Math 134 Homework 4

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4.1

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A number c is called a fixed point of f if $f(c) = c$. Prove the proposition.

Proposition. *If f is differentiable on an interval I and $f'(x) < 1$ for all $x \in I$, then f has at most one fixed point in I.*

Proof. For a contradiction, assume that f has two fixed points on I.

Define $g(x) = f(x) - x$ so that when x is a fixed point of f, $g = 0$.

Since f is differentiable, on I, it is also continuous on I. So, g as the sum of a two continuous functions is also continuous.

Similarly, since f and $-x$ are both differentiable, g is differentiable.

So,

$$g'(x) = f'(x) - 1.$$

But, $f'(x) < 1$ was given. So,

$$g'(x) < 0.$$

Then, by the assumption, if f has two fixed points on I, then g has two zeros on I.

Let $a, b \in I$ be defined such that $g(a) = g(b) = 0$.

Then, there exists c in I between a and b such that $g'(c) = 0$ by Rolle's Theorem.

But, $g'(x) < 0$ and $g'(c) = 0$ is a contradiction.

So the proposition is true. □

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Show that the proposition (2) holds.

Proposition. *If $f(x) = x^m + ax + b$, and $m = 2n+1 : n \in \mathbb{Z}^+$, then f has at most three distinct real roots.*

Proposition (1)**.** *If g is a differentiable function and g' has no more than k roots, then g has no more than $k+1$ roots.*

Proof of (1). For a contradiction, assume that g has $k+2$ roots.

Then, by Rolle's Theorem, g' has $k+1$ roots, contradiction the assumption that g' had k roots. □

Proof. First, we will differentiate f twice.

$$\begin{aligned} f(x) &= x^m + ax + b \\ f'(x) &= mx^{m-1} + a \\ f''(x) &= m(m-1)x^{m-2} \end{aligned}$$

If $m = 1$, then f is a linear function $(1+a)x + b$, which only has one root, so the proposition holds. So, we continue with all positive odd integers such that $m > 1$.

We see that f'' can only have one root at $x = 0$.

Therefore, by proposition (1), f' has no more than two roots.

Similarly, since f' has no more than two roots, f has no more than three roots, again by proposition (1).

Therefore f has at most three distinct roots.

□

4.2

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Show that the proposition holds.

Proposition. *If $1 < n \in \mathbb{Z}^+$ and $x > 0$, then $(1+x)^n > 1 + nx$.*

Proof by induction. We will first prove the proposition by induction on n.

For the base case, $n = 2$.

$$\begin{aligned} (1+x)^2 &> 1 + 2x \\ x^2 + 2x + 1 &> 1 + 2x \\ x^2 &> 0 \end{aligned}$$

So, the base case holds because $x > 0$ was given.

Then, we assume that the proposition holds for $n = k$ by the inductive hypothesis.

For the inductive step, we will show that $n = k + 1$ also holds.

$$\begin{aligned}(1+x)(1+x)^k &> (1+kx)(1+x)\\ (1+x)^{k+1} &> 1 + x + kx + x^2, \quad x^2 > 0\\ (1+x)^{k+1} &> 1 + (1+k)x + x^2 > 1 + (k+1)x\\ (1+x)^{k+1} &> 1 + (k+1)x\end{aligned}$$

Which is the proposition when $n = k + 1$. So, by induction, the proposition is true. □

Proof. We will define two functions f and g such that,

$$f(x) = (1+x)^n, \quad g(x) = 1 + nx.$$

Assume that, for all $x > 0$, $f' > g'$.

$$f' - g' > 0, \quad x > 0.$$

We differentiate $f(x)$ and $g(x)$,

$$f'(x) = n(1+x)^{n-1}, \quad g'(x) = n.$$

So, by the assumption and the fact that $n \geq 2$ was given,

$$\begin{aligned}n(1+x)^{n-1} &> n\\ (1+x)^{n-1} &> 1\end{aligned}$$

Which holds since $x > 0$ by the assumption and $n - 1 > 1$ for all n valid for the proposition.

Therefore the assumption holds.

If $x = 0$, then $f(0) = (1+0)^n = 1^n = 1$ and $g(0) = 1 + (0)x = 1$.

Since, $f(0) = g(0) = 1$, then $f(0) - g(0) = 0$.

For a contradiction, assume that, for all $x > 0$, $f < g$. So, $f - g < 0$.

Then, by the assumption, and that, at $x = 0$, $f(0) - g(0) = 0$, there exists an $a > 0$ such that $f'(a) - g'(a) < 0$ and $f'(a) < g'(a)$ by the Mean Value Theorem.

But, this contradicts the assumption that, for all $x > 0$, $f'(x) > g'(x)$.

Therefore, by contradiction, $f(x) > g(x)$ when $x > 0$.

So, by the definitions of f and g, the proposition holds.

$$f(x) = (1+x)^n > 1 + nx = g(x).$$

□

4.3

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Let $y = f(x)$ be differentiable and suppose that the graph of f does not pass through the origin. The distance D from the origin to a point $P(x, f(x))$ of the graph is given by

$$D = \sqrt{x^2 + [f(x)]^2}.$$

Show that if D has a local extreme value at c, then the line through $(0,0)$ and $(c, f(c))$ is perpendicular to the line tangent to the graph of f at $(c, f(c))$.

Proof. Since $(0,0)$ does not lie on f, then D, being the distance from f to $(0,0)$, is never zero.

Since D has a local extreme at c, then $D'(c) = 0$ or $D'(c)$ does not exist.

First, we find D',

$$\begin{aligned}
\frac{d}{dx}D &= \frac{d}{dx}\left[\sqrt{x^2 + [f(x)]^2}\right] \\
&= \frac{1}{2\sqrt{x^2 + [f(x)]^2}} \cdot \frac{d}{dx}\left[x^2 + [f(x)]^2\right] \\
&= \frac{2x + 2f(x)f'(x)}{2\sqrt{x^2 + [f(x)]^2}} \\
&= \frac{x + ff'}{\sqrt{x^2 + f^2}} \\
&= \frac{x + ff'}{D}
\end{aligned}$$

Then, given that $D'(c) = 0$ (we assume D to be differentiable at c and so its critical point exists), and that $D \neq 0$,

$$\begin{aligned}
D'(c) = \frac{c + f(c)f'(c)}{D(c)} &= 0 \\
c + f(c)f'(c) &= 0 \\
f'(c) &= \frac{-c}{f(c)}
\end{aligned}$$

Two lines are perpendicular if their slopes are negative reciprocals of each other, such that,

$$m_1 = \frac{-1}{m_2}.$$

The line through $(0,0)$ and $(c, f(c))$ has a slope of $\frac{f(c)}{c}$.

The line tangent to f at $(c, f(c))$ has a slope of $f'(c)$.

Since $f'(c) = \frac{-c}{f(c)}$, which is the slope of the second line, then its negative reciprocal is $\frac{-1}{f'(c)} = \frac{f(c)}{c}$, which is the slope of the first line.

So, the lines are perpendicular and the statement holds.

□

4.4

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Show that the proposition holds.

Proposition. *If f is continuous on $[a, b]$ and $f(a) = f(b)$, then f has at least one critical point in (a, b).*

Proof. We will prove the proposition by two cases, either f is differentiable on (a, b), or f is not differentiable on the interval.

Recall that a critical number is defined as some c on a differentiable f such that $f'(c) = 0$ or $f'(c)$ does not exist.

If f is differentiable on (a, b), then, since $f(a) = f(b)$, by Rolle's Theorem, there exists a $c \in (a, b)$ such that $f'(c) = 0$. In this case, c is a critical point.

If f is not differentiable on (a, b), then there is at least one $\gamma \in (a, b)$ such that $f'(\gamma)$ does not exist. Therefore γ is a critical point is this case as well. □

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Give an example of a non-constant function that takes on both its absolute maximum and absolute minimum on every interval.

Use the Dirichlet function, $f(x) = \begin{cases} \alpha, & x \in \mathbb{Q} \\ \beta, & x \notin \mathbb{Q} \end{cases}$ where $x \in \mathbb{R}$ and $\alpha < \beta$ so that α is the absolute minimum and β is the absolute maximum.

For any interval $I \subseteq \mathbb{R}$, there will be some $a, b \in I$ such that $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$. Therefore, f will attain the value of both α and β on any interval, which are the absolute minimum and maximum respectively.

4.6

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Show that if a cubic polynomial $p(x) = x^3 + ax^2 + bx + c$ has a local maximum and a local minimum, then the midpoint of the line segment that connects the local high point to the local low point is a point of inflection.

Proof. First, we find the local minimum and maximum of p via setting $p' = 0$.

$$\begin{aligned}
p' = 3x^2 + 2ax + b &= 0 \\
x^2 + \frac{2a}{3}x + \frac{b}{3} &= 0 \\
\left(x + \frac{a}{3}\right)^2 - \frac{a^2}{9} + \frac{b}{3} &= 0 \\
\left(x + \frac{a}{3}\right)^2 &= \frac{a^2}{9} - \frac{b}{3} \\
x + \frac{a}{3} &= \pm\sqrt{\frac{a^2 - 3b}{9}} \\
x &= -\frac{a}{3} \pm \frac{\sqrt{a^2 - 3b}}{3}
\end{aligned}$$

Then, it is clear that the midpoint m of the local extrema is $-\frac{a}{3}$.

We verify this value by solving for x when $f''(x) = 0$ to determine the infection point.

$$\begin{aligned}
p''(x) = 6x + 2a &= 0 \\
6x &= -2a \\
x &= \frac{-a}{3}
\end{aligned}$$

We see that the infection point and the midpoint of the line segment between the extrema are equal.

Next, we recognize that p is symmetrical about the midpoint m. So for an $h \geq 0$, we assume that,

$$p(m+h) + p(m-h) = 2p(m)$$

Since, $m = \frac{-a}{3}$, then $3m + a = 0$. So, $2h^2(3m + a) = 0$.

Then, by expanding,

$$\begin{aligned}
2p(m) &= p(m+h) + p(m-h) \\
&= (m+h)^3 + a(m+h)^2 + b(m+h) + c \\
&+ (m-h)^3 + a(m-h)^2 + b(m-h) + c \\
&= (m^3 + 3m^2h + 3mh^2 + h^3) + (am^2 + 2amh + ah^2) + (bm + bh) + c \\
&+ (m^3 - 3m^2h + 3mh^2 - h^3) + (am^2 - 2amh + ah^2) + (bm - bh) + c \\
&= (m^3 + 3mh^2) + (am^2 + ah^2) + (bm) + c \\
&+ (m^3 + 3mh^2) + (am^2 + ah^2) + (bm) + c \\
&= 2\left[(m^3 + am^2 + bm + c) + (3mh^2 + ah^2)\right] \\
&= 2p(m) + 2h^2(3m + a) \\
&= 2p(m)
\end{aligned}$$

We see that the assumption holds.

So, with $h = \frac{\sqrt{a^2 - 3b}}{3}$, we see that the the two extrema can be represented by $m - h$ and $m + h$.

Then, the middle value of the line segment between the extrema, given by the mean, is equal to the function p at the midpoint m by the assumption.

$$\frac{p(m+h)+p(m-h)}{2} = p(m)$$

Finally, we compute $p\left(-\frac{a}{3}\right)$ to find the value of the function p at the midpoint.

$$\begin{aligned} p\left(\frac{-a}{3}\right) &= \left(\frac{-a}{3}\right)^3 + a\left(\frac{-a}{3}\right)^2 + b\left(\frac{-a}{3}\right) + c \\ &= \frac{-a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c \\ &= \frac{2a^3}{27} - \frac{ab}{3} + c \end{aligned}$$

Thus, the midpoint and point of inflection is,

$$\left(\frac{-a}{3}, \frac{2a^3}{27} - \frac{ab}{3} + c\right)$$

$\square$

4.7

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Sketch the graph of the function showing all vertical and oblique asymptotes.

$$f(x) = \frac{1+x-3x^2}{x}$$

For convenience, we rewrite f as $f(x) = \frac{1}{x} + \frac{x}{x} - \frac{3x^2}{x} = \frac{1}{x} + 1 - 3x$

We will see that there is a vertical asymptote at $x = 0$ by taking the limit as x approaches zero from both directions.

$$\lim_{x\to 0}\left[\frac{1}{x} + 1 - 3x\right]$$

As x approaches zero, the magnitude of f increases without bound by the term $\lim\limits_{x\to 0} \frac{1}{x}$ whose limit does not exist.

We recognize that, $\lim\limits_{x\to 0^-} \frac{1}{x}$ with result in increasingly large negative values while $\lim\limits_{x\to 0^+}$ will produce increasingly large positive values.

For the end behavior, we take the limit as x gets very large in both the positive and negative directions.

$$\lim_{x\to -\infty}\left[\frac{1}{x} + 1 - 3x\right] = 0 + 1 - \lim_{x\to -\infty} 3x$$

We see that the term $\lim\limits_{x\to -\infty} 3x$ will become very large in the negative direction. Since the sign of this term is negative, f will get very large as x gets very large in the negative direction.

$$\lim_{x\to \infty}\left[\frac{1}{x} + 1 - 3x\right] = 0 + 1 - \lim_{x\to \infty} 3x$$

Similarly, we see that the term $\lim_{x \to -\infty} 3x$ will grow very large. However, since this term in negated in the sum in f, then f will get very large in the negative direction as x becomes very large in the positive direction.

We notice that these two end behaviors fall on the oblique asymptote $y = -3x$.

For the roots, we find all x such that $f(x) = 0$. Note that $f(x)$ is not defined at zero, so we can begin with $x \neq 0$.

$$\begin{aligned}
\frac{1 + x - 3x^2}{x} &= 0 \\
1 + x - 3x^2 &= 0 \\
-3x^2 + x &= -1 \\
x^2 - \frac{x}{3} &= \frac{1}{3} \\
\left(x - \frac{1}{6}\right)^2 - \frac{1}{36} &= \frac{1}{3} \\
\left(x - \frac{1}{6}\right)^2 &= \frac{13}{36} \\
x - \frac{1}{6} &= \pm\frac{\sqrt{13}}{6} \\
x &= \frac{1 \pm \sqrt{13}}{6}
\end{aligned}$$

So, the two roots are $x_1 = \frac{1-\sqrt{13}}{6}$ and $x_2 = \frac{1+\sqrt{13}}{6}$.

We differentiate f with respect to x to see that f is decreasing for all $x \in \mathbb{R}$.

$$\frac{d}{dx}\left[\frac{1}{x} + 1 - 3x\right] = \frac{-1}{x^2} - 3$$

Since $x^2 > 0$ for all $x \in \mathbb{R}$, then $f'(x)$ will always be negative.

Once more, we differentiate f' to check the concavity of f.

$$f''(x) = \frac{d}{dx}\left[\frac{-1}{x^2} - 3\right] = \frac{2}{x^3}$$

For $x < 0$, $f'' < 0$. So f is concave down to the left of its asymptote at $x = 0$.

For $x > 0$, $f'' > 0$. So f is concave up on the right of its vertical asymptote.

We combine these features in a sketch of the graph.

4.9

48

To estimate the height of a bridge, a man drops a stone into the water below. How high is the bridge (a) if the stone hits the water 3 seconds later? (b) if the man hears the splash 3 seconds later? (Use 1080 feet per second as the speed of sound.)

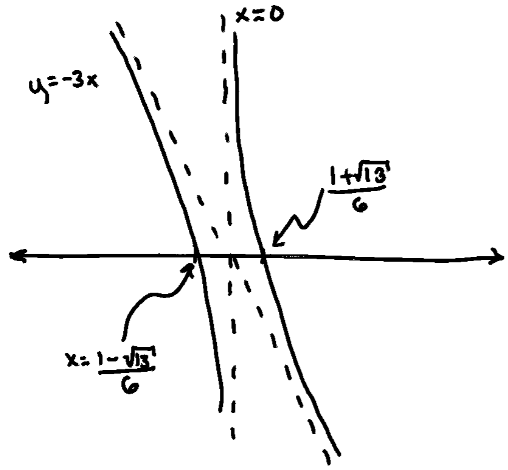


Figure 1: Sketch of f.

We will use the gravitational constant g as 32 feet per second per second.

Recall that the equation for vertical motion in the presence of gravity is $y(t) = \frac{-1}{2}gt^2 + v_0t + y_0$ where v_0 is the initial upward velocity and y_0 is the initial vertical position.

Given that the stone was dropped, it has no initial velocity.

For (a), to compute the initial height y_0, we use the fact that it took three seconds for the rock to reach the water, which we will consider as the height zero. So, with $y(3) = 0$,

$$\begin{aligned} -\frac{32}{2}(3)^2 + y_0 &= 0 \\ -16(9) &= -y_0 \\ 144 &= y_0 \end{aligned}$$

So, in (a), the height of the bridge is 144 feet.

For (b), we know that the the splash occurs some time three seconds have elapsed, and then some t seconds occur before the sound reaches the top of the bridge. As such, we can represent the height of the bridge as $1080t$ feet, and the duration of time for the rock to fall and splash as $3 - t$.

So, we equate the height of the bridge and the distance fallen by the rock until the splash,

$$\begin{aligned}
1080t &= \frac{32}{2}(3-t)^2 \\
\frac{135}{2}t &= 9 - 6t + t^2 \\
-9 &= t^2 - \frac{147}{2}t \\
-9 &= \left(t - \frac{147}{4}\right)^2 - \left(\frac{147}{4}\right)^2 \\
\frac{147^2 - 144}{16} &= \left(t - \frac{147}{4}\right)^2 \\
\pm\frac{\sqrt{147^2 - 144}}{4} &= t - \frac{147}{4} \\
\frac{147 \pm \sqrt{147^2 - 144}}{4} &= t \\
\frac{147 \pm 9\sqrt{265}}{4} &= t
\end{aligned}$$

Since the height of the bridge was given by $y_0 = 1080t$, then $y_0 = 1080 \cdot \frac{147 \pm 9\sqrt{265}}{4} = 39690 \pm 2430\sqrt{265}$

Given that part (a) estimated that the bridge was 144 feet, (b) should give a similar answer. In this case, we compute $39690 - 2430\sqrt{265}$ to see that the height of the bridge is about 132.466 feet.

4.11

20

View the earth as a sphere of radius 4000 miles. The volume of ice that covers the north and south poles is estimated to be 8 million cubic miles. Suppose that this ice melts and the water produced distributes itself uniformly over the surface of the earth. Estimate the depth of this water.

First, we find the volume of the earth.

$$V_E = \frac{4}{3}\pi(4000)^3 = \frac{256}{3}\pi \cdot 10^9$$

Then, we find the radius of the sphere comprised of the total volume of both the earth and all of its melted ice.

$$\begin{aligned}
\frac{4}{3}\pi r^3 &= \frac{256}{3}\pi \cdot 10^9 + 8 \cdot 10^6 \\
r^3 &= \frac{3}{4\pi}\left(\frac{256}{3}\pi \cdot 10^9 + 8 \cdot 10^6\right) \\
r^3 &= 64 \cdot 10^9 + \frac{6 \cdot 10^6}{\pi}
\end{aligned}$$

$$\begin{aligned}
r &= \sqrt[3]{64 \cdot 10^9 + \frac{6 \cdot 10^6}{\pi}} \\
r &\approx 4000.04
\end{aligned}$$

Finally, we find the height of the ice by taking the radius of the earth from r, the earth and melted ice radius.

$$4000.04 - 4000 = 0.04$$

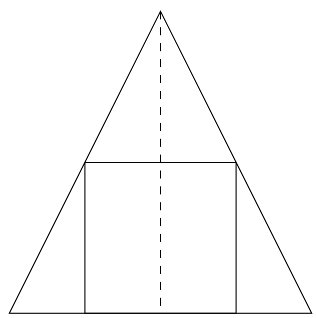
So, the added depth of water covering the earth would be about 0.04 miles or around 211 feet.

Worksheet

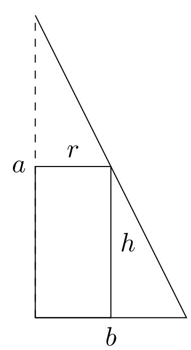
4

Find the maximum volume of a cylinder inscribed in a cone of radius b and height a.

We take a cross section of the cone and cylinder about the center axis.



We will look at just the right half of the cross section and notice a set of similar triangles. We label the radius and height of the inner cylinder cross section as r and h respectively.



We can equate by the ratio of the heights to the bases,

$$\frac{h}{b-r} = \frac{a}{b}$$

Solving for h in terms of r, we see that,

$$h = \frac{a(b-r)}{b}$$

So, we can create a function for the volume of the cylinder with $r > 0$ as,

$$V(r) = \pi r^2 \frac{a(b-r)}{b} = a\pi r^2 - \frac{a\pi r^3}{b}$$

We begin to optimize by finding the derivative of V with respect to its radius r and setting V' to zero.

$$\begin{aligned}
V' &= 2a\pi r - \frac{3a\pi r^2}{b} \\
0 &= 2a\pi r - \frac{3a\pi r^2}{b} \\
0 &= a\pi r\left(2 - \frac{3r}{b}\right) \\
0 &= (r)\left(2 - \frac{3r}{b}\right)
\end{aligned}$$

We note that a cylinder of radius $r = 0$ would have no volume. So, we proceed by assuming that $h, r \neq 0$ (where a height h of zero would have no volume either).

$$\begin{aligned}
0 &= 2 - \frac{3r}{b} \\
3r &= 2b \\
r &= \frac{2}{3}b
\end{aligned}$$

We now check the second derivative of V to determine whether $r = \frac{2}{3}b$ is a local minimum or maximum.

$$\begin{aligned}
V''(r) &= 2a\pi - \frac{6a\pi r}{b} \\
V''\left(\frac{2}{3}b\right) &= 2a\pi - \frac{6a\pi\left(\frac{2}{3}b\right)}{b} \\
&= 2a\pi - 4a\pi \\
&= -2a\pi < 0, \quad a > 0
\end{aligned}$$

Since $V''(r)$ at $r = \frac{2}{3}b$ is less than zero, V is concave down at r, so the given r is a local maximum.

Then, we calculate $V\left(\frac{2}{3}b\right)$ to determine the maximum volume.

$$\begin{aligned}
V\left(\frac{2}{3}b\right) &= a\pi\left(\frac{2}{3}b\right)^2 - \frac{a\pi\left(\frac{2}{3}b\right)^3}{b} \\
&= \frac{4ab^2\pi}{9} - \frac{8ab^2\pi}{27} \\
&= \frac{4ab^2\pi}{27}
\end{aligned}$$

So, the maximum volume of the inset cylinder is $\frac{4ab^2\pi}{27}$.

We will now check the end behavior of V when the radius of the cylinder r is the radius of the cone b and also at $r = 0$.

$$V(b) = ab^2\pi - \frac{ab^3\pi}{b} = 0$$

And,

$$V(0) = a(0)^2\pi - \frac{a(0)^3\pi}{b} = 0$$

Since $\frac{4ab^2\pi}{27} > 0$, then $r = \frac{2}{3}b$ is the global maximum.

The minimum possible volume is zero, which is attained when either the height or radius of the cone is zero.