Math 135 Homework 8

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February 26, 2024

1. Solve $y'' + 3ty + 3y = 0$ using series. We begin by obtaining y, y', and y'', being mindful of the starting index to avoid zero terms in the sum,

$$\begin{aligned} y &= \sum_{k=0}^{\infty} a_k t^k, \\ y' &= \sum_{k=1}^{\infty} k a_k t^{k-1}, \\ y'' &= \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2}. \end{aligned}$$

Then we use these series definitions in place of y and its derivatives in the differential equation,

$$\sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} + 3t \sum_{k=1}^{\infty} k a_k t^{k-1} + 3 \sum_{k=0}^{\infty} a_k t^k = 0.$$

We make a substitution or change of variables such that all t terms are to the same power,

$$\sum_{k=0}^{\infty} (m+2)(m+1) a_{m+2} t^m + 3 \sum_{m=1}^{\infty} m a_m t^m + 3 \sum_{m=0}^{\infty} a_m t^m = 0.$$

We then set all series to begin at the same index $m = 1$ by extracting initial terms as well as grouping the coefficients of t^m,

$$2a_2 + 3a_0 + \sum_{m=1}^{\infty} \left[(m+2)(m+1) a_{m+2} + 3m a_m + 3a_m \right] t^m = 0.$$

Given that the sum of the series and the initial coefficients is identically zero, then both the sum initial coefficients themselves and the coefficients of the series must all be zero such such that the equation holds for all t.

So,

$$\begin{aligned} 2a_2 + 3a_0 &= 0 \\ a_2 &= -\frac{3}{2} a_0. \end{aligned}$$

Then, we can establish our recursive relationship,

$$\begin{aligned} a_{m+2} &= \frac{-3(1-m)}{(m+2)(m+1)} a_m \\ &= \frac{-3}{m+2} a_m. \end{aligned}$$

We can use this relationship to verify a_2,

$$a_2 = \frac{-3}{0+2} a_m = -\frac{3}{2} a_0.$$

We proceed to write several coefficients,

$$\begin{aligned} a_2 &= \frac{-3}{2} a_0, \\ a_3 &= \frac{-3}{3} a_1, \\ a_4 &= \frac{-3}{4}\left(\frac{-3}{2}\right) a_0 = \frac{3^2}{2^3} a_0, \\ a_5 &= \frac{-3}{5}\left(\frac{-3}{3}\right) a_1 = \frac{3^2}{3 \cdot 5} a_1, \\ a_6 &= \frac{-3}{6}\left(\frac{3^2}{2^3}\right) a_0 = \frac{-3^3}{2^4 \cdot 3} a_0, \\ a_7 &= \frac{-3}{7}\left(\frac{2^2}{3 \cdot 5}\right) a_1 = \frac{-2^2 \cdot 3}{3 \cdot 5 \cdot 7} a_1, \\ a_8 &= \frac{-3}{8}\left(\frac{-3^3}{2^4 \cdot 3}\right) a_0 = \frac{3^4}{2^7 \cdot 3} a_0, \\ a_9 &= \frac{-3}{9}\left(\frac{-2^2 \cdot 3}{3 \cdot 5 \cdot 7}\right) a_0 = \frac{-2^2 \cdot 3^3}{3^3 \cdot 5 \cdot 7} a_0, \\ a_{10} &= \frac{-3}{10}\left(\frac{3^4}{2^7 \cdot 3}\right) = \frac{-3^5}{2^8 \cdot 3 \cdot 5} a_0, \end{aligned}$$

First, we will consider just the even coefficients a_{2k}, we see that they have a power of $(-3)^k$, in addition to the product of all even integers up to $2n$. This can be written in terms of factorials as $2^k k!$.

So, the rule for even coefficients is,

$$a_{2k} = \frac{(-3)^k}{2^k k!}.$$

Next, the odd coefficients a_{2k+1} also contain the k^{th} multiple of -3, but their denominator is comprised of the product of all odd integers up to $2k+1$. This can be expressed using the double factorial,[1].

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot \dots \cdot (2k+1) = \frac{(2k+1)!}{2^k k!}.$$

[1] The double factorial definition and expression with the regular factorial, `https://en.wikipedia.org/wiki/Double_factorial`.

So, the rule for odd coefficients is,

$$a_{2k+1} = \frac{(-3)^k}{\frac{(2k+1)!}{2^k k!}} = \frac{(-6)^k k!}{(2k+1)!}.$$

2. Find the general solution to $(1+t^2)y'' - 4ty' + 6y = 0$ using the series method.

First, we rewrite the equation using series,

$$\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} + \sum_{k=2}^{\infty} k(k-1)a_k t^k - 4\sum_{k=1}^{\infty} k a_k t^k + 6\sum_{k=0}^{\infty} a_k t^k = 0.$$

Then, we shift each term to match in the power of t,

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m + \sum_{k=2}^{\infty} m(m-1)a_m t^m - 4\sum_{m=1}^{\infty} m a_m t^m + 6\sum_{m=0}^{\infty} a_m t^m = 0.$$

Next, we start all summations at the latest index $m = 2$ by bringing out prior terms,

$$2a_2 + 6a_3 t - 4a_1 t + 6a_0 + 6a_1 t + \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} + m(m-1)a_m - 4ma_m + 6a_m\right] t^m = 0.$$

We will first confider the initial coefficients by separating them into their respective powers of t, t^0 and t^1.

Recalling that the equation is identically for all t, we see that,

$$\begin{aligned} 2a_2 + 6a_0 &= 0 \\ a_2 &= -3a_0; \end{aligned}$$

$$\begin{aligned} (6a_3 - 4a_1 + 6a_1)t &= 0 \\ (6a_3 + 2a_1)t &= 0 \\ a_3 &= -\frac{a_1}{3}. \end{aligned}$$

We proceed with the recursive relation,

$$\begin{aligned} (m+2)(m+1)a_{m+2} + m(m-1)a_m - 4ma_m + 6a_m &= 0 \\ a_{m+2} &= -\frac{m(m-1) - 4m + 6}{(m+1)(m+2)} a_m \\ &= -\frac{m^2 - 5m + 6}{(m+1)(m+2)} a_m \\ &= -\frac{(m-2)(m-3)}{(m+1)(m+2)} a_m. \end{aligned}$$

We will evaluate a few terms to see that, for $n \geq 4$, all coefficients $a_n = 0$.

For $m = 2$,

$$a_4 = -\frac{(2-2)(2-3)}{(2+1)(2+2)} a_2 = 0.$$

For $m = 3$,

$$a_5 = -\frac{(3-2)(3-3)}{(3+1)(3+2)} a_3 = 0.$$

Since all higher coefficients are multiples of a_4 and a_5, then they are all zero.

So, we are left with

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3,$$

which can be expressed from the two arbitrary constants–from the second order differential equation–a_0 and a_1,

$$a_0 + a_1 t - 3a_0 t^2 - \frac{a_1 t^3}{3}.$$

We relabel using $a_0 = c_1$ and $a_1 = c_2$ and collect terms,

$$y(t) = c_1(1 - 3t^2) + c_2 \left(t - \frac{t^3}{3} \right).$$

3. Find the general solution to $(1 - 4t^2)y'' + 8y = 0$.

We replace y and its derivatives with their power series form and combine on the same power of t.

$$\begin{aligned}
\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} - 4\sum_{k=2}^{\infty} k(k-1)a_k t^k + 8\sum_{k=0}^{\infty} a_k t^k &= 0 \\
\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} t^m - 4\sum_{m=2}^{\infty} m(m-1)a_m t^m + 8\sum_{m=0}^{\infty} a_m t^m &= 0 \\
2a_2 + 6a_3 t + 8a_0 + 8a_1 t + \sum_{m=2}^{\infty} \left[(m+2)(m+1)a_{m+2} - 4m(m-1)a_m + 8a_m \right] t^m &= 0.
\end{aligned}$$

For the standalone coefficients,

$$\begin{aligned}
2a_2 + 8a_0 &= 0 \\
a_2 &= -4a_0;
\end{aligned}$$

$$\begin{aligned}
(6a_3 + 8a_1)t &= 0 \\
a_3 &= \frac{-4}{3} a_1.
\end{aligned}$$

For the recurrence relation,

$$\begin{aligned}(m+2)(m+1)a_{m+2} - 4m(m-1)a_m + 8a_m &= 0\\ a_{m+2} &= \frac{4m(m-1)a_m - 8a_m}{(m+1)(m+2)}\\ &= 4a_m\frac{m^2-m-2}{(m+1)(m+2)}\\ &= 4a_m\frac{(m+1)(m-2)}{(m+1)(m+2)}\\ &= 4a_m\frac{m-2}{m+2}.\end{aligned}$$

We will consider some even coefficients, starting with a_4 at $m = 2$,

$$\begin{aligned}a_4 &= 4a_2\frac{2-2}{2+2} = 0,\\ a_6 &= 4a_4\frac{4-2}{2+2} = 0,\\ a_8 &= 4a_6(\cdots) = 0,\\ &\vdots\\ a_{2n} &= 0.\end{aligned}$$

In fact, all even coefficients for $n > 1$ reduce to zero.

We will then consider some odd coefficients, starting with a_5 at $m = 3$,

$$\begin{aligned}a_5 &= 4a_3\frac{1}{5} = \frac{-4^2}{3\cdot 5}a_1,\\ a_7 &= 4a_5\frac{3}{7} = \frac{-4^3}{5\cdot 7}a_1,\\ a_9 &= 4a_7\frac{5}{9} = \frac{-4^4}{7\cdot 9}a_1,\\ a_{11} &= 4a_9\frac{7}{11} = \frac{-4^5}{9\cdot 11}a_1.\end{aligned}$$

We see that the explicit form for all of the odd coefficients can be written as,

$$a_{2k+1} = \frac{-4^k}{(2k-1)(2k+1)}a_1.$$

Given all coefficients, we can express the solution to the differential equation as a series[2] noting that the coefficient for t^2, $a_2 = -4a_0$; so,

$$y = a_0(1-4t^2) + a_1\sum_{k=0}^{\infty}\frac{-4^k}{(2k-1)(2k+1)}t^{2k+1}.$$

[2]Since the explicit form holds for a_1 and $a_3 = \frac{-4}{3}a_1$, we incorporate these two coefficients into the series and let the starting index be zero.

We evaluate this series by itself,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{-4^k}{(2k-1)(2k+1)} t^{2k+1} &= -\frac{1}{2}\left[\sum_{k=0}^{\infty} \frac{4^k t^{2k+1}}{2k-1} - \frac{4^k t^{2k+1}}{2k+1}\right] \\
&= -\frac{1}{2}\left[t^2 \sum_{k=0}^{\infty} \frac{4^k t^{2k-1}}{2k-1} - \sum_{k=0}^{\infty} \frac{4^k t^{2k+1}}{2k+1}\right] \\
&= -\frac{1}{2}\left[t^2 \int \sum_{k=0}^{\infty} 4^k t^{2k-2}\, dt - \int \sum_{k=0}^{\infty} 4^k t^{2k}\, dt\right] \\
&= -\frac{1}{2}\left[4t^2 \int \sum_{k=0}^{\infty} \left((2t)^2\right)^{k-1} dt - \int \sum_{k=0}^{\infty} \left((2t)^2\right)^{k} dt\right] \\
&= -\frac{1}{2}\left[4t^2 \int \left[\frac{1}{4t^2} + \sum_{k=0}^{\infty} \left((2t)^2\right)^{k}\right] dt - \int \sum_{k=0}^{\infty} \left((2t)^2\right)^{k} dt\right] \\
&= -\frac{1}{2}\left[4t^2 \left(-\frac{1}{4t}\right) + \left(4t^2 - 1\right) \int \sum_{k=0}^{\infty} \left((2t)^2\right)^{k} dt\right] \\
&= \frac{1}{2}\left[t + \left(1 - 4t^2\right) \int \frac{dt}{1 - (2t)^2}\right] \\
&= \frac{1}{2}\left[t + \left(1 - 4t^2\right) \cdot \frac{1}{2} \int \left(\frac{1}{1+2t} + \frac{1}{1-2t}\right) dt\right] \\
&= \frac{1}{2}\left[t + \left(1 - 4t^2\right) \cdot \frac{1}{4} \left(\ln(1+2t) - \ln(1-2t)\right)\right].
\end{aligned}$$

We then use this along with $c_1 = a_0$ and $c_2 = 8a_1$ to find the general solution,

$$y(t) = c_1\left(1 - 4t^2\right) + c_2\left(4t + \left(1 - 4t^2\right)\left[\ln(1+2t) - \ln(1-2t)\right]\right).$$

4. We will consider the system when there is y centimeters of mercury above equilibrium on one side and y centimeters below equilibrium on the other side.

   So, if we remove the $2y$ column of mercury, the system is at equilibrium and the net force is equal to zero.

   Thus the net force in the system, net mass times net acceleration, is given by the weight of the column of mercury of height $2y$. The downward force of the weight of the column of mercury opposes the direction of acceleration of the system.

   The acceleration is modeled by the second derivative of y.

   The volume of this column is $2y$ times the cross sectional area of the cylindrical tube $A = \pi$.

So,

$$
\begin{aligned}
-2yA\rho_L g &= m_L y'' \\
m_L y'' + 2yA\rho_L g &= 0 \\
y'' + \frac{2A\rho_L g}{m_L} y &= 0 \\
y'' + \frac{2 \cdot \pi \cdot 13.5 \cdot 9.8}{500} y &= 0 \\
y'' + \frac{1323\pi}{2500} y &= 0
\end{aligned}
$$

For the characteristic equation,

$$
\begin{aligned}
r^2 + \frac{1323\pi}{2500} = 0 \\
r = \pm \frac{21\sqrt{3\pi} i}{50}.
\end{aligned}
$$

Let $\omega = \frac{21\sqrt{3\pi}}{50}$.

Then,

$$y = c_1 \cos \omega t + c_2 \sin \omega t.$$

To find the period T, we consider when $\omega(t+T) = \omega t + 2\pi$, which is the period of the sine and cosine functions.

So,

$$
\begin{aligned}
\omega(t+T) &= \omega t + 2\pi \\
\omega T &= 2\pi \\
T &= \frac{2\pi}{\omega}, \\
T &= \frac{100\pi}{21\sqrt{3\pi}} \\
T &= \frac{100\sqrt{\pi}}{21\sqrt{3}}.
\end{aligned}
$$