Calcul numérique des solides et structures non linéaires

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Numerical solutions of non linear problems in structural dynamics

Overview

1. Equations of motion
2. Variational formulation
3. Discretisation
4. Solution by implicit methods
5. Solution by explicit methods
6. Convergence of the methods
7. Conclusion

1. Equations of motion

Different possibilities

- Variational principles as for non linear elasticity
  Hamilton's principle
  (also principle of least action, principle of stationnary action, ...)

$$S(\underline{q}) = \int_{t_1}^{t_2} L(\underline{q}(t), \dot{\underline{q}}(t), t)\, dt \qquad \left(L = W_{\mathrm{ext}} + W_{\mathrm{kin}} - W_{\mathrm{int}}\right)$$

$$\frac{\delta S}{\delta \underline{q}(t)} = 0$$

- Principle of virtual power

$$P_{\mathrm{int}} + P_{\mathrm{ext}} = P_{\mathrm{a}}$$

In the following, we will use this principle of virtual power

Can be easier for non conservative forces

conservation laws and partial differential equations

Mass conservation (Lagrangian)

$$\int_{D(t)} \rho(\underline{x},t)\,dV = \int_{D(0)} \rho(\underline{X},t)\,J(\underline{X},t)\,dV_0$$
$$= \int_{D(0)} \rho_0(\underline{X})\,dV_0$$

$$\rho(X,t)\,J(X,t) = \rho_0(X)$$

Current value Reference value

Conservation of momentum (Lagrangian)

$$\underline{f}(t) \quad = \quad \frac{D\underline{p}}{Dt}(t) \quad (Newton's \;\; law)$$

with $\quad \underline{f}(t) \quad = \quad \int_{D(0)} \rho_0(\underline{X})\underline{f}(\underline{X},t)dV_0 + \int_{\partial D(0)} \underline{\underline{P}}(\underline{X},t).Nda_0$

$$= \quad \int_{D(0)} \rho_0(\underline{X})\underline{f}(\underline{X},t)dV_0 + \int_{D(0)} div\,\underline{\underline{P}}(\underline{X},t)dV_0$$

and $\quad \frac{D\underline{p}}{Dt}(t) \quad = \quad \frac{D}{Dt}\int_{D(0)} \rho_0(\underline{X})\underline{v}(\underline{X},t)dV_0$

$$= \quad \int_{D(0)} \rho_0(\underline{X})\frac{\partial \underline{v}}{\partial t}(\underline{X},t)dV_0$$

$$\rho_0(\underline{X})\frac{\partial \underline{v}}{\partial t}(\underline{X},t) = \rho_0(\underline{X})\underline{f}(\underline{X},t) + div\,\underline{\underline{P}}(\underline{X},t)$$

Static or dynamic ?

By making the equation dimensionless

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = \rho_0 \underline{f} + div\, \underline{\underline{P}}$$

$$\frac{\rho_0 L^2}{E\, T^2} \frac{\partial \underline{\tilde{v}}}{\partial \tilde{t}} = \frac{L}{E} \rho_0 \underline{f} + div\, \underline{\underline{\tilde{P}}}$$

The problem can be considered as static when

$$\frac{\rho_0 L^2}{E\, T^2} \ll 1$$

Or also

$$\left(\frac{L}{c\, T}\right)^2 \ll 1$$

So when the wave propagation is very fast over the distance L

2. Variational formulation

Two types of boundary conditions

Displacement conditions $\quad u_i = u_i^d \quad on \quad \partial D_{u_i}$

Cinematically admissible displacement fields

Force conditions $\quad T_i = T_i^d \quad on \quad \partial D_{T_i}$

At each point of the boundary and in each direction only one condition

Principle of virtual power

Vector space of virtual velocities

$$V = \left\{ \delta \underline{v} \;\middle|\; \delta v_i = 0 \;\; on \;\; \partial D_{u_i} \right\}$$

$$P_{\text{int}} + P_{\text{ext}} = P_a$$

And for all rigid motions $\qquad P_{\text{int}} = 0$

Lagrangian description

$$P_a = \int_{D(0)} \rho_0 \frac{\partial \underline{v}}{\partial t} . \delta \underline{v} \, dV_0$$

$$= \int_{D(0)} \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} . \delta \underline{v} \, dV_0$$

Power of acceleration

$$P_{\text{ext}} = \int_{D(t)} \rho \underline{f} . \delta \underline{v} \, dV + \int_{\partial D_T(t)} \underline{T}^d . \delta \underline{v} \, da$$

$$= \int_{D(0)} \rho_0 \underline{f} . \delta \underline{v} \, dV_0 + \int_{\partial D_T(0)} \underline{T}^d . \delta \underline{v} \, dA_0$$

Power of external forces

$$P_{\text{int}} = -\int_{D(t)} \underline{\underline{\sigma}} : \underline{\underline{\epsilon}}(\delta \underline{v}) \, dV$$

$$= -\int_{D(0)} \underline{\underline{P}} : \underline{\underline{\nabla \delta v}} \, dV_0$$

Power of internal forces

Principle of virtual power

$$P_{\text{int}} + P_{\text{ext}} = P_a$$

$$\forall\, \delta \underline{v} \in V$$

$$\int_{D(0)} \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} . \delta \underline{v}\, dV_0 \quad = \quad \int_{D(0)} \rho_0 \underline{f} . \delta \underline{v}\, dV_0 + \int_{\partial D_T(0)} \underline{T}^d . \delta \underline{v}\, dA_0$$

$$- \int_{D(0)} \underline{\underline{P}} : \underline{\underline{\nabla \delta v}}\, dV_0$$

Equivalent to the partial differential equation with the boundary conditions

Find $\underline{u}$ regular enough such that

$$P_{\text{int}} + P_{\text{ext}} \quad = \quad P_a$$

$$\forall\, \delta \underline{v} \in V \qquad - K\left(\underline{u}, \delta \underline{v}\right) + F\left(\delta \underline{v}\right) = M\left(\ddot{\underline{u}}, \delta \underline{v}\right)$$

with

$$K\left(\underline{u}, \delta \underline{v}\right) \quad = \quad \int_{D(0)} \underline{\underline{P}} : \underline{\underline{\nabla \delta \underline{v}}}\, dV_0$$

$$F\left(\delta \underline{v}\right) \quad = \quad \int_{D(0)} \rho_0 \underline{f} . \delta \underline{v}\, dV_0 + \int_{\partial D_T(0)} \underline{T}^d . \delta \underline{v}\, dA_0$$

$$M\left(\ddot{\underline{u}}, \delta \underline{v}\right) \quad = \quad \int_{D(0)} \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} . \delta \underline{v}\, dV_0$$

K bilinear for a linear problem, otherwise only linear relatively to $\delta \underline{v}$

F and M linear in $\delta \underline{v}$

3. Discretisation

Discrete functions

Choose a basis $\{N_i\}_{i=1}^{i=N}$ then

$$u_h(\underline{x},t)=\sum_{j=1}^{j=N} \underline{u}_j(t)N_j(\underline{x}), \qquad \underline{u}_j(t) \text{ unknowns to be found}$$

$$\delta \underline{v}_h(\underline{x})=\sum_{i=1}^{i=N} \delta \underline{v}_i N_i(\underline{x}), \qquad \delta \underline{v}_i \text{ arbitrary parameters, virtual velocity}$$

Insert this into the variational formulation

$$-K(\underline{u}_h,\delta \underline{v}_h)+F(\delta \underline{v}_h)=M(\ddot{\underline{u}}_h,\delta \underline{v}_h) \qquad \forall\, \delta \underline{v}_h \in V_h$$

Form of the discrete equation

$$
\begin{aligned}
-K\left(\underline{u}_h, \delta \underline{v}_h\right) + F\left(\delta \underline{v}_h\right) &= M\left(\underline{\ddot{u}}_h, \delta \underline{v}_h\right) \\
-K\left(\underline{u}_h, \sum_{i=1}^{i=N} \delta \underline{v}_i N_i\left(\underline{x}\right)\right) + F\left(\sum_{i=1}^{i=N} \delta \underline{v}_i N_i\left(\underline{x}\right)\right) &= M\left(\sum_{j=1}^{j=N} \underline{\ddot{u}}_j N_j\left(\underline{x}\right), \sum_{i=1}^{i=N} \delta \underline{v}_i N_i\left(\underline{x}\right)\right) \\
-\sum_{i=1}^{i=N} \delta \underline{v}_i K\left(\underline{u}_h, N_i\left(\underline{x}\right)\right) + \sum_{i=1}^{i=N} \delta \underline{v}_i F\left(N_i\left(\underline{x}\right)\right) &= \sum_{i=1}^{i=N} \sum_{j=1}^{j=N} \delta \underline{v}_i \underline{\ddot{u}}_j M\left(N_j\left(\underline{x}\right), N_i\left(\underline{x}\right)\right) \\
\sum_{i=1}^{i=N} \delta \underline{v}_i \underline{F}_i^{\text{int}} + \sum_{i=1}^{i=N} \delta \underline{v}_i \underline{F}_i^{\text{ext}} &= \sum_{i=1}^{i=N} \sum_{j=1}^{j=N} \delta \underline{v}_i \underline{\ddot{u}}_j M_{ij}
\end{aligned}
$$

$$
\underline{F}^{\text{int}} + \underline{F}^{\text{ext}} = \underline{\underline{M}}\, \underline{\ddot{u}}
$$

$$
\underline{F}^{\text{int}} + \underline{F}^{\text{ext}} = 0 \qquad \textit{for static problems}
$$

Computation of interior forces

$$
\begin{aligned}
-\int_{D(0)} \underline{\underline{P}} : \underline{\underline{\nabla \delta v}} \, dV_0 &= -\int_{D(0)} P_{ik} \frac{\partial \left(\sum_{j=1}^{j=n} \delta v_{kj} N_j \right)}{\partial X_i} dV_0 \\
&= -\sum_{j=1}^{j=n} \int_{D(0)} P_{ik} \frac{\partial \left(\delta v_{kj} N_j \right)}{\partial X_i} dV_0 \\
&= -\sum_{j=1}^{j=n} \int_{D(0)} B_{ji} P_{ik} \delta v_{kj} \, dV_0 \\
&= \sum_{j=1}^{j=n} f_{kj}^{\text{int}} \delta v_{kj}
\end{aligned}
$$

with

$$
f_{kj}^{\text{int}} = -\int_{D(0)} P_{ik} \frac{\partial N_j}{\partial X_i} dV_0 = -\int_{D(0)} B_{ji} P_{ik} \, dV_0
$$

$$
B_{ji} = \frac{\partial N_j}{\partial X_i}
$$

Computation of external forces

$$\int_{D(0)} \rho_0 \underline{f} . \delta \underline{v}\, dV_0 + \int_{\partial D_T(0)} \underline{T}^d . \delta \underline{v}\, dA_0$$

$$= \int_{D(0)} \rho_0 f_k \sum_{j=1}^{j=n} \delta v_{kj} N_j(\underline{X})\, dV_0 + \int_{\partial D_T(0)} T_k^d \sum_{j=1}^{j=n} \delta v_{kj} N_j(\underline{X})\, dA_0$$

$$= \sum_{j=1}^{n} f_{kj}^{ext} \delta v_{kj}$$

with

$$f_{kj}^{ext} = \int_{D(0)} \rho_0 f_k N_j(\underline{X})\, dV_0 + \int_{\partial D_T(0)} T_k^d N_j(\underline{X})\, dA_0$$

Computation of acceleration forces

$$\int_{D(0)} \rho_0 \frac{\partial^2 \underline{u}}{\partial t^2} . \delta \underline{v} \, dV_0 = \int_{D(0)} \rho_0 \sum_{l=1}^{l=n} \ddot{u}_{kl} N_l(\underline{X}) \sum_{j=1}^{j=n} \delta v_{kj} N_j(\underline{X}) \, dV_0$$

$$= \sum_{j=1}^{n} f_{kj}^{a} \delta v_{kj}$$

with

$$f_{kj}^{a} = \int_{D(0)} \rho_0 \sum_{l=1}^{l=n} \ddot{u}_{kl} N_l(\underline{X}) N_j(\underline{X}) \, dV_0$$

$$= \sum_{l=1}^{l=n} \int_{D(0)} \rho_0 N_l(\underline{X}) N_j(\underline{X}) \, dV_0 \, \ddot{u}_{kl} = \sum_{l=1}^{l=n} M_{jl} \ddot{u}_{kl}$$

Damping

- Can be included into interior forces
- Many possibilities
- For linear viscous damping the force is

$$\underline{F}^{damp} = \underline{\underline{C}}\,\underline{\dot{u}}$$

Several possibilities for the damping matrix $\underline{\underline{C}}$

One of them is the Rayleigh damping

$$\underline{\underline{C}} = a\,\underline{\underline{M}} + b\,\underline{\underline{K}}$$

a and b are constants computed from the values of the damping at the two extremities of the frequency band of interest

Final equations

$$\underline{F}^{int} + \underline{F}^{ext} = \underline{\underline{M}}\,\ddot{\underline{u}}$$

$$\underline{u}(0) = \underline{u}_0$$

$$\dot{\underline{u}}(0) = \dot{\underline{u}}_0$$

The damping force has been added to the interior force

Remark

- The mass matrix is constant

$$
\begin{aligned}
M_{jl} &= \int_{D(t)} \rho\, N_l(\underline{x}) N_j(\underline{x})\, dV \\
&= \int_{D(0)} \rho_0 N_l(\underline{X}) N_j(\underline{X})\, dV_0
\end{aligned}
$$

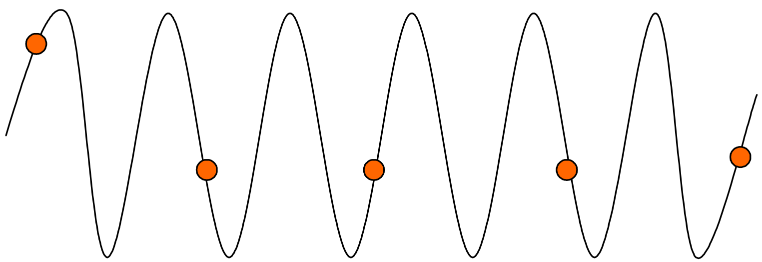
- Calculation of derivatives of shape functions

$$
\frac{\partial N_j}{\partial X_i} = \frac{\partial N_j}{\partial \xi_k} \cdot \frac{\partial \xi_k}{\partial X_i} = \frac{\partial N_j}{\partial \xi_k} \cdot G_{ki}^{-1}
$$

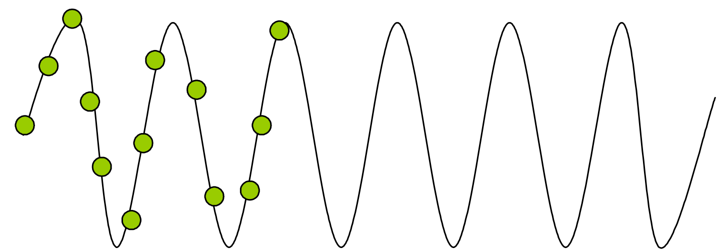
$$
with \quad G_{ki} = \frac{\partial X_k}{\partial \xi_i}
$$

Mesh refinement

One needs between 5 and 10 nodes by wavelength



Not dense enough



Correct

4. Solution by implicit methods

Integrate equations like

$$\underline{F}^{int} + \underline{F}^{ext} = \underline{\underline{M}}\,\underline{\ddot{u}}$$

$$\underline{u}(0) = \underline{u}_0$$

$$\underline{\dot{u}}(0) = \underline{\dot{u}}_0$$

Discretise in time $\underline{u}_n = \underline{u}(n\Delta t)$

How to go from $\{\underline{u}_n, \underline{\dot{u}}_n, \underline{\ddot{u}}_n\}$ to $\{\underline{u}_{n+1}, \underline{\dot{u}}_{n+1}, \underline{\ddot{u}}_{n+1}\}$?

The system to solve can be written as

$$0=\underline{r}\left(\underline{u}^{n+1},t^{n+1}\right)=s\,\underline{\underline{M}}\,\underline{a}^{n+1}-\underline{f}^{int}\left(\underline{u}^{n+1},t^{n+1}\right)-\underline{f}^{ext}\left(\underline{u}^{n+1},t^{n+1}\right)$$

$$with$$

$$s=\begin{cases} 0 \;\; for\ a\ statical\ problem \\ 1 \;\; for\ a\ dynamical\ problem\ (and\ the\ damping\ is\ included \\ in\ the\ interior\ force) \end{cases}$$

$\underline{r}\left(\underline{u}^{n+1},t^{n+1}\right)$ is the residue which must equal zero for the solution

Integration scheme

- There are many possibilities for the integration scheme of these equations
- A popular method is the Newmark scheme

$$\underline{u}^{n+1} = \underline{\tilde{u}}^{n+1} + \beta \Delta t^2 \underline{a}^{n+1}$$

$$\underline{v}^{n+1} = \underline{\tilde{v}}^{n+1} + \gamma \Delta t \underline{a}^{n+1}$$

$$\Delta t = t^{n+1} - t^n$$

$$\underline{\tilde{u}}^{n+1} = \underline{u}^n + \Delta t \underline{v}^n + \Delta t^2 \left(\frac{1}{2} - \beta \right) \underline{a}^n$$

$$\underline{\tilde{v}}^{n+1} = \underline{v}^n + (1 - \gamma) \Delta t \underline{a}^n$$

$$0 \leq \beta \leq 1/2$$

$$0 \leq \gamma \leq 1$$

$$\textit{if} \quad \beta = 0\,, \gamma = \frac{1}{2} \quad \textit{central difference}$$

$$\textit{if} \quad \beta = \frac{1}{4}\,, \gamma = \frac{1}{2} \quad \textit{non damped trapezoidal rule}$$

$$\textit{if} \quad \gamma > \frac{1}{2} \quad \textit{numerical damping proportionnal to} \quad \gamma - \frac{1}{2}$$

$$\textit{If} \quad \beta \geq \frac{\gamma}{2} \,\text{and}\, \gamma \geq \frac{1}{2}\, \textit{unconditionnaly stable}$$

One can solve the new accelerations by

$$\underline{u}^{n+1} = \underline{\tilde{u}}^{n+1} + \beta \Delta t^2 \underline{a}^{n+1}$$

$$\underline{a}^{n+1} = \frac{1}{\beta \Delta t^2} \left(\underline{u}^{n+1} - \underline{\tilde{u}}^{n+1} \right)$$

What leads to

$$\begin{aligned} 0 &= \underline{r}\left(\underline{u}^{n+1}, t^{n+1}\right) \\ &= \frac{s}{\beta \Delta t^2} \underline{\underline{M}} \left(\underline{u}^{n+1} - \underline{\tilde{u}}^{n+1} \right) - \underline{f}^{int}\left(\underline{u}^{n+1}, t^{n+1}\right) - \underline{f}^{ext}\left(\underline{u}^{n+1}, t^{n+1}\right) \end{aligned}$$

Non linear system of equations relative to $\underline{u}^{n+1}$

Solving by the Newton's method

(also Newton-Raphson)

Iterative method on the displacement between $t^n \; and \; t^{n+1}$

very close to the non linear elastic case

And starting at $\underline{u}_0 = \underline{u}^n$

Development of the residue relatively to the displacement around its current value

$$\underline{r}(u_v) + \frac{\partial \underline{r}}{\partial \underline{u}}(u_v)\Delta u + O\left(|\Delta u|^2\right) = 0$$

$$\underline{\underline{A}}(u_v) = \frac{\partial \underline{r}}{\partial \underline{u}}(u_v)$$

Jacobian matrix

(tangent stiffness)

Ignoring the second order terms

$$\begin{aligned}
\underline{r}\left(\underline{u}_v\right)+\underline{\underline{\underline{A}}}\left(\underline{u}_v\right)\Delta\,\underline{u} &= 0\\
\underline{\underline{\underline{A}}}\left(\underline{u}_v\right)\Delta\,\underline{u} &= -\underline{r}\left(\underline{u}_v\right)\\
\Delta\,\underline{u} &= -\underline{\underline{\underline{A}}}^{-1}\left(\underline{u}_v\right)\underline{r}\left(\underline{u}_v\right)\\
\underline{u}_{v+1} &= \underline{u}_v-\underline{\underline{\underline{A}}}^{-1}\left(\underline{u}_v\right)\underline{r}\left(\underline{u}_v\right)
\end{aligned}$$

One goes on until convergence

Computation of the jacobian matrix

$$\underline{\underline{A}}\left(\underline{u}_v\right) = \frac{\partial \underline{r}}{\partial \underline{u}}\left(\underline{u}_v\right)$$

$$= \frac{s}{\beta\left(\Delta t\right)^2}\underline{\underline{M}} - \frac{\partial \underline{f}^{int}}{\partial \underline{u}} - \frac{\partial \underline{f}^{ext}}{\partial \underline{u}}$$

$$\underline{\underline{K}}^{int}\left(\underline{u}_v\right) = \frac{\partial \underline{f}^{int}}{\partial \underline{u}} \quad \text{tangent stiffness matrix}$$

$$\underline{\underline{K}}^{ext}\left(\underline{u}_v\right) = \frac{\partial \underline{f}^{ext}}{\partial \underline{u}} \quad \text{loading stiffness matrix}$$

$$\underline{\underline{A}}\left(\underline{u}_v\right) = \frac{s}{\beta\left(\Delta t\right)^2}\underline{\underline{M}} - \underline{\underline{K}}^{int}\left(\underline{u}_v\right) - \underline{\underline{K}}^{ext}\left(\underline{u}_v\right)$$

Use implicit methods for slow dynamics

- Vibration
- Seismic
- Large deformation of elasto-plastic structures
- Response involving a small number of modes

5. Solution by explicit methods

Central difference method

Simulation time $0 \leq t \leq t_E$

Divided in time steps $\Delta t, \quad t^n = n \Delta t$

The displacement at time step *n* is denoted $\underline{u}^n = \underline{u}\left(t^n\right)$

For the velocity

$$\dot{\underline{u}}^{n+1/2} = \underline{v}^{n+1/2} = \frac{\underline{u}^{n+1} - \underline{u}^n}{\Delta t}$$

What is equivalent to

$$\underline{u}^{n+1} = \underline{u}^n + \Delta t \, \underline{v}^{n+1/2}$$

For the acceleration

$$\underline{\ddot{u}}^n = \underline{a}^n = \frac{\underline{v}^{n+1/2} - \underline{v}^{n-1/2}}{\Delta t}$$

$$\underline{v}^{n+1/2} = \underline{v}^{n-1/2} + \Delta t\, \underline{a}^n$$

$$\underline{\ddot{u}}^n = \underline{a}^n = \frac{\underline{u}^{n+1} - 2\,\underline{u}^n + \underline{u}^{n-1}}{\Delta t^2}$$

Formula of the central difference for the second derivative

Integration of the equation of motion

$$\underline{\underline{M}}\,\underline{a}^n = \underline{f}^{ext}\left(\underline{u}^n, t^n\right) + \underline{f}^{int}\left(\underline{u}^n, t^n\right) = \underline{f}^n$$

With the displacement boundary conditions

$$\underline{g}_I\left(\underline{u}^n\right) = 0, \quad I = 1..n_c$$

Updating the velocity

$$\underline{v}^{n+1/2}=\underline{v}^{n-1/2}+\Delta t\,\underline{\underline{M}}^{-1}\underline{f}^{n}$$

At each time step the velocity is known, then

$$\underline{u}^{n+1}=\underline{u}^{n}+\Delta t\,\underline{v}^{n+1/2}$$

The nodal forces $\underline{f}^{n}$ can be computed from the constitutive relation and the external forces and obtained from $\underline{u}^{n}$

$$\underline{\underline{M}}\,\underline{a}^n = \underline{f}^{ext}\left(\underline{u}^n, t^n\right) + \underline{f}^{int}\left(\underline{u}^n, t^n\right) = \underline{f}^n$$

- If the mass matrix is diagonal one can compute the acceleration without solving any equation
- Idem for the updating of the velocity and the displacement
- The price to pay is that for the method to be stable one has to satisfy

$$\Delta t \leq \Delta t_{crit}$$

Otherwise the solution increases without limit

One has to take $\Delta t \leq \alpha \, \Delta t_{crit}$ with for instance

$$\Delta t_{crit} = \frac{2}{\omega_{\max}} \leq \min_{e,I} \frac{2}{\omega_I^e} = \min_e \frac{l_e}{c_e}$$

With :

- $\omega_{\max}$ maximal pulsation of the linearized system
- l_e characteristic length of element e
- c_e current wave velocity in element e

$0.8 \leq \alpha \leq 0.98$ To take into account the destabilizing effect of non linearities

summary

$$\underline{f}^{n} = \underline{f}^{ext}\left(\underline{u}^{n}, t^{n}\right) + \underline{f}^{int}\left(\underline{u}^{n}, t^{n}\right)$$

$$\underline{v}^{n+1/2} = \underline{v}^{n-1/2} + \Delta t\, \underline{\underline{M}}^{-1} \underline{f}^{n}$$

$$\underline{u}^{n+1} = \underline{u}^{n} + \Delta t\, \underline{v}^{n+1/2}$$

Use explicit methods for fast dynamics

- Shock
- Wave propagation
- Response involving medium and high frequencies

Simulation
of explosion

Vibration of building with contact

6. Convergence of the methods

Conservation of energy

$$\underline{\underline{M}}\,\underline{\ddot{u}} + \underline{\underline{K}}\,\underline{u} = 0$$

Case of a linear problem

$$^{t}\underline{\dot{u}}\;\underline{\underline{M}}\,\underline{\ddot{u}} + {}^{t}\underline{\dot{u}}\;\underline{\underline{K}}\,\underline{u} = 0$$

This can be written as

$$\frac{d}{dt}\left[\mathcal{K}(t) + \mathcal{W}(t)\right] = 0$$

Is the conservation satisfied for the discrete system ?

with

$$\mathcal{K}(t) = \frac{1}{2}\,{}^{t}\underline{\dot{u}}\,\underline{\underline{M}}\,\underline{\dot{u}}$$

$$\mathcal{W}(t) = \frac{1}{2}\,{}^{t}\underline{u}\,\underline{\underline{K}}\,\underline{u}$$

From the relations

$$\mathcal{K}(t) = \frac{1}{2}\,{}^{t}\underline{\dot{u}}\,\underline{\underline{M}}\,\underline{\dot{u}}$$

$$\mathcal{W}(t) = \frac{1}{2}\,{}^{t}\underline{u}\,\underline{\underline{K}}\,\underline{u}$$

At the discrete level one gets

$$\mathcal{K}(t_{n+1}) - \mathcal{K}(t_n) = 2\underline{\dot{u}}_n^s\,\underline{\underline{M}}\,\underline{\dot{u}}_n^d$$

$$\mathcal{W}(t_{n+1}) - \mathcal{W}(t_n) = 2\underline{u}_n^s\,\underline{\underline{K}}\,\underline{u}_n^d$$

with

$$\underline{u}_n^s = \frac{1}{2}\left(\underline{u}_{n+1} + \underline{u}_n\right)$$

$$\underline{u}_n^d = \frac{1}{2}\left(\underline{u}_{n+1} - \underline{u}_n\right)$$

Relations for the Newmark's scheme

$$0 = -2\underline{u}_n^d + \Delta t\,\dot{\underline{u}}_n^s - \Delta t\,\dot{\underline{u}}_n^d + \frac{1}{2}\Delta t^2\left[\ddot{\underline{u}}_n^s + (4\beta - 1)\ddot{\underline{u}}_n^d\right]$$

$$0 = -2\dot{\underline{u}}_n^d + \Delta t\left[\ddot{\underline{u}}_n^s + (2\gamma - 1)\ddot{\underline{u}}_n^d\right]$$

By defining the energy like

$$\mathcal{E}(t) = \mathcal{K}(t) + \mathcal{W}(t) + \frac{1}{2}\Delta t^2\left(\beta - \frac{\gamma}{2}\right)^t\ddot{\underline{u}}\,\underline{\underline{M}}\,\ddot{\underline{u}}$$

And after some calculations (exercise), one gets

$$\mathcal{E}(t_{n+1}) - \mathcal{E}(t_n) = 2(1 - 2\gamma)\,^t\underline{u}_n^d\,\underline{\underline{K}}\,\underline{u}_n^d + \Delta t^2(\gamma - 2\beta)(2\gamma - 1)\,^t\ddot{\underline{u}}_n^d\,\underline{\underline{M}}\,\ddot{\underline{u}}_n^d$$

Stability of the discrete scheme

$$\mathcal{E}(t_{n+1}) - \mathcal{E}(t_n) = 2(1-2\gamma)\,{}^t\underline{u}_n^d \underline{\underline{K}}\,\underline{u}_n^d + \Delta t^2(\gamma-2\beta)(2\gamma-1)\,{}^t\underline{\ddot{u}}_n^d \underline{\underline{M}}\,\underline{\ddot{u}}_n^d$$

- If $\gamma = 1/2$, the energy $\varepsilon(t)$ is constant
- If $\gamma \geq \frac{1}{2}$ and $2\beta - \gamma = 0$, the total energy $\mathcal{K}(t) + \mathcal{W}(t)$ is decreasing
- If $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ the total energy is conserved
- If $\gamma \geq \frac{1}{2}$ and $2\beta > \gamma$ the energy $\varepsilon(t)$ is decreasing
- If $\gamma < \frac{1}{2}$ the energy is increasing if $\gamma \leq 2\beta$

  and the scheme can diverge

7. Conclusion

Essentially two possibilities:

- Implicit methods:
    - ✔ can be inconditionaly stable
    - ✔ large time steps
    - ✔ complex in each iteration
    - ✔ for slow dynamics

- Explicit methods:
    - ✔ conditionaly stable
    - ✔ small time steps
    - ✔ simple in each iteration
    - ✔ for fast dynamics

- Several algorithms in each family

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