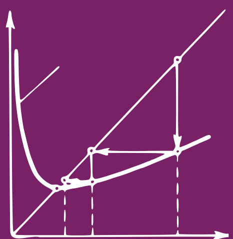

Learn Limits Through Problems!

S. I. Gelfand M. L. Gerver A. A. Kirillov
N. N. Konstantinov A.G. Kushnirenko

Workbook 2

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LEARN LIMITS THROUGH PROBLEMS!

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Contents

Preface

This is the second work book in the Pocket Mathematical Library. It is essentially a programmed text inviting you to learn about limits (a key concept of modern mathematics) by solving a series of 56 problems and reading a little interspersed text. The problems are for the most part equipped with hints and answers (or both), enough for you to get the hang of them (harder problems are marked with asterisks). Moreover, the problems fall into three groups. The first, called "Preliminaries," puts you into the right frame of mind for absorbing the limit concept. The second, called "Concepts," presents the irreducible amount of theoretical material needed to understand limits. The third, called "Calculations," shows you how to evaluate limits once you know what they are.

The core of the book is really the section called "Solutions," where all 56 problems are worked out in full detail. This section should be read carefully after you have tried solving the problems on your own. Please do not give up too soon, since this will only defeat the purpose of the book.

When you are satisfied that you have mastered the subject matter of the book, try solving the 11 problems in the section called "Test Problems." These problems make up a little open-book examination, on which you should easily get a passing grade. Otherwise figure out where things went wrong and fill in the gaps in your knowledge. Don't despair, because nobody finds the notion of a limit easy the first time around. Bon voyage!

Problems

Group 1 (Preliminaries)

1. As a classroom project, two students keep a calender of the weather, according to the following scheme: Days on which the weather is good are marked with the sign $+$, while days on which the weather is bad are marked with the sign $-$. The first student makes three observations daily, one in the morning, one in the afternoon and one in the evening. If it rains at the time of any of these observations, he writes $-$, but otherwise he writes $+$. The second student makes observations at the same times as the first student, writing $+$ if the weather is fair at any of these times and $-$ otherwise. Thus it would seem that the weather on any given day might be described as $++$, $+-$, $-+$ or $--$ (the first symbol made the first student, the second symbol by the second student). Are these four cases all actually possible?

Answer. The case $+-$ is impossible. The others are possible.

2. Suppose a third student joins the two who are making the weather calendar described in the preceding problem. This student makes observations at the same times as the other two, but he writes $-$ if it is raining at the times of at least two observations and $+$ otherwise. Which of the eight cases $+++$, $++-$, $+-+$, $-++$, $-+-$, $--+$, $+--$, $---$ (the third symbol due to the third student, the others as before) are actually possible?

Answer. The cases $+++$, $-++$, $-+-$ and $---$ are possible.

3. Three hundred men are arranged in 30 rows and 10 columns. The tallest man is chosen from each row and then the shortest man is chosen from these 30 men. On another occasion, the shortest man is chosen from each column and then the tallest man is chosen from these 10 men. Who is taller, the tallest of the short men or the shortest of the tall men?

Answer. The shortest of the tall men.

4. Several groups of mathematics students take the same examination. The examination is said to be "easy" if there is a student in each group who solves all the problems. Define a "difficult" examination.

Hint. "Difficult" means the same thing as "not easy."

5. Consider the following definitions of an "easy" examination (taken by several groups of students):

1) Every problem is solved by at least one student in each group;
2) At least one student in each group solves all the problems.

Can the examination be easy in the sense of definition 1 and difficult in the sense of definition 2?

Answer. Yes.

6. Which of the following theorems is true?

1) If each term of a sum is divisible by 7, then the sum is divisible by 7.
2) If none of the terms of a sum is divisible by 7, then the sum is not divisible by 7.
3) If at least one term of a sum is divisible by 7, then each of its terms is divisible by 7.
4) If a sum is divisible by 7, then each of its terms is divisible by 7.
5) If a sum is not divisible by 7, then none of its terms is divisible by 7.
6) If a sum is not divisible by 7, then at least one of its terms is not divisible by 7.

Answer. Theorems 1 and 6 are true, the rest are false.

**7.* Let A and B denote any two statements, and let $\bar{A}$ and $\bar{B}$ denote their negatives. For example, if A is the statement "It is raining," then $\bar{A}$ is the statement "It is not raining." Consider the following 8 theorems:

1) A implies B;
2) $\bar{A}$ implies B;
3) A implies $\bar{B}$;
4) $\bar{A}$ implies $\bar{B}$;
5) B implies A;
6) $\bar{B}$ implies A;
7) B implies $\bar{A}$;
8) $\bar{B}$ implies $\bar{A}$.

Suppose Theorem 1 is known to be true. Then divide the other three theorems into three groups,

1) True theorems;
2) False theorems;
3) Indeterminate theorems, i.e., theorems which may be either true or false. (Here we agree not to let A and B be statements which are always true like "The medians of a triangle intersect in a single point" or always false like "All the angles of a triangle are right angles.")

Answer. Theorem 8 is true, Theorems 2, 3, 6 and 7 are false, and Theorems 4 and 5 are indeterminate.

8. The quantity $|x|$, called the *absolute value* of x, is defined by

$$|x| = \begin{cases} x \text{ if } x > 0, \\ 0 \text{ if } x = 0, \\ -x \text{ if } x < 0. \end{cases}$$

Solve the inequalities

a) $x + 2|x| = 3$; b) $x^2 + 3|x| - 4 = 0$;

c) $|2x + 2| + |2x - 1| = 2$.

Hint. a, b) Consider the cases $x \geqslant 0$ and $x < 0$ separately; c) Consider the cases $x \leqslant -\frac{1}{2}$, $-\frac{1}{2} < x < \frac{1}{2}$ and $x \geqslant \frac{1}{2}$ separately.

Answer. a) $x = 1, \ -3$; b) $x = 1, \ -1$; c) x is any number in the closed interval $[-\frac{1}{2}, \frac{1}{2}]$.†

9. Prove the inequalities

a) $|x + y| \leqslant |x| + |y|$; b) $|x - y| \geqslant |x| - |y|$;

c) $|x - y| \geqslant ||x| - |y||$.

In each case find out when the inequality becomes an equality.

Answer. Equality is possible in case a) only if $x \geqslant 0, y \geqslant 0$ or $x \leqslant 0, y \leqslant 0$, in case b) only if $x \geqslant y \geqslant 0$ or $x \leqslant y \leqslant 0$ and in case c) only if $x \geqslant 0, y \geqslant 0$ or $x \leqslant 0, y \leqslant 0$.

10. Does there exist a positive integer n such that

a) $\sqrt[n]{1000} < 1.001$; *b) $\sqrt[n]{n} < 1.001$;

c) $\sqrt{n + 1} - \sqrt{n} < 0.1$; d) $\sqrt{n^2 + n} - n < 0.1$?

Answer. Yes in cases a), b) and c), no in case d).

11. Does there exist a number C such that

$$\left|\frac{k^2 - 2k + 1}{k^2 - 3}\right| < C$$

for every integer k?

Answer. Yes.

**12.* Is it true that for any number C there are infinitely many integers k for which the inequality

$$k \sin k > C$$

holds?

Answer. Yes.

Let $\{x_n\}$ be an infinite sequence.† Suppose the points of the sequence are plotted along the real line. (It may turn out, of course, that several terms of the sequence correspond to the

---

† By the *(closed) interval* $[a, b]$ where $b > a$ is meant the set of all x such that $a \leqslant x \leqslant b$. The numbers a and b are called the *end points* of the interval.

same point; for example, in the sequence $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \dots$ all the terms with odd subscripts lie at the point 1.) An interval $[a, b]$ on the real line will be called a "trap" for the sequence $\{x_n\}$ if only a finite number of terms of $\{x_n\}$ (possibly no terms at all) lie outside $[a, b]$, while an interval $[a, b]$ will be called a "lure" for $\{x_n\}$ if an infinite number of terms of $\{x_n\}$ lie inside $[a, b]$ (see Fig. 1).

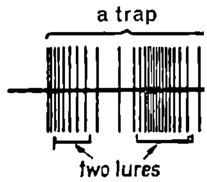


Fig. 1

13. a) Prove that every trap is a lure.

b) Find a sequence $\{x_n\}$ and an interval which is a lure but not a trap for $\{x_n\}$.

Hint. b) See the next problem.

14. Consider the sequences

a) $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots,$

b) $1, 2, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{3}, \dots, \frac{1}{n}, \frac{n+1}{n}, \dots,$

c) $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \dots, 2n-1, \frac{1}{2n}, \dots$

and the intervals

A) $[-\frac{1}{2}, \frac{1}{2}]$,

B) $[-1, 1]$,

C) $[-2, 2]$.

Which intervals are traps or lures for which sequences?

15. Does there exist a sequence for which each of the intervals $[0, 1]$ and $[2, 3]$ is

a) *A* lure; b) *A* trap?

Answer. a) Yes; b) No.

† $\{x_n\}$ is shorthand for $x_1, x_2, \dots, x_n \dots$ The terms *sequence* and *infinite sequence* are synonymous.

16. Suppose each of the intervals [0, 1] and [9, 10] is a lure for a certain sequence. Does this sequence have

a) A trap of length 1;
b) A trap of length 9?

Answer. a) No; b) Perhaps.

17. a) Is there a sequence with no lures at all?

*b) Is there a sequence for which every interval is a lure?

Answer. a) Yes; b) Yes.

Group 2 (Concepts)

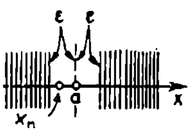


Fig. 2

A number a is said to be the *limit* of a sequence $\{x_n\}$ if given any positive ε (the Greek letter "epsilon"), there is a number k such that

$$|x_n - a| < \varepsilon$$

for all n greater than k (see Fig. 2). The fact that a is the limit of the sequence $\{x_n\}$ is expressed by writing

$$\lim_{n\to\infty} x_n = a$$

(read "a is the limit of x_n as n approaches infinity") or

$$x_n \to a \text{ as } n \to \infty$$

(read "x_n approaches a as n approaches infinity).

✓ *18.* Consider the sequences with general terms

a) $x_n = \dfrac{1}{n}$,

b) $x_n = \dfrac{1}{n^2 + 1}$,

c) $x_n = (-\frac{1}{2})^n$,

d) $x_n = \log_n 2$

($\log_n$ denotes the logarithm to the base n). In each case find a number k such that the following inequalities hold for $n > k$:

A) $|x_n| < 1$,

B) $|x_n| < 0.001$,

C) $|x_n| < 0.000001$.

19. a) Prove that if $x_n \to a$ as $n \to \infty$, then any interval with midpoint a is a trap for the sequence $\{x_n\}$.
b) Is the converse true?

Answer. b) Yes.

**20.* a) Prove that if $x_n \to a$ as $n \to \infty$, then any interval with midpoint a is a lure for the sequence $\{x_n\}$ while no interval failing to contain a is a lure for $\{x_n\}$.
b) Suppose any interval with midpoint a is a lure for a given sequence $\{x_n\}$ while no interval failing to contain a is a lure for $\{x_n\}$. Can it be asserted that $x_n \to a$ as $n \to \infty$?

Answer. b) No, as shown by sequence (c of Problem 14.

21. Prove that if an interval I is a lure for a sequence $\{x_n\}$, then no number outside I can be a limit of $\{x_n\}$.

Hint. Use the result of Problem 15b.

✓ *22.* Find the limit (if any) of the each of the following sequences:

a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, \frac{1}{2n-1}, -\frac{1}{2n}, \ldots;$

b) $0, \frac{2}{3}, \frac{8}{9}, \frac{26}{27}, \ldots, \frac{3^n - 1}{3^n}, \ldots;$

c) $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8},$

$1 + \frac{1}{2} + \cdots + \frac{1}{2^n}, \ldots;$

d) $1, 2, 3, 4, \ldots, n, \ldots;$

e) $1, 1, 1, 1, \dots, 1, \dots;$

f) $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \dots, 0, \frac{1}{n}, \dots;$

g) $0.2, 0.22, 0.222, 0.2222, \dots, 0.\underbrace{22\dots2}_{n \text{ times}}, \dots;$

h) $\sin 1°, \sin 2°, \sin 3°, \dots, \sin n°, \dots;$

i) $\frac{\cos 1°}{1}, \frac{\cos 2°}{2}, \frac{\cos 3°}{3}, \dots, \frac{\cos n°}{n}, \dots;$

j) $0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}, \dots, (-1)^n + \frac{1}{n}, \dots?$

Answer. a) 0; b) 1; c) 2; d) No limit; e) 1; f) 0; g) 2/9; h) No limit; i) 0; j) No limit.

23. Can two different numbers be limits of the same sequence?

Answer. No (cf. Prob. 21).

24. A number a is said to be a *limit point* of a sequence $\{x_n\}$ if given any positive number ε and any number k, there is an integer $n > k$ such that

$$|x_n - a| < \varepsilon.$$

a) Prove that if a is a limit point of $\{x_n\}$, then any interval with midpoint a is a lure for $\{x_n\}$.

b) Prove the converse theorem.

Hint. b) What does it mean to say that a is not a limit point of $\{x_n\}$?

25. Prove that the limit of a sequence $\{x_n\}$ (if such exists) is a limit point of $\{x_n\}$.

26. Find all the limit points of each of the following sequences:

a) $2, \frac{3}{2}, \frac{4}{3}, \dots, \frac{n+1}{n}, \dots;$

b) $-1, 1, -1, \dots, (-1)^n, \dots;$

c) $\sin 1°, \sin 2°, \sin 3°, \ldots, \sin n°, \ldots$;

d) $-1, \frac{1}{2}, -\frac{1}{3}, \ldots, n^{(-1)^n}, \ldots$;

e) $1, 2, 3, \ldots, n, \ldots$;

f) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$

Answer. a) 1; b) 1, -1; c) 0, $\pm\sin 1°$, $\pm\sin 2°$, ..., $\pm\sin 89°$, ± 1;
d) 0; e) No limit points; f) Every point in the interval [0, 1].

27. A sequence $\{x_n\}$ is said to be *bounded* if there is a positive number C such that $|x_n| \leqslant C$ for all n. Define an unbounded sequence.

28. a) Prove that a sequence is bounded if it has a limit.
b) Is the converse true?

Answer. b) No (cf. Prob. 14b).

✓ *29.* A sequence $\{x_n\}$ is said to *approach infinity* (a fact expressed by writing $x_n \to \infty$ as $n \to \infty$) if given any positive number C, there is a number k such that $|x_n| > C$ for all $n > k$. Which of the sequences $\{x_n\}$ approach infinity and which are unbounded if

a) $x_n = n$;

b) $x_n = (-1)^n n$;

c) $x_n = n^{(-1)^n}$;

d) $x_n = \begin{cases} n \text{ for even } n, \\ \sqrt{n} \text{ for odd } n; \end{cases}$

e) $x_n = \dfrac{100n}{100 + n^2}$?

Answer. a) $x_n \to \infty$; b) $x_n \to \infty$; c) $\{x_n\}$ is unbounded; d) $x_n \to \infty$; e) $\{x_n\}$ is bounded.

✓ *30.* Find a bounded sequence $\{x_n\}$ which has

a) A largest and a smallest term;

b) A largest but no smallest term;

c) A smallest but no largest term;
d) Neither a largest nor a smallest term.

**31.* Consider the following 16 assertions, where $\exists$ means "there exists," $\forall$ means "for any" and $\ni$ means "such that":

1) $\exists \varepsilon > 0\, \exists k\, \exists n > k \ni |x_n - a| < \varepsilon$;

2) $\exists \varepsilon > 0\, \exists k\, \exists n > k \ni |x_n - a| \geqslant \varepsilon$;

3) $\exists \varepsilon > 0\, \exists k\, \forall n > k \ni |x_n - a| < \varepsilon$;

4) $\exists \varepsilon > 0\, \exists k\, \forall n > k \ni |x_n - a| \geqslant \varepsilon$;

5) $\exists \varepsilon > 0\, \forall k\, \exists n > k \ni |x_n - a| < \varepsilon$;

6) $\exists \varepsilon > 0\, \forall k\, \exists n > k \ni |x_n - a| \geqslant \varepsilon$;

7) $\exists \varepsilon > 0\, \forall k\, \forall n > k \ni |x_n - a| < \varepsilon$;

8) $\exists \varepsilon > 0\, \forall k\, \forall n > k \ni |x_n - a| \geqslant \varepsilon$;

9) $\forall \varepsilon > 0\, \exists k\, \exists n > k \ni |x_n - a| < \varepsilon$;

10) $\forall \varepsilon > 0\, \exists k\, \exists n > k \ni |x_n - a| \geqslant \varepsilon$;

11) $\forall \varepsilon > 0\, \exists k\, \forall n > k \ni |x_n - a| < \varepsilon$;

12) $\forall \varepsilon > 0\, \exists k\, \forall n > k \ni |x_n - a| \geqslant \varepsilon$;

13) $\forall \varepsilon > 0\, \forall k\, \exists n > k \ni |x_n - a| < \varepsilon$;

14) $\forall \varepsilon > 0\, \forall k\, \exists n > k \ni |x_n - a| \geqslant \varepsilon$;

15) $\forall \varepsilon > 0\, \forall k\, \forall n > k \ni |x_n - a| < \varepsilon$;

16) $\forall \varepsilon > 0\, \forall k\, \forall n > k \ni |x_n - a| \geqslant \varepsilon$.

Interpret these assertions in terms of familiar properties of sequences or their negatives.

Answer. 1) Every sequence $\{x_n\}$ has this property; 2) Not every term of $\{x_n\}$ equals a; 3) The sequence $\{x_n\}$ is bounded; 4) The point a is not a limit point of $\{x_n\}$; 5) The sequence $\{x_n\}$ does not approach infinity; 6) The number a is not the limit of $\{x_n\}$;

7) The sequence $\{x_n\}$ is bounded; 9) The number a is either one of the terms of the sequence or a limit point of the sequence.

The remaining assertions are the negatives of those just given. In fact, assertion k is the negative of assertion $17 - k$. For example, assertion 10 says that assertion 7 does not hold, i.e., the sequence is unbounded, assertion 15 says that assertion 2 does not hold, i.e., all the terms of the sequence equal a, and so on.

32. Consider the following five properties of a sequence $\{x_n\}$:

a) $x_n = a$ for all n;
b) a is the limit of $\{x_n\}$;
c) a is a limit point of $\{x_n\}$;
d) $\{x_n\}$ is bounded;
e) $\{x_n\}$ approaches infinity.

Correspondingly, every sequence $\{x_n\}$ can be characterized by an array of five plus or minus signs. For example, $-+++-$ means that the sequence has properties 2, 3 and 4, but not properties 1 and 5. Some arrays are meaningless. For example, $+++++$ makes no sense, since a sequence obviously cannot have both properties 1 and 5.

a) Write all meaningful arrays of plus and minus signs, and in each case write a sequence $\{x_n\}$ characterized by the array.
b) Prove that the remaining sets are meaningless.

Answer. a) $++++-,\ x_n = a;$

$$-+++-,\ x_n = a + \frac{1}{n};$$

$$--++-,\ x_n = a + 1 + (-1)^n;$$

$$--+--,\ x_n = a + n\,[1 + (-1)^n];$$

$$---+-,\ x_n = a + (-1)^n;$$

$$----+,\ x_n = n;$$

$$-----,\ x_n = a + 1 + n\,[1 + (-1)^n].$$

A sequence $\{x_n\}$ is said to be *increasing* if

$$x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_n \leqslant \ldots, \tag{1}$$

decreasing if

$$x_1 \geqslant x_2 \geqslant x_3 \geqslant \cdots \geqslant x_n \geqslant \ldots, \tag{2}$$

and *monotone* if either (1) or (2) holds. Similarly, $\{x_n\}$ is said to be *strictly increasing* if

$$x_1 < x_2 < x_3 < \cdots < x_n < \ldots, \tag{3}$$

strictly decreasing if

$$x_1 > x_2 > x_3 > \cdots > x_n > \ldots, \tag{4}$$

and *strictly monotone* if either (3) or (4) holds.

33. Prove that if a sequence $\{x_n\}$ has a limit, then there are just three possibilities:

a) $\{x_n\}$ has a largest term but no smallest term;
b) $\{x_n\}$ has a smallest term but no largest term;
c) $\{x_n\}$ has both a largest and a smallest term.

Give examples illustrating each case.

Hint. If $\{x_n\}$ has neither a largest nor a smallest term, find both a strictly increasing subsequence and a strictly decreasing subsequence of $\{x_n\}$.†

Answer. a) $x_n = \dfrac{1}{n}$; b) $x_n = \dfrac{n-1}{n}$; c) $x_n = \left(-\dfrac{1}{2}\right)^n$.

34. Prove that any sequence contains a monotone subsequence.

---

† Given a sequence $x_1, x_2, \ldots, x_n, \ldots$, by a *subsequence* of $\{x_n\}$ is meant any sequence of the form $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$, where $n_1, n_2, \ldots, n_k, \ldots$ is a strictly increasing sequence of positive integers (so that $n_1 < n_2 < \cdots < n_k < \cdots$). For example, $x_2, x_4, \ldots, x_{2k}, \ldots$ and $x_1, x_3, \ldots, x_{2k-1}, \ldots$ are subsequences of $x_1, x_2, \ldots, x_n, \ldots$, the first consisting of the even-numbered terms of $\{x_n\}$, the second of the odd-numbered terms.

The following important property of the real numbers, usually taken as an axiom, plays an important role in the theory of limits:

Bolzano-Weierstrass axiom. Every bounded monotone sequence has a limit.

This axiom expresses the "completeness" property of the real numbers, which, figuratively speaking, means that there are no "gaps" or "holes" in the number line.

Remark. In a course on mathematical analysis, it is usually shown that the Bolzano-Weierstrass axiom is equivalent to each of the following two assertions:

1) Let $I_1, I_2, I_3, \ldots, I_n, \ldots$ be an infinite sequence of closed intervals, each containing the next (thus I_1 contains I_2, I_2 contains I_3, and so on). Then the intervals have at least one point in common.
2) Every real number can be written as an infinite (repeating or nonrepeating) decimal, and every such decimal corresponds to a real number.

If either of these assertions is taken as an axiom, the other assertion and the Bolzano-Weierstrass axiom become *theorems* which can be proved.

**35.* Prove that the Bolzano-Weierstrass axiom does not hold if only rational numbers are considered, i.e., prove that there exists a bounded monotone sequence of rational numbers which does not have a *rational* limit.

Hint. Show that the sequence

$$1.4,\ 1.41,\ 1.414,\ 1.4142,\ \ldots$$

of decimal approximations to $\sqrt{2}$ (from below) is bounded and monotone, but does not have a rational limit.

36. Prove that every bounded sequence has at least one limit point.

Hint. Use the result of Problem 34 and the Bolzano-Weierstrass axiom.

37. Prove that each of the following sequences has a limit:

a) $1,\ 1 + \frac{1}{4},\ 1 + \frac{1}{4} + \frac{1}{9},\ \ldots,\ 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2},\ \ldots;$

b) $1,\ 1 - \frac{1}{3},\ 1 - \frac{1}{3} + \frac{1}{5},\ \ldots,$

$$1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^{n-1}}{2n - 1},\ \ldots$$

Hint. a) Use the inequality

$$\frac{1}{n^2} < \frac{1}{n(n-1)}$$

to prove that the sequence is bounded; b) Examine separately the subsequences consisting of the even and odd-numbered terms.

Group 3 (Calculations)

38. Prove that if $\lim\limits_{n\to\infty} x_n = a$ and $\lim\limits_{n\to\infty} y_n = b$, then

a) $\lim\limits_{n\to\infty} (x_n + y_n) = a + b;$

b) $\lim\limits_{n\to\infty} (x_n - y_n) = a - b;$

c) $\lim\limits_{n\to\infty} x_n y_n = ab;$

d) $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}$ if $b \neq 0$ and $y_n \neq 0$.

39. Find sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim\limits_{n\to\infty} x_n = 0$, $\lim\limits_{n\to\infty} y_n = 0$ and

a) $\frac{x_n}{y_n} \to 0$ as $n \to \infty$;

b) $\dfrac{x_n}{y_n} \to 1$ as $n \to \infty$;

c) $\dfrac{x_n}{y_n} \to \infty$ as $n \to \infty$;

d) $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n}$ does not exist.

40. Find the limit of the sequence $\{x_n\}$ where

a) $x_n = \dfrac{2n+1}{3n-5}$;

b) $x_n = \dfrac{10n}{n^2+1}$;

c) $x_n = \dfrac{n(n+2)}{(n+1)(n+3)}$;

d) $x_n = \dfrac{2^n+1}{2^n-1}$.

Answer. a) $\frac{2}{3}$; b) 0; c) 1; d) 1.

**41*. Find the limit of the sequence $\{x_n\}$ if

$$x_n = \frac{1}{n^{k+1}}(1^k + 2^k + \cdots + n^k),$$

where k is a fixed positive integer.

Hint. First use mathematical induction in k to show that $1^k + 2^k + \cdots + n^k$ is a polynomial of degree $k+1$ in n with leading coefficient $\dfrac{1}{k+1}$.

Answer. $\dfrac{1}{k+1}$.

Comment. The sequence $\{x_n\}$ has the following geometric interpretation: Let R be the plane region bounded by the graph

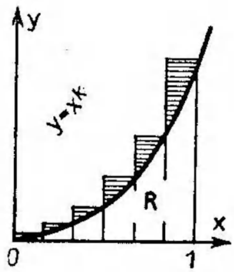


Fig. 3

of the function $y = x^k$, the x-axis and the line $x = 1$. Divide the interval [0, 1] of the x-axis into n equal segments, and on each segment construct a rectangle such that the upper right-hand vertex lies on the graph (see Fig. 3). Then the sum of the areas of all the rectangles is just

$$x_n = \frac{1}{n^{k+1}} (1^k + 2^k + \cdots + n^k).$$

The limit of this quantity as $n \to \infty$ is by definition the area of R.

42. Prove that

$$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n-1}\right) = 0.$$

Hint. Use the identity

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}.$$

43. Prove that

$$\lim_{n\to\infty} \frac{n}{2^n} = 0.$$

Hint. Write 2^n as $(1 + 1)^n$ and use the binomial theorem.

44. Prove that

$$\lim_{n\to\infty} \frac{n}{a^n} = 0$$

if $a > 1$.

Hint. Write a^n as $[1 + (a - 1)]^n$ and use the binomial theorem.

**45.* Prove that

$$\lim_{n\to\infty} \frac{\log_2 n}{n} = 0.$$

Hint. Use the result of Problem 43.

**46.* Prove that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

So far we have considered the limits of sequences of numbers. But numbers can be used to specify various geometric objects. For example, the direction of a straight line in the plane can be specified by its slope, a point in the plane can be specified by its coordinates, and so on. Thus, whenever the words "limit" and "approaches" are applied to a sequence of geometric objects, we have in mind a sequence of numbers characterizing the objects. For example, "the sequence of points M_n approaches the point M" means that the coordinates of the points M_n approach those of the point M.

47. A snail crawls along a piece of graph paper in the following fashion: Starting from any point of the paper, it moves first one unit to the right, then one unit upward, then another unit to the right, then another unit upward, and so on, as shown in Figure 4 (each little square of the graph paper is 1 unit on a side). A second snail stays still and observes the progress of the first snail through a telescope. Does the direction of the telescope approach a limit?

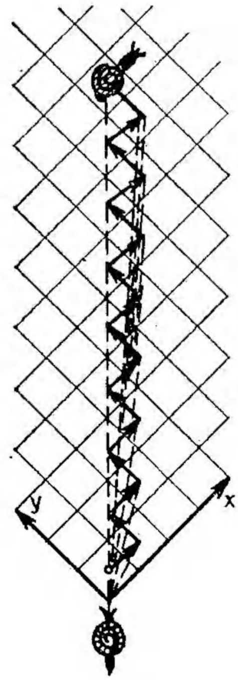


Fig. 4

Hint. Choose the axes of a rectangular coordinate system along two perpendicular lines of the graph paper, with the origin at the position of the second snail. Suppose the first snail starts from the point (a, b). Find the point (a_n, b_n) reached by the snail after n steps. Then

$$k_n = \frac{b_n}{a_n}$$

is the slope of the line along which the telescope points. Show that

$$\lim_{n \to \infty} k_n = 1.$$

48. How does the answer to the preceding problem change if the snail moves as follows:

a) 1 unit to the right, 2 units upward, 1 unit to the right, 2 units upward, and so on;

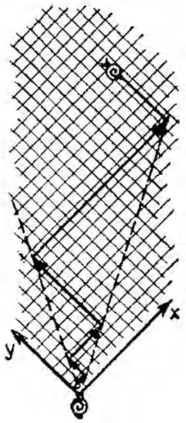


Fig. 5

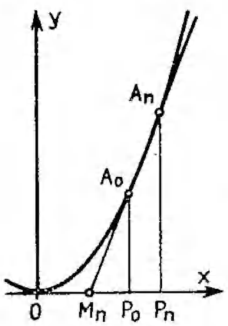


Fig. 6

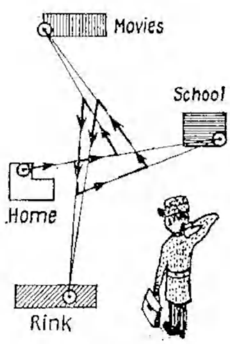


Fig. 7

b) 1 unit to the right, 2 units upward, 3 units to the right, 4 units upward, 5 units to the right, 6 units upward, and so on;
c) 1 unit to the right, 2 units upward, 4 units to the right, 8 units upward, 16 units to the right, 32 units upward, and so on (see Fig. 5)?

Hint. Use the same notation as in the solution to the preceding problem.

Answer. a) $\lim\limits_{n\to\infty} k_n = 2$; b) $\lim\limits_{n\to\infty} k_n = 1$; c) The sequence $\{k_n\}$ has no limit.

49. On the parabola which is the graph of the function $y = x^2$, choose a point A_0 with abscissa a and let A_n be the point with abscissa $a + \dfrac{1}{n}$. Let M_n be the point of intersection of the x-axis with the secant drawn through the points A_0 and A_n (see Fig. 6). Prove that the sequence of points $\{M_n\}$ has a limit M_0 as $n \to \infty$, and find M_0. (The line A_0M_0 is called the *tangent* to the parabola at the point A_0.)

Hint. Drop perpendiculars from the points A_0 and A_n to the x-axis, and let P_0 and P_n be the feet of these perpendiculars (see Fig. 6). Use the fact that the triangles $M_nA_0P_0$ and $M_nA_nP_n$ are similar to calculate the length of the segment M_nP_0.

Answer. The point M_0 is the midpoint of the segment OP_0, where O is the origin of coordinates.

50. Johnny leaves home and heads for school. Halfway to school he decides to play hooky and heads for the movies. After going halfway to the movies, he changes his mind and decides to go skating instead. After heading for the skating rink and going halfway, he feels guilty and decides to go back to school. But after heading back to school and going half of the remaining distance, he again changes his mind and heads for the movies again (see Fig. 7). Describe Johnny's eventual motion if he keeps on changing his mind like this.

Hint. Choose three points denoting the school, the movie theatre and the skating rink. Then construct the polygonal line

$A_1A_2A_3A_4 \ldots$ representing Johnny's path. Compare the segments A_1A_4, A_2A_5, A_3A_6, A_4A_7, etc.

51. Suppose a sequence of points $\{M_n\}$ on a straight line is constructed as follows: The first two points M_1 and M_2 are chosen arbitrarily, and every subsequent point is the midpoint of the segment joining the preceding two points. Prove that $\{M_n\}$ has a limit M. What is the point M?

Answer. M divides the segment M_1M_2 in the ratio 2:1.

By an *infinite series* is meant an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots, \tag{1}$$

where the numbers $a_1, a_2, \ldots, a_n, \ldots$ are the terms of an infinite sequence. Let $S_n = a_1 + a_2 + \cdots + a_n$ be the nth partial sum of (1), i.e., the sum of the first n terms of (1), and suppose the sequence $\{S_n\}$ has a limit S. Then the series (1) is said to *converge* and S is called the *sum* of (1). However, if $\{S_n\}$ has no limit, (1) is said to *diverge* (with no sum at all).

52. Find the nth partial sum S_n and the sum S of each of the following series:

a) $1 + a + a^2 + \cdots + a^n + \cdots;$

b) $a + 2a^2 + 3a^3 + \cdots + na^n + \cdots;$

c) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} + \cdots;$

d) $$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} + \cdots.$$

Hint. b) Write S_n as the sum of n geometric progressions.

Answer.

a) $S_n = \dfrac{1 - a^{n+1}}{1 - a}, \quad S = \dfrac{1}{1 - a}$ if $|a| < 1$.

The series is divergent if $|a| \geqslant 1$.

b) $S_n = \dfrac{a - (n + 1 - na)\, a^{n+1}}{(1 - a^2)^2}$, $\quad S = \dfrac{a}{(1 - a)^2}$ if $|a| < 1$.

The series is divergent if $|a| \geqslant 1$.

c) $S_n = 1 - \dfrac{1}{n + 1}$, $\quad S = 1$.

d) $S_n = \dfrac{1}{2}\left[\dfrac{1}{2} - \dfrac{1}{(n + 1)(n + 2)}\right]$, $\quad S = \dfrac{1}{4}$.

**53.* Prove that the *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent.

Hint. Use the fact that

$$\frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n} > \frac{1}{2}.$$

Fig. 8

54. Given infinitely many identical bricks (shaped like rectangular parallelepipeds), suppose the bricks are laid one on top of another with displacements such that the pile never collapses (see Fig. 8). How long a "roof" can be built up in this way?

Hint. Suppose the bricks are of unit length. Prove by induction that the pile of bricks will not collapse if the $(n + 1)$st brick (counting from the top of the pile) is shifted a distance $1/2n$ relative to the nth brick. Then use the result of the preceding problem.

Answer. A roof of any length whatsoever!

55. Let $\{x_n\}$ be the sequence

$$2,\ 2 + \frac{1}{2},\ 2 + \frac{1}{2 + \frac{1}{2}},\ 2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}},\ \ldots$$

Prove that

$$\lim_{n \to \infty} x_n = 1 + \sqrt{2}.$$

56. The following method of "successive approximations" can be used to calculate the square root of a positive number a: Take any number x_0 and construct a sequence $\{x_n\}$ according to the rule

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Prove that

$$\lim_{n \to \infty} x_n = \sqrt{a}$$

if $x_0 > 0$, while

$$\lim_{n \to \infty} x_n = -\sqrt{a}$$

if $x_0 < 0$ ($\sqrt{a}$ denotes the positive square root of a). How many successive approximations are needed (i.e., how many terms of the sequence $\{x_n\}$ must be calculated) in order to find the value of $\sqrt{10}$ to within 0.00001 if $x_0 = 3$ is chosen as the initial value?

Answer. It is enough to calculate x_2.

Solutions

1. If it doesn't rain at any of the observation times, both students write $+$ and the day is described as $++$. If it rains at just one observation time, the first student writes $-$ and the second $+$, giving the case $-+$. If it rains exactly twice, the first student writes $-$ and the second $+$, giving the case $-+$ again. If it rains at all the observation times, both students write $-$ and the weather is described as $--$. Thus we have accounted for every possible type of weather without the case $+-$ ever occurring.

2. If is doesn't rain at any of the observation times, all three students write $+$ and the day is described as $+++$. If it rains at just one observation time, the first student writes $-$, the second $+$ and the third $+$, giving the case $-++$. If it rains exactly twice, the first student writes $-$, the second $+$ and the third $-$, giving the case $-+-$. Finally, if it rains at all the observation times, all three students write $-$ and the weather is described as $---$. Thus we have accounted for every possible type of weather without the cases $+-+$, $++-$, $+--$ and $--+$ ever occurring.

3. Let A be the shortest of the tall men and B the tallest of the short men. Compare A and B with C, the man in the same row as A and in the same column as B. Since A is the tallest man in his row, he is taller than C, and since B is the shortest man in his column, he is shorter than C. Therefore A is taller than B.

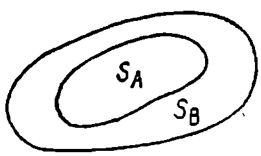
4. An examination is said to be "difficult" if every student in at least one group of students has failed to solve at least one problem.

5. Suppose each student solves only one problem, but every problem is solved by some student in each group. Then the examination is easy in the sense of definition 1, but not in the sense of definition 2.

6. The examples $3 + 4 = 7$ and $2 + 7 = 9$ show that Theorems 2, 3, 4 and 5 are false. Theorem 1 is obviously true, and Theorem 6 is easily proved by contradiction (explained in the solution to the next problem).

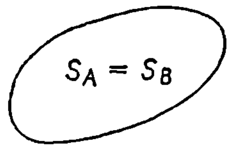
7. If Theorem 1 is true, then Theorem 8 is easily proved by contradiction. In fact, suppose Theorem 8 is false, i.e., suppose $\bar{B}$ does not imply $\bar{A}$. Then $\bar{B}$ can hold without $\bar{A}$ holding, i.e., $\bar{B}$ can hold and so can A. But this contradicts Theorem 1, according to which A implies B. Therefore Theorem 8 must be true.

In the case of Theorems 4 and 5, we can easily find statements A and B for which the theorems are true as well as statements A and B for which the theorems are false. The simplest way of doing this is the following: Let S_A and S_B be two sets of points in the plane, let A be the statement "The point p belongs to the set S_A" and let B be the statement "The point p belongs to the set S_B." Then Theorem 1 says that the set S_A contains the set S_B, Theorem 2 says that the complement of S_A contains S_B, Theorem 3 says that S_A contains the complement of S_B, and so on.† (Formulate the rest of the theorems in this language, and give examples of sets S_A and S_B for which the theorems are true as well as examples for which the theorems are false.) Then



† By the *complement* of a set in the plane we mean the set of all points of the plane which do not belong to the set. For example, the complement of the upper half-plane (including the x-axis) is the lower half-plane (excluding the x-axis).

is an example where Theorem 1 holds but Theorems 4 and 5 do not, while



is an example where Theorems 1, 4 and 5 all hold. In other words, if Theorem 1 is true, then Theorems 4 and 5 are indeterminate.

Finally, we show that if Theorem 1 is true, then Theorems 2, 3, 6 and 7 are false. If Theorems 1 and 2 were both true, then

$$A \xrightarrow{\text{Thm 1}} B \xleftarrow{\text{Thm 2}} \bar{A}$$

where the statement at the tail of the arrow implies the statement at the head of the arrow (because of the theorem labelling the arrow). But then B holds regardless of whether A is true or false, and we have agreed to exclude statements of this kind. Similarly, if Theorems 1 and 3 were both true, then

$$\bar{B} \xleftarrow{\text{Thm 1}} A \xrightarrow{\text{Thm 3}} B,$$

i.e., B is both true and false if A is true, and hence A is always false. But such statements have been excluded from the outset. In the same vein, if Theorems 1 and 6 were both true, then

$$\bar{B} \xrightarrow{\text{Thm 6}} A \xrightarrow{\text{Thm 1}} B,$$

i.e., if B is false, then B is true, and hence B is always true which is impossible. Finally, if Theorems 1 and 7 were both true, then

$$A \xrightarrow{\text{Thm 1}} B \xrightarrow{\text{Thm 7}} \bar{A}.$$

In other words, if A is true, then A is false, and hence A is always false which is impossible.

8. a) First let $x \geqslant 0$. Then $|x| = x$ and hence

$$x + 2|x| = 3 \tag{1}$$

becomes

$$3x = 3$$

with solution $x = 1$. Next let $x \leqslant 0$. Then $|x| = -x$ and (1) becomes

$$-x = 3$$

with solution $x = -3$. Therefore $x = 1$ or $x = -3$.
b) For $x \geqslant 0$ we have $|x| = x$ and hence

$$x^2 + 3|x| - 4 = 0 \tag{2}$$

becomes

$$x^2 + 3x - 4 = 0$$

with solutions $x = 1$ and $x = -4$, but only the first solution $x = 1$ satisfies the condition $x \geqslant 0$. For $x \leqslant 0$ we have $|x| = -x$ and (2) becomes

$$x^2 - 3x - 4 = 0$$

with solutions $x = -1$ and $x = 4$, but only the first solution $x = -1$ satisfies the condition $x \leqslant 0$. Therefore $x = 1$ or $x = -1$.
c) If $x \leqslant -\frac{1}{2}$, then

$$|2x + 1| = -(2x + 1), \quad |2x - 1| = -(2x - 1),$$

and hence

$$|2x + 1| + |2x - 1| = 2 \tag{3}$$

becomes

$$-(2x + 1) - (2x - 1) = 2$$

or

$$x = -\tfrac{1}{2}.$$

Similarly, if $x \geqslant \frac{1}{2}$, then

$$|2x + 1| = 2x + 1, \quad |2x - 1| = 2x - 1$$

and (3) becomes

$$(2x + 1) + (2x - 1) = 2$$

or

$$x = \tfrac{1}{2}.$$

Finally let $-\frac{1}{2} < x < \frac{1}{2}$. Then

$$|2x + 1| = 2x + 1, \quad |2x - 1| = -(2x - 1),$$

and (3) becomes

$$(2x + 1) - (2x - 1) = 2$$

which reduces to the identity

$$2 = 2.$$

Combining all three cases, we find that the solutions of (3) consist of all x such that $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$, i.e., all x in the closed interval $[-\frac{1}{2}, \frac{1}{2}]$.

9. a) If either x or y (or both) equals zero, then

$$|x + y| \leqslant |x| + |y| \tag{1}$$

becomes an obvious equality. Now let $x > 0$, $y > 0$. Then

$$|x| = x, \quad |y| = y, \quad |x + y| = x + y$$

and (1) becomes

$$x + y \leqslant x + y,$$

which obviously holds (with the sign $=$). Next let $x > 0, y < 0$. Then there are two possibilities $x + y \geqslant 0$ and $x + y < 0$. In the first case

$$|x| = x, \quad |y| = -y, \quad |x + y| = x + y,$$

and (1) becomes

$$x + y \leqslant x - y$$

which obviously holds (with the sign $<$) since y is negative. In the second case

$$|x| = x, \quad |y| = -y, \quad |x + y| = -(x + y),$$

and (1) becomes

$$-x - y \leqslant x - y,$$

which obviously holds (with the sign $<$) since x is positive.

The remaining possibilities ($x < 0, y < 0$ and $x < 0, y > 0$) are obtained from those just discussed by changing the sign of both x and y. Since this has no effect on $|x|$, $|y|$ and $|x + y|$, the inequality (1) remains valid. A moment's thought shows that $x \geqslant 0$, $y \geqslant 0$ and $x \leqslant 0$, $y \leqslant 0$ are the only cases in which (1) becomes an equality.

b) As in case a), we could examine the various possibilities obtained by giving x and y different signs, but we might as well use the fact that the inequality (1) has already been proved. Let $z = x - y$, so that $x = y + z$. Then

$$|y + z| \leqslant |y| + |z| \tag{2}$$

becomes

$$|x| \leqslant |y| + |x - y|$$

or

$$|x - y| \geqslant |x| - |y|. \tag{3}$$

Since equality can occur in (2) only if $y \geqslant 0$, $z \geqslant 0$ or $y \leqslant 0$, $z \leqslant 0$, it can occur in (3) only if $x \geqslant y \geqslant 0$ or $x \leqslant y \leqslant 0$.

c) Again we could examine the various possibilities, but it is simpler to use the fact that the inequality (3) has already been proved. If $|x| \geqslant |y|$, the inequality

$$|x - y| \geqslant ||x| - |y|| \tag{4}$$

coincides with (3). However, if $|x| < |y|$, then (4) becomes

$$|x - y| \geqslant |y| - |x|$$

or

$$|y - x| \geqslant |y| - |x|,$$

which is identical with (3) if the roles of x and y are interchanged. Since equality can occur in (3) only if $x \geqslant y \geqslant 0$ or $x \leqslant y \leqslant 0$, it can occur in (4) only if $x \geqslant y \geqslant 0$, $y \geqslant x \geqslant 0$ or $x \leqslant y \leqslant 0$, $y \leqslant x \leqslant 0$, i.e., only if $x \geqslant 0$, $y \geqslant 0$ or $x \leqslant 0$, $y \leqslant 0$.

Remark. A more intuitive solution can be based on the fact that $|x|$ is the distance between the points x and 0 on the real line, while $|x - y|$ is the distance between the points x and y.

In this regard, see "The Coordinate Method" by I.M. Gelfand et al., another title in The Pocket Mathematical Library.

10. a) Writing the inequality

$$\sqrt[n]{1000} < 1.001 \tag{1}$$

in the form

$$1000 < (1 + 0.001)^n$$

and using the binomial theorem to expand the right-hand side, we obtain

$$(1 + 0.001)^n = 1 + \frac{n}{1000} + \frac{n(n-1)}{2 \cdot 1000^2} + \cdots + \frac{1}{1000^n}, \tag{2}$$

from which it is apparent that $(1 + 0.001)^n$ is greater than $1 + \dfrac{n}{1000}$ if $n > 1$. Hence, for sufficiently large n, e.g., for $n = 1{,}000{,}000$, we have

$$1000 < (1 + 0.001)^n.$$

It follows that the original inequality (1) also holds for $n = 1{,}000{,}000$.

b) Just as in part a), we write the inequality

$$\sqrt[n]{n} < 1.001 \tag{3}$$

in the form

$$n < (1 + 0.001)^n$$

and use the binomial theorem to expand the right-hand side. According to (2),

$$(1.001)^n > 1 + \frac{n}{1000} + \frac{n(n-1)}{2 \cdot 1000^2}$$

if $n > 2$. But for sufficiently large n, this expression is larger than n itself. In fact, the last term alone is larger than n if $n - 1 > 2 \cdot 1000^2$. Therefore the inequality

$$n < (1 + 0.001)^n$$

holds for $n = 2 \cdot 1000^2 + 2$, and hence so does the original inequality (3).

c) It follows from the identity

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

that

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}.$$

Hence

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \frac{1}{10}$$

if $n > 25$.

d) There is no positive integer n such that

$$\sqrt{n^2 + n} - n < 0.1. \tag{4}$$

In fact, (4) implies

$$\sqrt{n^2 + n} < n + 0.1$$

and hence

$$n^2 + n < (n + 0.1)^2 = n^2 + \frac{n}{5} + 0.01$$

or

$$n < \frac{n}{5} + 0.01. \tag{5}$$

But (5) is impossible since actually

$$n > \frac{n}{5} + 0.01$$

for every positive integer n.

11. First we note that

$$\left|\frac{k^3 - 2k + 1}{k^4 - 3}\right| = \left|\frac{k^3\left(1 - \dfrac{2}{k^2} + \dfrac{1}{k^3}\right)}{k^4\left(1 - \dfrac{3}{k^4}\right)}\right|$$

$$= \frac{1}{|k|}\left|\frac{\left|1 - \dfrac{2}{k^2} + \dfrac{1}{k^3}\right|}{\left|1 - \dfrac{3}{k^4}\right|}\right|. \tag{1}$$

If $|k| \geqslant 2$, then

$$\left|1 - \frac{2}{k^2} + \frac{1}{k^3}\right| \leqslant 1 + \frac{2}{k^2} + \frac{1}{|k|^3} \leqslant 1 + \frac{1}{2} + \frac{1}{8} < 2,$$

$$\left|1 - \frac{3}{k^4}\right| \geqslant 1 - \frac{3}{k^4} \geqslant 1 - \frac{3}{16} > \frac{1}{2},$$

and hence the right-hand side of (1) is less than

$$\frac{1}{2}\,\frac{2}{\frac{1}{2}} = 2.$$

Therefore the left-hand side of (1) does not exceed 2 in absolute value if $|k| \geqslant 2$. Moreover, for $k = -1, 0, 1$ it takes the values, $1, \frac{1}{3}, 0$, respectively. Hence there is a number C such that

$$\left|\frac{k^3 - 2k + 1}{k^4 - 3}\right| < C$$

for every integer k. In fact, we can choose $C = 2$.

Remark. Actually, the same estimates made a bit more carefully show that we can choose $C = 1$, since the left-hand side of (1) takes its largest value (equal to 1) for $k = -1$ (prove this).

12. Consider the product $k \sin k$. The first factor can be made arbitrarily large, and then the product will be large if the second factor is not too small. For example, suppose $\sin k$ is larger than $\frac{1}{2}$. The set of all points x such that $\sin x > \frac{1}{2}$ consists of infinitely many intervals of the form

$$2\pi n + \frac{\pi}{6} < x < 2\pi n + \frac{5\pi}{6}, \tag{1}$$

where n is an arbitrary integer (see Fig. 9) and each interval is of length $2\pi/3$. Since this length exceeds 1, there is at least one integer in each such interval. It follows that given any number C, there are infinitely many integers for which k sin $k > C$. In fact, sin $k > \frac{1}{2}$ for every k belonging to an interval (1), and hence if k exceeds $2C$ and belongs to such an interval, we have

$$k \sin k = 2C \cdot \tfrac{1}{2} = C.$$

Obviously, there are infinitely many such integers k.

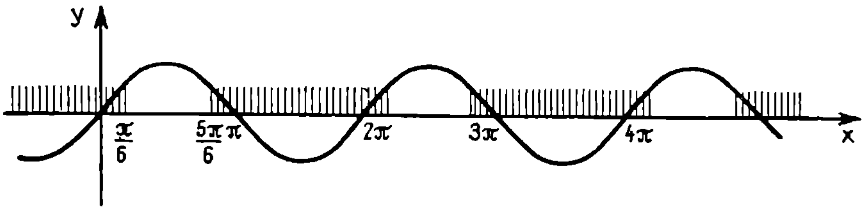


Fig.9

13. a) If $[a, b]$ is a trap for a sequence $\{x_n\}$, then only finitely many terms of $\{x_n\}$ lie outside $[a, b]$. If $[a, b]$ were not also a lure for $\{x_n\}$, then only finitely many terms of $\{x_n\}$ would lie inside $[a, b]$. But the whole sequence contains infinitely many terms. This contradiction shows that $[a, b]$ must also be a lure for $\{x_n\}$.

b) Use the sequence b) and the interval B) given in the next problem. The interval contains the first term, third term, fifth term, ... of the sequence, but not the second term, fourth term, sixth term, ... Hence the interval is a lure but not a trap.

14. Each of the intervals is a trap for sequence a). A) and B) are lures and C) a trap for sequence b). Each of the intervals is a lure for sequence c).

15. a) Each of the intervals [0, 1] and [2, 3] is a lure for the sequence

$$1, 3, \frac{1}{2}, \frac{5}{2}, \ldots, \frac{1}{n}, \frac{2n+1}{n}, \ldots$$

b) There is no such sequence. In fact, suppose the interval [0, 1] is a trap for some sequence $\{x_n\}$. Then only a finite number of terms of $\{x_n\}$ lie outside [0, 1]. But then only a finite number of terms of $\{x_n\}$ can lie inside [2, 3], and hence [2, 3] is not a lure for $\{x_n\}$, much less a trap (recall Prob. 13a).

16. a) No interval of length 1, regardless of its location, can intersect more than one of the intervals [0, 1] and [9, 10]. For example, suppose an interval fails to intersect [0, 1]. If the interval were a trap, then only a finite number of terms of the sequence would lie outside the interval, in particular inside [0, 1]. But this contradicts the fact that [0, 1] is a lure, by hypothesis.
b) Consider the two sequences

$$1, 9, \frac{1}{2}, \frac{19}{2}, \frac{1}{3}, \frac{29}{3}, \ldots, \frac{1}{n}, \frac{10n-1}{n}, \ldots$$

and

$$0, 10, \frac{1}{2}, \frac{19}{2}, \frac{2}{3}, \frac{28}{3}, \ldots, \frac{n-1}{n}, \frac{9n+1}{n}, \ldots,$$

each with the intervals [0, 1] and [9, 10] as lures. There is no trap of length 9 for the first sequence. (The intervals $[0, \frac{1}{3}]$ and $[\frac{29}{3}, 10]$ are lures for this sequence, and hence, as in the solution of part a, there can be no trap of length 9.) On the other hand, the second sequence has the interval $[\frac{1}{2}, \frac{19}{2}]$ as a trap of length 9, since the interval contains all the terms of the sequence starting from the third.

17. a) The sequence

$$1, 2, \ldots, n, \ldots$$

has no lures at all, since no more than $l + 1$ terms of the sequence belong to any interval of length l.
b) A sequence containing all the rational numbers would have every interval as a lure, since every interval contains infinitely many rational numbers. To construct such a sequence, we take a piece of graph paper and move along the path shown in Figure 10. As we go through each node of the lattice (such points are indicated by little circles in the figure), we write a term of the sequence according to the following rule: The term

corresponding to the node with coordinates x and y equals x/y (as shown by the figure, x and y are integers with $y > 0$). This procedure leads to the sequence

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{0}{2}, \frac{-1}{2}, \frac{-1}{1}, \frac{-2}{1}, \frac{-2}{2}, \frac{-2}{3}, \dots \qquad (1)$$

This sequence contains all the rational numbers. In fact, by definition every rational number can be written in the form p/q, where p and q are integers and $q > 0$. But clearly the path shown in the figure eventually arrives at the point with coordinates p and q, and we then write the term p/q.

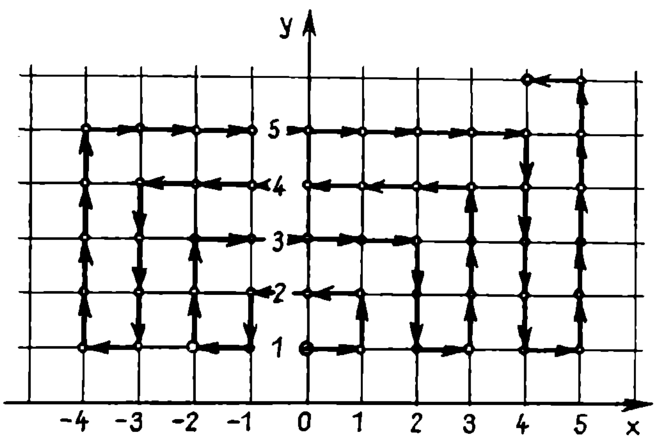


Fig. 10

Remark. Actually every rational number appears infinitely many times in the sequence (1), since there are infinitely many ways of writing any given rational number as a fraction p/q, e.g.,

$$\tfrac{2}{3} = \tfrac{4}{6} = \tfrac{6}{9} = \tfrac{8}{12} = \cdots$$

However, it would be hard to give an explicit rule for finding just where a given fraction appears in the sequence. For example, in what positions do the terms $\frac{4}{3}$, $-\frac{5}{7}$ and 10 appear, and what is the hundredth term of the sequence?

18. The answer is contained in the following table:

	A	B	C
a	1	1000	1000000
b	0	$\sqrt{999}$	$\sqrt{999999}$
c	0	$\dfrac{3}{\log_{10} 2}$	$\dfrac{6}{\log_{10} 2}$
d	2	2^{1000}	$2^{1000000}$

19. a) Consider an interval I with midpoint a, and let 2ε be the length of I. By the definition of a limit, there is a number k such that $|x_n - a| < \varepsilon$ for all $n > k$, i.e., such that x_n belongs to I for all $n > k$. In other words, no more than k terms of the sequence lie outside I. Hence I is a trap for $\{x_n\}$.
b) Given any positive number ε, consider an interval I of length less than 2ε with midpoint a. By hypothesis, I is a trap for the sequence $\{x_n\}$ and hence only a finite number of terms of $\{x_n\}$ lie outside I. Let k be the largest subscript of these terms (let $k = 0$ if there are no terms at all outside I). Then for any $n > k$, x_n belongs to I or equivalently $|x_n - a| < \varepsilon$. In other words, a is the limit of $\{x_n\}$.

20. Let I be an interval with midpoint a, and let 2ε be the length of I. By the definition of a limit, there is a number k such that $|x_n - a| < \varepsilon$ for all $n > k$. But then all the terms of the sequence $\{x_n\}$ with subscripts greater than k lie inside I, and hence I is a lure (in fact, a trap) for $\{x_n\}$.

Now consider an interval I which does not contain a, and let ε be the distance from a to the nearest end point of I. By the definition of a limit, there is a number k such that $|x_n - a| < \varepsilon$ for all $n > k$. But then all the terms of the sequence $\{x_n\}$ with subscripts greater than k are closer to a than to the nearest end point of I, i.e., these terms all lie outside I. Therefore I cannot contain infinitely many terms of $\{x_n\}$, and hence I cannot be a lure for $\{x_n\}$.

b) Any interval with midpoint 0 is a lure for the sequence

$$1, \tfrac{1}{2}, 3, \tfrac{1}{4}, \ldots, n^{(-1)^{n-1}}, \ldots \tag{1}$$

[this is sequence c) of Prob. 14]. In fact, the subsequence of (1) consisting of the terms with even subscripts, i.e., the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2n}, \ldots \tag{2}$$

obviously approaches 0 as $n \to \infty$. Therefore, by part a), any interval with midpoint 0 is a lure for (1) and hence also a lure for the original sequence (1). Moreover, no interval failing to contain 0 can be a lure for (1). In fact, let I be such an interval. Then by part a, I is not a lure for (2). Furthermore, no interval can be a lure for the subsequence of (1) consisting of the terms with odd subscripts, i.e., for the sequence

$$1, 3, 5, 7, \ldots \tag{3}$$

(prove this). Therefore I contains only finitely many terms of (2) and only finitely many terms of (3) and hence only finitely many terms of (1). It follows that I is not a lure for (1).

Thus the sequence (1) satisfies the conditions of the problem, with $a = 0$. But (1) does not have 0 as a limit, since if it did, the inequality $|x_n| < 1$ would hold for all n starting from some value, which is impossible since $|x_n| < 1$ does not hold for any term x_n with an odd subscript.

Remark. The sequence (1) can be regarded as a superposition (more exactly, an "interlacing") of the two simpler sequences (2) and (3). This way of forming sequences is often useful. For example, to construct a sequence for which each of two given intervals I_1 and I_2 is a lure (see Prob. 15b), we can put together two sequences, one with I_1 as a lure, the other with I_2 as a lure.

21. Let a be a number lying outside the given interval I which is a lure for the sequence $\{x_n\}$. Then x_n cannot approach a as

$n \to \infty$, since otherwise, as proved in the solution to Problem 20a, I could not be a lure for $\{x_n\}$.

22. To prove that a sequence $\{x_n\}$ has a as a limit, we must show that given any positive number ε, there is a number k such that $|x_n - a| < \varepsilon$ for all $n > k$. It is often possible to find an explicit formula for k in terms of ε.

a) If
$$x_n = \frac{(-1)^{n-1}}{n},$$
then
$$\lim_{n\to\infty} x_n = 0.$$
Since
$$|x_n| = \frac{1}{n},$$
we can choose k to be $1/\varepsilon$. In fact, if $n > 1/\varepsilon$, then
$$|x_n| = \frac{1}{n} < \varepsilon.$$

Remark. Before going on, we note that the number k is not uniquely determined by the value of ε. In the example just considered, we could take k to be
$$\frac{2}{\varepsilon} \quad \text{or} \quad \frac{1}{\varepsilon} + 1,$$
or for that matter, any number greater than $1/\varepsilon$. Rather than the "most economical" value of k (which may involve ε in a very complicated way), it is often best to choose a "cruder" value. For example, to prove that
$$\lim_{n\leftarrow\infty} x_n = 0,$$
where
$$x_n = \frac{1}{n^3 + 0.6n + 3.2},$$

we can use the fact that

$$x_n < \frac{1}{n^3} \leqslant \frac{1}{n}$$

and then take k to be $1/\varepsilon$. This is much more convenient than the most economical value

$$k = \sqrt[3]{-1.6 + \frac{1}{2\varepsilon} + \sqrt{2.568 - \frac{1.6}{\varepsilon} + \frac{1}{4\varepsilon^2}}}$$
$$+ \sqrt[3]{-1.6 + \frac{1}{2\varepsilon} - \sqrt{2.568 - \frac{1.6}{\varepsilon} + \frac{1}{4\varepsilon^2}}}$$

obtained by solving the cubic equation

$$n^3 + 0.6n + 3.2 = \frac{1}{\varepsilon}.$$

There are also cases where an explicit formula for the most economical value of k cannot be found, whereas it is an easy matter to find a suitable "overestimate" of k.

b) If

$$x_n = \frac{3^n - 1}{3^n} = 1 - \frac{1}{3^n},$$

then

$$\lim_{n\to\infty} x_n = 1.$$

In fact, since

$$|x_n - 1| = \frac{1}{3^n},$$

we need only choose

$$k = \log_3 \frac{1}{\varepsilon}.$$

c) Using the formula for the sum of a geometric progression, we have

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}.$$

Hence

$$\lim_{n\to\infty} x_n = 2,$$

and we can choose

$$k = \log_2 \frac{1}{\varepsilon}.$$

d) The sequence has no limit, since given any number a and any ε, the inequality $|x_n - a| < \varepsilon$ holds for only finitely many values of n.

e) In this case

$$\lim_{n\to\infty} x_n = 1,$$

since $|x_n - 1|$ equals zero for all n, so that the inequality $|x_n - 1|$ holds trivially for arbitrary positive ε and all n.

f) Consider the cases of even and odd n separately. For even $n = 2m$ we have

$$x_n = \frac{1}{m},$$

and hence $|x_n| < \varepsilon$ for $m > \frac{1}{\varepsilon}$, i.e., for $n > \frac{2}{\varepsilon}$. For odd n, $x_n = 0$ and hence $|x_n| < \varepsilon$ holds trivially for all positive ε. It follows that

$$\lim_{n\to\infty} x_n = 0,$$

and we can choose

$$k = \frac{2}{\varepsilon}.$$

g) By the formula for the sum of a geometric progression, we have

$$\underbrace{0.22\ldots 2}_{n \text{ times}} = \frac{2}{10} + \frac{2}{10^2} + \cdots + \frac{2}{10^n} = \frac{2}{9}\left(1 - \frac{1}{10^n}\right).$$

Therefore

$$\lim_{n\to\infty} x_n = \frac{2}{9}$$

and for k we can choose $\log_{10} \dfrac{1}{9\varepsilon}$ or simply $\log_{10} \dfrac{1}{\varepsilon}$ [recall the remark made after part a)].

h) The sequence has no limit. In fact, $x_n = 0$ if $n = 180m$, while $x_n = 1$ if $n = 90 + 360m$. If the sequence had a limit a, then

$$|x_n - a| < \tfrac{1}{4}$$

for n greater than some number k. But this would in turn imply

$$|x_{n_1} - x_{n_2}| \leqslant |x_{n_1} - a| + |x_{n_2} - a| < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

if $n_1 > k$, $n_2 > k$, while, as just noted

$$|x_{180m} - x_{90+360m}| = 1$$

for arbitrary m, no matter how large. Contradiction!

i) Since $|\cos n°| \leqslant 1$, we have

$$|x_n| \leqslant \frac{1}{n}.$$

Therefore

$$\lim_{n\to\infty} x_n = 0,$$

and we can choose

$$k = \frac{1}{\varepsilon}.$$

j) The sequence has no limit. In fact, if a were the limit of the sequence, then

$$|x_n - a| < \tfrac{1}{2}$$

for all sufficiently large n, and hence

$$|x_n - x_{n+1}| \leqslant |x_n - a| + |a - x_{n+1}| < \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

But the difference between consecutive terms of the sequence exceeds 1 in absolute value.

23. Suppose two different numbers a and b are limits of the same sequence $\{x_n\}$, and let $2\varepsilon = |a - b|$. Since $\{x_n\}$ has a as a limit, there is a number k_1 such that $|x_n - a| < \varepsilon$ for all $n > k_1$. Similarly, since $\{x_n\}$ has b as a limit, there is a number

k_2 such that $|x_n - b| < \varepsilon$ for all $n > k_2$. Therefore, if n exceeds both k_1 and k_2, we have

$$|x_n - a| < \varepsilon \quad \text{and} \quad |x_n - b| < \varepsilon.$$

But this impossible, since $|a - b| = 2\varepsilon$.

24. a) Given that a is a limit point of the sequence $\{x_n\}$, let I be any interval with midpoint a and let 2ε be the length of I. We want to show that I is a lure for $\{x_n\}$, which means that infinitely many terms of $\{x_n\}$ fall in I, i.e., satisfy the inequality $|x_n - a| \leqslant \varepsilon$. Suppose this is not true, so that only finitely many terms of $\{x_n\}$ fall in I. Let k be the largest subscript of these terms (let $k = 0$ if there are no terms at all in I). Then all terms with subscripts greater than k lie outside I. But this contradicts the definition of a limit point, according to which given any k, there is an integer $n > k$ such that $|x_n - a| < \varepsilon$, i.e., such that x_n lies inside I.

b) Suppose to the contrary that a is not a limit point of the sequence x_n. Then for some ε and k, there is no integer $n > k$ such that $|x_n - a| < \varepsilon$. In other words, there is some interval with midpoint a which contains only finitely many (no more than k) terms of $\{x_n\}$. But this contradicts the hypothesis that any interval with midpoint a is a lure for $\{x_n\}$ and hence contains infinitely many terms of $\{x_n\}$.

25. The result follows from the results of Problems 19a, 13a and 24b. Give another proof without using the notion of traps and lures.

26. a) If

$$x_n = \frac{n+1}{n},$$

then

$$\lim_{n\to\infty} x_n = 1,$$

and hence, by Problem 25, 1 is a limit point of $\{x_n\}$. The sequence has no other limit points (cf. Probs. 20a and 24a).

b) The points 1 and -1 are obviously limit points of the sequence $\{x_n\} = \{(-1)^n\}$. Given any other point a, we can construct an interval with midpoint a containing no terms at all of $\{x_n\}$. Therefore $\{x_n\}$ has no limit points other than ± 1.
c) The function $\sin x°$ has period $360°$, i.e., $\sin(360 + x)° = \sin x°$ for all x. Therefore each of the 181 numbers

$$0, \pm\sin 1°, \pm\sin 2°, \ldots, \pm\sin 89°, \pm 1$$

appears infinitely many times in the sequence $\{x_n\} = \{\sin n°\}$, and hence they are all limit points of $\{x_n\}$. If a is not one of the numbers (1), then we can construct an interval with midpoint a which does not contain a single term of $\{x_n\}$. In fact, it is only necessary to make the length of the interval smaller than the distance from a to the nearest of the numbers (1). Hence $\{x_n\}$ has no other limit points.
d) If

$$x_n = n^{(-1)^n},$$

the sequence $\{x_n\}$ can be thought of as a superposition of the two sequences $\{y_n\}$ and $\{z_n\}$, where

$$y_n = \frac{1}{2n-1}, \quad z_n = 2n$$

(recall the remark on p. 35). The sequence $\{y_n\}$ has the unique limit point 0, while the sequence $\{z_n\}$ has no limit points (verify these assertions). But a point a is a limit point of the original sequence $\{x_n\}$ if and only if it is a limit point of at least one of the sequences $\{y_n\}$ and $\{z_n\}$. In fact, according to the result of Problem 24, we need only prove the following almost obvious assertion: An interval is a lure for the sequence $\{x_n\}$ if and only if it is a lure for one of the sequences $\{y_n\}$ and $\{z_n\}$. Therefore $\{x_n\}$ has the unique limit point 0.

Remark. Precisely the same argument leads to the following more general result: Suppose $\{x_n\}$ is the "union" of the sequences $\{y_n\}, \{z_n\}, \ldots, \{t_n\}$ in the sense that every term of $\{x_n\}$ is a term of at least one of the sequences $\{y_n\}, \{z_n\}, \ldots, \{t_n\}$, while every term of each of the sequences $\{y_n\}, \{z_n\}, \ldots, \{t_n\}$ is a term of $\{x_n\}$. Then the set of all limit points of $\{x_n\}$ is the union of the sets of all limit points of $\{y_n\}, \{z_n\}, \ldots, \{t_n\}$ (by the *union* of two sets, we mean the set of all points belonging to at least one of the sets).

e) The sequence $\{x_n\} = \{n\}$ has no limit points at all since no interval contains more than a finite number of terms of $\{x_n\}$.

f) Infinitely many terms of the sequence

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \ldots \tag{1}$$

belong to any interval with midpoint a, provided that $0 \leqslant a \leqslant 1$. In fact, if the interval I is of length 2ε and if $n > \dfrac{1}{\varepsilon}$, then at least one of the terms

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$$

belongs to I. On the other hand, if $a < 0$ or $a > 1$, we can construct an interval with midpoint a which contains no terms of the sequence (1). It follows that all the points of the interval $[0, 1]$ are limit points of (1), and only such points.

27. A sequence $\{x_n\}$ is said to be *unbounded* if given any number C, there is an integer n such that $|x_n| > C$.

28. a) Suppose x_0 is the limit of a sequence $\{x_n\}$, and consider any interval $[a, b]$ with midpoint x_0. Since $[a, b]$ is a trap for $\{x_n\}$ (see Prob. 19a), only finitely many terms of $\{x_n\}$ lie outside $[a, b]$. Let x_k be the largest of these terms in absolute value, and let C denote the largest of the numbers $|a|$, $|b|$ and $|x_k|$. Then $|x_n| \leqslant C$ for all n. In fact, if x_n belongs to $[a, b]$, then $|x_n|$ cannot exceed the larger of the numbers $|a|$ and $|b|$, while if x_n does not belong to $[a, b]$, then $|x_n| \leqslant |x_k|$ by the very definition of x_k.

b) The sequence

$$1, 2, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{3}, \ldots, \frac{1}{n}, \frac{n+1}{n}, \ldots \tag{3}$$

(see Prob. 14b) is bounded since $|x_n| \leqslant 2$ for all n. However, it is easy to see that (1) has no limit. In fact, the intervals $[0, \frac{1}{4}]$ and $[1, \frac{5}{4}]$ are both lures for (1) and the distance between these intervals is $\frac{3}{4}$. It follows (see the solution to Prob. 16a) that (1) has no trap of length less than $\frac{3}{4}$. But if (1) had a limit a, then any interval with midpoint a would be a trap, i.e., there would be traps of arbitrarily small length. This contradiction shows that (1) can have no limit. Of course, the same thing can be proved without recourse to the language of "traps" and "lures," as in the solutions to Problems 22h and 22k (give such a proof).

29. a) If $x_n = n$, then given any positive number C, we have $|x_n| > C$ if $n > C$. Therefore $\{x_n\}$ approaches infinity.

b) The solution is the same as in part a).

c) If $x_n = (-1)^n n$, then $\{x_n\}$ is unbounded since given any positive number C, there is an integer n such that $|x_n| > C$ (for n take any even number greater than C). But this sequence does not approach infinity since the inequality $|x_n| > 1$ does not hold for any x_n with odd n.

d) Suppose

$$x_n = \begin{cases} n \text{ for even } n, \\ \sqrt{n} \text{ for odd } n, \end{cases}$$

and let C be any positive number. Then $x_n > C$ for all $n > C^2$ and hence $x_n \to \infty$ as $n \to \infty$.

e) If

$$x_n = \frac{100n}{100 + n^2},$$

then the sequence $\{x_n\}$ is bounded, and in fact $|x_n| \leqslant 5$. Since $x_n > 0$, we need only show that $x_n \leqslant 5$. But this follows from

$$5 - x_n = 5 - \frac{100n}{100 + n^2} = \frac{5}{100 + n^2}(100 + n^2 - 20n)$$

$$= \frac{5(10 - n)^2}{100 + n^2} \geqslant 0.$$

30. a) $x_n = \left(-\frac{1}{2}\right)^n$; b) $x_n = \frac{1}{n}$; c) $x_n = \frac{n-1}{n}$;

d) $x_n = (-1)^n \frac{n-1}{n}$.

31. We confine ourselves to a detailed analysis of two cases:

1) Any sequence $\{x_n\}$ satisfies assertion 1. In fact, we need only find a positive number ε, a number k and an integer $n > k$ such that $|x_n - a| < \varepsilon$. But the following choice of ε, k and n is clearly suitable:

$$\varepsilon = |x_1 - a| + 1, \quad k = 0, \quad n = 1.$$

2) Assertion 6 means that the sequence $\{x_n\}$ does not have a as a limit. In fact, to say that "a is the limit of the sequence $\{x_n\}$" means that "for any $\varepsilon > 0$, there is a k such that $|x_n - a| < \varepsilon$ for all $n > k$". The negative of this assertion reads "for not every $\varepsilon > 0$ is there a k such that $|x_n - a| < \varepsilon$ for all $n > k$." This last statement can be paraphrased as "for some $\varepsilon > 0$ there is no k such that $|x_n - a| < \varepsilon$ for all $n > k$," which in turn can be stated as "there is an $\varepsilon > 0$ such that for any k, $|x_n - a| < \varepsilon$ does not hold for all $n > k$." Finally, we can paraphrase our assertion as "there is an $\varepsilon > 0$ such that for any k there is an $n > k$ such that $|x_n - a| \geqslant \varepsilon$," and this is precisely assertion 6. Note that this assertion can be obtained from the definition of a limit by the following rule:

a) Instead of "there is ... such that" write "for any";
b) Instead of "for any" or "for all" write "there is ... such that";
c) Instead of $|x_n - a| < \varepsilon$ write $|x_n - a| \geqslant \varepsilon$.

This same rule is applicable in general. Prove that it causes every assertion to be replaced by its negative (symbolically, the rule replaces $\exists$, $\forall$ and $<$ by $\forall$, $\exists$ and $\geqslant$, respectively).

32. If the array of plus and minus signs begins with $+$, then all the terms of the sequence equal a and the remaining signs are uniquely determined, i.e., the sequence has a as a limit and as a limit point, is it bounded and does not approach infinity.

Next consider arrays beginning with $-+$. Then the sequence approaches a (without every term being equal to a), and the remaining signs are uniquely determined, i.e., the sequence has a as a limit point (see Prob. 25), is bounded (see Prob. 28a) and does not approach infinity.

Finally consider arrays ending in $+$. Then the sequence approaches infinity and this determines all the other signs, i.e. not every term equals a, the sequence does not approach a, does not have a as a limit point and is unbounded.

The remaining arrays have minus signs in the first, second and fifth positions. In this case, as shown in the answer on p. 11, the third and fourth positions can be occupied by either a plus or a minus sign.

33. The answer on p. 12 gives examples of three sequences, each with a limit. The first sequence has a largest term but no smallest term, the second has a smallest term but no largest term, and the third has both a largest and a smallest term. It must still be shown that there is no sequence with a limit which has neither a largest nor a smallest term. Suppose $\{x_n\}$ is such a sequence. Since $x_{n_1} = x_1$ is not the largest term of $\{x_n\}$, there is a term $x_{n_2} > x_{n_1}$. But x_{n_2} cannot be larger than any of the subsequent terms of the sequence, since otherwise the largest of the first n_2 terms of $\{x_n\}$ would be the largest of all the terms of $\{x_n\}$, contrary to the assumption that $\{x_n\}$ has no largest term. Hence there is an integer $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$. But x_{n_3} cannot be larger than any of the subsequent terms of $\{x_n\}$, and so on. Thus $\{x_n\}$ contains a strictly increasing subsequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$, i.e., a subsequence $\{x_{n_k}\}$ such that

$$x_{n_1} < x_{n_2} < \cdots < x_{n_k} < \cdots$$

A similar argument, based on the fact that $\{x_n\}$ has no smallest term, shows that $\{x_n\}$ contains a strictly decreasing subsequence

$x_{m_1}, x_{m_2}, \ldots, x_{m_k}, \ldots$, i.e., a subsequence $\{x_{m_k}\}$ such that

$$x_{m_1} > x_{m_2} > \cdots > x_{m_k} > \cdots$$

Setting $\varepsilon = x_{n_1} - x_{m_1}$, we find that

$$x_{n_k} - x_{m_k} > x_{n_1} - x_{m_1} = \varepsilon$$

for all $k > 1$. It follows that $\{x_n\}$ has no limit. In fact, if $\{x_n\}$ had a limit, than $x_{n_k} - x_{m_k}$ could be made smaller than any $\varepsilon > 0$ for sufficiently large k, since

$$|x_{n_k} - x_{m_k}| \leqslant |x_{n_k} - a| + |a - x_{m_k}|$$

and both terms on the right can be made arbitrarily small for sufficiently large k.

34. If either the sequence $\{x_n\}$ or one of its subsequences has no largest term, then, as shown in the solution to the preceding problem, $\{x_n\}$ contains a monotone subsequence (in fact, a strictly increasing subsequence). Now consider the case where some subsequence of $\{x_n\}$ has a largest term. Let x_{n_1} be this largest term, let x_{n_2} be the largest of the terms following x_{n_1}, let x_{n_3} be the largest of the terms following x_{n_2}, and so on. Then obviously

$$x_{n_1} \geqslant x_{n_2} \geqslant \cdots \cdots \geqslant x_{n_k} \geqslant \ldots,$$

and hence $\{x_{n_k}\}$ is a monotone subsequence of $\{x_n\}$.

35. Consider the sequence

$$1.4, 1.41, 1.414, 1.4142, \ldots,$$

whose nth term x_n approximates the value of $\sqrt{2}$ (from below) to within 10^{-n}. It follows from the very definition of the sequence $\{x_n\}$ that

$$\lim_{n\to\infty} x_n = \sqrt{2}$$

$\left(\text{in fact } \left|x_n - \sqrt{2}\right| < \varepsilon \text{ if } n > \log_{10}\dfrac{1}{\varepsilon}\right)$. But $\sqrt{2}$ is irrational and a sequence can only have one limit. Therefore $\{x_n\}$ cannot have a rational limit.

Remark. We have just used the fact that there is a real number with square equal to 2 which can be written as an infinite decimal. The proof can be modified in such a way that it involves only rational numbers. This is done by defining x_n as the largest of the numbers which can be written as finite decimals with n digits after the decimal point and whose squares are less than 2, and then showing that $a^2 = 2$ where a is the limit of $\{x_n\}$ (guaranteed by the Bolzano-Weierstrass axiom). The idea of the proof is the following: The number x_n has the property that $x_n^2 < 2$ but $\left(x_n + \dfrac{1}{10^n}\right)^2 > 2$. Therefore x_n^2 is very close to 2 for large n. On the other hand, x_n is very close to a for large n. Hence the numbers x_n^2 and a^2 are very close together for large n. It follows that the difference between a^2 and 2 can be made arbitrarily small. But this difference is a constant, independent of n, and hence $a^2 = 2$. Try to make a rigorous proof out of these qualitative considerations. Also prove that $\sqrt{2}$ is actually irrational.

36. Let $\{x_n\}$ be a bounded sequence and choose a monotone subsequence $\{x_{n_k}\}$ from $\{x_n\}$ as in the solution to Problem 34. The sequence $\{x_{n_k}\}$ is bounded, and hence has a limit a, by the Bolzano-Weierstrass axiom. But a is a limit point of the original sequence $\{x_n\}$. In fact, every interval I with midpoint a is a trap for $\{x_{n_k}\}$ and hence contains infinitely many terms of $\{x_n\}$, i.e., I is a lure for $\{x_n\}$. It follows from Problem 24b that a is a limit point of $\{x_n\}$.

37. a) The sequence with general term

$$x_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

is obviously strictly increasing. Moreover, it is bounded since

$$\frac{1}{n^2} < \frac{1}{n(n-1)} \quad (n > 1)$$

implies

$$x_n < 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n-1)}$$

$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots$$

$$+ \left(\frac{1}{n-1} - \frac{1}{n}\right) < 2 - \frac{1}{n} < 2.$$

Therefore $|x_n| < 2$ for all n, and hence, by the Bolzano-Weierstrass axiom, $\lim_{n\to\infty} x_n$ exists.

Remark. It can be shown by methods beyond the scope of this book that

$$\lim_{n\to\infty} = \frac{\pi^2}{6}.$$

b) If

$$x_n = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{n-1},$$

then the sequence $\{x_{2n}\}$ made up of the even-numbered terms of $\{x_n\}$ is strictly increasing, since

$$x_{2n} = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots$$

$$+ \left(\frac{1}{4n-3} - \frac{1}{4n-1}\right) > x_{2n-2}.$$

Moreover $\{x_{2n}\}$ is bounded, since

$$x_{2n} = 1 - \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{7} - \frac{1}{9}\right) - \cdots$$

$$- \left(\frac{1}{4n-5} - \frac{1}{4n-3}\right) - \frac{1}{4n-1} < 1,$$

and hence $\lim_{n\to\infty} \{x_{2n}\}$ exists. Denoting this limit by a, we now show that a is the limit of the whole sequence $\{x_n\}$. In fact, given any $\varepsilon > 0$, there is a number k such that

$$|x_{2n} - a| < \frac{\varepsilon}{2}.$$

But

$$|x_{2n+1} - x_{2n+2}| < \frac{\varepsilon}{2}$$

if $2n + 1 > \dfrac{1}{\varepsilon}$. It follows that

$$|x_n - a| < \varepsilon$$

if n exceeds the larger of the two numbers k and $\dfrac{1}{\varepsilon}$.

Remark. It turns out that $a = \dfrac{\pi}{4}$.

38. a) Since $\lim\limits_{n\to\infty} x_n = a$ by hypothesis, then, given any $\varepsilon > 0$, there is a number k_1 such that

$$|x_n - a| < \frac{\varepsilon}{2}$$

if $n > k_1$. Similarly, since $\lim\limits_{n\to\infty} y_n = b$, there is a number k_2 such that

$$|y_n - b| < \frac{\varepsilon}{2}$$

if $n > k_2$. Let k be the larger of the numbers k_1 and k_2. Then, given any $\varepsilon > 0$,

$$|x_n + y_n - a - b| \leqslant |x_n - a| + |y_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if $n > k$. In other words,

$$\lim_{n\to\infty} (x_n + y_n) = a + b.$$

b) The proof is the same as for part a).
c) The proof makes use of the formula

$$x_n y_n - ab = x_n(y_n - b) + b(x_n - a).$$

Since the sequence $\{x_n\}$ has a limit, it is bounded (see Prob. 28a), i.e., there is a positive number C such that $|x_n| \leqslant C$ for all n.

Let M be the larger of the numbers C and $|b|$, where $b = \lim_{n\to\infty} y_n$, and suppose we are given any $\varepsilon > 0$. Then, since $\lim_{n\to\infty} x_n = a$, there is a number k_1 such that

$$|x_n - a| < \frac{\varepsilon}{2M}$$

if $n > k_1$. Similarly, there is a number k_2 such that

$$|y_n - b| < \frac{\varepsilon}{2M}$$

if $n > k_2$. Let k be the larger of the numbers k_1 and k_2. Then

$$|x_n y_n - ab| = |x_n (y_n - b) + b(x_n - a)|$$

$$\leqslant |x_n|\,|y_n - b| + |b|\,|x_n - a| < M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon$$

if $n > k$. In other words,

$$\lim_{n\to\infty} x_n y_n = ab,$$

since $\varepsilon > 0$ is arbitrary.

d) The proof makes use of the formula

$$\frac{x_n}{y_n} - \frac{a}{b} = \frac{x_n - a}{y_n} + \frac{a(b - y_n)}{by_n}.$$

Suppose first that we have managed to show that the sequence $\left\{\frac{1}{y_n}\right\}$ is bounded, so that there is a positive number C such that

$$\left|\frac{1}{y_n}\right| \leqslant C$$

for all n. Then we have

$$\left|\frac{x_n}{y_n} - \frac{a}{b}\right| = \left|\frac{x_n - a}{y_n}\right| + \left|\frac{a(b - y_n)}{by_n}\right| \leqslant C\,|x_n - a|$$

$$+ C\left|\frac{a}{b}\right|\,|y_n - b|.$$

Given any $\varepsilon > 0$, there is a number k_1 such that

$$|x_n - a| < \frac{\varepsilon}{2C}$$

if $n > k_1$ and a number k_2 such that

$$|y_n - b| < \left|\frac{b}{a}\right| \frac{\varepsilon}{2C}$$

if $n > k_2$. Let k be the larger of the numbers k_1 and k_2. Then

$$\left|\frac{x_n}{y_n} - \frac{a}{b}\right| \leqslant C\frac{\varepsilon}{2C} + C\left|\frac{a}{b}\right|\left|\frac{b}{a}\right|\frac{\varepsilon}{2C} = \varepsilon$$

if $n > k$ (what happens if $a = 0$?). In other words,

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b},$$

since ε is arbitrary.

Thus the proof reduces to showing that the sequence $\left\{\frac{1}{y_n}\right\}$ is bounded if

$$\lim_{n\to\infty} y_n = b, \quad b \neq 0, \quad y_n \neq 0.$$

Suppose, to be explicit, that $b > 0$. Then there is a number k such that

$$|y_n - b| < \frac{b}{2}$$

if $n > k$, and hence all the terms of the sequence $\{y_n\}$ with subscripts $n > k$ lie in the interval $\left[\frac{b}{2}, \frac{3b}{2}\right]$. Let C be the largest of the numbers

$$\frac{2}{b}, \frac{1}{|y_1|}, \frac{1}{|y_2|}, \ldots, \frac{1}{|y_k|}$$

(assume that k is an integer). Then obviously

$$\frac{1}{|y_n|} \leqslant C$$

for all n. The case $b < 0$ is left as an exercise.

39. a) $x_n = \dfrac{1}{n^2}, y_n = \dfrac{1}{n}$; b) $x_n = \dfrac{1}{n}, y_n = \dfrac{1}{n}$;

c) $x_n = \dfrac{1}{n}, y_n = \dfrac{1}{n^2}$; d) $x_n = \dfrac{1}{n}, y_n = \dfrac{(-1)^n}{n}$.

40.

a) $$\lim_{n\to\infty} \frac{2n+1}{3n-5} = \lim_{n\to\infty} \frac{2+\dfrac{1}{n}}{3-\dfrac{5}{n}} = \frac{\lim\limits_{n\to\infty}\left(2+\dfrac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(3-\dfrac{5}{n}\right)} = \frac{2}{3};$$

b) $$\lim_{n\to\infty} \frac{10n}{n^2+1} = \lim_{n\to\infty} \frac{\dfrac{10}{n}}{1+\dfrac{1}{n^2}} = \frac{\lim\limits_{n\to\infty}\dfrac{10}{n}}{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n^2}\right)} = 0;$$

c) $$\lim_{n\to\infty} \frac{n(n+2)}{(n+1)(n+3)} = \lim_{n\to\infty} \frac{1+\dfrac{2}{n}}{\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{3}{n}\right)}$$

$$= \frac{\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)}{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)\lim\limits_{n\to\infty}\left(1+\dfrac{3}{n}\right)} = 1;$$

d) $$\lim_{n\to\infty} \frac{2^n-1}{2^n+1} = \lim_{n\to\infty} \frac{1-\dfrac{1}{2^n}}{1+\dfrac{1}{2^n}} = \frac{\lim\limits_{n\to\infty}\left(1-\dfrac{1}{2^n}\right)}{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{2^n}\right)} = 1.$$

41. We begin by using mathematical induction in k to show that $1^k + 2^k + \cdots + n^k$ is a polynomial of degree $k+1$ in n

with leading coefficient $\dfrac{1}{k+1}$. The assertion is obvious for $k = 1$, since

$$1^0 + 2^0 + \cdots + n^0 = n.$$

Suppose the assertion is true for all $k \leqslant k_0$. Then it is true for $k = k_0 + 1$. In fact, by the binomial theorem,

$$\frac{(n+1)^{k_0+2} - n^{k_0+2}}{k_0+2}$$

$$= \frac{n^{k_0+2} + (k_0+2)\,n^{k_0+1} + \frac{1}{2}(k_0+2)\,(k_0+1)\,n^{k_0} + \cdots + 1 - n^{k_0+2}}{k_0+2}$$

$$= n^{k_0+1} + P(n),$$

where $P(n)$ is a polynomial of degree k_0 in n. It follows that

$$n^{k_0+1} = \frac{(n+1)^{k_0+2} - n^{k_0+2}}{k_0+2} + Q(n),$$

where $Q(n) = -P(n)$, and hence

$$1^{k_0+1} + 2^{k_0+1} + \cdots + n^{k_0+1} = \frac{2^{k_0+2} - 1^{k_0+2}}{k_0+2}$$

$$+ \frac{3^{k_0+2} - 2^{k_0+2}}{k_0+2} + \cdots + \frac{n^{k_0+2} - (n-1)^{k_0+2}}{k_0+2}$$

$$+ \frac{(n+1)^{k_0+2} - n^{k_0+2}}{k_0+2}$$

$$+ Q(1) + Q(2) + \cdots + Q(n)$$

$$= \frac{(n+1)^{k_0+2}}{k_0+2} - \frac{1}{k_0+2}$$

$$+ Q(1) + Q(2) + \cdots + Q(n). \tag{1}$$

Suppose

$$Q(n) = a_0 n^{k_0} + a_1 n^{k_0-1} + \cdots + a_{k_0}.$$

Then

$$Q(1) + Q(2) + \cdots + Q(n) = a_0(1^{k_0} + 2^{k_0} + \cdots + n^{k_0})$$
$$+ a_1(1^{k_0-1} + 2^{k_0-1} + \cdots + n^{k_0-1}) + \cdots + a_{k_0}$$

is a polynomial of degree $k_0 + 1$ in n by hypothesis. Therefore (1) implies that

$$1^{k_0+1} + 2^{k_0+1} + \cdots + n^{k_0+1}$$

is a polynomial of degree $k_0 + 2$ in n with leading coefficient $\dfrac{1}{k_0+2}$. This completes the induction and shows that for every k, $1^k + 2^k + \cdots + n^k$ is a polynomial of degree $k + 1$ in n with leading coefficient $\dfrac{1}{k+1}$.

It is now a simple matter to evaluate

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n^{k+1}}(1^k + 2^k + \cdots + n^k).$$

In fact

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\dfrac{1}{k+1}n^{k+1} + b_0 n^k + b_1 n^{k-1} + \cdots + b_k}{n^{k+1}},$$

where $b_0, b_1, \ldots, b_k$ are suitable constants, and hence

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{1}{k+1} + \frac{b_0}{n} + \frac{b_1}{n^2} + \cdots + \frac{b_k}{n^{k+1}}\right) = \frac{1}{k+1}.$$

42. Since

$$\sqrt{n+1} - \sqrt{n-1} = \frac{2}{\sqrt{n+1} + \sqrt{n-1}},$$

we have

$$|x_n| = \frac{2}{\sqrt{n+1} + \sqrt{n-1}} \leqslant \frac{1}{\sqrt{n-1}}.$$

Therefore $|x_n| < \varepsilon$ if

$$n > \frac{1}{\varepsilon^2} + 1,$$

and hence

$$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n-1}\right) = 0.$$

43. Since

$$2^n = (1+1)^n = 1 + n + \frac{n(n-1)}{2} + \cdots > n + \frac{n(n-1)}{2}$$

$$= \frac{n}{2} + \frac{n^2}{2} + \cdots > \frac{n^2}{2},$$

we have

$$|x_n| = \frac{n}{2^n} < \frac{2}{n}.$$

Therefore $|x_n| < \varepsilon$ if

$$n > \frac{2}{\varepsilon},$$

and hence

$$\lim_{n\to\infty} \frac{n}{2^n} = 0.$$

44. Since

$$a^n = [1 + (a-1)]^n = 1 + n(a-1) + \frac{n(n-1)}{2}(a-1)^2 + \cdots$$

$$> \frac{n(n-1)}{2}(a-1)^2,$$

we have

$$|x_n| = \frac{n}{a^n} < \frac{n}{\frac{n(n-1)}{2}(a-1)^2} = \frac{2}{(n-1)(a-1)^2}.$$

Therefore $|x_n| < \varepsilon$ if

$$n > \frac{2}{(a-1)^2\,\varepsilon} + 1,$$

and hence

$$\lim_{n\to\infty} \frac{n}{a^n} = 0.$$

45. If n lies between 2^{m-1} and 2^m, then

$$\frac{\log_2 n}{n} < \frac{m}{2^{m-1}}.$$

But according to Problem 43, given any $\varepsilon > 0$ there is a number k such that

$$\frac{m}{2^m} < \frac{\varepsilon}{2} \quad \text{or} \quad \frac{m}{2^{m-1}} < \varepsilon$$

if $m > k$. If $n > 2^k$, then n lies between 2^{m-1} and 2^m for some $m > k$, and hence

$$\frac{\log_2 n}{n} < \frac{m}{2^{m-1}} < \varepsilon.$$

It follows that

$$\lim_{n\to\infty} \frac{\log_2 n}{n} = 0.$$

46. We must show that given any $\varepsilon > 0$, there is a number k such that

$$\left|\sqrt[n]{n} - 1\right| < \varepsilon$$

if $n > k$. Since $\sqrt[n]{n} \geqslant 1$, we have

$$\left|\sqrt[n]{n} - 1\right| = \sqrt[n]{n} - 1.$$

Consider the values of n for which the inequality

$$\sqrt[n]{n} - 1 < \varepsilon$$

does not hold, i.e., the values of n such that

$$\sqrt[n]{n} - 1 \geqslant \varepsilon \tag{1}$$

or

$$\sqrt[n]{n} \geqslant 1 + \varepsilon$$

or finally

$$n \geqslant (1 + \varepsilon)^n. \tag{2}$$

Since

$$(1 + \varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2}\varepsilon^2 + \cdots > \frac{n(n-1)}{2}\varepsilon^2,$$

(2) implies

$$n > \frac{n(n-1)}{2}\varepsilon^2,$$

which in turn implies

$$n < \frac{2}{\varepsilon^2} + 1.$$

Therefore the inequality (1) is false if

$$n > \frac{2}{\varepsilon^2} + 1. \tag{3}$$

In other words,

$$|\sqrt[n]{n} - 1| = \sqrt[n]{n} - 1 < \varepsilon$$

if (3) holds, and hence

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

47. After the first step, the snail arrives at the point $(a + 1, b)$, after the second step at the point $(a + 1, b + 1)$, and so on (in the notation of the hint on p. 17). After the $(2n)$th step, the snail is at the point $(a + n, b + n)$ and after the $(2n + 1)$st step it is at the point $(a + n + 1, b + n)$. Therefore

$$k_{2n} = \frac{b+n}{a+n}, \quad k_{2n+1} = \frac{b+n}{a+n+1}.$$

To determine the limit of the sequence k_n, we write

$$k'_n = k_{2n}, \quad k''_n = k_{2n+1}.$$

Clearly both $\{k'_n\}$ and $\{k''_n\}$ approach 1. Therefore given any $\varepsilon > 0$, there is a number l_1 such that $|k'_n - 1| < \varepsilon$ if $n > l_1$ and a number l_2 such that $|k''_n - 1| < \varepsilon$ if $n > l_2$. Let l be the larger of the numbers $2l_1$ and $2l_2 + 1$. Then $|k_n - 1| < \varepsilon$ if $n > l$, i.e.,

$$\lim_{n\to\infty} k_n$$

exists and equals 1. In other words, the "limiting direction" of the telescope is the line making an angle of 45° with the positive x-axis.

48. a) In this case

$$k_n' = k_{2n} = \frac{b + 2n}{a + n}, \; k_n'' = k_{2n+1} = \frac{b + 2n}{a + n + 1},$$

and hence

$$\lim_{n\to\infty} k_n' = \lim_{n\to\infty} k_n'' = 2.$$

Therefore, by the same argument as in the solution to Problem 47, $\lim\limits_{n\to\infty} k_n$ exists and equals 2.

b) We now have

$$k_n' = k_{2n} = \frac{b + 2 + 4 + \cdots + 2n}{a + 1 + 3 + \cdots + (2n - 1)} = \frac{b + n^2 + n}{a + n^2},$$

$$k_n'' = k_{2n+1} = \frac{b + 2 + 4 + \cdots + 2n}{a + 1 + 3 + \cdots + (2n + 1)} = \frac{b + n^2 + n}{b + (n + 1)^2}.$$

It follows that

$$\lim_{n\to\infty} k_n' = \lim_{n\to\infty} k_n'' = 1,$$

and hence, as before, that $\lim\limits_{n\to\infty} k_n$ exists and equals 1.

c) Finally we have

$$k_n' = k_{2n} = \frac{b + 2 + 8 + \cdots + 2^{2n-1}}{a + 1 + 4 + \cdots + 2^{2n-2}} = \frac{b + \frac{2}{3}(4^n - 1)}{a + \frac{1}{3}(4^n - 1)},$$

$$k_n'' = k_{2n+1} = \frac{b + 2 + 8 + \cdots + 2^{2n-1}}{a + 1 + 4 + \cdots + 2^{2n}} = \frac{b + \frac{2}{3}(4^n - 1)}{a + \frac{1}{3}(4^{n+1} - 1)},$$

and hence

$$\lim_{n\to\infty} k_n' = 2, \quad \lim_{n\to\infty} k_n'' = \tfrac{1}{2}.$$

It follows that the sequence $\{k_n\}$ has no limit. In fact, if a sequence has a limit, then any of its subsequences has the same limit (prove this). But in our case the sequences $\{k_{2n}\}$ and $\{k_{2n+1}\}$ have different limits.

49. It follows from the similarity of the triangles $A_0M_nP_0$ and $A_nM_nP_0$ that

$$\frac{M_nP_0}{M_nP_n} = \frac{P_0A_0}{P_nA_n}. \tag{1}$$

Let x_n be the abscissa of the point M_n. Then (1) implies

$$\frac{a - x_n}{a + \frac{1}{n} - x_n} = \frac{a^2}{\left(a + \frac{1}{n}\right)^2},$$

and hence

$$x_n = \frac{a(an + 1)}{2an + 1} = \frac{a\left(a + \frac{1}{n}\right)}{2a + \frac{1}{n}}.$$

Therefore

$$\lim_{n\to\infty} x_n = \frac{\lim\limits_{n\to\infty} a\left(a + \frac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(2a + \frac{1}{n}\right)} = \frac{a^2}{2a} = \frac{a}{2}.$$

Geometrically, this fact can be stated as follows: The tangent to a parabola at any point bisects the segment joining the vertex of the parabola to the projection of the point of tangency onto the x-axis.

50. Suppose Johnny starts from the point A_1, and let B, C and D denote the school, the movie theater and the skating rink (see Fig. 11). Suppose Johnny's path is represented by the polygonal line $A_1A_2A_3A_4 \ldots$, where A_2 is the midpoint of the segment A_1B, A_3 is the midpoint of the segment A_2C, A_4 is the midpoint of the segment A_3D, etc. Then the points $A_1, A_4, A_7, \ldots, A_{3n+1}, \ldots$ all lie on the same line. This can be seen as follows: The segment A_2A_5 joins the midpoints of the sides A_1B and BA_4 of the triangle A_1BA_4 and hence is parallel to A_1A_4 and one half as long as A_1A_4. Similarly, examining the triangle A_2CA_5, we find that the segment A_3A_6 is parallel to

A_2A_5 and one half as long, while A_4A_7 is parallel to A_3A_6 and one half as long (consider the triangle A_3DA_6). Comparing these results, we see that each of the segments A_1A_4 and A_4A_7 is an extension of the other, with A_4A_7 eight times shorter than A_1A_4. In just the same way, we find that A_7A_{10} is the extension of A_4A_7, that $A_{10}A_{13}$ is the extension of A_7A_{10}, and so on. Therefore the points $A_1, A_4, A_7, \ldots, A_{3n+1}, \ldots$ all lie on the same line, as asserted. At the same time, we have shown that the distances between the points form a geometric progression with ratio $\frac{1}{8}$.

It is now an easy matter to show that the sequence A_1, A_4, $A_7, \ldots, A_{3n+1}, \ldots$ has a limit. In fact, let A_1 be the origin on the line passing through the points $A_1, A_4, A_7, \ldots, A_{3n+1}, \ldots$, and let the length of the sequence A_1A_4 be the unit of distance. Then the coordinate of the point A_{3n+1} equals

$$x_n = 1 + \frac{1}{8} + \frac{1}{8^2} + \cdots + \frac{1}{8^{n-1}} = \frac{1 - \left(\dfrac{1}{8}\right)^n}{1 - \dfrac{1}{8}}.$$

Therefore $\lim\limits_{n \to \infty} x_n$ exists and equals $\frac{8}{7}$.

Thus we have shown that the sequence of points A_1, A_4, A_7, $A_{3n+1}, \ldots$ approaches a point M. In just the same way, it can be shown that the sequence $A_2, A_5, A_8, \ldots, A_{3n+2}, \ldots$ approaches a point N and that the sequence $A_3, A_6, A_9, \ldots, A_{3n}, \ldots$ approaches a point P. Therefore Johnny's motion very soon approximates motion along the sides of the triangle MNP. As an exercise show that the position of the points M, N and P depends only on the position of the points B, C and D and not on the point A_1 from which Johnny's motion started.

51. First solution

On the given straight line, choose M_1 as the origin and the length of the segment M_1M_2 as the unit of distance. Then the coordinate x_n of the point M_n is related to the coordinates of the preceding two points by the formula

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}$$

(why?). As we now show,

$$x_n = \tfrac{2}{3}\,[1 + 2\,(-\tfrac{1}{2})^n], \tag{1}$$

so that in particular the sequence $\{x_n\}$ approaches $\frac{2}{3}$. To prove (1), we use induction noting first that (1) holds for $n = 1$ and $n = 2$ since

$$x_1 = \tfrac{2}{3}\,[1 + 2(-\tfrac{1}{2})] = 0,$$

$$x_2 = \tfrac{2}{3}\,[1 + 2(-\tfrac{1}{2})^2] = 1.$$

Suppose (1) holds for all $n \leqslant k$. Then it holds for $n = k + 1$, thereby completing the induction, since

$$\begin{aligned} x_{k+1} &= \frac{\frac{2}{3}\,[1 + 2(-\frac{1}{2})^k] + \frac{2}{3}\,[1 + 2(-\frac{1}{2})^{k-1}]}{2} \\ &= \tfrac{2}{3}\,[1 + (-\tfrac{1}{2})^k + (-\tfrac{1}{2})^{k-1}] \\ &= \tfrac{2}{3}\,[1 + (-\tfrac{1}{2})^k\,(1 - 2)] \\ &= \tfrac{2}{3}\,[1 + 2\,(-\tfrac{1}{2})^{k+1}]. \end{aligned}$$

Second solution

Consider the point N which divides the segment M_1M_2 in the ratio 2:1. Prove by induction that N divides all the segments M_2M_3, M_3M_4, ..., M_nM_{n+1}, ... in the same ratio. This immediately implies that the length of the segment M_nN is 2^{n-1} times the length of the segment M_1N. But then the sequence M_1, M_2, ..., M_n, ... has the point N as its limit.

52. a) By the formula for the sum of a geometric progression,

$$S_n = \frac{1 - a^{n+1}}{1 - a}.$$

If $|a| < 1$, then

$$\lim_{n\to\infty} S_n = \frac{\lim\limits_{n\leftarrow\infty}(1 - a^{n+1})}{1 - a} = \frac{1}{1 - a}.$$

If $|a| > 1$, then $S_n \to \infty$ as $n \to \infty$. Finally, if $|a| = 1$, we find that $S_n = n$ for $a = 1$ or

$$S_n = \frac{1 + (-1)^{n-1}}{2}$$

for $a = -1$, and in both cases S_n approaches no limit as $n \to \infty$.
b) Writing S_n in the form

$$S_n = (a + a^2 + \cdots + a^n) + (a^2 + a^3 + \cdots + a^n) + \cdots + (a^{n-1} + a^n) + a^n$$

and summing each of the geometric progressions in parentheses, we obtain

$$S_n = \frac{a - a^{n+1}}{1 - a} + \frac{a^2 - a^{n+1}}{1 - a} + \cdots + \frac{a^n - a^{n+1}}{1 - a}$$

$$= \frac{a + a^2 + \cdots + a^n - na^{n+1}}{1 - a} = \frac{\dfrac{a - a^{n+1}}{1 - a} - na^{n+1}}{1 - a}$$

$$= \frac{a - (n + 1 - na)\, a^{n+1}}{(1 - a)^2}.$$

It follows from this formula (cf. Prob. 44) that S_n approaches

$$\frac{a}{(1 - a)^2}$$

if $|a| < 1$, but that the series diverges if $|a| \geqslant 1$.
c) Since

$$\frac{1}{n(n + 1)} = \frac{1}{n} - \frac{1}{n + 1},$$

then

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1},$$

and hence

$$S = \lim_{n\to\infty} S_n = 1.$$

d) Since

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right],$$

then

$$S_n = \frac{1}{2}\left[\left(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right) + \left(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}\right)\right.$$

$$\left. + \cdots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{1\cdot 2} - \frac{1}{(n+1)(n+2)}\right],$$

and hence

$$S = \lim_{n\to\infty} S_n = \tfrac{1}{4}.$$

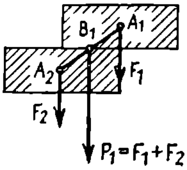
53. Clearly

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2},$$

since the left-hand side is the sum of n terms, the smallest of which equals $1/2n$. But then

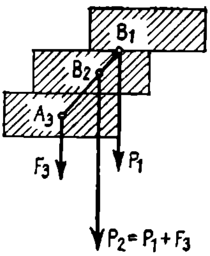
$$S_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$+ \cdots + \left(\frac{1}{2^{2n-1}+1} + \cdots + \frac{1}{2^n}\right) > 1 + \frac{n}{2},$$



$\frac{A_1B_1}{A_2B_1} = \frac{F_1}{F_2} = 1$

(a)



$\frac{B_1B_2}{A_3B_2} = \frac{F_3}{P_1} = \frac{1}{2}$

b)

Fig. 12

since each term in parentheses exceeds $\frac{1}{2}$. It follows that $S_n \to \infty$ as $n \to \infty$, and hence the series diverges.

54. The second brick can be shifted a distance $\frac{1}{2}$ relative to the first brick (the bricks are of unit length). The resultant P_1 of the forces F_1 and F_2 acts at a distance $\frac{1}{4}$ from the edge of the second brick (see Fig. 12a), and hence the third brick can be shifted a distance $\frac{1}{4}$ relative to the second brick. The point of application of the resultant of the forces F_3 and P_1 divides the segment between the points of application of the forces F_3 and P_1 in the ratio 2:1 (see Fig. 12b), since P_1 is twice as large as F_3, and hence the fourth brick can be shifted a distance $\frac{1}{6}$ relative to the third brick. By continuing this argument, it can be verified (do so, using mathematical induction) that the $(n + 1)$st brick can be shifted a distance $1/2n$ relative to the nth brick. Thus n bricks can be used to construct a "roof" of length

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2(n-1)}.$$

But the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges (see Prob. 53), and hence the roof can be made as long as we please.

55. Obviously the sequence $\{x_n\}$ satisfies the condition

$$x_{n+1} = 2 + \frac{1}{x_n}. \tag{1}$$

Suppose $\{x_n\}$ has a limit a. Then the left-hand side of (1) approaches a while the right-hand side approaches $2 + \frac{1}{a}$ (recall Prob. 38). It follows that

$$a = 2 + \frac{1}{a},$$

and hence

$$a = 1 \pm \sqrt{2},$$

where the minus sign is excluded since all the x_n are greater than 2. Therefore the limit of $\{x_n\}$ equals $1 + \sqrt{2}$, provided the limit exists.

To prove that the limit of $\{x_n\}$ actually exists, we now show that

$$\lim_{n\to\infty} y_n = 0,$$

where

$$y_n = x_n - (1 + \sqrt{2}). \tag{2}$$

Substituting (2) into (1), we find that

$$1 + \sqrt{2} + y_{n+1} = 2 + \frac{1}{1 + \sqrt{2} + y_n},$$

which implies

$$\begin{aligned} y_{n+1} &= \frac{2\left(1 + \sqrt{2} + y_n\right) + 1 - \left(1 + \sqrt{2} + y_n\right)}{1 + \sqrt{2} + y_n} \\ &= \frac{\left(1 - \sqrt{2}\right) y_n}{1 + \sqrt{2} + y_n}. \end{aligned} \tag{3}$$

But

$$|1 - \sqrt{2}| < \tfrac{1}{2},$$

while

$$1 + \sqrt{2} + y_n > 1$$

(the last inequality follows from $|y_n| < \frac{1}{2}$, which is easily proved by induction). Therefore

$$|y_{n+1}| = \frac{|1 - \sqrt{2}|\,|y_n|}{|1 + \sqrt{2} + y_n|} < \frac{1}{2}\,|y_n|,$$

which implies

$$|y_n| < \frac{|y_1|}{2^{n-1}} < \frac{1}{2^n}.$$

Therefore $\{y_n\}$ approaches zero, as asserted.

Remark. It can be shown that the sequence

$$n_1,\, n_1 + \frac{1}{n_2},\, n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3}},\, n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4}}},\, \ldots, \tag{4}$$

where $n_1, n_2, n_3, n_4, \ldots$ are arbitrary positive integers, always has some irrational number a as its limit. The limit of (4) is called a *continued fraction.* If the sequence $n_1, n_2, n_3, n_4, \ldots$ is *periodic*†, then a is of the form

$$r_1 + \sqrt{r_2}, \tag{5}$$

where r_1 and r_2 are rational numbers. The converse is also true: Every irrational number of the form (5) can be written as an infinite periodic continued fraction.

56. Suppose the sequence $\{x_n\}$ has a limit b. Then taking the limit as $n \to \infty$ of the relation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \tag{1}$$

we find that

$$b = \frac{1}{2}\left(b + \frac{a}{b}\right),$$

which implies

$$b^2 = a, \quad b = \pm\sqrt{a}.$$

If $x_0 > 0$, all the terms of the sequence are positive, while if $x_0 < 0$, all the terms of the sequence are negative. Therefore, in the first case

$$\lim_{n\to\infty} x_n = \sqrt{a},$$

† I.e., if $\{n_k\}$ is of the form

$$n_1, \ldots, n_{r-1}, n_r, n_{r+1}, \ldots, n_{r+s}, n_r, n_{r+1}, \ldots, n_{r+s}, \ldots,$$

where the block $n_r, n_{r+1}, \ldots, n_{r+s}$ repeats itself over and over again. In Problem 55, $\{n_k\}$ is obviously periodic, since $\{n_k\} = 2, 2, 2, \ldots$

while in the second case

$$\lim_{n\to\infty} x_n = -\sqrt{a}.$$

We must still prove that the limit of $\{x_n\}$ actually exists. To be explicit, let $x_0 > 0$ as in Figure 13, and let

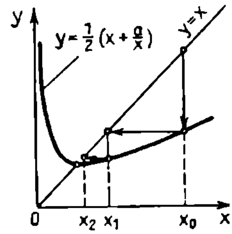


Fig. 13

$$y_n = \frac{x_n - \sqrt{a}}{\sqrt{a}}. \tag{2}$$

Substituting (2) into (1), we find that

$$\sqrt{a}(1 + y_{n+1}) = \frac{1}{2}\left[\sqrt{a}\,(1 + y_n) + \frac{a}{\sqrt{a}\,(1 + y_n)}\right],$$

which implies

$$y_{n+1} = \frac{y_n^2}{2(1 + y_n)}.$$

Since

$$1 + y_0 = 1 + \frac{x_0 - \sqrt{a}}{\sqrt{a}} = \frac{x_0}{\sqrt{a}} > 0,$$

the numbers y_n are all positive for $n \geqslant 1$, and therefore

$$|y_{n+1}| = y_{n+1} = \frac{y_n^2}{2(1 + y_n)} < \frac{y_n}{2}.$$

It follows that

$$|y_n| < \frac{y_1}{2^{n-1}}$$

and hence

$$\lim_{n\to\infty} y_n = 0$$

or

$$\lim_{n\to\infty} x_n = \sqrt{a}$$

because of (2).

Turning to the numerical example, let

$$a = 10, \quad x_0 = 3.$$

Then

$$y_0 = \frac{3 - \sqrt{10}}{\sqrt{10}}.$$

Since

$$(3.2)^2 = 10.24 < 10,$$

we have

$$\sqrt{10} < 3.2$$

Therefore

$$|y_0| = \frac{3 - \sqrt{10}}{\sqrt{10}} < \frac{0.2}{3} = \frac{1}{15}$$

and

$$|y_1| = \frac{y_0^2}{2\,(1 + y_0)} < \frac{\left(\frac{1}{15}\right)^2}{2\left(1 - \frac{1}{15}\right)} < \frac{1}{400}.$$

Moreover

$$|y_2| = \frac{y_1^2}{2\,(1 + y_0)} < \frac{\left(\frac{1}{400}\right)^2}{2} = \frac{1}{320000},$$

and hence

$$|x_2 - \sqrt{10}| = \sqrt{10}\,y_2 < \frac{\sqrt{10}}{320000} < 0.00001.$$

Thus to find $\sqrt{10}$ to within 0.00001, we need only determine x_2 as follows:

$$x_1 = \frac{1}{2}\left(x_0 + \frac{10}{x_0}\right) = \frac{1}{2}\left(3 + \frac{10}{3}\right) = \frac{19}{6} = 3.1666666\ \ldots,$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{10}{x_1}\right) = \frac{1}{2}\left(\frac{19}{6} + \frac{60}{19}\right) = \frac{721}{228}$$

$$= 3\,\frac{37}{228} = 3.1622807\ldots$$

Actually

$$\sqrt{10} = 3.1622776\ \ldots,$$

and hence our value of $\sqrt{10}$ differs from the true value by less than 0.00001, as asserted.

Test Problems

1. Prove that the sequence

$$1, 3, \frac{1}{2}, \frac{5}{2}, \frac{1}{3}, \frac{7}{3}, \ldots, \frac{1}{n}, \frac{2n+1}{n}, \ldots$$

does not have 2 as a limit.

2. Given that $x_n \to 1$ as $n \to \infty$, find the limit of the sequence $\{y_n\}$ if

a) $y_n = \dfrac{2x_n - 1}{x_n + 1}$;

b) $y_n = \dfrac{x_n^2 + 2x_n - 2}{x_n - 1}$;

c) $y_n = \dfrac{x_n^{10} - 1}{x_n - 1}$;

d) $y_n = \sqrt{x_n}$.

3. Prove that if a is the limit of a sequence $\{x_n\}$, then a is also the limit of an infinite subsequence of $\{x_n\}$.

4. Prove that if a is the limit of a sequence $\{x_n\}$, then a is also the limit of any sequence obtained by rearranging the terms of $\{x_n\}$.

5. Prove that if a is a limit point of a sequence $\{x_n\}$, then $\{x_n\}$ contains a subsequence with a as its limit. Is the converse true?

6. Find a sequence all of whose limit points are integers and for which every integer is a limit point.

7. Prove that if $x_n \to 0$ for all n and if $\lim\limits_{n\to\infty} x_n = a$, then $a \geqslant 0$.

8. Find the following limits:

a) $\lim\limits_{n\to\infty} \dfrac{n^2 + 3n - 2}{1 + 2 + \cdots + n}$;

b) $\lim\limits_{n\to\infty} \dfrac{n^2}{4^n}$;

c) $\lim\limits_{n\to\infty} \dfrac{n + \log_{10} n + 2^n}{n^2 - \log_{10} n - 2^n}$;

d) $\lim\limits_{n\to\infty} \left(\sqrt{n^2 + n} - \sqrt{n^2 - n}\right)$.

9. A sequence $\{x_n\}$ is constructed by the following rule: The first term is chosen arbitrarily and every subsequent term is expressed in terms of the preceding term by the formula

$$x_{n+1} = ax_n + b,$$

where a and b are constants. For what values of a and b does $\{x_n\}$ have a limit?

**10.* Prove that if all numbers containing the digit 3 (like 31, 223, etc.) are deleted from the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots,$$

then the sum of the remaining terms is finite.

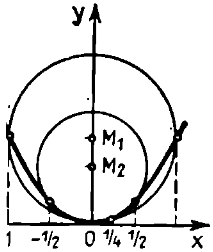


Fig. 14

11. On the graph of the function $y = x^2$, consider the points A_n and B_n with abscissas $\dfrac{1}{n}$ and $-\dfrac{1}{n}$, respectively. Draw the circle through A_n, B_n and the origin of coordinates, and let M_n be the center of this circle (see Fig. 14). Prove that the sequence of points $\{M_n\}$ has a limit. What is this limit?