Math 135 Homework 10

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1. A vector-valued function $\mathbf{G}$ is called an *antiderivative* for $\mathbf{f}$ on $[a, b]$ provided that

   (i) $\mathbf{G}$ is continuous on $[a, b]$ and

   (ii) $\mathbf{G}'(t) = \mathbf{f}(t)$ for all $t \in (a, b)$.

   Show that:

   (a) If $\mathbf{f}$ is continuous on $[a, b]$ and $\mathbf{G}$ is an antiderivative for $\mathbf{f}$ on $[a, b]$, then

   $$\int_a^b \mathbf{f}(t)\, dt = \mathbf{G}(b) - \mathbf{G}(a).$$

   (b) If $\mathbf{f}$ is continuous on an interval I and $\mathbf{F}$ and $\mathbf{G}$ are antiderivatives for $\mathbf{f}$, then

   $$\mathbf{F} = \mathbf{G} + \mathbf{C}$$

   for some constant vector $\mathbf{C}$.

   For (a), we will consider a vector in three space. The argument holds for a vector of n dimensions as well.

   Let $\mathbf{f}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$.

   Then, by 14.1.8,

   $$\int_a^b \mathbf{f}(t)\, dt = \left\langle \int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \int_a^b f_2(t)dt \right\rangle.$$

   We then let each component of $\mathbf{G}$, $\mathbf{G}_i$, be the antiderivative of the corresponding component of $\mathbf{f}$, $\mathbf{f}_i$, on $[a, b]$.

   So, by the Fundamental Theorem of Calculus the above becomes,

   $$\int_a^b \mathbf{f}(t)\, dt = \langle \mathbf{G}_1(b) - \mathbf{G}_1(a), \mathbf{G}_2(b) - \mathbf{G}_2(a), \mathbf{G}_3(b) - \mathbf{G}_3(a)\rangle$$

   But, each $\mathbf{G}_i$ is a component of $\mathbf{G}$, so we can rewrite the above as,

   $$\int_a^b \mathbf{f}(t)\, dt = \mathbf{G}(b) - \mathbf{G}(a).$$

For (b), let $\mathbf{F} = \mathbf{G} + \mathbf{C}$.

By 14.1.6, the derivative of a constant vector is the zero vector.

Since $\mathbf{G}' = \mathbf{f}$, then $\mathbf{F}' = (\mathbf{G} + \mathbf{C})' = \mathbf{f} + \mathbf{0} = \mathbf{f}$.

2. Let $\mathbf{f}$ be a differentiable vector-valued function. Show that, where $||\mathbf{f}(t)|| \neq 0$,

$$\frac{d}{dt}\left[\frac{\mathbf{f}}{||\mathbf{f}||}\right] = \frac{\mathbf{f}'}{||\mathbf{f}||} - \frac{\mathbf{f} \cdot \mathbf{f}'}{||\mathbf{f}||^3}\mathbf{f}.$$

Let $f = ||\mathbf{f}||$ so $\mathbf{f} \cdot \mathbf{f} = f^2$.

By 14.2.4,

$$\left(\frac{\mathbf{f}}{f}\right)' = \frac{1}{f^3}\left((\mathbf{f} \times \mathbf{f}') \times \mathbf{f}\right), \quad f \neq 0.$$

By 13.4.11, the above becomes,

$$\begin{aligned} &\frac{1}{f^3}\left((\mathbf{f} \cdot \mathbf{f})\,\mathbf{f}' - (\mathbf{f} \cdot \mathbf{f}')\,\mathbf{f}\right) \\ =&\frac{f^2}{f^3}\mathbf{f}' - \frac{\mathbf{f} \cdot \mathbf{f}'}{f^3}\mathbf{f} \\ =&\frac{\mathbf{f}'}{f} - \frac{\mathbf{f} \cdot \mathbf{f}'}{f^3}\mathbf{f} \\ =&\frac{\mathbf{f}'}{||\mathbf{f}||} - \frac{\mathbf{f} \cdot \mathbf{f}'}{||\mathbf{f}||^3}\mathbf{f}. \end{aligned}$$

3. The curvature of the curve traced by the vector function $\mathbf{r}(t)$ is given by

$$\kappa = \frac{||\mathbf{T}'||}{||\mathbf{r}'||}$$

where $\mathbf{T}$ is the unit tangent vector. Prove

$$\kappa = \frac{||\mathbf{v} \times \mathbf{a}||}{\left(\frac{ds}{dt}\right)^3}$$

(formula 14.5.9) using question 2 above and formula 13.4.11.

Proof. We will first rewrite 14.5.9 using $\mathbf{r}' = \mathbf{v}$, $\mathbf{r}'' = \mathbf{a}$, and

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} ||\mathbf{r}'(u)||\,du = ||\mathbf{r}'||.$$

This gives,

$$\kappa = \frac{||\mathbf{r}' \times \mathbf{r}''}{||\mathbf{r}'||^3}.$$

Since $\mathbf{T} = \frac{\mathbf{r}'}{||\mathbf{r}'||}$, then $\mathbf{T}' = \left(\frac{\mathbf{r}'}{||\mathbf{r}'||}\right)'$.

By 14.2.4 as used in question 2, this becomes,

$$\frac{(\mathbf{r}'' \times \mathbf{r}') \times \mathbf{r}'}{||\mathbf{r}'||^3}.$$

Then,

$$||\mathbf{T}'|| = \frac{||\,(\mathbf{r}'' \times \mathbf{r}') \times \mathbf{r}'||}{||\mathbf{r}'||^3}.$$

With the the fact that $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||||\mathbf{b}|| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, we expand the above,

$$||\mathbf{T}'|| = \frac{1}{||\mathbf{r}'||^3}||\mathbf{r}'||||\mathbf{r}'' \times \mathbf{r}'|| \sin\theta.$$

Since $\mathbf{r}'$ will be orthogonal to any vector that is produced from a cross product of any vector and itself. So, the angle θ between $\mathbf{r}'$ and $\mathbf{r}'' \times \mathbf{r}'$ will be $\pi/2$. So, $\sin\theta = 1$.

Thus,

$$||\mathbf{T}'|| = \frac{1}{||\mathbf{r}'||^3}||\mathbf{r}'||||\mathbf{r}'' \times \mathbf{r}'|| \sin\theta = \frac{||\mathbf{r}'' \times \mathbf{r}'||}{||\mathbf{r}'||^2}.$$

Then it immediately follows from our first definition for κ that,

$$\kappa = \frac{||\mathbf{r}'' \times \mathbf{r}'||}{||\mathbf{r}'||^3}.$$

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