin (1969: 147,149) argues that Peirce's lack of sharp delineation between the '$\in$' and '$\subset$' relations is deliberate, and is a consequence of this conception of the extension of relations.

This idea also recalls Peirce's criticism of Aristotle, which we noted in the last chapter:

"Aristotle supposes that a general term is equal to a sum of singulars, which is a doctrine that Peirce cannot admit, for 'the extension of a universal term consists in the total of possible things to which it is applicable and not merely to those that are found to occur (W 1:263)" (Bellucci, 34).

The extension of a *gath* can only consist of those objects that are found to occur. But the extension of a *sam* can offer an intensional criterion for membership, which applies to all *possible* objects, not just those that are found to exist.[125]

So, while we might view the referent of an unsaturated relation extensionally from a God's-eye-view, in actual practice of reasoning about relations, this is not what happens. What happens in actual reasoning is that we employ intension.

But, in addition to arguing that Peirce has an intensional view in mind, I wish to add a further claim, which I introduced in the previous chapter. This further claim is that, in actual reasoning, we take collections of individual relatives (i.e. collections of individual ordered n-tuples, which in aggregate are a *subset* of the extension of the

[125] See Hawkins, 130-1 for an account of how this distinction between *sam* and *gath* impacts understanding of the identity relation.

complete God's-eye-view relation) as *an index* of the complete relation. This notion, where a subset functions as an index of the whole, is roughly the same idea that we discussed in the account of Induction—where a collection of individuals are an index of the whole set (which is not accessible to us).

Recall that, according to Peirce, an inductive argument takes the following form (EP 1.9):

> S', S'', S''', and S^{iv} are taken as samples of the collection M
> S', S'', S''', and S^{iv} are P
> All M is P

As I explained in Chapter Two the set of samples $\{S', S'', S''', S^{iv}\}$ is an index of the class M, which, in its entirety, is not accessible to us. But since all of the samples possess the property P, we infer inductively that the whole set has the property P. If we then discover that some other member of M, say S^{v}, does not have the property P, then this reveals that it is false that all Ms are Ps. But this conclusion would employ deductive reasoning.

In the case of the extension of relations, I wish to suggest that something similar is going on. We have an unsaturated relational predicate ϕ defined not as *being* a set of n-tuples (in the actual world, or any other world) but as embodying some quality or set of qualities. We then determine if a new given n-tuple, *p*, is a member of the class (*sam*) ϕ by seeing whether *p* possesses the criteria defined by ϕ or other properties deducible from ϕ. If we determine that *p* is a member of the *sam* defined by ϕ, then two things happen: we gain determinate information about *p* (now we know it is a ϕ), and we gain determinate information about ϕ (now we *know* that its extension includes *p*, which we didn't know before).

This may sound like blending of Induction with Deduction. But, as Peirce argues, "no 'necessary' conclusion is any more apodictic than inductive reasoning becomes from the moment when experimentation can be multiplied *ad libitum* at no more cost than a summons before the imagination" (CP 4.531). Peirce wants to retain both concepts of the referents of relations (the extensional and intensional), but even at the purely formal level, he wants to account for how semeiosis is involved. In this way, I think he is attempting to construct a more honest logic, which incorporates within itself how it is to be mapped to Nature.[126]

My account also allows us to make sense of Peirce's claim that, "denotation essentially takes a part for its whole" (EP, 2.322). He conceives of denotation in the ostensive sense, where we denote by means of an index—i.e. a sign that is in *direct physical* contact with its objet.

Here, we recall the discussion from the previous chapter about logical depth and logical breadth. I believe that Peirce's notion of logical depth (all the predicates that fall under a logical subject) is analogous to Frege's notion of "comprehensive knowledge" (Frege, 153). Frege notes, "Comprehensive knowledge of the *Bedeutung* would require us to be able to say immediately whether any given sense attaches to it. To such knowledge we will never attain" (ibid). But herein lies Peirce's pragmatism. I believe he would respond to Frege that *we might actually be able* to define precisely how to go about getting (indexical) acquaintance with the objects that we are talking about, and we might be able to characterize the dispositional habits of those objects in a precise way. Peirce's example of the "definition" of lithium is one such example. So Peirce's criticism of Frege would be to say that he is only thinking in terms of Secondness, when there is more hope scientifically to define essences by Firstness and Thirdness as well.[127]

Relative Product as Determinative

What does this theory of relations have to do with relative product?

I set out in this chapter aiming to argue that relative product shows us that the restriction on bonding in EG is not arbitrary, but is a genuine discovery that comes out of a study of logic itself.

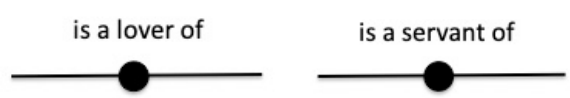
I believe that if we understand the reference of relations in the way that I have described, then we can begin to see how mutual determination occurs. When a logical subject and logical predicate are joined, they fill each other's lack, since the extension of

[126] See CP 3.419, 3.420 and 3.423 for Peirce's own account of how his logic is concerned with its own connection to Nature.
[127] I also think part of Peirce's (imagined) criticism of Frege would be to say that Frege is assuming that all Deduction has to be corollarial, when in fact, Deduction can be either corollarial or theorematic. We will explain these terms in Chapter Five.

the predicate is increased by means of the subject, and the intension of the subject is increased by means of the predicate. It is also important to reiterate that in EG there is no semantic feature where relations are conceived as sets of n-tuples, which only further supports the idea that Peirce understands the referents of unsaturated relations in a non-extensional way.

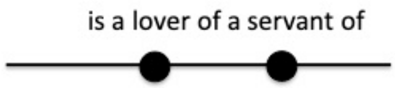
In ALPT, Peirce notes that, "[Since] the logic of relatives considers statements involving two or more individuals at once…indices here are required" (CP 3.392). These indices (in the quantificational system of ALPT) are the variables or coefficients that attach to the relational term—for example the *i* in l_{ij}.[128] The new information—i.e. greater determination—that we get from Relative Product is the assertion that two indices are signs of the same object. So, importantly, we see that relative product does for a proposition what the interpretant does for a term.

It is useful to also consider how Relative Product appears in EG. As Roberts explains, in EG every rhema must have lines of identity connected at each hook. Thus "formulas with free individual variables cannot be expressed in EG" (Roberts, 49). So in EG we might express the two rhemata '__is a lover of__' and '__is a servant of__' by the following graph:



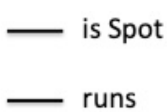
When we take the relative product of these we get the following graph, which has not four ligatures but three:

[128] In "New Elements" Peirce explains why these are to be understood as indices. There he uses the example of the signs 'Socrates' and 'is wise', claiming that the sign that unifies them is an index. "But, it may be objected, an index has for its object a thing *hic et nunc*, while a sign is not such a thing. This is true, if under 'thing' we include singular events, which are the only things that are strictly *hic et nunc*. But it is not the two signs 'Socrates' and 'is wise' that are connected, but the *replicas* of them used in this sentence" (EP 2.310). These replicas are physically connected *on the page*, and thus "They form a pair of reacting things which the index of connection denotes in their present reaction" (ibid). The new sign formed through their connection (the proposition) is a symbol.



As Herzberger explains (52), and as we can see in this example, relative product obeys the valency rule of bonding algebra, which says: “The union of any μ-ad with any ν-ad gives $[\mu+\nu-2\lambda]$-ad, where λ is the number of bonds of union” (Herzberger, 52). As we will discuss in the next chapter, the Reduction Thesis hinges on this idea. But, the question I have been asking is why? Why can only two loose ends be bonded rather than three? The answer comes back to Relative Product. It is because this is how conceptual combination and conceptual determination actually takes place. The purpose of a sign is to express a fact (EP 2.304). But, as we have explained, facts themselves have the structure of propositions. So the sign (proposition) that expresses this fact must do so through combining logical subject (or subjects) and logical predicate (rheme).

As noted above, Peirce hold that, “No cognition and no Sign is absolutely precise, not even a Percept; and indefiniteness is of two kinds, indefiniteness as to what is the Object of the Sign, and indefiniteness as to its Interpretant, or indefiniteness in Breadth and in Depth” (CP 4.543). Let us take another example we discussed earlier—the proposition ‘Spot runs’. This proposition consists of a monadic predicate and one logical subject. If we considered these separately—that is to say, if we left out the assertion that they are signs of the same object (the fact)—then we would graph them in the following way:



This is because proper names in EG are treated themselves as monadic predicates. But when we assert that they are unified, they mutually determine one another since now we know that Spot is included in the logical breadth of the predicate ‘__runs’ and that the predicate ‘__runs’ is included in the logical depth of Spot. That graph in EG would be the following:

The following lengthy passage from Peirce is worth quoting in full:

"A mystery, or paradox, has always overhung the question of the Composition of Concepts. Namely, if two concepts, A and B, are to be compounded, their composition would seem to be necessarily a third ingredient, Concept C, and the same difficulty will arise as to the Composition of A and C. But the Method of Existential Graphs solves this riddle instantly by showing that, as far as propositions go, and it must evidently be the same with Terms and Arguments, there is but one general way in which their composition can possibly take place; namely, each component must be indeterminate in some respect or another; and in their composition each determines the other. On the [sheet of assertion] this is obvious: 'Some man is rich' is composed of 'Something is a man' and 'something is rich,' and the two somethings merely explain each other's vagueness in a measure…The composition of a Conditional Proposition is to be explained in the same way. The Antecedent is a Sign which is Indefinite as to its Interpretant; the Consequent is a Sign which is Indefinite as to its Object. They supply each the other's lack" (CP 4.572).[129]

This idea was also germinal (albeit in a very remote way) in De Morgan, who claimed that, "in the consideration of the proposition, *identification* of objects is in truth a relation of concepts" (DeMorgan 214-215). Identification of objects is accomplished generally by something like the relative product operation. In the context of quantificational logic, objects would be identified either by relations being "saturated" by individual constants, or they would be identified in an indefinite way through quantification.

As we noted in the previous chapter, existentially quantified propositions, on Peirce's view, assert that if one filled the blank of a rhema with the right sort of index, then the proposition would be true. As such, the proposition itself has the same structure as it does in the case when its blank is filled by an individual constant. And we see this clearly in EG. The graph of the proposition 'Spot runs' clearly consists of three things—the two rhemata and the one line of identity, which asserts that these are signs of the same object. In EG unsaturated rhemata clearly display their own lack or indeterminateness.

[129] This idea of mutual determination is also emphasized by Brunning (258), and Herzberger (44).

Chapter Five

The Reduction Thesis

Logic After 1885

With ALPT, in 1885, Peirce's quantificational logic achieves full expression. There, he presents a complete first-order system, with quantifiers that work conceptually just like the ones we use today. In that paper he also gives examples of complex propositions that employ many nested quantifiers (e.g. CP 3.395), and he delves into second-order logic—what Peirce called "Second-intentional logic" (CP 3.398) (Brady, 173, 189).

But the very next year, while Schröder was setting to work within Peirce's system, writing his landmark work *Vorlesungen über die Algebra der Logik*,[130] Peirce turned away from the quantificational project. A significant reason for this was Alfred Bray Kempe's 1886 publication of his "Memoir on the Theory of Mathematical Form." Kempe, former president of the London Mathematical Society (EP 2.169-70) was a colleague of Peirce's, and sent Peirce his paper. The paper was an attempt to represent mathematical relations by means of diagrams using spots and lines (see CP 3.423; CP 3.468; EP 2.170). Kempe "dressed his ideas in geometrical and (proto-) topological clothing. This excited Peirce, who saw in the work a means of developing the logic of relations (which Kempe himself had not done)" (Guinness, 36). According to Peirce, Kempe's work had "the 'intrinsic value [...] of taking us out of the logician's rut, and showing us how the mathematician conceives of logical objects' (CP5.505 (1905)" (Atkin, 189).

Peirce was deeply inspired by this work. But it also posed a "formidable objection" to his own views (CP 3.423), even causing him to reconsider his commitment

[130] "This book, which appeared in three volumes, has a third volume on the logic of relations (*Algebra und Logik der Relative*, 1895). The three volumes were the best-known logic text in the world among advanced students, and they can safely be taken to represent what any mathematician interested in the study of logic would have had to know, or at least become acquainted with, in the 1890s" (Putnam, RHF, 256). The first volume of Schröder's work was published in 1890.

to the categories.[131] In the 1892 essay "The Critic of Arguments", Peirce discusses how Kempe "represents every possible relationship by a diagram consisting of only *two* different kinds of elements, namely, spots and lines between pairs of spots" (CP 3.423). This fact is significant. For, if a mathematical system can rigorously demonstrate that only two elements are required for constructing any relational proposition in mathematics, this would seem to suggest that there are not three fundamental logical elements, but rather only two. Triadicity, then, seems to be derivative and not elemental. Peirce notes that Kempe thought as much (EP 2.174), and admits (in 1903), "To mathematical minds this will probably seem a formidable objection. It seemed so to me when I read it, I confess" (ibid). In the 1892 paper, he claims that Kempe's paper "causes me somewhat to modify my position, but not to surrender it" (CP 3.423).

However, in 1892 Peirce was attempting to defend his categories against Kempe's blow in the wrong way. He attempted to rescue the categories by claiming that, "the spots, or units, as he calls them, involve the idea of firstness; the two-ended lines that of secondness; the attachment of lines to spots, that of mediation" (CP 3.423). But in 1903 he describes that this view was in error. In writing the 1903 lecture "On the Categories Defended", Peirce claims to have realized that Kempe's spots, "each of which ties together any number of lines…far from representing the Category of Unity [i.e. Firstness], plainly embody the Category of Plurality—the Third Category—and it is the Surface upon which the graph is written as *one* whole which in its Unity represents the Category of Unity" (EP 2.175). It would be easy to overlook the fact that a necessary ingredient of any diagram is the space itself upon which the diagram is written. Peirce claims that, "until I came to write this lecture, it had never occurred to me to examine [EG] in respect to its relation to the categories" (EP 2.176). However, on considering the graphs in relation to the categories, Peirce saw a way to remedy the "vexatious inelegance" (ibid) that had plagued the Entitative Graphs. This inelegance was the fact that, in that system,

"Propositions written on the sheet together were not understood to be independently asserted but to be *alternatively* [i.e. disjunctively] asserted. The consequence was that a

[131] Though he was still working out important logical discoveries, Peirce was convinced of the three logical categories, and thus the three metaphysical categories, at least by 1867.

blank sheet instead of expressing only what was taken for granted had to be interpreted as an absurdity. One system seems to be about as good as the other, except that unnaturalness and aniconicity haunt every part of the system of entitative graphs…" (CP 4.434).

I maintain that it is philosophically significant that, in EG, the blank sheet of assertion represents 'the true' in an entirely vague way—or, similarly, that it represents what is taken for granted as true between graphist and interpreter (CP 4.551), yet is not specified. Peirce suggested around 1873, in an outline of an unpublished book, that one chapter was to be on the necessity of the concept of space for logic.[132] He writes that all mathematical diagrams involve space; "but space is a matter of real experience" (NEM IV.xv). In EG the blank sheet of assertion is true but utterly vague, and undergoes determination as new propositions (both "saturated" and "unsaturated") are asserted and transformed through rules of transformation. In this sense, EG iconizes what actually happens in thought—relational concepts (relations) undergo determination either by being saturated with indices (their relata), *or* by being combined with other relations (asserting equality of indices). And objects (i.e. logical subjects) are determined by certain predicates being attached to them.

The Reduction Thesis

"Accordingly, all rhemata higher than the dual may be considered as belonging to one and the same order; and we may say that all rhemata are either singular, dual, or plural. [422] Such at least is the doctrine I have been teaching for twenty-five years, and which, if deeply pondered, will be found to enwrap an entire philosophy" (CP 3.421-2) (1892).

As Peirce suggests here, his entire philosophy is wrapped up in the division of rhemata into three classes—single, dual and plural. Herzberger cites Peirce's pronouncement that, "nothing in philosophy is more important' (CP 1.298) (Herzberger, 42-43). But on one level, it is peculiar to hear Peirce say in 1892 that he has been teaching this doctrine for twenty-five years. For, as we have cited before, he claimed in 1907 that it wasn't until about 1892 that he had reached a "provisionally final result" about the three fundamental classes of n-adic predicates (EP 2.425). Yet, as Herzberger cites in his Appendix of Peirce's references to the Reduction Thesis, he was convinced of the idea at least as early

[132] W3.81; Bellucci, 81

as 1870 in DNLR, and his later reformulations of the thesis are more refinement than wholesale change. This should give us some hesitation. For a perpetual worry in thinking about the Reduction Thesis and Peirce's categories in general is the thought that Peirce is making claims about logic that are rooted in prior commitments from metaphysics or phenomenology.

Yet I wish to defend Peirce. The categories are much more than logical concepts, and thus they can be discussed and defended within other realms of philosophy, like phenomenology (as Peirce himself does—particularly in the 1903 lectures.[133] But the Reduction Thesis is a claim about formal logic. I have been arguing that the discovery of the Reduction Thesis, and thus the discovery of the categories, comes out of a thorough study of logic itself. Further, I believe that an argument to defend the Reduction Thesis is best understood *not* as a phenomenological or metaphysical argument, but as a logical argument—or, at least a semiotic argument.

This last addition should give every philosopher pause. How can I claim that the argument is logical, but then go on to moderate that bold claim with the concession that perhaps it is a semiotic argument. There is much to say on this topic, but it cannot be addressed here. But, as several thinkers—perhaps most notably, Bellucci—emphasize, Peirce came to view logic and semiotics as intimately connected, even stating in 1903 that, "Logic…develops into a general theory of signs" (EP 2.272). Logic, as we typically understand it today, is one of three divisions of logic, as Peirce understands it. The question whether the Reduction Thesis is best understood as a semiotic claim or a metalogical claim (in the sense that we understand metalogic today) must be left for another project. But in what follows, I will attempt to lay some groundwork for how we should understand it.

The Reduction Thesis is a two-part claim:

The **first** part asserts that from three distinct types of logical predicates—namely monads, dyads and triads—all predicates of higher addicities are formable.

[133] See particularly the essays "On Phenomenology" (1903) and "The Basis of Pragmaticism in Phaneroscopy" (1906)—both included in EP Vol. 2.

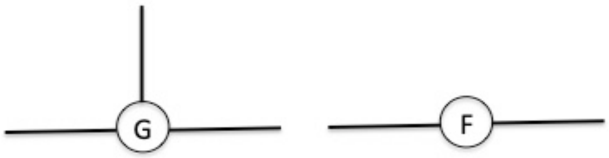
The **second** part asserts that these three logical types are irreducible. That is to say that dyads cannot be constructed from monads, and triads cannot be constructed from monads and dyads.[134]

This amounts to saying (rather informally) that there are three primitive logical categories, and only three.

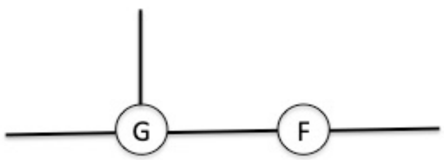
Herzberger rightly emphasizes the importance of the Reduction Thesis for Peirce: Peirce claimed that, " 'nothing in philosophy is more important' [CP 1.298]" (Herzberger, 42-43). Peirce also claimed that "his own contributions to pragmatism [were] 'entirely the fruit of this outgrowth from formal logic' [CP 5.469]" (ibid, 43). And yet, as Brunning points out, "the reduction of polyads to dyads is commonplace in present logical theory" (Brunning, 252), which would seem to refute the key pillar of the Reduction Thesis, which claims that triads cannot be reduced. So how are we to understand Peirce's "remarkable theorem"?[135] I will first give a brief sketch of the relevant literature on the Reduction Thesis, and then I will discuss my own view for how we should understand it.

Aside from Peirce's own writings, scholarship on the Reduction Thesis effectively begins with Herzberger's 1981 essay "Peirce's Remarkable Theorem." Herzberger offers a formal proof, which is set within a formal "bonding algebra" that he defines. I will not go into any details of his proof except to note that the relative product operation is central. "One definitional process stands out as central to Peirce's reduction methods and to his whole thinking on the combination of concepts. This is the operation of *relative product*…" (ibid, 43). He defines relative product algebraically, but it can perhaps best be understood with a graphical example, where we take the triadic rhema '__gives__to__' and the dyadic rhema '__is a friend of__' (Herzberger, 44).

[134] As mentioned in the Introduction, one might think that this should say that monads cannot be constructed from *dyads and triads*; dyads cannot be constructed from monads *and triads*; and triads cannot be constructed from monads and dyads. But Peirce shows that all addicities can be constructed from triads (EP 2.346), which might seem to suggest that only triads are primitive. But Peirce thinks that triads presuppose monads and dyads.
[135] This phrase comes from Peirce: "Now I call your attention to a remarkable theorem. Every polyad higher than a triad can be analyzed into triads, though not every triad can be analyzed into dyads" (NEM IV.338). Herzberger borrows the phrase for the title of his essay.



The relative product of these two rhemata combines the third loose end of the rhema G with the first loose end of the rhema F, producing the following graph:



This graph (in quantificational form) asserts: $\exists w\exists x\exists y\exists z(\mathrm{G}wxy \wedge \mathrm{F}yz)$.[136]

As Herzberger points out, the relative product operation, as defined in this bonding algebra (and as understood by Peirce) obeys the following "valency rule": "The union of any μ-ad with any ν-ad gives $[\mu+\nu-2\lambda]$-ad, where λ is the number of bonds of union" (Herzberger, 52). We can see this intuitively in the example above, where in the first graph we count the five loose ends, but in the second graph we see that two loose ends have been combined, leaving the whole graph with three loose ends. As I will discuss below, most of the technical literature on the Reduction Thesis effectively takes this bonding formula as an axiom within the system in which the Reduction Thesis is proved.

But, as I have emphasized, my overall task in this paper is to ask *why* this type of conceptual combination is privileged. *That* the Reduction Thesis holds *if* this type of conceptual combination is privileged seems clear enough to me (though I admit there may be mathematical layers to the question, which are beyond my expertise). But I find the literature about the Reduction Thesis wanting on the question of *why*. Perhaps one reason for this neglect is that this seems like a philosophical question that is not strictly

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[136] Here one may wonder what determines the order of quantifiers. As Shin explains (Shin, 122), there is a stipulated clockwise order for cases like this. When ligatures cross cuts, scope of quantifiers is determined by the outermost part of the ligature, though the clockwise stipulation also still holds. Burch points out that a central problem in translating graphical notation to algebraic notation (PAL) is how to account for order.

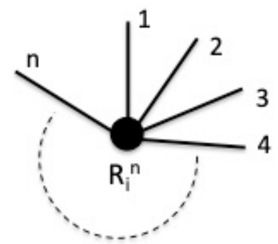
logical or mathematical. But, as I suggest below—following Brunning, Burch, and Hintikka—this philosophical question may have a metalogical answer. More will be discussed below.

The next major step in scholarship on the Reduction Thesis is Robert Burch's 1991 book *A Peircean Reduction Thesis*. As previously noted, Burch constructs a system called "Peircean Algebraic Logic" (PAL) (Burch, APRT, 5), where relations are the primitive logical terms and individuals are defined in a derivative way from them. This is in contrast to "quantification theory [which] places individuals at the fundamental level, by virtue of its variables and constants. This, in Peirce's view, is a mistake." (Burch, APRT, 1-2). In that vein, Burch defines an intensional semantics, which he claims can be understood in a way that is analogous to Kripke's possible world semantics[137] He differentiates two notions of interpretation: an Enterpretation (extensional interpretation) and Interpretation (intensional interpretation) (ibid, 27). This strategy aligns with my discussion in the previous two chapters about Peirce's two views of the referents of relations. But, again, Burch attempts to formalize the notion of an intensional semantics using the notion of possible worlds.[138]

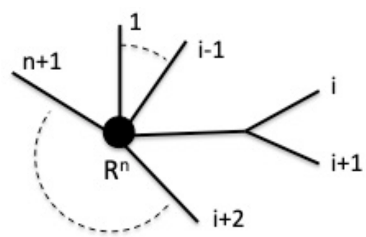
Burch is largely concerned with trying to precisely formulate (in an algebraic way) the different operations through which loose ends can bond to each other. He gives a formal account of these operations in PAL, and presents them with analogous formulations in quantificational logic.

An example will, perhaps, clarify in very broad strokes the kind of project Burch is undertaking. Take a primitive relational term R_i^n where *i* is a natural number to differentiate that primitive relational term from a set of such terms, and *n* is the addicity of that term (APRT, 22; PRT, 236). Thus we could assign the dyadic 'father of' relation the number 7 (supposing it is the seventh relation in a list of particular relations relevant to the interpretation). Thus, that relation would indicated by R_7^2. Burch represents the term R_i^n graphically in the following way, where "loose ends" represent unsaturated places of potential bonding and the addicity of the relation is *n* (Burch, PRT, 237):

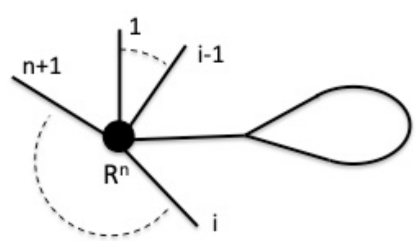
[137] Burch, APRT 27; also see Burch, PRT, 235-6
[138] I am suspicious of this strategy, that it may fall into the tendency (mentioned in Chapter Four) of favoring Secondness at the expense of Thirdness and Firstness, but I am not familiar enough with Burch's complete argument to criticize it.



Continuing with the example, one operation that Burch incorporates into PAL is (an algebraic analog of) the operation in EG where a line of identity branches. So Burch introduces a PAL operation for this called COMMA. The graphical representation looks like this (Burch, PRT, 244):



Another distinct operation defined in PAL would permit joining two loose ends of the same relational term to each other—e.g. (ibid, 245). This sort of operation would be required if the same individual were to fill two relevant slots—e.g. if someone were a lover of herself:



These examples are simply meant to show that Burch is aiming to give an exhaustive account of the way that lines of identity may branch and be joined. He has far more operations than those cited above. The result of his 1991 paper is a proof that generalizes beyond Herzberger, by eliminating a stipulation that Herzberger had to make about the size of the relevant domain in a reduction (see Burch, PRT, 251). We will not go into any detail on this generalization though.

Two other works on the Reduction Thesis are worth mentioning. One is Kenneth Ketner's essay "Peirce's NonReduction and Relational Completeness Claims in the Context of First-Order Predicate Logic" (2011), which is a rather approachable paper. It seems intended simply to clarify why the Reduction Thesis holds in a system where bonding is restricted in the way that we have suggested above—i.e. where bonding is restricted to Herzberger's valency rule [$\mu+\nu-2\lambda$].

The second paper worth mentioning is extremely technical, and, regrettably, not well understood by me. It is called "The Teridentity and Peircean Algebraic Logic" by Joachim Hereth Correia and Reinhard Pöschel, two mathematicians. I have been assured by Frithjof Dau—another mathematician, who has written on EG and the Reduction Thesis, and who is a colleague of Correia and Pöschel—that this paper is a major advance, but since I do not understand it, I am in no place to comment on it.

So scholars of various stripes are doing work on the Reduction Thesis. But how can we make sense of these proofs, since Quine (1953) has proved that triads can be reduced to dyads in quantificational logic?[139] Quine's paper "Reduction to a Dyadic Predicate" considers "any interpreted theory Θ formulated in the notation of quantification theory (or lower predicate calculus) with interpreted predicate letters" (Quine, RDP, 225). These predicate letters may be of any addicity. Quine then offers a proof that Θ is translatable into a theory Θ', which consists of only one predicate letter, F, which is dyadic (ibid). In regard to this predicate, we must clarify a few points. First, Quine notes, "let us construe the ordered pair $x;y$ in Kuratowski's fashion, viz. as $\{\{x\},\{x,y\}\}$, and then construe $x;y;z$ as $x;(y;z)$, and $x;y;z;w$ as $x;(y;z;w)$, and so on" (ibid, 224). Secondly, he clarifies that we must understand x as being distinct from $\{x,y\}$ and $\{\{x\}\}$, though we can understand x to be equivalent to $\{x,x\}$ (ibid, 224). Finally, "the universe of Θ' is to comprise all objects of the universe of Θ and, in addition, $\{x,y\}$ for every x and y in the universe of Θ' " (ibid, 225). Quine's dyadic predicate (translated into more familiar notation is the following (Quine, RDP 225):

$Fxy \leftrightarrow (\exists z)(x=\{y,z\}) \vee [(x=y) \wedge (\exists n)(\exists w_1)\ldots(\exists w_{dn})(F_n w_1\ldots w_{dn} \wedge x=nw_1;\, w_1;\, w_1;\, w_2;\, w_3;\ldots;\, w_{dn})]$

[139] Burch also cites Löwenheim, 1915 (Burch, APRT, 120)

By this proof, Quine demonstrates that (in a quantificational system) we can "reduce" any triadic relation into a dyadic one. This reduction would be accomplished by translating the triadic relation into Quine's predicate *Fxy*. In this case, it would seem, contrary to what Peirce claims, that triads are reducible, and therefore not primitive in the same way that dyads and monads are. And indeed, mainstream scholars (at least those who know of the Reduction Thesis) have taken Quine's proof to be a deathblow to it—writing off Peirce's "theorem" as "one more uncollectable debt" (Herzberger, 41).

Could Quine and Peirce both be right? And if so, how? As Brunning notes, "The Beta part of the graphs has the same theorems as first-order predicate calculus with identity. However, theorem isomorphism is a weak condition and the structures embedded in the graphs make this a very different system" (Brunning, 257). Herzberger similarly acknowledges that the counterargument to his proof "holds within standard notions of definability…[underscoring] the novelty of this kind of logical framework" (Herzberger, 42).

In Burch's SEP article on Peirce, he notes that Quine and Peirce were both right; the apparent conflict simply comes from the fact that Peirce's "constructive resources are to be understood to include only negation, a generalization of de Morgan's relative product operation, and the use of a particular triadic relation that Peirce called 'the teridentity relation'…[which] we might today write as $x=y=z$" (Burch, SEP). So if we restrict the logical toolkit from the beginning, then we end up with a different logical system. Burch has proved (1991) that, in that system, the Reduction Thesis holds.

But I am convinced that to simply leave things here would not be good enough for Peirce. He was convinced that the Reduction Thesis had broader import, and, I believe, did not understand it in the more narrow sense as being simply a consequence of one (admittedly, rather peculiar) logical system. Indeed, merely understood in that sense, one can simply argue that the logical system itself—i.e. EG—is gerrymandered in a way to ensure something like the Reduction Thesis holds. And in matter of fact, this objection would not be far off base. After all, as Brunning notes, citing Peirce's own account, the categories taught Peirce how to construct the graphs (MS 439) (Brunning, 253). And, as we have noted above, Peirce was initially defending the wrong interpretation of how the categories are manifest in the Entitative graphs, which suggests that he surely had a prior commitment to the categories.

But I want to suggest, instead, that Peirce's discovery of the Reduction Thesis is a genuine (logical/semiotic) discovery, which emerges from a careful study of logic. If we take the Reduction Thesis in the more narrow sense, then we must be relativists about which logical system we *should* be using for philosophical discovery. Admittedly, Peirce did believe that quantificational logic was likely the most effective when used as a calculus to aid reasoning. But he thought that EG was far superior for logical discovery—that is to say, he saw EG as capable of revealing hidden relations implicit in our reasoning because the restrictions it depends on capture something true about the way that we reason.

Furthermore, if we see the Reduction Thesis in the more limited way, it does not have the force that Peirce thought it did in bringing out philosophical and metaphysical consequences, like his Pragmatic Realism (e.g. CP 4.1). I am convinced that, on Peirce's view, more is going on. But how might we understand the Reduction Thesis in this more far-reaching way?

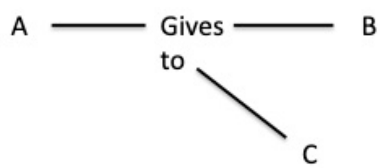
One way is by defending the legitimacy of the restriction on conceptual combination that it makes, which I have already attempted in the previous chapters. But I believe there is *also another way* that we can understand the Reduction Thesis in a more far-reaching way. I believe we get a clue from the following passage (in a letter from Peirce to Victoria Welby, 1904; quoted by Hawkins):

"…Mr. Russell's idea that there is a *fourthness*, etc. is natural; but I prove absolutely that all systems of more than three elements are reducible to compounds of triads; and he will see that it is so on reflection. The point is that triads evidently cannot be so reduced since the very relation of a whole to two parts is a triadic relation" (Hawkins, 134).

This recalls a distinction emphasized earlier in this paper—namely the distinction between the structure of an argument in the object language and the (perhaps implicit) structure of an argument in the metalanguage. Peirce seems to be saying here that any argument that purports to reduce triads to dyads itself *employs* triads in the meta-argument. He had already made the discovery as early as 1866 (in Memoranda) that a "reduction", whose procedure is specified in an object language, might *itself implicitly employ* (in the meta-level argument) the very structure it claims to reduce. I believe that this discovery was the key to Peirce unlocking the categories (albeit in a rather hazy form, which needed to be ironed out through decades of logical work).

This idea—that triads cannot be reduced without using triads (in the metalogical argument)—is also emphasized by Brunning: "Peirce often repeated a criticism of the algebra of dual relatives, noting that: 'the very triadic relations which [it] does not recognize, it does itself employ' (8.331). He made this complaint a bit more explicit when he said: 'A triadic relation cannot be reduced without the use of a triadic relation' (1.346)" (Brunning, 261).

Brunning helps clarify this idea with an example of Peirce's. Peirce explained the example in response to Kempe's objection that one could express the triadic fact 'A gives B to C' by three sets of dyads. The triadic fact would be graphed in the following way (Brunning, 260):

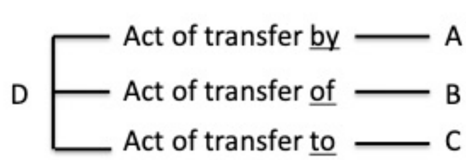


But, according to Peirce, "Mr. Kempe (§330) virtually shows that my algebra is perfectly adequate to expressing that A gives B to C; since I can express each of the following relations:

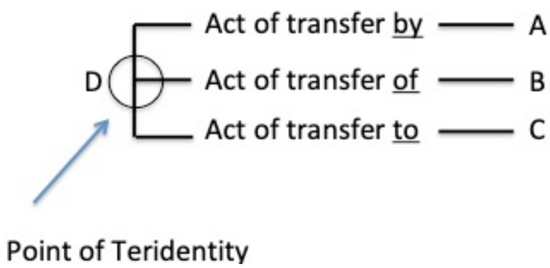
> In a certain act, D, something is given by A;
> In the act, D, something is given to C;
> In the act, D, to somebody is given B." (CP 3.424)

Peirce's point here is that we can only accomplish this reduction of the triadic fact to sets of dyads by "adding to the universe of concrete things the abstraction 'this action'" (ibid).

Brunning further clarifies things when she considers how we might graph the three dyads above in EG. They would be graphed in the following way (Brunning, 260):



In this diagram, we see that *not only* have we added a new term (D) to the logical universe; but *the very term we have added represents a triadic act*. So triadicity has not been analyzed away; it has merely shown up in a new way. When this is graphed in EG, the triadicity is made explicit through the point of teridentity (identified below).



In this discussion, we ought to recall our earlier discussion of *Memoranda*, and Peirce's statement to James, in 1909:

" 'I find myself bound, in a way which I discovered in the sixties, to recognize that there are concepts which, however we may attempt to analyze them, will always be found to enter intact into one or the other or both of the components into which we may fancy that we have analyzed them' (NEM 3:851)"

In this example above, it seems that the statement that 'A gives B to C' is reduced, or translated, into three sets of dyads, but that translation is only possible by adding a new term to the domain, which is *itself* a triadic term. So triadicity "enter[s] intact" into the translated expression. The only relations in the object-language argument are dyads, but the *metalogical* argument must give an account of what this new object D is. In doing that, it must *itself employ* a triadic relation.

Burch makes a similar point, arguing that any reduction of triadic relations to dyadic relations "involves Thirdness" (Burch, APRT, 118). Later he discusses Quine's and Löwenheim's respective reduction arguments, claiming that, "Quine's methods for constructing relations from relations involve Thirdness throughout. These methods are formulated by using devices like quantification and the identification of free variables" (Burch, 121). Obviously a major question here is how we should precisely define Thirdness. Burch does offer a technical definition, but it involves other technical terms, the explanation of which would take us too far into his argument. But, using notions we have already discussed, perhaps we can make some sense of the idea that "Thirdness", understood logically as triadicity, is involved in quantification and the identification of free variables. As we discussed in Chapter Three, an account (in a metalogical argument) of any proposition—even a monadic one—requires speaking of *three things*—the logical subject, the rhema and their union.

I will not delve fully into Quine's argument in RDP; that must be left for another time. But, my suggestion, which will be elaborated a bit below (with help from Hintikka), is that *any* metalogical proof, like Quine's, that reduces triads to dyads *must itself* employ triadicity. To give a proper proof of this would go beyond my current scope; here I only wish to lay some of the groundwork for what a proof like that *might* look like. In Quine's case, it would require examining the features of the dyadic predicate F in the theory Θ', and considering the meaning of these. Then, of course, one would have to generalize the conclusion to apply to any such metalogical reduction. In talking of meaning, it is unclear whether this claim—that any reduction of triads to dyads must employ triadicity—is a logical claim about metalogical arguments, or whether it is a semiotic claim about metalogical arguments. The answer to that question depends on how one is able to precisify the notion of how triadicity is employed in such a metalogical proof.

There is a significant worry, though. The worry, as I have emphasized before, is that the idea of Thirdness smuggles in metaphysics. In that case, the Reduction Thesis would depend on extra-logical premises, and also would not have the logical thrust that Peirce seems to think it does. That would mean that either (i) we have to give up the idea that metaphysics is built on logic and not vice versa (and Peirce would certainly not want to do that), or (ii) we have to admit that the Reduction Thesis is something extra-logical—e.g. metaphysical or phenomenological.

Perhaps a third view is that the Reduction Thesis is semiotic. As I have explained, for Peirce, the relationship between logic and semiotics is complex. I maintain that it is still an open question whether, with the Reduction Thesis, Peirce is making a *logical* criticism of any metalogical argument that reduces triads to dyads, or a *semiotic* criticism. In answering this question—which is a topic for further research—the pivotal question will always be: How can we make logically precise the idea that reduction of triads to dyads employs triadicity? In this project—which is left for a future time—we should keep in mind, however, that, while "Thirdness" is an extra-logical notion, triadicity is a strictly logical notion.

Corollarial and Theorematic Deduction

One possibility for making this idea more precise is offered by Jaakko Hintikka. In the discussion of Hintikka that follows, I aim simply to gesture toward a possible method that *might* be promising in making Peirce's metalogical critique precise.

Hintikka considers Peirce's "first real discovery", which is the distinction between two different forms of deductive reasoning—corollarial deduction and theorematic deduction:

"*Corollarial deduction* is where it is only necessary to imagine any case in which the premises are true in order to perceive immediately that the conclusion holds in that case…*Theorematic deduction* is deduction in which it is necessary to experiment in imagination upon the image of the premiss in order for the result of such experiment to make corollarial deductions to the truth of the conclusion' (Eisel [NEM], vol. 4, p. 38) (Hintikka, 107).

Elsewhere, Peirce characterizes Theorematic deduction as a process of reasoning where, from a "general condition" certain results follow, but "not without imagining something more than what the condition supposes to exist" (NEM 4.288-9) (Hintikka, 108). These definitions, in speaking of the imagination sound decidedly psychologistic and imprecise. But Hintikka (following Peirce) provides a way to make it logically precise.

Hintikka points out that the distinction of these two forms of reasoning originates with Euclid, and was a distinction that geometers commonly drew in Peirce's time. Peirce was a great admirer of Euclid's *Elements*, and studied it closely; praising it for the value it could have for logic (e.g. EP 2.301). Peirce observed that Euclid's proofs often involve "constructions" in preparation for the demonstration, and steps in the demonstration that refer to these constructions (CP 4.616). Thus, in geometry, as embodied in the *Elements*, "we can say that geometrical reasoning is corollarial if no constructions (no further or auxiliary constructions) are needed in it, and theorematic if they are indispensible" (Hintikka, 109). Peirce's "brilliant insight" is to generalize the distinction from geometry to all deductive reasoning (ibid).

However, there is a dominant school of thought (Hintikka cites Russell as the best known proponent), which holds that these constructions involved in a proof are merely a heuristic, which is only indispensible if the geometric system is not fully axiomatized.

However, Hintikka praises "the sharpness of Peirce's insight [in realizing their] mistake" (ibid, 110). He goes on,

"What makes a deduction theorematic according to Peirce is that in it we must envisage other individuals than those needed to instantiate the premise of the argument. The new individuals do not have to be visualized, as the geometrical objects introduced by an Euclidean construction are. They have to be mentioned and considered in the argument however" (ibid, 110).

As we have already noted in Chapter Three, Peirce saw all reasoning—at least, all reasoning in which one could conceivably doubt that the conclusion follows validly from the premises—as requiring diagrams to exhibit the relations involved in the argument. Though his invention of EG came far after his discovery of this distinction, there is clearly a consistent narrative thread.

Hintikka goes on to ask, "how are such new individuals introduced?" (ibid, 110). His answer is either through introduction of individual constants in the proof (through steps of instantiation), or introduction of a new layer of quantifiers. He says,

"each new layer of quantifiers adds a new individual (geometrical object) to the configurations of individuals we are considering…a valid deductive step is theorermatic if it increases the number of layers of quantifiers in the propositions in question" (Hintikka, 110).

Hintikka also claims that this corresponds precisely to his own distinction "between non-trivial and trivial logical arguments (surface tautologies and depth tautologies)" (ibid). He goes on later to discuss instantiation, which would make use of new individual constants (ibid, 111). We can see how this happens when we convert "traditional semi-formal geometrical argument[s]" into proofs in modern logic (ibid, 111). In that case, wherever the traditional geometric proof adds a new geometrical object, the modern logical proof exhibits a step of instantiation. Thus, "what Peirce realized was in effect that the geometrical distinction does not disappear even when geometrical arguments are 'formalized'" (ibid, 111).[140] It may be that in a proof, multiple individual constants are

[140] Hintikka's paper provides additional points of interest, suggesting how "we can turn theorematicity into a matter of degree" by measuring the number of new individuals introduced and/or the number of layers of quantifiers. This goes beyond Peirce since "Peirce does not envisage such a quantification of his distinction" (Hintikka, 113). He also points towards Rantala's urn models as a related advance in how to explicitly measure deductive information (ibid) and ends the paper by gesturing toward "a 'model

required. For example, the proof of the following sequent in Gentzen's NK requires the use of two individual constants: $\forall x\ (Fx \rightarrow \forall yGy) \vdash_{NK} \forall x\forall y\ (Fx \rightarrow Gy)$. "Note that use of a new individual constant…is essential" (Forbes, IML, 44). Since two individual constants are required in the proof, an additional (unstated) conclusion seems to be that there are at least two things in the domain.

I believe Peirce's discovery also has important relevance for the Reduction Thesis, though I leave it open whether this relevance is only by way of analogy, or something more significant. Considered analogically, Hintikka (articulating Peirce's own distinction) shows a way that we can make precise the idea that a proof *must* make use of certain features. In the case of identifying when Deduction is theorematic vs. corollarial, those features are the new individual objects, which are identified explicitly through layers of quantifiers, new free variables or new individual constants. In the case of the Reduction Thesis, we would aim to look for some necessary feature of a metalogical reduction that exhibits triadicity.

But perhaps this very idea of layers of quantifiers, free variables or individual constants is exactly what Peirce had in mind. After all, De Morgan held that, "in the consideration of the proposition, *identification* of objects is in truth a relation of concepts" (DeMorgan 214-215). We have already explained how identification of objects in logic either comes indefinitely—through an operation like relative product, or existential quantification—or particularly—through use of individual constants attached to predicates. If triadic relations are required (in any metalogical proof) to refer to those steps of determination, then perhaps that is the crux of proving the Reduction Thesis in the general way that Peirce was aiming at.

Concluding Remarks

I have argued that the restriction on conceptual combination where only two loose ends can be joined is not arbitrary. This is because in conceptual combination, the two rhemata being joined mutually determine one another, in the same way that logical

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theory of proof theory,' that is, a systematic model-theoretic interpretation of the basic proof-theoretic concepts and results" (ibid, 116). Perhaps such a project would unify the two disparate traditions in formal logic.

subject and predicate mutually determine one another in the formation of any proposition. On being joined, it becomes determinate that *this* logical subject is included in the logical breadth of the predicate, and that *this* logical predicate (rhema) is included in the logical depth of the subject. But this idea that concepts mutually determine one another is obfuscated if we take a God's-eye-view and imagine the referents of relations as already fully determinate. In actual reasoning, this is not how reference works. But if we understand reference in the more nuanced sense, where a subset of a class functions as an index of the entire class, and *stands* for it, then we can begin to see how logical depth and breadth become more determinate and how knowledge grows.

This scheme is also tied to the Reduction Thesis. In this view of conceptual combination and determination, we can see how relations involve three things—that is to say, an account of this sort of combination must refer to the two things being combined as well as their combination in a new whole—the proposition, which is capable of being true or false. When conceptual combination is restricted in this way in a formal system like EG (where only two loose ends can be joined, and when lines of identity can only branch in threes, which is related to this view), the metalogical result is that the Reduction Thesis holds in that system. If one is convinced that this view of relations and of conceptual combination captures something true about our reasoning, then it is very significant that the Reduction Thesis holds as a result. But the Reduction Thesis may have even broader import. If it shows that any metalogical proof that purports to reduce triads to dyads itself must employ triads in the metalogical proof itself, then that would be an even more compelling reason to think that the division of relations into monads, dyads, and triads captures something logically primitive. I believe this is how Peirce understood the Reduction Thesis, and this insight is something he clued into very early in his career. That proof must be left for some other project. But if there are three logically primitive forms of predication, then it seems to me just a metaphysical corollary that there are three metaphysical categories. If this is true, then Peirce's Pragmatic Realism becomes a compelling metaphysical and scientific view, with important relevance to contemporary metaphysics and philosophy of science.

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