Math 334 Homework 3

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October 11, 2024

Problem (1)**.** Give an example of an open cover of the open unit interval $(0, 1)$ which does not admit a finite subcover.

Proposition. One such cover S is the union of expanding covers formed by the one-ball centered at $\frac{1}{2}$ with radius given by the sequence $x_k = \frac{1}{2} - \frac{1}{2^k}$ which converges to $\frac{1}{2}$.

$$S = \lim_{n \to \infty} S_n, \quad S_n = \bigcup_{k=1}^{n} B_{x_k}\left(\frac{1}{2}\right), \quad x_k = \frac{1}{2} - \frac{1}{2^k}.$$

Proof. Suppose, for a contradiction, that there exists a finite subcover S_m which covers the open unit interval.

By the convergence of x_k, $\exists N,\ \forall \epsilon > 0,\ \forall k \geq N,\ \left|x_k - \frac{1}{2}\right| < \epsilon$. So, $\left|\frac{1}{2} - \frac{1}{2^k} - \frac{1}{2}\right| < \epsilon \implies \frac{1}{2^k} < \epsilon$. Then,

$$k > \log_2\left(\frac{1}{\epsilon}\right).$$

Since $\lim\limits_{x \to 0} \log_2\left(\frac{1}{x}\right) \to \infty$ and ϵ was arbitrarily small, then k must be larger than any number.

But, we wished to find a finite subcover S_m, yet □

Problem.

Proof. □

Problem.

Proof. □

Problem (4)**.** Suppose $f : \mathbb{S}^n \longrightarrow \mathbb{R}^n$ is continuous. Note that $x \in \mathbb{S}^n \implies -x \in \mathbb{S}^n$. Prove $\exists x \in \mathbb{S}^n \ : \ f(x) = f(-x)$.

Proof. Let $g(x) = f(x) - f(-x)$. We wish to find an x such that $g(x) = 0$.

Since g is the difference of two continuous functions f, then g is also continuous.

Since $\mathbb{S}^n$ connected by Problem 3, then $g : \mathbb{S}^n \longrightarrow \mathbb{R}^n$, a continuous function on a connected domain, is connected.

Notice that $g(-x) = f(-x) - f(x) = -g(x)$. So, g is odd.

If $g(x) = 0$ identically, then we're done. So, there must be an x where $g(x) > 0$. Then, since g odd, $g(-x) < 0$. But, g is connected, so $\exists a,\ -x < a < x,\ g(a) = 0$. □

Problem.

Proof. □