Math 462 Homework 7

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Problem 1. Modify Euler's formula so that it holds for any planar graph, not just connected planar graphs. Prove your formula.

Proof. Since G is planar and not connected, then G has k connected components.

We can connect align these components in a row and connect them with $k-1$ edges.

Now, the new graph is connected and planar, so $V - (E + (k-1)) + F = 2$ holds.

Hence, we have $V - E + F = 3 - k$ for a disconnected planar graph of k connected components. $\square$

Problem 2. (i) Prove that if G is a simple planar graph with no triangles and at least three vertices, then $E \leq 2V - 4$.

(ii) Use this to prove that $K_{3,3}$ is not planar.

Proof of (a). Since G is simple and planar with no triangles, then every face must have at least length 4.

So,

$$4F \leq \sum_{f \in F} \text{length}(f) = 2E \implies F \leq \frac{1}{2}E.$$

Then, substituting the above into Euler's formula we have,

$$2 = V - E + F \leq V - \frac{1}{2}E \implies E \leq 2V - 4.$$

$\square$

Proof of (b). We have that, for $K_{3,3}$, $V = 6$ and $E = 9$.

But, $V > 2V - 4$ as $9 > 2 \cdot 6 - 4 = 8$.

Therefore, $K_{3,3}$ is not planar. $\square$

Problem 3.

Let G be a simple graph (not necessarily planar). Let d be the maximum degree of a vertex in G.

(i) Prove that $\chi(G) \leq d+1$. *Hint:* Use induction.

(ii) Find an infinite family of non-isomorphic graphs for which equality holds in the above inequality.

Proof of (a). We will prove the statement by induction on d.

For the base case, when $d = 0$, the all vertices must have degree zero and must therefore be not connected. Hence $\chi(G) = 0 + 1 = 1$ color for each vertex.

Assume, that $\chi(G) = d$ for the maximum vertex degree in G of $d - 1$.

Now, for the maximum degree of d, we have by the inductive hypothesis that the chromatic number must be at least d.

Then, the vertex with degree d, call it v is adjacent to d other vertices, each of which can be given one of d colors.

But, v must have a different color from the other d vertices; hence the graph must be $(d+1)$-colorable.

Therefore $\chi(G) \leq d+1$. □

Proof of (b). Consider the complete graphs K_n. The degree of each vertex in K_n is $n-1$, so $\chi(K_n) = n$.

Each K_n is not isomorphic to K_{n-1} as these graphs have a different number of vertices.

So, for all n, we have that the infinite family of complete graphs has their chromatic number equal to one more than their maximum degree. □

Problem 4.

A polyhedron has 26 faces, 20 of which are triangles and 6 of which are quadrilaterals. Find the number of vertices and edges of this polyhedron.

Proof. We have that,

$$2E = \sum_{f \in F} \text{length}(f) = 20 \cdot 3 + 6 \cdot 4 = 84 \implies E = 42.$$

Then, by Euler's formula,

$$V - E + F = V - 42 + 26 = 2 \implies V = 18.$$

□

Problem 5.

An *n-gonal pyramid* is a polyhedron formed by connecting each vertex of an n-sided polygon with one additional vertex.

(i) What is the dual polyhedron of an n-gonal pyramid?

(ii) What is the chromatic number of an n-gonal pyramid?

Proposition 1. An n-gonal pyramid is its own dual.

Proof of Proposition and (a). Let N be the n-gonal pyramid and N^* be its dual.

Then, for N, $V = n + 1$ for the vertices in the base n-gon and the apex.

$F = n + 1$ for the n-gon base and the n triangular faces.

$E = 2n$ for the n edges in the base n-gon and the n edges connecting the base vertices to the apex.

Next, for N^*, there is a vertex at each face of N, so $V^* = F = n + 1$.

Similarly, there is a face on each vertex of N, so $F^* = V = n + 1$.

Each edge in N corresponds to an edge in the dual N^*, so $E^* = N = 2n$.

In fact, N^* has the same structure as N.

The vertex in the face of the base n-gon in N is the apex in N^*.

The n vertices on triangular faces of N form the vertices of the n-gon base in N^*.

Thus, N^* implies that an n-gonal pyramid is its own dual. □

Proof of (b). By Problem 3, since the maximum degree in N belongs to the apex vertex with degree n, then $\chi(N) = n + 1$. □

Problem 6.

Let G_n be the graph with vertex set $\{0, 1\}^n$ and an edge between x, $y \in \{0, 1\}^n$ if x and y differ at exactly 2 positions.

(i) Prove that $\chi(G_3) = 4$ and $\chi(G_4) = 4$.

(ii) Prove that $\chi(G_n) \geq n$ for all positive integers n.

Proof. □