Math 136 Homework 6

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1.

Problem. Let A be a matrix such that A^*A is invertible. With $P_{\operatorname{Im} A}$ given by $A(A^*A)^{-1}A^*$, find $P_{\ker A}$, $P_{\ker A^*}$, and $P_{\operatorname{Im} A^*}$.

We can express any vector $z \in V$ uniquely by expressing its components as projections onto the linearly independent subspaces $E, E^\perp \subset V$,

$$z = P_E(z) + P_{E^\perp}(z).$$

Given the $P_{\operatorname{Im} A}$ and the above, we can find the projection onto $(\operatorname{Im} A)^\perp = \ker A^*$.

$$\begin{aligned} z &= P_{\operatorname{Im} A}(z) + P_{\ker A^*}(z) \\ z - P_{\operatorname{Im} A}(z) &= P_{\ker A^*}(z) \\ z(I - A(A^*A)^{-1}A^*) &= P_{\ker A^*}(z) \\ I - A(A^*A)^{-1}A^* &= P_{\ker A^*}. \end{aligned}$$

Then, we will consider the same with the linearly independent pair of orthogonal subspaces $\ker A$ and $(\ker A)^\perp = \operatorname{Im} A^*$. Since A^*A is invertible, $\ker A = \vec{0}$, then the image of any vector under the operator A projected into the kernel of A will be the zero vector. So,

$$\begin{aligned} z &= P_{\ker A} + P_{\operatorname{Im} A^*} \\ Az &= AP_{\ker A} + AP_{\operatorname{Im} A^*} \\ Az &= AP_{\operatorname{Im} A^*} \\ I &= P_{\operatorname{Im} A^*} \end{aligned}$$

and

$$0 = P_{\ker A}.$$

2.

Problem. Let $\langle \cdot\,,\,\cdot \rangle$ be a positive definite inner product on the vector space V. Let $L : V \longmapsto V$ be a linear transformation that satisfies the condition,

$$\langle u, L(v) \rangle = \langle L(u), v \rangle, \quad \forall u, v \in V.$$

Such an operator is said to be *self-adjoint*.

Let v_λ and v_μ be eigenvectors associated to the eigenvalues λ and μ of L, with $\lambda \neq \mu$.

Show that $v_\lambda \perp v_\mu$

Lemma. *Self-adjoint matrices have real eigenvalues.*

Proof. Let z be an eigenvector with eigenvalue λ.

First, we will use the left eigenvalue,

$$\begin{aligned}
\langle Lz, z\rangle &= \langle z, Lz\rangle \\
\langle \lambda z, z\rangle &= \langle z, \lambda z\rangle \\
\lambda \langle z, z\rangle &= \overline{\lambda} \langle z, z\rangle .
\end{aligned}$$

Since $z = \overline{z}$, then $z \in \mathbb{R}$. $\square$

Eigenvectors with different eigenvalues from a self-adjoint matrix are always orthogonal.

Since L is self adjoint, by the Lemma, we will only take out real eigenvalues from either position in the inner product.

So,

$$\begin{aligned}
\langle Lv_\mu, v_\lambda\rangle &= \langle v_\mu, Lv_\lambda\rangle \\
\langle \mu v_\mu, v_\lambda\rangle &= \langle v_\mu, \lambda v_\lambda\rangle \\
\mu \langle v_\mu, v_\lambda\rangle &= \lambda \langle v_\mu, v_\lambda\rangle \\
(\mu - \lambda) \langle v_\mu, v_\lambda\rangle &= 0, \qquad \mu \neq \lambda \\
\langle v_\mu, v_\lambda\rangle &= 0.
\end{aligned}$$

Therefore $v_\mu \perp v_\lambda$.

3.

Problem. Let V be the vector space of continuous functions on the closed interval $[-1, 1]$, with scalar product defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

(a) Apply the Gram-Schmidt orthogonalization process to the set $\left\{1, x, x^2, x^3\right\}$ to obtain an orthogonal set of four polynomials, $\{p_0(x), p_1(x), p_2(x), p_3(x)\}$.

(b) Verify that p_k is a solution of the differential equation

$$(1 - x^2)y'' - 2xy' + \lambda y = 0, \text{ with } \lambda = k(k + 1)$$

for $k = 0, 1, 2, 3$.

(a) Let $v_k = x^k$ for $k = 0, 1, 2, 3$.

Gram-Schmidt orthogonalization gives the following formula where P_{E_k} gives the projection onto the k^{th} subspace defined by the first k vectors from the p_k set,

$$p_{k+1} = v_{k+1} - P_{E_k} v_{k+1} = v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, p_i\rangle}{\langle p_i, p_i\rangle} p_i.$$

We let $p_1 = v_1 = 1$.

Then, $p_2 = v_2 - \frac{\langle v_2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1$.

But $\langle v_2, p_1 \rangle = \int_{-1}^{1} x \cdot 1 \, dx = 0$ given that x is odd.

So, $p_2 = x$.

Then, $p_3 = v_3 - \left(\frac{\langle v_3, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1 + \frac{\langle v_3, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2 \right)$.

For the first term in the sum, $\langle v_3, p_1 \rangle = \int_{-1}^{1} x^2 \cdot 1 \, dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$ and $\langle p_1, p_1 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2$ since the integrand is a rectangle.

But, $\langle v_3, p_2 \rangle = \int_{-1}^{1} x^2 \cdot x \, dx = 0$ since the integrand x^3 is odd, so the second term is zero.

So, $p_3 = x^2 - \frac{2/3}{2} \cdot 1 = x^2 - \frac{1}{3}$.

Then $p_4 = v_4 - \left(\frac{\langle v_4, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1 + \frac{\langle v_4, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2 + \frac{\langle v_4, p_3 \rangle}{\langle p_3, p_3 \rangle} p_3 \right)$.

But, the first and third terms become zero since $\langle v_4, p_1 \rangle$ and $\langle v_4, p_3 \rangle$ both produce integrands of odd functions by the integral definition of the inner product, and odd function have zero signed area under the curve on symmetrical regions like $[-1, 1]$. So, these terms reduce to zero.

So, we will consider the second term, where $\langle v_4, p_2 \rangle = \int_{-1}^{1} x^3 \cdot x \, dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5}$ and $\langle p_2, p_2 \rangle = \int_{-1}^{1} x \cdot x \, dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$.

Thus, $p_4 = x^3 - \left(\frac{2/5}{2/3} \right) x = x^3 - \frac{3}{5} x$.

(b) For $k = 0$, $y = p_0 = 1$, so the coefficient of the D^0 term is $\lambda = 0(0+1) = 0$.

So we will verify that the chosen $y = 1$, $y' = y'' = 0$ holds for the equation

$$(1 - x^2)y'' - 2xy' = 0.$$

Clearly, if both the first and second derivatives of y are zero, then the left side of the equation is zero and the identity holds.

For $k = 1$, $y = p_1 = x$ and $\lambda = 1(1+1) = 2$.

So, we get the equation

$$(1 - x^2)y'' - 2xy' + 2y = 0.$$

With $y' = 1$ and $y'' = 0$, we need only to compare the D^1 and D^0 terms,

$$(1 - x^2)(0) - 2x(1) + 2(x) = 0.$$

We quickly see that this equation holds.

For $k = 2$, we get $\lambda = 2(2+1) = 6$, $y = p_2 = x^2 - \frac{1}{3}$, $y' = 2x$, and $y'' = 2$.

We will check if the following holds:

$$(1-x^2)(2) - 2x(2x) + 6\left(x^2 - \frac{1}{3}\right) = 0.$$

We verify by expanding,

$$2 - 2x^2 - 4x^2 + 6x^2 - 2 = 0.$$

We see that this is zero because of term cancellation.

Finally, for $k = 3$, we get $\lambda = 3(3+1) = 12$, $y = p_3 = x^3 - \frac{3}{5}x$, $y' = 3x^2 - \frac{3}{5}$, and $y'' = 6x$.

Then, we will simply the following and see that it holds:

$$\begin{aligned}(1-x^2)(6x) - 2x\left(3x^2 - \frac{3}{5}\right) + 12\left(x^3 - \frac{3}{5}x\right) &= 0\\ 6x - 6x^3 - 6x^3 + \frac{6}{5}x + 12x^3 - \frac{36}{5}x &= 0\\ \frac{30}{5}x + \frac{6}{5}x - \frac{36}{5}x - 2\cdot 6x^3 + 12x^3 &= 0\\ &.\end{aligned}$$

Again, we see that, through cancellation of terms, the differential equation condition holds.

So, p_0, p_1, p_2, p_3 satisfy the given differential equation.