Math 335 Homework 4

Alexandre Lipson

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Problem 1. Suppose the radius of convergence of $\sum_0^\infty a_n x^n$ is R. What is the radius of convergence of $\sum_0^\infty a_n x^{kn}$, $\forall k \in \mathbb{Z}, k \geq 2$.

Proof. Let $y = x^k$. Then the radius of convergence of $\sum_0^\infty a_n y^n$ is R. Hence,

$$|y| = |x^k| < R \implies |x| < R^{\frac{1}{k}}.$$

$\square$

Problem 2. Show that for all sequences a_n, the radius of convergence of $\sum_0^\infty a_n x^n = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1}$.

Proof. Consider the root test on power series. Let $L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. We wish to show that the radius of convergence $R = \frac{1}{L}$.

We have that,

$$\limsup_{n\to\infty} |a_n x^n|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} |x| = L|x|.$$

So, the given series converges when $L|x| < 1 \implies |x| < \frac{1}{L}$ by the root test.

But, we had $|x| < R$ as the radius of convergence as well.

Thus, we must have $R = \frac{1}{L}$. $\square$

Problem 3. Show that the following functions have a power series expansion centered at the origin. Find the expansion and give the interval of validity.

a) $\int_0^x e^{-t^2}\,dt$.

We have that e^x is uniformly convergent for all x. So,

$$e^{-t^2} = \sum_0^\infty \frac{(-t^2)^n}{n!} = \sum_0^\infty (-1)^n \frac{t^{2n}}{n!}.$$

Then, we can integrate termwise,

$$\sum_0^\infty \frac{(-1)^n}{n!} \int_0^x t^{2n}\,dt = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)n!},$$

which holds for all x.

b) $\int_0^x \cos t^2\, dt$.

Since $\cos x$ is uniformly convergent for all x, then

$$\cos t^2 = \sum_0^\infty (-1)^n \frac{t^{4n}}{(2n)!}.$$

So, we can integrate termwise. This gives, for all x,

$$\int_0^x \cos t^2\, dt = \sum_0^\infty \frac{(-1)^n}{(2n)!} \int_0^x t^{4n}\, dt = \sum_0^\infty \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}.$$

c) $\int_0^x t^{-1} \log(1+2t)\, dt$.

We have that $\log(1+x)$ is uniformly convergent for all $x \in (-1, 1]$. So,

$$t^{-1} \log(1+2t) = \sum_1^\infty \frac{(-1)^{n+1} 2^n t^{n-1}}{n}$$

is valid for all $t \in \left(-\frac{1}{2}, \frac{1}{2}\right]$.

Thus,

$$\int_0^x t^{-1} \log(1+2t)\, dt = \sum_1^\infty \frac{(-1)^{n+1} 2^n}{n} \int_0^x t^{n-1}\, dt = \sum_1^\infty \frac{(-1)^{n+1}(2x)^n}{n^2},$$

which is valid for all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ by comparison with n^{-2}.

Problem 4. Use series expansions from the previous problem to compute the following with an error less than 10^{-3}.

a) $\int_0^1 e^{-t^2}\, dt$.

We have that, $\forall x \in [0,1]$, $|R_k(x)| \leq \sup |f^{(k+1)}(x)| \frac{x^{k+1}}{(k+1)!}$.

As in Homework 1 Question 7, we will let $g(x) = e^{-x}$. Then, $\forall x \in [0,1]$,

$$|R_k(x)| \leq \left|\sup g^{(k+1)}(x)\right| \frac{x^{k+1}}{(k+1)!} \leq \frac{|\sup e^{-x}|}{(k+1)!} = \frac{1}{(k+1)!}.$$

Then,

$$\left|\int_0^1 \left(e^{-t^2} - P_k(t)\right) dt\right| \leq \int_0^1 |R_k(t^2)|\, dt \leq \int_0^1 \frac{dt}{(k+1)!} = \frac{1}{(k+1)!} \leq 10^{-3}.$$

So, we have that $(k+1)! \geq 10^3 \implies k \geq 7$. Then, using the 7^{th} order Taylor Expansion in a calculator, we have that, at $x = 1$,

$$\sum_0^7 \frac{(-1)^n}{(2n+1)n!} \approx 0.7468.$$

b) $\int_0^1 \cos t^2 \, dt$.

Similar to the above, $g(x) = \cos x$ has all derivatives bounded by 1. So, $|R_k(x)| \leq \frac{1}{(k+1)!}$ and $k \geq 7$ will work as well.

Then, at $x = 1$,

$$\sum_0^7 \frac{(-1)^n}{(4n+1)(2n)!} \approx 0.9045.$$

c) $\int_0^{\frac{1}{2}} t^{-1} \log(1+2t) \, dt$.

At $x = \frac{1}{2}$, we have $\sum_1^\infty \frac{(-1)^{n+1}}{n^2}$ [1]

Note that, by AST, the difference between the sum of the series and the N^{th} partial sum is bounded by $\frac{1}{(N+1)^2}$.

So,

$$\frac{1}{(N+1)^2} \leq 10^3 \implies N \geq 10^{\frac{3}{2}} - 1 \approx 31.$$

So, the 31^{st} partial sum will give an error within the desired bound,

$$\sum_1^{31} \frac{(-1)^{n+1}}{n^2} \approx 0.8230.$$

Problem 5. Let $f(x) = \sum_0^\infty a_n x^n$ with radius of convergence $R > 0$. For all x in the radius of convergence, show that

i) $f(-x) = f(x)$, f even iff $\forall a_n = 0$ when n is odd, and

ii) $f(-x) = -f(x)$, f odd iff $\forall a_n = 0$ when n is even.

Proof. Note that even functions satisfy $f(-x) = f(x)$ and have only even powers.

If a_n for all n odd, then the series must only have even powers. So, by definition, the series must be an even function satisfying $f(-x) = f(x)$.

Similarly, $f(-x) = -f(x)$ for odd functions, which only have odd powers.

If $a_n = 0$ for all even n, then the series must have only odd powers, so it must be an odd function where $f(-x) = -f(x)$ is satisfied. □

Problem 6. For $k \in \mathbb{Z}_{\geq 0}$, let the Bessel function of order k be defined as

$$J_k(x) = \sum_0^\infty \frac{(-1)^n}{n!(n+k)!} \left(\frac{x}{2}\right)^{2n+k}.$$

[1]This is Dirichlet $\eta(2) = \frac{\pi^2}{12}$.

a) Verify $J_k(x)$ converges for all x.

Using the ratio test, we see that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^2}{4(n+1)(n+k+1)}\right| = 0$$

for all fixed x since the ratio is of order $\frac{1}{n^2}$.

b) Show that $(x^k J_k(x))' = x^k J_{k-1}(x)$.

We have that

$$x^k J_{k-1}(x) = \sum_1^\infty \frac{(-1)^n x^{2n+2k-1}}{2^{2n+k-1} n!(n+k-1)!}.$$

We will take termwise derivatives to show that the above is the same as the left-hand side of the given equation.

$$\sum_0^\infty \frac{d}{dx}\frac{(-1)^n x^{2n+2k}}{2^{2n+k} n!(n+k)!} = \sum_1^\infty \frac{(-1)^n 2(n+k) x^{2n+2k-1}}{2^{2n+k} n!(n+k)!} = \sum_1^\infty \frac{(-1)^n x^{2n+2k-1}}{2^{2n+k-1} n!(n+k-1)!},$$

which indeed matches the above.

c) Show that $(x^{-k} J_k(x))' = -x^{-k} J_{k+1}(x)$.

First, we have that

$$-x^{-k} J_{k+1}(x) = \sum_0^\infty \frac{(-1)^{n+1} x^{2n+1}}{2^{2n+k+1} n!(n+k+1)!}.$$

Then, performing differentiation on the left hand side of the given equation,

$$\begin{aligned}
\sum_0^\infty \frac{d}{dx}\frac{(-1)^n x^2 n}{2^{2n+k} n!(n+k)!} &= \sum_1^\infty \frac{2n(-1)^n x^{2n-1}}{2^{2n+k} n!(n+k)!} \\
&= \sum_{n=1}^\infty \frac{(-1)^n x^{2n-1}}{2^{2n+k-1}(n-1)!(n+k)!} \\
&= \sum_{n=0}^\infty \frac{(-1)^{n+1} x^{2(n+1)-1}}{2^{2(n+1)+k-1}(n+1-1)!(n+k+1)!} \\
&= \sum_0^\infty \frac{(-1)^{n+1} x^{2n+1}}{2^{2n+k+1} n!(n+k+1)!},
\end{aligned}$$

which is the same as the above, so the given equation holds.

d) Show that $u = J_k(x)$ satisfies the differential equation

$$x^2 u'' + xu' + (x^2 - k^2)u = 0.$$

By part b, we have

$$\frac{d}{dx}[x^k J_k(x)] = kx^{k-1} J_k(x) + x^k J_k'(x) = x^k J_{k-1}(x).$$

This gives,

$$J_k'(x) = x^{-k}(x^k J_{k-1}(x) - kx^{k-1}J_k(x)) = J_{k-1}(x) - \frac{k}{x}J_k(x).$$

By part c, we have

$$\frac{d}{dx}[x^{-k}J_k(x)] = -kx^{-k-1}J_k(x) + x^{-k}J_k'(x) = -x^{-k}J_{k+1}(x).$$

This gives,

$$J_k'(x) = x^k(kx^{-k-1}J_k(x) - x^{-k}J_{k+1}(x)) = \frac{k}{x}J_k(x) - J_{k+1}(x).$$

Differentiating one form of $J_k'(x)$, we have

$$\begin{aligned} J_k''(x) &= J_{k-1}'(x) + \frac{k}{x^2}J_k(x) - \frac{k}{x}J_k'(x) \\ &= \frac{k-1}{x}J_{k-1}(x) - J_k(x) + \frac{k}{x^2}J_k(x) - \frac{k}{x}\left(J_{k-1}(x) - \frac{k}{x}J_k(x)\right) \\ &= -\frac{1}{x}J_{k-1}(x) + \frac{k^2+k-x^2}{x^2}J_k(x). \end{aligned}$$

Now, we wish to show that

$$x^2J_k''(x) = -xJ_k'(x) + (k^2 - x^2)J_k(x).$$

Using the derived equations above,

$$\begin{gathered} x^2\left(-\frac{1}{x}J_{k-1}(x) + \frac{k^2+k-x^2}{x^2}J_k(x)\right) = -x\left(J_{k-1}(x) - \frac{k}{x}J_k(x)\right) + (k^2-x^2)J_k(x) \\ -xJ_{k-1}(x) + (k^2+k-x^2)J_k(x) = -xJ_{k-1}(x) + (k^2+k-x^2)J_k(x), \end{gathered}$$

which is indeed true, so the differential equation holds.

Problem 7. Snow that the series

$$1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots + \frac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)}$$

converges for all x, and that the sum satisfied $f''(x) = xf(x)$.

Proof. Let $f(x) = \sum_0^\infty a_n x^{3n}$ where $a_0 = 1$ and $a_n = \left(\prod_1^n (3k-1)(3k)\right)^{-1}$.

By the ratio test, we have that

$$\left|\frac{a_{n+1}x^{3(n+1)}}{a_n x^{3n}}\right| = \left|\frac{a_{n+1}}{a_n}\right||x|^3.$$

Then,

$$a_{n+1} = \left(\prod_1^{n+1}(3k-1)(3k)\right)^{-1} = \frac{a_n}{(3(n+1)-1)(3(n+1))} = \frac{a_n}{(3n+2)(3n+3)}.$$

So,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \approx \lim_{n\to\infty} O\left(\frac{1}{9n^2}\right) = 0.$$

Thus, the series converges for all x.

Then we have $f''(x) = \sum_1^\infty (3n)(3n-1)a_n x^{3n-2}$ and $xf(x) = \sum_0^\infty a_n x^{3n+1}$. We will shift the index of the second series by 1, then we have $\sum_1^\infty a_{n-1}x^{3n-2}$.

But, $a_{n-1} = (3n)(3n-1)a_n$. So,

$$f''(x) = \sum_1^\infty (3n)(3n-1)a_n x^{3n-2} = xf(x).$$

□

Problem 8. Express the following series in terms of elementary functions and their antiderivatives.

a) $\sum_1^\infty \frac{nx^n}{(n+1)!}$.

$$\begin{aligned}\sum_0^\infty \frac{x^{n+1}}{(n+1)!} &= e^x - 1\\ \sum_0^\infty \frac{x^n}{(n+1)!} &= \frac{e^x-1}{x}\\ \sum_1^\infty \frac{nx^{n-1}}{(n+1)!} &= \left(\frac{e^x-1}{x}\right)'\\ \sum_{n=1}^\infty \frac{nx^n}{(n+1)!} &= x\left(\frac{e^x-1}{x}\right)'.\end{aligned}$$

Now,

$$x\left(\frac{e^x-1}{x}\right)' = x\left(\frac{xe^x-(e^x-1)}{x^2}\right) = \frac{xe^x-e^x+1}{x}.$$

b) $\sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+2)!}$.

$$\begin{aligned}\sum_0^\infty \frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!} &= \cos x - 1\\ \sum_0^\infty \frac{(-1)^n x^{2n}}{(2n+2)!} &= \frac{1-\cos x}{x^2}\\ \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+2)!} &= \int_0^x \frac{1-\cos t}{t^2}\,dt.\end{aligned}$$

c) $\sum_0^\infty \frac{x^n}{(n+1)^2 n!}$.

$$\sum \frac{x^n}{n!} = e^x$$
$$\sum \frac{x^{n+1}}{(n+1)n!} = \int_0^x e^t\,dt = e^x - 1$$
$$\sum \frac{x^n}{(n+1)n!} = \frac{e^x - 1}{x}$$
$$\sum \frac{x^n}{(n+1)^2 n!} = \frac{1}{x}\int_0^x \frac{e^t - 1}{t}\,dt.$$

Problem 9. Consider $f(x) = \int_0^x \arctan t\,dt$.

a) Integrate to evaluate f in terms of elementary functions.

By IBP,

$$\int_0^x \arctan t\,dt = \left[t\arctan t - \frac{1}{2}\log(1+t^2)\right]_0^x = x\arctan x - \frac{1}{2}\log(1+x^2).$$

b) Using the arctangent series, compute the Taylor Series of f and show that it converges when $|x| \le 1$.

We have that,

$$\sum_0^\infty \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan x,\ \forall |x| < 1.$$

Then,

$$\int_0^x \arctan t\,dt = \sum_0^\infty \int_0^x \frac{(-1)^n t^{n+1}}{2n+1}\,dt = \sum_0^\infty \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)}.$$

Note that, for all fixed $|x| \le 1$, this series is of order $\frac{1}{4n^2}$, which converges.

c) Conclude that

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

We have that

$$\int_0^1 \arctan t\,dt = \arctan 1 - \frac{1}{2}\log(1+1^2) = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

.

We also have that

$$\sum_0^\infty \frac{(-1)^n}{(2n+1)(2n+2)} = \sum_0^\infty (-1)^n\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots.$$

Thus, the statement holds.