Math 334 Homework 4

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Problem (1)**.** Prove $f : \mathbb{R} \longrightarrow \mathbb{R}$, $\exists c \in \mathbb{R}$, $\forall x \in \mathbb{R}$, $|f'(x)| \leq c \implies f$ uniformly continuous.

Proof. Let $\epsilon = c\delta$. For $|x - y| < \delta$, we must also have $y \to x$. So, the limit definition of the derivative can be expressed as $\left|\frac{f(y)-f(x)}{y-x}\right| = |f'(x)| \leq c$. Then,

$$\begin{aligned} c|x-y| &< c\delta = \epsilon \\ \left|\frac{f(y)-f(x)}{y-x}\right| |x-y| &< \epsilon \\ |f(x)-f(y)| &< \epsilon. \end{aligned}$$

Thus, $|x - y| < \delta \implies |f(y) - f(x)| < \epsilon$, so f is uniformly continuous. □

Problem (2)**.** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuous at (0,0),

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

i) Show $\forall v \in \mathbb{S}$, $\exists \lim\limits_{t\to 0} \frac{f(tv)-f(0)}{t}$.

ii) Prove f not differentiable at (0,0).

Proof of i. Let $v = (x, y)$. Since $f(0) = 0$ and $x^2 + y^2 = |v|^2 = 1$,

$$\begin{aligned} \lim_{t\to 0} \frac{f(tv)}{t} &= \lim_{t\to 0} \frac{t^3 x^2 y}{t|v|^2} \\ &= \lim_{t\to 0} t^2 x^2 y \\ &= 0. \end{aligned}$$

□

Proof of ii. □

Problem (3)**.** Let $f : U \longrightarrow \mathbb{R}$, $U \subset \mathbb{R}^n$ open. Show $\forall i$, $\exists \frac{\partial f}{\partial x_i} : U \longrightarrow \mathbb{R}$ bounded $\implies f$ continuous on U.

Proof. Since $U \subset \mathbb{R}^n$ and $\left|\frac{\partial f}{\partial x_i}\right| \leq c$, then we can apply problem (1) to each partial of f. Since f is continuous in all component directions, it is also everywhere continuous. □

Problem (4)**.** Use the linear approximation of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ to approximate the distance between $p = (0.99, -0.97, 2.02)$ and $q = (4.02, 0.98, 8.01)$.

Proof. The distance between p and q is given by $f(q - p)$. So, we will approximate f near $q - p$ for some nice integer-valued vector $r = (3, 2, 4) \approx q - p$.

Then, $f(r) = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{9 + 4 + 16} = \sqrt{29}$. Note that our approximation of the distance between p and q should be close to this value.

Then, $\nabla f(x) = \frac{1}{f(x)}(x, y, z)$, so $\nabla f(r) = \frac{1}{\sqrt{29}(3,2,4)}$.

Our linear approximation L at r is given by,

$$L_r(x) = f(r) + \nabla f(r) \cdot (x - r).$$

So, at the point $x = q - p = (3.03, 1.95, 5.99)$,

$$\begin{aligned}
L_r(x) &= \sqrt{29} + \frac{1}{\sqrt{29}}(3, 2, 4) \cdot (0.03, -0.05, -0.01) \\
&= \sqrt{29} + \frac{1}{\sqrt{29}}(0.09 - 0.1 - 0.04) \\
&= \sqrt{29} - \frac{0.05}{\sqrt{29}} \\
&= \frac{20\sqrt{29} - \frac{1}{\sqrt{29}}}{20} \\
&= \frac{29(20)\sqrt{29} - \sqrt{29}}{29(20)} \\
&= \frac{29(20) - 1}{29(20)}\sqrt{29} \\
&= \frac{579}{580}\sqrt{29}.
\end{aligned}$$

So, the approximate distance between p and q is $\frac{579}{580}\sqrt{29}$, which is indeed close to $\sqrt{29}$. □

Problem (5)**.** Let $S_1 = \{(x, y, z) \mid y + z^3 = 2\}$ and $S_2 = \{(x, y, z) \mid x^2 + xy + y^4 = 21\}$. Let $C = S_1 \cap S_2$, C smooth.

(a) Sketch S_1, S_2, and C on the same diagram.

(b) Find a parametric equation for the tangent line to C at $p = (4, 1, 1)$.

For (a), we note that S_1 is a cylinder in x, and S_2 is a cylinder in z. So, we can construct a level set diagram for S_1...

Proposition (b)**.** The tangent line to C at p is given by $\ell(t) = (4 - 8t, 1 + 9t, 1 - 3t)$.

Proof. First, we will consider the change in z of C at p. This is given solely by S_1. So, with the we will use the $\mathbb{R}^2$ function for cylinder surface of S_1, $y = 2 - z^3$, we will consider $\frac{dy}{dz} = -3z^2$ at the z-coordinate of p, $z = 1$, $y'(1) = -3$. So the change in z is -3.

Second, we will consider the changes in x and y. We will form the function $f(x, y) = x^2 + xy + x^4 - 21$, which parametrizes the surface S_2. Then, $\nabla f(x, y) = (2x + y, x + 4y^3)$. At the x and y coordinates of p, $(x, y) = (4, 1)$, $\nabla f(4, 1) = (9, 8)$. But, recalling that $\nabla f \perp f$, we know that the tangent line will change in x and y by -8 and 9 respectively. So, the tangent line will have the direction vector of $(-8, 9, -3)$ at the point $(4, 1, 1)$. Thus, the tangent line is parametrized by $(4, 1, 1) + (-8, 9, -3)t$. $\square$