Math 334 Homework 2

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Problem (1).

a. Prove that an infinite union of open sets is open. Where U_i is an open subset of $\mathbb{R}^n$, $\cup_{i=1}^{\infty} U_i$ is open.

   Is the countable size of the collection of sets important?

b. Give an example of an infinite collection of closed sets S_i whose union $\cup_{i=1}^{\infty} S_i$ is not closed.

Proposition. The union of any two open sets $S_1, S_2 \in \mathbb{R}^n$ is open.

Proof of Proposition. We wish to show that, for S_1, S_2 open, $\partial(S_1 \cup S_2) \cap (S_1 \cup S_2) = \emptyset$.

Since S_1 open, $\forall x_1 \in S_1$. $\exists r > 0$. $B_r(x) \subset S_1 \implies B_r(x) \subset S_1 \cup S_2$.

Since S_2 open, $\forall x_2 \in S_2$. $\exists r > 0$. $B_r(x) \subset S_2 \implies B_r(x) \subset S_1 \cup S_2$.

Therefore, $\forall x \in S_1 \cup S_2$. $\exists r > 0$. $B_r(x) \subset S_1 \cup S_2$, which means that $S_1 \cup S_2$ is open. □

Proof of a. We will prove the statement by induction on $m \in \mathbb{Z}^+$.

For the base case, if we choose m as one, we see that the single union of an open set will produce itself, an already open set. Thus, we choose $m = 2$, $\cup_{i=1}^{2} U_i = U_1 \cup U_2$, which is open by the proposition.

Assume the $m = k$ case holds, that is, $\cup_{i=1}^{k} U_i$ in open.

Then, for the $m = k + 1$ case,

$$\cup_{i=1}^{k+1} U_i = \left(\cup_{i=1}^{k} U_i\right) \cup U_{k+1}.$$

But, we see that the left hand side is open by the I.H., and the right hand side is open by the statement. So, $\cup_{i=1}^{k+1} U_i$ is open by the proposition and the $k + 1$ case holds. □

If we had an uncountable infinity, we could not have performed induction. Is it possible that a sufficiently large infinite union of open sets is no longer open?

Proof for b. Consider last week's problem using a set of rationals.

Let S_i for some index i be a set with a single vector with rational components, $x \in \mathbb{Q}^n$. The singleton S_i is closed because, for the only value $x \in S_i$,

$$\forall r > 0.\ B_r(x) \cap S_i = \{x\} \neq \emptyset \wedge B_r(x) \cap S_i^c = \mathbb{R}^n \setminus \{x\} \neq \emptyset.$$

But, the infinite (or even finite) union of such closed singletons produces a set whole boundary contains irrationals, as we have seen that the interior of such a union is $\emptyset$.

Such irrationals are not contained within any S_i as they contain only rational-valued components.

Thus, we have that $\partial S \not\subset S$ where $S = \cup_{i=1}^{\infty} S_i$, which means that such a union is not closed. $\square$

Problem (2)**.** Let $f(x) = \frac{1}{q}$ where $\forall p, q \in \mathbb{Z}.\ x = \frac{p}{q},\ q > 0$ such that p, q coprime, and $f(x) = 0$ where $x \in \mathbb{R} \setminus \mathbb{Q}$.

Determine all x for which $f(x)$ is continuous.

Proposition. f is continuous for all irrationals and discontinuous for all rationals.

Proof. First, we will consider all a irrational, where $f(a) = 0$. We will consider the interval of rationals containing a, $\left(\frac{n}{m}, \frac{n+1}{m}\right)$ for some integers m, n.

We will then define $\delta = \min\left\{a - \frac{n}{m}, a - \frac{n+1}{m}\right\}$. Note that $\delta < \frac{n+1}{m} - \frac{n}{m} = \frac{1}{m}$. Now, let $\epsilon > 0$ be given and choose an m such that $1/m < \epsilon$.

We will now consider x inside of the interval. If x rational, then let x be of the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ coprime.

We see that $|x - a| < \delta$ implies $q > m$ since $\frac{n}{m} < \frac{p}{q} < \frac{n+1}{m}$. So,

$$0 < f(x) = \frac{1}{q} < \frac{1}{m} < \epsilon,$$

which also means $|f(x) - f(a)| < \epsilon$.

If x irrational, then $f(x) = 0$, so $|f(x) - f(a)| = 0 < \epsilon$.

Thus, $\forall x$, $\forall \epsilon > 0$, $\exists \delta > 0$, $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$. So, f is continuous at all irrational numbers.

Second, we will consider all $a = \frac{p}{q}$ rational, with $f(a) = \frac{1}{q}$. Then, for $k \in \mathbb{Z}$, let $x = a + \frac{\sqrt{2}}{k}$ be an irrational number.

Now, $|x - a| = \frac{\sqrt{2}}{k}$ and $f(x) = 0$. Then, $\forall \delta > 0$, $\exists \epsilon > 0$, $|x - a| = \frac{\sqrt{2}}{k} < \delta \wedge |f(x) - f(a)| = \frac{1}{q} \geq \epsilon$. Choose $\epsilon = \frac{1}{2q}$ and we're done: f is not continuous for any rational number.

$\square$

Problem (3)**.** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{y(y-x^2)}{x^4} & 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Determine all points s.t. f not continuous.

Proposition. f is continuous everywhere but at $(0,0)$.

Proof. First, we will show that $\lim\limits_{(x,y)\to(0,0)} \neq 0 = f(0)$.

Consider approaching the origin along the path $y = \frac{x^2}{2}$,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{y(y-x^2)}{x^4} &= \lim_{x\to 0} \frac{\frac{x^2}{2}\left(\frac{x^2}{2} - x^2\right)}{x^4} \\ &= \lim_{x\to 0} \frac{\frac{x^4}{4} - \frac{x^4}{2}}{x^4} \\ &= \lim_{x\to 0} \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= -\frac{1}{4} \neq 0. \end{aligned}$$

Since $f(0) = 0$ but the limit approaching zero along the path $y = \frac{x^2}{2}$ is not zero, then f is not continuous at zero.

Since $0 < y < x^2 \implies 0 < x^4$, and both the numerator and denominator of f are continuous, then their quotient with non-zero denominator is also continuous outside of the parabola x^2 for positive y.

Along the boundary path $y = x^2$, f is continuous as it achieves the value zero,

$$\lim_{y\to x^2} \frac{y(y-x^2)}{x^4} = \frac{x^2(x^2-x^2)}{x^4} = 0 = f(S),$$

where $S = \{(x,y) \in \mathbb{R}^2 \mid y \geq x^2\}$, the compliment of the main region excluding non-positive y. $\square$

Problem (4)**.** Let $x_1 = \sqrt{2}, x_{n+1} = \sqrt{2 + x_n}$

a Prove $\forall n,\ x_n < 2$ and $x_{n+1} > x_n$.

b Prove $(x_n)_{n=1}^{\infty}$ converges, and find the limit.

Proof of a. We will prove the statement by induction on n. Let $n = 1$, then $x_1 = \sqrt{2} < 2$. Next,

$$x_2 = \sqrt{2 + x_1} = \sqrt{2 + \sqrt{2}} > \sqrt{2} = x_1,$$

which implies that $x_2 > x_1$.

Now, assume that the statement holds for $n = k$,

$$x_k < 2 \text{ and } x_{k+1} > x_k.$$

Then,

$$x_{k+1} = \sqrt{2 + x_k} < \sqrt{2+2} = 2.$$

So $x_{k+1} < 2$.

Also,

$$2x_{k+1} > (x_{k+1})^2 = 2 + x_k > 2x_k.$$

So $x_{k+1} > x_k$.

Since the base case $n = 1$ holds, and $n = k + 1$ holds where $n = k$ holds, then the statement holds for all n. □

Proof of b. Since $\forall n,\ x_n < 2$, then (x_n) is bounded above by 2. Since $\forall n,\ x_{n+1} > x_n$, (x_n) is monotonically increasing. Thus, (x_n) converges by the MBST to its upper boundary. □

Proposition. (x_n) converges to 2, $\sup(x_n) = 2$.

Proof of Proposition. Suppose, for a contradiction, $\exists\, b < 2$ such that $\sup(x_n) = b$.

Then, $\forall \epsilon > 0,\ \exists x_n,\ b - \epsilon < x_n < b < 2$. But, since b is ϵ close to 2 for any small ϵ, then $b = 2$, contradicting the assumption that there was a supremum b smaller than 2. □

Problem (5)**.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Suppose $(x_n)_{n=1}^{\infty}$ converges to x.

Prove

$$\lim_{n\to\infty} \frac{f(x_1) + \cdots + f(x_n)}{n} = f(x).$$

Proof. Since (x_n) converges to x, $\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ |x_n - x| < \epsilon$.

So, we will split the limit into two parts, where f maps on x_n which are ϵ close to x, and on x_n which are far from x.

$$\lim_{n\to\infty} \left[\sum_{i=1}^{N-1} \frac{f(x_i)}{n} + \sum_{i=N}^{n} \frac{f(x_i)}{n} \right] = f(x).$$

But, the left sum is a finite value by n which exceeds any number, so this quantity will become to zero. Then, since the x_i for $i \geq N$ approach x, $f(x_i)$ will approach $f(x)$ by sequential continuity. So,

$$\begin{aligned}
f(x) &= \lim_{n\to\infty} \sum_{i=N}^{n} \frac{f(x_i)}{n} \\
&= \lim_{n\to\infty} \sum_{i=N}^{n} \frac{f(x)}{n} \\
&= \lim_{n\to\infty} \left(\frac{n-N}{n} \right) f(x) \\
&= \lim_{n\to\infty} \left(1 - \frac{N}{n} \right) f(x) \\
&= f(x).
\end{aligned}$$

□