Math 335 Homework 3

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February 9, 2025

Problem 1. Compute $\lim\limits_{k\to\infty} f_k$ on the given interval I Is f_k uniformly convergent on I? If not, is it uniformly convergent on a smaller interval?

i) $f_k(x) = x^{\frac{1}{k}}$, $x \in [0,1]$.

ii) $f_k(x) = kxe^{-kx}$, $x \in [0, infty)$.

iii) $f_k(x) = \frac{x}{k}e^{-\frac{x}{k}}$, $x \in [0,\infty)$.

iv) $f_k(x) = \frac{x^k}{1+x^{2k}}$, $\forall x \in [0,\infty)$.

Proof of i. We see that at $x = 0$, then $\forall k$, $x^{\frac{1}{k}} = 0$. For $x \in (0,1]$,

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} e^{\log f_k(x)} = \lim_{k\to\infty} e^{\log x^{1/k}} = \lim_{k\to\infty} e^{\frac{1}{k}\log x} = e^0 = 1.$$

So $f_k \to \begin{cases} 0, & x = 0, \\ 1, & x \in (0,1]. \end{cases}$

We will now check if f_k converges uniformly. Already, since f_k converges to a discontinuous function, then f_k will not converge uniformly.

We see that the convergence of $x^{\frac{1}{k}}$ to 1 will have k depend on x. If we fix $k =$ so that $\sqrt[k]{x} = 1$, then we can consider $\frac{x}{2}$ where $\left(\frac{x}{2}\right)^{1/k} = \left(\frac{1}{2}\right)^{1/k} < 1$. Thus, k depends on x so $\{f_k\}$ is not uniformly convergent.

Now, consider $\forall \epsilon > 0$, $x \in [\epsilon, 1]$,

$$\sup_{x\in[\epsilon,1]} |f_k(x) - f(x)| = \sup_{x\in[\epsilon,1]} |x^{\frac{1}{k}} - 1|.$$

We can fix k such that $\epsilon^{\frac{1}{k}} = 1$, and this same k will work for all $\epsilon < x \leq 1$ as well. □

Proof of ii. First, we see that

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} e^{\log\left(kxe^{-kx}\right)} = \lim_{k\to\infty} e^{-kx\log(kxe)} = 0.$$

For convergence, we will find the maximum of f_k.

$$f_k'(x) = ke^{-kx}(1-kx) = 0 \implies x = \frac{1}{k}.$$

So, the max of f_k is $f_k\left(\frac{1}{k}\right) = e^{-1}$.

Note that, as $k \to \infty$, $\frac{1}{k} \to 0$.

So, we have that, as $k \to \infty$,

$$\sup_{x\in[\epsilon,\infty)} \left|kxe^{-kx}\right| \to 0$$

where $\epsilon > \frac{1}{k} \to 0$ as we have a fixed k. ☐

Proof of iii. We see that, as $k \to \infty$, $f_k \to 0$. Then, for the max of f_k,

$$f_k'(x) = \frac{1}{k}e^{-\frac{x}{k}}\left(1 - \frac{x}{k}\right) = 0 \implies x = k.$$

Since $f_k(0) = 0$, $f_k \to 0$, and $f_k > 0$ for all x in the given domain, then the critical point $x = k$ is the supremum of $\frac{x}{k}e^{-\frac{x}{k}}$ will occur at $x = k$.

But, as $k \to \infty$,

$$\sup_{x\in[0,\infty)} \left|\frac{x}{k}e^{-\frac{x}{k}}\right| \to 0.$$

So, $f_k \rightrightarrows 0$. ☐

Proof of iv. Note that $f_k(0) = 0$. Then, as $k \to \infty$,

$$\begin{aligned} x < 1 &\implies f_k(x) \approx x^k \to 0, \\ x = 1 &\implies f_k(1) = \tfrac{1}{2}, \\ x > 1 &\implies f_k(x) \approx x^{-k} \to 0. \end{aligned}$$

So, we see that f_k will converge uniformly on $x \in \left[0, \frac{1}{2}\right)$ and $x \in \left(\frac{1}{2}, \infty\right)$. ☐

Problem 2. Test for absolute and uniform convergence. State the intervals of convergence. Check for the continuity of the sum.

i) $\sum_0^\infty \frac{x^n}{n^2+n+1}$.

ii) $\sum_1^\infty \frac{\cos nx}{n^3}$.

iii) $\sum_1^\infty n^{-x}$.

Proof of i. Let $f_n(x) = \frac{x^n}{(n+1)^2}$. Then, $\forall |x| \leq 1$, $|f_n(x)| \leq \frac{1}{(n+1)^2}$.

Since $\sum_0^\infty \frac{1}{(n+1)^2}$ converges, then f_n will converge absolutely and uniformly by Weierstrass.

Since $f_n \rightrightarrows f$ and f_n is a polynomial and therefore continuous, then f is continuous by Theorem 7.8. ☐

Proof of ii. Note that,

$$\forall x \in \mathbb{R}, \left|\frac{\cos nx}{n^3}\right| \leq \frac{1}{n^3}.$$

Since $\sum \frac{1}{n^3}$ converges, then (ii) will converge absolutely and uniformly by Weierstrass.

Since (ii) converges uniformly and $\frac{\cos nx}{n^3}$ is continuous for all n, then the sum is also continuous by Theorem 7.8. □

Proof of iii. We need $n^{-x} \to 0$ as $n \to \infty$. This occurs when $x > 1$ by comparison to

$$\int_1^\infty x^{-p}\, dx,$$

which converges $\forall x > 1$ and $p > 1$.

So, we have that (iii) converges when $x > 1$. Then, $|n^{-x}| = \frac{1}{n^x} \to 0$ as $n \to \infty$, we also have that (iii) converges absolutely and uniformly by Weierstrass.

Since we have uniform converge and n^{-x} is continuous for all n, x, then the sum is also continuous by Theorem 7.8. □

Problem 3. Let $f_k(x) = g(x)x^k$ where g is continuous on $[0, 1]$ and $g(1) = 0$. Show $f_k \rightrightarrows 0 on [0, 1]$.

Proof. Note that $\forall x \in [0, 1)$, $x^k \to 0$.

We have that $f_k(1) = 0$. So we can also write $\forall x \in [0, 1]$, $f_k \to 0$.

For uniform convergence, we wish to show that, as $k \to \infty$,

$$\sup_{x\in[0,1]} f_k(x) - 0| = \sup_{x\in[0,1]} \left|g(x)x^k\right| \to 0.$$

Since g is continuous on a closed interval, then g is bounded.

Thus, $\exists M > 0 \ : \ \forall x \in [0, 1]$, $|g(x)| \leq M$.

So,

$$\sup_{x\in[0,1]} \left|g(x)x^k\right| \leq M \sup_{x\in[0,1]} x^k.$$

But, $x^k = 1$ when $x = 1$ for all k, so we will consider the above supremum on two intervals, $[0, 1 - \delta]$ and $[1 - \delta, 1]$.

Since $1 - \delta < 1$, then $M \sup_{x\in[0,1-\delta]} x^k = M(1 - \delta)^k \to 0$ as $k \to \infty$.

Next, since $\sup_{[1-\delta,1]} x^k = 1$, g continuous, and $g(1) = 0$, then $\forall \epsilon > 0$, $\exists \delta \ : \ \forall x \in [1 - \delta, 1]$, $|g(x)| < \epsilon$.

Then, $\sup_{[1-\delta,1]} \left|g(x)x^k\right| \leq \delta$.

But δ was arbitrary, so this vanishes as $k \to \infty$.

Hence,

$$\sup_{x\in[0,1]} f_k(x) \to 0 \text{ as } k \to \infty \implies f_k \rightrightarrows 0.$$

□

Problem 4. Show that $\sum_1^\infty \frac{1}{x^2-n^2}$ converges uniformly on any compact interval that does not contain $x \in \mathbb{Z} \setminus \{0\}$. Conclude that the sum is continuous on $\mathbb{R} \setminus (\mathbb{Z} \setminus \{0\})$.

Proof. First, note that, if $n = x_{\in}\mathbb{Z} \setminus \{0\}$, then $f_n(n) = \frac{1}{n^2-n^2}$, which is not defined.

Then, for a fixed $0 < \epsilon < \frac{1}{2}$, we will consider x in $I = [k+\epsilon, k+1-\epsilon]$, $\forall k \in \mathbb{Z}$, or $x \in [-1+\epsilon, 1-\epsilon]$.

We wish to find a convergent sequence M_n which bounds $f_n(x)$ for all n.

We see that, when x is close to n, then $x^2 - n^2$ will be small, so f_n will be large.

We will consider f_n for n in three cases, $n < k$, $n = k$ (the important case), and $n > k$.

First, $n < k \implies 1 < k^2 - n^2 < x^2 - n^2$. So $\frac{1}{|x^2-n^2|} < 1$.

Next, $n = k \implies (n+\epsilon)^2 \leq x^2 \implies 2n\epsilon + \epsilon^2 \leq x^2 - n^2$. So, $\frac{1}{|x^2-n^2|} \leq \frac{1}{2n\epsilon - \epsilon^2}$ for some fixed epsilon.

Finally, $n > k \implies x^2 < n^2$. Then $|x^2 - n^2| \geq n^2 - (k+1)^2 \approx n^2$ for large n.

So $\frac{1}{|x^2-n^2|} \leq \frac{1}{n^2-(k+1)^2} \approx \frac{1}{n^2}$.

But, we only have finitely many finite terms of M_n where $n \leq k$.

Since the tail $\sum_{k+1}^\infty \frac{1}{n^2}$ converges, then the entire $\sum M_n$ will converge as well.

Thus, f_n is uniformly convergent by Weierstrass. $\square$

Problem 5. Prove $\sum_1^\infty \frac{(-1)^{n-1}}{x^2+n}$ converges uniformly on $x \in \mathbb{R}$ but conditionally at all points.

Proof. By the Alternating Series Test, the series will converge if $\frac{1}{x^2+n} \to 0$ as $n \to \infty$.

Then, $\forall x \in \mathbb{R}$, $\lim_{n\to\infty} \frac{1}{x^2+n} = 0$.

So the series converges.

But, the absolute value of the terms in the series are $\frac{1}{x^2+n} \approx O\left(\frac{1}{n}\right)$ on the order of $\frac{1}{n}$, which diverges. So, the series converges conditionally and not absolutely at any given x.

Next, for uniform convergence, we will consider

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{(-1)^{n-1}}{x^2+n} - \right| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Thus, the series converges uniformly. $\square$

Problem 6. Let f_k be continuous on $[a,b]$. Prove $f_k \rightrightarrows$ on $(a,b) \implies f_k \rightrightarrows f$ on $[a,b]$.

Proof. We will consider the endpoints $x = a, b$.

$f_k \rightrightarrows f$ on $(a, b) \implies \forall \epsilon, \exists K \ : \ \forall x \in (a, b), \forall k > K,\ |f_k(x) - f(x)| < \epsilon$.

f_k continuous $\implies |f_k(a) - f_k(x)| < \frac{\epsilon}{2} \forall x \in [a, a+\delta] \forall \delta > 0$.

Similarly, continuity gives $|f_k(b) - f_k(x)| < \frac{\epsilon}{2} \forall x \in [b - \delta, b] \forall \delta > 0$.

Since $a + \delta \in (a, b)$, then $\forall k > K, \forall x \in (a, b)$,

$$|f(x) - f_k(x)| + |f_k(x) - f_k(a)| < \epsilon \implies |f(x) - f_k(a)| < \epsilon \implies f_k \rightrightarrows f$$

at $x = a$.

We can repeat the same for $x = b$ to conclude that $f_k \rightrightarrows f$ on $[a, b]$. □

Problem 7. Let $f(x) = \sum_1^\infty \frac{\sin nx}{n^2}$. Prove f continuous on $\mathbb{R}$ and

$$\int_0^{\frac{\pi}{2}} f(x)\, dx = \forall k \in \mathbb{Z}_{\geq 0}, \sum_{n=2k+1}^{\infty} \frac{1}{n^3} + 2 \sum_{n=4k+2}^{\infty} \frac{1}{n^3}.$$

Proof of continuity. Since sine is bounded by 1, then f_n converges uniformly by Weierstrass with $M_n = \frac{1}{n^2}$.

Since f_n is continuous $\forall x \in \mathbb{R}$ and f_n converges uniformly, then f is continuous on $\mathbb{R}$ by Theorem 7.10. □

Proof of sum identity. Note that $\cos\left(\frac{\pi}{2} n\right) = \forall k \in \mathbb{Z}_{\geq 0}, \begin{cases} 0, & n = 2k+1, \\ 1, & n = 4k, \\ -1, & n = 4k+2. \end{cases}$

Since f_n is continuous an uniformly convergent, then, by Theorem 7.13,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \left[\sum \frac{\sin nx}{n^2} \right] dx &= \sum \int_0^{\frac{\pi}{2}} \frac{\sin nx}{n^2}\, dx \\ &= \sum_1^\infty \left[\frac{-\cos nx}{n^3} \right]_{x=0}^{x=\pi/2} \\ &= \sum_1^\infty \left(\frac{1 - \cos\left(\frac{n\pi}{2}\right)}{n^3} \right). \end{aligned}$$

Then, with the cosine table table above, we have that the above sum becomes,

$$\sum_{k=1}^{\infty} \left[\frac{1}{(2k+1)^3} + \frac{2}{(4k+2)^3} \right]$$

as desired. □

Problem 8. Let $f_k(x) = x \arctan kx$. Prove the following:

a) $\lim_{k\to\infty} f_k(x) = \frac{\pi}{2}|x|$.

b) $\exists \lim_{k\to\infty} f_k'(x)$ for all x including zero, but that f_k' does not converge uniformly on any interval containing zero.

Proof of a. Note that $\lim_{y\to\infty} \arctan y = \frac{\pi}{2}$ and $\lim_{y\to-\infty} \arctan y = -\frac{\pi}{2}$.

So, $\lim_{k\to\infty} \arctan kx = \begin{cases} \frac{\pi}{2}, & x > 0, \\ 0, & x = 0, \\ -\frac{\pi}{2}, & x < 0. \end{cases}$

Multiplying by x will produce a positive produce for all x, which is $\frac{\pi}{2}|x|$ as desired. □

Proof of b. Note that $f_k'(x) = \arctan kx + \frac{kx}{1+(kx)^2}$.

So, $f_k \in C^1$. We also have that f_k converges at pointwise from (a).

We wish to show that f_k' converges for every x, but only uniformly for intervals not containing zero.

We have that, $\forall x > 0$, $\lim_{k\to\infty} f_k'(x) = \frac{\pi}{2} + O(1/kx) = \frac{\pi}{2}$.

Similarly, for $x < 0$, $\lim_{k\to\infty} f_k'(x) = -\frac{\pi}{2}$.

Also, for $x = 0$, $\lim_{k\to\infty} f_k'(0) = \arctan 0 + \frac{0}{1+0} = 0$.

So, we now have that $f_k'(x) \to \begin{cases} \frac{\pi}{2}, & x > 0, \\ 0, & x = 0, \\ -\frac{\pi}{2}, & x < 0, \end{cases}$ which supports the fact that $f(x) = \frac{\pi}{2}|x|$.

However, uniform convergence preserves continuity, so we cannot have $f_k'(x)$ continuous converge uniformly to a discontinuous function $f'(x)$.

Thus, $f_k'(x)$ cannot converge uniformly on an interval containing the discontinuity $x = 0$.

Then, we will show that $f_k'(x)$ converges uniformly for $x > 0$ or $x < 0$.

From above, we have

$$\lim_{k\to\infty} \sup_{x>0} \left| \arctan kx + \frac{kx}{1+(kx)^2} - \frac{\pi}{2} \right| \to 0.$$

So, $f_k'(x)$ converges uniformly for $x > 0$; this holds for $x < 0$ as well. □

Problem 9. $\forall x \notin \mathbb{Z} \setminus \{0\}$, let $f(x) = 2x \sum_1^\infty \frac{1}{x^2-n^2}$. Show $f \in C^1$ on the given domain and

$$f'(x) = -\sum_1^\infty \left[\frac{1}{(x-n)^2} + \frac{1}{(x+n)^2} \right].$$

Proof. We have that, for $x \neq n \in \mathbb{Z} \setminus \{0\}$,

$$f_n'(x) = \frac{2(x^2 - n^2) - 4x^2}{(x^2 - n^2)^2} = \frac{-2(x^2 + n^2)}{(x^2 - n^2)^2} = -\frac{(x-n)^2 + (x+n)^2}{((x-n)(x+n))^2} = -\left(\frac{1}{(x-n)^2} + \frac{1}{(x+n)^2}\right).$$

So, $f_n \in C^1$ on such a domain.

We have seen that $\sum_1^\infty \frac{1}{x^2 - n^2}$ will converge on this domain in Problem 4.

So, $2x \sum_1^\infty \frac{1}{x^2-n^2}$ will converge as well.

We now wish to show that $f_n'(x)$ converges uniformly to $-\left(\frac{1}{(x-n)^2} + \frac{1}{(x+n)^2}\right)$.

Consider $x \in [k + \epsilon, k + 1 - \epsilon]$, $\forall 0 < \epsilon < \frac{1}{2}, \forall k \in \mathbb{Z} \setminus \{0\}$ as in Problem 4.

Then, we will construct a bounding sequence $|f_n| \leq M_n$, $\forall n$ such that $\sum M_n$ converges.

Since $x \in [k + \epsilon, k + 1 - \epsilon]$, we have $|x| \leq \epsilon < 1$.

So, $|x \pm n| \geq n - |x| > n - 1$. Then, $(x \pm n)^2 > (n-1)^2$.

Thus, $\left|\frac{1}{(x+n)^2} + \frac{1}{(x-n)^2}\right| < \frac{2}{(n-1)^2} \approx 1/n^2$.

Since $\sum M_n = \sum \frac{2}{(n-1)^2}$ converges by comparison to $\sum n^{-2}$, then $f_n'(x)$ converges uniformly by Weierstrass.

Thus, by Theorem 7.13, we have that

$$f'(x) = -\sum_1^\infty \left[\frac{1}{(x-n)^2} + \frac{1}{(x+n)^2}\right],$$

which is what we wanted to prove. $\square$