Math 136 Homework 8

Alexandre Lipson

June 2, 2024

1.

Problem. Let $r = \|\vec{r}\|$ where $\vec{r} = (x, y, z)$. If f is a continuously differentiable function of r, show that, where $r \neq 0$

$$\nabla f(r) = f'(r)\frac{\vec{r}}{r}.$$

With $\|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$,

$$\nabla f(r) = \nabla f\left(\sqrt{x^2 + y^2 + z^2}\right) = \left(\frac{df}{dr}\frac{\partial r}{\partial x} + \frac{df}{dr}\frac{\partial r}{\partial y} + \frac{df}{dr}\frac{\partial r}{\partial z}\right).$$

We find $\frac{\partial r}{\partial x}$,

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}.$$

The same holds for y and z.

So, the above becomes

$$\nabla f(r) = \left(\frac{df}{dr}\frac{x}{r} + \frac{df}{dr}\frac{y}{r} + \frac{df}{dr}\frac{z}{r}\right) = \frac{df}{dr}\left(\frac{1}{r}\right) \cdot (x, y, z) = f'(r)\frac{\vec{r}}{r},$$

which is equivalent to the statement that we wished to show.

2.

Problem. Assume $u(x, y)$ has continuous second partials. Show that the Laplace polar form holds,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

Let $x = r\cos\theta$ and $y = r\sin\theta$.

So, $\frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial y}{\partial r} = \sin\theta$; and $\frac{\partial x}{\partial \theta} = -r\sin\theta$ and $\frac{\partial y}{\partial \theta} = r\cos\theta$.

We use to chain rule to compute to the r partials,

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial x}{\partial y} \\ &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.\end{aligned}$$

We deploy the chain rule again,

$$
\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= \cos\theta \frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin\theta \frac{\partial}{\partial r}\frac{\partial u}{\partial y} \\
&= \cos\theta \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y \partial x}\frac{\partial y}{\partial r} \right) + \sin\theta \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r} \right) \\
&= \cos^2\theta \frac{\partial^2 u}{\partial x^2} + \cos\theta\sin\theta \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \right) + \sin^2\theta \frac{\partial^2 u}{\partial y^2}.
\end{aligned}
$$

We will apply the same for θ,

$$
\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\
&= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta).
\end{aligned}
$$

And, for the second order partial as well,

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \theta^2} &= -r\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial u}{\partial x}\right] + r\frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial u}{\partial y}\right] \\
&= -r\left(\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial}{\partial \theta}\frac{\partial u}{\partial x} \right) + r\left(-\sin\theta \frac{\partial u}{\partial y} + \cos\theta \frac{\partial}{\partial \theta}\frac{\partial u}{\partial y} \right) \\
&= -r\left(\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} \right) - r\sin\theta \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y \partial x}\frac{\partial y}{\partial \theta} \right) + r\cos\theta \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta} \right) \\
&= -r\left(\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} \right) \\
&\quad - r\sin\theta \left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y \partial x}(r\cos\theta) \right) \\
&\quad + r\cos\theta \left(\frac{\partial^2 u}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta) \right) \\
&= -r\left(\cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} \right) + r^2\left(\sin^2\theta \frac{\partial^2 u}{\partial x^2} - \cos\theta\sin\theta \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \right) + \cos^2\theta \frac{\partial^2 u}{\partial y^2} \right).
\end{aligned}
$$

We will assemble the terms, starting with $\frac{\partial^2 u}{\partial \theta^2}$, noting that we can substitute part the first term with $\frac{\partial u}{\partial r}$.

$$
\frac{1}{r^2}\frac{\partial^2 u}{\partial q^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta \frac{\partial^2 u}{\partial x^2} - \cos\theta\sin\theta \left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \right) + \cos^2\theta \frac{\partial^2 u}{\partial y^2}.
$$

Then, adding $\frac{\partial^2 u}{\partial r^2}$, we get cancellation of this mixed partials and the product of a cosine squared and sine squared term for each second partial of u in x and y respectively. These reduce to a coefficient of one.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

which is the identity we wished to show.

3.

Problem. Suppose that $f : \mathbb{R}^2 \longmapsto \mathbb{R}$ is C^2 (has continuous second partials). Fix $\vec{x}, \vec{h} \in \mathbb{R}^2$. Let $g : \mathbb{R} \longmapsto \mathbb{R}$, $g(t) = f(\vec{x} + t\vec{h})$. Find $g'(t)$ and $g''(t)$.

Define f as $f(x, y)$ and let $\vec{x} = (x_1, x_2)$ and $\vec{h} = (h_1, h_2)$.

Then, let $x = x_1 + th_1$ and $y = x_2 + th_2$, so $\frac{dx}{dt} = h_1$ and $\frac{dy}{dt} = h_2$.

Then,

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h_1\frac{\partial f}{\partial x} + h_2\frac{\partial f}{\partial y}.$$

Next,

$$\begin{aligned}
\frac{d^2 g}{dt^2} &= \frac{d}{dt}\left[h_1\frac{\partial f}{\partial x} + h_2\frac{\partial f}{\partial y}\right] \\
&= h_1\left(\frac{\partial^2 f}{\partial x^2}\frac{dt}{dx} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}\right) + h_2\left(\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right) \\
&= {h_1}^2 f_{xx} + h_1 h_2(f_{xy} + f_{yx}) + {h_2}^2 f_{yy}.
\end{aligned}$$