# Math 135 Homework 5

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## Probation Practice

1. Solve the differential equations and check the solution.

   (a) $y' = x - 4xy$

   We begin by reorganizing the equation to the form $y' + p(x)y = q(x)$ in order to arrive at the solution via the method of integrating factors. S.H.E. 9.1.2 Provides that, with $H(x) = \int p(x)\,dx$,

   $$y(x) = e^{-H(x)}\left[\int e^{H(x)}q(x)\,dx\right],$$

   with a constant of integration after the evaluation of the indefinate integral.

   So, $y' + 4xy = x$ gives

   $$H(x) = \int 4x\,dx = 2x^2.$$

   Then,

   $$\begin{aligned} y(x) &= e^{-2x^2}\left[\int xe^{2x^2}\,dx\right] \\ &= e^{-2x^2}\left[\frac{1}{4}\int e^u\,du\right] \\ &= e^{-2x^2}\left(\frac{e^{2x^2}}{4} + c\right) \\ &= ce^{-2x^2} + \frac{1}{4}. \end{aligned}$$

   We verify this result by taking the derivative of our solution,

   $$y' = \frac{d}{dx}\left[ce^{-2x^2} + \frac{1}{4}\right] = -4cxe^{-2x^2},$$

and then matching it with the equation for $y$ above,

$$\begin{aligned} y' &= x - 4xy \\ -4cxe^{-2x^2} &= x - 4x\left(ce^{-2x^2} + \frac{1}{4}\right) \\ &= x - 4cxe^{-2x^2} - \frac{4x}{4} \\ &= -4cxe^{-2x^2}. \end{aligned}$$

This statement is true, and therefore we have arrived at a valid general solution.

(b) $y' = \csc x + y \cot x$

Again, we will put the first order linear differential equaition into standard form such that we can use the form provided by S.H.E. 9.1.2.

With $y' - y\cot x = \csc x$, we will need $H(x) = \int -\cot x\, dx = -\ln \sin x$.

Then,

$$\begin{aligned} y(x) &= e^{\ln \sin x}\left[\int \csc x e^{-\ln \sin x}\, dx\right] \\ &= \sin x \left[\int -\frac{dx}{\sin x e^{\ln \sin x}}\right] \\ &= \sin x \left[\int -\csc^2 x\, dx\right] \\ &= \sin x\left(-\cot x + c\right) \\ &= c\sin x - \cos x. \end{aligned}$$

We diffentiate our solution,

$$y'(x) = c\cos x + \sin x.$$

We compare this to the given form for $y'(x)$,

$$\begin{aligned} y'(x) &= \csc x + y\cot x \\ c\cos x + \sin x &= \csc x + (c\sin x - \cos x)\cot x \\ &= \frac{1}{\sin x} + c\cos x - \frac{\cos^2 x}{\sin x} \\ &= \frac{1 - \cos^2 x}{\sin x} + c\cos x \\ &= \frac{\sin^2 x}{\sin x} + c\cos x \\ &= \sin x + c\cos x. \end{aligned}$$

We see that the statement is true and so the general solution is valid.

(c) $x^2y' + 2xy = 8x^3$

We put this equation into standard form and proceed with the methods used in the previous problems,

$$y' + \frac{2}{x}y = 8x.$$

With, $\int p(x)\,dx = \int \frac{2}{x}\,dx = 2\ln x$,

$$\begin{aligned} y(x) &= e^{-2\ln x}\left[\int 8xe^{2\ln x}\,dx\right] \\ &= \frac{1}{x^2}\left[\int 8x^3\,dx\right] \\ &= \frac{1}{x^2}\left(2x^4 + c\right) \\ &= 2x^2 + \frac{c}{x^2}. \end{aligned}$$

To check this $y$, we differentiate,

$$y'(x) = 4x - \frac{2c}{x^3}.$$

Then we compare, with the given differential equation where $y'$ is isolated,

$$\begin{aligned} 4x - \frac{2c}{x^3} &= 8x - \frac{2}{x}\left(2x^2 + \frac{c}{x^2}\right) \\ &= 8x - 4x - \frac{2c}{x^3} \\ &= 4x - \frac{2c}{x^3}, \end{aligned}$$

which is true, so the general solution holds as well.

(d) $y' = xe^{y-x^2}, \quad y(0) = 0$

This is a separable differential equation. We also have an initial condition that will pin the single constant at a fixed number.

We rewrite the equation with regards to differentials $\frac{dy}{dx}$,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{xe^y}{e^{x^2}} \\
\int \frac{dy}{e^y} &= \int \frac{x\,dx}{e^{x^2}} \\
-e^{-y} &= -\frac{1}{2}\int e^u\,du \\
e^{-y} &= \frac{1}{2}e^u + c \\
e^{-y} &= \frac{1}{2}e^{-x^2} + c \\
-y &= \ln\left(\frac{1}{2}e^{-x^2} + c\right) \\
y &= -\ln\left(\frac{1}{2}e^{-x^2} + c\right).
\end{aligned}$$

We then consider the initial condition $y(0) = 0$,

$$\begin{aligned}
0 &= -\ln\left(\frac{1}{2}e^{-0^2} + c\right) \\
0 &= -\ln\left(\frac{1}{2} + c\right) \\
1 &= \frac{1}{2} + c \\
\frac{1}{2} &= c.
\end{aligned}$$

So, we have the solution,

$$\begin{aligned}
y(x) &= -\ln\left(\frac{1}{2}(1 + e^{-x^2})\right) \\
&= \ln 2 - \ln\left(1 + e^{-x^2}\right).
\end{aligned}$$

We can verify this solution in the same manner as previous problems.

First, we differentiate our solution,

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left[\ln 2 - \ln\left(1 + e^{-x^2}\right)\right] \\
&= -\frac{-2xe^{-x^2}}{1 + e^{-x^2}} \\
&= \frac{2x}{1 + e^{x^2}}.
\end{aligned}$$

Then we compare it to the given $y'$ with our derived solution $y$,

$$\begin{aligned} y'(x) &= \frac{xe^{\left(\ln 2 - \ln\left(1+e^{-x^2}\right)\right)}}{e^{x^2}} \\ &= \frac{xe^{\ln 2}}{e^{x^2} e^{\ln\left(1+e^{-x^2}\right)}} \\ &= \frac{2x}{e^{x^2}\left(1+e^{-x^2}\right)} \\ &= \frac{2x}{1+e^{x^2}}. \end{aligned}$$

Since both $y'$ match, we confirm that we have arrived at a valid general solution.

(e) $(x + yx)\,dx = (x^2y^2 + x^2 + y^2 + 1)\,dy$

We rearrange the terms to reveal that this equation is separable.

$$\begin{aligned} (x+yx)\,dx &= (x^2y^2 + x^2 + y^2 + 1)\,dy \\ x(y+1)\,dx &= (x^2+1)(y^2+1)\,dy \\ \int \frac{x\,dx}{x^2+1} &= \int \frac{y^2+1}{y+1}\,dy \\ \frac{1}{2}\int \frac{du}{u} &= \int \frac{(y+1)^2 - 2y}{y+1}\,dy \\ \frac{\ln u}{2} + c/2 &= \int \left(y + 1 - \frac{2(y+1)-2}{y+1}\right)\,dy \\ \frac{\ln\left(x^2+1\right)}{2} + c/2 &= \int \left(y + 1 - 2 + \frac{2}{y+1}\right)\,dy \\ &= \int \left(y - 1 + \frac{2}{y+1}\right)\,dy \\ &= \frac{y^2}{2} - y + 2\ln(y+1) \\ \ln\left(x^2+1\right) + c &= y^2 - 2y + 4\ln(y+1) \end{aligned}$$

2. Find the first four Picard approximations of $y' = e^x + y$ with $y(0) = 0$.

Since $y(x_0) = y_0$, then $(x_0, y_0) = (0, 0)$.

So, $y_0(x) = y_0 = 0$.

The $n^{\text{th}}$ Picard approximation is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f[x, y_{n-1}(x)]\,dx,$$

where $y' = f(x, y(x))$.

So, for $y_1$,

$$\begin{aligned} y_1(x) &= 0 + \int_0^x f(x, y_0(x))\,dx \\ &= \int_0^x (e^x + 0)\,dx \\ &= e^x\big|_0^x \\ &= e^x - 1. \end{aligned}$$

Next, for $y_2$,

$$\begin{aligned} y_2(x) &= 0 + \int_0^x \left(e^x + (e^x - 1)\right)\,dx \\ &= \int_0^x (2e^x - 1)\,dx \\ &= [2e^x - x]_0^x \\ &= 2e^x - x - 2. \end{aligned}$$

For $y_3$,

$$\begin{aligned} y_3(x) &= 0 + \int_0^x \left(e^x + (2e^x - x - 2)\right)\,dx \\ &= \int_0^x (3e^x - x - 2)\,dx \\ &= \left[3e^x - \frac{x^2}{2} - 2x\right]_0^x \\ &= 3e^x - \frac{x^2}{2} - 2x - 3. \end{aligned}$$

Lastly, for $y_4$,

$$\begin{aligned} y_4(x) &= 0 + \int_0^x \left(e^x + \left(3e^x - \frac{x^2}{2} - 2x - 3\right)\right)\,dx \\ &= \int_0^x \left(4e^x - \frac{x^2}{2} - 2x - 3\right)\,dx \\ &= \left[4e^x - \frac{x^3}{6} - x^2 - 3x\right]_0^x \\ &= 4e^x - \frac{x^3}{6} - x^2 - 3x - 4. \end{aligned}$$

3. First the first three Picard approximations of the system of differential equations,

$$\begin{aligned} x'(t) &= t + y^2 & x(0) &= 0, \\ y'(t) &= x - t & y(0) &= 1. \end{aligned}$$

We first notice that $t_0 = 0$, $x_0 = 0$, and $y_0 = 1$.

For a parametric system of differential equations, the Picard approximations are given by

$$\begin{aligned} x_n &= x_0 + \int_{t_0}^{t} f_1[t, x_{n-1}(t) y_{n-1}(t)]\, dt, \\ y_n &= y_0 + \int_{t_0}^{t} f_2[t, x_{n-1}(t) y_{n-1}(t)]\, dt, \end{aligned}$$

where $x'(t) = f_1(t, x(t), y(t))$ and $y'(t) = f(t, x(t), y(t))$.

We start with the first pair of approximations,

$$\begin{aligned} x_1(t) &= 0 + \int_0^t (t + 1^2)\, dt \\ &= \left[ t + \frac{t^2}{2} \right]_0^t \\ &= t + \frac{t^2}{2} + t, \end{aligned}$$

and

$$\begin{aligned} y_1(t) &= 1 + \int_0^t (0 - t)\, dt \\ &= 1 - \frac{t^2}{2}{}_0^t \\ &= 1 - \frac{t^2}{2}. \end{aligned}$$

We continue recursively as before,

$$\begin{aligned} x_2(t) &= 0 + \int_0^t \left( t + \left( 1 - \frac{t^2}{2} \right)^2 \right)\, dt \\ &= \int_0^t \left( 1 + t - t^2 + \frac{t^4}{4} \right)\, dt \\ &= \left[ t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20} \right]_0^t \\ &= t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20}, \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= 10y \int_0^t \left( t + \frac{t^2}{2} - t \right)\, dt \\ &= 1 + \frac{t^3}{6} \bigg|_0^t \\ &= 1 + \frac{t^3}{6}. \end{aligned}$$

Finally,

$$\begin{aligned} x_3(t) &= 0 + \int_0^t \left(t + \left(1 + \frac{t^3}{6}\right)^2\right) dt \\ &= \int_0^t \left(1 + t + \frac{t^3}{3} + \frac{t^6}{36}\right) dt \\ &= t + \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^7}{252}, \end{aligned}$$

and

$$\begin{aligned} y_3(t) &= 1 + \int_0^t \left(t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20} - t\right) dt \\ &= 1 + \left[\frac{t^3}{6} - \frac{t^4}{12} + \frac{t^6}{120}\right]_0^x. \end{aligned}$$

# Homework

1. The equation $y' + P(x)y = Q(x)y^n$ is called the Bernoulli equation. It becomes a linear equation after the change of variable $y^{1-n} = z$. Solve the equation $y(6y^2 - x - 1)\,dx + 2x\,dy = 0$ using this idea.

First, we rearrange the equation to the given form,

$$\begin{aligned} y(6y^2 - x - 1)\,dx + 2x\,dy = 0 \\ -2x\frac{dy}{dx} &= 6y^3 - y(x+1) \\ y' - \frac{x+1}{2x}y &= -\frac{3}{x}y^3. \end{aligned}$$

We see that this is a Bernoulli differential equation of degree three.

We will perform the substitution with $y^{1-n} = y^{1-3} = y^{-2} = z$ to find $y$ and $y'$ in $z$,

$$\begin{aligned} y^{-2} &= z \\ y &= z^{-1/2} \\ y' &= -\frac{1}{2}z^{-3/2}z'. \end{aligned}$$

We then rewrite the DE in $z$,

$$\begin{aligned} -\frac{1}{2}z^{-3/2}z' - \frac{x+1}{2x}\left(z^{-1/2}\right) &= -\frac{3}{x}\left(z^{-1/2}\right)^3 \\ -\frac{z'}{2} - \frac{x+1}{2x}z &= -\frac{3}{x} \\ z' + \frac{x+1}{x}z &= \frac{6}{x}. \end{aligned}$$

We can use the method of an integrating factor to solve this first order linear differential equation.

Let
$$\mu = e^{\int \frac{x+1}{x}\,dx} = e^{x+\ln x} = xe^x.$$

Then,
$$\begin{aligned}
\mu z' + \mu \frac{x+1}{x} z &= \mu \frac{6}{x} \\
xe^x z' + e^x(x+1)z &= 6e^x \\
\frac{d}{dx}\left[xe^x z\right] &= 6e^x \\
xe^x z &= \int 6e^x\,dx \\
xe^x z &= 6e^x + c \\
z &= \frac{6e^x + c}{xe^x}.
\end{aligned}$$

But recall that $y^{-2} = z$. So,
$$\begin{aligned}
y^{-2} &= \frac{6e^x + c}{xe^x} \\
y^2 &= \frac{xe^x}{6e^x + c} \\
y &= \pm\sqrt{\frac{xe^x}{6e^x + c}} \\
y &= \pm\sqrt{\frac{x}{6 + ce^{-x}}},
\end{aligned}$$

which is the general solution to our differential equation.

2. Let $I = [0,1]$ and $Y_n(t) = t^n$. Show that the sequence $(Y_n)$ is not Cauchy by computing $||Y_n - Y_m||$.

3. Let $I = [-\pi, \pi]$, and consider the function $f_0 : I \to \mathbb{R}$ defined by $f_0(t) = e^t$. Let $f_n, n \in \mathbb{Z}^+$ be the sequence of functions on $I$ defined inductively by the formula
$$f_{n+1}(t) = \cos(t) + \frac{1}{2}\sin(t) f_n(t).$$

So $f_1(t) = \cos(t) + \frac{1}{2}\sin(t)e^t, \quad f_2(t) = \cos(t) + \frac{1}{2}\sin(t)\left(\cos(t) + \frac{1}{2}\sin(t)e^t\right)$, etc.

Show that $f_n$ converges uniformly to a continuous function and find the limit $\lim\limits_{n\to\infty} f_n$.