

Math 135 Homework 5

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Probation Practice

1. Solve the differential equations and check the solution.

(a) $y' = x - 4xy$

We begin by reorganizing the equation to the form $y' + p(x)y = q(x)$ in order to arrive at the solution via the method of integrating factors. S.H.E. 9.1.2 Provides that, with $H(x) = \int p(x) dx$,

$$y(x) = e^{-H(x)} \left[\int e^{H(x)} q(x) dx \right],$$

with a constant of integration after the evaluation of the indefinite integral.

So, $y' + 4xy = x$ gives

$$H(x) = \int 4x dx = 2x^2.$$

Then,

$$\begin{aligned} y(x) &= e^{-2x^2} \left[\int x e^{2x^2} dx \right] \\ &= e^{-2x^2} \left[\frac{1}{4} \int e^u du \right] \\ &= e^{-2x^2} \left(\frac{e^{2x^2}}{4} + c \right) \\ &= ce^{-2x^2} + \frac{1}{4}. \end{aligned}$$

We verify this result by taking the derivative of our solution,

$$y' = \frac{d}{dx} \left[ce^{-2x^2} + \frac{1}{4} \right] = -4cxe^{-2x^2},$$

and then matching it with the equation for y above,

$$\begin{aligned}
 y' &= x - 4xy \\
 -4cxe^{-2x^2} &= x - 4x \left(ce^{-2x^2} + \frac{1}{4} \right) \\
 &= x - 4cxe^{-2x^2} - \frac{4x}{4} \\
 &= -4cxe^{-2x^2}.
 \end{aligned}$$

This statement is true, and therefore we have arrived at a valid general solution.

(b) $y' = \csc x + y \cot x$

Again, we will put the first order linear differential equation into standard form such that we can use the form provided by S.H.E. 9.1.2.

With $y' - y \cot x = \csc x$, we will need $H(x) = \int -\cot x \, dx = -\ln \sin x$.

Then,

$$\begin{aligned}
 y(x) &= e^{\ln \sin x} \left[\int \csc x e^{-\ln \sin x} \, dx \right] \\
 &= \sin x \left[\int -\frac{dx}{\sin x e^{\ln \sin x}} \right] \\
 &= \sin x \left[\int -\csc^2 x \, dx \right] \\
 &= \sin x (-\cot x + c) \\
 &= c \sin x - \cos x.
 \end{aligned}$$

We differentiate our solution,

$$y'(x) = c \cos x + \sin x.$$

We compare this to the given form for $y'(x)$,

$$\begin{aligned}
 y'(x) &= \csc x + y \cot x \\
 c \cos x + \sin x &= \csc x + (c \sin x - \cos x) \cot x \\
 &= \frac{1}{\sin x} + c \cos x - \frac{\cos^2 x}{\sin x} \\
 &= \frac{1 - \cos^2 x}{\sin x} + c \cos x \\
 &= \frac{\sin^2 x}{\sin x} + c \cos x \\
 &= \sin x + c \cos x.
 \end{aligned}$$

We see that the statement is true and so the general solution is valid.

(c) $x^2 y' + 2xy = 8x^3$

We put this equation into standard form and proceed with the methods used in the previous problems,

$$y' + \frac{2}{x}y = 8x.$$

With, $\int p(x) dx = \int \frac{2}{x} dx = 2 \ln x$,

$$\begin{aligned} y(x) &= e^{-2 \ln x} \left[\int 8x e^{2 \ln x} dx \right] \\ &= \frac{1}{x^2} \left[\int 8x^3 dx \right] \\ &= \frac{1}{x^2} (2x^4 + c) \\ &= 2x^2 + \frac{c}{x^2}. \end{aligned}$$

To check this y , we differentiate,

$$y'(x) = 4x - \frac{2c}{x^3}.$$

Then we compare, with the given differential equation where y' is isolated,

$$\begin{aligned} 4x - \frac{2c}{x^3} &= 8x - \frac{2}{x} \left(2x^2 + \frac{c}{x^2} \right) \\ &= 8x - 4x - \frac{2c}{x^3} \\ &= 4x - \frac{2c}{x^3}, \end{aligned}$$

which is true, so the general solution holds as well.

(d) $y' = xe^{y-x^2}, \quad y(0) = 0$

This is a separable differential equation. We also have an initial condition that will pin the single constant at a fixed number.

We rewrite the equation with regards to differentials $\frac{dy}{dx}$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{xe^y}{e^{x^2}} \\ \int \frac{dy}{e^y} &= \int \frac{x dx}{e^{x^2}} \\ -e^{-y} &= -\frac{1}{2} \int e^u du \\ e^{-y} &= \frac{1}{2} e^u + c \\ e^{-y} &= \frac{1}{2} e^{-x^2} + c \\ -y &= \ln \left(\frac{1}{2} e^{-x^2} + c \right) \\ y &= -\ln \left(\frac{1}{2} e^{-x^2} + c \right).\end{aligned}$$

We then consider the initial condition $y(0) = 0$,

$$\begin{aligned}0 &= -\ln \left(\frac{1}{2} e^{-0^2} + c \right) \\ 0 &= -\ln \left(\frac{1}{2} + c \right) \\ 1 &= \frac{1}{2} + c \\ \frac{1}{2} &= c.\end{aligned}$$

So, we have the solution,

$$\begin{aligned}y(x) &= -\ln \left(\frac{1}{2} (1 + e^{-x^2}) \right) \\ &= \ln 2 - \ln \left(1 + e^{-x^2} \right).\end{aligned}$$

We can verify this solution in the same manner as previous problems.

First, we differentiate our solution,

$$\begin{aligned}y'(x) &= \frac{d}{dx} \left[\ln 2 - \ln \left(1 + e^{-x^2} \right) \right] \\ &= -\frac{-2xe^{-x^2}}{1 + e^{-x^2}} \\ &= \frac{2x}{1 + e^{x^2}}.\end{aligned}$$

Then we compare it to the given y' with our derived solution y ,

$$\begin{aligned} y'(x) &= \frac{x e^{\left(\ln 2 - \ln(1+e^{-x^2})\right)}}{e^{x^2}} \\ &= \frac{x e^{\ln 2}}{e^{x^2} e^{\ln(1+e^{-x^2})}} \\ &= \frac{2x}{e^{x^2} (1+e^{-x^2})} \\ &= \frac{2x}{1+e^{x^2}}. \end{aligned}$$

Since both y' match, we confirm that we have arrived at a valid general solution.

(e) $(x + yx) dx = (x^2 y^2 + x^2 + y^2 + 1) dy$

We rearrange the terms to reveal that this equation is separable.

$$\begin{aligned} (x + yx) dx &= (x^2 y^2 + x^2 + y^2 + 1) dy \\ x(y + 1) dx &= (x^2 + 1)(y^2 + 1) dy \\ \int \frac{x dx}{x^2 + 1} &= \int \frac{y^2 + 1}{y + 1} dy \\ \frac{1}{2} \int \frac{du}{u} &= \int \frac{(y + 1)^2 - 2y}{y + 1} dy \\ \frac{\ln u}{2} + c/2 &= \int \left(y + 1 - \frac{2(y + 1) - 2}{y + 1} \right) dy \\ \frac{\ln(x^2 + 1)}{2} + c/2 &= \int \left(y + 1 - 2 + \frac{2}{y + 1} \right) dy \\ &= \int \left(y - 1 + \frac{2}{y + 1} \right) dy \\ &= \frac{y^2}{2} - y + 2 \ln(y + 1) \\ \ln(x^2 + 1) + c &= y^2 - 2y + 4 \ln(y + 1) \end{aligned}$$

2. Find the first four Picard approximations of $y' = e^x + y$ with $y(0) = 0$.

Since $y(x_0) = y_0$, then $(x_0, y_0) = (0, 0)$.

So, $y_0(x) = y_0 = 0$.

The n^{th} Picard approximation is given by

$$y_n(x) = y_0 + \int_{x_0}^x f[x, y_{n-1}(x)] dx,$$

where $y' = f(x, y(x))$.

So, for y_1 ,

$$\begin{aligned} y_1(x) &= 0 + \int_0^x f(x, y_0(x)) \, dx \\ &= \int_0^x (e^x + 0) \, dx \\ &= e^x \Big|_0^x \\ &= e^x - 1. \end{aligned}$$

Next, for y_2 ,

$$\begin{aligned} y_2(x) &= 0 + \int_0^x (e^x + (e^x - 1)) \, dx \\ &= \int_0^x (2e^x - 1) \, dx \\ &= [2e^x - x]_0^x \\ &= 2e^x - x - 2. \end{aligned}$$

For y_3 ,

$$\begin{aligned} y_3(x) &= 0 + \int_0^x (e^x + (2e^x - x - 2)) \, dx \\ &= \int_0^x (3e^x - x - 2) \, dx \\ &= \left[3e^x - \frac{x^2}{2} - 2x \right]_0^x \\ &= 3e^x - \frac{x^2}{2} - 2x - 3. \end{aligned}$$

Lastly, for y_4 ,

$$\begin{aligned} y_4(x) &= 0 + \int_0^x \left(e^x + \left(3e^x - \frac{x^2}{2} - 2x - 3 \right) \right) \, dx \\ &= \int_0^x \left(4e^x - \frac{x^2}{2} - 2x - 3 \right) \, dx \\ &= \left[4e^x - \frac{x^3}{6} - x^2 - 3x \right]_0^x \\ &= 4e^x - \frac{x^3}{6} - x^2 - 3x - 4. \end{aligned}$$

3. First the first three Picard approximations of the system of differential equations,

$$\begin{aligned} x'(t) &= t + y^2 & x(0) &= 0, \\ y'(t) &= x - t & y(0) &= 1. \end{aligned}$$

We first notice that $t_0 = 0$, $x_0 = 0$, and $y_0 = 1$.

For a parametric system of differential equations, the Picard approximations are given by

$$\begin{aligned}x_n &= x_0 + \int_{t_0}^t f_1[t, x_{n-1}(t), y_{n-1}(t)] dt, \\y_n &= y_0 + \int_{t_0}^t f_2[t, x_{n-1}(t), y_{n-1}(t)] dt,\end{aligned}$$

where $x'(t) = f_1(t, x(t), y(t))$ and $y'(t) = f_2(t, x(t), y(t))$.

We start with the first pair of approximations,

$$\begin{aligned}x_1(t) &= 0 + \int_0^t (t + 1^2) dt \\&= \left[t + \frac{t^2}{2} \right]_0^t \\&= t + \frac{t^2}{2} + t,\end{aligned}$$

and

$$\begin{aligned}y_1(t) &= 1 + \int_0^t (0 - t) dt \\&= 1 - \frac{t^2}{2} \Big|_0^t \\&= 1 - \frac{t^2}{2}.\end{aligned}$$

We continue recursively as before,

$$\begin{aligned}x_2(t) &= 0 + \int_0^t \left(t + \left(1 - \frac{t^2}{2} \right)^2 \right) dt \\&= \int_0^t \left(1 + t - t^2 + \frac{t^4}{4} \right) dt \\&= \left[t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20} \right]_0^t \\&= t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20},\end{aligned}$$

and

$$\begin{aligned}y_2(t) &= 10y \int_0^t \left(t + \frac{t^2}{2} - t \right) dt \\&= 1 + \frac{t^3}{6} \Big|_0^t \\&= 1 + \frac{t^3}{6}.\end{aligned}$$

Finally,

$$\begin{aligned} x_3(t) &= 0 + \int_0^t \left(t + \left(1 + \frac{t^3}{6} \right)^2 \right) dt \\ &= \int_0^t \left(1 + t + \frac{t^3}{3} + \frac{t^6}{36} \right) dt \\ &= t + \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^7}{252}, \end{aligned}$$

and

$$\begin{aligned} y_3(t) &= 1 + \int_0^t \left(t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20} - t \right) dt \\ &= 1 + \left[\frac{t^3}{6} - \frac{t^4}{12} + \frac{t^6}{120} \right]_0^t. \end{aligned}$$

Homework

1. The equation $y' + P(x)y = Q(x)y^n$ is called the Bernoulli equation. It becomes a linear equation after the change of variable $y^{1-n} = z$. Solve the equation $y(6y^2 - x - 1) dx + 2x dy = 0$ using this idea.
2. Let $I = [0, 1]$ and $Y_n(t) = t^n$. Show that the sequence (Y_n) is not Cauchy by computing $\|Y_n - Y_m\|$.
3. Let $I = [-\pi, \pi]$, and consider the function $f_0 : I \rightarrow \mathbb{R}$ defined by $f_0(t) = e^t$. Let $f_n, n \in \mathbb{Z}^+$ be the sequence of functions on I defined inductively by the formula

$$f_{n+1}(t) = \cos(t) + \frac{1}{2} \sin(t) f_n(t).$$

So $f_1(t) = \cos(t) + \frac{1}{2} \sin(t)e^t$, $f_2(t) = \cos(t) + \frac{1}{2} \sin(t) (\cos(t) + \frac{1}{2} \sin(t)e^t)$, etc.

Show that f_n converges uniformly to a continuous function and find the limit $\lim_{n \rightarrow \infty} f_n$.