Math 135 Homework 7

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Practice

1. Compute the Laplace transform of each of the following functions.

   (a) (i) $f_1(t) = te^{4t}\cos(-2t)$ First, take

   $$\mathcal{L}\{t\cos(-2t)\} = \frac{s^2-4}{(s^2+4)^2},$$

   then shift the Laplace function in the frequency domain by the exponential e^{4t} so that we obtain,

   $$\mathcal{L}\left\{te^{4t}\cos(-2t)\right\} = \frac{(s-4)^2-4}{\left((s-4)^2+4\right)^2}.$$

   (ii) $f_2(t) = \cos^2 t$ We will use the identity,

   $$\cos^2 t = \frac{1+\cos 2t}{2}.$$

   So,

   $$\begin{aligned}\mathcal{L}\left\{\cos^2 t\right\} &= \mathcal{L}\left\{\frac{1}{2}(1+\cos 2t)\right\}\\ &= \frac{1}{2}\left[\mathcal{L}\{1\} + \mathcal{L}\{\cos 2t\}\right]\\ &= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right).\end{aligned}$$

   (iii) $f_3(t) = \sqrt{t}e^t$ We will consider $\mathcal{L}\left\{\sqrt{t}\right\}$ shifted by e^t. Given that

   $$\mathcal{L}\left\{\sqrt{t}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{s^3}},$$

   then,

   $$\mathcal{L}\left\{\sqrt{t}e^t\right\} = \frac{1}{2}\sqrt{\frac{\pi}{(s-1)^3}}.$$

(b) $f(t) = \begin{cases} 4, & t < 2, \\ t+2, & 2 \le t \le 5, \\ e^{-t}, & 5 < t. \end{cases}$

We begin by rewriting the piecewise function in terms of the step function H, where $H_c(t) = H(t-c)$.

So,

$$\begin{aligned} f(t) &= 4(1-H_2) + (t+2)(H_2 - H_5) + e^{-t}H_5 \\ &= t + (t+2-4)H_2 + (e^{-t} - t - 2)H_5 \\ &= 4 + (t-2)H_2 - (t-5)H_5 + (e^{-t} - 7)H_5. \end{aligned}$$

The Laplace transform of f is then,

$$\begin{aligned} \mathcal{L}\left\{f\right\} &= \frac{4}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-5s}}{s^2} + e^{-5s}\mathcal{L}\left\{e^{-(t+5)} + 7\right\} \\ &= \frac{4}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-5s}}{s^2} + e^{-5s}\left(\frac{e^5}{s+1} + \frac{7}{s}\right) \\ &= \frac{4 + 7e^{-5s}}{s} + \frac{e^{-2s} - e^{-5s}}{s^2} + \frac{e^{5(1-s)}}{s+1}. \end{aligned}$$

2. Compute the inverse Laplace transform of the following functions.

(a) (i)

$$\begin{aligned} F_1(s) &= \tfrac{1}{s^2+2s+10} \\ &= \tfrac{1}{(s+1)^2+3^2} \qquad \Big\downarrow \mathcal{L}^{-1} \\ &= \tfrac{1}{3}e^{-t}\sin 3t. \end{aligned}$$

(ii)

$$\begin{aligned} F_2(s) &= \tfrac{3s}{s^2+4s+13} \\ &= \tfrac{3(s+2)-6}{(s+2)^2+3^2} \\ &= 3\tfrac{s+2}{(s+2)^2+3^2} - 2\tfrac{3}{(s+2)^2+3^2} \qquad \Big\downarrow \mathcal{L}^{-1} \\ &= e^{-2t}(3\cos 3t - 2\sin 3t). \end{aligned}$$

(iii)

$$\begin{aligned} F_3(s) &= \tfrac{2s+7}{s^2+6s+9} \\ &= \tfrac{2(s+3)+1}{(s+3)^2} \\ &= \tfrac{2}{s+3} + \tfrac{1}{(s+3)^2} \qquad \Big\downarrow \mathcal{L}^{-1} \\ &= e^{-3t}(t+2). \end{aligned}$$

(b) (i)

$$\begin{aligned}
F_1(s) &= \tfrac{s^2-6}{s^3+4s^2+3s} \\
&= \tfrac{s^2-6}{s(s+1)(s+3)} \\
&= \tfrac{-6/3}{s} + \tfrac{-5/2}{s+1} + \tfrac{3/6}{s+3} \\
&= \tfrac{1}{6}\left(\tfrac{3}{s+3} - \tfrac{-15}{s+1} - \tfrac{12}{s}\right) && \Big\downarrow \mathcal{L}^{-1} \\
&= \tfrac{1}{6}\left(3e^{-3t} - 15e^{-t} - 12\right) \\
&= \tfrac{1}{2}e^{-3t} - \tfrac{5}{2}e^{-t} - 2t.
\end{aligned}$$

(ii)

$$\begin{aligned}
F_2(s) &= \tfrac{16}{s(s^2+4)} \\
&= 4\left(\tfrac{1}{s} - \tfrac{s}{s^2+4}\right) && \Big\downarrow \mathcal{L}^{-1} \\
&= 4\left(1 - \cos 2t\right).
\end{aligned}$$

(iii)

$$\begin{aligned}
F_3(s) &= \tfrac{6s-3}{s(s+1)^2} \\
&= 3\left(\tfrac{-1}{s} + \tfrac{(s+1)+3}{(s+1)^2}\right) \\
&= 3\left(\tfrac{3}{(s+1)^2} + \tfrac{1}{s+1} - \tfrac{1}{s}\right) && \Big\downarrow \mathcal{L}^{-1} \\
&= 3(3te^{-t} + e^{-t} - 1) \\
&= 3e^{-t}(3t + 1 - e^{t}).
\end{aligned}$$

(c)

$$\begin{aligned}
F(s) &= \tfrac{(1-e^{-2s})(1-3e^{-2s})}{s^2} \\
&= \tfrac{1-4e^{-2s}+3e^{-4s}}{s^2} \\
&= \tfrac{1}{s^2} - 4\tfrac{e^{-2s}}{s^2} + 3\tfrac{e^{-4s}}{s^2} && \Big\downarrow \mathcal{L}^{-1} \\
&= t + 3(t-4)H_4 - 4(t-2)H_2.
\end{aligned}$$

3. The following problems are from page 311 in the TP ODE textbook.

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$$\begin{aligned}
y' - y &= 0, \quad y(0) = 1 && \Big\downarrow \mathcal{L} \\
sY - y(0) - Y &= 0 \\
(s-1)Y &= 1 \\
Y &= \tfrac{1}{s-1} && \Big\downarrow \mathcal{L}^{-1} \\
y &= e^{t}.
\end{aligned}$$

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$$\begin{aligned}
y' - y &= e^{x}, \quad y(0) = 1 && \Big\downarrow \mathcal{L} \\
(s-1)Y - 1 &= \tfrac{1}{s-1} \\
Y &= \tfrac{1}{(s-1)^2} + \tfrac{1}{s-1} && \Big\downarrow \mathcal{L}^{-1} \\
y &= e^{t}(t+1).
\end{aligned}$$

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$$
\begin{aligned}
y' + y &= e^x, \quad y(0) = 1 && \downarrow \mathcal{L} \\
sY - y(0) + Y &= \tfrac{1}{s-1} \\
(s+1)Y &= \tfrac{1}{s-1} + 1 \\
Y &= \tfrac{1}{(s+1)^2} + \tfrac{1}{s+1} && \downarrow \mathcal{L}^{-1} \\
y &= e^{-t}(t+1).
\end{aligned}
$$

Problems

1. Show that e^{t^2} is not of exponential order.

   Since $t^2 > at$ for all $t > a > 0$,

   Then, $e^{t^2} > e^{at}$ for all $t > a > 0$.

   Since $e^x > 0$ for all x, then $e^x = |e^x|$ for all x.

   So $\left|e^{t^2}\right| > e^{at}$ for all $t > a > 0$.

   Since $a > 0$, then e^{t^2} is not of exponential order as it is the negation of the definition for some f of exponential order that $|f(x)| \leq Ae^{bt}$ for $b > 0$.

2. (a) We are given that

$$\int \operatorname{Re}\left[e^{(a+bi)t}\right]\, dt = \operatorname{Re}\int e^{(a+bi)t}\, dt. \tag{*}$$

   We expand $e^{(a+bi)t}$ according to Euler's formula,

$$e^{(a+bi)t} = e^{at}e^{bit} = e^{at}\left(\cos bt + i\sin bt\right).$$

   So,

$$\operatorname{Re}\left[e^{(a+bi)t}\right] = e^{at}\cos bt.$$

   Then, by (*),

$$
\begin{aligned}
\int e^{at}\cos bt &= \operatorname{Re}\int e^{(a+bi)t}\, dt \\
&= \operatorname{Re}\left[\frac{e^{(a+bi)t}}{a+bi}\right] \\
&= \operatorname{Re}\left[\frac{(a-bi)e^{at}(\cos bt + i\sin bt)}{a^2+b^2}\right] \\
&= \operatorname{Re}\left[\frac{e^{at}\left((a-bi)\cos bt + (ai+b)\sin bt\right)}{a^2+b^2}\right] \\
&= \frac{e^{at}(a\cos bt + b\sin bt)}{a^2+b^2}. \qquad (**)
\end{aligned}
$$

(b) So, for $\mathcal{L}\{\cos bt\}$, with the definition of the Laplace transform,

$$\begin{aligned}
\mathcal{L}\{\cos bt\} &= \int_0^\infty e^{-st}\cos bt\, dt \\
&= \lim_{n\to\infty}\left[\frac{e^{-st}\left(-s\cos bt + b\sin bt\right)}{\left(-s\right)^2+b^2}\Bigg|_0^n\right] \\
&= \lim_{n\to\infty}\frac{e^{-sn}(-s\cos bn + b\sin bn) - s}{s^2+b^2} \\
&= \lim_{n\to\infty}\frac{s - \frac{b\sin bn - s\cos bn}{e^{sn}}}{s^2+b^2} \\
&= \frac{s}{s^2+b^2}.
\end{aligned}$$

(c) For $\mathcal{L}\{\sin bt\}$, we will let $f(t) = \frac{-\cos bt}{b}$. So $f'(t) = \sin bt$. Then we consider the Laplace of a derivative,

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\
\mathcal{L}\{\sin bt\} &= s\mathcal{L}\left\{\frac{-\cos bt}{b}\right\} - \left(\frac{-\cos(b\cdot 0)}{b}\right) \\
&= \frac{-s}{b}\mathcal{L}\{\cos bt\} + \frac{1}{b} \\
&= \frac{1}{b}\left(1 - s\left(\frac{s}{s^2+b^2}\right)\right) \\
&= \frac{1}{b}\left(\frac{b^2}{s^2+b^2}\right) \\
&= \frac{b}{s^2+b^2}.
\end{aligned}$$

(d) For $\mathcal{L}\{t\sin bt\}$, we consider the derivative of the Laplace transform,

$$\frac{d^n}{ds^n}F(s) = \mathcal{L}\{(-t)^n f(t)\}, \quad F(s) = \mathcal{L}\{f(t)\}. \tag{**}$$

Let $f(t) = -\sin bt$. Then $F(s) = \frac{-b}{s^2+b^2}$ from above.

With (**) and $n = 1$,

$$\begin{aligned}
\mathcal{L}\{-tf(t)\} &= \frac{d}{ds}F(s) \\
\mathcal{L}\{t\sin bt\} &= \frac{d}{ds}\left[\frac{-b}{s^2+b^2}\right] \\
&= \frac{2bs}{(s^2+b^2)^2}.
\end{aligned}$$

(e) For $\mathcal{L}\{e^{at}\sin bt\}$, we note that the product of a function f with the exponential function produces a horizontal shift in the Laplace frequency domain,

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a), \quad \mathcal{L}\{f\} = F.$$

Since

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2} = F(s),$$

Then

$$\mathcal{L}\left\{e^{at}\sin bt\right\} = F(s-a) = \frac{b}{(s-a)^2+b^2}.$$

(f) With $\mathcal{L}\{te^{at}\sin bt\}$, we consider both (d) and (e), where

$$F(s) = \mathcal{L}\{t\sin bt\} = \frac{2bs}{(s^2+b^2)^2},$$

such that

$$F(s-a) = \mathcal{L}\left\{te^{at}\sin bt\right\} = \frac{2b(s-a)}{\left((s-a)^2+b^2\right)^2}.$$

(g) For $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{a^2}{(s^2+a^2)^2}\right\}$, we will rewrite the Laplace function argument $F(s)$ into terms that can be inverted easily from our given Laplace table.

First,

$$\begin{aligned}\frac{s^2}{(s^2+a^2)^2} &= \frac{1}{2}\cdot\frac{s^2+a^2+s^2-a^2}{(s^2+a^2)^2}\\ &= \frac{1}{2}\left(\frac{s^2+a^2}{(s^2+a^2)^2}+\frac{s^2-a^2}{(s^2+a^2)^2}\right)\\ &= \frac{1}{2}\left(\frac{1}{a}\cdot\frac{a}{s^2+a^2}+\frac{s^2-a^2}{(s^2+a^2)^2}\right).\end{aligned}$$

Then, we are ready to take the inverse Laplace,

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{1}{a}\cdot\frac{a}{s^2+a^2}-\frac{s^2-a^2}{(s^2+a^2)^2}\right)\right\} = \frac{1}{2}\left(\frac{1}{a}\sin at + t\cos at\right).$$

Next,

$$\begin{aligned}\frac{a^2}{(s^2+a^2)^2} &= \frac{1}{2}\cdot\frac{s^2+a^2-(s^2-a^2)}{(s^2+a^2)^2}\\ &= \frac{1}{2}\left(\frac{s^2+a^2}{(s^2+a^2)^2}-\frac{s^2-a^2}{(s^2+a^2)^2}\right)\\ &= \frac{1}{2}\left(\frac{1}{a}\cdot\frac{a}{s^2+a^2}-\frac{s^2-a^2}{(s^2+a^2)^2}\right).\end{aligned}$$

So,

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{s^2+a^2}{(s^2+a^2)^2}-\frac{s^2-a^2}{(s^2+a^2)^2}\right)\right\} = \frac{1}{2}\left(\frac{1}{a}\sin at - t\cos at\right).$$

3. Solve
$$y''+' = f(t), \quad y(0) = y'(0) = 0,$$
where
$$f(t) = \begin{cases} 4, & 0 \leq t \leq 2, \\ t+2, & 2 < t. \end{cases}$$

First, we rewrite the piecewise function f using the step function H, where
$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0; \end{cases}$$
we will use $H_c = H(t-c)$.

The forcing function becomes
$$\begin{aligned} f(t) &= 4(1 - H_2) + (t+2)H_2 \\ &= 4 + (t+2-4)H_2 \\ &= 4 + (t-2)H_2. \end{aligned}$$

Let $\mathcal{L}\{y\} = Y$. Then, the Laplace transform of
$$y'' + y = 4 + (t-2)H_2$$
becomes
$$\begin{aligned} s^2 Y - sy(0) - y'(0) + Y &= \frac{4}{s} + \frac{e^{-2s}}{s^2} \\ (s^2+1)Y &= \frac{4}{s} + \frac{e^{-2s}}{s^2} \\ Y &= \frac{4}{s(s^2+1)} + \frac{e^{-2s}}{s^2(s^2+1)}. \end{aligned}$$

Let us take a moment to consider the partial fraction expansions of $\frac{1}{s(s^2+1)}$ and $\frac{1}{s^2(s^2+1)}$. We will show that the following hold:

first,
$$\begin{aligned} \frac{1}{s(s^2+1)} &= \frac{1}{s} - \frac{s}{s^2+1} \\ &= \frac{s^2+1}{s(s^2+1)} - \frac{s^2}{s(s^2+1)} \\ &= \frac{1}{s(s^2+1)}. \end{aligned}$$

Next,

$$\begin{aligned}\frac{1}{s^2(s^2+1)} &= \frac{1}{s^2} - \frac{1}{s^2+1} \\ &= \frac{s^2+1}{s^2(s^2+1)} - \frac{s^2}{s^2(s^2+1)} \\ &= \frac{1}{s^2(s^2+1)}.\end{aligned}$$

We continue from above, substituting the partial fractions in place of the product denominators,

$$Y = 4\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right).$$

We find the general solution y by taking the inverse Laplace transform,

$$y = 4(1-\cos t) + H_2\left((t-2) - \sin(t-2)\right).$$

4. Compute the sawtooth function $\mathcal{L}\{h(t)\}$, where

$$h(t) = \begin{cases} t, & 0 \le t < 1, \\ h(t-1), & 1 \le t. \end{cases}$$

We begin by writing h in terms of the step function H,

$$h(t) = t(1-H_1) + h(t-1)H_1.$$

We then take the Laplace transform of this equation with the fact that

$$\mathcal{L}\{f(t-c)H_c\} = e^{-cs}\mathcal{L}\{f(t)\},$$

$$\begin{aligned}\mathcal{L}\{h(t)\} &= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + e^{-s}\mathcal{L}\{h(t)\} \\ (1-e^{-s})\mathcal{L}\{h(t)\} &= \frac{1-e^{-s}}{s^2} - \frac{e^{-s}}{s} \\ \mathcal{L}\{h(t)\} &= \frac{1}{s^2} - \frac{e^{-s}}{s\left(1-e^{-s}\right)} \\ \mathcal{L}\{h(t)\} &= \frac{1}{s^2} + \frac{1}{s\left(1-e^{s}\right)}.\end{aligned}$$