Math 134 Homework 8

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November 20, 2023

7.3

66

Find a formula for the nth derivative.

$$\frac{d^n}{dx^n}[\ln(1-x)]$$

Proposition. $\frac{d^n}{dx^n}[\ln(1-x)] = (n-1)!(-1)^n(1-x)^{-n}$.

Proof. We will prove that the proposition holds by induction on n.

For the base case, when $n = 1$,

$$(1-1)(-1)^1(1-x)^{-1} = \frac{-1}{1-x}.$$

We verify by differentiating $\ln(1-x)$,

$$\frac{d}{dx}[\ln(1-x)] = -1 \cdot \frac{1}{1-x} = \frac{-1}{1-x}.$$

So, the base case holds.

By the inductive hypothesis, we assume that the formula holds for $n = k$.

For the inductive step, we will find $\frac{d^{k+1}}{dx^{k+1}}[\ln(1-x)]$ by differentiating $\frac{d}{dx}[\ln(1-x)]$.

$$\begin{aligned}
&\frac{d}{dx}\left[(k-1)!(-1)^k(1-x)^{-k}\right] \\
=&(k-1)!(-1)^k\frac{d}{dx}\left[(1-x)^{-k}\right] \\
=&(k-1)!(-1)^k \cdot \left((-k)(1-x)^{-k-1}\right) \\
=&k(k-1)!(-1)^{k+1}(1-x)^{-(k+1)} \\
=&((k+1)-1)!(-1)^{k+1}(1-x)^{-(k+1)},
\end{aligned}$$

which is the case when $n = k+1$.

So, the proposition holds.

$\square$

7.4

72

Prove the proposition.

Proposition. *For all $x > 0$ and $n \in \mathbb{Z}^+$,*

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Proof. Since $e^x > 1$ for all $x > 0$, then, by 5.8.4,

$$\int e^t \, dt > \int dt.$$

So,

$$1 + \int_0^x e^t \, dt > 1 + \int_0^x dt.$$

But

$$e^x = 1 + \int_0^x e^t \, dt > 1 + \int_0^x dt = 1 + x,$$

So

$$e^x > 1 + x.$$

Then,

$$1 + \int_0^x e^t \, dt > 1 + \int_0^x 1 + t dt.$$

So,

$$e^x > 1 + x + \frac{x^2}{2}.$$

If we continue on in this manner, after n times we will achieve

$$e^x > 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

So,

$$
\begin{aligned}
1 + \int_0^x &> 1 + \int_0^x \sum_{k=0}^{n} \frac{t^k}{k!}\, dt \\
e^x &> \sum_{k=0}^{n} \int_0^x \frac{t^k}{k!}\, dt, \quad \text{by 5.4.4, linearity of the integral} \\
e^x &> \sum_{k=0}^{n} \frac{t^{k+1}}{(k+1)!} \\
e^x &> \sum_{k=0}^{n+1} \frac{x^k}{k!}.
\end{aligned}
$$

So, we have shown that $e^x > \sum_{k=0}^{n} \frac{x^k}{k!}$ for $n = 1$, where $e^x > 1 + x$, for n, which covers the inductive hypothesis, and finally for $n + 1$.

Therefore, by induction, the proposition holds.

$\square$

73

Prove the proposition.

Proposition. *If n is a positive integer, then for all x sufficiently large,*

$$e^x > x^n.$$

Proof. From 72, we note that,

$$e^x > \sum_{k=0}^{n} \frac{x^k}{k!}.$$

We will show that

$$\sum_{k=0}^{n} \frac{x^k}{k!} > x^n.$$

Since the proposition concerns only large x, then with $x > 0$,

$$\sum_{k=0}^{n} \frac{x^k}{k!} > 0.$$

So, omitting all but the $n + 1^{\text{th}}$ term of the sum,

$$e^x > \frac{x^{n+1}}{(n+1)!},$$

which we claim is also $\frac{x^{n+1}}{(n+1)!} > x^n$ for sufficiently large x.

Then, with $x > 0$ and thus $x^n > 0$,

$$\begin{aligned} \frac{x^{n+1}}{(n+1)!} &> x^n \\ \frac{x}{(n+1)!} &> 1 \\ x &> (n+1)!. \end{aligned}$$

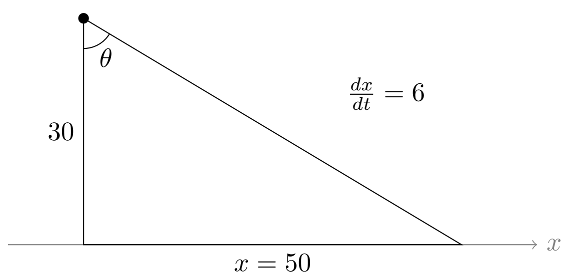
So, for all $x > (n+1)!$, x is sufficiently large such that,

$$e^x > x^n.$$

$\square$

7.7

A person walking along a straight path at the rate of 6 feet per second is followed by a spotlight that is located 30 feet from the path. How fast is the spotlight turning at the instant the person is 50 feet past the point on the path that is closest to the spotlight?



From the figure, we can see that

$$\begin{aligned} \tan\theta &= \frac{x}{30} \\ \theta &= \arctan\frac{x}{30}, \end{aligned}$$

So, with the given conditions of $x = 50$ and $\frac{dx}{dt} = 6$,

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{\left(\frac{x}{30}\right)^2 + 1}\left(\frac{1}{30}\right)\left(\frac{dx}{dt}\right) \\ &= \frac{1}{\left(\frac{5}{3}\right)^2 + 1}\left(\frac{1}{5}\right) \\ &= \frac{1}{5\left(\frac{25}{9} + 1\right)} \\ &= \frac{1}{5\left(\frac{34}{9}\right)} \\ &= \frac{9}{170}. \end{aligned}$$

So, at 50 feet past the perpendicular of the spotlight on the path, $\frac{d\theta}{dt} = \frac{9}{170}$ radians per second.

8.2

78

We are familiar with the identity

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

(a) Assume that f has a continuous second derivative. Use integration by parts to derive the identity

$$f(b) - f(a) = f'(a)(b-a) - \int_a^b f''(x)(x-b)\, dx.$$

We rearrange the terms so that we wish to show

$$\int_a^b f''(x)(x-b)\, dx = f'(a)(b-a) - (f(b) - f(a)).$$

Then, we evaluate the integral with integration by parts, using $u(x) = (x-b)$ and $dv = f''(x)dx$,

$$\begin{aligned}
&\int_a^b f''(x)(x-b)\, dx \\
=&\left[f'(x)(x-b) - \int f'(x)\, dx \right]_a^b \\
=&[f(x)(x-b) - f(x)]_a^b \\
=&\,(f'(b)(b-b) - f(b)) - (f'(a)(a-b) - f(a)) \\
=&f'(a)(b-a) - (f(b) - f(a)),
\end{aligned}$$

which matches the identity above.

So, the assumption (a) holds.

(b) Assume that f has a continuous third derivative. Use the result in part (a) and integration by parts to derive the identity

$$f(b) - f(a) = f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \int_a^b \frac{f'''(x)}{2}(x-b)^2\, dx.$$

Again, we rearrange the terms such that,

$$\int_a^b \frac{f'''(x)}{2}(x-b)^2\, dx = (f(b) - f(a)) - f'(a)(b-a) - \frac{f''(a)}{2}(b-a)^2.$$

Then, we compute with integration by parts, using $u = (x-b)^2$ and $dv = f'''(x)dx$.

$$\begin{aligned}
&\frac{1}{2}\int_a^b f'''(x)(x-b)^2\,dx \\
=&\left[f''(x)(x-b)^2 - \int f''(x)(2(x-b))\,dx\right]_a^b \\
=&\left[\frac{f''(x)}{2}(x-b)^2 - \int_a^b f''(x)(x-b)\,dx\right]_a^b.
\end{aligned}$$

We notice that the second term was already evaluated in part (a); so we continue,

$$\begin{aligned}
=&\left[\frac{f''(x)}{2}(x-b)^2 - (f'(x)(x-b) - f(x))\right]_a^b \\
=&\left[f(x) + \frac{f''(x)}{2}(x-b)^2 - f'(x)(x-b)\right]_a^b \\
=&f(b) - f(a) - \frac{f''(x)}{2}(a-b)^2 - f'(a)(b-a).
\end{aligned}$$

So, we have achieved the desired identity.

8.3

53

(a) Use integration by parts to show that for $n > 2$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

We will let $u(x) = \sin^{n-1}x$ and $\frac{dv}{dx} = \sin x$, so that $\frac{du}{dx} = (n-1)\sin^{n-2}x\cos x$ and $v(x) = -\cos x$.

Then,

$$\begin{aligned}
\int \sin^n x\,dx &= -\cos x \sin; n-1 x + \int (n-1)\cos^2 x \sin^{n-2}x\,dx \\
\int \sin^n x\,dx &= -\cos x \sin^{n-1}x + (n-1)\int \left(1-\sin^2 x\right)\sin^{n-2}x\,dx \\
\int \sin^n x\,dx &= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx \\
(1+(n-1))\int \sin^n x\,dx &= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx \\
\int \sin^n x\,dx &= -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.
\end{aligned}$$

So, the statement holds.

(b) Then show that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx.$$

We evaluate the identity from part (a),

$$\int_0^{\pi/2} \sin^n x\, dx$$
$$= \left[-\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x\, dx \right]_0^{\frac{\pi}{2}},$$

but $\sin 0 = 0$ and $\cos \frac{\pi}{2} = 0$, so the first term will reduce to zero.

So, we are left with,

$$\int_0^{\pi/2} \sin^n x\, dx = \left[\int \frac{n-1}{n} \sin^{n-2} x\, dx \right]_0^{\frac{\pi}{2}} = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx.$$

(c) Verify the *Wallis sine formulas*:

We will use the notation $x!!$, "double factorial"[1] to represent the product of the integers up to n with the same parity as n.

for even $n \geq 2$,

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{(n-1)\cdots 5 \cdot 3 \cdot 1}{n \cdots 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{(n-1)!!}{n!!};$$

for odd $n \geq 3$,

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{(n-1)\cdots 6 \cdot 4 \cdot 2}{n \cdots 5 \cdot 3 \cdot 1} = \frac{(n-1)!!}{n!!}.$$

Since

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx,$$

then

$$\int_0^{\pi/2} \sin^{n-2} x\, dx = \left(\frac{(n-2)-1}{(n-2)} \right) \int_0^{\pi/2} \sin^{n-4} x\, dx.$$

So,

$$\int_0^{\pi/2} \sin^n x\, dx = \left(\frac{n-1}{n} \right) \left(\frac{(n-2)-1}{(n-2)} \right) \int_0^{\pi/2} \sin^{n-4} x\, dx.$$

First, we consider even n, such that continued subtraction by 2 k times, $n - 2k$, is zero when $k = n/2$.

We take a total of k substitutions for the $\int_0^{\pi/2} \sin^{n-2k} x\, dx$ term so that there will be $k-1$ coefficients before the sine integral term.

[1] https://en.wikipedia.org/wiki/Double_factorial

So, for n even,

$$\begin{aligned}\int_0^{\pi/2} \sin^n x\,dx &= \frac{(n-1)(n-3)\cdots(n-(2(k-1)-1))}{(n)(n-2)\cdots(n-2(k-1))}\cdot\int_0^{\pi/2}\sin^0 x\,dx\\ &= \frac{(n-1)(n-3)\cdots(n-2k+3)}{(n)(n-2)\cdots(n-2k+2)}\cdot\int_0^{\pi/2} dx\\ &= \frac{(n-1)(n-3)\cdots 3}{(n)(n-2)\cdots 2}\cdot\frac{\pi}{2}\\ &= \frac{\pi(n-1)!!}{2n!!}.\end{aligned}$$

Next, we consider odd n, such that for the same k substitutions, $n-2k=1$.

Now, the remaining sine integral term becomes $\int_0^{\pi/2}\sin^1 x\,dx = [-\cos x]_0^{\pi/2} = (0-(-1)) = 1$.

So, for n odd,

$$\begin{aligned}\int_0^{\pi/2} \sin^n x\,dx &= \frac{(n-1)(n-3)\cdots(n-(2(k-1)-1))}{(n)(n-2)\cdots(n-2(k-1))}\cdot\int_0^{\pi/2}\sin^1 x\,dx\\ &= \frac{(n-1)(n-3)\cdots 2}{(n)(n-2)\cdots 3}\\ &= \frac{(n-1)!!}{n!!}.\end{aligned}$$

Therefore, the Wallis sine formulas hold.

54

Use Exercise 53 to show that

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^n x\,dx.$$

First, we note that $\cos x = \sin\frac{\pi}{2} - x$.

So,

$$\begin{aligned}\int_0^{\pi/2}\cos^n x\,dx &= \int_0^{\pi/2}\sin^n\left(\frac{pi}{2}-x\right)\\ &= -\int_{\pi/2}^0 \sin^n u\,du\\ &= \int_0^{\pi/2}\sin^n u\,du.\end{aligned}$$

Therefore

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^n x\,dx.$$

Additionally, we will follow the same procedure as in 53.

First, we evaluate $\cos^n x$ using integration by parts,

$$
\begin{aligned}
\int_0^{\pi/2} \cos^n x\,dx &= \left[\sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx\right]_0^{\pi/2} \\
\int_0^{\pi/2} \cos^n x\,dx &= \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\pi/2} \left(1-\cos^2 x\right)\cos^{n-2} x\,dx \\
\int_0^{\pi/2} \cos^n x\,dx &= 0 + (n-1)\left[\int_0^{\pi/2} \cos^{n-2} x\,dx - \int_0^{\pi/2} \cos^n x\,dx\right] \\
1 + (n-1)\int_0^{\pi/2} \cos^n x\,dx &= (n-1)\int_0^{\pi/2} \cos^{n-2} x\,dx \\
\int_0^{\pi/2} \cos^n x\,dx &= \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx.
\end{aligned}
$$

Then, following 53,

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)}{(n)(n-2)}\int_0^{\pi/2} \cos^{n-4} x\,dx.$$

So, we will apply the same argument.

For n even,

$$\int_0^{\pi/2} \cos^0 x\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

For n odd,

$$\int_0^{\pi/2} \cos^1 x\,dx = \int_0^{\pi/2} \cos x\,dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = (1-0) = 1.$$

Then, after k substitutions of the cosine term,

For n even,

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{\pi(n-1)!!}{2n!!}.$$

For n odd,

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)!!}{n!!}$$

Since these match $\int_0^{\pi/2} \sin^n x\,dx$, then

$$\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \sin^n x\,dx.$$

8.4

51

Let Ω be the region under the curve $y = \sqrt{x^2 - a^2}$ from $x = a$ to $x = \sqrt{2}a$.

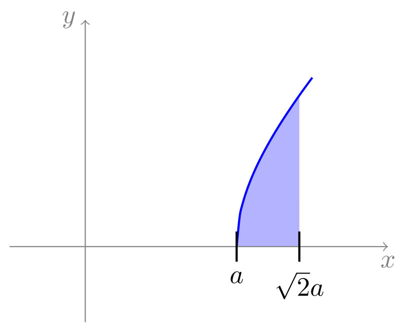


Figure 1: Sketch of Ω.

(a) Sketch Ω.

(b) Find the area of Ω.

The area A is given by the integral,

$$A = \int_a^{\sqrt{2}a} \sqrt{x^2 - a^2}\, dx.$$

We evaluate this integral using the trigonometric substitution

$$x = a \sec \theta.$$

So,

$$dx = a \tan \theta \sec \theta d\theta.$$

Then,

$$x^2 - a^2 = (a \sec \theta)^2 - a^2 = (a \tan \theta)^2.$$

So,

$$\begin{aligned} \int \sqrt{x^2 - a^2}\, dx &= \int \sqrt{(a \tan \theta)^2} \cdot a \tan \theta \sec \theta \, d\theta \qquad = \int a^2 \tan^2 \theta \sec \theta \, d\theta \\ &= a^2 \int \left(\sec^2 \theta - 1\right) \sec \theta \, d\theta \\ &= a^2 \left[\int \sec^3 \theta \, d\theta - \int \sec \theta \, d\theta \right]. \end{aligned}$$

We have previously computed,

$$\begin{aligned} \int \sec \theta \, d\theta &= \ln |\tan \theta + \sec \theta| \text{ and} \\ \int \sec^3 \theta \, d\theta &= \frac{1}{2} \left(\ln |\tan \theta + \sec \theta| + \tan \theta \sec \theta \right). \end{aligned}$$

So, we continue from above,

$$
\begin{aligned}
&= a^2\left(\frac{1}{2}\left(\ln|\tan\theta+\sec\theta| + \tan\theta\sec\theta\right) - \ln|\tan\theta+\sec\theta|\right)\\
&= \frac{a^2}{2}\left(\tan\theta\sec\theta - \ln|\tan\theta+\sec\theta|\right)
\end{aligned}
$$

Using trigonometric identities, we see that $\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$.

Then, we return to the integral in terms of x, using $\sec\theta = \frac{x}{a}$ and $\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$,

$$
\begin{aligned}
\int \sqrt{x^2-a^2} &= \frac{a^2}{2}\left(\left(\frac{\sqrt{x^2-a^2}}{a}\right)\left(\frac{x}{a}\right) - \ln\left|\frac{\sqrt{x^2-a^2}}{a} + \frac{x}{a}\right|\right)\\
&= \frac{a^2}{2}\left(\frac{x\sqrt{x^2-a^2}}{a^2} - \ln\left|\frac{\sqrt{x^2-a^2}+x}{a}\right|\right).
\end{aligned}
$$

We evaluate the integral of the bounds of the region Ω, from $x = a$ to $x = \sqrt{2}a$,

$$
\begin{aligned}
A &= \int_a^{\sqrt{2}a} \sqrt{x^2-a^2}\\
&= \left[\frac{a^2}{2}\left(\frac{x\sqrt{x^2-a^2}}{a^2} - \ln\left|\frac{\sqrt{x^2-a^2}+x}{a}\right|\right)\right]_a^{\sqrt{2}a}\\
&= \frac{a^2}{2}\left[\frac{x\sqrt{x^2-a^2}}{a^2} - \ln\left|\frac{\sqrt{x^2-a^2}+x}{a}\right|\right]_a^{\sqrt{2}a}\\
&= \frac{a^2}{2}\left(\left(\frac{\sqrt{2}a\cdot a}{a^2} - \ln\left|\frac{a+\sqrt{2}a}{a}\right|\right) - \left(\frac{a(0)}{a^2} - \ln\left|\frac{0+a}{a}\right|\right)\right)\\
&= \frac{a^2}{2}\left(\sqrt{2} - \ln(1+\sqrt{2}) + \ln 1\right)\\
&= \frac{a^2}{2}\left(\sqrt{2} - \ln(1+\sqrt{2})\right)
\end{aligned}
$$

So,

$$
A = \frac{a^2}{2}\left(\sqrt{2} - \ln(1+\sqrt{2})\right).
$$

(c) Locate the centroid of Ω.

For the x coordinate, we evaluate

$$
\frac{1}{A}\int_a^{\sqrt{2}a} x\sqrt{x^2-a^2}\,dx.
$$

We make the substitution $u(x) = x^2 - a^2$,

$$\begin{aligned}
\frac{1}{2A}\int_{u(\sqrt{2}a)}^{u(a)} \sqrt{u}\,du &= \frac{1}{2A}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{a^2} \\
&= \frac{a^3}{3A} \\
&= \frac{a^3}{3\left(\frac{a^2}{2}\left(\sqrt{2} - \ln\left(1+\sqrt{2}\right)\right)\right)} \\
&= \frac{2a}{3(\sqrt{2} - \ln\left(\sqrt{2}+1\right))}.
\end{aligned}$$

For the y coordinate, we evaluate

$$\begin{aligned}
\frac{1}{A}\int_a^{\sqrt{2}a} \frac{1}{2}\left(\sqrt{x^2-a^2}\right)^2 dx &= \frac{1}{2A}\int_a^{\sqrt{2}a} x^2 - a^2\,dx \\
&= \frac{1}{2A}\left[\frac{x^3}{3} - a^2 x\right]_a^{\sqrt{2}a} \\
&= \frac{1}{2A}\left(\left(\frac{2\sqrt{2}a^3}{3} - \sqrt{2}a^3\right) - \left(\frac{a^3}{3} - a^3\right)\right) \\
&= \frac{1}{2A}\left(\left(\frac{-\sqrt{2}a^3}{3}\right) + \left(\frac{2a^3}{3}\right)\right) \\
&= \frac{1}{2A}\left(\frac{a^3}{3}(2-\sqrt{2})\right) \\
&= \frac{a^3(2-\sqrt{2})}{6\left(\frac{a^2}{2}\right)\left(\sqrt{2} - \ln\left(1+\sqrt{2}\right)\right)} \\
&= \frac{a(2-\sqrt{2})}{3(\sqrt{2} - \ln\left(\sqrt{2}+1\right))}.
\end{aligned}$$

So, the centroid of the region Ω is

$$\left(\frac{2a}{3(\sqrt{2} - \ln\left(\sqrt{2}+1\right))}, \frac{a(2-\sqrt{2})}{3(\sqrt{2} - \ln\left(\sqrt{2}+1\right))}\right).$$

52

Find the volume of the solid generated by revolving Ω about the x-axis and determine the centroid of that solid.

We determine the volume of Ω rotated about the x-axis using the disk method.

Then,

$$V = \int_a^{\sqrt{2}a} \pi\left(\sqrt{x^2-a^2}\right)^2 dx,$$

which is the same integral that we evaluated in 51.

So,

$$V = \frac{\pi a^3}{3}(2 - \sqrt{2}).$$

Then, the centroid of the solid of revolution about the x-axis has its y coordinate given by the equation,[2]

$$\overline{x} = \frac{\pi}{V}\int_a^{\sqrt{2}a} x\left(\sqrt{x^2 - a^2}\right)^2 dx.$$

Then, we evaluate,

$$\begin{aligned}
\frac{\pi}{V}\int_a^{\sqrt{2}a} x\left(\sqrt{x^2 - a^2}\right)^2 dx &= \frac{\pi}{V}\int_a^{\sqrt{2}a} x^3 - a^3 x\, dx \\
&= \frac{\pi}{V}\left[\frac{x^4}{4} - \frac{a^2x^2}{2}\right]_a^{\sqrt{2}a} \\
&= \frac{\pi}{V}\left(\left(\frac{4a^4}{4} - \frac{2a^4}{2}\right) - \left(\frac{a^4}{4} - \frac{a^4}{2}\right)\right) \\
&= \frac{\pi}{V}\left(\frac{a^4}{4}\right) \\
&= \frac{\pi a^4}{4\left(\frac{\pi a^3}{3}(2 - \sqrt{2})\right)} \\
&= \frac{3a}{4(2 - \sqrt{2})}.
\end{aligned}$$

So, the centroid of the volume of rotation about the x-axis is

$$\left(\frac{3a}{4(2 - \sqrt{2})}, 0\right).$$

53

Find the volume of the solid generated by revolving Ω about the y-axis and determine the centroid of that solid.

Following the procedure in 52, we will first find the volume the region Ω rotated about the y-axis.

[2]The equations for the centroids of volumes of revolution were derived in a previous assignment.

We find the volume using the shell method and the substitution $u(x) = x^2 - a^2$[3],

$$
\begin{aligned}
V &= \int_a^{\sqrt{2}a} 2\pi x \sqrt{x^2 - a^2}\, dx \\
&= \pi \int_{u(a)}^{u(\sqrt{2}a)} \sqrt{u}\, du \\
&= \pi \left[\frac{2}{3} u^{\frac{3}{2}} \right]_0^{a^2} \\
&= \frac{2\pi a^3}{3}.
\end{aligned}
$$

Then, the y coordinate of the centroid about the y axis is given by, the same equation for x coordinate of a rotation about the x axis, but with the corresponding volume of rotation.

So,

$$
\begin{aligned}
\overline{y} &= \frac{1}{V} \int_a^{\sqrt{2}a} \pi x \left(\sqrt{x^2 - a^2} \right)^2 \\
&= \frac{\pi a^4}{4V} \\
&= \frac{\pi a^4}{4\left(\frac{2\pi a^3}{3}\right)} \\
&= \frac{3a}{8}.
\end{aligned}
$$

Therefore, the centroid of the volume of rotation about the y-axis is

$$\left(0, \frac{3a}{8}\right).$$

[3] We have already evaluated this integral above as well. However, it is simple enough to repeat.