Homework 2

1. How many ways can 10 people sit in a row of 30 chairs, if there must be at least one chair between each person?

   Proof. First, we will place 10 people in chairs with 9 gaps in between them, so there are 20 chairs remaining. Note that there are 10! ways to permute these people.

   Since there must be at least 1 chair between each person, we will use 9 chairs to accomplish this, leaving 20-9=11 chairs remaining.

   Now, we must distribute 11 chairs between the now 11 gaps (the former 9 plus the outside 2 for the first and last person) where each gap could have 0 or more chairs. This is a weak completion of 11 into 11 parts, or $\binom{11+11-1}{11} = \binom{21}{11}$.

   Given the 10! ways to permute the 10 people, which is independent of choosing our gaps, the two choices produce

   $$10!\binom{21}{11} = 10!\frac{21!}{11!(21-11)!} = \frac{21!}{11!}$$

   ways to place 10 people into a row of 30 chairs with at least one chair between each person. □

2. For each positive integer n, how many ways can you write n as a sum of positive integers, if the order of the sum matters? For example, for $n = 4$ there are 8 ways: 4, $3+1$, $1+3$, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$.

   Proof. We will establish a bijection between a set with size 2^{n-1} and the number of ways to produce the sums described above.

   First, we will describe the domain set; consider n "stars" with $n-1$ gaps.

   In each gap, insert either a comma or a plus; e.g.,

   $$* \quad * \quad * \quad * \; \to \; * + * + *, *.$$

   This gives $n-1$ independent choices of 2, or 2^{n-1}.

   Now, consider each star to represent 1, grouping all stars connected by pluses into larger integers; e.g.,

   $$* + * + *, * \; \to 3, 1.$$

   This will produce an ordered list of terms which sum to n where $n-1, 1$ is different than $1, n-1$.

   Now, we will show that this is a bijection to our desired set of sums by describing the inverse.

   With an ordered sum of n, replace all addition with commas; e.g.,

   $$3 + 1 \to 3, 1.$$

   Then represent all integers greater than 1 with stars and pluses, and all 1s with stars; e.g.,

   $$3, 1 \to * + * + *, *.$$

   Since we have arrived at our domain set once more, we have established an inverse mapping and therefore shown that our original construction was a bijection.

   Since our map is a bijection with a domain cardinality of 2^{n-1}, then the number of ways to write n as a sum of positive integers where order matters is also 2^{n-1}.

   □

3. Find the number of 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ where each a_i is an odd positive integer and $\sum_{i=1}^{5} a_i = 25$.

Proof. We will use a stars and bars argument with 25 stars and 4 bars. Since each a_i must be an odd positive integer, then $\forall i$, $a_i \geq 1$.

So, we will first place 5 stars with 4 bars in between.

We have $25 - 5 = 20$ remaining stars, but we must add these stars in groups of 2 to ensure that all a_i remain odd (odd + even = odd); so we have 10 units to add.

So, this process is equivalent to a weak completion of 10 into 5 parts, each of the a_i. This is

$$\binom{10+5-1}{5-1} = \binom{14}{4}.$$

□

4. Let n and k be positive integers with $k \leq n/2$. Prove that

$$\binom{n}{k-1} < \binom{n}{k}.$$

What if $k > n/2$?

Proof. (i) We will consider the ratio

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}} = \frac{n-k+1}{k}.$$

Then, $k \leq \frac{n}{2} \implies 2k \leq n \implies k \leq n-k+1$.

Therefore,

$$\frac{n-k+1}{k} \geq 1 \implies \binom{n}{k-1} \leq \binom{n}{k}.$$

This inequality is strict unless $n = \frac{k}{2}$, where n is even and the binomial coefficients are equal for $k = \frac{n}{2}, \frac{n}{2}+1$.

(ii) When $k > \frac{n}{2}$, we have $n-k+1 < k$.

So the order flips,

$$\binom{n}{k-1} > \binom{n}{k}.$$

□

5. Find a closed formula for

$$\sum_{k=1}^{n} k^2 \binom{n}{k}.$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x^k &= (x+1)^n \\
\sum_{k=1}^{n} \binom{n}{k} k x^{k-1} &= n(x+1)^{n-1} \\
\sum_{k=1}^{n} \binom{n}{k} k x^{k} &= nx(x+1)^{n-1} \\
\sum_{k=1}^{n} \binom{n}{k} k^2 x^{k-1} &= n(x+1)^{n-1} + n(n-1)x(x+1)^{n-2} \\
\sum_{k=1}^{n} \binom{n}{k} k^2 1^{k-1} &= n(1+1)^{n-1} + n(n-1)x(1+1)^{n-2} \\
\sum_{k=1}^{n} \binom{n}{k} k^2 &= n \cdot 2^{n-1} + n(n-1)2^{n-2} \\
&= n \cdot 2^{n-2} \cdot (2+n-1) \\
&= (n+1)n2^{n-2}.
\end{aligned}$$

□

6. Let n and k be positive integers with $k \leq n$. Prove that

$$\sum_{i=0}^{k} \binom{n}{i} (-1)^i = \binom{n-1}{k} (-1)^k.$$

Proof. By the alternating sum identity, we have

$$\begin{aligned}
0 = \sum_{i=0}^{n} \binom{n}{i} (-1)^i &= \sum_{i=0}^{k} \binom{n}{i} (-1)^i + \sum_{i=k+1}^{n} \binom{n}{i} (-1)^i \\
\implies \sum_{i=0}^{k} \binom{n}{i} (-1)^i &= - \sum_{i=k+1}^{n} \binom{n}{i} (-1)^i.
\end{aligned}$$

By the Pascal's triangle summation identity, we have

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}.$$

So,

$$\begin{aligned}
\sum_{i=k+1}^{n} \binom{n}{i} (-1)^i &= \sum_{i=k+1}^{n} \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] (-1)^i \\
&= \sum_{i=k+1}^{n} \binom{n-1}{i-1} (-1)^i + \sum_{i=k+1}^{n} \binom{n-1}{i} (-1)^i.
\end{aligned}$$

We will reindex the left sum by $i = j + 1$ and split off the first summand on the left and then final summand on the right term; the above becomes,

$$\begin{aligned}
&= \sum_{j=k}^{n-1} \binom{n-1}{j} (-1)^{j+1} + \left(\sum_{i=k+1}^{n} \binom{n-1}{i} (-1)^i + \binom{n-1}{n} (-1)^n \right) \\
&= -\left(\binom{n-1}{k} (-1)^k + \sum_{j=k+1}^{n-1} \binom{n-1}{j} (-1)^{j+1} \right) + \left(\sum_{i=k+1}^{n} \binom{n-1}{i} (-1)^i + 0 \right) \\
&= -\binom{n-1}{k} (-1)^k.
\end{aligned}$$

So,

$$\sum_{i=0}^{k} \binom{n}{i} (-1)^k = -\sum_{i=k+1}^{n} \binom{n}{i} (-1)^k = \binom{n-1}{k} (-1)^k.$$

□