Math 335 Homework 6

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Problem 1. Prove the generalized Parseval identity: If f, g are Reimann integrable with complex Fourier coefficients c_n, d_n, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)\overline{g(x)}+\overline{f(x)}g(x))\,dx=\sum_{-\infty}^{\infty}(c_n\overline{d_n}+\overline{c_n}d_n).$$

Proof. We will apply the Parseval identity to $f+g$ and $f-g$.

We can begin considering both $f \pm g$ at once,

$$\begin{aligned}
\sum_{-\infty}^{\infty}|c_n|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2\,dx\\
\sum_{-\infty}^{\infty}|c_n \pm d_n|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f+g|^2\,dx\\
\sum_{-\infty}^{\infty}(c_n \pm d_n)\overline{(c_n \pm d_n)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f \pm g)\overline{(f \pm g)}\,dx.
\end{aligned}$$

Now, we will consider the difference of the addition and subtractions cases,

$$\begin{aligned}
\sum_{-\infty}^{\infty}(c_n\overline{c_n}+c_n\overline{d_n}+\overline{c_n}d_n+d_n\overline{d_n}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(f\overline{f}+\overline{f}g+f\overline{g}+g\overline{g})\,dx\\
\sum_{-\infty}^{\infty}(c_n\overline{c_n}-c_n\overline{d_n}-\overline{c_n}d_n-d_n\overline{d_n}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(f\overline{f}-\overline{f}g-f\overline{g}-g\overline{g})\,dx\\
\implies 2\sum_{-\infty}^{\infty}(c_n\overline{d_n}+\overline{c_n}d_n) &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\overline{f}g+f\overline{g})\,dx\\
\sum_{-\infty}^{\infty}(c_n\overline{d_n}+\overline{c_n}d_n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)\overline{g(x)}+\overline{f(x)}g(x))\,dx.
\end{aligned}$$

□

Problem 2. With the same conditions as in Problem 1, prove

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\overline{g(x)}\,dx=\sum_{-\infty}^{\infty}c_n\overline{d_n}.$$

Proof. Note that $f\overline{g} = \overline{f}g = 2\text{Re}(f\overline{g})$.

By Problem 1,

$$\sum_{-\infty}^{\infty}(c_n\overline{d_n} + \overline{c_n}d_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\text{Re}(f\overline{g})\,dx.$$

Now, with $f = if$, we have

$$\sum_{-\infty}^{\infty}(c_n\overline{d_n} - \overline{c_n}d_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2i\text{Im}(f\overline{g})\,dx.$$

So, summing both of the above, we have

$$2\sum_{-\infty}^{\infty} c_n\overline{d_n} = \frac{1}{\pi}\int_{-\pi}^{\pi}(\text{Re}(f\overline{g}) + i\text{Im}(f\overline{g}))\,dx \implies \sum_{-\infty}^{\infty} c_n\overline{d_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$$

□

Problem 3. Find the Fourier series of the 2π-periodic function $f_{\frac{\pi}{4}}(x) = \left(x - \frac{\pi}{4}\right)^2$ on the interval $\left[-\frac{3}{4}\pi, \frac{5}{4}\pi\right]$.

Proof. We will consider the Fourier series of $f_0(x) = x$ on $[-\pi, \pi]$ shifted by $x \to x - \frac{\pi}{4}$. Note that this substitution achieves both the desired function and interval.

Since $f_0(x) = x^2$ is an even function, then its sine Fourier coefficients b_n are zero.

Then, for the constant term,

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{3\pi}x^3|_0^\pi = \frac{\pi^2}{3}.$$

Next, we have,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx & &= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2\sin nx}{n^3} + \frac{2x\cos nx}{n^2}\right]_0^\pi \\ &= 2\left(\frac{2\pi\cos n\pi}{n^2}\right) \\ &= \frac{4\cos n\pi}{n^2} \\ &= \frac{4(-1)^n}{n^2}.. \end{aligned}$$

Therefore we have that

$$f_0(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx.$$

Now, shifting by $x \to x - \frac{\pi}{4}$, we have that

$$f_{\frac{\pi}{4}}(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos\left(n\left(x - \frac{\pi}{4}\right)\right).$$

□

Problem 4. Find the sum of the following series using the series for x^2 and choosing the appropriate value of x.

(i) $\sum_1^\infty \frac{1}{n^2}$.

(ii) $\sum_1^\infty \frac{(-1)^{n+1}}{n^2}$.

From Problem 2, we have that, on $x \in [-\pi, \pi]$,

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_1^\infty \frac{4(-1)^n}{n^2} \cos nx.$$

Proof of (i). If $x = \pi$, then $\cos nx = (-1)^n$. So, since f is continuous at π,

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} + \sum_1^\infty \frac{4(-1)^n}{n^2}(-1)^n \implies \frac{\pi^2}{6} = \sum_1^\infty \frac{1}{n^2}.$$

□

Proof of (ii). Since f is continuous at zero and $x = 0 \implies \cos nx = 1$, then

$$f(0) = 0 = \frac{\pi^2}{3} + \sum_1^\infty \frac{4(-1)^n}{n^3} \implies \frac{\pi^2}{12} = \sum_1^\infty \frac{(-1)^{n+1}}{n^2}.$$

□

Problem 5. Using the Fourier series of $f(x) = x^2$, show that

$$x^3 - \pi^2 x = 12 \sum_1^\infty \frac{(-1)^n \sin nx}{n^3}, \qquad |x| \le \pi.$$

Proof. Let $F(x) = \frac{1}{3}(x^3 - \pi^2 x)$, so $F' = x^2 - \frac{\pi^2}{3} = f - \frac{\pi^2}{3}$, which is the Fourier series of x^2 without its constant term.

Since $f - \frac{\pi^2}{3}$ is continuous and piecewise smooth on $|x| \le \pi$, and the mean value of F on $|x| \le \pi$ is zero because F is odd, then

$$F(x) = \sum_1^\infty \frac{4(-1)^n}{n^3} \sin nx.$$

Therefore, we have that

$$x^3 - \pi^2 x = \sum_1^\infty \frac{12(-1)^n}{n^3} \sin nx$$

on $|x| \le \pi$ as desired. □

Problem 6. Suppose f is piecewise continuous on $[0, 2\ell]$, satisfies $f(x) = f(2\ell - x)$, and is symmetrical about $x = \ell$. Let a_n, b_n be the Fourier coefficients of f. Show that $a_n = 0$ for odd n, and $b_n = 0$ for even n.

Proof. Note that $\cos(n\pi - x) = (-1)^n \cos x$ and $\sin n\pi - x = (-1)^{n+1} \sin x$.

We have that,

$$\begin{aligned}
a_n &= \frac{1}{\ell} \int_0^{2\ell} f(x) \cos \frac{n\pi x}{2\ell}\, dx \\
&= \frac{1}{\ell} \left(\int_0^{\ell} f(x) \cos \frac{n\pi x}{2\ell}\, dx + \int_{\ell}^{2\ell} f(x) \cos \frac{n\pi x}{2\ell}\, dx \right) \\
&= \frac{1}{\ell} \left(\int_0^{\ell} f(x) \cos \frac{n\pi x}{2\ell}\, dx + \int_0^{\ell} f(2\ell - x) \cos \frac{n\pi(2\ell - x)}{2\ell}\, dx \right) \\
&= \frac{1}{\ell} \int_0^{\ell} f(x) \left(\cos \frac{n\pi x}{2\ell} + \cos \left(n\pi - \frac{n\pi x}{2\ell} \right) \right) dx \\
&= \frac{1}{\ell} \int_0^{\ell} f(x) \left(\cos \frac{n\pi x}{2\ell} \right) (1 + (-1)^n)\, dx,
\end{aligned}$$

which vanishes for odd n.

Similarly, we have,

$$\begin{aligned}
b_n &= \frac{1}{\ell} \int_0^{2\ell} f(x) \sin \frac{n\pi x}{2\ell}\, dx \\
&= \frac{1}{\ell} \int_0^{\ell} f(x) \left(\sin \frac{n\pi x}{2\ell} + \sin \frac{n\pi(2\ell - x)}{2\ell} \right) dx \\
&= \frac{1}{\ell} \int_0^{\ell} f(x) \left(\sin \frac{n\pi x}{2\ell} \right) (1 + (-1)^{n+1})\, dx,
\end{aligned}$$

which vanishes for all even n. □

Problem 7. Determine the constants a, b, c so that the functions

$$f_0(x) = 1, \qquad f_1(x) = x + a, \qquad f_2(x) = x^2 + bx + c$$

form an orthogonal set on $[0, 1]$.

Proof. We must have $\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx = 0$ for all pairs of f_0, f_1, f_2.

First,

$$0 = \langle f_0, f_1 \rangle = \int_0^1 (x + a)\, dx = \frac{1}{2} + a \implies a = -\frac{1}{2}.$$

Next,

$$0 = \langle f_0, f_2 \rangle = \int_0^1 (x^2 + bx + c)\, dx = \frac{1}{3} + \frac{b}{2} + c.$$

Now,

$$
\begin{aligned}
0 = \langle f_1, f_2 \rangle &= \int_0^1 (x+a)(x^2+bx+c)\,dx \\
&= \int_0^1 (x^3 + bx^2 + cx + ax^2 + abx + ac)\,dx \\
&= \frac{1}{4} + \frac{b+a}{3} + \frac{c+ab}{2} + ac \\
&= \frac{1}{4} + \frac{b-\frac{1}{2}}{3} + \frac{c-\frac{b}{2}}{2} - \frac{c}{2} \\
&= \frac{1}{4} + \frac{4b-2}{12} - \frac{b}{4} \\
&= \frac{3-3b+4b-2}{12} \\
&= \frac{1+b}{12} \\
\implies b = -1.
\end{aligned}
$$

Lastly,

$$c = -\frac{1}{3} - \frac{b}{2} = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.$$

Thus,

$$f_0(x) = 1, \qquad f_1(x) = x - \frac{1}{2}, \qquad f_2(x) = x^2 - x + \frac{1}{6}.$$

□

Problem 8. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$ using Parseval's identity.

Proof. For $f(x) = x^2$, we had that

$$a_n = \frac{4(-1)^n}{n^2} = c_n + c_{-n} \implies c_n = \frac{2(-1)^n}{n^2} \implies |c_n|^2 = \frac{4}{n^4}.$$

Also, we have, $c_0 = \frac{1}{2}a_0 = \frac{\pi^2}{3} \implies |c_0|^2 = \frac{\pi^4}{9}$.

So, by Parseval with f, we have that,

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{\pi^4}{5}.$$

Then, with $c_n = c_{-n}$,

$$\begin{aligned}
\sum_{-\infty}^{\infty} |c_n|^2 = |c_0|^2 + 2\sum_{1}^{\infty} |c_n|^2 &= \frac{\pi^4}{5} \\
\frac{\pi^4}{9} + 2\sum_{1}^{\infty} \frac{4}{n^4} &= \frac{\pi^4}{5} \\
\sum_{1}^{\infty} \frac{1}{n^4} &= \frac{1}{2}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) \\
\sum_{1}^{\infty} \frac{1}{n^4} &= \frac{1}{2}\left(\frac{4\pi^4}{45}\right) \\
\sum_{1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}.
\end{aligned}$$

□