To prove the theorem, we shall use the following lemma, whose proof is given in Appendix B.

Lemma 1: For any $\alpha \in \mathbb{R}_+$ independent of N, we have

$$\mathbb{P}\left[\widehat{\mathsf{k}}_b^{\text{MAP}} \neq \mathsf{k}_b\right] = o\left(\frac{1}{N^\alpha}\right). \tag{15}$$

From Lemma 1, we have the following corollary.

Corollary 1: Using (15), we have

$$\mathbb{E}\left[\mathbb{P}\left[\widehat{\mathsf{k}}_b^{\text{MAP}} \neq \mathsf{k}_b|\mathbf{y}\right]\right] = o\left(\frac{1}{N^\alpha}\right), \tag{16}$$

$$\mathbb{E}\left[\mathbb{P}\left[\widehat{\mathsf{k}}_b^{\text{MAP}} = k|\mathbf{y}\right]\right] = o\left(\frac{1}{N^\alpha}\right), \ \forall k \in \mathcal{S}_{K_b} \backslash \mathsf{k}_b. \tag{17}$$

The roadmap for the proof of the theorem is as follows:

- Step 1: Express the optimality gap between the MMSE and MAP-based QLMMSE estimators as a function of the error probability of the MAP synchronizer $\widehat{\mathsf{k}}_b^{\text{MAP}}$.
- Step 2: Express the MMSE (11) as a sum of the MAP-based QLMMSE (12) and the expected squared norm of the optimality gap, also known as the "regret".
- Step 3: Show that the regret is upper bounded by terms that decay polynomially fast, for any fixed polynomial rate (using Lemma 1).

Proof of Theorem 1: Let us write the the MMSE estimator (6), explicitly, using (7), in terms of the MAP-based QLMMSE estimator (10), as (recall $\mathsf{k}_s = 0$, by assumption),

$$\begin{aligned}\widehat{\mathbf{s}}_{\text{MMSE}} &= \sum_{m_b=1}^{K_b} \mathbb{P}[\mathsf{k}_b = m_b|\mathbf{y}]\, \widehat{\mathbf{s}}_{\text{LMMSE}}(m_b) \\ &= \underbrace{\sum_{\substack{m_b=1 \\ m_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{P}[\mathsf{k}_b = m_b|\mathbf{y}]\, \widehat{\mathbf{s}}_{\text{LMMSE}}(m_b)}_{\triangleq \boldsymbol{\delta}(\mathbf{y})} \\ &\quad + \mathbb{P}\left[\mathsf{k}_b = \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \widehat{\mathbf{s}}_{\text{LMMSE}}(\widehat{\mathsf{k}}_b^{\text{MAP}}).\end{aligned} \tag{18}$$

Using (18), we define the optimality gap (vector),

$$\boldsymbol{\Delta}(\mathbf{y}) \triangleq \widehat{\mathbf{s}}_{\text{MMSE}} - \widehat{\mathbf{s}}_{\text{LMMSE}}(\widehat{\mathsf{k}}_b^{\text{MAP}}) = \widehat{\mathbf{s}}_{\text{MMSE}} - \widehat{\mathbf{s}}_{\text{MAP-QLMMSE}} \tag{19}$$

$$= \boldsymbol{\delta}(\mathbf{y}) - \mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}. \tag{20}$$

We proceed to the second step of the proof. For shorthand, let $\mathbf{e}_{\text{MAP-QLMMSE}} \triangleq \widehat{\mathbf{s}}_{\text{MAP-QLMMSE}} - \mathbf{s}$, and let us first write the MMSE in terms of the estimation error $\mathbf{e}_{\text{MAP-QLMMSE}}$ and the optimality gap $\boldsymbol{\Delta}(\mathbf{y})$ as,

$$\mathbb{E}\left[\|\widehat{\mathbf{s}}_{\text{MMSE}} - \mathbf{s}\|_2^2\right] = \mathbb{E}\left[\|\widehat{\mathbf{s}}_{\text{MMSE}} - \widehat{\mathbf{s}}_{\text{MAP-QLMMSE}} + \widehat{\mathbf{s}}_{\text{MAP-QLMMSE}} - \mathbf{s}\|_2^2\right] \tag{21}$$

$$= \varepsilon^2_{\text{MAP-QLMMSE}}(N) - \mathbb{E}\left[\|\boldsymbol{\Delta}(\mathbf{y})\|_2^2\right], \tag{22}$$

where we have used (19) in (21), and the well-known orthogonality property of the estimation error in MMSE estimation to any function of the measurements in (22). Expanding the first term, we have,

$$\begin{aligned}&\mathbb{E}\left[\|\boldsymbol{\Delta}(\mathbf{y})\|_2^2\right] = \\ &\mathbb{E}\left[\|\boldsymbol{\delta}(\mathbf{y})\|_2^2\right] + \mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]^2 \|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^2\right] \\ &\quad - 2\Re\left\{\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right]\right\}.\end{aligned} \tag{23}$$

We now show that (the magnitude of) each of the terms in (23) is bounded. It will then follow that the expected squared norm of the optimality gap, $\mathbb{E}\left[\|\boldsymbol{\Delta}(\mathbf{y})\|_2^2\right]$, is also bounded.

Starting with the first term in (23), we have,

$$\mathbb{E}\left[\|\boldsymbol{\delta}(\mathbf{y})\|_2^2\right] = \sum_{n=1}^{N} \mathbb{E}\left[\delta_n^2(\mathbf{y})\right] = \tag{24}$$

$$\sum_{n=1}^{N} \mathbb{E}\left[\left(\sum_{\substack{m_b=1 \\ m_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{P}[\mathsf{k}_b = m_b|\mathbf{y}]\, \widehat{\mathsf{s}}_{\text{LMMSE},n}(m_b)\right)^2\right]. \tag{25}$$

Focusing on one element of the sum in (25), we have,

$$\mathbb{E}\left[\left(\sum_{\substack{m_b=1 \\ m_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{P}[\mathsf{k}_b = m_b|\mathbf{y}]\, \widehat{\mathsf{s}}_{\text{LMMSE},n}(m_b)\right)^2\right] \leq \tag{26}$$

$$\sum_{\substack{m_1=1 \\ m_1 \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \sum_{\substack{m_2=1 \\ m_2 \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_1|\mathbf{y}]^2\, \mathbb{P}[\mathsf{k}_b = m_2|\mathbf{y}]^2\right]^{\frac{1}{2}} \cdot$$

$$\mathbb{E}\left[\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_1)\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_2)\right]^{\frac{1}{2}} \leq \tag{27}$$

$$\sum_{\substack{m_1=1 \\ m_1 \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \sum_{\substack{m_2=1 \\ m_2 \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_1|\mathbf{y}]^4\right]^{\frac{1}{4}} \mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_2|\mathbf{y}]^4\right]^{\frac{1}{4}} \cdot$$

$$\mathbb{E}\left[\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_1)\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_2)\right] \leq \tag{28}$$

$$\sum_{\substack{m_1=1 \\ m_1 \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \sum_{\substack{m_2=1 \\ m_2 \neq \widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \underbrace{\mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_1|\mathbf{y}]\right]^{\frac{1}{4}}}_{o\left(\frac{1}{N^{4\alpha}}\right)} \underbrace{\mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_2|\mathbf{y}]\right]^{\frac{1}{4}}}_{o\left(\frac{1}{N^{4\alpha}}\right)} \cdot \tag{29}$$

$$\underbrace{\mathbb{E}\left[\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_1)\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_2)\right]}_{\mathcal{O}(1)} = \tag{30}$$

$$o\left(\frac{1}{N^\alpha}\right), \tag{31}$$

where we have used the Cauchy-Schwarz inequality repeatedly in (26) and (27), the following, almost trivial, observation,

$$\mathbb{P}[\mathbf{z} = \mathbf{z}]^\beta \leq \mathbb{P}[\mathbf{z} = \mathbf{z}], \quad \forall \beta \geq 1. \tag{32}$$

in (28), and (17) in (29). Since (25) is a sum of N terms as in (26), we obtain

$$\mathbb{E}\left[\|\boldsymbol{\delta}(\mathbf{y})\|_2^2\right] = o\left(\frac{1}{N^{\alpha-1}}\right). \tag{33}$$

Moving to the second term in (23), we have,

$$\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]^2 \|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^2\right] \leq \tag{34}$$

$$\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^2\right] \leq \tag{35}$$

$$\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]^2\right]^{\frac{1}{2}} \mathbb{E}\left[\|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^4\right]^{\frac{1}{2}} \leq \tag{36}$$

$$\underbrace{\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]\right]^{\frac{1}{2}}}_{o\left(\frac{1}{N^{2\alpha}}\right)} \underbrace{\mathbb{E}\left[\|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^4\right]^{\frac{1}{2}}}_{\mathcal{O}(N)} = o\left(\frac{1}{N^{\alpha-1}}\right), \tag{37}$$

where we have used (32) in (34) and (36), the Cauchy-Schwarz inequality in (35), and (16) in (37). As for the magnitude of the last term in (23), we similarly obtain,

$$\left|\Re\left\{\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right]\right\}\right| \leq \tag{38}$$

$$\left|\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right]\right| \leq \tag{39}$$

$$\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right)^2\right]^{\frac{1}{2}} \leq \tag{40}$$

$$\underbrace{\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]\right]^{\frac{1}{2}}}_{o\left(\frac{1}{N^{2\alpha}}\right)} \underbrace{\mathbb{E}\left[\left(\boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right)^2\right]^{\frac{1}{2}}}_{\mathcal{O}(N)} = \tag{41}$$

$$o\left(\frac{1}{N^{\alpha-1}}\right), \tag{42}$$

where we have used, again, the Cauchy-Schwarz inequality in (39), (32) in (40), and (16) in (41). We note in passing that the term on the right in (41) may be bound more tightly, but this is not necessary for the following steps of this proof.

We have established upper bounds on the magnitudes of the terms in (23). Hence, using (33), (37) and (42), we now have

$$\mathbb{E}\left[\|\boldsymbol{\Delta}(\mathbf{y})\|_2^2\right] = o\left(\frac{1}{N^{\alpha-1}}\right), \tag{43}$$

which, together with (22), yields

$$\varepsilon^2_{\text{MMSE}}(N) = \varepsilon^2_{\text{MAP-QLMMSE}}(N) + o\left(\frac{1}{N^{\alpha-1}}\right). \tag{44}$$

By the definition of the MMSE estimator, the (trivial) upper bound

$$\varepsilon^2_{\text{MMSE}}(N) \leq \varepsilon^2_{\text{MAP-QLMMSE}}(N) \implies \frac{\varepsilon^2_{\text{MMSE}}(N)}{\varepsilon^2_{\text{MAP-QLMMSE}}(N)} \leq 1 \tag{45}$$

holds for any $N \in \mathbb{N}_+$. Therefore, and since (44) hold for any $\alpha \in \mathbb{R}_+$, we can always choose some α to have

$$\frac{\varepsilon^2_{\text{MMSE}}(N)}{\varepsilon^2_{\text{MAP-QLMMSE}}(N)} = 1 - o\left(\frac{1}{N^{\alpha}}\right), \tag{46}$$

where we used $\varepsilon^2_{\text{MAP-QLMMSE}}(N) = \mathcal{O}(N)$, proving the theorem. ∎

Appendix B
Proof of Lemma 1

In the proof of Lemma 1, we shall use the following lemma.

Lemma 2: For $\psi_N(\mathbf{y}, k)$, in Definition 1 (TDC), we have,

$$\mathbb{E}\left[e^{\tau\psi_N(\mathbf{y},\mathsf{k}_b)}\right] = \left(1 - \frac{2\tau}{N}\right)^{-\frac{N}{2}} \cdot e^{-\tau}, \quad \forall\tau < \frac{N}{2}. \tag{47}$$

Proof of Lemma 2: First, recall $\mathbf{y}|\mathsf{k}_b \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{yy}(\mathsf{k}_b))$, where $\mathbf{C}_{yy}(\mathsf{k}_b) = \mathbf{C}_{ss}(0) + \mathbf{C}_{vv}(\mathsf{k}_b)$. Using the Cholesky decomposition, we write $\mathbf{C}_{yy}(\mathsf{k}_b) \triangleq \mathbf{\Gamma}_y(\mathsf{k}_b)\mathbf{\Gamma}_y^{\mathsf{H}}(\mathsf{k}_b)$, where $\mathbf{\Gamma}_y(\mathsf{k}_b) \in \mathbb{C}^{N\times N}$. Then, conditioned on k_b, we have

$$\psi_N(\mathbf{y}, \mathsf{k}_b) + 1 = \frac{1}{N}\mathbf{y}^{\mathsf{H}}\mathbf{C}_{yy}^{-1}(\mathsf{k}_b)\mathbf{y} \tag{48}$$

$$= \frac{1}{N}\mathbf{y}^{\mathsf{H}}\mathbf{\Gamma}_y^{-\mathsf{H}}(\mathsf{k}_b)\mathbf{\Gamma}_y^{-1}(\mathsf{k}_b)\mathbf{y} \tag{49}$$

$$= \frac{1}{N}\Big(\underbrace{\mathbf{\Gamma}_y^{-1}(\mathsf{k}_b)\mathbf{y}}_{\triangleq\mathbf{u}(\mathsf{k}_b)}\Big)^{\mathsf{H}} \underbrace{\mathbf{\Gamma}_y^{-1}(\mathsf{k}_b)\mathbf{y}}_{=\mathbf{u}(\mathsf{k}_b)} \tag{50}$$

$$= \frac{1}{N}\|\mathbf{u}(\mathsf{k}_b)\|_2^2, \tag{51}$$

where $\mathbf{u}(\mathsf{k}_b)|\mathsf{k}_b \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ is a white Gaussian vector. Thus,

$$\mathbb{E}\left[e^{\tau\psi_N(\mathbf{y},\mathsf{k}_b)}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\tau\psi_N(\mathbf{y},\mathsf{k}_b)}|\mathsf{k}_b\right]\right] \tag{52}$$

$$= \mathbb{E}\left[\mathbb{E}\left[e^{\tau\left(\frac{1}{N}\|\mathbf{u}(\mathsf{k}_b)\|_2^2-1\right)}|\mathsf{k}_b\right]\right] \tag{53}$$

$$= \mathbb{E}\left[\mathbb{E}\left[e^{\frac{\tau}{N}\sum_{n=1}^{N}\mathsf{u}_n^2(\mathsf{k}_b)}|\mathsf{k}_b\right]\right] e^{-\tau} \tag{54}$$

$$= \mathbb{E}\left[\prod_{n=1}^{N}\mathbb{E}\left[e^{\frac{\tau}{N}\mathsf{u}_n^2(\mathsf{k}_b)}|\mathsf{k}_b\right]\right] e^{-\tau} \tag{55}$$

$$\overset{\forall\tau<\frac{N}{2}}{=} \mathbb{E}\left[\prod_{n=1}^{N}\left(1-\frac{2\tau}{N}\right)^{-\frac{1}{2}}\right] e^{-\tau} \tag{56}$$

$$= \left(1-\frac{2\tau}{N}\right)^{-\frac{N}{2}} \cdot e^{-\tau}, \tag{57}$$

where we have used the law of total expectation in (52); the conditional statistical independence of the elements of $\mathbf{u}(\mathsf{k}_b)$ (given k_b) in (55); the fact that $\{\mathsf{u}_n^2(\mathsf{k}_b) \sim \chi_1^2\}_{n=1}^N$, namely all the squared elements of $\mathbf{u}(\mathsf{k}_b)$ given k_b are chi-squared random variables with one degree of freedom; and, accordingly, that the moment generating function of a random variable $\mathsf{q} \sim \chi_1^2$ is $\mathbb{E}[e^{\widetilde{\tau}\mathsf{q}}] = (1-2\widetilde{\tau})^{-\frac{1}{2}}$, for some $\widetilde{\tau} < \frac{1}{2}$, in (56), where in our case $\widetilde{\tau} = \tau/N$, hence the condition on τ in (55) ∎

Equipped with Lemma 2, we now prove Lemma 1.

By definition, the MAP estimator has the lowest error probability. Therefore, to show (15), it is sufficient to show that there exists another estimator of k_b, whose error probability is $o(N^{-\alpha})$ for any finite $\alpha \in \mathbb{R}_+$, independent of N. For this, let us consider the estimator,

$$\widehat{\mathsf{k}}_b \triangleq \underset{m\in\mathcal{S}_{K_b}}{\arg\min}\,|\psi_N(\mathbf{y}, m)|. \tag{58}$$

In words, as $N \to \infty$, the error probability of (58) is governed by how far is $|\psi_N(\mathbf{y}, \mathsf{k}_b)|$ from zero, since from the TDC, $\nexists k \in \mathcal{S}_{K_b}/\mathsf{k}_b : \lim_{N\to\infty} |\psi_N(\mathbf{y}, k)| = 0$, whereas

$$\lim_{N\to\infty} \psi_N(\mathbf{y}, \mathsf{k}_b) = \mathbb{E}\left[\psi_N(\mathbf{y}, \mathsf{k}_b)\right] \tag{59}$$

$$= \mathbb{E}\left[\mathbb{E}\left[\psi_N(\mathbf{y}, \mathsf{k}_b)|\mathsf{k}_b\right]\right] \tag{60}$$

$$= \frac{1}{N}\,\mathbb{E}\left[\mathbb{E}\left[\|\mathbf{u}(\mathsf{k}_b)\|_2^2|\mathsf{k}_b\right]\right] - 1 = 0, \tag{61}$$

where we have used (51), $\mathbf{u}(\mathsf{k}_b)|\mathsf{k}_b \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$, and (59) follows from the fact that $\mathbb{V}\mathrm{ar}(\psi_N(\mathbf{y}, \mathsf{k}_b)) = 2/N$, which can be shown in a similar fashion to (59)–(61).

Formally, the error probability of this estimator is given by,

$$\mathbb{P}\left[\widehat{\mathsf{k}}_b \neq \mathsf{k}_b\right] = \mathbb{P}\left[|\psi_N(\mathbf{y}, \mathsf{k}_b)| > \min_{m\in\mathcal{S}_{K_b}/\mathsf{k}_b} |\psi_N(\mathbf{y}, m)|\right]. \tag{62}$$

We now show that the probability that $\psi_N(\mathbf{y}, \mathsf{k}_b)$ is bounded away from zero decreases in the desired rate. Clearly, for any $a > 0$, we have

$$\mathbb{P}[|\psi_N(\mathbf{y}, \mathsf{k}_b)| > a] = \mathbb{P}[\psi_N(\mathbf{y}, \mathsf{k}_b) > a] \tag{63}$$

$$+ \mathbb{P}[\psi_N(\mathbf{y}, \mathsf{k}_b) < -a]. \tag{64}$$

Using the Chernoff bound, we have

$$\mathbb{P}[\psi_N(\mathbf{y}, \mathsf{k}_b) > a] \leq \mathbb{E}\left[e^{t\psi_N(\mathbf{y}, \mathsf{k}_b)}\right] e^{-ta} \triangleq B_1(t, a), \tag{65}$$

$$\mathbb{P}[\psi_N(\mathbf{y}, \mathsf{k}_b) < -a] \leq \mathbb{E}\left[e^{-t\psi_N(\mathbf{y}, \mathsf{k}_b)}\right] e^{-ta} \triangleq B_2(t, a). \tag{66}$$

Using Lemma 2, it follows that

$$B_1(t, a) = \left(1 - \frac{2t}{N}\right)^{-\frac{N}{2}} \cdot e^{-t(1+a)}, \quad \forall t < \frac{N}{2}, \tag{67}$$

$$B_2(t, a) = \left(1 + \frac{2t}{N}\right)^{-\frac{N}{2}} \cdot e^{t(1-a)}, \quad \forall t > -\frac{N}{2}. \tag{68}$$

Minimizing $B_1(t, a)$ and $B_2(t, a)$ with respect to t and choosing $a = \log^{-1}(N)$, we obtain

$$\min_{t<\frac{N}{2}} B_1(t, \log^{-1}(N)) = \left(\frac{1 + \log(N)}{\log(N)}\right)^{\frac{N}{2}} e^{-\frac{N}{2\log(N)}} \tag{69}$$

$$\triangleq B_1^*(N), \tag{70}$$

$$\min_{t>-\frac{N}{2}} B_2(t, \log^{-1}(N)) = \left(\frac{\log(N) - 1}{\log(N)}\right)^{\frac{N}{2}} e^{\frac{N}{2\log(N)}} \tag{71}$$

$$\triangleq B_2^*(N). \tag{72}$$

Finally, as for any $\alpha \in \mathbb{R}_+$ and any $\delta > 0$ independent of N,

$$\lim_{N\to\infty} N^{\alpha+\delta} B_1^*(N) = \lim_{N\to\infty} N^{\alpha+\delta} B_2^*(N) = 0, \tag{73}$$

it follows that for any $\alpha \in \mathbb{R}_+$ independent of N,

$$\mathbb{P}\left[|\psi_N(\mathbf{y}, \mathsf{k}_b)| > \frac{1}{\log(N)}\right] = o\left(\frac{1}{N^\alpha}\right) \tag{74}$$

$$\implies \mathbb{P}\left[\widehat{\mathsf{k}}_b \neq \mathsf{k}_b\right] = o\left(\frac{1}{N^\alpha}\right). \tag{75}$$