Math 462 Homework 4

Alexandre Lipson

February 14, 2025

Problem 1. Let W be a walk with at least two vertices which begins and ends at the same vertex. Prove that W either contains a cycle or has an edge that repeats immediately (i.e., at some point the walk goes across an edge and then immediately back across the same edge).

Proof. We will consider two cases whether or not W contains a cycle. A cycle is a subwalk in which a vertex is visited more than once without immediate backtracking.

First, if W revisits a vertex (at least the starting and ending one) without immediate backtracking, then W must contain a cycle.

Second, if W does not contain a cycle, then it must backtrack each time it revisits a vertex; i.e., whenever W reaches a vertex that it has already seen, then it will immediately return to the previous vertex along the same edge. If W uses a parallel edge instead of backtracking, we have a cycle, which is a contradiction. This means, at some point, W must traverse an edge and backtrack immediately. Thus, if W does not contain a cycle, it must contain an edge that repeats immediately.

Since these two cases cover all possibilities, W must either contain a cycle or have an edge that is repeated immediately $\square$

Problem 2. Let G be the graph whose vertex set is $\{0, 1\}^n$, with an edge between x and y if x and y differ at exactly one position.

1. What is the degree of each vertex of G?

2. How many edges does G have?

Proof of a. There are n ways that a vertex in $\{0, 1\}^n$ can change by one position, so each vertex in G must have degree n. $\square$

Proof of b. Note that there are $|\{0, 1\}^n| = 2^n$ vertices in G. Then, by the handshake Lemma, we have, for all vertices v and edge set E,

$$2^n \deg v = 2|E| => n2^{n-1} = |E|.$$

So there are $n2^{n-1}$ edges in G. $\square$

Problem 3. Let G be the graph whose vertex set is $\{0, 1\}^n$, with an edge between x and y if x and y differ at exactly two positions. How many connected components does G have?

Proof. Each vertex in G can be represented by an n-bit binary string. Such strings haves either an even or odd parity, given by an even or odd number of 1s respectively. Note that flipping two bits does not change the parity of a string.

We claim that G is bipartite between even and odd parity vertices.

We can reach every vertex in each subgraph by flipping any two bits and maintaining parity.

Thus, each parity class subgraph forms a connected component as adding any vertex of opposing parity would make the graph disconnected as there is no way to reach opposing parity by flipping two bits.

Since there are two parity subgraphs, each of which is connected, then G has two connected components. □

Problem 4. Consider a league with two divisions of 13 teams each. Is it possible to schedule a season with each team playing nine games against teams within its division and four games against teams in the other division?

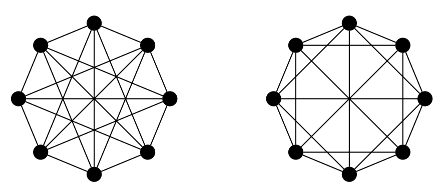
Proof. Consider only the intra-division games. We can create a subgraph graph of of such games for one division where each of the 13 teams in a vertex and each game played is an edge.

If each team must play 9 games within the division, then each edge must have a degree of 9.

However, the total number of edges in this graph is $13 \cdot 9 = 117$, which is odd. By the handshake Lemma, the number of edges must be even, which is a contradiction.

Therefore, this setup is not possible. □

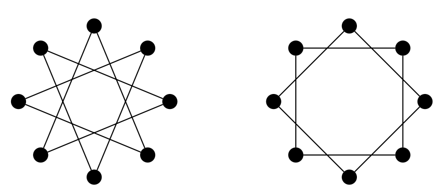
Problem 5. Are the following two graphs isomorphic?



Proof. Since isomorphism preserves all structured properties of a graph, then we first note that both graphs have all 8 vertices of degree 5, and therefore also both have 20 edges.

Since both graphs have their vertices arranged in the same way, we can expose the differences in each graph by removing edges that are the same.

Now, we have



But, we see that the left graph has one 8 cycle, while the right has two 4 cycles. Since these graphs do not have the same number of cycles, then they are not isomorphic. □

Problem 6. Prove that every n vertex graph with at least n edges contains a cycle.

Proposition 1. A tree with n vertices has $n-1$ edges.

Proof of Proposition. We will proceed by induction. A tree with 1 vertex has 0 edges. Suppose that a tree with n vertices has $n-1$ edges.

Now, construct a tree with $n+1$ vertices from the previous. The new vertex must be a leaf because a tree cannot have cycles. Since the new vertex is a leaf, it adds one edge to the tree. Thus, a tree with $n+1$ vertices has $n-1+1=n$ edges. □

Proof of Problem. Let G be a graph with n vertices and n edges that has no cycles. We will consider two cases, whether or not G is connected.

First, assume G is connected. Since G is connected and has no cycles, then G is a tree of n vertices. By the Proposition, it must have $n-1$ edges. But G has n edges, so the n^{th} edge must connect two already connected vertices, which makes a cycle. So, when G is connected, G must contain a cycle.

Second, assume G is not connected, so G is a forest. Suppose that G has k connected components, and none of which have a cycle. So, each connected component of G must be a tree.

Since G is composed of k trees, each with n_i vertices where $1 \leq i \leq k$ and $\sum_k n_i = n$, then it must have $\sum_1^k (n_i - 1) = n - k$ edges.

But, G has at least n edges, so $n \leq n-k \implies k \leq 0$, which is a contradiction since G must have at least one component to have n vertices.

So, G is either connected, and has a cycle by the first case, or the assumption that each connected component was acyclic was false, so G has a cycle.

Thus, a graph G with n vertices and n edges must have a cycle. □