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Hadamard topology in quantum circuits

The Hadamard gate is the one-qubit unitary operation $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
Consider a quantum circuit on n qubits of the following form:

- The circuit takes as input the state $|0\rangle^{\otimes n}$.
- Each gate is either a Hadamard gate or a diagonal gate on k qubits, where $k = O(1)$.
- The circuit can be divided into phases $D1, H1, D2, H2, \ldots DN, HN, D(N+1)$, which occur one after another.
- A phase Dl consists of $O(\text{poly}(n))$ diagonal gates and $O(\log(n))$ Hadamard gates.
- A phase Hl consists only of Hadamard gates.
- The output is measured in the computational basis.

Let's say a circuit of the above form has a N-layer Hadamard topology. Note that, since $H^2 = I$, a phase Hl takes the form $H^{\otimes m} \otimes I^{\otimes (n-m)}$ for some subset of m qubits.

Theorem 1. *The N-layer Hadamard topology quantum circuit is universal.*

Proof. The gate set { Hadamard H, phase gate $Z(\theta)$, controlled phase flip gate CZ } is universal. $Z(\theta)$ and CZ are both diagonal, on one and two qubits respectively. We can get a circuit composed of these gates into the N-layer Hadamard topology form by trivially assigning a Hl layer for each individual Hadamard gate in the circuit. □

Theorem 2. *We can efficiently classically simulate 1-layer Hadamard topology quantum circuits.*

Proof. I think I can prove this. □

Theorem 3. *We cannot efficiently classically simulate 2-layer Hadamard topology quantum circuits.*

Proof. A special case of the 2-layer Hadamard topology quantum circuit is the IQP circuit (instantaneous quantum computation). Bremner, Josza, Shepherd 2010 showed that IQP circuits cannot be efficiently classically simulated in general, unless the polynomial hierarchy collapses to the third level.
Reference: https://arxiv.org/pdf/1005.1407.pdf □

This shows that Theorem 2 is tight in some sense.