Oriented Projective Geometry

(extended abstract)

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Oriented projective geometry is a model for geometric computation that combines the elegance of classical projective geometry with the ability to talk about oriented lines and planes, signed angles, line segments, convex figures, and many other concepts that cannot be defined within the classical version. Classical projective geometry is the implicit framework of many geometric computations, since it underlies the well-known homogeneous coordinate representation. It is argued here that *oriented* projective geometry — and its analytic model, based on *signed* homogeneous coordinates — provide a better foundation for computational geometry than their classical counterparts.

The differences between the classical and oriented versions are largely confined to the mathematical formalism and its interpretation. Computationally, the changes are minimal and do not increase the cost and complexity of geometric algorithms. Geometric algorithms that use homogeneous coordinates can be easily converted to the oriented framework at little cost. The necessary changes are largely a matter of paying attention to the order of operands and to the signs of coordinates, which are frequently ignored or left unspecified in the classical framework.

1. Classical projective geometry

The classical projective plane $\mathbf{P}_2$ is usually defined by means of three *models*.[2] The *straight model* consists of the Euclidean plane $\mathbf{E}_2$, augmented by a *line at infinity* Ω, and by an *infinity point* $d\infty$ for each pair of opposite directions $\{d, -d\}$. The point $d\infty = (-d)\infty$ is by definition on the line Ω and also on every line that is parallel to the direction d. The *spherical model* consists of the surface of a sphere, with diametrally opposite points identified. Lines of $\mathbf{P}_2$ are represented by great circles of the sphere, again with opposite points identified. See figure 1.

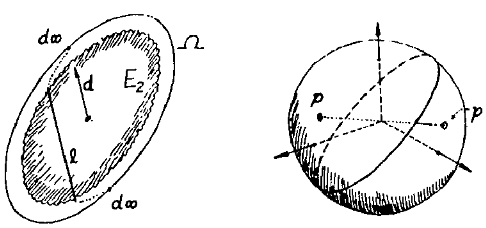

Figure 1. The straight and spherical models of $\mathbf{P}_2$.

The *analytic model* represents points and lines of $\mathbf{P}_2$ by their *homogeneous coordinates*. A point is by definition a non-zero triplet of real numbers $[w, x, y]$, with the convention that $[w, x, y]$ and $[\lambda w, \lambda x, \lambda y]$ are the same point, for all $\lambda \neq 0$. A line is also represented by a non-zero real triplet $\langle W, X, Y\rangle$, which by definition is incident to all points $[w, x, y]$ such that $Ww + Xx + Yy = 0$. Note that $\langle W, X, Y\rangle$ and $\langle \lambda X, \lambda Y, \lambda Z\rangle$ are the same line for all $\lambda \neq 0$.

The analytic and straight models of $\mathbf{P}_2$ are connected by the familiar homogeneous-to-Cartesian coordinate transformation, whereby the homogeneous triplet $[w, x, y]$ is mapped to the point $(x/w, y/w)$ of the Cartesian plane (the first coordinate w being called the *weight* of the triplet). That same triplet corresponds to the pair of antipodal points

$$\pm(w, x, y)/\sqrt{w^2 + x^2 + y^2}$$

of the spherical model. Geometrically, these mappings corresponds to *central projection* of $\mathbf{R}^3$ onto the unit sphere, or onto the plane π tangent to the sphere at $(1, 0, 0)$. See figure 2. Observe how this correspondence preserves points, lines, and their incidence relationships.

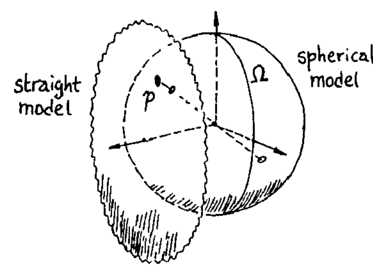

Figure 2. Central projection between the models of $\mathbf{P}_2$.

Projective geometry and homogeneous coordinates have several well-known advantages over its Cartesian counterparts. For one thing, they generally lead to simpler formulas that require no division operations; this reduces processing costs and makes it possible to do exact geometric computations with all-integer arithmetic. They are also able to represent points and lines at infinity, which are useful in algorithms as "sentinels" and in the elimination of special cases. For example, when computing the intersection of two lines we need not worry about them being parallel.

Duality is another important notion that is available only in projective geometry. Consider the one-to-one mapping $*$ from the points to the lines of $\mathbf{P}_2$, and vice-versa, that exchanges the point $[w, x, y]$ with the line $\langle w, x, y\rangle$. This mapping preserves incidence: if point p is on line l, then line p^* passes through point l^*. It follows that every definition, theorem, or algorithm of projective geometry has a *dual*, obtained by exchanging the word "point" with the word "line," and any previously defined concepts by their duals. For example, the assertion "there is a unique line incident to any two distinct points" dualizes to "there is a unique point incident to any two distinct lines." Duality is extremely useful in theory and practice; thanks to it, every proof automatically establishes the correctness of two very different theorems, and every geometrical algorithm automatically solves two very different problems. It is worth pointing out that in Cartesian geometry one cannot define such a perfect duality; at best, one has to leave out a subset of the lines (e.g., the vertical ones, or those passing through the origin). This leads to unnecessarily complicated theorems, and to algorithms with lots of annoying special cases.[3,4]

1.1. Drawbacks of classical projective geometry

In spite of its advantages, the projective plane has a few peculiar features that are annoying from the viewpoint of computational geometry.[5] Here are some of them:

- *The projective plane is not orientable.* There is no way to distinguish between "clockwise" or "counterclockwise" turns that is consistent over the whole plane $\mathbf{P}_2$. For the same reason, it is impossible to tell whether two triangles have the same or opposite handedness (which is a basic test in most geometric algorithms).

- *Lines have only one side.* If we remove a straight line from the projective plane, what remains a *single* connected set of points, topologically equivalent to a disk. Therefore, we cannot meaningfully ask whether two points are on the same side of a given line.

- *Segments are ambiguous.* In projective geometry we cannot consistently define *the* line segment connecting two points. Two points divide the line passing through them in *two* simple arcs, and there is no consistent way to distinguish one from the other.

- *Directions are ambiguous.* By the same token, we cannot define *the* direction from point p to point q. In particular, each point at infinity lies simultaneously in two opposite directions, as seen from a finite point. This property often makes it hard to use points at infinity as "sentinels" in geometric algorithms and data structures.

- *No convex figures.* Since we cannot define the segment between two points, it is not possible to define convex sets in a way that preserves their essential properties, such as closure under intersection.

2. Oriented projective geometry

Oriented projective geometry retains most advantages of the classical theory, but avoids the problems listed in the previous section. Its primitive objects are points and *oriented flats*: oriented lines, oriented planes, and so on. In particular, every straight line has an intrinsic *longitudinal orientation*, which determines a "forward" direction along the line at every one of its points.

Every line of the classical projective plane is thus replaced by two coincident but oppositely oriented (hence distinct) lines. In order to maintain the exact duality between points and lines, each point must also be replaced by two "oppositely oriented" copies. Algebraically, this means treating $[w, x, y]$ and $[-w, -x, -y]$ as distinct points, and similarly treating $\langle W, X, Y\rangle$ and $\langle -W, -X, -Y\rangle$ as distinct lines. The resulting set of points is topologically a double covering of the projective plane. (Accordingly, I will use *two-sided* as a synonym of *oriented projective*.) The set of points is in fact topologically equivalent to a sphere, with straight lines corresponding to oriented great circles; therefore, oriented projective geometry is simply an oriented version of spherical geometry.

The double covering makes it possible to postulate an intrinsic *circular orientation* for the whole plane, which defines the "positive sense of turning" at every point, in a consistent way. This allows us to talk about the orientation of other objects in absolute terms: we can say that a triangle is positively oriented, without having to specify a "reference" triangle every time. The global orientation of the plane also makes it possible to use the "forward" direction of a line to define its "left" and "right" sides.

In oriented projective geometry, a pair of points p, q generally determines not one but *two* distinct lines, with same position but opposite orientations. We can still unambiguously speak of *the* line from p to q, if we pay attention to the order of those two points, distinguishing between the line joining p to q and the one joining q to p. Dually, two lines l and m on the plane have generally two points in common, so we must distinguish the point where l meets m from the point where m meets l. We will see that the two can be unambiguously defined by taking into account the orientations of l and m, *and* the intrinsic orientation of the plane.

The previously mentioned advantages of projective geometry are retained in the oriented version. In particular, we are still able to define an exact duality between points and lines that preserves not only the incidence properties of all objects, but also their relative orientations. In addition, the oriented version allows us to define the concept of convexity in a truly projective way. Unlike the Cartesian definition, the new one is unaffected by arbitrary projective maps and duality: we can finally say that the problem of intersecting n half-planes is *exactly* dual to finding the convex hull of n points, and not *approximately* so. Indeed, the ability to support both convexity and duality is perhaps the greatest advantage of the new framework.

All of this extends quite nicely to spaces of higher dimension. Once we leave the plane, the orientation of objects becomes much harder to visualize and to reason about. A great advantage of oriented projective geometry is that it gives us effective and reliable tools for doing this. Oriented projective geometry can be viewed as the marriage of projective geometry with an *algebra of orientations.*

2.1. Relation to previous work

Hermann Grassmann seems to have been the first to consider a geometric calculus based on two dual products (what we call join and meet), about a hundred years ago. His ideas were explored and reformulated by several other mathematicians since then, notably Clifford, Schröder, Witehead, Cartan, and Peano. For an example of a recent work along the same general lines, see the book by Hestenes and Sobczyk.[7]

For some reason, the geometric calculus developed by those authors was relegated to relative obscurity, and its usefulness for practical computations has been largely ignored so far. Part of the reason may be the highly abstract language, excessive generality, and heavy mathematical formalism used in most expositions, which make the fundamental ideas seem much more complicated than what they really are.

The notation used in this paper is quite similar to the one used in a recent paper by Barnabei, Bieri and Rota,[1] although it was developed independenly from their work. The notion of an oriented flat as defined below is clearly connected to what they call an *extensor*, or decomposable antisymmetric tensor. More precisely, the flats of oriented projective geometry are the equivalence classes we obtain by considering two extensors equivalent iff they differ by a positive scalar factor. Compared to their paper, the present one gives more emphasis to the geometric (as opposed to algebraic) aspects of the calculus, and in particular to its suitability as the common language of computational geometry.

3. Oriented projective spaces

Rather than defining oriented projective geometry by a list of axioms, I will construct a *canonical two-sided space* $\mathbf{T}_\nu$ for each dimension ν. That object will consist of an oriented manifold $\mathbf{U}_\nu$ (whose elements are called *points*), a collection $\mathcal{F}_\nu$ of oriented submanifolds of $\mathbf{U}_\nu$ (the *flats*), and two operations $\vee_\nu$ and $\wedge_\nu$ (*join* and *meet*) acting on $\mathcal{F}_\nu$. Then I will be able to define a generic oriented projective space as any algebraic structure isomorphic to $(\mathcal{F}_\nu, \vee_\nu, \wedge_\nu)$ for some ν. Actually, I will construct three equivalent versions of $\mathbf{T}_\nu$, analogous to the straight, spherical, and analytic models of $\mathbf{P}_2$.

The *spherical model* of $\mathbf{T}_\nu$ consists of the unit sphere $\mathbf{S}_\nu$ of $\mathbf{R}^{\nu+1}$, that is, the set of all points $(x_0, .. x_\nu)$ of $\mathbf{R}^{\nu+1}$ such that $\sum x_i{}^2 = 1$. Note that diametrally opposite points are *not* identified. The *straight model* of $\mathbf{T}_\nu$ consists of two copies U^+ and U^- of $\mathbf{R}^\nu$, and one *point at infinity* $d\infty$ for every direction vector d in $\mathbf{R}^\nu$. The sets U^+ and U^- are called the *front* and *back ranges* of $\mathbf{T}_\nu$.

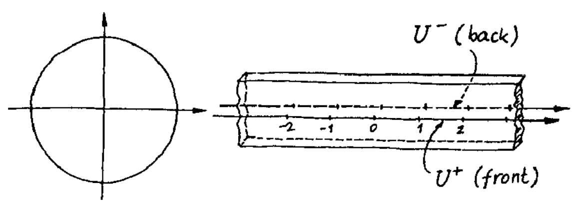

Figure 3. The spherical and straight models of $\mathbf{T}_1$.

For example, the spherical model of $\mathbf{T}_1$ (the *two-sided line*) is modeled by the unit circle of $\mathbf{R}^2$. The straight model of $\mathbf{T}_1$ consists of two copies of the real line $\mathbf{R}$, and two points at infinity $+\infty$ and $-\infty$. We can visualize this model as as an infinitely long ruler with graduated scales on both sides. See figure 3.

Similarly, the spherical model of $\mathbf{T}_2$ (the *two-sided plane*) is the unit sphere of $\mathbf{R}^3$. The straight model of $\mathbf{T}_2$ consists of two copies of $\mathbf{R}^2$, and an infinity point $d\infty$ for every direction d of $\mathbf{R}^2$. We can visualize U^+ and U^- as the front and back sides of an infinite sheet of paper. Figure 4 is a sketch of this model, where the front and back ranges are represented by two copies of the open unit disk. The infinity point $d\infty$ is represented by point d on the boundary of the front disk, and point $-d$ on the boundary of the back one.

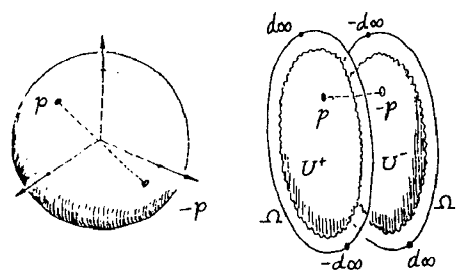

Figure 4. The spherical and straight models of $\mathbf{T}_2$.

Note that the infinity points $d\infty$ and $(-d)\infty$ are *not* identified (unlike the conventions of standard projective geometry). Each infinity point is a limit point of both faces, but the connection is somewhat peculiar: by definition, $d\infty$ is a limit point of the front face in the direction d, and of the back face in the *opposite* direction $-d$. The straight model of $\mathbf{T}_3$ is entirely analogous.

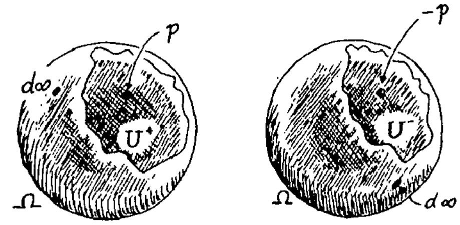

Figure 5. The straight model of $\mathbf{T}_3$.

The *analytic model* of $\mathbf{T}_\nu$ consists of the non-zero vectors of $\mathbf{R}^{\nu+1}$, where two vectors are considered to be the same point if one is a *positive* multiple of the other. I will denote by $[u_0, .. u_\nu]$ the point represented by the vector $(u_0, .. u_\nu)$ and its positive multiples; any of those vectors is called the (*signed*) *homogeneous coordinates* of that point. Obviously, $[u_0, .. u_\nu] = [v_0, .. v_\nu]$ if and only if $u_i = \alpha v_i$ for for all i and some *positive* real α. Note that $[u_0, .. u_\nu]$ and $[-u_0, .. -u_\nu]$ are distinct points of $\mathbf{T}_\nu$.

The three models are related by *central projection* from the origin of $\mathbf{R}^{\nu+1}$. In algebraic terms, a point $[w, x, y, \ldots, z]$ of the analytic model corresponds to the points

$$(w, x, y, \ldots, z)/\sqrt{w^2 + x^2 + \cdots + z^2}$$

of the spherical model and $(x/w,\ y/w, \ldots,\ z/w)$ of the straight model. By definition, the latter is on the front range if $w > 0$, on the back range if $w < 0$, and on the line at infinity in the direction $(x, y, \ldots, z)$ if $w = 0$. See figure 6. I will use this standard correpondence to jump back and forth between the three models, and I will usually treat them as if they were the same object.

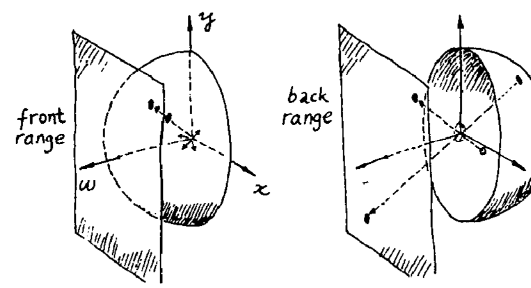

Figure 6. Central projection between the models of $\mathbf{T}_2$.

4. Flats

A *great subsphere* of $\mathbf{S}_\nu$ is the intersection C of $\mathbf{S}_\nu$ and some linear (vector) subspace V of $\mathbf{R}^{\nu+1}$. From elementary linear algebra we know that the bases of V can be divided in two equivalence classes, with two bases being in the same class if and only if they are related by a matrix with positive determinant. I will identify those two classes with the two *orientations* of the subsphere C. By naming one of these classes the set of *positive* bases, we get an *oriented great subsphere* of $\mathbf{S}_\nu$.

By definition, a *flat of dimension* κ of $\mathbf{T}_\nu$ (for $0 \le \kappa \le \nu$) is a κ-dimensional oriented great subsphere of $\mathbf{S}_\nu$. The *set* of points in a flat, without regard to its orientation, is an *unoriented flat*. Flats of dimension 1,2,3,... are called *lines*, *planes*, *three-spaces*, and so on. The flats of dimension zero can be identified with the points of $\mathbf{T}_\nu$.

The *opposite* $-f$ of a flat f is the same subsphere taken with the opposite orientation. If p is a point, $-p$ is also called its *antipode*; it is the point diametrally opposite to p in the spherical model, or the point of the straight model with same position as p but on the opposite range. In addition to the flats defined above, I will postulate two flats of dimension minus one, the *positive vacuum* $\mathbf{1}$ and its opposite, the *negative vacuum* $-\mathbf{1}$. They should be regarded as oriented versions of the empty set.

If we reduce every vector of a basis to unit length we get a sequence $s = (s^0, .. s^\kappa)$ of $\kappa + 1$ points of $\mathbf{T}_\nu$ that do not lie in any flat with dimension lower than κ. I call such an object a *κ-dimensional simplex*. Therefore, the orientation of a flat f consists of a class of equivalent simplices, the *positive simplices* of f. A κ-dimensional simplex s determines a unique flat of $\mathbf{T}_\nu$, consisting of the smallest subsphere containing s, oriented so that s is a positive simplex. I denote that flat by $<s> = <s^0, .. s^\kappa>$. By definition, the *universe* $\mathbf{U}_\nu$ of $\mathbf{T}_\nu$ is the flat determined by the standard basis of $\mathbf{R}^{\nu+1}$ (the *standard simplex* of $\mathbf{T}_\nu$). The universe $\mathbf{U}_\nu$ and its opposite $-\mathbf{U}_\nu$ are the only two flats of dimension ν.

In particular, a line (1-dimensional flat) is an oriented great circle of the unit sphere $\mathbf{S}_\nu$. The orientation can be visualized as an arrow that tells which direction along the circle is *positive* ("forward"). In the straight model, a line is either an oriented "circle at infinity," or two copies of the same Euclidean line, one on each range, directed the same way. See figure 7. The universe of $\mathbf{T}_1$ is oriented counter-clockwise, that is, from $(1, 0)$ to $(0, 1)$ by the shortest route.

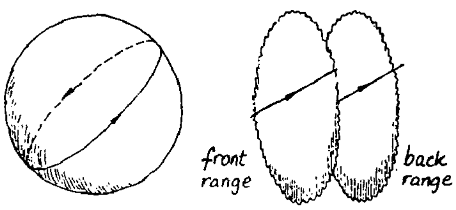

Figure 7. Lines of $\mathbf{T}_2$.

A plane (2-dimensional flat) is an oriented great 2-sphere of $\mathbf{S}_\nu$. An orientation for a 2-sphere can be visualized as a small "circular arrow" painted on it. By sliding that arrow over the sphere, we can tell whether a turn at any point is *positive* (agreeing with the arrow) or *negative*. In the straight model, a plane is either an oriented "sphere at infinity," or two copies of the same Euclidean plane, one on each range, oriented in the *opposite* way. See figure 8. The universe of $\mathbf{T}_2$ is oriented counter-clockwise on the front and clockwise on the back.

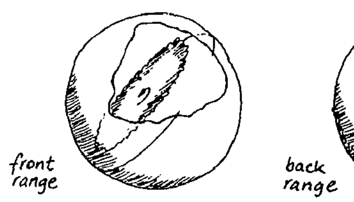

Figure 8. Planes of $\mathbf{T}_3$.

Observe that if we restrict our attention to the front range of $\mathbf{T}_\nu$ and to the front part of every flat, we get the geometric strucuture of the ν-dimensional Euclidean space. Therefore, oriented projective geometry is able to simulate all the constructions and algorithms of affine, Euclidean and Cartesian geometry.

The *rank* of a flat is its dimension plus one. In order to simplify the formulas, I will adopt the following convention: the Greek letters $\kappa, \mu, \nu, \rho, \sigma, \tau$ will usually denote dimensions, and the the corresponding italic letters k, m, n, r, s, t will usually denote the corresponding ranks. The identities $k = \kappa + 1$, $m = \mu + 1$, and so on will be assumed throughout. Since I will be usually working in a fixed space of dimension ν, I will also write $\mathbf{T}$, $\mathbf{U}$, $\vee$, etc. instead of in $\mathbf{T}_\nu$, $\mathbf{U}_\nu$, $\vee_\nu$, etc.

5. The join operation

The join operation of classical projective geometry produces the smallest flat containing two given flats. In general, the join of two points is the line passing through them; the join of a point and a line is the plane containing both; and so on. This is mostly true in oriented projective geometry, too, except that arguments and result are oriented flats.

Let's begin with the join of two points in the spherical model of $\mathbf{T}_\nu$. Two points p, q generally determine a unique great circle of $\mathbf{S}_\nu$, and divide that circle in two unequal arcs. The *segment pq* is, by definition, the shorter of these two arcs (the one that does not contain any antipodal pairs). If we orient the circle from p to q along the segment pq, we get the *join of p to q*, denoted by $p \vee q$. See figure 9(a). In the straight model, the segment pq may be a single Euclidean segment (if both points are on the same range), a ray (if only one of them is finite), an arc on Ω (if both are infinite), or a a pair of oppositely directed rays (if the points lie one on each range). See figure 9(b).

Figure 9. The join of two points.

The line $p \vee q$ is defined if and only if the two points are distinct and not antipodal. Note that $q \vee p$ is oriented in the direction opposite to that of $p \vee q$. That is, the join of two points is *anticommutative*: for all independent p, q,

$$q \vee p = -(p \vee q) \tag{1}$$

Note also that

$$p \vee (-q) = (-p) \vee q = -(p \vee q) \tag{2}$$

5.1. The join of general flats

I will now generalize the join operation to arbitrary flats. Recall that a flat f of rank k is completely determined by a proper simplex with k vertices that spans f and has positive orientation in f. By definition, the join of two flats determined by simplices $(u^0, .. u^\kappa)$ and $(v^0, .. v^\mu)$ is the flat defined by their concatenation, that is

$$<u^0, .. u^\kappa> \vee <v^0, .. v^\mu> = <u^0, .. u^\kappa, v^0, .. v^\mu>. \tag{3}$$

With a little linear algebra we can prove that the concatenation of two proper simplices is a proper simplex if and only if the corresponding flats have no point in common. If they do, their join is undefined. It is easy to see that the result of the join is the same no matter which simplices we choose to represent the two flats.

The join of flats with rank 0 is defined explicitly, by letting $\mathbf{1} \vee a = a \vee \mathbf{1} = a$ and $(-\mathbf{1}) \vee a = a \vee (-\mathbf{1}) = -a$ for all a. In other words, $\mathbf{1}$ is the left and right identity of join. Note that the flats 1 and -1 are disjoint from any other flat, including themselves; they behave like oriented empty sets (hence their names). Note that every flat of rank $k \geq 1$ is the join of of the k vertices of any of its positive simplices. Obviously,

$$\operatorname{rank}(a \vee b) = \operatorname{rank}(a) + \operatorname{rank}(b) \tag{4}$$

whenever the join is defined.

For an example, consider the join of a line l and a point p not on l. If (r, s) is a positive simplex of l (that is, if $l = r \vee s$) then $p \vee l$ will be the plane containing p and l, oriented so that the triangle prs is positive. In other words, l turns in the positive sense as seen from p. See figure 10.

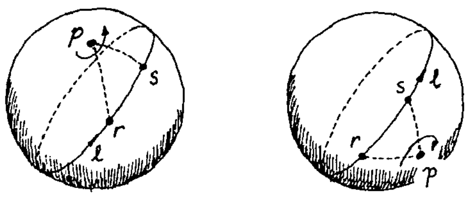

Figure 10. Join of a point to a line

The associativity of $\vee$ follows immediately from the definition: for any flats a, b, c,

$$a \vee (b \vee c) \;=\; (a \vee b) \vee c \tag{5}$$

$$(-a) \vee b \;=\; a \vee (-b) \;=\; -(a \vee b) \tag{6}$$

whenever either side of the equality is defined.

Observe that transposing the order of two vectors in a simplex reverses its orientation. Observe also that in order to go from $<u^0, .. u^\kappa, v^0, .. v^\mu>$ to $<v^0, .. v^\mu, u^0, .. u^\kappa>$ we have to transpose $(\kappa + 1)(\mu + 1)$ adjacent pairs of vectors. We conclude that for any two flats a, b,

$$a \vee b \;=\; (-1)^{\operatorname{rank}(a)\operatorname{rank}(b)}(b \vee a) \tag{7}$$

Therefore, $a \vee b = b \vee a$ if a or b has even rank, and $a \vee b = -(b \vee a)$ if both have odd rank. Thus, for example, the join of a point and a line is commutative, but that of a point and a plane is not.

6. The meet operation

Another fundamental operation of classic geometry is computing the *meet* (intersection point) of two lines. One can also intersect a line and a plane to get a point, two planes to get a line, and in general two flats to get a flat.

In classic geometry, flats are just sets of points, and meet is just set-theoretical intersection. In two-sided geometry, the the meet operation must specify also an orientation for the result. For example, two lines of $\mathbf{T}_2$ generally intersect on a pair of antipodal points. See figure 11. To choose an orientation for the intersection means to pick one of the two points as *the* meeting point of the two lines.

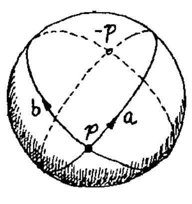

Figure 11.The meet of two lines in the plane.

Note that the shortest turn from the direction of a to that of b is positive (counterclockwise) at one of the two points, and negative at the other. By definition, the first one is the *point where a meets b*, denoted by $a \wedge b$. We can also define this point as follows: for any three points p, q, r of $\mathbf{T}_2$,

$$p \vee q \vee r = \mathbf{U}_2 \;\Leftrightarrow\; (p \vee q) \wedge (q \vee r) = q \tag{8}$$

The meet $a \wedge b$ is not defined when $a = b$ or $a = -b$. Like the join of two points, the meet of two lines is anticommutative, and depends on the orientation of its operands. For any two lines a, b of $\mathbf{T}_2$, we have

$$\begin{gathered} b \wedge a = -(a \wedge b) \\ (-a) \wedge b = a \wedge (-b) = -(a \wedge b) \end{gathered} \tag{9}$$

Note that the meet of two lines as defined above depends strongly on the global orientation of $\mathbf{T}_2$, as well as on those of the operands. This is not an accident; it turns out that it is not possible to consistently select one of the intersection points without a reference orientation for the whole plane. For that reason, the meet of two lines it cannot be extended to a pair of coplanar lines in $\mathbf{T}_3$ or in a higher-dimensional space, since there is no consistent way to pick an orientation for the plane containing them.

6.1. Meet of general flats

The meet of arbitrary flats can be defined by a straightforward extension of formula (8). Namely, for any flats a, b, c of $\mathbf{T}_\nu$,

$$a \vee b \vee c = \mathbf{U} \;\Leftrightarrow\; (a \vee b) \wedge (b \vee c) = b \tag{10}$$

It can be shown that the result of $(a \vee b) \wedge (b \vee c)$ does not depend on a and c, as long as $a \vee b$ and $b \vee c$ are the given flats. That is, from $a' \vee b' \vee c' = \mathbf{U} = a \vee b \vee c$, $a' \vee b' = a \vee b$, and $b' \vee c' = b \vee c$, we can deduce $b' = b$. Note that equation (10) determines $f \wedge g$ only if $\mathbf{U}$ is the smallest space enclosing both f and g; which is to say, if and only if $\operatorname{rank}(f) + \operatorname{rank}(g) - \operatorname{rank}(f \cap g) = \operatorname{rank}(\mathbf{U})$. When that is not the case, $f \wedge g$ is undefined. Note also that the orientation of the result depends on that of the universe $\mathbf{U}$.

For example, consider the meet of a plane π and a line l in $\mathbf{T}_3$. A line and a plane generally have two antipodal points in common. According to the definition, $l \wedge \pi = x$ if and only if there are points p, q on π and r on l such that (p, q, x) is a positive triangle of π, (x, r) is a positive pair on l, and (p, q, x, r) is a positive tetrahedron of $\mathbf{T}_3$. We conclude that $\pi \wedge l$ is the point where the circular arrow of π and the longitudinal arrow of l are like the fingers and thumb of the right hand. See fig. 12.

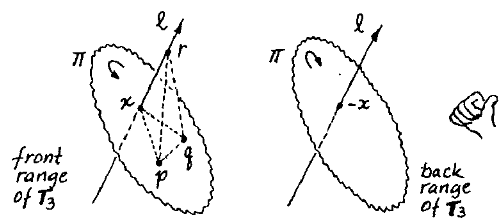

Figure 12. The meet of a line and a plane in $\mathbf{T}_3$.

The intersection of two planes π, σ in $\mathbf{T}_3$ is a pair of oppositely oriented lines. According to (10), we must find a positive tetrahedron (p, q, r, s) of $\mathbf{T}_3$ such that (p, q, r) is a positive triangle of π and (q, r, s) is a positive triangle of σ. Then $\pi \wedge \sigma$ will be the line from q to r. See figure 13. Note that $\sigma \wedge \pi = -(\pi \wedge \sigma)$; that is, the meet of two planes in $\mathbf{T}_3$ is anticommutative.

When dealing with the meet operation, it is convenient to classify the flats of $\mathbf{T}_\nu$ by their *complementary rank* or *co-rank*, defined by $\operatorname{corank}(f) = \operatorname{rank}(\mathbf{U}) - \operatorname{rank}(f)$. The co-rank of f is how many points must be joined to f to get the universe. Clearly,

$$\operatorname{corank}(a \wedge b) = \operatorname{corank}(a) + \operatorname{corank}(b) \tag{11}$$

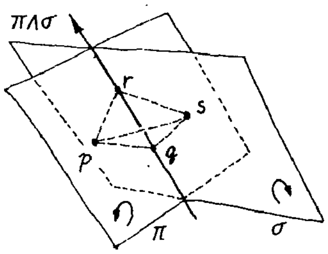

Figure 13. The meet of two planes in $\mathbf{T}_3$.

and

$$\operatorname{rank}(a \wedge b) \;=\; \operatorname{rank}(a) - \operatorname{corank}(b) \;=\; \operatorname{rank}(b) - \operatorname{corank}(a) \quad (12)$$

Since hyperplanes have co-rank equal to one, it follows that the co-rank of a flat f is also the number of hyperplanes we have to intersect in order to get f.

It follows from the definition that $a \wedge \mathbf{U} = \mathbf{U} \wedge a = a$ for all a, and that $(-a) \wedge b = a \wedge (-b) = -(a \wedge b)$ whenever one of the meets is defined. Finally, for any flats a, b of $\mathbf{T}_\nu$ we have

$$b \wedge a = (-1)^{\operatorname{corank}(a)\operatorname{corank}(b)}\,(a \wedge b) \qquad (13)$$

That is, $\wedge$ is anti-commutative if and only if both operands have odd co-rank. In three dimensions and less, these "odd" cases are: two points in $\mathbf{T}_1$; two lines in $\mathbf{T}_2$; two planes, or a point and a plane, in $\mathbf{T}_3$.

6.2. Null objects

Recall that $a \vee b$ is undefined if a and b have a common point. It may be tempting to plug this hole, and extend the definition so that $a \vee b$ is always one of the two flats of smallest dimension containing a and b. Unfortunately this extension cannot be made to work in the oriented framework. If a and b are not disjoint, it is impossible to define the orientation of $a \vee b$ in a consistent way. Similarly, we cannot extend $a \wedge b$ to all cases, since there is no consistent way to orient the intersection of the two sets of points if they do not span the whole universe.

Nevertheless, partially defined operations are quite a nuisance, especially from the programmer's point of view. To make life easier, I will postulate for every k a dummy object $\mathbf{0}^k$, the *null* or *indeterminate object of rank k* (which is *not* to be considered a flat of $\mathbf{T}_\nu$). I will let $a \vee b$ be $\mathbf{0}^{\operatorname{rank}(a)+\operatorname{rank}(b)}$ and $a \wedge b = \mathbf{0}^{\operatorname{rank}(a)-\operatorname{corank}(b)}$, whenever those operations would be undefined by the rules given above. I will also postulate $-\mathbf{0}^k = \mathbf{0}^k$, $\mathbf{0}^k \vee a = a \vee \mathbf{0}^k = \mathbf{0}^{k+\operatorname{rank}(a)}$, and $\mathbf{0}^k \wedge a = a \wedge \mathbf{0}^k = \mathbf{0}^{k-\operatorname{corank}(a)}$, for all a. As we shall see, this extension is quite natural from the computational point of view, and can be implemented at zero or negative cost. With these rules, all properties of join and meet listed so far are always true, even when the operands are not disjoint and/or are null objects.

6.3. Relative orientation

One advantage of two-sided geometry is that it allows us to consistently define the two sides of a line in the plane, or of a plane in three-space. For example, a line l divides the two-sided plane in two halves. I call these the *left* and *right* (or *positive* and *negative*) sides of l, as they would be seen by an ant crawling along the line on the outside of the unit sphere. See figure 14.

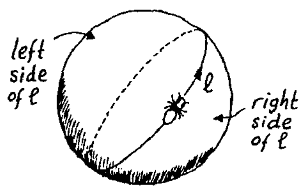

Figure 14. The two sides of a line.

We can express these concepts in terms of $\vee$, by observing that p is on the left side of l if $p \vee l = +\mathbf{U}$, on the right side if $p \vee l = -\mathbf{U}$, and on l itself if $p \vee l = \mathbf{0}^n$. In terms of $\wedge$, these three cases correspond to $p \wedge l = +\mathbf{1}$, $p \wedge l = -\mathbf{1}$, and $p \wedge l = \mathbf{0}^0$, respectively. Notice how this definition derives the "transversal" (left-right) orientation of l from the "longitudinal" orientation of l and the intrinsic "counterclockwise" orientation of the universe $\mathbf{U}$.

The same idea can be used to test the relative positions of a point and a plane in $\mathbf{T}_3$, or, in general, of any two flats a, b of $\mathbf{T}_\nu$ whose ranks add to n. We define the *relative orientation of a and b* as the sign-valued function

$$a \diamond b = \left\{\begin{array}{c} +1 \\ 0 \\ -1 \end{array}\right\} \text{ iff } a \vee b = \left\{\begin{array}{c} +\mathbf{U} \\ \mathbf{0}^n \\ -\mathbf{U} \end{array}\right\} \text{ iff } a \wedge b = \left\{\begin{array}{c} +\mathbf{1} \\ \mathbf{0}^0 \\ -\mathbf{1} \end{array}\right\}$$

When $a \diamond b = +1$, I say that *a is positively oriented with respect to b*. In particular, when a is a point and b is an hyperplane, we say that a is on the *positive side of b*. Note that the order of a and b matters if and only if $\operatorname{rank}(a)\operatorname{rank}(b)$ is odd, that is, if and only if the space has odd dimension and one of the operands has even dimension. This is the case when testing a pair of points on a line, or a point against a plane in three-space.[1]

For example, in the two-sided line $\mathbf{T}_1$ the positive side of a point (= hyperplane) b is the half-line *before* b, i.e. the arc from $-b$ to b. In the three-dimensional space $\mathbf{T}_3$, the positive side of a plane b can be visualized as follows. Imagine the orientation of b as a circular arrow painted somewhere on its front range. Now imagine a "normal" arrow on the same spot, perpendicular to π and oriented according to the right-hand rule (figure 15). This arrow will point *out* of the positive side of b and into its negative side. (Note that on the back range both the circular arrow on b and the normal one are reversed.)

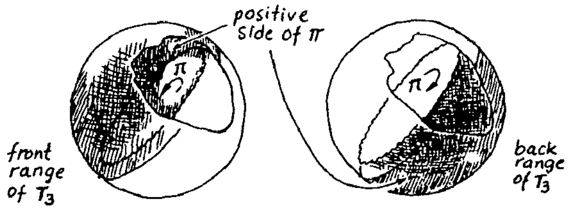

Figure 15. The two sides of a plane.

If l and m are two lines of $\mathbf{T}_3$, $l \diamond m = +1$ if and only if m "turns around" the line l according to the right-handed rule.

[1] Observe that we could in priciple replace $\vee$ and $\wedge$ by a single operation $\diamond$, where $a \diamond b$ is $a \vee b$ if $\operatorname{rank}(a) + \operatorname{rank}(b) \le n$, and $a \wedge b$ if $\operatorname{rank}(a) + \operatorname{rank}(b) \ge n$. However, there doesn't seem to be a good geometric justification for this amalgamated operation. Therefore, I will let this opportunity pass, and restrict the use of $\diamond$ to the case $\operatorname{rank}(a) = \operatorname{rank}(b) = n$.

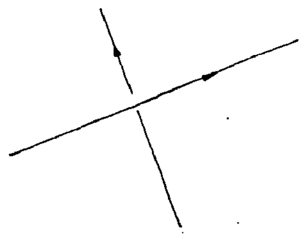

Figure 16. Two positively oriented lines of $\mathbf{T}_3$.

7. Duality

The operations of meet and join have very similar properties. Compare for example the formulas

$$\begin{array}{ll} a \vee \mathbf{1} = a & a \wedge \mathbf{U} = a \\ a \vee (-b) = -(a \vee b) & a \wedge (-b) = -(a \wedge b) \\ b \vee a = (-1)^{rs}(a \vee b) & b \wedge a = (-1)^{r's'}(a \wedge b) \end{array} \qquad (14)$$

where $r = \text{rank}(a)$, $s = \text{rank}(b)$, $r' = \text{corank}(a)$, $s' = \text{corank}(b)$. This phenomenon is the principle of *projective duality*. In what follows, I will make this concept more precise.

I will say that two points of $\mathbf{T}_\nu$ are *orthogonal* if they are represented by orthogonal vectors in the spherical model. In general, two flats a, b are *orthogonal*, denoted by $a \perp b$, if every point on one of them is orthogonal to every point on the other. The *right orthogonal complement* of a flat a of $\mathbf{T}_\nu$ is the flat $a^\vdash$ such that

$$a \perp a^\vdash \quad \text{and} \quad a \vee a^\vdash = \mathbf{U} \qquad (15)$$

It is not hard to see that $a^\vdash$ always exists and is unique, is a continuous function of a, is disjoint from a, and satisfies $\text{rank}(a^\vdash) = \text{rank}(\mathbf{U}) - \text{rank}(a) = \text{corank}(a)$. The inverse of $\vdash$ is the *left orthogonal complement* $\dashv$, defined by $a^\dashv \perp a$ and $a^\dashv \vee a = \mathbf{U}$. If $rank(a) = r$ and $\text{rank}(\mathbf{U}) = n$, it is easy to see that

$$a^\vdash = (-1)^{r(n-r)}\, a^\dashv. \qquad (16)$$

and therefore $(a^\vdash)^\vdash = (a^\dashv)^\dashv = (-1)^{r(n-r)}\, a$. Note that when n is odd, the product $r(n-r)$ is always even. Therefore, in spaces of odd rank (even dimension) $\dashv$ and $\vdash$ are the same function, which will denote by $\perp$. In spaces of even rank (odd dimension), we have $a^\vdash = a^\dashv$ or $a^\vdash = -a^\dashv$, depending on whether the rank of a is even or odd.

In the spherical model of the two-sided plane (rank 3), the single orthogonal complement $\perp$ takes every oriented great circle of $\mathbf{S}_2$ to the apex of its left hemisphere, and, conversely, every point of $\mathbf{S}_2$ to the oriented great circle whose left hemisphere has that point at the apex. See figure 17(a). In the straight model, the image $l^\perp$ of a line l passing at distance $d > 0$ from the origin O is the point p such that the vector Op is perpendicular to l, directed away from l, and with length $1/d$. The point $l^\perp$ will be on the front face if and only if the line is directed counterclockwise as seen from the front origin, i.e. if the origin lies on the positive side of l. See figure 17(b). If l passes through the origin, $l^\perp$ is the point at infinity in the direction 90° counterclockwise from the direction of l. Finally, if $l = \pm\Omega$, then $l^\perp = \pm O$ (the origin on the front and back range).

It can be shown that $\dashv$ and $\vdash$ have the following properties:

$$\begin{array}{ll} \mathbf{1}^\vdash = \mathbf{U} & \mathbf{1}^\dashv = \mathbf{U} \\ \mathbf{U}^\vdash = \mathbf{1} & \mathbf{U}^\dashv = \mathbf{1} \\ (-a)^\vdash = -(a^\vdash) & (-a)^\dashv = -(a^\dashv) \\ (a \vee b)^\vdash = a^\vdash \wedge b^\vdash & (a \vee b)^\dashv = a^\dashv \wedge b^\dashv \\ (a \wedge b)^\vdash = a^\vdash \vee b^\vdash & (a \wedge b)^\dashv = a^\dashv \vee b^\dashv \end{array} \qquad (17)$$

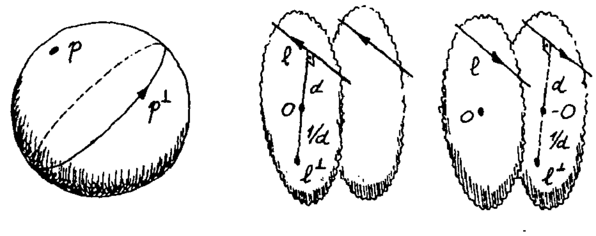

Figure 17. Orthogonal complement in $\mathbf{T}_2$.

That is, $\vdash$ and $\dashv$ are isomorphisms between the algebraic structures $(\mathcal{F}, -, \vee, \wedge)$ and $(\mathcal{F}, -, \wedge, \vee)$. This shows that $\vee$ and $\wedge$ have the same abstract structure. Therefore, if we take any theorem T of oriented projective geometry, construct its *formal dual* T^* by exchanging every occurrence of $\vee$ with $\wedge$, and $\mathbf{1}$ with $\mathbf{U}$, the result must also be a theorem. Of course, we must swap also all "implicit" occurrences, i.e. we must replace all concepts that can be defined in terms of meet and join by their formal duals. That includes swapping rank with co-rank, the word "point" with "hyperplane," the predicate $a \subseteq b$ (for flats) with $a \supseteq b$, and so on.

In particular, let's consider the predicate $a \diamond b = +1$. To construct its dual, we first rewrite it in terms of join, which gives $a \vee b = +\mathbf{U}$. The formal dual of this is $a \wedge b = +\mathbf{1}$. But this too is the same as $a \diamond b = +1$; we conclude that $\diamond$ is its own dual.

8. Projective maps

Projective maps are isomorphisms between two projective spaces. In particular the projective maps of $\mathbf{T}_\nu$ to itself are essentially the linear maps of $\mathbf{R}^n$ to itself.

More precisely, let X and Y be flats of $\mathbf{T}_\nu$ (in the spherical model), and U, V be the linear subspaces of $\mathbf{R}^n$ containing them. A linear map L from U to V *induces* a map M from X to Y defined by $M(\langle u \rangle) = \langle L(u) \rangle$ for all $u \in U$. By definition, a *projective map from X to Y* is a map of this form that takes positive simplices of X to positive simplices of Y. I will say that the linear map L *induces* M, and denote that by $\langle L \rangle = M$. It can be shown that two linear maps K, L induce the same projective map if and only if $K = \alpha L$ for some $\alpha > 0$.

A projective map $M = \langle L \rangle$ can be extended to arbitrary sub-flats of its domain X by the equations $M(\pm\mathbf{1}) = \pm\mathbf{1}$ and $M(\langle u^0, .. u^\kappa \rangle) = \langle L(u^0), .. L(u^\kappa) \rangle$ for every flat $\langle u^0, .. u^\kappa \rangle \subseteq X$. It can be shown that the image of a flat does not depend on which positive basis $\mathbf{u}^0, .. u^\kappa$ is used to represent it. Note that the positive universe of X is mapped to the positive universe of Y. Figure 18 shows the effect on $\mathbf{T}_2$ of the projective map induced by the linear transformation of $\mathbf{R}^3$ to itself given by the matrix A shown. Note that a portion of the front range is mapped to the back.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

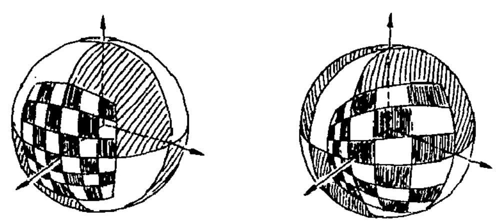

Figure 18. A projective map of $\mathbf{T}_2$.

Every projective map $M = <L>$ has an inverse $\overleftarrow{M}$, induced by the inverse $\overleftarrow{L}$ of L. Projective maps are continuous, one-to-one, and closed under composition. The restriction of a projective map M to any flat Z contained in its domain is a projective map from Z to $M(Z)$.

From the definitions it follows immediately that for every projective map M and any flats a, b in its domain,

$$\begin{aligned} M(-a) &= -M(a) \\ M(a \vee b) &= M(a) \vee M(b) \\ M(a \wedge b) &= M(a) \wedge M(b) \\ a \diamond b &= M(a) \diamond M(b) \end{aligned} \tag{18}$$

In other words, a projective map is an isomorphism between projective spaces. Note however that $\perp$ and the orthogonal complements need not be preserved by projective maps.

It can be shown that a projective map between two subspaces of $\mathbf{T}_\nu$ is completely characterized by an $n \times n$ real matrix, and, conversely, every $n \times n$ matrix determines such a map. The domain and range of the map are the flats determined by the subspaces of $\mathbf{R}^n$ spanned by the columns and rows of the matrix, respectively. In particular, a map from $\mathbf{T}_\nu$ to itself is given by a matrix with positive determinant.

A *projective frame* for a flat X of rank k consists of a simplex of X, plus an extra point or hyperplane of X (For example, a projective frame for a plane has either four points, or three points and a line.) A convenient way to specify a projective map is by a pair of projective frames, one on its domain and one on its range. It can be shown that there is at most one projective map that takes the first to the second. In oriented geometry, the map exists only if corresponding elements of the two frames have the same relative orientation.

9. Convexity

In affine geometry, a convex figure is one which contains the segment connecting any two of its points. This definition is meaningless in classical projective geometry, essentially because one cannot define unambiguously *the* segment connecting two given points. In two-sided geometry the segment pq is well defined and unique, as long as the two points are not antipodal. Moreover, it is a purely projective notion, that can be defined in terms of $\vee$: point x is on the segment pq if and only if $p \vee x = x \vee q = p \vee q \neq 0^2$. Therefore, the resulting notion of convexity is preserved by projective maps, and can be dualized in a straightforward way.

The usual definition of convexity can be extended to the oriented projective world in two ways, depending on how we handle antipodal pairs. A set of points X is *quasi-convex* if any line segment with endpoints in X is contained in X. A set X is *convex* if it is quasi-convex, non-empty, and contains no antipodal pairs of points.

Note that if X is contained in the front range of $\mathbf{T}_\nu$, the two definitions agree, and describe precisely those subsets of the front range that are convex in the classical sense. Examples of convex subsets of $\mathbf{T}_2$ are: a single point; a proper line segment; a bounded convex polygon on the front face; an open half-plane; an open wedge (the intersection of two half-planes); and half of a straight line without one or both of its endpoints. Some quasi-convex sets that are not convex are: a straight line; one half of a straight line with both its endpoints; a closed half-plane; a pair of antipodal points; and the whole two-sided plane.

Analytically, the segment pq consists of all points whose homogeneous coordinates interpolate between those of p and q. More precisely, if $p = [p_0, .. p_\nu]$, $q = [q_0, .. q_\nu]$ and $r = [r_0, .. r_\nu]$, then r is in the segment pq if and only if there are $\alpha, \beta > 0$ such that $r_i = \alpha p_i + \beta q_i$ for all i. It follows that a subset X of $\mathbf{T}_\nu$ is quasi-convex if and only if it is the central projection on $\mathbf{S}_2$ of $Y - \{(0, .. 0)\}$, where Y is a convex subset of $\mathbf{R}^n$. Furthermore, X is convex if and only if there is such an Y that does not include the origin $(0, .. 0)$. Most properties of convex sets of $\mathbf{T}_\nu$ follow trivially from this characterization.

Convex sets in $\mathbf{T}_\nu$ have many of the properties of classical ones. For example, the intersection of any family of quasi-convex subsets is quasi-convex. The same is true for convex sets, provided the intersection is not empty. A subset X of $\mathbf{T}_\nu$ is quasi-convex (resp. convex) if and only if its intersection with every line of is quasi-convex (resp. either empty or convex). Some properties of convex sets have to be modified: for example, in $\mathbf{T}_\nu$ the common convex hull of two convex sets X and Y exists only if $X \cup Y$ contains no antipodal points, that is, $X \cap (-Y) = \emptyset$. Also, in $\mathbf{T}_\nu$ a convex set is always contained in some half-space (which may not be the case in $\mathbf{E}_\nu$).

If we represent the boundary of a convex set by its *tracing*,[8] that is, as the set of all pairs of the form (boundary point, tangent hyperplane at that point), we find that the boundary of *every* convex set is topologically equivalent to a sphere (Note that this is not true for unbounded convex sets in $\mathbf{E}_\nu$). The image of any such tracing by the dual mapping $\vdash$ is the boundary tracing of another convex set. Among other things, this correspondence transforms the common convex hull of two convex figures into their intersection. See figure 19. This correspondence is exact, and is free from the exceptions, special cases, and ambiguities that arise when this dualization is attempted in other spaces. [3,4]

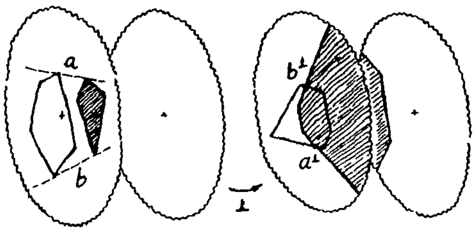

Figure 19. Convex hull and convex intersection.

10. Analytic projective geometry

Homogeneous coordinates can be extended to lines and higher-dimensional flats in several ways. An hyperplane h can be represented instead by its *coefficients*, a real vector $\langle h_0, .. h_\nu \rangle$ such that $x \diamond h = \text{sign}(x_0 h_0 + \cdots x_\nu h_\nu)$, for all points $x = [x_0, .. x_\nu]$. Obviously, $\langle \lambda h_0, .. \lambda h_\nu \rangle$ is the same hyperplane for all $\lambda > 0$, and $\langle -h_0, .. -h_\nu \rangle$ is the opposite hyperplane $-h$.

A flat a with rank $k \in \{2 .. n-2\}$ can be represented by a $k \times n$ real matrix,

$$s = \begin{pmatrix} s_0^0 & s_1^0 & \dots & s_\nu^0 \\ \vdots & \vdots & & \vdots \\ s_0^\kappa & s_1^\kappa & \dots & s_\nu^\kappa \end{pmatrix} \tag{19}$$

whose rows are the homogeneous coordinates of the vertices of any positive simplex of a.

To compute $a \vee b$, we just stack the matrix of a on top of that of b. Other operations are substantially harder, however. To check whether two matrices s, t represent the same flat, we have to test whether there is a $k \times k$ real matrix A with positive determinant such that $s = At$. To compute the meet of two flats, we have to find a suitably oriented basis for the intersection of the two vector spaces spanned by the rows of their matrices. These problems can be solved in practice by Gaussian elimination and similar numerical methods, with roughly

$O(nk^2)$ running time. The algorithms are identical to those of unoriented projective geometry, except that we must be careful to preserve the orientations of matrices all along. For example, when swapping two rows of a simplex we must multiply one of them by -1.

The matrix representation is highly redundant, especially when k is close to n. The condition for equivalence stated above implies that the set of all flats of rank k in $\mathbf{T}_\nu$ has dimension $kn - k^2$, and yet the matrix (19) above has kn coefficients. To decrease the cost of this representation, we can switch to the dual one (a matrix with the coefficients of $n-k$ hyperplanes whose meet is a) when $k > n/2$. Alternatively, we can use a Gaussian elimination method to transform every simplex into an equivalent one with some coordinates equal to zero. By storing only the non-zero coordinates, together with the positions of the zeros in a compacted form, we can bring the storage cost down to about $k(n-k)+2$ words. More details will be reported esewhere.

10.1. Plücker coordinates

For spaces of low rank, a more elegant alternative is to represent flats by their *Plücker coordinates.* Let a be an arbitrary flat of rank k in $\mathbf{T}_\nu$, and let $s = (s^0; \ldots s^\kappa)$ be one of its positive simplices. Let's write the homogeneous coordinates of its vertices in the form of a $k \times n$ matrix, as in (19). By definition, the *Plücker coordinates* of a are all the $\binom{n}{k}$ minor determinants of order k of this matrix,

$$a_{\{i_0, i_1, \ldots, i_\kappa\}} = \begin{vmatrix} s^0_{i_0} & \cdots & s^0_{i_\kappa} \\ \vdots & \ddots & \vdots \\ s^\kappa_{i_0} & \cdots & s^\kappa_{i_\kappa} \end{vmatrix} \tag{20}$$

The *label* of each coordinate is the set $\{i_0, i_1, \ldots, i_\kappa\}$ of the columns included in the determinant. The labels ranges over all k-element subsets of $\mathrm{N} = \{0, \ldots \nu\}$.

For example, the line l determined by the simplex

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix}$$

has coordinates

$$\begin{aligned} l_{\{0\,1\}} &= \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 & \quad l_{\{0\,2\}} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \\ l_{\{1\,2\}} &= \begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} = 2 & \quad l_{\{0\,3\}} &= \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0 \\ l_{\{1\,3\}} &= \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} = -6 & \quad l_{\{2\,3\}} &= \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3 \end{aligned} \tag{21}$$

It can be shown that the Plücker coordinates of a flat are not affected by the choice of the representative simplex, except for a positive scale factor. More precisely, *two flats of $\mathbf{T}_\nu$ are the same flat if and only if their Plücker coordinates differ only by a positive factor.*

The Plücker representation of a flat of rank k has $\binom{n}{k}$ coordinates, which for large n and k is a lot more than the $\min\{kn, (n-k)n\}$ of the matrix representation. However, for $n \leq 5$ the two representations have the same size, except for lines in three-space ($k = 2, n = 3$) where the Plücker form is actually smaller: six numbers instead of eight. Therefore, for two-, three-, and four-dimensional geometry Plücker coordinates is an attractive option.

10.2. Formulas for Plücker coordinates

With Plücker coordinates, the operations of join, meet, relative orientation, and orthogonal complement are given by relatively simple formulas. If $\operatorname{rank}(a) = r$ and $\operatorname{rank}(b) = s$, then

$$(a \vee b)_\mathrm{K} = \sum_{\substack{\mathrm{I} \cup \mathrm{J} = \mathrm{K} \\ \mathrm{I} \cap \mathrm{J} = \emptyset \\ |\mathrm{I}| = r,\, |\mathrm{J}| = s}} (-1)^{|\mathrm{I} > \mathrm{J}|} a_\mathrm{I} b_\mathrm{J} \tag{22}$$

$$(a \wedge b)_\mathrm{K} = \sum_{\substack{\mathrm{I} \cap \mathrm{J} = \mathrm{K} \\ \mathrm{I} \cup \mathrm{J} = \mathrm{N} \\ |\mathrm{I}| = r,\, |\mathrm{J}| = s}} (-1)^{|\overline{\mathrm{J}} > \overline{\mathrm{I}}|} a_\mathrm{I} b_\mathrm{J} \tag{23}$$

$$a \diamond b = \operatorname{sign}\Bigg(\sum_{\substack{\mathrm{K} \subseteq \mathrm{N} \\ |\mathrm{K}| = r}} (-1)^{|\mathrm{K} > \overline{\mathrm{K}}|} a_\mathrm{K} b_{\overline{\mathrm{K}}} \Bigg) \tag{24}$$

$$(a^{\vdash})_\mathrm{K} = (-1)^{|\overline{\mathrm{K}} > \mathrm{K}|} a_{\overline{\mathrm{K}}} \tag{25}$$

$$(a^{\dashv})_\mathrm{K} = (-1)^{|\mathrm{K} > \overline{\mathrm{K}}|} a_{\overline{\mathrm{K}}} \tag{26}$$

where a_I and b_J are the Plücker coordinates of a and b, $\overline{\mathrm{K}}$ denotes the complement of K with respect to N, and $|\mathrm{I} > \mathrm{J}|$ is the number of pairs $i \in \mathrm{I}, j \in \mathrm{J}$ such that $i > j$.

In formulas (and in computer programs) it is more convenient to write the homogeneous coordinates of a flat in some canonical order, so that the labels can be omitted. A natural choice is to enumerate the labels in increasing order of their binary weight, where the binary weight of a set X is $\sum_{x \in \mathrm{X}} 2^x$. For example, the natural order of the sets of size 3 is

$\{0\,1\,2\}$
$\{0\,1\,3\}$ $\{0\,2\,3\}$ $\{1\,2\,3\}$
$\{0\,1\,4\}$ $\{0\,2\,4\}$ $\{1\,2\,4\}$ $\{0\,3\,4\}$ $\{1\,3\,4\}$ $\{2\,3\,4\}$
$\{0\,1\,5\}$ ...

I will use the notation $[z_0,\, z_1,\, \ldots]^k$ for the flat of $\mathbf{T}_\nu$ with rank k whose Plücker coordinates are z_0, z_1, ..., listed in the natural order of their labels. For example, I will denote the line of example (21) as

$$\begin{array}{cccccccc} l = [& 2, & 1, & 2, & 0, & -6, & -3 &]^2 \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ & \{0\,1\} & \{0\,2\} & \{1\,2\} & \{0\,3\} & \{1\,3\} & \{2\,3\} & \end{array} \tag{27}$$

Flats of rank n (the universe and its antipode) have a single coordinate, whose label is the entire set $\{0 \ldots \nu\}$. It is easy to see that $\mathbf{T}_\nu = [+1]^n$, and $-\mathbf{T}_\nu = [-1]^n$. By convention, the flats of zero rank (the vacua) also have a single coordinate, whose label is the empty set and whose sign is that of the flat itself. That is, $\mathbf{1} = [+1]^0$ and $-\mathbf{1} = [-1]^0$.

In practice, a procedure that computes a geometric operation such as $c \leftarrow a \vee b$ will be given the Plücker coordinates of the operands in natural order, as two arrays $[a_0, a_1 \ldots]^r$ and $[b_0, b_1, \ldots]^s$, and is expected to return the result in the same format. Since Plücker coordinates are adequate only for spaces of dimension four or less, the best way to implement equations (22–26) is to write a separate routine for each combination of operand ranks, and expand the formulas by hand. For example, here are the formulas for $c \leftarrow a \vee b$, $c \leftarrow a \wedge b$, and $c \leftarrow a^{\vdash}$ in three-dimensional space:

line $\leftarrow$ point $\vee$ point
$c_0 \leftarrow a_0b_1 - a_1b_0$
$c_1 \leftarrow a_0b_2 - a_2b_0$
$c_2 \leftarrow a_1b_2 - a_2b_1$
$c_3 \leftarrow a_0b_3 - a_3b_0$
$c_4 \leftarrow a_1b_3 - a_3b_1$
$c_5 \leftarrow a_2b_3 - a_3b_2$

line $\leftarrow$ plane $\wedge$ plane
$c_0 \leftarrow a_0b_1 - a_1b_0$
$c_1 \leftarrow a_0b_2 - a_2b_0$
$c_2 \leftarrow a_0b_3 - a_3b_0$
$c_3 \leftarrow a_1b_2 - a_2b_1$
$c_4 \leftarrow a_1b_3 - a_3b_1$
$c_5 \leftarrow a_2b_3 - a_3b_2$

space $\leftarrow$ plane $\vee$ point space $\leftarrow$ point $\vee$ plane vacuum $\leftarrow$ point $\wedge$ plane vacuum $\leftarrow$ plane $\wedge$ point sign $\leftarrow$ point $\diamond$ plane sign $\leftarrow$ plane $\diamond$ point
$c_0 \leftarrow a_0b_3 - a_1b_2 + a_2b_1 - a_3b_0$

plane $\leftarrow$ line $\vee$ point
$c_0 \leftarrow a_0b_2 - a_1b_1 + a_2b_0$
$c_1 \leftarrow a_0b_3 - a_3b_1 + a_4b_0$
$c_2 \leftarrow a_1b_3 - a_3b_2 + a_5b_0$
$c_3 \leftarrow a_2b_3 - a_4b_2 + a_5b_1$

point $\leftarrow$ line $\wedge$ plane
$c_0 \leftarrow a_0b_2 - a_1b_1 + a_3b_0$
$c_1 \leftarrow a_0b_3 - a_2b_1 + a_4b_0$
$c_2 \leftarrow a_1b_3 - a_2b_2 + a_5b_0$
$c_3 \leftarrow a_3b_3 - a_4b_2 + a_5b_1$

space $\leftarrow$ line $\vee$ line vacuum $\leftarrow$ line $\wedge$ line sign $\leftarrow$ line $\diamond$ line
$c_0 \leftarrow a_0b_5 - a_1b_4 + a_2b_3 + a_3b_2 - a_4b_1 + a_5b_0$

plane $\leftarrow$ point$^{\vdash}$ point $\leftarrow$ plane$^{\vdash}$
$c_0 \leftarrow -a_3$
$c_1 \leftarrow a_2$
$c_2 \leftarrow -a_1$
$c_3 \leftarrow a_0$

line $\leftarrow$ line$^{\vdash}$ line $\leftarrow$ line$^{\dashv}$
$c_0 \leftarrow a_5$
$c_1 \leftarrow -a_4$
$c_2 \leftarrow a_3$
$c_3 \leftarrow a_2$
$c_4 \leftarrow -a_1$
$c_5 \leftarrow a_0$

plane $\leftarrow$ point$^{\dashv}$ point $\leftarrow$ plane$^{\dashv}$
$c_0 \leftarrow a_3$
$c_1 \leftarrow -a_2$
$c_2 \leftarrow a_1$
$c_3 \leftarrow -a_0$

The remaining combinations are either trivial (join with a vacuum, meet with a universe, etc.) or can be derived from the above by the commutativity rules.

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12. References

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