Math 136 Homework 1

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1. Write down a basis for the space of

   (a) 3×3 symmetric matrices;

   (b) $n \times n$ symmetric matrices;

   (c) $n \times n$ antisymmetric matrices $A^T = -A$ matrices;

   a) We note that a symmetric matrix is given by $A^T = A$. So, each entry a_{ij} must be equal to a_{ji} for $i, j \leq 3$.

   Thus, we can construct a basis of six matrices that are symmetric about the diagonal,

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \right\}.$$

   b) Let M_{ij} be the matrix of all zeros with a 1 in position i, j.

   Then, the basis for the space of $n \times n$ symmetric matrices is given by the set

$$\{M_{ij} + M_{ji}, i \geq j\}.$$

   We restrict the indices to $i \geq j$ such that we will not create any linearly dependent duplicates.

   c) Using the same definition of M_{ij} as above, we consider that the middle diagonal the any antisymmetric matrix must be zero because, for zero only, $0 = -0$. Thus, our basis can be defined as follows,

$$\{M_{ij} - M_{ji}, i > j\}.$$

   This time, we do not include the cases where $i = j$ because our middle row must be zero.[1]

2. Prove that $\text{trace}(AB) = \text{trace}(BA)$.

   First, we will consider how the diagonals of the matrix AB, $(AB)_{ii}$ are created.

   With $A_{m\times n}$ and $B_{n\times m}$, the product AB will be a $m \times m$ square matrix. We will fix some i as the index of m and note that we take the dot product of the i^{th} row of A and the i^{th} column of B. We will iterate over n with the index j.

[1] The trace of an antisymmetric matrix must be zero.

This gives,

$$\sum_{j=1}^{n} a_{ij} b_{ji} = (AB)_{ii}.$$

In order to obtain the trace of AB, we need to sum over the index i from 1 to n.

Thus,

$$\text{trace}(AB) = \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji}.$$

We then notice that BA produces an $n \times n$ matrix. We then see that,

$$\text{trace}(BA) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} a_{ji}.$$

We can use the fact that we can rearrange sums (linearity of addition) and that multiplication is commutative to see that this is the same as

$$\sum_{j=1}^{m} \sum_{i=1}^{n} b_{ij} a_{ji}.$$

We can swap labels where $i = j$ and $j = i$ to make this the same as above.

So, the statement holds.