Math 136 Homework 7

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1.

Problem. Let $g : \mathbb{R}^2 \longmapsto \mathbb{R}$ be the function defined by

$$g(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that

$$g_y(x,0) = \lim_{h\to 0} \frac{g(x,h)-g(x,0)}{h} = x$$

and similarly that $g_x(0,y) = -y$. Hence, show that $g_{yx}(0,0) = 1$ and $g_{xy}(0,0) = -1$.

First, we compute g_y with differentiation and evaluate at $y = 0$,

$$\begin{aligned} g_y(x,y) &= \frac{\partial}{\partial y}\left[\frac{xy(x^2-y^2)}{x^2+y^2}\right] \\ &= \frac{\left[x(x^2-y^2)\right](x^2+y^2) - \left[xy(x^2-y^2)\right](2y)}{(x^2+y^2)^2} \\ g_y(x,0) &= \frac{x^5}{x^4} \\ &= x. \end{aligned}$$

We then compute g_y using the limit definition at $y = 0$,

$$\begin{aligned} \lim_{h\to 0} \frac{g(x,h)-g(x,0)}{h} &= \lim_{h\to 0} \frac{\frac{xh(x^2-h^2)}{x^2+h^2} - \frac{x(0)(x^2-0^2)}{x^2+0^2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{xh(x^2-h^2)}{x^2+h^2}}{h} \\ &= \lim_{h\to 0} \frac{x(x^2-h^2)}{x^2+h^2} \\ &= \frac{x^3}{x^2} \\ &= x. \end{aligned}$$

So, $g_y(x,0) = x$ by both the limit definition and the derivative computation. For simplicity, we will use the latter method of finding the partials of g.

For $g_x(0, y)$,

$$\begin{aligned}
g_x(x, y) &= \frac{[y(x^2 - y^2) + 2x^2y](x^2 + y^2) - (xy(x^2 - y^2))[2x]}{(x^2 + y^2)^2} \\
g_x(0, y) &= \frac{y(-y^2)y^2}{y^4} \\
&= -y.
\end{aligned}$$

Then, for $g_{yx}(0, 0)$, we take the derivative of $g_y(x, 0)$ in x, $\frac{d}{dx}x = 1$.

For $g_{xy}(0, 0)$, similarly, we take the derivative of $g_x(0, y)$ in y, $\frac{d}{dx} - y = -1$.

2. (a) Recall and write down the definition of continuity for $f : \mathbb{R} \longmapsto \mathbb{R}$ at $x = a$.

   (b) Modify it using the norm on $\mathbb{R}^n$ (instead of absolute value) to write down the definition of continuity for $f : \mathbb{R}^n \longmapsto \mathbb{R}$ being continuous at $x = a$ in $\mathbb{R}^n$.

   (c) We say $f : \mathbb{R}^n \longmapsto \mathbb{R}$ is continuous if it is continuous at every $x \in \mathbb{R}^n$. Now assume $f : \mathbb{R}^n \longmapsto \mathbb{R}$ is a continuous function and let I be an open interval of the form $I = (a, b)$ where $a < b$. Show that the preimage of I

$$f^{-1}(I) = \{\vec{x} \in \mathbb{R}^n : f(\vec{x}) \in I\}$$

   is an open set. (This becomes an alternative definition of continuity in analysis or topology: We say $f : \mathbb{R}^n \longmapsto \mathbb{R}$ is continuous if the preimage of every open set in $\mathbb{R}^m$ is open in $\mathbb{R}^n$.)

   (a)

$$\forall \epsilon > 0, \exists \delta > 0 : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

   (b)

$$\forall \epsilon > 0, \exists \delta > 0 : \|\vec{x} - \vec{a}\| < \delta \implies |f(\vec{x}) - f(\vec{a})| < \epsilon.$$

   (c) Given $x \in f^{-1}(I)$, we wish to find a $\delta > 0$ such that we can form a δ-sized ball around $\vec{x}$ which is a subset of $f^{-1}(I)$. Then, for any vector $\vec{y}$ δ-close to $\vec{x}$, $f(\vec{y})$ will be in the open interval I

$$\|\vec{y} - \vec{x}\| < \delta \implies f(\vec{y}) \in I.$$

   Since f is continuous, we choose $\epsilon = \min\{f(\vec{x}) - a, b - f(\vec{x})\}$ where $a < f(\vec{x}) < b$ and are given the desired δ.