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# Solution to HW 1, Problem 1 *akashj, Jain, Akash*

1. $\ln n < \log n$ for $n > 1$ because $\log n$ has a smaller base. To see this, realise that $\log n = \frac{\ln n}{\ln 2}$. Since $\ln 2 < 1, \Rightarrow \ln n < \log n$. Since the functions grow at the same rate (within a constant factor), $\ln n = \Theta(\log n)$

2. $\log n < \log n^2 = 2 \log n$ for $n > 1$. Again, since they grow at the same rate (within a constant rate), $\log n = \Theta(\log n^2)$.

3. We now show $\log n^2 < 2^{\log n}$. Since $2^{\log n} = n$, this is equivalent to showing $n > 2 \log n$, which is true for all $n > 2$. We can show this rigorously using L'Hopital's Rule:

$$\lim_{n\to\infty} \frac{2 \log n}{n} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1} = 0$$

   Therefore, $\log n^2 = o(2^{\log n})$.

4. We now show that $\binom{n}{10} > 2^{\log n}$. Since $2^{\log n} = n$, this amounts to showing $\binom{n}{10} > n$, which is true by inspection (for $n > 11$). This can be more clearly seen if you expand $\binom{n}{10}$ out to $n \times (n-1) \times \ldots \times (n-9)$. So $n = o(\binom{n}{10})$.

5. We now show that $n^{10} > \binom{n}{10} = \frac{n!}{(n-10)! \cdot 10!}$. To make our lives easier, we shall instead show $n^{10} > \frac{n!}{(n-10)!}$ and we can do this because $\frac{n!}{(n-10)!} > \frac{n!}{(n-10)! \cdot 10!}$ and the 10! term is just a constant.

   It is important to note that each side of the inequality is the multiplication of 10 terms. On the LHS, it is $n \times n \times \ldots \times n$, and on the RHS it is $n \times (n-1) \times \ldots \times (n-9)$. Since each term on the LHS is $\geq$ than the RHS, the LHS will be larger for all $n > 10$. However, both sides of the equation are of order $n^{10}$, so $\binom{n}{10} = \Theta(n^{10})$

6. We now show that $2^{\ln^2 n} > n^{10}$ by taking logs of both sides to give $\ln^2 n > 10 \log n$. We show this by using L'Hopital's Rule:

$$\lim_{n\to\infty} \frac{10 \log n}{\ln^2 n} = \lim_{n\to\infty} \frac{\frac{10}{n}}{\frac{2 \log n}{n}} = 0$$

   Thus $n^{10} = o(2^{\ln^2 n})$

7. We now show that $n^{\log n} > 2^{\ln^2 n}$ by taking logs of both sides to give $\log^2 n > \ln^2 n$. Square rooting both sides ($n > 1$) gives $\log n > \ln n$, which we showed to be true in number 1. However, since the distance is a constant factor, $n^{\log n} = \Theta(2^{\ln^2 n})$.

8. We now show that $(\log n)^n > n^{\log n}$. First, we take the log of both sides to give us $\log(\log^n n) > \log(n^{\log n})$. This simplifies to $n \log(\log n) > \log^2 n$. To make our lives easier, we now will show that $n > \log^2 n$, which is a stronger result because $n < n \log(\log n)$. Taking limits and applying L'Hopital's Rule twice,

$$\lim_{n \to \infty} \frac{\log^2 n}{n} = \frac{2 \log n}{n} = \lim_{n \to \infty} \frac{2}{n} = 0$$

Therefore $\log^2 n = o(n)$ and so $n^{\log n} = o((\log n)^n)$.

9. We now finally show that $n! > (\log n)^n$. Taking logs of both sides gives $\log n! > n \log \log n$. Using LL notes p137, we can use Stirling's Approximation to say that $\log n! = \Theta(n \log n)$.

   Now we show $n \log n > n \log \log n$. This simplifies to showing $\log n > \log \log n$. We show this using L'Hopital's Rule:

$$\lim_{n \to \infty} \frac{\log \log n}{\log n} = \lim_{n \to \infty} \frac{\frac{1}{n \log n}}{\frac{1}{n}} = 0$$

   Thus $(\log n)^n = o(n!)$

   ...

   So the final order (in descending order) is:

   $n!, (\log n)^n, n^{\log n}, 2^{\ln^2 n}, n^{10}, \binom{n}{10}, 2^{\log n}, \log n^2, \log n, \ln n$

   For the relationships between adjacent pairs, see the explanations above.