Math 334 Homework 7

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Problem (1)**.** Show $\chi_{\mathbb{Q}}$ is not Reimann integrable on [0,1].

Proof. Consider any partition P of the closed unit interval. Then, the lower sums $L_P\chi_{\mathbb{Q}}$ will be equal to infimum of zero, while the upper sums $U_P\chi_{\mathbb{Q}}$ will be equal to the supremum of one. Since the upper and lower sums do not converge, then the indicator function for the rationals is not Reimann integrable.

Furthermore, from class, we have that $\chi_{\mathbb{Q}}$ is not Jordan measurable as the boundary $\partial\chi_{\mathbb{Q}} = \{x \in [0,1],\, x \in \mathbb{R} \setminus \mathbb{Q}\}$ does not have measure zero. Since the indicator set is not Jordan measurable, is it also not Reimann integrable. □

Problem (2)**.** Let $\mathcal{C}$ be the Cantor set where $\mathcal{C}_0 = [0,1]$, $C_{i+1} = \frac{1}{3}\mathcal{C}_i \cup \left(\frac{1}{3}\mathcal{C}_i + \frac{2}{3}\right)$ (which removes the middle third of each interval in the previous set), and $\mathcal{C} = \bigcap_{i=0}^{\infty} \mathcal{C}_i$.

Prove that the indicator χ_C is integrable on [0,1].

Proof. We will show that $\mathcal{C}$ is both closed and has measure zero, and then conclude that $\chi_{\mathcal{C}}$ must be integrable as the measure zero set $\mathcal{C}$ will not affect integrability.

First, since each $\mathcal{C}_i$ is the finite union of closed sets, then each $\mathcal{C}_i$ is closed. Then, as the intersection of only closed sets, $\mathcal{C}$ must also be closed.

Second, the length of each $\mathcal{C}_k$ is $\left(\frac{2}{3}\right)^k$ since one third of the length of every interval is removed at each iteration. So, $\forall\epsilon > 0$, $\exists K$ such that $\operatorname{len}(\mathcal{C}_K) = \left(\frac{2}{3}\right)^K < \epsilon$. So, $\mathcal{C}_K$ can be covered by an open set with total length less than epsilon. But, $\forall k$, $\mathcal{C} \subset \mathcal{C}_k$ since $\mathcal{C}$ is an intersection of at least $\mathcal{C}_k$, so any cover of $\mathcal{C}_k$ will also cover $\mathcal{C}$. Thus, C has measure zero.

Since $\chi_{\mathcal{C}}$ is bounded on [0,1] and continuously zero on $[0,1] \setminus \mathcal{C}$, then it is integrable on [0,1]. □

Problem (3)**.** Let $f : [0,1]^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{1}{y^2}, & 0 < x < y < 1, \\ -\frac{1}{x^2}, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show f not integrable on the square $[0,1]^2$.

(b) Show, for any fixed y, that $x \mapsto f(x,y)$ is integrable on $[0,1]$, and, similarly, that, for any fixed x, $y \mapsto f(x,y)$ is integrable on [0,1].

(c) Show by explicit computation that

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy \neq \int_0^1 \int_0^1 f(x,y)\,dx\,dy.$$

Proof of a. Since f is unbounded, it cannot have convergent upper and lower sums. In particular, around the origin, the infimum of f is $-\infty$, while the supremum is ∞. □

Proof of b, c. First, we will fix y to find the function $g(x) = \int_0^1 f(x,y)\,dy$.

$$\begin{aligned} g(x) &= \int_0^1 -\frac{1}{x^2}\,dy + \int_x^1 \frac{1}{y^2}\,dy \\ &= -\frac{1}{x} + 1 + \frac{1}{x} \\ &= 1. \end{aligned}$$

Next, we will fix x to find $h(y) = \int_0^1 f(x,y)\,dx$.

$$\begin{aligned} h(y) &= \int_0^1 \frac{1}{x^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx \\ &= \frac{1}{y} - 1 + \frac{1}{y} \\ &= -1. \end{aligned}$$

So, we see that both $f(x,y)$ can be integrated for fixed x or y respectively, but that these sums are not the same, thus the statement in (c) holds. □

Problem (4)**.** Let E be the square pyramid in $\mathbb{R}^3$ with vertices at $(\pm1,\pm1,0)$ and apex (0,0,1). Find $\iiint_E f\,dV$ where $f(x,y,z) = e^{x+y}$.

First, we note that f is symmetrical about the plane $x = y$, so we will evaluate f on half of E divided by this plane, and then double the result.

We will integrate over the half of E where $y < x$ by slicing out triangles of E in the z axis. So, we have the range for $z \in [0,1]$. We will further slice these triangular sections in the y axis. Thus, we wish to find bounds for x which depend on z, and bounds for y which depend on both x and z.

Since E is a square pyramid with apex (0,0,1) and edges along $(\pm1,\pm1,0)$, then its sides have a slope of 1. So, considering the projection of E in y-z plane, we see that $y(z)$ (for y¡x) is given by $y = z - 1$. Thus, we have that $y \in [z-1, x]$.

Similarly, for the projection of E in the x-z plane, we have the range of $x \in [z-1, 1-z]$. So, we can construct our iterated integral, $2\int_0^1 \int_{z-1}^{1-z} \int_{z-1}^{x} f(x,y,z)\,dy\,dx\,dz$.

Finally, we will compute using the given f,

$$\begin{aligned}
\iiint_E e^{x+y}\,dV &= 2\int_0^1\int_{z-1}^{1-z}\int_{z-1}^{x} e^{x+y}\,dy\,dx\,dz \\
&= 2\int_0^1\int_{z-1}^{1-z} e^x\left[e^y\right]_{y=z-1}^{y=x}\,dx\,dz \\
&= 2\int_0^1\int_{z-1}^{1-z} e^{2x}-e^{x+z-1}\,dx\,dy \\
&= 2\int_0^1\left[\frac{e^{2x}}{2}-e^{x+z-1}\right]_{x=z-1}^{x=1-z}dz \\
&= 2\int_0^1\left(\frac{e^{2(1-z)}}{2}-1\right)-\left(\frac{e^{2(z-1)}}{2}-e^{2(z-1)}\right)dz \\
&= 2\int_0^1\frac{e^{2(1-z)}}{2}-1+\frac{e^{2(z-1)}}{2}\,dz \\
&= \int_0^1 e^{2(1-z)}+e^{2(z-1)}-2\,dz \\
&= \left[-\frac{e^{2(1-z)}}{2}+\frac{e^{2(z-1)}}{2}-2z\right]_0^1 \\
&= -2-\left(-\frac{e^2}{2}+\frac{e^{-2}}{2}\right) \\
&= e^2/2-2-e^{-2}/2\approx 1.6.
\end{aligned}$$

Problem (5)**.** Rewrite $\int_0^2\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$ in the order $dx\,dy\,dz$.

Since x and y depend solely on one other variable, we can construct the reordered integral by looking at the y-z and x-y planes.

First, for the x-y plane, we have that $0<y<x^2$ and $0<x<2$. We wish to have x depend on y instead as dx is on the inside of dy in the desired order. So, we can rearrange these to give $\sqrt{y}<x<2$ and $0<y<4$.

But, we wish to have y dependent on z. So, for the y-z plane, we see that $0<z<y$. Then, switching the order and recalling that $0<y<4$, we get $z<y<4$ and $0<z<4$.

So, we can construct our iterated integral as

$$\int_0^4\int_z^4\int_{\sqrt{y}}^2 f(x,y,z)\,dx\,dy\,dz.$$