

Math 335 Homework 5

a lipson

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For Problems 1, 2, and 3, find the Fourier series of the 2π -periodic function $f(x)$ on $(-\pi, \pi)$.

Problem 1. (i) The square wave $f(x) = \begin{cases} -1 & (-\pi, 0), \\ 1 & (0, \pi). \end{cases}$

(ii) $f(x) = \sin^2 x$.

Proof of (i). We have that f is an odd function, so $a_n = 0$ for all n .

We will proceed to find the b_n Fourier coefficients,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx &&= \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx - \frac{1}{\pi} \int_{-\pi}^0 \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx \\ &= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left(\frac{1 - \cos n\pi}{n} \right), \end{aligned}$$

which vanishes for all even n , and is $\frac{4}{n\pi}$ for all odd n .

Thus, we have that, for n odd,

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin nx}{n},$$

which is

$$\sum_{k=0}^{\infty} \frac{4 \sin(2k+1)x}{\pi(2k+1)}.$$

□

Proof of (ii). We have that

$$f(x) = \sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2} \cos 2x.$$

□

Problem 2. (i) $f(x) = e^{bx}$, $b > 0$.

(ii) $f(x) = x(\pi - |x|)$.

Proof of (i). We will consider the complex Fourier coefficient,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(b-in)x} dx \\ &= \frac{1}{\pi} \left[\frac{e^{(b-in)x}}{b-in} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi(b-in)} \left(e^{(b-in)\pi} - e^{-(b-in)\pi} \right) \\ &; \end{aligned}$$

by Euler's identity, at $x = \pi$, we have that $e^{\pm in\pi} = (-1)^n$. So, with the hyperbolic sine identity $2 \sinh b\pi = e^{b\pi} - e^{-b\pi}$, the above becomes,

$$\frac{(-1)^n \sinh b\pi}{\pi(b-in)}.$$

Thus, we have

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{\sinh b\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n}{b-in} e^{inx}.$$

□

Proof of (ii). Since f is an odd function, then $a_n = 0$ for all n .

So,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x(\pi - |x|) \sin nx dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 x(\pi - (-x)) \sin nx dx + \frac{1}{\pi} \int_0^{\pi} x(\pi - x) \sin nx dx \\ &= \frac{1}{\pi} \int_{\pi}^0 (-u)(\pi - u) \sin(-nu)(-1) du + \frac{1}{\pi} \int_0^{\pi} u(\pi - u) \sin nu du \\ &= \frac{1}{\pi} \int_0^{\pi} u(\pi - u)(\sin nu - \sin(-nu)) du \\ &= \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx dx \\ &= 2 \int_0^{\pi} x \sin nx dx - \frac{2}{\pi} \int_0^{\pi} x^2 \sin nx dx \\ &= 2 \left(\frac{-x \cos nx}{n} \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx \right) - \frac{2}{\pi} \left(\frac{-x^2 \cos nx}{n} \Big|_0^{\pi} + \int_0^{\pi} \frac{2x \cos nx}{n} dx \right) \\ &= 2 \left(\frac{-\pi \cos n\pi}{n} + \left[\frac{\sin nx}{n} \right]_0^{\pi} \right) - \frac{2}{\pi} \left(\frac{-\pi^2 \cos n\pi}{n} + 2 \left(\frac{x \sin nx}{n^2} \Big|_0^{\pi} + \int_0^{\pi} \frac{\sin nx}{n^2} dx \right) \right) \\ &= 2 \left(\frac{\sin n\pi}{n^2} - \frac{\pi \cos n\pi}{n} \right) + \left(\frac{2\pi \cos n\pi}{n} - \frac{4}{\pi} \left(\frac{\pi \sin n\pi}{n^2} + \left[\frac{\cos nx}{n^3} \right]_0^{\pi} \right) \right) \\ &= \frac{2 \sin n\pi}{n^2} - \frac{2\pi \cos n\pi}{n} + \frac{2\pi \cos n\pi}{n} - \frac{4 \sin n\pi}{n^2} - \frac{4}{\pi} \left(\frac{\cos n\pi - 1}{n^3} \right) \\ &= -\frac{2 \sin n\pi}{n^2} - \frac{4}{\pi} \left(\frac{\cos n\pi - 1}{n^3} \right). \end{aligned}$$

This quantity vanishes for even n , and $b_n = \frac{8}{\pi n^3}$ for odd n .

So, for odd n ,

$$f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^3},$$

which is

$$\frac{8}{\pi} \sum_0^{\infty} \frac{\sin(2k+1)x}{(2k+1)^3}.$$

□

Problem 3. (i) $f(x) = \begin{cases} \frac{1}{a} & |x| < a, \\ -\frac{1}{\pi-a} & a < |x| < \pi, \end{cases}$ for $a \in (0, \pi)$.

The values of f are chosen to make the area under the curve of f on $[0, a]$ and $[a, \pi]$ both equal to one.

(ii) $f(x) = \begin{cases} \frac{a-|x|}{a^2} & |x| < a, \\ 0 & a < |x| < \pi, \end{cases}$ for $a \in (0, \pi)$.

The constraints on f are chosen such that the area under the triangle in the graph of f is equal to one.

Proof of (i). Since f is an even function, then $b_n = 0$ for all n .

Since the graph of f has equal positive and negative areas between the x -axis (both one) between 0 and π , and f is an even function, then

$$\frac{1}{2}a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = 0.$$

Next,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{-a} -\frac{1}{\pi-a} \cos nx dx + \frac{1}{\pi} \int_a^{\pi} -\frac{1}{\pi-a} \cos nx dx + \frac{1}{\pi} \int_{-a}^a \frac{1}{a} \cos nx dx \\ &= \frac{1}{\pi} \int_{\pi}^a -\frac{1}{\pi-a} \cos(-nx)(-1) dx + \frac{1}{\pi} \int_a^{\pi} -\frac{1}{\pi-a} \cos nx dx + \frac{1}{a\pi} \left[\frac{\sin nx}{n} \right]_{-a}^a \\ &= \frac{1}{\pi} \int_a^{\pi} -\frac{1}{\pi-a} (\cos nx + \cos nx) dx + \frac{1}{a\pi n} (\sin na - \sin(-na)) \\ &= -\frac{2}{\pi(\pi-a)} \left[\frac{\sin nx}{n} \right]_a^{\pi} + \frac{2 \sin na}{a\pi n} \\ &= -\frac{2}{\pi(\pi-a)} \left(\frac{\sin n\pi - \sin na}{n} \right) + \frac{2 \sin na}{a\pi n} \\ &= \frac{2 \sin na}{\pi(\pi-a)n} + \frac{2 \sin na}{a\pi n} \\ &= \frac{2a \sin na + 2(\pi-a) \sin na}{a\pi(\pi-a)n} \\ &= \frac{(2a + 2\pi - 2a) \sin na}{a\pi(\pi-a)n} \\ &= \frac{2 \sin na}{a(\pi-a)n}. \end{aligned}$$

Thus,

$$f(x) = \frac{2}{a(\pi - a)} \sum_{n=1}^{\infty} \frac{\sin na}{n} \cos nx.$$

□

Proof of (ii). Since f is even, then $b_n = 0$ for all n .

Since the area under the curve of the graph of f on $(-\pi, \pi)$ is one, then

$$\frac{1}{2}a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi}.$$

Next,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \int_{-a}^a \frac{a - |x|}{a^2} \cos nx dx \\ &= \frac{2}{\pi} \int_0^a \frac{a - x}{a^2} \cos nx dx \\ &= \frac{2}{\pi} \int_0^a \left(\frac{\cos nx}{a} - \frac{x \cos nx}{a^2} \right) dx \\ &= \frac{2}{\pi} \left[\frac{\sin nx}{an} \Big|_0^a - \left(\frac{x \sin nx}{a^2 n} \Big|_0^a - \int_0^a \frac{\sin nx}{a^2 n} dx \right) \right] \\ &= \frac{2}{\pi} \left[\frac{\sin na}{an} - \frac{a \sin na}{a^2 n} - \left[\frac{\cos nx}{a^2 n^2} \right]_0^a \right] \\ &= \frac{2}{a^2 \pi} \left(\frac{1 - \cos na}{n^2} \right). \end{aligned}$$

Thus,

$$f(x) = \frac{1}{2\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos na}{a^2 n^2} \cos nx.$$

□

Problem 4. Let the Fourier coefficients of f be a_n and b_n . Find the Fourier coefficients A_n and B_n of $g(x) = f(x) \sin x$.

Proof. We have that,

$$f(x) \sin x = \left(\frac{1}{a} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right) \sin x = \frac{\sin x}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx \sin x + b_n \sin nx \sin x).$$

By the product identities, the above series becomes

$$\frac{1}{2} \sum_{n=1}^{\infty} (a_n (\sin(n+1)x - \sin(n-1)x) + b_n (\cos(n-1)x - \cos(n+1)x)).$$

Considering the Fourier series of $g(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$ and collecting the sine and cosine terms, we have that

$$A_n = \frac{1}{2}(b_{n-1} - b_{n+1}) \text{ and } B_n = \frac{1}{2}(a_{n+1} - a_{n-1}),$$

which are the Fourier coefficients of $f(x) \sin x$. □

Problem 5. Let f be 2π -periodic and monotonously decreasing on $(0, 2\pi)$. Prove that the Fourier coefficient $b_n \geq 0$ for all n .

Proof. Note that $\sin n(2\pi - x) = -\sin nx$. We will change the region of integration according to this identity.

$$\begin{aligned} b_n &= \frac{1}{\pi} \left(\int_0^\pi f(x) \sin nx \, dx + \int_\pi^{2\pi} f(x) \sin nx \, dx \right) \\ &= \frac{1}{\pi} \left(\int_0^\pi f(x) \sin nx \, dx + \int_0^\pi f(2\pi - x) \sin n(2\pi - x) \, dx \right) \\ &= b_n \frac{1}{\pi} \int_0^\pi (f(x) - f(2\pi - x)) \sin nx \, dx \end{aligned}$$

Since f is monotonously decreasing on $(0, 2\pi)$, then, for all $x \in (0, \pi)$, we have that $x \leq 2\pi - x \implies f(x) \geq f(2\pi - x)$, or $f(x) - f(2\pi - x) \geq 0$.

Then, with $\sin nx \geq 0$ for all $x \in (0, \pi)$ and $n \geq 0$, we have that

$$b_n = \frac{1}{\pi} \int_0^\pi (f(x) - f(2\pi - x)) \sin nx \, dx \geq 0 \implies b_n \geq 0.$$

□

Problem 6. Let $f \in C^2([- \pi, \pi])$, $f(-\pi) = f(\pi)$, and $f'(-\pi) = f'(\pi)$. Prove $c_n = O(n^{-2})$ as $n \rightarrow \infty$.

Proof. We can perform IBP twice on c_n because $f \in C^2$.

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^\pi f(x) e^{-inx} \, dx \\ &= \frac{1}{2\pi} \left(f(x) \left(\frac{-e^{-inx}}{in} \right) \Big|_{-\pi}^\pi + \int_{-\pi}^\pi f'(x) \frac{e^{-inx}}{in} \, dx \right) \\ &\text{the left summand vanishes because } f(\pi) - f(-\pi) = 0 \\ &= \frac{1}{2\pi} \left(f'(x) \left(\frac{e^{-inx}}{n^2} \right) \Big|_{-\pi}^\pi - \int_{-\pi}^\pi f''(x) \frac{e^{-inx}}{n^2} \, dx \right) \\ &\text{the left summand vanishes because } f'(\pi) - f'(-\pi) = 0 \\ &= \frac{1}{2\pi n^2} \int_{-\pi}^\pi f''(x) e^{-inx} \, dx \sim O(n^{-2}). \end{aligned}$$

Thus, $c_n \sim O(n^{-2})$. □

Problem 7. Find the following limits.

$$(i) \lim_{\lambda \rightarrow \infty} \int_0^\infty \frac{\cos^2 \lambda x}{1+x^2} dx.$$

$$(ii) \lim_{\lambda \rightarrow \infty} \int_{-\pi}^\pi \sin^2 \lambda x dx.$$

Proof of (i). We have that $\cos^2 \lambda x = \frac{1+\cos 2\lambda x}{2}$. So,

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty \frac{1+\cos 2\lambda x}{2(1+x^2)} dx = \frac{1}{2} \lim_{\lambda \rightarrow \infty} \int_0^\infty \left(\frac{1}{1+x^2} + \frac{\cos 2\lambda x}{1+x^2} \right) dx.$$

since $\frac{1}{1+x^2}$ is integrable over $(0, \infty)$, then the right summand vanishes by the Reimann-Lebesgue Lemma.

So, we have that the above limit becomes

$$\frac{1}{2} \int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2} \arctan x \Big|_0^\infty = \frac{\pi}{4}.$$

□

Proof of (ii). We have that $\sin^2 \lambda x = \frac{1-\cos 2\lambda x}{2}$. So,

$$\frac{1}{2} \lim_{\lambda \rightarrow \infty} \int_{-\pi}^\pi (1 - \cos 2\lambda x) dx = \pi - \frac{1}{2} \lim_{\lambda \rightarrow \infty} \int_{-\pi}^\pi \cos 2\lambda x dx.$$

Since the unit function 1 is also integrable on $[-\pi, \pi]$, then the right summand vanishes by the Reimann-Lebesgue Lemma.

Hence, the limit in (ii) is π .

□

Problem 8. (i) Prove

$$\sum_{k=0}^{n-1} \sin \left(k + \frac{1}{2} \right) x = \frac{\sin^2 \left(\frac{nx}{2} \right)}{\sin \left(\frac{x}{2} \right)}, \quad \forall n \geq 1.$$

(ii) Use (i) to show

$$\int_0^\pi \frac{\sin^2 \left(\frac{nx}{2} \right)}{\sin \left(\frac{x}{2} \right)} dx = \pi, \quad \forall n \geq 1.$$

Proof of (i). We will induct on n .

For $n = 1$,

$$\sum_{k=0}^{1-1=0} \sin \left(\frac{2k+1}{2} x \right) = \sin \frac{x}{2} = \frac{\sin^2 \frac{x}{2}}{\sin \frac{x}{2}}.$$

Assume that (i) holds for n . Now, for $n+1$,

$$\sum_{k=0}^n \sin \frac{2k+1}{2} x = \frac{\sin^2 \frac{nx}{2}}{\sin \frac{x}{2}} + \sin \frac{2n+1}{2} x.$$

Consider the right summand; using the product of sines,

$$\begin{aligned}\frac{\sin\left(\frac{2n+1}{2}x\right) \cdot \sin\frac{x}{2}}{\sin\frac{x}{2}} &= \frac{1}{2\sin\frac{x}{2}} \left(\cos\left(\frac{2n+1}{2}x - \frac{x}{2}\right) - \cos\left(\frac{2n+1}{2}x + \frac{x}{2}\right) \right) \\ &= \frac{1}{2\sin\frac{x}{2}} (\cos nx - \cos(n+1)x).\end{aligned}$$

Then, for left summand, using square sine identity,

$$\frac{\sin^2\frac{nx}{2}}{\sin\frac{x}{2}} = \frac{1 - \cos nx}{2\sin\frac{x}{2}}.$$

So, the above sum in the inductive step simplifies to,

$$\frac{1 - \cos(n+1)x}{2\sin\frac{x}{2}} = \frac{\sin^2\left(\frac{n+1}{2}x\right)}{\sin\frac{x}{2}},$$

which is our desired result with $n+1$. □

Proof of (ii). We will induct on n .

For $n=1$,

$$\int_0^\pi \frac{\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}} dx = \int_0^\pi dx = 1 \cdot \pi.$$

Assume (ii) holds for n . Now, for $n+1$, by part (i),

$$\int_0^\pi \frac{\sin^2\left(\frac{n+1}{2}x\right)}{\sin^2\frac{x}{2}} dx = \int_0^\pi \sum_{k=0}^n \frac{\sin\left(k + \frac{1}{2}\right)x}{\sin\frac{x}{2}} dx = \sum_{k=0}^n \int_0^\pi \frac{\sin\left(k + \frac{1}{2}\right)x}{\sin\frac{x}{2}} dx.$$

Here, we have the k^{th} Dirichlet kernel $D_k(x)$, where

$$D_k(x) = \frac{\sin\left(k + \frac{1}{2}\right)x}{2\pi \sin\frac{x}{2}}, \quad \int_0^\pi D_k(x) dx = \frac{1}{2}.$$

So, the above is

$$\sum_{k=0}^n \int_0^\pi 2\pi D_k(x) dx = \sum_{k=0}^n 2\pi \cdot \frac{1}{2} = n\pi.$$

□

Problem 9. Let f be Riemann integrable on $[a, b]$. Let g be a continuous T -periodic function on \mathbb{R} . Prove

$$\lim_{\lambda \rightarrow \infty} \int_a^b f(x)g(\lambda x) dx = \frac{1}{T} \int_0^T g(x) dx \int_a^b f(x) dx.$$

Proof. Since f is Riemann integrable on $[a, b]$, then, given any $\epsilon > 0$, there exists a step function h such that

$$\int_a^b |f(x) - h(x)| dx < \epsilon.$$

So, we have that

$$\int_a^b f(x)g(\lambda x) dx = \int_a^b h(x)g(\lambda x) dx + \int_a^b (f(x) - h(x))g(\lambda x) dx. \quad (*)$$

Since h is a step function, then we can write it as

$$h(x) = \sum_{i=0}^n c_i \chi_{[x_i, x_{i+1}]}(x),$$

where $\chi_{[x_i, x_{i+1}]}(x)$ is the indicator function of the interval $[x_i, x_{i+1}]$.

Note that $\int_a^b h(x) dx = \sum_{i=0}^n c_i (x_{i+1} - x_i)$.

Then, the left summand of $(*)$ becomes,

$$\int_a^b h(x)g(\lambda x) dx = \sum_{i=0}^n c_i \int_{x_i}^{x_{i+1}} g(\lambda x) dx.$$

Since g is continuous and T -periodic, then the integral of g over large intervals approaches the average over T as λ grows large.

$$\int_{x_i}^{x_{i+1}} g(\lambda x) dx = \frac{1}{\lambda} \int_{\lambda x_i}^{\lambda x_{i+1}} g(u) du \approx \frac{x_{i+1} - x_i}{T} \int_0^T g(u) du.$$

Thus, summing over i and for λ large,

$$\int_a^b h(x)g(\lambda x) dx \approx \frac{1}{T} \int_0^T g(x) dx \sum_{i=0}^n c_i (x_{i+1} - x_i) \rightarrow \frac{1}{T} \int_0^T g(x) dx \int_a^b h(x) dx.$$

Since g is continuous and T -periodic, then it is bounded, and $|g(\lambda x)| \leq M$. Hence,

$$\left| \int_a^b (f(x) - h(x))g(\lambda x) dx \right| \leq M \int_a^b |f(x) - h(x)| dx < M\epsilon.$$

Since ϵ was arbitrary, then the right summand of $(*)$ vanishes.

Thus,

$$\lim_{\lambda \rightarrow \infty} \int_a^b f(x)g(\lambda x) dx = \frac{1}{T} \int_0^T g(x) dx \int_a^b h(x) dx.$$

□