Math 462 Homework 5

Alexandre Lipson

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Problem 1. Let T be a tree which has a vertex of degree of d. Prove that T has at least d leaves.

Proof. Since T is a tree, then it is connected and acyclic. So, if we start at vertex v with degree d and follow any edge, we must eventually reach a leaf. Since T is a cyclic, all paths must terminate.

We have that v is connected to d subtrees. Since these are also acyclic, they must end in at least one leaf.

Specifically, $T - v$ has d connected components, each of which has at least one vertex and no cycles. If the component has exactly one vertex, then it is already a leaf adjacent to v. If the component has more than one vertex, then it has at least two leaves, where one of which is adjacent to v.

Thus, there must be at least d leaves in T because each of the d subtrees adjacent to v must contribute at least one leaf. $\square$

Problem 2. Let G be a connected graph, and let T and T' be spanning trees of G. Let e be an edge of T not in T'. Prove there is an edge e' of T' not in T such that $T - e + e'$ is a spanning tree of G.

Proof. Consider $T - e$. Deleting $e \in T$ from T disconnects T into two connected components which are also trees; label these components C_1 and C_2. Let the endpoints of e be $u \in C_1$ and $v \in C_2$.

Since T' is a tree, then between any two vertices in T', there is a unique path. So, there must be a unique path from u to v, call it P. Since u, v are in different components, then there must be an edge e' in P that connects the components.

If every edge of P were also in T, then adding e to that path would create a new path between u and v, and thus a cycle in T. But T is a tree and thus cannot have a cycle.

So, at least one edge of P is not in T, let this edge be e'. Then, by construction, we have that $e' \in T' \setminus T$.

Now, consider $T - e + e'$. $T - e$ had two connected components C_1 and C_2. But e' goes between C_1 and C_2, so $T - e + e'$ connects all vertices in G.

Since we have only replaced one edge with another, the total edge count remains the same. Therefore, $T - e + e'$ is a connected subtree of G with the same number of edges as a spanning tree T, thus it is a spanning tree as well. $\square$

Problem 3. Let $G = (V, E)$ be a connected graph with a weighting $w : E \to \mathbb{R}_{\geq 0}$ on its edges. Prove that every minimum spanning tree of G has the same multiset of edge weights.

Proof. Suppose $T = (V, E)$ and $T = (V, E')$ are two different minimum spanning trees (MST) of G. Then, they differ by at least one edge. Let $e \in E \setminus E'$ and $e' \in E' \setminus E$.

WE must be able to pick such edges since T and T' both span G, otherwise $T = T'$, so there is only one spanning which must already be minimum with one multiset.

By the announcement, both $T - e + e'$ and $T' - e' + e$ are spanning trees of G.

Since T and T' are both MSTs, then they must both have the same total weight, call it w.

$$w(E) = w(E') = w.$$

The new spanning trees cannot have a smaller weight than the MST weight w, otherwise T and T' would not have been MSTs. So, we have

$$w(E) \leq w(E - e + e') = w(E) - w(e) + w(e') \implies w(e) \leq w(e').$$

We also have

$$w(E') \leq w(E' - e' + e) = w(E') - w(e') + w(e) \implies w(e') \leq w(e).$$

Thus,

$$w(e) = w(e').$$

Hence, whenever T and T' differ by an edge, the differing edges must have the same weight. Repeating this for all differing edges implies that T and T' share the same multiset of weights.

Since T and T' were arbitrary MSTs, then every MST of G must have the same multiset of edge weights. □

Problem 4. Determine whether the graphs from Problems 2 and 3 of Homework 4 are bipartite.

Let $d_H(x, y)$ be the Hamming distance between two binary strings, which represents the number of bit positions in which x and y differ.

We can label each binary string with an even or odd parity based on the sum of bits mod 2 (0 is even parity and 1 is odd parity).

a) $G_1 = \{\{0,1\}^n, \{xy \mid x, y \in V,\ d_H(x, y) = 1\}\}$.

b) $G_2 = \{\{0,1\}^n, \{xy \mid x, y \in V,\ d_H(x, y) = 2\}\}$.

Proof of a. We claim that G is bipartite by vertex parity.

Since every edge connects vertices of opposite parity, we must have a bipartition $\{X, Y\}$ where X contains vertices of even parity and Y contains vertices of odd parity.

Since there are no edges between vertices of the same parity, we have established a bipartition. □

Proof of b. We will first consider $n = 1$. 0 and 1 are the only vertices. Since they do not differ by two bits, they have no edge between them and are thus bipartite.

For $n = 2$, we have that 01 and 10 are connected, as well as 00 and 11. So, we have a bipartition $\{\{01, 00\}, \{10, 11\}\}$ of G_2 when $n = 2$.

For $n \geq 3$, we have a three cycle or vertices that contain the following substrings: 001, 010, and 100, each of which differ by two bits, so they form an odd cycle. This set of substrings has the same parity, they must be contained in at least one of the connected components of G_2 when $n \geq 3$.

Since there is an odd cycle in G_2, then G_2 is not bipartite for $n \geq 3$. □

Problem 5. Prove that a connected bipartite graph has exactly one bipartition.

Note that the bipartition $\{X, Y\}$ is the same as $\{Y, X\}$ since we can just swap labels.

Proof. Suppose that there were two distinct set partitions that are not just exchanges as in the note.

Then, there must be at least one vertex v whose membership differs between the two set partitions; i.e., we can move v between partitions and we will still have a connected bipartite graph G.

Let $\{X, Y\}$ be a bipartition of $G - v$. Now, if we set $v \in X$, then there must be no edges between v and any $x \in X$. Similarly, if $v \in Y$, then there are no edges between v and any $y \in Y$.

But, since X and Y contain all vertices of $G - v$, then v is not connected to any other vertex of $G - v$, which contradicts the assumption that G was connected.

Thus, we cannot have two distinct set partitions of a connected graph. □

Problem 6. Let G be a bipartite graph with bipartition $\{X, Y\}$. Suppose every vertex of G has the same degree. Prove that $|X| = |Y|$.

Proof. Since each vertex has the same degree, call it d, then the sum of the degrees in X is $d|X|$. Since G is bipartite, then every vertex in X has an edge to a vertex in Y.

$$N(X) = Y.$$

Hence, the total number of edges is exactly this count,

$$|E| = d|X|.$$

By the same reasoning, $N(Y) = X \implies |E| = d|Y|$, which counts the same edges as before, just from the other side.

Thus,

$$d|X| = |E| = d|Y| \implies |X| = |Y|.$$

□