if luakeys == nil then luakeys = require('luakeys')() luakeys.depublish$_{f}unctions(luakeys)end$

penlight = require'penlight'

penlight.stringx.import()penlight.stringx.format$_{o}perator()penlight.utils.import(penlight.func)$

require'penlightplus'

YAMLvars = require('YAMLvars')

YAMLvars.setts2default()

Math 136 Homework 1

Alexandre Lipson

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1. Write down a basis for the space of

   (a) 3×3 symmetric matrices;

   (b) $n \times n$ symmetric matrices;

   (c) $n \times n$ antisymmetric matrices $A^T = -A$ matrices;

   a) We note that a symmetric matrix is given by $A^T = A$. So, each entry a_{ij} must be equal to a_{ji} for $i, j \leq 3$.

   Thus, we can construct a basis of six matrices that are symmetric about the diagonal,

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \right\}.$$

   b) Let M_{ij} be the matrix of all zeros with a 1 in position i, j.

   Then, the basis for the space of $n \times n$ symmetric matrices is given by the set

$$\{M_{ij} + M_{ji}, i \geq j\}.$$

   We restrict the indices to $i \geq j$ such that we will not create any linearly dependent duplicates.

   c) Using the same definition of M_{ij} as above, we consider that the middle diagonal the any antisymmetric matrix must be zero because, for zero only, $0 = -0$. Thus, our basis can be defined as follows,

$$\{M_{ij} - M_{ji}, i > j\}.$$

   This time, we do not include the cases where $i = j$ because our middle row must be zero.[1]

2. Prove that $\text{trace}(AB) = \text{trace}(BA)$.

   First, we will consider how the diagonals of the matrix AB, $(AB)_{ii}$ are created.

   With $A_{m \times n}$ and $B_{n \times m}$, the product AB will be a $m \times m$ square matrix. We will fix some i as the index of m and note that we take the dot product of the i^{th} row of A and the i^{th} column of B. We will iterate over n with the index j.

[1] The trace of an antisymmetric matrix must be zero.

This gives,

$$\sum_{j=1}^{n} a_{ij}b_{ji} = (AB)_{ii}.$$

In order to obtain the trace of AB, we need to sum over the index i from 1 to n.

Thus,

$$\text{trace}(AB) = \sum_{i=1}^{m}(AB)_{ii} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji}.$$

We then notice that BA produces an $n \times n$ matrix. We then see that,

$$\text{trace}(BA) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}a_{ji}.$$

We can use the fact that we can rearrange sums (linearity of addition) and that multiplication is commutative to see that this is the same as

$$\sum_{j=1}^{m}\sum_{i=1}^{n} b_{ij}a_{ji}.$$

We can swap labels where $i = j$ and $j = i$ to make this the same as above.

So, the statement holds.

3. Let $T : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ be the **projection** of the points on the xyz-space to the plane through the origin given by the equation $\alpha x + \beta y + \gamma z = 0$. Find the matrix of this transformation with respect to the standard basis on $\mathbb{R}^3$.

   Let $\vec{n} = \langle \alpha, \beta, \gamma \rangle$ be the normal vector to the plane.

   We find the transformation T of any point $\vec{P} = \langle x, y, z \rangle$ on the plane by the component of $\vec{P}$ that is normal to the plane's normal vector $\vec{n}$.

   In order to find the component of $\vec{P}$ normal to $\vec{n}$, we subtract the projection of $\vec{P}$ on $\vec{n}$ from $\vec{P}$.

$$T(\vec{P}) = \vec{P} - \text{proj}_{\ \vec{n}}\vec{P}.$$

   We will consider the transformation of the $\mathbb{R}^3$ basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

First,

$$\begin{aligned}
T(\hat{i}) &= \hat{i} - \operatorname{proj}_{\vec{n}} \hat{i} \\
&= \langle 1,0,0\rangle - \frac{\vec{n}\cdot\langle 1,0,0\rangle}{\|\vec{n}\|^2}\vec{n} \\
&= \langle 1,0,0\rangle - \frac{\alpha}{\|\vec{n}\|^2}\vec{n} \\
&= \left\langle 1 - \frac{\alpha^2}{\|\vec{n}\|^2}, \frac{-\alpha\beta}{\|\vec{n}\|^2}, \frac{-\alpha\gamma}{\|\vec{n}\|^2} \right\rangle \\
&= \frac{1}{\|\vec{n}\|^2}\left\langle \|\vec{n}\|^2 - \alpha^2, -\alpha\beta, \alpha\gamma \right\rangle.
\end{aligned}$$

Then,

$$\begin{aligned}
T(\hat{j}) &= \langle 0,1,0\rangle - \frac{\beta}{\|\vec{n}\|^2}\vec{n} \\
&= \left\langle \frac{-\alpha\beta}{\|\vec{n}\|^2}, 1 - \frac{\beta^2}{\|\vec{n}\|^2}, \frac{-\beta\gamma}{\|\vec{n}\|^2} \right\rangle \\
&= \frac{1}{\|\vec{n}\|^2}\left\langle -\alpha\beta, \|\vec{n}\|^2 - \beta^2, -\beta\gamma \right\rangle
\end{aligned}$$

Lastly,

$$\begin{aligned}
T(\hat{k}) &= \langle 0,0,1\rangle - \frac{\gamma}{\|\vec{n}\|^2}\vec{n} \\
&= \frac{1}{\|\vec{n}\|^2}\left\langle -\alpha\gamma, -\beta\gamma, \|\vec{n}\|^2 - \gamma^2 \right\rangle
\end{aligned}$$

So,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = -\frac{1}{\|\vec{n}\|^2}\begin{bmatrix} \alpha^2 - \|\vec{n}\|^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 - \|\vec{n}\|^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 - \|\vec{n}\|^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

4. Let $T : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ be the **reflection** of the points on the xyz-space to the plane through the origin given by the equation $\alpha x + \beta y + \gamma z = 0$. Find the matrix of this transformation with respect to the standard basis on $\mathbb{R}^3$.