Math 334 Homework 3

Alexandre Lipson

October 15, 2024

Problem (1)**.** Give an example of an open cover of the open unit interval $(0, 1)$ which does not admit a finite subcover.

Proposition. One such cover S is the union of expanding covers formed by the one-ball centered at $\frac{1}{2}$ with radius given by the sequence $x_k = \frac{1}{2} - \frac{1}{2^k}$ which converges to $\frac{1}{2}$.

$$S = \lim_{n\to\infty} S_n, \quad S_n = \bigcup_{k=1}^{n} B_{x_k}\left(\frac{1}{2}\right), \quad x_k = \frac{1}{2} - \frac{1}{2^k}.$$

Proof. Suppose, for a contradiction, that there exists a finite subcover S_m which covers the open unit interval.

By the convergence of x_k, $\exists N,\ \forall \epsilon > 0,\ \forall k \geq N,\ \left|x_k - \frac{1}{2}\right| < \epsilon$. So, $\left|\frac{1}{2} - \frac{1}{2^k} - \frac{1}{2}\right| < \epsilon \implies \frac{1}{2^k} < \epsilon$. Then,

$$k > \log_2\left(\frac{1}{\epsilon}\right).$$

Since $\lim\limits_{x\to 0} \log_2\left(\frac{1}{x}\right) \to \infty$ and ϵ was arbitrarily small, then k must be larger than any number.

But, we wished to find a finite subcover $S_m = \bigcup\limits_{k=1}^{m} B_{x_k}(1/2) \subset S_n$, yet we have seen that B_{x_k} will cover the open unit interval only when k exceeds any number. Thus m must also exceed any number, so we are unable to produce a finite subcover of S. □

Problem (2)**.** Given $U, V \subset \mathbb{R}^n$, define the distance metric,

$$d(U, V) = \inf\{|x - y| \mid x \in U,\ y \in V\}.$$

a) Show $(\overline{U} \cap V) \neq \emptyset \vee (U \cap \overline{V}) \neq \emptyset \implies d(U, V) = 0$.

b) Show U compact, V closed, $U \cap V = \emptyset \implies d(U, V) > 0$.

c) Give an example for $U, V \subset \mathbb{R}^2$ closed, disjoint, and $d(U, V) = 0$.

Proof of a. First, we will consider $\overline{U} \cap V \neq \emptyset$. Then,

$$\overline{U} \cap V \neq \emptyset \implies \exists a \in \overline{U} \cap V \implies a \in \overline{U} \wedge a \in V.$$

Choose $y = a,\ y \in V$. If $a \in U \subset \overline{U}$, then choose $x = a,\ x \in U$, so $x = y$. Thus, $d(U, V) = 0$.

If $a \in \overline{U}$ but $a \notin U$, then $a \in \partial U$. So, there must be a ball with radius $\epsilon > 0$ around a which contains some $x_0 \in U$. Thus, a must be ϵ close to x_0 for some arbitrarily small ϵ, so choose $x_0 = x$ such that $x - y = \epsilon$. Then, $d(U, V)$ is this ϵ, so $d(U, V) = 0$. □

Proof of b. Since U compact and in R^n, then U is closed, so U contains all of its limit points. Since V is closed, it contains all of its limit points. For a contradiction, assume $U \cap V = \emptyset$ and $d(U, V) = 0$. So, $\exists u_n \in U,\ \exists v_n \in V,\ |u_n - v_n| \to 0$ as $n \to \infty$.

Since U compact, then all sequences (u_n) have a convergence subsequence (u_{n_k}). Let $u_{n_k} \to u \in U$. Let the sequence $v_{n_k} \in V$ correspond to u_{n_k}. Since $|u_n - v_n| \to 0$, then $|u_{n_k} - v_{n_k}| \to 0$. So, $\exists v \in V$ such that $v_{n_k} \to v$.

Since u_{n_k}, v_{n_k} converge to the same limit point, then $u = v$, which implies $U \cap V \neq \emptyset$, contradicting the assumption that U and V were disjoint. □

Proof of c. Let $U = \{(x, 0) \mid x \geq 1\}$. U is closed because it contains all of its limits points $(x_0, 0)$, $x_0 > 1$.

Let $V = \{(x, \frac{1}{x}) \mid x \geq 1\}$ because it contains all of its limit points.

But, the sets are disjoint as $\forall x \in \mathbb{R}$, $(x, 0) \neq (x, \frac{1}{x})$, so $U \cap V = \emptyset$. Then, as $x \to \infty$, $|(x, \frac{1}{x}) - (x, 0)| = |(0, \frac{1}{x})| = \frac{1}{x} \to 0$. So, $d(U, V) \to 0$ as well. □

Problem (3)**.** Let $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$. Prove $\mathbb{S}^n$ is connected.

Proof. We will show that $\mathbb{S}^n$ is path connected. For all $p, q \in \mathbb{S}^n$, let $\ell : [0, 1] \longrightarrow \mathbb{R}^{n+1}$, $\ell(t) = (1 - t)p + tq$ be a linear interpolation between function p and q. Clearly, $\ell(0) = p$ and $l(1) = q$.

But, we wish to show that there is a function that maps to $\mathbb{S}^n$, and we might have $\exists t \in [0, 1],\ |l(t)| \neq 1$. To ensure that our path connecting p and q maps to $\mathbb{S}^n$ for all t, then we can normalize $\ell(t)$.

Now, let $\gamma(t) = \frac{\ell(t)}{|\ell(t)|} = \frac{(1-t)p+tq}{|(1-t)p+tq|}$. Thus, $\forall t$, $|\gamma| = 1$, so $\gamma : [0, 1] \longrightarrow \mathbb{S}^n$.

Next, the linear interpolation function is continuous and the norm function is continuous. So, their composition is continuous. Since their composition is not zero for all t, then their quotient γ is continuous. Furthermore, $\gamma(0) = \frac{p}{|p|} = p$ and $\gamma(1) = \frac{q}{|q|} = q$ still holds. So, $\mathbb{S}^n$ is path connected for any points p, q. Thus, $\mathbb{S}^n$ is also connected. □

Problem (4)**.** Suppose $f : \mathbb{S}^n \longrightarrow \mathbb{R}^n$ is continuous. Note that $x \in \mathbb{S}^n \implies -x \in \mathbb{S}^n$. Prove $\exists x \in \mathbb{S}^n\ :\ f(x) = f(-x)$.

Proof. Let $g(x) = f(x) - f(-x)$. We wish to find an x such that $g(x) = 0$.

Since g is the difference of two continuous functions f, then g is also continuous.

Since $\mathbb{S}^n$ connected by Problem 3, then $g : \mathbb{S}^n \longrightarrow \mathbb{R}^n$, a continuous function on a connected domain, is connected.

Notice that $g(-x) = f(-x) - f(x) = -g(x)$. So, g is odd.

If $g(x) = 0$ identically, then we're done. So, there must be an x where $g(x) > 0$. Then, since g odd, $g(-x) < 0$. But, g is connected, so $\exists a,\ -x < a < x,\ g(a) = 0$. $\square$

Problem (5)**.**

a) Prove S connected $\implies \overline{S}$ connected.

b) Let $S = \{(x, y) \mid x > 0,\, y = \sin\left(\frac{1}{x}\right)\} \cup \{(0, y) \mid y \in [-1, 1]\}$ be the topologists' sine curve. Prove S connected.

Proof of a. For a contradiction, assume S connected but $\overline{S}$ not connected. Then, $\overline{S} = A \cup B$ with $\overline{A} \cup B = A \cup \overline{B} = \emptyset$. So, A and B are disjoint. Then, we can write S as,

$$S = S \cap \overline{S} = S \cap (A \cup B) = (S \cap A) \cup (S \cap B).$$

But, $S \cap A$ and $S \cap B$ are subsets of the disjoint A and B respectively. So, S is a union of disjoint sets, contradicting the assumption that S was connected. So, by contradiction, $\overline{S}$ must be connected as well. $\square$

Proof of b. Let the left hand side of the union defining S be A and the right and side be B. We know that A is path connected because $\sin(1/x)$ is continuous. Thus, A is also connected.

Next, we will show that $\overline{A} = A \cup B = S$. First, $\forall k > 0$, $\forall x \geq k$, A closed because it is the image of the continuous function $\sin(1/x)$ over the closed interval $[k, \infty)$ as a subset of the extended reals.

Next, we will now consider A for $x < k$, which is open. B is the boundary of of this part of A because, $\forall r > 0$, $\forall b \in B$, $B_r(b)$ will contain some point in A for $x > 0$ and some point in A^c for $x < 0$. Thus, $A \cup B = \overline{A}$.

Since A connected, then $\overline{A} = S$ connected by part a. $\square$