# Math 335 Homework 1

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**Problem** (1)**.** Determine whether the following integrals converge.

i) $\int_0^\infty x^2 e^{-x^2}\,dx$

Using IBP, we see that the above integral becomes $-\frac{x}{2}e^{-x^2}\Big|_0^\infty - \int_0^\infty -\frac{1}{2}e^{-x^2}\,dx$.

The left term is zero as $\lim\limits_{x\to\infty} -\frac{x}{2e^{x^2}} = 0$. The right term can be split into $\int_0^\infty e^{-x^2}\,dx$, which has no singularities and is finite; and $\int_1^\infty e^{-x^2}\,dx$, which converges by comparison with $x^{-2}$ on $x \in [1, \infty)$. So (i) converges.

ii) $\int_3^\infty \frac{\sin 4x}{x^2-x-2}\,dx$

Note that the integrand is less than $\frac{1}{x^2-x-2} = f(x)$. Let $g(x) = x^{-2}$.

Then, $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{x^2}{x^2-x-2} = 1$.

Since $\int_3^\infty x^{-2}\,dx$ converges and the limit as $x \to \infty$ of the ratio of $f$ and $g$ is finite, then $\int f$ converges by the comparison theorem.

Since $f$ converges and (ii) is bounded above by $f$, then (ii) converges as well.

iii) $\int_1^\infty \tan\frac{1}{x}\,dx$

Note that $\forall\theta \in [0, \frac{\pi}{2})$, $\tan\theta > \theta$.

Therefore, for large $x$, say $x > N$, $\tan\frac{1}{x} > \frac{1}{x}$.

So, $\int_N^\infty \tan\frac{1}{x}\,dx > \int_N^\infty \frac{dx}{x}$. Thus, by comparison with $x^{-1}$, (iii) diverges.

**Problem** (2)**.** Determine whether the following integrals converge.

i) $\int_{\frac{\pi}{2}}^{\pi} \cot x\,dx$

Note that there is a singularity of cotangent at $\pi$. Around $x = \pi$, $\sin x \approx x - \pi$ and $\cos x \approx -1$.

So, $\cot x \approx \frac{1}{\pi - x}$, which means that, near $x = \pi$, $\cot x \approx O(\frac{1}{x})$, which diverges.

Thus, (i) diverges by comparison.

ii) $\int_0^1 \frac{dx}{x^{1/2}(x^2+x)^{1/3}}$

We see that (ii) can also be expressed as $\int_0^1 \frac{dx}{x^{5/6}(x+1)^{1/3}}$.

Let $g(x) = \frac{1}{x^{5/6}}$. $g$ converges on $x \in [0, 1]$.

Then, $\lim_{x\to\infty} \frac{x^{5/6}}{x^{5/6}(x+1)^{1/3}} = 0$.

Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ and $g$ converges on the given interval, then, by the comparison test, (ii) converges as well.

iii) $\int_0^1 \frac{1-\cos x}{\sin^3 2x}\, dx$

Note that there is a singularity at zero, and this is the only such value given that the next root of sine occurs at $\frac{\pi}{2} > 1$.

Near $x = 0$, $\sin 2x \approx 2x$ and $1 - \cos x \approx \frac{x^2}{2}$.

So, the integrand is approximately $\frac{\frac{x^2}{2}}{(2x)^3} = \frac{1}{16x}$, which diverges at zero.

So, (iii) diverges by comparison.

**Problem** (3)**.** Determine whether the following integrals converge.

i) $\int_0^\infty \frac{\sqrt{x}}{e^x - 1}\, dx$

Note that (i) has a singularity at zero.

So, we will split the integral into two regions $[0, \epsilon]$ and $[\epsilon, \infty)$.

Near $x = 0$, $e^x - 1 \approx x$, so $\frac{\sqrt{x}}{e^x-1} = O(x^{-1/2})$, which converges on $[0, \epsilon]$.

For large $x \to \infty$,

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x - 1} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{e^x} = 0.$$

So, the integral of $[\epsilon, \infty)$ converges.

Since the integral converges on both subregions, then it converges on their union which was the given interval.

ii) $\int_0^\infty x^{-1/5} \sin\left(\frac{1}{x}\right) dx$

First, we will split the integral into two regions $[0, 1]$ and $[0, \infty)$. Note that $|\sin\left(\frac{1}{x}\right)| \leq 1$.

For the first interval, let $y = \frac{1}{x}$. So $x \to 0 \implies y \to \infty$. Then, $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2}\, dy$.

So,

$$\begin{aligned}\int_0^1 x^{-1/5} \sin\left(\frac{1}{x}\right) dx &= \int_\infty^1 y^{1/5} \sin y \left(-\frac{1}{y^2}\right) dy \\ &= \int_1^\infty y^{-9/5} \sin y\, dy,\end{aligned}$$

which converges by comparison with $y^{-9/5}$ (given that the sine term is bounded) on $y \in [1, \infty)$ because $\frac{9}{5} > 1$.

For the second interval, since the sine term is bounded by one, then $|x^{-1/5} \sin\left(\frac{1}{x}\right)| \leq x^{-\frac{1}{5}}$.

Therefore the second interval converges on $[0, 1]$ by comparison with $x^{-\frac{1}{5}}$ since $\frac{1}{5} < 1$.

Since (ii) converges on both subintervals, then it also converges on their union.

iii) $\int_{-\infty}^\infty \frac{e^x}{e^x + x^2}\, dx$

We will split the integral of (iii) into two regions about $x = 0$. We will now show that the integral over the positive interval diverges.

For large $x$, $\frac{e^x}{e^x + x^2} = 0(1)$. In particular, using the Taylor expansion of $e^x$, we see that $\forall x > 0$, $e^x > x^2 \implies \frac{e^x}{e^x + x^2} > \frac{e^x}{2e^x} = \frac{1}{2}$.

Then, for large $x \geq N$, $\int_N^\infty \frac{e^x}{e^x + x^2}\, dx > \int_N^\infty \frac{dx}{2} = \infty$.

Since (iii) diverges on the positive subinterval, then (iii) diverges across the union of both subintervals.

**Problem** (4)**.** Let $f_p(x) = x^{-1}(\log x)^{-p}$ with $p > 0$.

a) Given $p, \epsilon > 0$, show for large $x$,

$$x^{-1-\epsilon} < f_p(x) < x^{-1}.$$

$\forall x > e$, $\forall p > 0$, $(\log x)^p > 1 \implies f_p(x) = x^{-1}(\log x)^{-p} < x^{-1}$.

With the same qualifications, $\lim_{\epsilon \to 0} x^\epsilon = 1 < \log x^p \implies f_p(x) = x^{-1}(\log x)^{-p} < \lim_{\epsilon \to 0} x^{-1-\epsilon}$.

b) For which $p$ does $\int_2^\infty f_p(x)\, dx$ converge?

Let $u = \log x$. So, $e^u = x$ and $e^u\,du = dx$.

Then, the above integral becomes $\int_{\log 2}^{\infty} e^{-u}u^{-p}e^u\,du = \int_{\log 2}^{\infty} u^{-p}\,du$, which converges when $p < 1$.

**Problem** (5)**.** Let $f(x) = 1$ on $[n, n + 2^{-n}]$ for $n \in \mathbb{Z}_{>0}$ and $f(x) = 0$ elsewhere.

a) Show $\int_0^\infty f(x)\,dx$ converges to one, yet $f(x) \not\to 0$ as $x \to \infty$.

The integral of $f$ can actually be expressed using a sum, since the area under the graph of $f$ consists of rectangles with height one and with $2^{-n}$, which is the length of each interval where $f(x) = 1$.

Thus, by the explicit formula for the geometric series, $\int_0^\infty f(x)\,dx = \sum_1^\infty 2^{-n} = \sum_0^\infty \left(\frac{1}{2}\right)^n - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 1$.

b) Modify $f$ to a function $g$ such that $\int_0^\infty g(x)\,dx$ converges yet $g$ is not bounded as $x \to \infty$.

If we desire $g$ to be unbounded, then we will need to shrink the intervals where $g \neq 0$ so that the increasing value of $g$ has a proportionally decreasing contribution to the final integral.

Let $g(x) = \begin{cases} x, & \forall n \in \mathbb{Z}_{>0},\ x \in [n, n + \frac{1}{2^n n}], \\ 0, \text{otherwise.} \end{cases}$

So, $\int_0^\infty g(x)\,dx \approx \sum_1^\infty 2^{-n} \cdot \frac{n}{n}$, which converges as seen above.

Furthermore, we can create an inequality by considering that each $n$th interval will have a value of $g$ less than $n+1$. This gives

$$\int_0^\infty g(x)\,dx < \sum_1^\infty 2^{-n}\left(\frac{n+1}{n}\right) < \infty.$$

So the integral of $g$ converges while $g$ grows without bound.

**Problem** (6)**.** Let $f(x) = x^2(x - \sin x)$ and $g(x) = (e^x - 1)(\cos 2x - 1)^2$.

a) Compute $T_5$ at $x_0 = 0$.

With the fifth order Taylor polynomial of sine, $\sum_{j=0}^{5} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$,

$$\begin{aligned} T_5(f) &= x^2\left(x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}\right)\right) \\ &= \frac{x^5}{3!} - \frac{x^7}{5!} + \frac{x^9}{7!} - \frac{x^{11}}{9!} + \frac{x^{13}}{11!}. \end{aligned}$$

Then, using the expansions of cosine and the exponential function,

$$\begin{aligned} T_5(g) &= \left(\sum_0^5 \frac{x^k}{k!} - 1\right)\left(\sum_0^5 \frac{(-1)^k (2x)^{2k}}{(2k)!} - 1\right)^2 \\ &= \left(\sum_0^5 \frac{x^k}{k!}\right)\left(\sum_1^5 \frac{(-1)^k (2x)^{2k}}{(2k)!}\right)^2. \end{aligned}$$

Since each of the leading terms in both sums if one, then we can expand the above to,

$$\left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)\left(-2x^2 + \frac{2x^4}{3} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \frac{(2x)^{10}}{10!}\right).$$

b) Use (a) to find $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ without L'Hopital.

To consider $x \to \infty$, we will use the largest order terms.

From $f$, this is $\frac{x^{13}}{11!}$. From $g$, this is $\frac{x^5}{5!}\left(-\frac{(2x)^{10}}{10!}\right) = \frac{-2^{10}x^{15}}{5!10!}$

So $\frac{f}{g} = O(\frac{1}{x^2})$, which goes to zero as $x \to \infty$. Thus, $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$ as well.

**Problem** (7)**.** Estimate $\int_0^1 e^{-x^2}\,dx$ with error less than $10^{-3}$.

Let $g(x) = e^{-x} = \sum_0^\infty \frac{(-1)^n x^n}{n!} = P_k(x) + R_k(x)$.

Corollary 2.61 $\implies \forall x \in [0,1]$,

$$\begin{aligned} |R_k(x)| &\le \left(\sup_{x\in[0,1]} g^{(k+1)}(x)\right)\frac{x^{k+1}}{(k+1)!} \\ &\le \left(\sup_{x\in[0,1]} e^{-x}\right)\frac{x^{k+1}}{(k+1)!} \\ &\le \frac{1}{(k+1)!}. \end{aligned}$$

Then,

$$\begin{aligned} \left|\int_0^1 \left(e^{-x^2} - P_k(x^2)\right)\,dx\right| &\le \int_0^1 |R_k(x^2)|\,dx \\ &\le \int_0^1 \frac{dx}{(k+1)!} \\ &= \frac{1}{(k+1)!} \\ &\le 10^{-3}. \end{aligned}$$

Since $(k+1)! \ge 10^3 \implies k \ge 7$, then we must use a Taylor polynomial of at least order seven.

Plugging in to a calculator, we see that $\int_0^1 e^{-x^2}\, dx \approx 0.7468$.

**Problem** (8)**.** Let $f \in C^k(I)$ with $a \in I$ open.

$$\forall i \in [1, k-1],\ f^{(i)}(a) = 0. \quad f^{(k)}(a) \neq 0..$$

Use Corollary 2.60 to show

i) $k$ even $\implies f$ attains local extrema at $a$ where

$$\begin{aligned} f^{(k)}(a) < 0 &\implies \text{max at } a \\ f^{(k)}(a) > 0 &\implies \text{min at } a. \end{aligned}$$

ii) $k$ odd $\implies f$ has no local extrema.

Since all derivatives except for the $k$th are zero, and $P_{a,k}(h)$ then

$$\begin{aligned} P_{a,k}(h) &= f(a) + \frac{f^{(k)}(a)}{h!}h^k \\ f(a+h) &= f(a) + \frac{f^{(k)}(a)}{h!}h^k + R_{a,k}(h) \\ \frac{f(a+h) - f(a)}{h^k} &= \frac{f^{(k)}(a)}{k!} + \frac{R_{a,k}(h)}{h^k}. \end{aligned}$$

Since $f \in C^k(I)$, then by Corollary 2.60,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h^k} = \frac{f^{(k)}(a)}{k!} + 0.$$

So these terms must have the same signs.

With $k$ even, then $h^k > 0$. Then, $f^{(k)}(a) < 0 \implies f(a) > f(a+h) \implies f(a)$ is a local max, and same with the local min.

With $k$ odd, then $h^k$ changes signs when approaching $h \to 0$, so $f(a)$ is not a local extrema.