To prove the theorem, we shall use the following lemma, whose proof is given in Appendix B.

Lemma 1: For any finite $\alpha \in \mathbb{R}_+$, we have

$$\mathbb{P}\left[\widehat{\mathsf{k}}_b^{\text{MAP}} \neq \mathsf{k}_b\right] = \mathcal{O}\left(\frac{1}{N^\alpha}\right). \tag{14}$$

From Lemma 1, we have the following corollary.

Corollary 1: Using (14), for any finite $\alpha \in \mathbb{R}_+$, we have

$$\mathbb{E}\left[\mathbb{P}\left[\widehat{\mathsf{k}}_b^{\text{MAP}} \neq \mathsf{k}_b|\mathbf{y}\right]\right] = \mathcal{O}\left(\frac{1}{N^\alpha}\right), \tag{15}$$

$$\mathbb{E}\left[\mathbb{P}\left[\widehat{\mathsf{k}}_b^{\text{MAP}} = k|\mathbf{y}\right]\right] = \mathcal{O}\left(\frac{1}{N^\alpha}\right), \ \forall k \in \mathcal{S}_{K_b}/\mathsf{k}_b. \tag{16}$$

The roadmap for the proof of the theorem is as follows:

- Step 1: Express the optimality gap between the MMSE and MAP-based QLMMSE estimators as a function of the error probability of the MAP synchronizer $\widehat{\mathsf{k}}_b^{\text{MAP}}$.
- Step 2: Upper and lower bound the MMSE (10) in terms of the MAP-based QLMMSE (11) and the optimality gap.
- Step 3: Show that the upper and lower bounds are either equal to or approach (11) as $N \to \infty$ (sufficiently fast), and apply the squeeze theorem to the ratio in (12).

Proof of Theorem 1: Let us write the the MMSE estimator (6), explicitly, using (7), in terms of the MAP-based QLMMSE estimator (9), as (recall $\mathsf{k}_s = 0$, by assumption),

$$\begin{aligned}\widehat{\mathbf{s}}_{\text{MMSE}} &= \sum_{m_b=1}^{K_b} \mathbb{P}[\mathsf{k}_b = m_b|\mathbf{y}]\, \widehat{\mathbf{s}}_{\text{LMMSE}}(m_b) \\ &= \underbrace{\sum_{\substack{m_b=1\\ m_b\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{P}[\mathsf{k}_b = m_b|\mathbf{y}]\, \widehat{\mathbf{s}}_{\text{LMMSE}}(m_b)}_{\triangleq \boldsymbol{\delta}(\mathbf{y})} \\ &\quad + \mathbb{P}\left[\mathsf{k}_b = \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \widehat{\mathbf{s}}_{\text{LMMSE}}(\widehat{\mathsf{k}}_b^{\text{MAP}}).\end{aligned} \tag{17}$$

Using (17), we define the optimality gap (vector),

$$\boldsymbol{\Delta}(\mathbf{y}) \triangleq \widehat{\mathbf{s}}_{\text{MMSE}} - \widehat{\mathbf{s}}_{\text{LMMSE}}(\widehat{\mathsf{k}}_b^{\text{MAP}}) = \widehat{\mathbf{s}}_{\text{MMSE}} - \widehat{\mathbf{s}}_{\text{MAP-QLMMSE}} \tag{18}$$

$$= \boldsymbol{\delta}(\mathbf{y}) - \mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}. \tag{19}$$

At this point, our strategy for the proof is to "sandwich" the ratio in the limit (12) by upper and lower bounds that are equal or converge to 1. The upper bound is trivially given by the definition of the MMSE estimator, since, for any $N \in \mathbb{N}_+$,

$$\varepsilon^2_{\text{MMSE}}(N) \leq \varepsilon^2_{\text{MAP-QLMMSE}}(N) \implies \frac{\varepsilon^2_{\text{MMSE}}(N)}{\varepsilon^2_{\text{MAP-QLMMSE}}(N)} \leq 1. \tag{20}$$

We proceed to the lower bound. For shorthand, let $\mathbf{e}_{\text{MAP-QLMMSE}} \triangleq \widehat{\mathbf{s}}_{\text{MAP-QLMMSE}} - \mathbf{s}$, and let us first write the MMSE in terms of the estimation error $\mathbf{e}_{\text{MAP-QLMMSE}}$ and the optimality gap $\boldsymbol{\Delta}(\mathbf{y})$, and further lower bound it, as,

$$\mathbb{E}\left[\|\widehat{\mathbf{s}}_{\text{MMSE}} - \mathbf{s}\|_2^2\right] = \tag{21}$$

$$\mathbb{E}\left[\|\mathbf{e}_{\text{MAP-QLMMSE}} + \boldsymbol{\Delta}(\mathbf{y})\|_2^2\right] \geq \tag{22}$$

$$\mathbb{E}\left[\left(\|\mathbf{e}_{\text{MAP-QLMMSE}}\|_2 - \|\boldsymbol{\Delta}(\mathbf{y})\|_2\right)^2\right] = \tag{23}$$

$$\varepsilon^2_{\text{MAP-QLMMSE}}(N) + \mathbb{E}\left[\|\boldsymbol{\Delta}(\mathbf{y})\|_2^2\right] - 2\,\mathbb{E}\left[\|\mathbf{e}_{\text{MAP-QLMMSE}}\|_2\|\boldsymbol{\Delta}(\mathbf{y})\|_2\right], \tag{24}$$

where we have used (18) in (21), $\|\mathbf{x}-\mathbf{y}\| \geq |\|\mathbf{x}\| - \|\mathbf{y}\||$ in (22), and (11) in (23). We now focus on the last two terms in (24). Starting with the term in the middle, we have,

$$\begin{aligned}&\mathbb{E}\left[\|\boldsymbol{\Delta}(\mathbf{y})\|_2^2\right] = \\ &\mathbb{E}\left[\|\boldsymbol{\delta}(\mathbf{y})\|_2^2\right] + \mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]^2 \|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^2\right] \\ &\qquad + 2\Re\left\{\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right]\right\}.\end{aligned} \tag{25}$$

We now show that (the magnitude of) each of the terms in (25) is bounded. It will then follow that the expected squared norm of the optimality gap, $\mathbb{E}\left[\|\boldsymbol{\Delta}(\mathbf{y})\|_2^2\right]$, is also bounded.

Starting with the first term in (25), we have,

$$\mathbb{E}\left[\|\boldsymbol{\delta}(\mathbf{y})\|_2^2\right] = \sum_{n=1}^{N} \mathbb{E}\left[\delta_n^2(\mathbf{y})\right] = \tag{26}$$

$$\sum_{n=1}^{N} \mathbb{E}\left[\left(\sum_{\substack{m_b=1\\ m_b\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{P}[\mathsf{k}_b = m_b|\mathbf{y}]\, \widehat{\mathsf{s}}_{\text{LMMSE},n}(m_b)\right)^2\right]. \tag{27}$$

Focusing on one element of the sum in (27), we have,

$$\mathbb{E}\left[\left(\sum_{\substack{m_b=1\\ m_b\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{P}[\mathsf{k}_b = m_b|\mathbf{y}]\, \widehat{\mathsf{s}}_{\text{LMMSE},n}(m_b)\right)^2\right] \leq \tag{28}$$

$$\sum_{\substack{m_1=1\\ m_1\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \sum_{\substack{m_2=1\\ m_2\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_1|\mathbf{y}]^2\, \mathbb{P}[\mathsf{k}_b = m_2|\mathbf{y}]^2\right]^{\frac{1}{2}} \cdot$$

$$\mathbb{E}\left[\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_1)\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_2)\right]^{\frac{1}{2}} \leq \tag{29}$$

$$\sum_{\substack{m_1=1\\ m_1\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \sum_{\substack{m_2=1\\ m_2\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_1|\mathbf{y}]^4\right]^{\frac{1}{4}} \mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_2|\mathbf{y}]^4\right]^{\frac{1}{4}} \cdot$$

$$\mathbb{E}\left[\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_1)\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_2)\right] \leq \tag{30}$$

$$\sum_{\substack{m_1=1\\ m_1\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \sum_{\substack{m_2=1\\ m_2\neq\widehat{\mathsf{k}}_b^{\text{MAP}}}}^{K_b} \underbrace{\mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_1|\mathbf{y}]\right]^{\frac{1}{4}}}_{\mathcal{O}\left(\frac{1}{N^{4\alpha}}\right)} \underbrace{\mathbb{E}\left[\mathbb{P}[\mathsf{k}_b = m_2|\mathbf{y}]\right]^{\frac{1}{4}}}_{\mathcal{O}\left(\frac{1}{N^{4\alpha}}\right)} \cdot \tag{31}$$

$$\underbrace{\mathbb{E}\left[\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_1)\widehat{\mathsf{s}}^2_{\text{LMMSE},n}(m_2)\right]}_{\mathcal{O}(1)} = \tag{32}$$

$$\mathcal{O}\left(\frac{1}{N^\alpha}\right), \tag{33}$$

where we have used the Cauchy-Schwarz inequality repeatedly in (28) and (29), the following, almost trivial, observation,

$$\mathbb{P}[\mathbf{z}=\mathbf{z}]^{\beta} \leq \mathbb{P}[\mathbf{z}=\mathbf{z}], \quad \forall \beta \geq 1. \tag{34}$$

in (30), and (16) in (31). Since (27) is a sum of N terms as in (28), we obtain

$$\mathbb{E}\left[\|\boldsymbol{\delta}(\mathbf{y})\|_2^2\right]=\mathcal{O}\left(\frac{1}{N^{\alpha-1}}\right). \tag{35}$$

Moving to the second term in (25), we have,

$$\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]^2 \|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^2\right] \leq \tag{36}$$

$$\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^2\right] \leq \tag{37}$$

$$\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]^2\right]^{\frac{1}{2}} \mathbb{E}\left[\|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^4\right]^{\frac{1}{2}} \leq \tag{38}$$

$$\underbrace{\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]\right]^{\frac{1}{2}}}_{\mathcal{O}\left(\frac{1}{N^{2\alpha}}\right)} \underbrace{\mathbb{E}\left[\|\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\|_2^4\right]^{\frac{1}{2}}}_{\mathcal{O}(N)}=\mathcal{O}\left(\frac{1}{N^{\alpha-1}}\right), \tag{39}$$

where we have used (34) in (36) and (38), the Cauchy-Schwarz inequality in (37), and (15) in (39). As for the magnitude of the last term in (25), we similarly obtain,

$$\left|\Re\left\{\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right]\right\}\right| \leq \tag{40}$$

$$\left|\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right] \boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right]\right| \leq \tag{41}$$

$$\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right)^2\right]^{\frac{1}{2}} \leq \tag{42}$$

$$\underbrace{\mathbb{E}\left[\mathbb{P}\left[\mathsf{k}_b \neq \widehat{\mathsf{k}}_b^{\text{MAP}}|\mathbf{y}\right]\right]^{\frac{1}{2}}}_{\mathcal{O}\left(\frac{1}{N^{2\alpha}}\right)} \underbrace{\mathbb{E}\left[\left(\boldsymbol{\delta}^{\mathsf{H}}(\mathbf{y})\widehat{\mathbf{s}}_{\text{MAP-QLMMSE}}\right)^2\right]^{\frac{1}{2}}}_{\mathcal{O}(N)} = \tag{43}$$

$$\mathcal{O}\left(\frac{1}{N^{\alpha-1}}\right), \tag{44}$$

where we have used, again, the Cauchy-Schwarz inequality in (41), (34) in (42), and (15) in (43). We note in passing that the term on the right in (43) may be bound more tightly, but this is not necessary for the following steps of this proof.

As an intermediate summary, we have established upper bounds on the magnitudes of the terms in (25). Hence, using (35), (39) and (44), we now have

$$\mathbb{E}\left[\|\boldsymbol{\Delta}(\mathbf{y})\|_2^2\right]=\mathcal{O}\left(\frac{1}{N^{\alpha-1}}\right). \tag{45}$$

Proceeding to the last term in (24), we have,

$$\mathbb{E}\left[\|\mathbf{e}_{\text{MAP-QLMMSE}}\|_2 \|\Delta(\mathbf{y})\|_2\right] \leq \tag{46}$$

$$\underbrace{\mathbb{E}\left[\|\mathbf{e}_{\text{MAP-QLMMSE}}\|_2\right]^{\frac{1}{2}}}_{\mathcal{O}(N)} \underbrace{\mathbb{E}\left[\|\Delta(\mathbf{y})\|_2\right]^{\frac{1}{2}}}_{\mathcal{O}\left(\frac{1}{N^{\alpha-1}}\right)}=\mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}-1}}\right) \tag{47}$$

$$\Longrightarrow \mathbb{E}\left[\|\mathbf{e}_{\text{MAP-QLMMSE}}\|_2 \|\Delta(\mathbf{y})\|_2\right]=\mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}-1}}\right). \tag{48}$$

where we have used the Cauchy-Schwarz inequality in (46), and (35) in (47). Now, with (45) and (48) established above, revisiting the lower bound (24), we have,

$$\varepsilon^2_{\text{MMSE}}(N) \geq \varepsilon^2_{\text{MAP-QLMMSE}}(N)+\mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}-1}}\right) \tag{49}$$

$$\Longrightarrow \frac{\varepsilon^2_{\text{MMSE}}(N)}{\varepsilon^2_{\text{MAP-QLMMSE}}(N)} \geq 1+\mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}}}\right). \tag{50}$$

By combining the upper (20) and lower (50) bounds, choosing, for example (since (14) is for any positive α), $\alpha=2$, and taking the limit $N \to \infty$, (12) is established. ∎

Appendix B
Proof of Lemma 1

In the proof of Lemma 1, we shall use the following lemma.

Lemma 2: For $\psi_N(\mathbf{y},k)$, in Definition 1 (TDC), we have,

$$\mathbb{E}\left[e^{\tau\psi_N(\mathbf{y},\mathsf{k}_b)}\right]=\left(1-\frac{2\tau}{N}\right)^{-\frac{N}{2}} \cdot e^{-\tau}, \quad \forall \tau<\frac{N}{2}. \tag{51}$$

Proof of Lemma 2: First, recall $\mathbf{y}|\mathsf{k}_b \sim \mathcal{CN}(\mathbf{0},\mathbf{C}_{yy}(\mathsf{k}_b))$, where $\mathbf{C}_{yy}(\mathsf{k}_b)=\mathbf{C}_{ss}(0)+\mathbf{C}_{vv}(\mathsf{k}_b)$. Using the Cholesky decomposition, we write $\mathbf{C}_{yy}(\mathsf{k}_b) \triangleq \mathbf{\Gamma}_y(\mathsf{k}_b)\mathbf{\Gamma}_y^{\mathsf{H}}(\mathsf{k}_b)$, where $\mathbf{\Gamma}_y(\mathsf{k}_b) \in \mathbb{C}^{N\times N}$. Then, conditioned on k_b, we have

$$\psi_N(\mathbf{y},\mathsf{k}_b)+1=\frac{1}{N}\mathbf{y}^{\mathsf{H}}\mathbf{C}_{yy}^{-1}(\mathsf{k}_b)\mathbf{y} \tag{52}$$

$$=\frac{1}{N}\mathbf{y}^{\mathsf{H}}\mathbf{\Gamma}_y^{-\mathsf{H}}(\mathsf{k}_b)\mathbf{\Gamma}_y^{-1}(\mathsf{k}_b)\mathbf{y} \tag{53}$$

$$=\frac{1}{N}\Big(\underbrace{\mathbf{\Gamma}_y^{-1}(\mathsf{k}_b)\mathbf{y}}_{\triangleq\mathbf{u}(\mathsf{k}_b)}\Big)^{\mathsf{H}}\underbrace{\mathbf{\Gamma}_y^{-1}(\mathsf{k}_b)\mathbf{y}}_{=\mathbf{u}(\mathsf{k}_b)} \tag{54}$$

$$=\frac{1}{N}\|\mathbf{u}(\mathsf{k}_b)\|_2^2, \tag{55}$$

where $\mathbf{u}(\mathsf{k}_b)|\mathsf{k}_b \sim \mathcal{CN}(\mathbf{0},\mathbf{I})$ is a white Gaussian vector. Thus,

$$\mathbb{E}\left[e^{\tau\psi_N(\mathbf{y},\mathsf{k}_b)}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\tau\psi_N(\mathbf{y},\mathsf{k}_b)}|\mathsf{k}_b\right]\right] \tag{56}$$

$$= \mathbb{E}\left[\mathbb{E}\left[e^{\tau\left(\frac{1}{N}\|\mathbf{u}(\mathsf{k}_b)\|_2^2-1\right)}|\mathsf{k}_b\right]\right] \tag{57}$$

$$= \mathbb{E}\left[\mathbb{E}\left[e^{\frac{\tau}{N}\sum_{n=1}^{N}\mathsf{u}_n^2(\mathsf{k}_b)}|\mathsf{k}_b\right]\right]e^{-\tau} \tag{58}$$

$$= \mathbb{E}\left[\prod_{n=1}^{N}\mathbb{E}\left[e^{\frac{\tau}{N}\mathsf{u}_n^2(\mathsf{k}_b)}|\mathsf{k}_b\right]\right]e^{-\tau} \tag{59}$$

$$\overset{\forall\tau\leq\frac{N}{2}}{=} \mathbb{E}\left[\prod_{n=1}^{N}\left(1-\frac{2\tau}{N}\right)^{-\frac{1}{2}}\right]e^{-\tau} \tag{60}$$

$$= \left(1-\frac{2\tau}{N}\right)^{-\frac{N}{2}} \cdot e^{-\tau}, \tag{61}$$

where we have used the law of total expectation in (56); the conditional statistical independence of the elements of $\mathbf{u}(\mathsf{k}_b)$ (given k_b) in (59); the fact that $\{\mathsf{u}_n^2(\mathsf{k}_b) \sim \chi_1^2\}_{n=1}^N$, namely all the squared elements of $\mathbf{u}(\mathsf{k}_b)$ given k_b are chi-squared random variables with one degree of freedom; and, accordingly, that the moment generating function of a random variable $\mathsf{q} \sim \chi_1^2$ is $\mathbb{E}[e^{\widetilde{\tau}\mathsf{q}}]=(1-2\widetilde{\tau})^{-\frac{1}{2}}$, for some $\widetilde{\tau}<\frac{1}{2}$, in (60), where in our case $\widetilde{\tau}=\tau/N$, hence the condition on τ in (59) ∎

Equipped with Lemma 2, we now prove Lemma 1.

By definition, the MAP estimator has the lowest error probability. Therefore, to show (14), it is sufficient to show that there exists another estimator of k_b, whose error probability is $\mathcal{O}(N^{-\alpha})$ for any finite $\alpha \in \mathbb{R}_+$. For this, let us consider the estimator,

$$\widehat{\mathsf{k}}_b \triangleq \arg\min_{m \in \mathcal{S}_{K_b}} |\psi_N(\mathbf{y}, m)| \,. \tag{62}$$

In words, as $N \to \infty$, the error probability of (62) is governed by how far is $|\psi_N(\mathbf{y}, \mathsf{k}_b)|$ from zero, since from the TDC, $\nexists k \in \mathcal{S}_{K_b}/\mathsf{k}_b : \lim_{N\to\infty} |\psi_N(\mathbf{y}, k)| = 0$, whereas

$$\lim_{N\to\infty} \psi_N(\mathbf{y}, \mathsf{k}_b) = \mathbb{E}\left[\psi_N(\mathbf{y}, \mathsf{k}_b)\right] \tag{63}$$

$$= \mathbb{E}\left[\mathbb{E}\left[\psi_N(\mathbf{y}, \mathsf{k}_b) | \mathsf{k}_b\right]\right] \tag{64}$$

$$= \frac{1}{N}\,\mathbb{E}\left[\mathbb{E}\left[\|\mathbf{u}(\mathsf{k}_b)\|_2^2 | \mathsf{k}_b\right]\right] - 1 = 0, \tag{65}$$

where we have used (55), $\mathbf{u}(\mathsf{k}_b)|\mathsf{k}_b \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$, and (63) follows from the fact that $\mathbb{V}\mathrm{ar}(\psi_N(\mathbf{y}, \mathsf{k}_b)) = 2/N$, which can be shown in a similar fashion to (63)–(65).

Formally, the error probability of this estimator is given by,

$$\mathbb{P}\left[\widehat{\mathsf{k}}_b \neq \mathsf{k}_b\right] = \mathbb{P}\left[|\psi_N(\mathbf{y}, \mathsf{k}_b)| > \min_{m \in \mathcal{S}_{K_b}/\mathsf{k}_b} |\psi_N(\mathbf{y}, m)|\right]. \tag{66}$$

We now show that the probability that $\psi_N(\mathbf{y}, \mathsf{k}_b)$ is bounded away from zero decreases in the desired rate. Clearly, for any $a > 0$, we have

$$\mathbb{P}[|\psi_N(\mathbf{y}, \mathsf{k}_b)| > a] = \mathbb{P}[\psi_N(\mathbf{y}, \mathsf{k}_b) > a] \tag{67}$$

$$+ \mathbb{P}[\psi_N(\mathbf{y}, \mathsf{k}_b) < -a] \,. \tag{68}$$

Using the Chernoff bound, we have

$$\mathbb{P}[\psi_N(\mathbf{y}, \mathsf{k}_b) > a] \leq \mathbb{E}\left[e^{t\psi_N(\mathbf{y}, \mathsf{k}_b)}\right] e^{-ta} \triangleq \bar{B}_1(t, a), \tag{69}$$

$$\mathbb{P}[\psi_N(\mathbf{y}, \mathsf{k}_b) < -a] \leq \mathbb{E}\left[e^{-t\psi_N(\mathbf{y}, \mathsf{k}_b)}\right] e^{-ta} = \bar{B}_2(t, a). \tag{70}$$

Using Lemma 2, it follows that

$$\bar{B}_1(t, a) = \left(1 - \frac{2t}{N}\right)^{-\frac{N}{2}} \cdot e^{-t(1+a)}, \quad \forall t < \frac{N}{2}, \tag{71}$$

$$\bar{B}_2(t, a) = \left(1 + \frac{2t}{N}\right)^{-\frac{N}{2}} \cdot e^{-t(1-a)}, \quad \forall t > -\frac{N}{2}. \tag{72}$$

Minimizing $\bar{B}_1(t, a)$ and $\bar{B}_1(t, a)$ with respect to t and choosing $a = \log^{-1}(N)$, we obtain

$$\min_{t < \frac{N}{2}} \bar{B}_1(t, \log^{-1}(N)) = \left(\frac{1 + \log(N)}{\log(N)}\right)^{\frac{N}{2}} e^{-\frac{N}{2\log(N)}} \tag{73}$$

$$\triangleq \bar{B}_1^*(N), \tag{74}$$

$$\min_{t > -\frac{N}{2}} \bar{B}_2(t, \log^{-1}(N)) = \left(\frac{\log(N) - 1}{\log(N)}\right)^{\frac{N}{2}} e^{\frac{N}{2\log(N)}} \tag{75}$$

$$\triangleq \bar{B}_2^*(N). \tag{76}$$

Finally, since for any $\alpha \in \mathbb{R}_+$,

$$\lim_{N\to\infty} N^{\alpha} \bar{B}_1^*(N) = \lim_{N\to\infty} N^{\alpha} \bar{B}_2^*(N) = 0, \tag{77}$$

it follows that for any $\alpha \in \mathbb{R}_+$,

$$\mathbb{P}\left[|\psi_N(\mathbf{y}, \mathsf{k}_b)| > \frac{1}{\log(N)}\right] = \mathcal{O}\left(\frac{1}{N^{\alpha}}\right) \tag{78}$$

$$\Longrightarrow \mathbb{P}\left[\widehat{\mathsf{k}}_b \neq \mathsf{k}_b\right] = \mathcal{O}\left(\frac{1}{N^{\alpha}}\right). \tag{79}$$

from which (14) follows, and this concludes the proof.