Math 135 Homework 7

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February 19, 2024

1. Show that e^{t^2} is not of exponential order.

   Since $t^2 > at$ for all $t > a > 0$,

   Then, $e^{t^2} > e^{at}$ for all $t > a > 0$.

   Since $e^x > 0$ for all x, then $e^x = |e^x|$ for all x.

   So $\left|e^{t^2}\right| > e^{at}$ for all $t > a > 0$.

   Since $a > 0$, then e^{t^2} is not of exponential order as it is the negation of the definition for some f of exponential order that $|f(x)| \leq Ae^{bt}$ for $b > 0$.

2. (a) We are given that

$$\int \mathrm{Re}\left[e^{(a+bi)t}\right] dt = \mathrm{Re}\int e^{(a+bi)t}\, dt. \tag{*}$$

We expand $e^{(a+bi)t}$ according to Euler's formula,

$$e^{(a+bi)t} = e^{at}e^{bit} = e^{at}\left(\cos bt + i\sin bt\right).$$

So,

$$\mathrm{Re}\left[e^{(a+bi)t}\right] = e^{at}\cos bt.$$

Then, by (*),

$$\begin{aligned}
\int e^{at}\cos bt &= \mathrm{Re}\int e^{(a+bi)t}\, dt \\
&= \mathrm{Re}\left[\frac{e^{(a+bi)t}}{a+bi}\right] \\
&= \mathrm{Re}\left[\frac{(a-bi)e^{at}(\cos bt + i\sin bt)}{a^2+b^2}\right] \\
&= \mathrm{Re}\left[\frac{e^{at}\left((a-bi)\cos bt + (ai+b)\sin bt\right)}{a^2+b^2}\right] \\
&= \frac{e^{at}(a\cos bt + b\sin bt)}{a^2+b^2}. \qquad (**)
\end{aligned}$$

(b) So, for $\mathcal{L}\{\cos bt\}$, with the definition of the Laplace transform,

$$\begin{aligned}
\mathcal{L}\{\cos bt\} &= \int_0^\infty e^{-st}\cos bt\,dt \\
&= \lim_{n\to\infty}\left[\frac{e^{-st}\left(-s\cos bt + b\sin bt\right)}{(-s)^2+b^2}\Bigg|_0^n\right] \\
&= \lim_{n\to\infty}\frac{e^{-sn}(-s\cos bn + b\sin bn) - s}{s^2+b^2} \\
&= \lim_{n\to\infty}\frac{s - \frac{b\sin bn - s\cos bn}{e^{sn}}}{s^2+b^2} \\
&= \frac{s}{s^2+b^2}.
\end{aligned}$$

(c) For $\mathcal{L}\{\sin bt\}$, we will let $f(t) = \frac{-\cos bt}{b}$. So $f'(t) = \sin bt$. Then we consider the Laplace of a derivative,

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\
\mathcal{L}\{\sin bt\} &= s\mathcal{L}\left\{\frac{-\cos bt}{b}\right\} - \left(\frac{-\cos(b\cdot 0)}{b}\right) \\
&= \frac{-s}{b}\mathcal{L}\{\cos bt\} + \frac{1}{b} \\
&= \frac{1}{b}\left(1 - s\left(\frac{s}{s^2+b^2}\right)\right) \\
&= \frac{1}{b}\left(\frac{b^2}{s^2+b^2}\right) \\
&= \frac{b}{s^2+b^2}.
\end{aligned}$$

(d) For $\mathcal{L}\{t\sin bt\}$, we consider the derivative of the Laplace transform,

$$\frac{d^n}{ds^n}F(s) = \mathcal{L}\{(-t)^n f(t)\}, \quad F(s) = \mathcal{L}\{f(t)\}. \tag{**}$$

Let $f(t) = -\sin bt$. Then $F(s) = \frac{-b}{s^2+b^2}$ from above.

With (**) and $n = 1$,

$$\begin{aligned}
\mathcal{L}\{-tf(t)\} &= \frac{d}{ds}F(s) \\
\mathcal{L}\{t\sin bt\} &= \frac{d}{ds}\left[\frac{-b}{s^2+b^2}\right] \\
&= \frac{2bs}{(s^2+b^2)^2}.
\end{aligned}$$

(e) For $\mathcal{L}\{e^{at}\sin bt\}$, we note that the product of a function f with the exponential function produces a horizontal shift in the Laplace frequency domain,

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a), \quad \mathcal{L}\{f\} = F.$$

Since
$$\mathcal{L}\left\{\sin bt\right\} = \frac{b}{s^2+b^2} = F(s),$$

Then
$$\mathcal{L}\left\{e^{at}\sin bt\right\} = F(s-a) = \frac{b}{\left(s-a\right)^2+b^2}.$$

(f) With $\mathcal{L}\left\{te^{at}\sin bt\right\}$, we consider both (d) and (e), where
$$F(s) = \mathcal{L}\left\{t\sin bt\right\} = \frac{2bs}{\left(s^2+b^2\right)^2},$$
such that
$$F(s-a) = \mathcal{L}\left\{te^{at}\sin bt\right\} = \frac{2b(s-a)}{\left(\left(s-a\right)^2+b^2\right)^2}.$$

(g) For $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{a^2}{(s^2+a^2)^2}\right\}$, we will rewrite the Laplace function argument $F(s)$ into terms that can be inverted easily from our given Laplace table.

First,
$$\begin{aligned}\frac{s^2}{\left(s^2+a^2\right)^2} &= \frac{1}{2}\cdot\frac{s^2+a^2+s^2-a^2}{\left(s^2+a^2\right)^2}\\ &= \frac{1}{2}\left(\frac{s^2+a^2}{\left(s^2+a^2\right)^2}+\frac{s^2-a^2}{\left(s^2+a^2\right)^2}\right)\\ &= \frac{1}{2}\left(\frac{1}{a}\cdot\frac{a}{s^2+a^2}+\frac{s^2-a^2}{\left(s^2+a^2\right)^2}\right).\end{aligned}$$

Then, we are ready to take the inverse Laplace,
$$\mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{1}{a}\cdot\frac{a}{s^2+a^2}-\frac{s^2-a^2}{\left(s^2+a^2\right)^2}\right)\right\} = \frac{1}{2}\left(\frac{1}{a}\sin at + t\cos at\right).$$

Next,
$$\begin{aligned}\frac{a^2}{\left(s^2+a^2\right)^2} &= \frac{1}{2}\cdot\frac{s^2+a^2-(s^2-a^2)}{\left(s^2+a^2\right)^2}\\ &= \frac{1}{2}\left(\frac{s^2+a^2}{\left(s^2+a^2\right)^2}-\frac{s^2-a^2}{\left(s^2+a^2\right)^2}\right)\\ &= \frac{1}{2}\left(\frac{1}{a}\cdot\frac{a}{s^2+a^2}-\frac{s^2-a^2}{\left(s^2+a^2\right)^2}\right).\end{aligned}$$

So,
$$\mathcal{L}^{-1}\left\{\frac{1}{2}\left(\frac{s^2+a^2}{\left(s^2+a^2\right)^2}-\frac{s^2-a^2}{\left(s^2+a^2\right)^2}\right)\right\} = \frac{1}{2}\left(\frac{1}{a}\sin at - t\cos at\right).$$

3. Solve
$$y''+' = f(t), \quad y(0) = y'(0) = 0,$$
where
$$f(t) = \begin{cases} 4, & 0 \le t \le 2, \\ t+2, 2 < t. \end{cases}$$

First, we rewrite the piecewise function f using the step function H, where
$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \ge 0; \end{cases}$$
we will use $H_c = H(t-c)$.

4. Compute the sawtooth function $\mathcal{L}\left\{h(t)\right\}$, where
$$h(t) = \begin{cases} t, & 0 \le t < 1, \\ h(t-1), & 1 \le t. \end{cases}$$

We begin by writing h in terms of the step function H,
$$h(t) = t(1 - H_1) + h(t-1)H_1.$$

We then take the Laplace transform of this equation with the fact that
$$\mathcal{L}\left\{f(t-c)H_c\right\} = e^{-cs}\mathcal{L}\left\{f(t)\right\},$$

$$\begin{aligned} \mathcal{L}\left\{h(t)\right\} &= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + e^{-s}\mathcal{L}\left\{h(t)\right\} \\ (1-e^{-s})\mathcal{L}\left\{h(t)\right\} &= \frac{1-e^{-s}}{s^2} - \frac{e^{-s}}{s} \\ \mathcal{L}\left\{h(t)\right\} &= \frac{1}{s^2} - \frac{e^{-s}}{s\left(1-e^{-s}\right)} \\ \mathcal{L}\left\{h(t)\right\} &= \frac{1}{s^2} - \frac{1}{s\left(e^{s}-1\right)}. \end{aligned}$$