Math 135 Homework 9

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1. Assume that the triangle identity holds,

$$||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||.$$

Note that $||\mathbf{x}-\mathbf{y}|| = ||\mathbf{y}-\mathbf{x}||$.

Then,

$$||\mathbf{x}|| = ||\mathbf{x}-\mathbf{y}+\mathbf{y}|| \leq ||\mathbf{x}-\mathbf{y}|| + ||\mathbf{y}||.$$

So,

$$||\mathbf{x}|| - ||\mathbf{y}|| \leq ||\mathbf{x}-\mathbf{y}||.$$

Then,

$$||\mathbf{y}|| = ||\mathbf{y}-\mathbf{x}+\mathbf{x}|| \leq ||\mathbf{y}-\mathbf{x}|| + ||\mathbf{x}||.$$

So,

$$||\mathbf{x}|| - ||\mathbf{y}|| \geq -||\mathbf{y}-\mathbf{x}|| = -||\mathbf{x}-\mathbf{y}||.$$

Thus,

$$\begin{aligned} -||\mathbf{x}-\mathbf{y}|| \leq ||\mathbf{x}|| - ||\mathbf{y}|| &\leq ||\mathbf{x}-\mathbf{y}|| \\ \Big| ||\mathbf{x}|| - ||\mathbf{y}|| \Big| &\leq ||\mathbf{x}-\mathbf{y}||. \end{aligned}$$

2. Since $||\mathbf{a}||^2 = \mathbf{a}\cdot\mathbf{a}$, then

$$\begin{aligned} &||\mathbf{a}+\mathbf{b}||^2 + ||\mathbf{a}-\mathbf{b}||^2 \\ =&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ =&\mathbf{a}\cdot\mathbf{a} + 2(\mathbf{a}\cdot\mathbf{b}) + \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a} - 2(\mathbf{a}\cdot\mathbf{b}) + \mathbf{b}\cdot\mathbf{b} \\ =&2(\mathbf{a}\cdot\mathbf{a}) + 2(\mathbf{b}\cdot\mathbf{b}) \\ =&2(||\mathbf{a}||^2 + ||\mathbf{b}||^2). \end{aligned}$$

3. The three points P_1, P_2, and Q form a plane that intersects the sphere on which they lie to form a circle.

Given that P_1 and P_2 are antipodal, we can instead consider a semicircle with P_1 and P_2 at the endpoints of the arc.

With these conditions, we will show that $\overrightarrow{P_1Q}$ is perpendicular to $\overrightarrow{P_2Q}$ in problem 5.

4. If **a**, **b**, and **c** are linearly independent, then, for any vector **d**,

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{d} + \gamma\mathbf{c}.$$

Since $\mathbf{b}, \mathbf{c}$ are linearly independent, they are not parallel. So, $\mathbf{b} \times \mathbf{c} \neq 0$.

Additionally, since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, then **a** is not a linear combination of **b** and **c**. So, **a** does not lie in the plane formed by **b** and **c**.

Thus, by 13.6.4,

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq 0. \tag{*}$$

To define α, we note that either **b** or **c** crossed with the vector $\mathbf{b} \times \mathbf{c}$ will always be zero since $\mathbf{b} \times \mathbf{c}$ is perpendicular to both **b** and **c**. Thus, we consider the following,

$$\begin{aligned} \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c}) &= (\alpha\mathbf{a} + \beta\mathbf{d} + \gamma\mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \alpha\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ \frac{\mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} &= \alpha. \end{aligned}$$

The denominator, being the triple product of **a**, **b**, and **c**, is not zero by (*), and is therefore non-zero for any arrangement of the vectors in the triple product by 13.4.7.

We repeat the computation above using $\mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ to retain only **b** or **c** respectively.

This yields the following definitions for α, β, and γ,

$$\alpha = \frac{\mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \qquad \beta = \frac{\mathbf{d} \cdot (\mathbf{a} \times \mathbf{c})}{\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})}, \qquad \gamma = \frac{\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}.$$

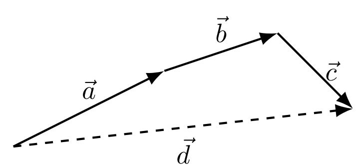


Figure 1: Linear combination of **a**, **b**, and **c** to form **d**.

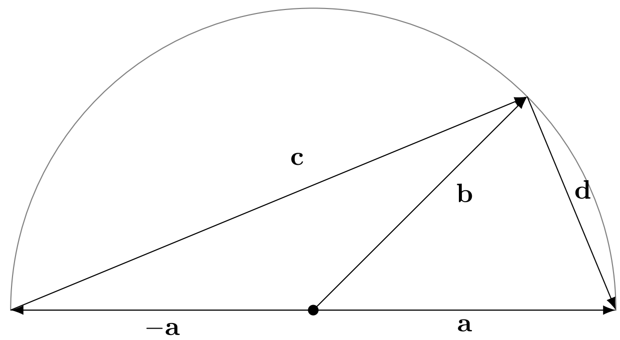
5.

Theorem. *Every angle inscribed in a semicircle is a right angle*

Proof. Let θ be the angle between **c** and **d**.

We will prove that $\mathbf{c} \cdot \mathbf{d} = 0$. This is an equivalent statement to the Theorem.

Since $\mathbf{c} - \mathbf{a} = \mathbf{b}$, then $\mathbf{c} = \mathbf{a} + \mathbf{b}$.



We see that $\mathbf{d} = \mathbf{a} - \mathbf{b}$.

We also notice that $||\mathbf{a}|| = ||\mathbf{b}||$, which is the radius of the semicircle.

Then,

$$\begin{aligned} \mathbf{c} \cdot \mathbf{d} &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\ &= ||\mathbf{a}||^2 - ||\mathbf{b}||^2 \\ &= 0. \end{aligned}$$

So $\mathbf{c} \cdot \mathbf{d} = 0$. $\square$