Conceptual Scaling

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1 Introduction

Scaling and measurement are usually based on numerical methods; this is, for instance, indicated by Stevens' definition: "measurement is the assignment of numerals to objects or events according to rules" (see [15], p. 22), or by Torgerson's statement: "measurement of a property involves the assignment of numbers to systems to represent that property" (see [16], p. 14). In contrast to this, conceptual scaling uses first of all set-theoretical methods to explore conceptual patterns in empirical data. Conceptual scaling has been developed in the frame of formal concept analysis, a theory based on a mathematization of conceptual hierarchies (see [18]). In this paper we give an introductory survey on conceptual scaling which concentrates on scales of ordinal type. First we recall basic notions and results of formal concept analysis and demonstrate them by an example (section 2). Then ideas of conceptual measurement are discussed in focussing on the question of measurability by standardized scales of ordinal type (section 3 and 4). These ideas are applied to the scaling of data contexts to derive conceptual hierarchies for the data (section 5). Finally, these scalings are used to introduce and to study a general notion of dependency between attributes which covers special notions like functional and linear dependency (section 6). Mathematically we presuppose some basic knowledge of order and lattice theory which can be found in [1] and [6].

2 Concept Lattices

Formal concept analysis is based on a set-theoretic model for conceptual hierarchies. This model mathematizes the philosophical understanding of a concept as a unit of thoughts consisting of two parts: the extension and the intension (comprehension); the extension covers all objects (or entities) belonging to the concept while the intension comprises all attributes (or properties) valid for all those objects (cf. [17]). In the set-theoretic model we fix a set G, the elements of which are called *objects*, and a set M, the elements of which are called *attributes*; furthermore we assume a binary relation I between G and M where $(g,m) \in I$ (resp. gIm) is read: the object g *has* the attribute m. The triple (G, M, I) which is called a *(formal) context* is the basic structure of formal concept analysis. For a context (G, M, I), the most frequently used operators are defined as follows:

$$A' := \{m \in M | gIm \text{ for all } g \in A\} \text{ for } A \subseteq G,$$

$$B' := \{g \in G | gIm \text{ for all } m \in B\} \text{ for } B \subseteq M.$$

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	Large Blade	Medium Blade	Small Blade	Manicure Blade	Screwdriver/Caplifter	Corkscrew	Can Opener/Screwdriver	Phillips Screwdriver	Fine Screwdriver	Scissors	Key Ring	Toothpick	Tweezers	Sew Blade/File	Wood Saw	Inch-Metric Rule/Fish Scaler	Reamer	Magnifier
CLASSIC			×	×						×	×	×	×					
SPARTAN	×	×			×	×	×				×						×	
NEW TINKER	×	×			×		×	×				×	×				×	
CAMPER	×	×			×	×	×				×	×	×		×		×	
CLIMBER	×	×			×	×	×			×	×	×	×				×	
EXPLORER	×	×			×	×	×	×		×	×	×	×				×	×
OUTDOORSMAN	×	×			×	×	×			×	×	×	×	×	×		×	
CHAMPION	×	×			×	×	×	×	×	×	×	×	×	×	×	×	×	×

Figure 1: Swiss Army Officers' Knives

Now, a *(formal) concept* of a context (G, M, I) is defined to be a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$; A and B are called the *extent* and the *intent* of the concept (A, B), respectively. The hierarchy of concepts is given by the relation *"subconcept-superconcept"* which has to be defined for a context by

$$(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2 \quad (\Leftrightarrow B_1 \supseteq B_2).$$

The set of all concepts of (G, M, I) with this order relation is a complete lattice called the *concept lattice* of (G, M, I) and denoted by $\underline{\mathfrak{B}}(G, M, I)$. Thus, to any set S of concepts of (G, M, I) there exists always a greatest subconcept, the *infimum* of S in $\underline{\mathfrak{B}}(G, M, I)$, and a smallest superconcept, the *supremum* of S in $\underline{\mathfrak{B}}(G, M, I)$. More precise information is given by the following theorem:

Basic Theorem on Concept Lattices (cf. [18]) *Let (G, M, I) be a context. Then $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice in which infimum and supremum can be described as follows:*

$$\bigwedge_{j\in J}(A_j, B_j) = (\bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)''), \quad \bigvee_{j\in J}(A_j, B_j) = ((\bigcup_{j\in J} A_j)'', \bigcap_{j\in J} B_j).$$

In general, a complete lattice L is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\gamma : G \longrightarrow L$ and $\mu : M \longrightarrow L$ such that γG is supremum-dense *in L (i.e. $L = \{\bigvee X | X \subseteq \gamma G\}$), μM is* infimum-dense *in L (i.e. $L = \{\bigwedge X | X \subseteq \mu M\}$), and $gIm \Leftrightarrow \gamma g \leq \mu m$ for all $g \in G$ and $m \in M$. In particular, $L \cong \underline{\mathfrak{B}}(L, L, \leq)$.*

We demonstrate the basic notions and results of formal concept analysis by an example. The table in fig.1 (from [2], p. 132) describes a context: its objects are

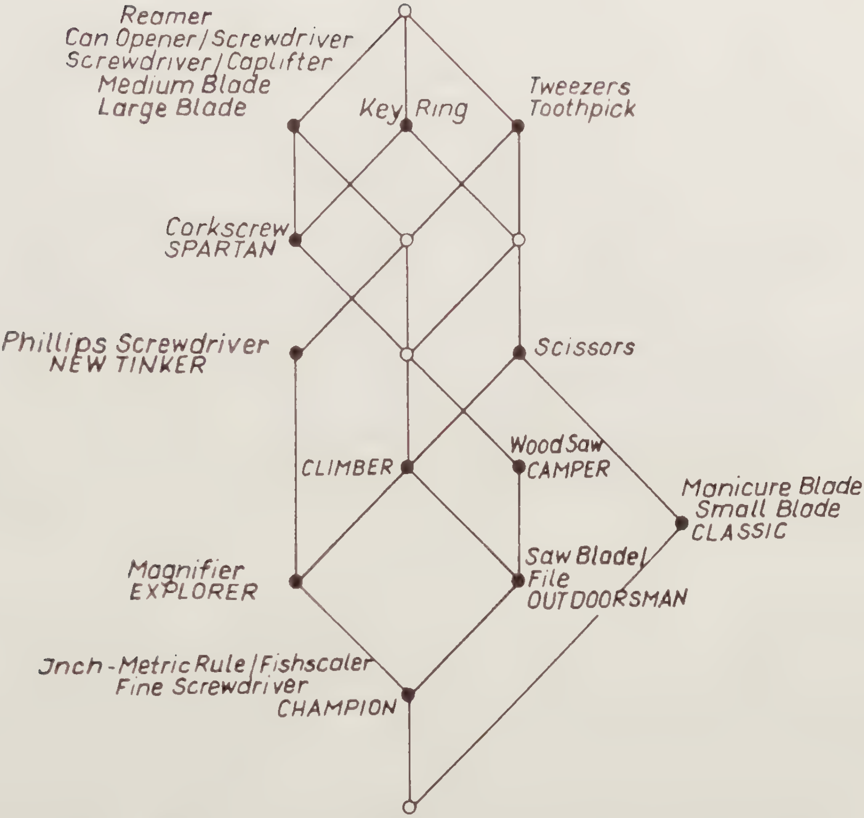


Figure 2: The concept lattice of the context in figure 1

types of Swiss Army Officers' Knives, its attributes are the possible components of the knives, and its relation is given by the crosses in the table. A concept of this context is for instance the pair consisting of the extent {Classic, Climber, Explorer, Outdoorsman, Champion} and the intent {Scissors, Key Ring, Toothpick, Tweezers}; a name of this concept may be "Knife with Scissors". All concepts of the context can be seen in fig.2 which shows the concept lattice of the context by a line diagram; a little circle labelled by a name of an object g represents $\gamma g := (\{g\}'', \{g\}')$, i.e. the smallest concept with g in its extent, and a little circle labelled by a name of an attribute m represents $\mu m := (\{m\}', \{m\}'')$, i.e. the greatest concept with m in its intent. The labelling allows to determine the extent and intent for any concept (A, B) since $A = \{g \in G | \gamma g \leq (A, B)\}$ and $B = \{m \in M | \mu m \geq (A, B)\}$ by the Basic Theorem. The equivalence $gIm \Leftrightarrow \gamma g \leq \mu m$ yields that the context can also be read from the diagram.

By the Basic Theorem, the lattice structure of a finite concept lattice $L := \underline{\mathfrak{B}}(G, M, I)$ is already determined by a *reduced context* $(G_r, M_r, I \cap (G_r \times M_r))$ with $G_r \subseteq G$ and $M_r \subseteq M$ so that the mappings γ and μ are bijections from G_r onto the set $J(L)$ of all join-irreducible elements of L and from M_r onto the set $M(L)$

	Large Blade	No Large Blade	Medium Blade	No Medium Blade	Small Blade	No Small Blade	Manicure Blade	No Manicure Blade	Screwdriver/Caplifter	No Screwdriver/Caplifter	Corkscrew	No Corkscrew	Can Opener/Screwdriver	No Can Opener/Screwdriver	Phillips Screwdriver	No Phillips Screwdriver	Fine Screwdriver	No Fine Screwdriver	Scissors	No Scissors	Key Ring	No Key Ring	Toothpick	No Toothpick	Tweezers	No Tweezers	Sew Blade/File	No Sew Blade/File	Wood Saw	No Wood Saw	Inch-Metric Rule/Fish Scaler	No Inch-Metric Rule/Fish Scaler	Reamer	No Reamer	Magnifier	No Magnifier
CLASSIC		×		×	×		×			×		×		×		×		×	×		×		×		×			×		×		×		×		×
SPARTAN	×		×			×		×	×		×		×			×		×		×	×			×		×		×		×		×	×			×
NEW TINKER	×		×			×		×	×			×	×		×			×		×		×	×		×			×		×		×	×			×
CAMPER	×		×			×		×	×		×		×			×		×		×	×		×		×			×	×			×	×			×
CLIMBER	×		×			×		×	×		×		×			×		×	×		×		×		×			×		×		×	×			×
EXPLORER	×		×			×		×	×		×		×		×			×	×		×		×		×			×		×		×	×		×	
OUTDOORSMAN	×		×			×		×	×		×		×			×		×	×		×		×		×		×		×			×	×			×
CHAMPION	×		×			×		×	×		×		×		×		×		×		×		×		×		×		×		×		×		×	

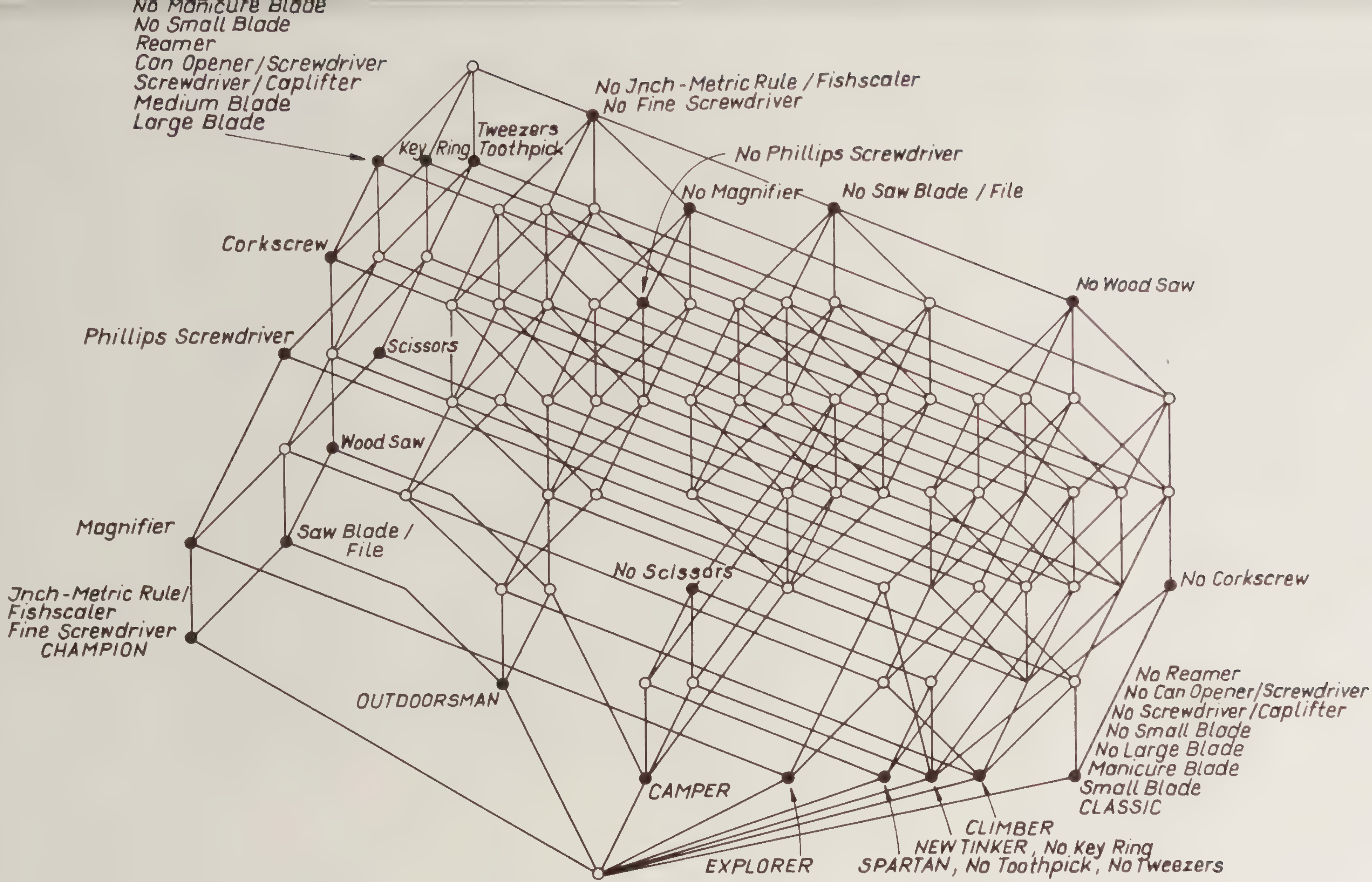
Figure 3: The dichotomic context

of all meet-irreducible elements of L, respectively. In our example such a reduced context is given by $G_r := \{$Classic, Spartan, New Tinker, Camper, Explorer, Outdoorsman, Champion$\}$ and $M_r := \{$Large Blade, Small Blade, Phillips Screwdriver, Scissors, Key Ring, Toothpick, Wood Saw$\}$. Notice that quite different contexts may have isomorphic concept lattices.

Not all information of the table in fig.1 is used to form concepts; for instance, the concept "Knife with no Scissors" does not occur in the concept lattice of fig.2. If one understands the table in fig.1 as a formal context, then the attributes are only used positively and combined by conjunction. If one wants to form concepts also with the negation of attributes, one has to extend the table by the negated attributes as it is done in fig.3. Of course, the dichotomic context in fig.3 has more concepts than the context in fig.1; this can be seen in fig.4. The derivation of the dichotomic context and its concept lattice is a simple example of conceptual scaling of a many valued context, which will be extensively discussed in section 5. Methods to determine the concept lattice of a given context and to draw adequate line diagrams can be found in [7, 19, 22]; there are also computer programs based on these methods (see [3, 8, 14]).

3 Scales and Scale Measures

Scaling is the development of formal patterns and their use for analyzing empirical data. In conceptual scaling these formal patterns consist of formal contexts and their concept lattices which have a clear structure and which reflect some meaning. Such a context is called a *scale* and, in general, denoted by $\mathbf{S} := (G_{\mathbf{S}}, M_{\mathbf{S}}, I_{\mathbf{S}})$; the elements of $G_{\mathbf{S}}$ and $M_{\mathbf{S}}$ are called *scale values* and *scale attributes*, respectively. In this section we restrict to the case where the empirical data are given in the form of a context


No Manicure Blade
No Small Blade
Reamer
Can Opener/Screwdriver
Screwdriver/Caplifter
Medium Blade
Large Blade
Key Ring
Tweezers
Toothpick
No Jnch-Metric Rule/Fishscaler
No Fine Screwdriver
No Phillips Screwdriver
No Magnifier
No Saw Blade / File
No Wood Saw
Corkscrew
Phillips Screwdriver
Scissors
Wood Saw
Magnifier
Saw Blade / File
No Scissors
No Corkscrew
Jnch-Metric Rule/
Fishscaler
Fine Screwdriver
CHAMPION
OUTDOORSMAN
CAMPER
No Reamer
No Can Opener/Screwdriver
No Screwdriver/Caplifter
No Small Blade
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EXPLORER

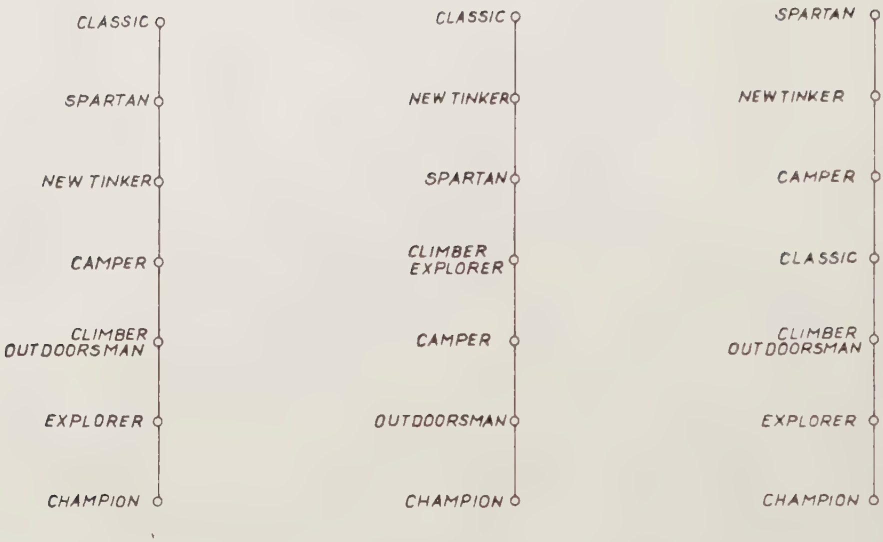

Figure 5: Three surjective measures onto the ordinal scale $\mathbb{O}_7$

$\mathbb{K} := (G, M, I)$. Empirical contexts are connected with scales by scale measures: a *scale measure* or, more precisely, an $\mathbb{S}$-*measure* of $\mathbb{K}$ is defined to be a mapping σ from G into $G_{\mathbb{S}}$ such that, for every extent A of $\mathbb{S}$, the preimage $\sigma^{-1}A$ is an extent of $\mathbb{K}$. An $\mathbb{S}$-measure σ of $\mathbb{K}$ is called *full* if every extent of $\mathbb{K}$ is the preimage of an extent of $\mathbb{S}$ under σ (cf. [18, 19, 9]). In measurement theory one usually assumes relational systems of a fixed type as empirical data so that scale measures are defined as homomorphisms between relational systems (cf. [13]); conceptual measurement, however, is based on a type-free notion of a scale measure which reflects the idea of a continuous map between topological spaces.

Conceptual scaling is in the first place of ordinal nature although other types of measurements may also be treated within conceptual scaling. In this paper we shall restrict our explanations to conceptual measurement with different scales of ordinal type. Let us begin with the discussion of some scale measures of the empirical data from section 2. Rankings of the objects are given by scale measures into the *one-dimensional ordinal scale*

$$\mathbb{O}_n := (\{1, 2, \ldots, n\}, \{1, 2, \ldots, n\}, \leq);$$

the concept lattice of $\mathbb{O}_n$ is an n-element chain. Since the preimages of the extents of a one-dimensional ordinal scale have to form a chain under set-inclusion, the scale measures into one-dimensional ordinal scales correspond to chains in the concept lattice of the empirical context. Thus, one can easily read from fig.2 that there are 22 surjective $\mathbb{O}_7$-measures of the context of fig.1, three of them are shown in fig.5. There does not exist a full measure into a one-dimensional scale because the empirical context is ordinally of higher dimension, i.e. its concept lattice is not a chain. To obtain full scale measures, we may use the *k-dimenssional grid scale*

$$\mathbb{G}_{n_1, n_2, \ldots, n_k} := (\mathbf{n}_1 \times \mathbf{n}_2 \times \cdots \times \mathbf{n}_k, \mathbf{n}_1 \times \{1\} \cup \mathbf{n}_2 \times \{2\} \cup \cdots \cup \mathbf{n}_k \times \{k\}, \nabla)$$

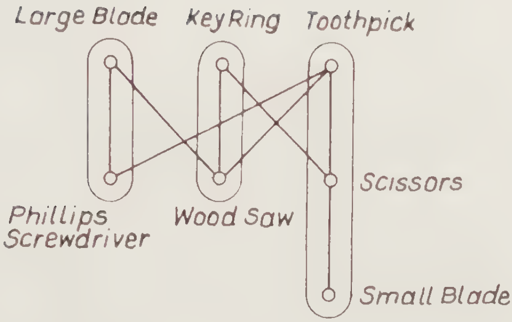

Figure 6: The ordered set of meet–irreducible concepts of the context in figure 1.

where $\mathbf{n}$ is the set $\{1, 2, \ldots, n\}$ with the natural order and

$$(v_1, v_2, \ldots, v_k)\nabla(m, j) :\Leftrightarrow v_j \leq m \,;$$

the concept lattice of $\mathbb{G}_{n_1,n_2,\ldots,n_k}$ is isomorphic to the direct product of the chains $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k$. To find full scale measures into grid scales, a general method is to decompose the ordered set of all meet-irreducible concepts of the empirical context into chains; then a full scale measure into a grid scale is given by assigning to an object g, for each chain, the number of concepts in the chain not containing g in their extent plus one. Fig. 6 shows the ordered set of all meet-irreducible concepts of our example with a partition into chains; the corresponding scale measure is visualized in fig.7 by a (join-) embedding of the concept lattice of fig.2 into a direct product of chains of natural numbers. The values of the grid scale which do not correspond to concepts of the empirical context indicate non-trivial implications between the attributes of the empirical context (cf. [4]). The value $(4, 2, 1)$, for instance, represents the attribute set {Phillips Screwdriver, Key Ring}; the greatest value below $(4, 2, 1)$ corresponding to a concept is the value $(2, 2, 1)$ which represents the attribute set {Phillips Screwdriver, Scissors, Key Ring}. This yields that {Phillips Srewdriver, Key Ring}$\Longrightarrow$ {Scissors}, i.e. every knife which has a Phillips screwdriver and a key ring has also scissors. The implications with a one-element premise, as {Small Blade}$\Longrightarrow${Key Ring} which is indicated by the scale value $(1, 3, 3)$, can already be read from the ordered set $M(L)$ in fig.6. If one restricts to the sublattice determined by the image of the scale measure as it is visualized in fig.8, only implications with more than one attribute in their premise are indicated by lattice elements not corresponding to a concept of the empirical context.

Classification is another basic meaning of data analysis besides ranking. Conceptual scaling yields classifications of objects by forming preimages of scale extents under scale measures. Such classifications consist of classes which are describable

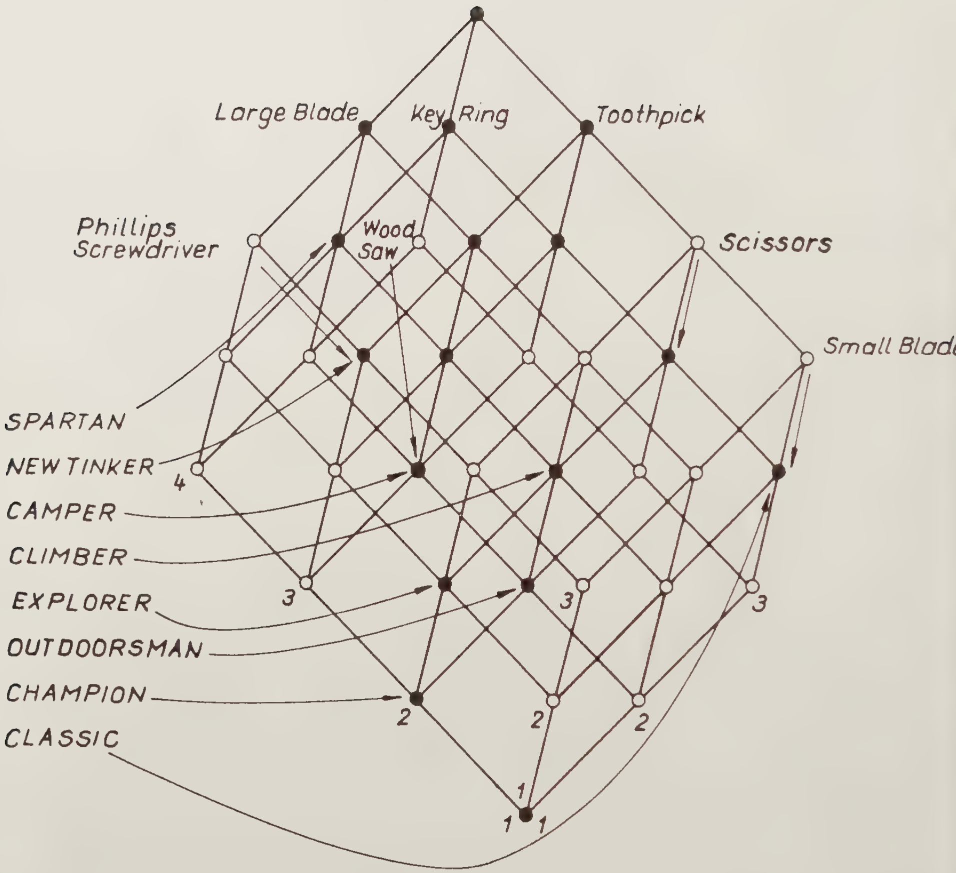

Figure 7: A full measure into the grid scale $G_{3,3,4}$

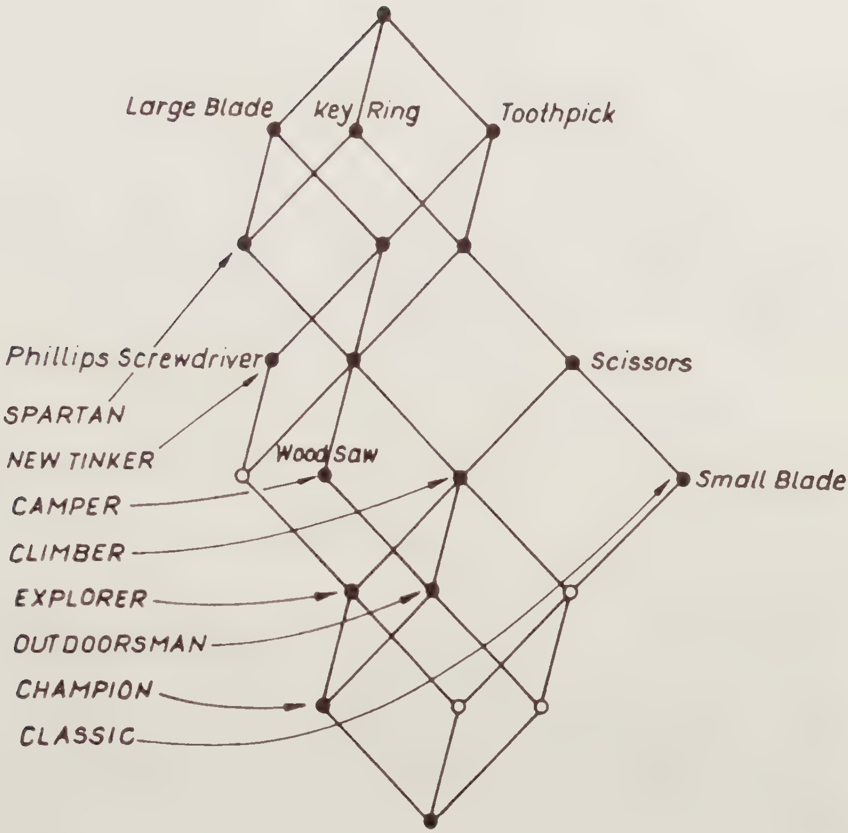

Figure 8: The sublattice of the concept lattice of $G_{3,3,4}$ generated by the image of the scale measure of fig.7

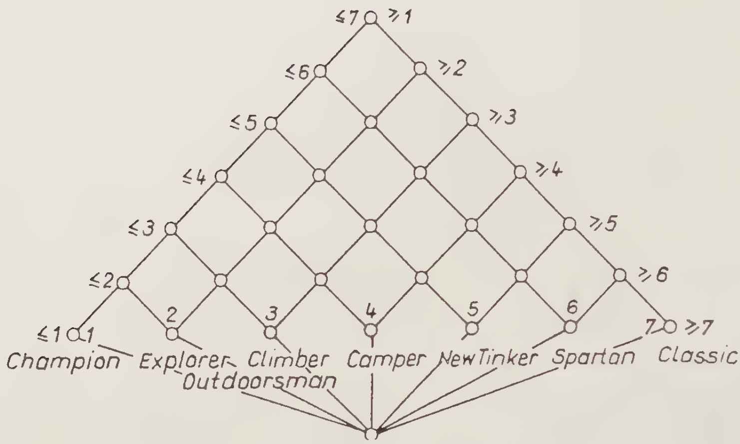

Figure 9: A surjective measure onto the interordinal scale $\mathbb{I}_7$

by the attributes of the empirical context. The most elementary scale measures for classification are the *nominal scales*

$$\mathbb{N}_n := (\{1,2,\ldots,n\},\{1,2,\ldots,n\},=);$$

the concept lattice of $\mathbb{N}_n$ consists of an n-element antichain with a common upper and lower bound. The scale measures into nominal scales correspond to partitions of the empirical objects into extents. Thus, one can read from fig.2 that the context of fig.1 admits only surjective scale measures into $\mathbb{N}_1$ and $\mathbb{N}_2$; in this way we obtain only one proper partition consisting of the set {Classic} and its complement. Many more partitions into extents has the dichotomic context of fig.3. In such a case it is more interesting to determine a hierarchical system of classification than only one single partition. This can be performed by measures into *one-dimensional interordinal scales*

$$\mathbb{I}_n := (\{1,2,\ldots,n\},\{1,2,\ldots,n\}\times\{1,2\},\diamond)$$

where $i\diamond(j,1) :\Leftrightarrow i \leq j$ and $i\diamond(j,2) :\Leftrightarrow i \geq j$; the concept lattice of $\mathbb{I}_7$ is shown in fig.9 together with a scale measure of the dichotomic context (the symbol $\leq j$ stands for the attribute $(j,1)$ and dually $\geq j$ for $(j,2)$.) This measure which is the dichotomic version of the first example in fig.5 emphasizes a left-right-structure with its inherent hierarchical classification of the empirical objects. A full scale measure of the dichotomic context into an interordinal scale of higher dimension can be constructed analogue to the measure described by fig.7.

Now, we introduce scales of ordinal type in general. Let us first define some context constructions which are used to form scales. For contexts $\mathbb{K} := (G,M,I)$ and $\mathbb{K}_j := (G_j,M_j,I_j)$ $(j=1,2,\ldots,k)$, we use the abbreviations $\dot{G}_j := G_j\times\{j\}$, $\dot{M}_j := M_j\times\{j\}$ and $\dot{I}_j := \{((g,j),(m,j))|(g,m)\in I_j\}$ to define

$$\begin{aligned}
\mathbb{K}^c &:= (G,M,(G\times M)\setminus I),\\
&\quad\ \text{the \textit{complementary context} of } \mathbb{K},\\
\mathbb{K}^d &:= (M,G,I^{-1}),\\
&\quad\ \text{the \textit{dual context} of } \mathbb{K},\\
\mathbb{K}_1\,\dot\cup\cdots\dot\cup\,\mathbb{K}_k &:= (\dot{G}_1\cup\cdots\cup\dot{G}_k,\\
&\quad\ \dot{M}_1\cup\cdots\cup\dot{M}_k, \dot{I}_1\cup\cdots\cup\dot{I}_k),\\
&\quad\ \text{the \textit{disjoint union} of } \mathbb{K}_1,\ldots,\mathbb{K}_k,
\end{aligned}$$

$$
\begin{aligned}
\mathsf{K}_1 + \cdots + \mathsf{K}_k &:= (\dot{G}_1 \cup \cdots \cup \dot{G}_k,\\
&\qquad \dot{M}_1 \cup \cdots \cup \dot{M}_k,\\
&\qquad \dot{I}_1 \cup \cdots \cup \dot{I}_k \cup \bigcup_{i\neq j} \dot{G}_i \times \dot{M}_j),\\
&\qquad \text{the } \textit{direct sum} \text{ of } \mathsf{K}_1,\ldots,\mathsf{K}_k,\\
\mathsf{K}_1 \ltimes \cdots \ltimes \mathsf{K}_k &:= (G_1 \times \cdots \times G_k, \dot{M}_1 \cup \cdots \cup \dot{M}_k, \nabla),\\
&\qquad \text{the } \textit{semiproduct} \text{ of } \mathsf{K}_1,\ldots,\mathsf{K}_k,\\
&\qquad \text{where } (g_1,\ldots,g_k)\nabla(m,j) \Leftrightarrow g_j I_j m,\\
\mathsf{K}_1 \times \cdots \times \mathsf{K}_k &:= (G_1 \times \cdots \times G_k, M_1 \times \cdots \times M_k, \nabla),\\
&\qquad \text{the } \textit{direct product} \text{ of } \mathsf{K}_1,\ldots,\mathsf{K}_k,\\
&\qquad \text{where } (g_1,\ldots,g_k)\nabla(m_1,\ldots,m_k)\\
&\qquad :\Leftrightarrow g_j I_j m_j \text{ for some } j,\\
\mathsf{K}^{\underline{0}} &:= \mathsf{K} + (\{\underline{0}\},\emptyset,\emptyset),\\
\mathsf{K}^{\underline{1}} &:= \mathsf{K} \;\dot{\cup}\; (\{\underline{1}\},\emptyset,\emptyset),\\
\mathsf{K}^{\overline{0}} &:= \mathsf{K} \;\dot{\cup}\; (\emptyset,\{\overline{0}\},\emptyset),\\
\mathsf{K}^{\overline{1}} &:= \mathsf{K} + (\emptyset,\{\overline{1}\},\emptyset),\\
&\qquad \text{and, if } G = G_1 = G_2,\\
\mathsf{K}_1 \mid \mathsf{K}_2 &:= (G, \dot{M}_1 \cup \dot{M}_2, \dot{I}_1 \cup \dot{I}_2),\\
&\qquad \text{the } \textit{apposition} \text{ of } \mathsf{K}_1 \text{ and } \mathsf{K}_2,\\
&\qquad \text{likewise, if } M = M_1 = M_2,\\
\frac{\mathsf{K}_1}{\mathsf{K}_2} &:= (\dot{G}_1 \cup \dot{G}_2, M, \dot{I}_1 \cup \dot{I}_2),\\
&\qquad \text{the } \textit{subposition} \text{ of } \mathsf{K}_1 \text{ and } \mathsf{K}_2.
\end{aligned}
$$

For defining and analyzing scales of ordinal type we use the following constructions of ordered sets (see [1]): $P_1 + \cdots + P_k$ is the *cardinal sum* of the ordered sets $P_1,\ldots,P_k$, $P_1 \times \cdots \times P_k$ is the *direct product* of $P_1,\ldots,P_k$, and $\mathfrak{P}(\mathbf{n})$ is the *power set* of $\mathbf{n} := \{1,2,\ldots,n\}$ ordered by set-inclusion.

The following table lists standardized scales of ordinal type which have been proved useful up to now.

Standardized Scales of Ordinal Type			
Symbol	Definition	Name	Basic Meaning
$\mathsf{O}_{\mathbf{P}}$	$(P, P, \leq)$	*ordinal scale*	hierarchy
O_n	$(\mathbf{n}, \mathbf{n}, \leq)$	*one-dimensional ordinal scale*	ranking
N_n	$(\mathbf{n}, \mathbf{n}, =)$	*nominal scale*	partition
$\mathsf{M}_{n_1, n_2, \ldots, n_k}$	$\mathsf{O}_{\mathbf{n}_1+\mathbf{n}_2+\cdots+\mathbf{n}_k}$	*multiordinal scale*	partition with rankings
$\mathsf{M}_{m,n}$	$\mathsf{O}_{\mathbf{n}+\mathbf{m}}$	*biordinal scale*	two–class ranking
B_n	$(\mathfrak{P}(\mathbf{n}), \mathfrak{P}(\mathbf{n}), \subseteq)$	*n-dimensional boolean scale*	attribute dependency
$\mathsf{G}_{n_1, n_2, \ldots, n_k}$	$\mathsf{O}_{n_1} \text{⧖} \cdots \text{⧖} \mathsf{O}_{n_k}$	*k-dimensional grid scale*	multiple ranking
$\mathsf{O}^{cd}_{\mathbf{P}}$	$(P, P, \not\geq)$	*contrary ordinal scale*	hierarchy and independence
N^c_n	$(\mathbf{n}, \mathbf{n}, \neq)$	*complementary nominal scale*	partition and independence
D	$(\{0,1\}, \{0,1\}, =)$	*dichotomic scale*	dichotomy
D_k	$\underbrace{\mathsf{D} \text{⧖} \cdots \text{⧖} \mathsf{D}}_{k\text{–times}}$	*k-dimensional dichotomic scale*	multiple dichotomy
$\mathsf{I}_{\mathbf{P}}$	$\mathsf{O}_{\mathbf{P}} \mid \mathsf{O}^d_{\mathbf{P}}$	*interordinal scale*	betweenness
I_n	$\mathsf{O}_n \mid \mathsf{O}^d_n$	*one–dimensional interordinal scale*	linear betweenness
$\mathsf{C}_{\mathbf{P}}$	$\mathsf{O}^{cd}_{\mathbf{P}} \mid \mathsf{O}^c_{\mathbf{P}}$	*convex ordinal scale*	convex ordering

For working with the standardized scales the following equalities can be useful:

$$\begin{aligned}
\mathsf{O}_{\mathbf{P}_1+\cdots+\mathbf{P}_k} &= \mathsf{O}_{\mathbf{P}_1} \dot\cup \cdots \dot\cup \mathsf{O}_{\mathbf{P}_k} \\
\mathsf{O}^{cd}_{\mathbf{P}_1+\cdots+\mathbf{P}_k} &= \mathsf{O}^{cd}_{\mathbf{P}_1} + \cdots + \mathsf{O}^{cd}_{\mathbf{P}_k} \\
\mathsf{O}^{cd}_{\mathbf{P}_1\times\cdots\times\mathbf{P}_k} &= \mathsf{O}^{cd}_{\mathbf{P}_1} \times \cdots \times \mathsf{O}^{cd}_{\mathbf{P}_k} \\
\mathsf{I}_{\mathbf{P}_1+\cdots+\mathbf{P}_k} &= \mathsf{I}_{\mathbf{P}_1} \dot\cup \cdots \dot\cup \mathsf{I}_{\mathbf{P}_k} \\
\mathsf{C}_{\mathbf{P}_1+\cdots+\mathbf{P}_k} &= \mathsf{C}_{\mathbf{P}_1} + \cdots + \mathsf{C}_{\mathbf{P}_k}. \\
\mathsf{C}_{\mathbf{P}_1\times\cdots\times\mathbf{P}_k} &= \mathsf{O}^{cd}_{\mathbf{P}_1} \times \cdots \times \mathsf{O}^{cd}_{\mathbf{P}_k} \mid \mathsf{O}^{c}_{\mathbf{P}_1} \times \cdots \times \mathsf{O}^{c}_{\mathbf{P}_k}
\end{aligned}$$

4 Measurability

Most important in measurement is the problem: by which scales can a given empirical structure be measured? In conceptual measurement we ask more specific whether an empirical context $\mathsf{K} := (G, M, I)$ admits (full) S-measures for significant scales $\mathsf{S} := (G_{\mathsf{S}}, M_{\mathsf{S}}, I_{\mathsf{S}})$. The definition of an S-measure yields directly that a mapping σ from G into G_{S} is an S-measure if and only if K has the same extents as $\mathsf{K} \mid \mathsf{K}_\sigma$ where $\mathsf{K}_\sigma := (G, M_{\mathsf{S}}, I_\sigma)$ with $gI_\sigma m :\Leftrightarrow \sigma(g) I_{\mathsf{S}} m$; σ is a full S-measure if and only if K and K_σ have the same extents. The context K_σ may be unterstood as another version of the subscale of S based on the image $\sigma(G)$; in general, a *subscale* S_T of S based on a subset T of G_{S} is the context $(T, M_{\mathsf{S}}, I_{\mathsf{S}} \cap (T \times M_{\mathsf{S}}))$. Since σ is an S-measure of K if and only if σ is an $\mathsf{S}_{\sigma(G)}$–measure of K, every scale measure can be replaced by a surjective one which has the same preimages of extents. This observation is basic for approaching the measurability problem; a first step is given by the following proposition:

Proposition 1 *Let σ be an* $\mathbb{S}$*-measure of* $\mathbb{K}$*. Then*

$$(A, A') \mapsto (\sigma^{-1}(A), \sigma^{-1}(A)')$$

describes a $\bigwedge$*-preserving map from* $\underline{\mathfrak{B}}(\mathbb{S})$ *into* $\underline{\mathfrak{B}}(\mathbb{K})$*; this map is injective if* σ *is surjective.*

From this proposition we obtain a lattice-theoretical characterization of scale measures using a basic result on Galois connections (cf. [6]):

Proposition 2 *(cf. [18]) Let* $\mathbb{S}$ *be a scale in which* $v \neq w$ *implies* $\{v\}' \neq \{w\}'$ *for all* $v, w \in G_{\mathbb{S}}$*. Then, for an* $\mathbb{S}$*-measure* σ *of* $\mathbb{K} := (G, M, I)$,

$$(A, A') \mapsto \bar{\sigma}(A, A') := (\sigma(A)'', \sigma(A)')$$

describes a $\bigvee$*-preserving map* $\bar{\sigma}$ *from* $\underline{\mathfrak{B}}(\mathbb{K})$ *into* $\underline{\mathfrak{B}}(\mathbb{S})$*; in particular,* $\bar{\sigma}(\gamma g) = \gamma_{\mathbb{S}}\sigma(g)$ *for all* $g \in G$*. Conversely, if* φ *is a* $\bigvee$*-preserving map from* $\underline{\mathfrak{B}}(\mathbb{K})$ *into* $\underline{\mathfrak{B}}(\mathbb{S})$ *such that for each* $g \in G$ *there is a* $\bar{\varphi}(g) \in G_{\mathbb{S}}$ *with* $\varphi(\gamma g) = \gamma_{\mathbb{S}}\bar{\varphi}(g)$*, then* $\bar{\varphi}$ *is an* $\mathbb{S}$*-measure of* $\mathbb{K}$*. There is a one-to-one correspondence between the* $\mathbb{S}$*-measures* σ *(resp.* $\bar{\varphi}$*) and the specific* $\bigvee$*-preserving maps* $\bar{\sigma}$ *(resp.* φ*).* σ *is full if and only if* $\bar{\sigma}$ *is injective.*

For the characterization of scale measures of a given context $\mathbb{K}$ it is useful to know of which type are the subscales of the considered scales. For comparing scales we introduce a notion of equivalence: two scales $\mathbb{S}_1$ and $\mathbb{S}_2$ are called *equivalent* if there is an isomorphism φ from $\underline{\mathfrak{B}}(\mathbb{S}_1)$ onto $\underline{\mathfrak{B}}(\mathbb{S}_2)$ such that φ induces a bijection from $\gamma_{\mathbb{S}_1} G_{\mathbb{S}_1}$ onto $\gamma_{\mathbb{S}_2} G_{\mathbb{S}_2}$. The following table describes equivalences for the subscales of some standardized scales of ordinal type:

scale	subscales equivalent to
$\mathbb{O}_n$	one-dimensional ordinal scales
$\mathbb{N}_n$	one-dimensional nominal scales
$\mathbb{M}_{n_1,\ldots,n_k}$	multiordinal scales
$\mathbb{O}^{cd}_{\mathbf{P}}$	contrary ordinal scales
$\mathbb{N}^c_n$	complementary nominal scales
$\mathbb{I}_n$	one-dimensional interordinal scales
$\mathbb{C}_{\mathbf{P}}$	convex ordinal scales

The next table gives necessary and sufficent conditions for a context $\mathbb{K} := (G, M, I)$ to admit surjective measures onto specific scales:

scale $\mathbb{S}$	condition for admitting a surjective $\mathbb{S}$-measure
$\mathbb{O}_n$	chain of n non-empty extents
$\mathbb{N}_n$	partition of G into n extents
$\mathbb{M}_{n_1,\ldots,n_k}$	partition of G into k chains of n_1 up to n_k extents
$\mathbb{O}^{cd}_{\mathbf{P}}$	isomorphic copy of $\mathbf{P}$ formed by extents which have as unions again extents; furthermore, for each object g there is a smallest of the fixed extents containing g and each of the fixed extents corresponds in this way to some object
$\mathbb{N}^c_n$	partition of G into n extents which have as unions again extents
$\mathbb{I}_n$	chain of n non-empty extents the complement of which are again extents
$\mathbb{C}_{\mathbf{P}}$	isomorphic copy of $\mathbf{P}$ formed by extents which have as unions and as complements again extents; furthermore, for each object g there is a smallest of the fixed extents containing g and each of the fixed extents corresponds in this way to some object

The conditions in the above table can be easily extended to obtain characterizations for full measures. Since subscales of ordinal scales need not to be ordinal again, the measurability into ordinal scales cannot be characterized via surjective measures. On the other hand, each context admits even full $\mathbb{O}_{\mathbf{P}}$-measures for suitable ordered sets $\mathbf{P}$ which can be determined via the ordered set of meet-irreducible concepts of the given context. This is demonstrated by the next two propositions.

Proposition 3 *Let $\mathbb{K} := (G, M, I)$ be a finite context. For a bijection $\iota : \mathbf{n} \longrightarrow M$ a full $\mathbb{B}_n$-measure σ of $\mathbb{K}$ is given by $\sigma(g) := \mathbf{n} \setminus \iota^{-1}\{g\}'$ for $g \in G$.*

The Boolean scales are equivalent to special grid scales; in general, $\mathbb{G}_{n_1,\ldots,n_k}$ is equivalent to $\mathbb{O}_{\mathbf{n}_1\times\cdots\times\mathbf{n}_k}$. A full measure into a grid scale offers a scheme of ordinal dimensions for the interpretation of an empirical context. For a finite context $\mathbb{K}$ it is especially interesting to determine the smallest number k such that $\mathbb{K}$ admits a full measure into a k-dimensional grid scale; k is called the *grid dimension* of $\mathbb{K}$.

Proposition 4 *(cf. [18]) Let $\mathbb{K} := (G, M, I)$ be a finite context. For an order-preserving bijection $\iota : \mathbf{n}_1 + \cdots + \mathbf{n}_k \longrightarrow M(\underline{\mathfrak{B}}(\mathbb{K}))$ a full $\mathbb{G}_{n_1+1,\ldots,n_k+1}$-measure σ_ι of $\mathbb{K}$ is given by $\sigma_\iota(g) := (v_1 + 1, \ldots, v_k + 1)$ for $g \in G$ where v_j is the greatest number in $\mathbf{n}_j$ such that g is not in the extent of $\iota(v_j)$. If τ is any full $\mathbb{G}_{m_1,\ldots,m_l}$-measure of $\mathbb{K}$ then there exist always a bijection $\iota : \mathbf{n}_1 + \cdots + \mathbf{n}_k \longrightarrow M(\underline{\mathfrak{B}}(\mathbb{K}))$ with $k \leq l$ and order-preserving surjections $\nu_j : \{1, \ldots, m_j\} \longrightarrow \{1, \ldots, n_j + 1\}$ for $j = 1, \ldots, k$ such that $\sigma_\iota(g) = (\nu_1(w_1), \ldots, \nu_1(w_k))$ for $g \in G$.*

Corollary 1 *The grid dimension of $\mathbb{K}$ equals the width of $M(\underline{\mathfrak{B}}(\mathbb{K}))$.*

The full measures into semiproducts of (one-dimensional) scales are also interesting for other types of scales. Besides grid scales we consider here only semiproducts of nominal scales, in particular dichotomic scales. For the characterization of full measures in such scales we introduce the following notion: a context $\mathbb{K} := (G, M, I)$ is called *atomistic* if γg is an atom of $\underline{\mathfrak{B}}(\mathbb{K})$ for all $g \in G$.

Proposition 5 *A finite context* $\mathbb{K}$ *admits a full scale measure into a semiproduct of nominal scales if and only if* $\mathbb{K}$ *is atomistic.* $\mathbb{K}$ *admits a full scale measure into the k-dimensional dichotomic scale if and only if* $\mathbb{K}$ *is atomistic and there are at most k pairs of complementary extents to which all extents of meet-irreducible concepts of* $\mathbb{K}$ *belong.*

Direct products of scales are also interesting in conceptual measurement as it becomes clearer in section 6. Here we give only a characterization of full measures into some type of direct product scales which indicate a connection to order dimension; let us recall that the *order dimension* of an ordered set $\mathbf{P}$ is the smallest number of chains the direct product of which allows an order embedding of $\mathbf{P}$.

Proposition 6 *(cf. [20]) A finite context* $\mathbb{K}$ *admits a full scale measure into a direct product of k contrary one-dimensional ordinal scales if and only if* $\mathbb{K}$ *is isomorphic to a context* $(P, P, \not\geq)$ *for some ordered set* $\mathbf{P}$ *of order dimension at most k.*

5 Scaled Contexts

A formal context, as defined in section 2, has a natural conceptual structure. Empirical data models, however, often arise in a form which does not *a priori* fall under this data type. They frequently use many-placed relations and operations. It seems natural to generalize the definition to that of a many-valued context (see [18, 10]): a *many-valued context* is a quadruple (G, M, W, I), where G, M and W are sets and I is a ternary relation between G, M and W (i.e. $I \subseteq G \times M \times W$) such that $(g, m, v) \in I$ and $(g, m, w) \in I$ always imply $v = w$; the elements of G, M and W are called *objects*, *(many-valued) attributes* and *attribute values*, respectively.

An attribute m of a many-valued context (G, M, W, I) may be considered as a partial map of G into W, which suggests to write $m(g) = w$ rather than $(g, m, w) \in I$, and to define the *domain* of m by

$$dom(m) := \{g \in G | (g, m, w) \in I \text{ for some } w \in W\}.$$

The attribute m is said to be *complete* if $dom(m) = G$, and a many-valued context is *complete* if all its attributes are. (G, M, W, I) is called an *n-valued context* if W has cardinality n. One-valued contexts correspond to the contexts defined in section 2.

Figure 10 shows an example taken from [11], it gives the rank-ordering of different Jazz styles according to personal constructs of a single test-person. We may interpret this data set as a many-valued context, with the styles (G = {New Orleans Jazz, Ragtime,...,Swing}) as object, the personal constructs (M = {cheerful to aggressive – sad to melancholy, ... , white music – black music}) as many-valued attributes, and with value set $W := \{1, 2, \ldots, 9\}$. The obvious interpretation is to read the ternary relation I from the table: the value written in row g and column m is just $m(g)$. This example yields a complete 9-valued context.

In general, there is no immediate, "automatic" way to associate a conceptual structure with a given many-valued context. The reason is that the notion is to general to reflect the structural information about the data set which is needed to do a conceptual analysis. Therefore a refined and enriched model is neccessary. This will be formalized below as a *scaled context*. We shall not abandon the notion of a many-valued context; our basic view is that empirical data are often represented in

Jazz styles	C1: cheerful to agressive — sad to melancholic	C2: wound up, wild — cool, damped	C3: emotional — intellectual	C4: regular rhythm — irregular rhythm	C5: typical of an area — typical of a breed of men	C6: suitable for film music — requires greater attention	C7: familiar, harmonic melody — unfamiliar, unharmonic melody	C8: short "songs" — longer compositions	C9: dance music — listening music	C10: long word — short word	C11: historically older — newer	C12: white music — black music
New Orleans Jazz	6	7	3	3	2	4	4	4	3	9	2	6
Ragtime	7	4	6	2	6	1	2	2	4	4	6	4
Free Jazz	1	6	8	9	8	9	9	9	9	5	9	1
Chicago Jazz	2	5	7	7	1	5	5	5	5	8	5	8
Bebop	4	2	2	5	4	7	6	6	6	2	3	9
Hard Bop	3	1	4	6	5	8	7	8	7	3	7	7
Cool Jazz	8	9	9	8	9	6	8	7	8	6	8	5
Dixieland	5	3	5	1	3	3	1	1	1	7	1	2
Swing	9	8	1	4	7	2	3	3	2	1	4	3

Figure 10: A ranking of Jazz styles by a non–expert

this form, from which in a formalized process of interpretation (called *scaling*) a scaled context can be derived. Among other advantages, this allows several interpretations of the same data set, e.g. "rougher" and "finer" analysis. The aim of the scaling process is to obtain a one-valued *derived context* with the same objects as (G, M, W, I), whose extents can be considered as the "meaningful" subsets of G with respect to an interpretation.

The first step of scaling is to uncover the structure of each attribute's value set. In the definition of a many-valued context, the values are just elements of a set; in practice, the values are often implictely structured, and sometimes it is tacitely assumed that some attribute values imply others. If e.g. a many-valued attribute has values "big" and "very big", two interpretations are possible: it may either be meant that "very big" implies "big", i.e. that every very big object ist also big. Or "big" is an abbreviation for "big, but not very big". In the second case, the concept of all big objects would not cover the very big ones. So, in the first step of scaling, our aim is to make precise which subsets T of the value-set $m(G)$ of an attribute m are "meaningful" or "concept-constituting" in the sense that the set of all objects having values in T is considered as an extent.

If we look at our example, we find that the values (of all attributes) are numbers, and it is stated that they indicate rankings of the objects. A natural assumption is that the values are taken from an ordinal scale, and that the extents of such a scale may be taken as concept-constituting entities. To be concrete, if we interprete the values $1, 2, \ldots, 9$ of the attribute "emotional – intellectual" as the objects of the one-dimensional ordinal scale $\mathbb{O}_9$, then we can form the concept of the "more emotional" Jazz styles (attribute value ≤ 4, say). But we cannot form the concept of the Jazz styles whose value for the emotional – intellectual attribute is one of $\{1, 2, 5, 7\}$, since $\{1, 2, 5, 7\}$ is not an extent of the ordinal scale.

Formally, the first step of scaling consists of assigning to each attribute $m \in M$ a scale $\mathbb{S}_m := (G_m, M_m, I_m)$ with $m(G) \subseteq G_m$. (Sometimes it is useful to remove the condition that $m(G) \subseteq G_m$ and to introduce a mapping $\nu_m : m(G) \longrightarrow \mathbb{S}_m$ instead. To keep notations short, we shall discuss only the simpler model.) The choice of these scales is a matter of interpretation, the task is to select $\mathbb{S}_m$ in such a way that every extent $U \subseteq G_m$ induces a meaningful set $U \cap m(G)$ of values of m and, conversely, that every meaningful set of attribute values is obtained in this way.

The second step of scaling is to decide how the different many-valued attributes can be combined to describe concepts. For qualitative data, as in our example, a simple conjuction of attributes may suffice. In a mathematical context, phrases such as "those quadrangles in which the height equals the width" must be permitted as more complex concept-constituting attribute combinations. Formally, we apply some *product operator* $\prod$ which composes the given scales to a common scale

$$\mathbb{S} := \prod_{m \in M} \mathbb{S}_m = \left(\bigtimes_{m \in M} G_m, N, J \right).$$

We shall not be very precise about the nature of such product operators, frequently used examples are the *semiproduct* and the *direct product*, introduced in section 3, but there is also the possibility of defining algebraic scale compositions, cf. [21]. It is assumed that the set of objects of the composed scale $\mathbb{S}$ is the cartesian product of the value sets of the scales $\mathbb{S}_m$, and, moreover, that for each $i \in M$ the i-th projection π_i, defined by $\pi_i((g_m)_{m \in M}) := g_i$, is an $\mathbb{S}_m$–measure of $\mathbb{S}$. The simple case of just allowing conjunctions of the attribute values, as mentioned above, is obtained when using the semiproduct operator. We then speak of *plain scaling*.

A many-valued context $\mathbb{K} := (G, M, W, I)$ together with the scale $\mathbb{S}$ is called a *scaled context* and is denoted by $(\mathbb{K}; \prod_{m\in M} \mathbb{S}_m)$. To understand the interplay between $\mathbb{K}$ and $\mathbb{S}$, it is helpful to think of the many-valued context as given by a rectangular table, the rows of which are indexed by the objects, columns by attributes, and where the entries are the values, as in figure 10. Suppose that $\mathbb{K}$ is complete, then every row can be read as a tuple $(m(g))_{m\in M}$ of attribute values, and each such tuple is an element of $\bigtimes_{m\in M} G_m$. But these are just the objects of the commen scale $\mathbb{S}$, so that to each row of $\mathbb{K}$ there corresponds an object of the scale $\mathbb{S}$. If each row of $\mathbb{K}$ is replaced by the corresponding row of $\mathbb{S}$, we derive a (formal) context for the scaled context $(\mathbb{K}; \prod_{m\in M} \mathbb{S}_m)$. This is made precise by the following definition:

Let $\mathbb{K}$ be a complete many-valued context scaled by

$$\mathbb{S} := \prod_{m\in M} \mathbb{S}_m = (\bigtimes_{m\in M} G_m, N, J).$$

We define the *derived context* as a one-valued context $(G, N, \tilde{J})$, where the objects are the objects of $\mathbb{K}$, the attributes are those of $\mathbb{S}$, and the incidence is given by

$$g\tilde{J}n :\Leftrightarrow (m(g))_{m\in M} J n \text{ in } \mathbb{S}.$$

The concepts of the derived context are called the *concepts of the scaled context* $(\mathbb{K}; \prod_{m\in M} \mathbb{S}_m)$. By $\mathfrak{U}(\mathbb{K}; \prod_{m\in M} \mathbb{S}_m)$ we denote the set of extents of the derived context.

In case that $\mathbb{K} := (G, M, W, I)$ is not complete, the definition is a little more technical. Again, the derived context will have the objects G of $\mathbb{K}$ and the attributes N of $\mathbb{S}$. We first give an intuitive description, and then a formal definition of the incidence $\tilde{J}$ between G and N. Let g be an object and n be an attribute. In the complete case, we have identified the g-row of (G, M, W, I) with the object $(m(g))_{m\in M}$ of $\mathbb{S}$. In the non-complete case, there may be empty cells in this row. If we complete the row by filling values in the empty cells, we obtain an object of $\mathbb{S}$ which has the attribute n, or has not. This will, of course, usually depend on the values which were filled in, but it may happen that *every* completion has n, independently of the supplemented values. Only in this case we define $g\tilde{J}n$. The formal definition is:

Let $\mathbb{K}$ be a many-valued context scaled by

$$\mathbb{S} := \prod_{m\in M} \mathbb{S}_m = (\bigtimes_{m\in M} G_m, N, J).$$

We define the *derived context* as a one-valued context $(G, N, \tilde{J})$, where the objects are the objects of G, the attributes are those of $\mathbb{S}$, and the incidence is given by

$$\begin{aligned} g\tilde{J}n :\Leftrightarrow hJn \text{ for all objects } h := (h_m)_{m\in M} \text{ of } \mathbb{S} \text{ satisfying} \\ h_m = m(g) \text{ whenever } m(g) \text{ is defined.} \end{aligned}$$

The concepts of the derived context are called the *concepts of the scaled context* $(\mathbb{K}; \prod_{m\in M} \mathbb{S}_m)$ and, as above, $\mathfrak{U}(\mathbb{K}; \prod_{m\in M} \mathbb{S}_m)$ denotes the set of extents of the derived context.

For defining dependencies we need to know how one may restrict a product operator Π to a subfamily of scales. For a subset R of M we define the derived context of $(\mathbb{K}; \prod_{m\in R} \mathbb{S}_m)$ to be the context $(G, N_R, \tilde{J}_R)$, where N_R is the set of all attributes $n \in N$ for which $(g_m)_{m\in M} Jn$ iff $(h_m)_{m\in M} Jn$ for all elements of $\bigtimes_{m\in M} G_m$ with $g_r = h_r$ for all $r \in R$. $\tilde{J}_R$ is defined as the restriction $\tilde{J}_R := \tilde{J} \cap G \times N_R$.

	$C10\leq6$	$C1\leq2$	$C12\leq8$	$C2\leq4$	$C4\leq8$	$C1\leq6$	$C2,C3\leq8$	$C1\leq8$	$C4\leq6$	$C11\leq8$	$C10\leq7$	$C8\leq7$	$C3\leq4$	$C3\leq5$	$C5\leq2$	$C11\leq5$	$C12\leq3$	$C12\leq6$	$C12\leq7$
New Orleans Jazz			×			×	×	×	×	×		×	×	×	×	×		×	×
Ragtime	×		×		×		×	×	×	×	×	×						×	×
Free Jazz	×	×	×	×	×	×	×	×			×						×	×	×
Chicago Jazz		×	×	×	×	×	×	×		×		×			×	×			
Bebop	×			×	×	×	×	×	×	×	×	×	×	×		×			
Hard Bop	×		×	×	×	×	×	×	×	×	×		×	×					×
Cool Jazz	×		×		×			×		×	×	×						×	×
Dixieland			×		×	×	×	×	×	×	×	×		×		×	×	×	×
Swing	×		×		×		×		×	×	×	×	×	×		×	×	×	×

Figure 11: The reduced derived context from plain ordinal scaling

	$C10\leq6$	$C1\leq2$	$C2\leq4$	$C4\leq8$	$C1\leq6$	$C2,C3\leq8$	$C1\leq8$	$C4\leq6$	$C10\leq7$	$C3\leq4$	$C3\leq5$
New Orleans Jazz					×	×	×	×		×	×
Ragtime	×			×		×	×	×	×		
Free Jazz	×	×	×	×	×	×	×		×		
Chicago Jazz		×	×	×	×	×	×				
Bebop	×		×	×	×	×	×	×	×	×	×
Hard Bop	×		×	×	×	×	×	×	×	×	×
Cool Jazz	×			×			×		×		
Dixieland				×	×	×	×	×	×		×
Swing	×			×		×		×	×	×	×

	$C12\leq8$	$C11\leq8$	$C8\leq7$	$C5\leq2$	$C11\leq5$	$C12\leq3$	$C12\leq6$	$C12\leq7$
New Orleans Jazz	×	×	×	×	×		×	×
Ragtime	×	×	×				×	×
Free Jazz	×					×	×	×
Chicago Jazz	×	×	×	×	×			
Bebop		×	×		×			
Hard Bop	×	×						×
Cool Jazz	×	×	×				×	×
Dixieland	×	×	×		×	×	×	×
Swing	×	×	×		×	×	×	×

Figure 12: Subcontexts obtained by splitting the attribute set

Now let us apply the definitions to the example of the Jazz styles. We have already mentioned that the twelve many-valued attributes are all of ordinal nature, which can naturally be interpreted by an one-dimensional ordinal or interordinal scale. For ordinal scaling, we use the scale $\mathbb{O}_9$ for all attributes. We decide for plain scaling, thus the composed scale will be the twelvefold semiproduct of $\mathbb{O}_9$:

$$\mathbb{S} = \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9 \bowtie \mathbb{O}_9.$$

The 9^{12} objects of this scale are the 12-tuples of integers between 1 and 9, and there are $9 \times 12 = 108$ attributes. The derived context thus has nine objects and 108 attributes. Most of these attributes are redundant, and the reduced context (see section 2) has only 19 attributes. This context is shown in figure 11. It has 190 concepts which is slightly too many to obtain a clear line diagram. However, there is a simple and precise way of representing a large concept lattice by *several* line diagrams (cf. [19]), by splitting the attribute set and drawing the concept lattices of the subcontexts separately. Our choice of subcontexts is shown in figure 12, the corresponding concept lattices have 38 and 18 elements, respectively (fig.13 and 14).

As a second plausible attempt we may unfold the Jazz styles data by plain interordinal scaling. There are, however, good reasons to be reluctant: we have already

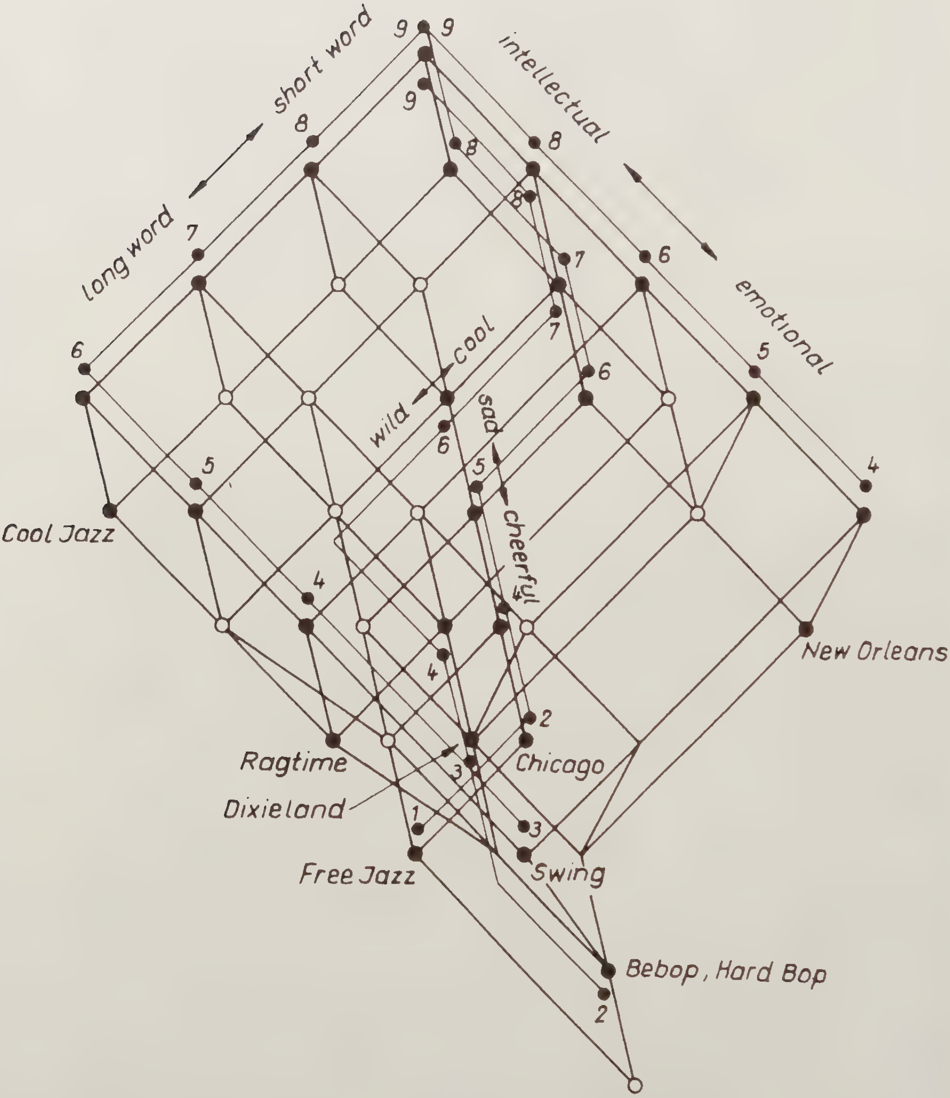

Figure 13: The concept lattice of the first subcontext in figure 12

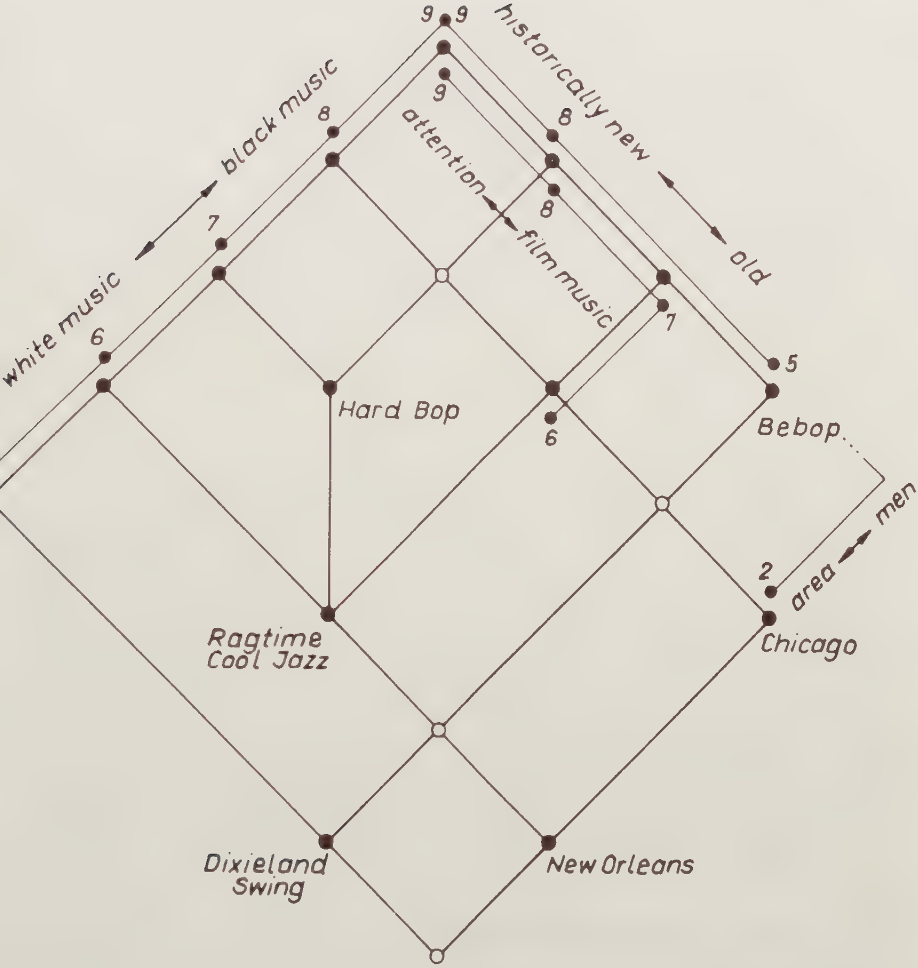

Figure 14: The concept lattice of the second subcontext in figure 12

	0	1
1	×	
2	×	
3	×	
4		
5		
6		
7		×
8		×
9		×

Figure 15: A dichotomic scale for threshold scaling

seen that plain ordinal scaling leads to a rather subtle analysis of the data set, and a comparison with the original data may suggest that we could overinterprete the data by analysing to a higher precision than justified by the data's meaning. The interordinal scale $\mathbb{I}_9$ is finer than the ordinal scale $\mathbb{O}_9$ in the sense that it admits an bijective $\mathbb{O}_9$–measure, and it can be shown that this is inherited to the semiproduct scale. Thus, plain $\mathbb{I}_9$-scaling would lead to a concept lattice which has far more than 190 elements, which is exaggerating. As a second scaling of this many-valued context we therefore discuss a much rougher approach, namely a threshold scaling into the dichotomic scale $\mathbb{D}^{\underline{0}}$. This scale has three scale values, while each of our many-valued attributes has nine attribute values. We therefore replace the scale by the equivalent scale shown in figure 15. The use of this scale has the following interpretation: we replace each many-valued attribute m, such as "emotional – intellectual", by two one-valued attributes (which we shall denote as "emotional" and "intellectual"). An object g is given the first attribute in the derived context, if $m(g) \leq 3$, and the second, if $m(g) \geq 7$. The derived context has 24 attributes, it is shown in figure 16. The 42-element concept lattice is given in figure 17.

6 Scaling and Dependency

A general notion of dependency between many-valued attributes has to include scalings of the attribute values. For instance, if we regard the values in fig.10 as unstructured we can only speak of functional dependency (cf. [12]), but in this case we obtain the useless result that each attribute functionally depends on each other attribute. For a meaningful analysis of dependencies we have to consider the ordinal nature of the attributes in fig.10. With the tools of formal concept analysis a general definition of dependency can be given based on the notion of a scaled context. The idea is that an attribute depends on other attributes if its conceptual contribution to the scaled context can already be furnished by the other attributes. Formally this is defined as follows (see [21]): in a scaled context $(\mathbb{K}; \prod_{m \in M} \mathbb{S}_m)$ a set Y of attributes *depends* on a set X of attributes if every extent of $(\mathbb{K}; \prod_{m \in X \cup Y} \mathbb{S}_m)$ is already an extent of

	cheerful to agressive	sad to melancholic	wound up, wild	cool, damped	emotional	intellectual	regular rhythm	irregular rhythm	typical of an area	typical of a breed of men	suitable for film music	requires greater attention	familiar, harmonic melody	unfamiliar, unharmonic melody	short "songs"	longer compositions	dance music	listening music	long word	short word	historically older	newer	white music	black music
New Orleans Jazz				×	×		×		×								×			×	×			
Ragtime		×					×				×		×		×									
Free Jazz	×					×		×		×		×		×		×		×				×	×	
Chicago Jazz	×					×		×	×											×				×
Bebop			×		×							×							×		×			×
Hard Bop	×		×									×		×		×		×	×			×		×
Cool Jazz		×		×		×		×		×				×		×		×				×		
Dixieland			×				×		×		×		×		×		×			×	×		×	
Swing		×		×	×					×	×		×		×		×		×				×	

Figure 16: The derived context, from dichotomic threshold scaling

$(\mathbb{K}; \prod_{m\in X} \mathbb{S}_m)$, i.e.

$$\mathfrak{U}(\mathbb{K}; \prod_{m\in X\cup Y} \mathbb{S}_m) = \mathfrak{U}(\mathbb{K}; \prod_{m\in X} \mathbb{S}_m).$$

A weaker notion of dependency is given by the condition: every extent of $(\mathbb{K}; \prod_{m\in Y} \mathbb{S}_m)$ is an extent of $(\mathbb{K}; \prod_{m\in X} \mathbb{S}_m)$; if this condition is fulfilled, Y is said to be *weakly dependent* on X. Dependency and weak dependency coincide in plain scaled contexts but not generally in many-valued contexts scaled by direct products of scales.

The definition of functional dependency as it is introduced in the theory of relational databases, reads in the language of formal concept analysis as follows: in a complete many-valued context (G, M, W, I) a set Y of attributes *functionally depends* on a set X of attributes if, for all $g, h \in G$, $x(g) = x(h)$ for all $x \in X$ implies $y(g) = y(h)$ for all $y \in Y$, i.e. there exists a function $f : W^X \longrightarrow W^Y$ such that $f(x(g))_{x\in X} = (y(g))_{y\in Y}$ for all $g \in G$. The next proposition shows by a scaling with complementary nominal scales that functional dependency can be understood as a special case of the general dependency defined above.

Proposition 7 *(see [21]) Let* $\mathbb{K}$ *be a complete many-valued context scaled by* $\bigtimes_{m\in M} \mathbb{N}^{c\bar{0}}_{n_m}$. *For* $X, Y \subseteq M$ *the following conditions are equivalent:*

1. *Y functionally depends on X in* $\mathbb{K}$.
2. *Y depends on X in* $(\mathbb{K}; \bigtimes_{m\in M} \mathbb{N}^{c\bar{0}}_{n_m})$.
3. *Y weakly depends on X in* $(\mathbb{K}; \bigtimes_{m\in M} \mathbb{N}^{c\bar{0}}_{n_m})$.

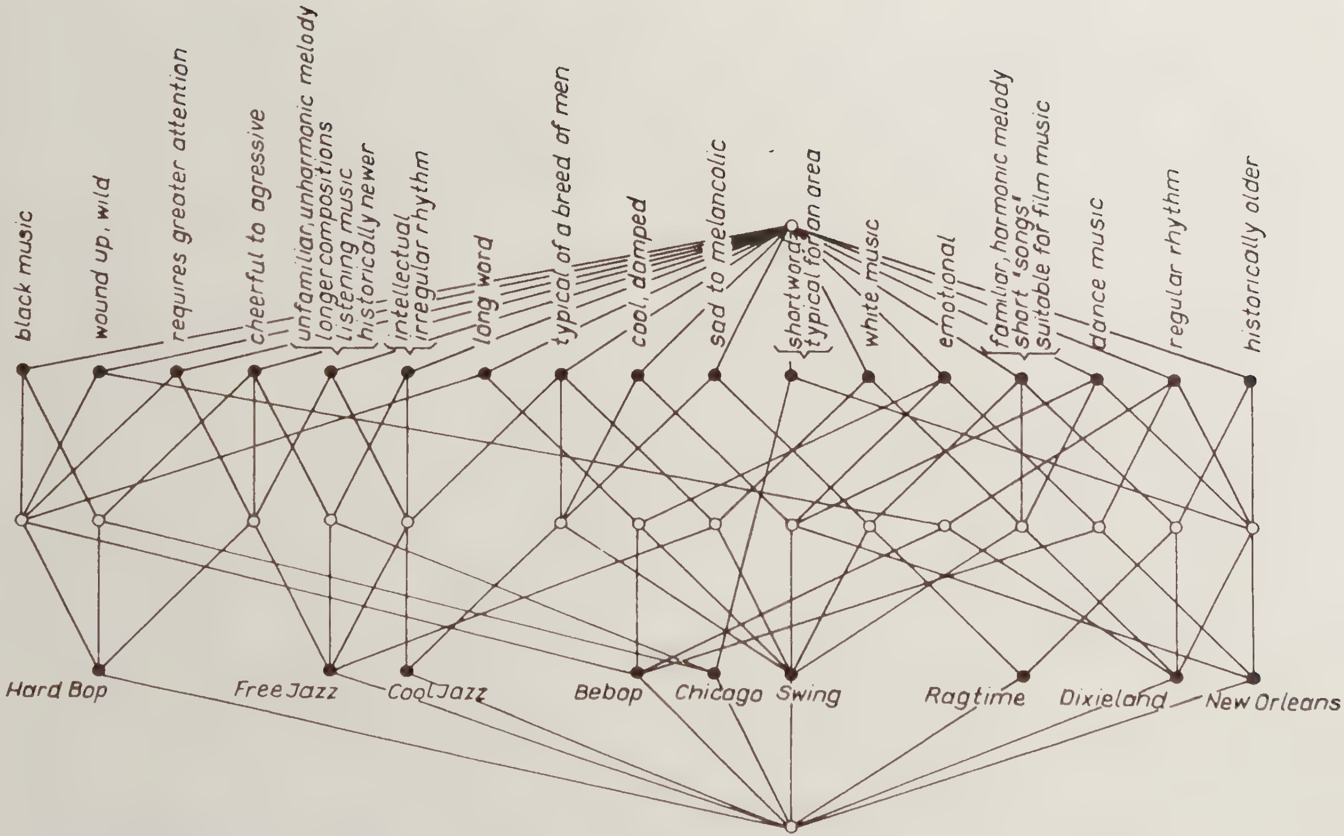

Figure 17: The concept lattice, from dichotomic treshold scaling

If the attribute values carry an order structure it is meaningful to introduce a notion of ordinal dependency (cf. [10]): in a complete many-valued context $(G, M, \mathbf{W}, I)$ with an ordered set $\mathbf{W} := (W, \leq)$ a set Y of attributes *ordinally depends* on a set X of attributes if, for $g, h \in G$, $x(g) \leq x(h)$ for all $x \in X$ implies $y(g) \leq y(h)$ for all $y \in Y$, i.e. there exists an order-preserving function $f : \mathbf{W}^X \longrightarrow \mathbf{W}^Y$ such that $f(x(g))_{x\in X} = (y(g))_{y\in Y}$ for all $g \in G$. It is also useful to introduce interordinal dependency considering the ternary relation *betweeness* which is defined by $[u, v, w] :\Leftrightarrow (u \leq v \leq w \text{ or } u \geq v \geq w)$. We say that Y **interordinally depends** on X if, for $g, h, k \in G$, $[x(g), x(h), x(k)]$ for all $x \in X$ implies $[y(g), y(h), y(k)]$ for all $y \in Y$, i.e. there exists a betweeness-preserving function $f : \mathbf{W}^X \longrightarrow \mathbf{W}^Y$ such that $f(x(g))_{x\in X} = (y(g))_{y\in Y}$ for all $g \in G$. Ordinal and interordinal dependency can also be understood in the frame of our general definition of dependency; for this we use scalings with contrary ordinal scales and convex ordinal scales, respectively.

Proposition 8 *(see [21]) Let* $\mathbb{K}$ *be a complete many-valued context scaled by* $\bigtimes_{m\in M} \mathbb{O}^{cd\bar{0}}_{\mathbf{P}_m}$ *so that the values of each attribute* m *of* $\mathbb{K}$ *are ordered by* $\mathbf{P}_m$. *For* $X, Y \subseteq M$ *the following conditions are equivalent:*

1. *Y ordinally depends on X in $\mathbb{K}$.*
2. *Y depends on X in* $(\mathbb{K}; \bigtimes_{m\in M} \mathbb{O}^{cd\bar{0}}_{\mathbf{P}_m})$.
3. *Y weakly depends on X in* $(\mathbb{K}; \bigtimes_{m\in M} \mathbb{O}^{cd\bar{0}}_{\mathbf{P}_m})$.

Proposition 9 *Let* $\mathbb{K}$ *be a complete many-valued context scaled by the apposition* $\bigtimes_{m\in M} \mathbb{O}^{cd\bar{0}}_{\mathbf{P}_m} \mid \bigtimes_{m\in M} \mathbb{O}^{c\bar{1}}_{\mathbf{P}_m}$ *so that the values of each attribute* m *of* $\mathbb{K}$ *are ordered by* $\mathbf{P}_m$. *For* $X, Y \subseteq M$ *the following conditions are equivalent:*

1. *Y interordinally depends on X in $\mathbb{K}$.*
2. *Y depends on X in* $(\mathbb{K}; \bigtimes_{m\in M} \mathbb{O}^{cd\bar{0}}_{\mathbf{P}_m} \mid \bigtimes_{m\in M} \mathbb{O}^{c\bar{1}}_{\mathbf{P}_m})$.
3. *Y weakly depends on X in* $(\mathbb{K}; \bigtimes_{m\in M} \mathbb{O}^{cd\bar{0}}_{\mathbf{P}_m} \mid \bigtimes_{m\in M} \mathbb{O}^{c\bar{1}}_{\mathbf{P}_m})$.

Ordinal and interordinal dependency can also be studied in many-valued contexts which are not complete (see [10]), here we restrict our explanations to the case of complete many-valued contexts. A basic question is how to determine and represent the dependencies of a scaled context. A solution for ordinal dependency is described in [10]. The idea is to translate the scaled context into a one-valued context with the same set of attributes so that the implications of the one-valued context are exactly the dependencies of the scaled context. Let us recall that in a (one-valued) context a set X of attributes *implies* a set Y of attributes if $X' \subseteq Y'$. In [8] a computer program is offered to determine functional, ordinal and interordinal dependencies; this program is based on the following proposition (notice that functional dependency is a special case of ordinal dependency).

Proposition 10 *Let* $\mathbb{K} := (G, M, \mathbf{W}, I)$ *be a complete many-valued context with an ordered set* $\mathbf{W} := (W, \leq)$ *of attribute values; furthermore, let* $\mathbb{K}_o := (G \times G, M, I_o)$ *be the context with* $(g, h)I_o m \Leftrightarrow m(g) \leq m(h)$ *and let* $\mathbb{K}_{io} := (G \times G \times G, M, I_{io})$ *be the context with* $(g, h, k)I_{io} m \Leftrightarrow [m(g), m(h), m(k)]$. *Then, for* $X, Y \subseteq M$,

1. $\{7,8,9,11,12\} \Rightarrow \{4\}$	29. $\{1,7,10\} \Rightarrow \{2,4,9,11\}$
2. $\{11,12\} \Rightarrow \{9\}$	30. $\{4,5,10\} \Rightarrow \{3,7,9,11\}$
3. $\{7,8,10,12\} \Rightarrow \{6\}$	31. $\{1,5,10\} \Rightarrow \{2\}$
4. $\{6,10,12\} \Rightarrow \{7,8\}$	32. $\{1,2,4,10\} \Rightarrow \{7,9,11\}$
5. $\{5,10,12\} \Rightarrow \{2,3\}$	33. $\{1,3,10\} \Rightarrow \{2\}$
6. $\{5,9,12\} \Rightarrow \{7,8\}$	34. $\{8,9\} \Rightarrow \{7\}$
7. $\{1,9,12\} \Rightarrow \{7,8\}$	35. $\{4,5,7,9\} \Rightarrow \{11\}$
8. $\{5,7,8,12\} \Rightarrow \{9\}$	36. $\{1,2,4,7,9\} \Rightarrow \{11\}$
9. $\{1,7,8,12\} \Rightarrow \{9\}$	37. $\{6,9\} \Rightarrow \{7,8\}$
10. $\{7,12\} \Rightarrow \{8\}$	38. $\{3,5,9\} \Rightarrow \{11\}$
11. $\{3,6,12\} \Rightarrow \{7,8,9\}$	39. $\{4,9\} \Rightarrow \{7\}$
12. $\{4,5,12\} \Rightarrow \{7,8,9,11\}$	40. $\{1,3,9\} \Rightarrow \{5,11\}$
13. $\{1,2,4,12\} \Rightarrow \{7,8,9,11\}$	41. $\{2,9\} \Rightarrow \{4,7\}$
14. $\{1,3,12\} \Rightarrow \{2,5\}$	42. $\{5,8\} \Rightarrow \{7\}$
15. $\{1,2,4,6,7,8,9,10,11\} \Rightarrow \{12\}$	43. $\{4,8\} \Rightarrow \{7\}$
16. $\{7,9,10,11\} \Rightarrow \{4\}$	44. $\{3,8\} \Rightarrow \{7,9\}$
17. $\{5,10,11\} \Rightarrow \{3\}$	45. $\{2,8\} \Rightarrow \{4,7\}$
18. $\{1,10,11\} \Rightarrow \{2\}$	46. $\{1,8\} \Rightarrow \{7\}$
19. $\{2,4,5,6,7,8,9,11\} \Rightarrow \{3\}$	47. $\{6,7\} \Rightarrow \{8\}$
20. $\{3,7,9,11\} \Rightarrow \{4\}$	48. $\{3,7\} \Rightarrow \{9\}$
21. $\{8,11\} \Rightarrow \{7,9\}$	49. $\{2,7\} \Rightarrow \{4\}$
22. $\{7,11\} \Rightarrow \{9\}$	50. $\{5,6\} \Rightarrow \{7,8\}$
23. $\{6,11\} \Rightarrow \{7,8,9\}$	51. $\{4,6\} \Rightarrow \{7,8\}$
24. $\{1,3,11\} \Rightarrow \{5\}$	52. $\{2,6\} \Rightarrow \{4,7,8\}$
25. $\{5,9,10\} \Rightarrow \{3,4,7,11\}$	53. $\{1,6\} \Rightarrow \{7,8\}$
26. $\{1,9,10\} \Rightarrow \{2,4,7,11\}$	54. $\{3,4,5\} \Rightarrow \{7,9,11\}$
27. $\{8,10\} \Rightarrow \{7\}$	55. $\{1,2,3,4\} \Rightarrow \{5,7,9,11\}$
28. $\{5,7,10\} \Rightarrow \{3,4,9,11\}$	56. $\{1,4\} \Rightarrow \{2\}$

Figure 18: A basis for the ordinal dependencies

1. *X ordinally depends on Y if and only if X implies Y in $\mathbb{K}_o$,*

2. *X interordinally depends on Y if and only if X implies Y in $\mathbb{K}_{io}$.*

Let us come back to our example from section 5. Since all rankings in fig.10 are different, each attribute functionally depends on each other attribute; but, by the same reason, no attribute ordinally depends on a single other attribute. All ordinal dependencies valid in our example can be derived from a minimal basis for these dependencies which is listed in fig.18 (cf. [5, 7]). For interpreting this analysis one should explicitely write down the dependencies in verbal form. For instance, dependency 48 of the list means that a Jazz style which is more emotional and more melodically familiar is judged rather to be dance music. Ordinal dependency compares the rankings only in the same direction, but in our example it makes also sense to compare the rankings in opposite directions. This is covered by the interordinal dependencies for which a minimal basis is given in fig.19.

1. $\{7,9,10,11,12\} \Rightarrow \{4\}$
2. $\{3,5,9,10,11,12\} \Rightarrow \{4,7\}$
3. $\{2,10,11,12\} \Rightarrow \{3,5\}$
4. $\{1,3,4,5,6,7,8,9,11,12\} \Rightarrow \{10\}$
5. $\{2,9,11,12\} \Rightarrow \{5\}$
6. $\{1,9,11,12\} \Rightarrow \{5\}$
7. $\{7,11,12\} \Rightarrow \{9\}$
8. $\{6,11,12\} \Rightarrow \{9\}$
9. $\{1,5,11,12\} \Rightarrow \{9\}$
10. $\{1,4,11,12\} \Rightarrow \{3\}$
11. $\{1,3,11,12\} \Rightarrow \{4\}$
12. $\{2,9,10,12\} \Rightarrow \{6,7,8\}$
13. $\{7,10,12\} \Rightarrow \{9\}$
14. $\{3,6,10,12\} \Rightarrow \{1,4,5,7,8,9,11\}$
15. $\{1,6,10,12\} \Rightarrow \{3,4,5,7,8,9,11\}$
16. $\{4,5,10,12\} \Rightarrow \{3\}$
17. $\{2,5,10,12\} \Rightarrow \{3\}$
18. $\{3,4,10,12\} \Rightarrow \{5\}$
19. $\{1,4,10,12\} \Rightarrow \{3,5\}$
20. $\{2,3,10,12\} \Rightarrow \{5\}$
21. $\{1,3,10,12\} \Rightarrow \{5\}$
22. $\{1,2,10,12\} \Rightarrow \{3,4,5,6,7,8,9,11\}$
23. $\{2,5,6,7,8,9,12\} \Rightarrow \{1,3\}$
24. $\{4,6,7,8,9,12\} \Rightarrow \{11\}$
25. $\{2,3,6,7,8,9,12\} \Rightarrow \{1\}$
26. $\{3,5,7,8,9,12\} \Rightarrow \{1\}$
27. $\{1,5,7,8,9,12\} \Rightarrow \{3\}$
28. $\{1,4,7,8,9,12\} \Rightarrow \{3,5,11\}$
29. $\{4,9,12\} \Rightarrow \{7\}$
30. $\{2,3,9,12\} \Rightarrow \{7,8\}$
31. $\{2,5,7,8,12\} \Rightarrow \{9\}$
32. $\{1,2,7,8,12\} \Rightarrow \{9\}$
33. $\{5,6,12\} \Rightarrow \{7,8\}$
34. $\{2,6,12\} \Rightarrow \{7,8,9\}$
35. $\{2,4,5,12\} \Rightarrow \{7,8,9,11\}$
36. $\{1,2,3,5,12\} \Rightarrow \{7,8,9\}$
37. $\{1,2,4,12\} \Rightarrow \{3,11\}$
38. $\{3,4,7,8,9,10,11\} \Rightarrow \{6\}$
39. $\{7,10,11\} \Rightarrow \{9\}$
40. $\{6,10,11\} \Rightarrow \{7,8,9\}$
41. $\{5,10,11\} \Rightarrow \{3\}$
42. $\{4,10,11\} \Rightarrow \{7,9\}$
43. $\{1,2,3,10,11\} \Rightarrow \{5\}$
44. $\{1,10,11\} \Rightarrow \{3\}$
45. $\{2,4,5,6,7,8,9,11\} \Rightarrow \{1\}$
46. $\{2,5,7,8,9,11\} \Rightarrow \{4\}$
47. $\{1,7,8,9,11\} \Rightarrow \{4\}$
48. $\{4,9,11\} \Rightarrow \{7\}$
49. $\{2,3,9,11\} \Rightarrow \{4,7,8\}$
50. $\{2,7,8,11\} \Rightarrow \{9\}$
51. $\{8,11\} \Rightarrow \{7\}$
52. $\{5,7,11\} \Rightarrow \{9\}$
53. $\{3,7,11\} \Rightarrow \{4,9\}$
54. $\{1,7,11\} \Rightarrow \{9\}$
55. $\{5,6,11\} \Rightarrow \{7,8,9\}$
56. $\{3,6,11\} \Rightarrow \{4,7,8,9\}$
57. $\{2,6,11\} \Rightarrow \{7,8,9\}$
58. $\{1,6,11\} \Rightarrow \{4,7,8,9\}$
59. $\{3,4,5,11\} \Rightarrow \{7,9\}$
60. $\{1,4,5,11\} \Rightarrow \{7,8,9\}$
61. $\{1,3,4,7,8,9,10\} \Rightarrow \{6\}$
62. $\{2,7,8,9,10\} \Rightarrow \{6\}$
63. $\{6,9,10\} \Rightarrow \{7,8\}$
64. $\{5,9,10\} \Rightarrow \{3,12\}$
65. $\{4,9,10\} \Rightarrow \{7\}$
66. $\{2,3,9,10\} \Rightarrow \{1,4,6,7,8\}$
67. $\{1,9,10\} \Rightarrow \{3\}$
68. $\{6,7,8,10\} \Rightarrow \{9\}$
69. $\{4,5,7,8,10\} \Rightarrow \{1\}$
70. $\{1,8,10\} \Rightarrow \{4,7\}$
71. $\{5,7,10\} \Rightarrow \{4\}$
72. $\{3,7,10\} \Rightarrow \{4,9\}$
73. $\{1,7,10\} \Rightarrow \{4,8\}$
74. $\{2,6,10\} \Rightarrow \{7,8,9\}$
75. $\{2,4,10\} \Rightarrow \{1,7,8\}$
76. $\{3,5,6,7,8,9\} \Rightarrow \{1\}$
77. $\{1,2,3,5,7,8,9\} \Rightarrow \{12\}$
78. $\{8,9\} \Rightarrow \{7\}$
79. $\{5,6,9\} \Rightarrow \{7,8\}$
80. $\{3,6,9\} \Rightarrow \{7,8\}$
81. $\{2,6,9\} \Rightarrow \{7,8\}$
82. $\{1,6,9\} \Rightarrow \{7,8\}$
83. $\{4,5,9\} \Rightarrow \{7,11\}$
84. $\{2,3,5,9\} \Rightarrow \{7,8\}$
85. $\{1,4,9\} \Rightarrow \{7,8\}$
86. $\{1,2,3,9\} \Rightarrow \{7,8\}$
87. $\{1,2,6,7,8\} \Rightarrow \{9\}$
88. $\{2,3,7,8\} \Rightarrow \{9\}$
89. $\{5,8\} \Rightarrow \{7\}$
90. $\{4,8\} \Rightarrow \{7\}$
91. $\{3,8\} \Rightarrow \{7\}$
92. $\{2,8\} \Rightarrow \{7\}$
93. $\{6,7\} \Rightarrow \{8\}$
94. $\{3,5,7\} \Rightarrow \{9\}$
95. $\{1,4,7\} \Rightarrow \{8\}$
96. $\{1,3,7\} \Rightarrow \{8\}$
97. $\{2,7\} \Rightarrow \{8\}$
98. $\{2,5,6\} \Rightarrow \{7,8\}$
99. $\{4,6\} \Rightarrow \{7,8\}$
100. $\{2,3,4,5\} \Rightarrow \{7,8,9,11\}$
101. $\{1,2,4,5\} \Rightarrow \{7,8\}$

Figure 19: A basis for the interordinal dependencies

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