Math 334 Take Home Quiz

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Problem (1)**.** Let $S = \{(x, y) \mid x > 0, y = \sin\left(\frac{1}{x}\right)\} \cup \{(0, y) \mid y \in [-1, 1]\}$ be the topologists' sine curve. Prove S not path connected.

Proof. We will show that part of S, for $x \in (0, \pi/2)$, is not path connected, so all of S is not path connected.

Let

$$\begin{aligned} f : [0,1] &\longrightarrow S \\ f(0) &\longmapsto (\pi/2, 1) \\ f(1) &\longmapsto (0, 1) \end{aligned}$$

be continuous.

Since $[0, 1]$ is a closed interval in $\mathbb{R}$, it is compact and connected by 1.21 and 1.25 respectively.

Since f is continuous, then the image of $f([0, 1]) \subset S$ is also compact and connected by 1.22 and 1.26 respectively.

Since the image of f is connected, then the first coordinate of f achieves all values $x \in (0, \pi/2)$ while $t \in [0, 1]$.

Since $\forall k \in \mathbb{Z}^+$, $\sin 2\pi k = 0$, then, for $x = \frac{1}{2\pi k}$, $y = \sin\left(\frac{1}{x}\right) = 0$. Thus, for all such k, $\exists t_k \in [0, 1]$ such that $f(t_k) = \left(\frac{1}{2\pi k}, 0\right)$. Since $[0, 1]$ compact, there is a convergent subsequence of t_k which converges to some $t_0 = \lim\limits_{k\to\infty} t_k$.

Then, note that, for t_{k+1}, $f(t_{k+1}) = \left(\frac{1}{2\pi(k+1)}, 0\right)$. Since $[0, 1]$ connected and t_{k+1} moves the sine function one more time through the period 2π, from 0 to 0, then there exists intermediary $t \in (t_k, t_{k+1})$ for which the sine function achieves values between -1 and 1. So, the second coordinate of f achieves all values $y \in (-1, 1)$ for such intermediary t.

But, by assumption, $f(t_0) = \lim\limits_{k\to\infty} \left(\frac{1}{2\pi k}, 0\right) = (0, 0)$, yet the second coordinate of f never settles as it ranges between -1 and 1 as t increases. So, by contradiction, S is not path connected. $\square$

Problem (2)**.**

(a) Define

$$
\begin{aligned}
f &: \mathbb{R}^2 \longrightarrow \mathbb{R} \\
f(x,y) &= \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}
\end{aligned}
$$

Determine whether f is continuous.

(b) Define

$$
\begin{aligned}
g &: B_{\frac{1}{4}}(0) \longrightarrow \mathbb{R} \\
g(x,y) &= \begin{cases} \frac{4xy}{(4x^2+y^2)\log(x^2+4y^2)}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}
\end{aligned}
$$

Determine whether g is continuous.

Proposition (a)**.** f is not continuous at the origin $(0,0)$

Proof of a. We will approach on the linear path $y = ax$.

So, given that $x \neq 0$,

$$f(x,ax) = \frac{ax^2}{x^2 + ax^2} = \frac{a}{1+a}.$$

But then,

$$\lim_{(x,y)\to(0,0)} \frac{a}{1+a} = \frac{a}{1+a}$$

depends solely on the chosen a. For example, if $a \neq 0$, then $\frac{a}{1+a} \neq 0$. So, f is not continuous at the origin, and the limit at the origin does not exist, so f is not everywhere continuous. □

Proposition (b)**.** g continuous.

Proof of b. We must ensure that the limit as $(x,y) \to (0,0)$ matches the function value of zero.

First, note that

$$
\begin{aligned}
(x+y)^2 &\geq 0 \\
x^2 + 2xy + y^2 &\geq 0 \\
|x^2 + y^2| &\geq |2xy| \\
2(x^2+y^2) &\geq |4xy| \\
2(4x^2+y^2) &> |4xy|.
\end{aligned}
$$

Thus, g is bounded above and below,

$$\frac{-2(4x^2+y^2)}{(4x^2+y^2)\log(x^2+4y^2)} < \frac{4xy}{(4x^2+y^2)\log(x^2+4y^2)} < \frac{2(4x^2+y^2)}{(4x^2+y^2)\log(x^2+4y^2)}.$$

With $(x,y) \neq (0,0)$, we see that g is bounded by two opposite terms,

$$\frac{-2}{\log(x^2+4y^2)} < g < \frac{2}{\log(x^2+4y^2)},$$

the denominator of which becomes large as $(x,y) \to (0,0)$, so the bounds go to zero. So, g must also go to zero by being bounded above and below by these two functions which go to zero. □

Problem (3)**.** The *limit superior* of a sequence $\{x_n\} \subset \mathbb{R}$ is defined by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} x_m.$$

Assume $\{x_n\}$ is bounded.

(a) Show that $\{s_n\}$ defined by $s_n := \sup_{m\geq n} x_m$ converges.

(b) (i) Show $x_{n_i} \to L$ convergent subsequence $\implies \forall n, \sup_{m\geq n} x_m \geq L$.

(ii) Given $\limsup_{n\to\infty} x_n = M$, find a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} x_{n_i} = M$.

(iii) Show that the limit superior is the superior of all subsequential limits

$$\limsup_{n\to\infty} x_n = \sup\left\{ \lim_{i\to\infty} x_{n_i} \;\middle|\; \{x_{n_i}\}_{i\in\mathbb{N}} \text{ is a convergent subsequence} \right\}.$$

Proof of a. Since the supremum is concerned only with with the x_m with $m \geq n$, then s_n cannot be an increasing sequence. If x_n was the supremum of $\{x_m\}$, then all other $x_i \in \{x_m\}$ must have been $x_i \leq x_n$. So, for s_{n+1}, the supremum could only stay the same or decrease, so s_n is decreasing.

Since $\{x_n\} \subset \mathbb{R}$ is bounded, then $\{s_()\}$, which contains all suprema of x_n, must also be bounded.

Since s_n is decreasing and bounded, then s_n must converge by 1.16. □

Proof of b. *(i)* Since s_n takes the supremum of a subset of $\{x_n\}$ starting with x_n, then,

$$\forall n,\, s_n \geq x_n.$$

So, $\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} x_n$.

But, if we choose a convergent subsequence $\{x_{n_i}\}$, then $\lim_{n\to\infty} s_n \geq \lim_{i\to\infty} x_{n_i}$ will still hold. Then, recalling that s_n is decreasing, $\forall n,\, s_n \geq \lim_{n\to\infty} s_n$, and that $x_{n_i} \to L$, we see that $s_n \geq L$.

(ii) We can construct a subsequence which converges to M by taking $x_{n_{i+1}}$ if $M - x_{n_i} \leq M - x_{n_{i+1}}$. Notice that this will ensure that we approach the limit superior from below, since $\lim_{n\to\infty} s_n \geq L$ for all subsequences x_{n_i} which converge to L.

(iii) Since $\limsup_{n\to\infty} x_n = M \geq \lim_{i\to\infty} x_{n_i} = L$, then take x_{n_i} with the largest L, obtained in (ii), where $L = M$. Thus,

$$\limsup_{n\to\infty} x_n = \sup\left\{ \lim_{i\to\infty} x_{n_i} \right\},$$

for all such convergent subsequences x_{n_i}.

□

Problem (4)**.** Let $S \subset \mathbb{R}^n$ be compact. Suppose $C_i \subset S$ are nonempty and closed, and are nested, $C_1 \supset C_2 \supset \cdots$. Prove

$$\bigcap_{i=1}^{\infty} C_i \neq \emptyset.$$

Proof. Let $U_i = C_i^c$, so each U_i is open.

Suppose, for a contradiction, that $\bigcap_{i=1}^{\infty} C_i = \emptyset$, then $\bigcup_{i=1}^{\infty} U_i = \mathbb{R}^n$.

So $\{U_i\}$ is an open cover of S, which has a finite subcover by compactness.

Thus, $\exists N$ such that $S \subset \bigcup_{i=1}^{N} U_i$, which implies $\bigcap_{i=1}^{N} C_i = \emptyset$.

But C_i nested, so $C_n = \bigcap_{i=1}^{N} C_i = \emptyset$, which is a contradiction. $\square$