Math 134 A Homework 2

Alexandre Lipson

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3.1

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Let $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $g(x) = xf(x)$

(a) Show that f and g are both continuous at 0.

(b) Show that f is not differentiable at 0.

(c) Show that g is differentiable at 0 and give $g'(0)$.

Proof of (a). We say that h is continuous at c if

$$\lim_{x \to c} h(x) = h(c).$$

This requires that $\lim_{x \to c^-} h(x) = \lim_{x \to c^+} h(x)$

First, for f,

$$\lim_{x \to 0^0} f(x) = \lim_{x \to 0^0} x \sin \frac{1}{x} = 0$$

which was proven by the Squeeze theorem in a previous assignment.

Since $f(0) = 0 = \lim_{x \to 0} f(x)$, then f is continuous.

For g,

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0}$$

□

Proof of (b). For a contradiction, assume f is differentiable at 0.

$\square$

Proof of (c). By 3.1.5,

$$g'(0) = \lim_{x \to 0} \frac{g(x) - 0}{x - 0} = \lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

$\square$

3.2

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Find A and B given that the derivate of $f(x) = \begin{cases} Ax^2 + B, & x < -1 \\ Bx^5 + Ax + 4, & x \geq -1 \end{cases}$ is everywhere continuous.

Proof. Since f is differentiable, it is continuous.

Therefore, given that f is continuous,

$$\begin{aligned} \lim_{x \to -1^-} Ax^2 + B &= \lim_{x \to -1^+} Bx^5 + Ax + 4 \\ A + B &= -B - A + 4 \\ 2B &= 4 - 2A \\ B &= 2 - A \end{aligned}$$

Then, we find $f'(x)$.

$$f(x) = \begin{cases} 2Ax & x < -1 \\ 5Bx^4 + A & x \geq -1 \end{cases}$$

Since f' is continuous, then,

$$\lim_{x \to -1^-} f' = \lim_{x \to -1^+} f' \text{and} \lim_{x \to c} f'(x) = f'(c).$$

So,

$$\begin{aligned} \lim_{x \to -1^-} 2Ax &= \lim_{x \to -1^+} 5Bx^4 + A \\ -2A &= 5B + A \\ -3A &= 5B \\ \frac{-3}{5}A &= B \end{aligned}$$

Then, using the relationship $B = 2 - A$ derived earlier, we can solve the system of equations.

$$\begin{aligned} 2 - A &= \frac{-3}{5}A \\ 2 &= \frac{2}{5}A \\ 5 &= A \end{aligned}$$

Lastly,

$$B = 2 - A = 2 - 5 = -3.$$

$\square$

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Set $f(x) = x^3$.

(a) Find an equation for the line tangent to the graph of f at $(c, f(c))$, $\quad c \neq 0$.

(b) Determine whether the tangent line found in (a) intersects the graph of f at a point other than (c, c^3).

Proof of (a). **Definition 3.1.2.** Let the tangent line to a point $(c, f(c))$ be given by the equation,

$$y - f(c) = f'(c)(x - c).$$

Let $f(c) = c^3$.

Then, $f'(c) = 3c^2$ by 3.2.7.

Next, we use f' to find the tangent line equation at $(c, f(c))$,

$$\begin{aligned} y - c^3 &= 3c^2(x - c) \\ y &= 3xc^2 - 3c^3 + c^3 \\ y &= 3xc^2 - 2c^3 \end{aligned}$$

Therefore, the tangent line at $(c, f(c))$ is $y = 3xc^2 - 2c^3$. $\square$

Proof of (b). We must check if the tangent line y found in (a) intersects the given curve $f(x) = x^3$.

So, we will find all the (x, x^3) on f that satisfy $y = 3xc^2 - 2c^3$

$$\begin{aligned} x^3 &= 3xc^2 - 2c^3 \\ x^3 - 3xc^2 + 2c^3 &= 0 \end{aligned}$$

We can begin to factor this polynomial using the fact that y is a tangent line to f at $x = c$ given the intersection point (c, c^3). Since a tangent line only touches the curve f but does not pass through it, it can be considered a second degree root of this equation. For this reason, we will begin to factor by $(x - c)^2$.

We will show that the resulting zero is $(2c + x)$.

$$\begin{aligned} x^3 - 3xc^2 + 2c^3 &= (x - c)^2(2c + x) \\ &= (x^2 - 2xc + c^2)(2c + x) \\ &= (2cx^2 - 4xc^2 + 2c^3 + x^3 - 2cx^2 + xc^2) \\ &= x^3 - 3xc^2 + 2c^3 \end{aligned}$$

With $(2c + x) = 0$, we see that $x = -2c$.

So, the valid x are $x = c$ and $x = -2c$.

With the first x, we reobtain the given point (c, c^3a).

However, with the latter x, we get the second intersection point on f,

$$(-2c, -8c^3).$$

□

3.3

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Set $g(x) = \begin{cases} x^3, & x \geq 0 \\ 0, x < 0. \end{cases}$

(a) Find $g'(0)$ and $g''(0)$.

(b) Determine $g'(x)$ and $g''(x)$ for all other x.

(c) Show that $g'''(0)$ does not exist.

(d) Sketch the graphs of g, g', g''.

Proof of (a). At $x = 0$, $g(x) = x^3$. So, $\frac{dg}{dx} = g'(x) = 3x^2$ by the power rule.

Then, $g'(0) = 0$.

Similarly, since $\frac{d^2g}{dx^2} = g''(x) = 6x$, then $g''(0) = 0$.

So, $g'(0) = g''(0) = 0$. □

Proof of (b). As shown in (a), at $x = 0$,

$$g'(x) = 3x^2 \text{ and } g''(x) = 6x.$$

This is the case for all $x \geq 0$ as well.

For $x < 0$, $g(x) = 0$, so $g'(x) = g''(x) = 0$.

This leaves us with both derivates defined for all $x \in \mathbb{R}$.

$$g'(x) = \begin{cases} 3x^2, & x \geq 0 \\ 0, & x < 0 \end{cases} \text{ and } g''(x) = \begin{cases} 6x, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

□

Proof of (c). For $g'''(0)$ to exist, g'' must de differentiable at 0.

We will show that g'' is not differentiable at zero because its limit as x approaches zero from positive x is different than the limit as x is approached from negative x.

Define $g''(x) = \begin{cases} 6x, & x \geq 0 \\ 0, & x < 0 \end{cases}$ from (b).

First, for the right limit, since $x > 0$, $g''(x) = 6x$.

$$\lim_{h \to 0^+} \frac{6(x+h) - 6x}{h} = \lim_{h \to 0^+} \frac{6h}{h} = 6.$$

Next, for the left limit, since $x < 0$, $g''(x) = 0$.

$$\lim_{h \to 0^-} \frac{0 - 0}{h} = \lim_{h \to 0^-} 0 = 0.$$

But,

$$\lim_{h \to 0^-} g''(x) = 0 \neq 6 = \lim_{h \to 0^+} g''(x)$$

Since these limits are not equal, the derivate at $x = 0$ does not exist, so neither does $g'''(0)$. □

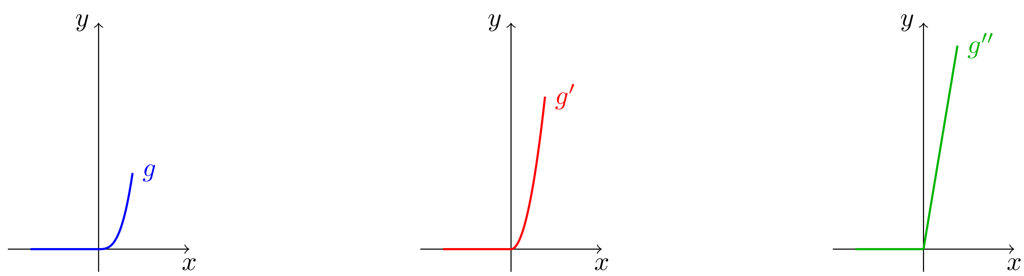

Figure 1: Sketches of g, g', and g'' for (d).

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Prove by induction that,

$$\text{if } y = x^{-1}\text{, then } \tfrac{d^n y}{dx^n} = (-1)^n n! x^{-n-1}.$$

Proof by induction. For the base case, for all $n \in \mathbb{Z}^+$, when $n = 1$,

$$\begin{aligned} \frac{d^1 y}{dx^1} &= (-1)^1 (1!) x^{-1-1} \\ \frac{dy}{dx} &= (-1)(1) x^{-2} \\ \frac{dy}{dx} &= -\frac{1}{x^2} \end{aligned}$$

We know that this is the first derivate of $\frac{1}{x}$ by the power rule, so the base case is true.

Then, we assume that the formula holds for $n = k$ by the inductive hypothesis.

Now, for the inductive step, we will show that the statement holds for $n = k + 1$.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{d^k y}{dx^k}\right] &= \frac{d}{dx}\left[(-1)^k k! x^{-k-1}\right] \\
\frac{d^{k+1}}{dx^{k+1}} &= (-1)^k k! \frac{d}{dx}\left[x^{-k-1}\right] \\
&= (-1)^k k!(-k-1)x^{-k-2} \\
&= (-1)^k k!(-1)(k+1)x^{-k-2} \\
&= (-1)^k (-1)k!(k+1)x^{-k-2} \\
&= (-1)^{k+1}(k+1)!x^{-k-2} \\
&= (-1)^{k+1}(k+1)!x^{-(k+1)-1}
\end{aligned}$$

Which we recognize to be the formula when $n = k + 1$. Therefore, by induction, the statement holds. □

3.5

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Let f be a differentiable function. Use the chain rule to show that:

(a) if f is even, then f' is odd.

(b) if f is odd, then f' is even.

Assume that an even function g defined such that it satisfies the property $g(x) = g(-x)$.

Assume that an odd function g defined such that it satisfies the property $g(x) = -g(-x)$.

Proof of (a). We differentiate both sides with respect to x.

$$\begin{aligned}
\frac{d}{dx}[f(x)] &= \frac{d}{dx}[f(-x)] \\
f'(x) &= -f'(-x)
\end{aligned}$$

We recognize this to be the form of an odd function where $f' = g$. □

Proof of (b). We differentiate both sides with respect to x.

$$\begin{aligned}
\frac{d}{dx}[f(x)] &= \frac{d}{dx}[-f(-x)] \\
f'(x) &= f'(-x)
\end{aligned}$$

We recognize this to be the form of an even function where $f' = g$. □

3.6

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Set $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ and $g(x) = xf(x)$. Both f and g are differentiable at each $x \neq 0$.

(a) Find $f'(x)$ and $g'(x)$ for $x \neq 0$.

(b) Show that g' is not continuous at 0.

Proof of (a). First, we will differentiate f where $x \neq 0$.

$$\begin{aligned} \frac{d}{dx}\left[f(x)\right] &= \frac{d}{dt}\left[x\sin\frac{1}{x}\right] \\ f'(x) &= \sin\frac{1}{x}\frac{d}{dx}\left[\frac{1}{x}\right]x\cos\frac{1}{x} \\ &= \sin\frac{1}{x} + x\left(\frac{-1}{x^2}\right)\cos\frac{1}{x} \\ &= \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} \end{aligned}$$

Next, we will do the same for g, again with the constraint $x \neq 0$.

$$\begin{aligned} \frac{d}{dx}\left[g(x)\right] &= \frac{d}{dx}\left[x^2\sin\frac{1}{x}\right] \\ g'(x) &= 2x\sin\frac{1}{x} + x^2\left(\frac{-1}{x^2}\right)\cos\frac{1}{x} \\ &= 2x\sin\frac{1}{x} - \cos\frac{1}{x} \end{aligned}$$

We have found f' and g' for f and g where $x \neq 0$. $\square$

Proof of (b) by contraction. For a contraction, we will assume that g' is continuous at zero.

Since $g(0) = 0$ and the derivative of zero is zero, $g'(0) = 0$.

Therefore, we assume that, if g' is continuous at $x = 0$, then,

$$\lim_{x\to 0} g' = 0.$$

We will use $g' = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ from (a).

$$\begin{aligned} &\lim_{x\to 0} 2x\sin\frac{1}{x} - \cos\frac{1}{x} \\ &= 2\lim_{x\to 0} x\sin\frac{1}{x} - \lim_{x\to 0}\cos\frac{1}{x} \end{aligned}$$

The first limit has been previously shown to be zero using the Squeeze Theorem. We will show it again briefly.

Proposition. $\lim_{x\to 0} x \sin \frac{1}{x} = 0$.

Proof. Assume that, for all $x \in \mathbb{R}$, $-|x| \leq x \leq |x|$.

Assume that the range of sine is $[-1, 1]$.

Which means that, $-1 \leq \sin\theta \leq 1$.

So, $-1 \leq \sin \frac{1}{x} \leq 1$.

Then, by the assumption, $-|x| \leq x \sin \frac{1}{x} \leq |x|$.

Since $\lim_{x\to 0} -|x| = 0$ and $\lim_{x\to 0} |x| = 0$, then $\lim_{x\to 0} x \sin \frac{1}{x} = 0$ by the Squeeze Theorem. □

So, the limit in the first term, $2 \lim_{x\to 0} x \sin \frac{1}{x}$, is zero.

For the second limit, we will prove that it does not exist, which implies that $\lim_{x\to 0} g'$ does not exist either.

Proposition. *The limit* $\lim_{h\to 0} \cos \frac{1}{h}$ *does not exist.*

Proof. For a contradiction, assume $\lim_{h\to 0} \cos \frac{1}{h} = L$.

For $\epsilon = 1$, there exists a $\delta > 0$ such that $|h| < \delta$ implies $\left| \cos \frac{1}{h} - L \right| < \epsilon = 1$.

Choose $n \in \mathbb{Z}^+$ such that $\frac{1}{m} < \delta$.

Then, let $h = \frac{1}{\pi\left(2n + \frac{1}{2}\right)}$.

So, $|h| = h < \frac{1}{m} < \delta$, which implies $\left| \cos \frac{1}{h} - L \right| < 1$.

But, $\sin \frac{1}{h} = \sin \pi \left(2n + \frac{1}{2}\right) = 1$.

So,

$$|1 - L| < 1.$$

Similarly, if $h = \frac{1}{\pi\left(2n + 1 + \frac{1}{2}\right)}$, then $|h| < \frac{1}{m} < \delta$ and $\cos \frac{1}{h} = -1$.

So,

$$|-1 - L| < 1.$$

Then, $-1 < 1 - L < 1$ and $-1 < -1 - L < 1$.

So, $-2 < -L < 0$ and $0 < -L < 2$.

Or, $0 < L < 2$ and $-2 < L < 0$.

But, $L > 0$ and $L < 0$ is a contradiction, so $\lim_{h\to 0} \cos \frac{1}{h}$ does not exist. □

Since $\lim_{x\to 0} g'(x)$ does not exist, then $\lim_{x\to 0} g' \neq 0$. Therefore g' is not continuous at zero at the assumption is false.

Therefore, we have shown that statement in (b) is true.

□

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A simple pendulum consists of a mass m swinging at the end of a rod or wire of negligible mass. The figure shows a simple pendulum of length L. The angular displacement θ at time t is given by a trigonometric expression:

$$\theta(t) = A \sin(\omega t + \phi)$$

where A, ω, ϕ are constants.

(a) Show that the function θ satisfies the equation

$$\frac{d^2\theta}{dt^2} + \omega^2 = 0.$$

(b) Show that θ can be written in the form

$$\theta(t) = A \sin \omega t + B \cos \omega t$$

where A, B, ω are constants.

Proof of (a). First, we differentiate both sides with respect to t,

$$\begin{aligned} \frac{d}{dt}\left[\theta(t)\right] &= \frac{d}{dt}\left[A \sin(\omega t + \phi)\right] \\ \frac{d\theta}{dt} &= A\omega \cos(\omega t + \phi) \end{aligned}$$

Then we repeat to obtain $\frac{d^2\theta}{dt^2}$,

$$\begin{aligned} \frac{d}{dt}\left[\frac{d\theta}{dt}\right] &= \frac{d}{dt}\left[A\omega \cos(\omega t + \phi)\right] \\ \frac{d^2\theta}{dt^2} &= -A\omega^2 \sin(\omega t + \phi) \end{aligned}$$

Next, we find $\omega^2\theta(t)$,

$$\omega^2\theta(t) = A\omega^2 \sin(\omega t + \phi)$$

We then see that,

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = -A\omega^2 \sin(\omega t + \phi) + A\omega^2 \sin(\omega t + \phi) = 0$$

□

Proof of (b). We rewrite $\theta(t)$ as,

$$\theta(t) = C \sin(\omega t + \phi).$$

Assume the identity for sine,

$$\sin \alpha + \beta = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

We can express θ according to this identity as,

$$C\left(\sin \omega t \cos \phi + \cos \omega t \cos \phi\right).$$

Then, we define $A = C\cos\phi$ and $B = C\sin\phi$ and simplify.

$$\begin{aligned} &C\left(\sin \omega t \cos \phi + \cos \omega t \cos \phi\right) \\ &= C \sin \omega t \cos \phi + C \cos \omega t \cos \phi \\ &= A \sin \omega t + B \cos \omega t \end{aligned}$$

Therefore, we can represent θ as the sum of a sine and cosine term of the same inner term ωt, $A\sin\omega t + B\cos\omega t$.

□

3.7

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Two families of curves are said to be *orthogonal trajectories* (of each other) if each member of one family is orthogonal to each member of the other family. Show that the families of curves are orthogonal trajectories.

The family of parabolas $x = ay^2$ and the family of ellipses $x^2 + \frac{1}{2}y^2 = b$ are orthogonal trajectories of each other.

We will first prove that the two curves intersect at two points.

Proposition. *There exist two real points where the curves $x = ay^2$ and $x^2 + \frac{1}{2}y^2 = b$ intersect.*

Proof. First, we isolate y^2 and combine the two equations.

Since, $y^2 = \frac{x}{a}$, then $x^2 + \frac{x}{2a} = b$.

We reorganize the terms and use the quadratic formula to see that,

$$\begin{aligned}
x &= \frac{-\frac{1}{2a} \pm \sqrt{\left(\frac{1}{2a}\right)^2 - 4(-b)}}{2} \\
&= \frac{-\frac{1}{2a} \pm \sqrt{\frac{1}{4a^2} + 4b}}{2} \\
&= \frac{-\frac{1}{2a} \pm \sqrt{\frac{1}{4a^2} + 4b}}{2} \left(\frac{2a}{2a}\right) \\
&= \frac{-1 \pm (2a)\sqrt{\frac{1}{4a^2} + 4b}}{4a} \\
&= \frac{-1 \pm \sqrt{\frac{4a^2}{4a^2} + 16a^2 b}}{4a} \\
&= \frac{-1 \pm \sqrt{1 + 16a^2 b}}{4a}
\end{aligned}$$

$$\text{For any } n \in \mathbb{R},\ n^2 \geq 0. \tag{1}$$

Since both x^2 and y^2 are non-negative by statement 1, and $\frac{1}{2} > 0$, then, in the second equation, b must also be non-negative.

Also, $a^2 \geq 0$ by statement 1.

So, $16a^2b$ must be non-negative.

Then, $\sqrt{1 + 16a^2b} \geq 1$.

Which means,

$$\frac{-1 \pm \sqrt{1 + 16a^2 b}}{4a} \geq \frac{-1 \pm 1}{4} a.$$

Since $x = ay^2$ and $y^2 \geq 0$ as above, then x and a must have the same sign.

So, $\frac{-1 \pm 1}{4} a$ must be non-negative.

Therefore, we choose only $\frac{-1+1}{4} = 0 \geq 0$.

So,

$$x = \frac{-1 + \sqrt{1 + 16a^2 b}}{4a}.$$

Therefore each of the two points will have the same x coordinate.

Solving for x with $x = ay^2$ gives $y = \pm\sqrt{\frac{x}{a}}$. So, there will be two values of y for the given x, which leaves us with two intersection points.

Therefore, the proposition holds. $\square$

We say that two lines are orthogonal is their slopes are negative reciprocals of each other, such that, for two slopes m_1 and m_2, $m_1 = -\frac{1}{m_2}$.

If $m_1 = m_2$, then $m_1 \cdot m_2 = -\frac{m_2}{m_2} = -1$.

Proposition. *The derivates of $x = ay^2$ and $x^2 + \frac{1}{2}y^2 = b$ are orthogonal to each other at all points that satisfy each equation.*

Proof. First, we evaluate the derivate with respect to x of the first equation.

$$\begin{aligned} \frac{d}{dx}[x] &= \frac{d}{dx}\left[ay^2\right] \\ 1 &= 2ay\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{2ay} \end{aligned}$$

Now, by the first equation $x = ay^2$, we know that $a = \frac{x}{y^2}$. So, we can modify the derivate to,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2y\frac{x}{y^2}} \\ &= \frac{1}{\frac{2x}{y}} \\ &= \frac{y}{2x} \end{aligned}$$

Next, we differentiate the second equation with respect to x as well.

$$\begin{aligned} \frac{d}{dx}\left[x^2 + \frac{1}{2}y^2\right] &= \frac{d}{dx}[b] \\ 2x + y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2x}{y} \end{aligned}$$

We recognize this to be the negative reciprocal of the derivate of the first equation.

To verify, we see that $-1 = \frac{y}{2x} \cdot -\frac{2x}{y}$.

Since their derivates are negative reciprocals of one another, the two equations are orthogonal for all x in their domains.

$\square$

By the first and second propositions, the statement holds.

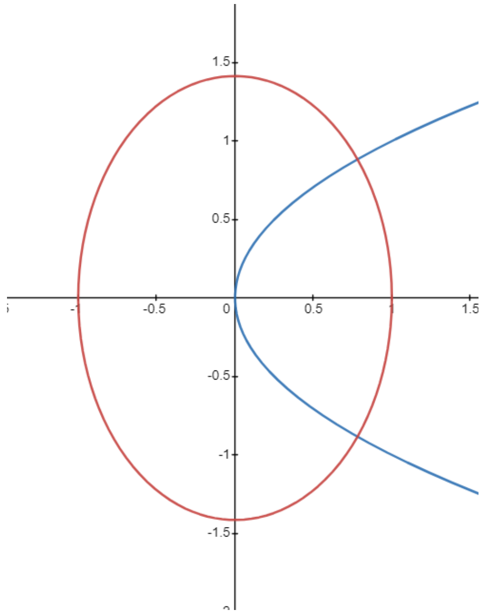

Figure 2: Graph where $a = b = 1$.

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The curve $\left(x^2+y^2\right)^2 = x^2 - y^2$ is called a *lemniscate*. Find the four points of the curve at which the tangent line is horizontal.

First, we differentiate the curve with respect to x, where we let $\frac{dy}{dx} = y'$.

$$\begin{aligned} \frac{d}{dx}\left[\left(x^2+y^2\right)^2\right] &= \frac{d}{dx}\left[x^2-y^2\right] \\ 2\left(x^2+y^2\right)\frac{d}{dx}\left[x^2+y^2\right] &= 2x-2yy' \\ 2\left(x^2+y^2\right)\left(2x+2yy'\right) &= 2x-2yy' \end{aligned}$$

We simplify this expression further by finding x and y only where the derivative y' is zero.

$$\begin{aligned} 2\left(x^2+y^2\right)\left(2x+2yy'\right) = 2x-2yy', \quad y' = 0 \\ 2\left(x^2+y^2\right)(2x) = 2x \\ 2x\left(x^2+y^2\right) - x = 0 \\ x\left(2\left(x^2+y^2\right)-1\right) = 0 \end{aligned}$$

This leaves us with two roots, $x = 0$ and $2\left(x^2+y^2\right) - 1 = 0$.

Considering the first root, $x = 0$, we can substitute this relationship into the lemniscate curve and

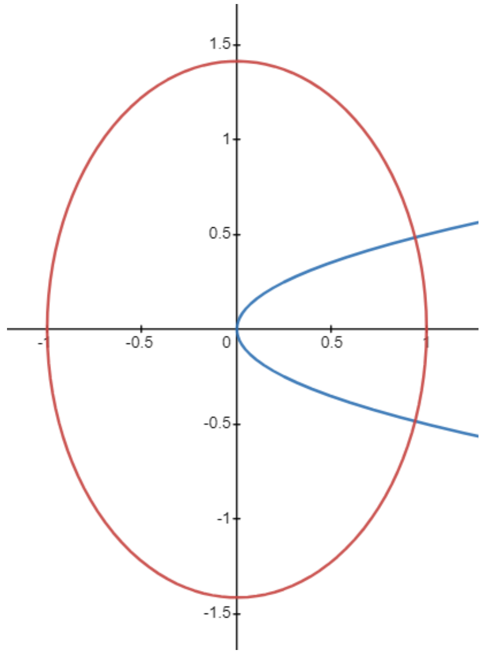

Figure 3: Graph where $a = 4$ and $b = 1$.

reduce,

$$\begin{aligned}(0 + y^2)^2 &= 0 - y^2\\ y^4 &= -y^2\\ y^4 + y^2 &= 0\\ y^2(y^2 + 1) &= 0\end{aligned}$$

We now have have two possible choices for y.

First, $y = 0$, which means that $(0, 0)$ is a location where the derivate is zero. However, upon inspection of the graph of the lemniscate, we can see that there is not a single slope at this location.

Moving onto the next potential y, we see that $y^2 + 1 = 0$, which does not have real solutions. Thus, we will not consider any potential points that follow from this relationship.

So, we will instead consider the other equation $x^2 + y^2 = \frac{1}{2}$. We recognize that this is an equation for a circle. So, the intersection points of this circle and the lemniscate will each have a derivate of zero. Additionally, per the figure, we see that there are four intersection points. These are the desired solutions where the derivate is zero.

We will use the relationship provided by differentiation of the original curve, $y^2 = \frac{1}{2} - x^2$ to find the

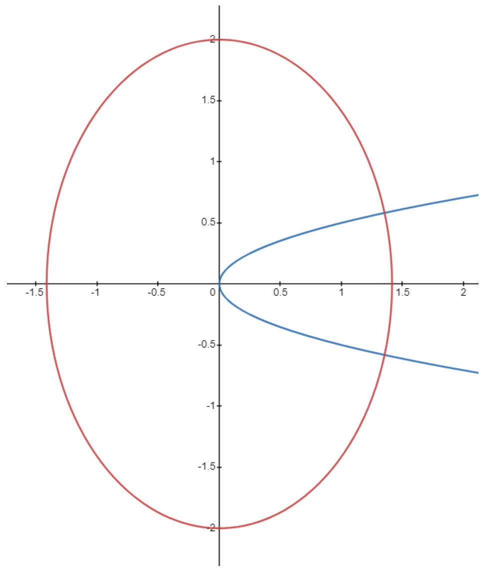

Figure 4: Graph where $a = 4$ and $b = 2$.

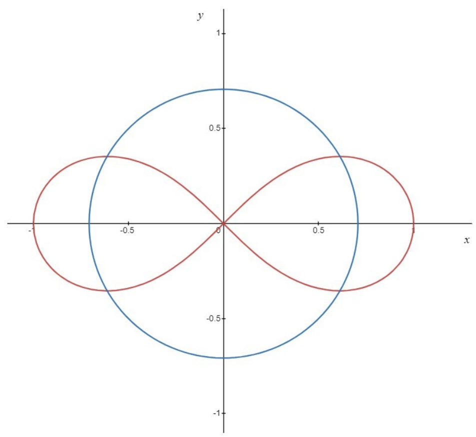

Figure 5: Constraints for $\frac{dy}{dx} = 0$.

x values where the derivative is zero on the curve.

$$\begin{aligned}
\left(x^2 + \left(\frac{1}{2} - x^2\right)\right)^2 &= x^2 - \left(\frac{1}{2} - x^2\right) \\
\left(\frac{1}{2}\right)^2 &= 2x^2 - \frac{1}{2} \\
\frac{1}{4} + \frac{1}{2} &= 2x^2 \\
\frac{3}{4} &= 2x^2 \\
\frac{3}{8} &= x^2 \\
x &= \pm\sqrt{\frac{3}{8}}
\end{aligned}$$

We then use x^2 and the relationship $y^2 = \frac{1}{2} - x^2$ to see that,

$$\begin{aligned} y^2 &= \frac{1}{2} - x^2 \\ y^2 &= \frac{1}{2} - \frac{3}{8} \\ y^2 &= \frac{1}{8} \\ y &= \pm\sqrt{\frac{1}{8}} \end{aligned}$$

We then can pair $y = \pm\sqrt{\frac{1}{8}}$ and $x = \pm\sqrt{\frac{3}{8}}$ to build the four points.

So, the four points that satisfy the statement are, $\left(\sqrt{\frac{3}{8}}, \sqrt{\frac{1}{8}}\right)$, $\left(-\sqrt{\frac{3}{8}}, \sqrt{\frac{1}{8}}\right)$, $\left(\sqrt{\frac{3}{8}}, -\sqrt{\frac{1}{8}}\right)$, and $\left(-\sqrt{\frac{3}{8}}, -\sqrt{\frac{1}{8}}\right)$.

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A circle of radius 1 with center on the y-axis is inscribed in the parabola $y = 2x^2$. Find the points of contact.

The equation of a circle of radius 1 with a vertical offset of a above the x-axis is $x^2 + (y-a)^2 = 1$.

The points of contact will have parallel tangent lines. Therefore, we seek a relationship of x and y from the equality of the derivates of the parabola and the circle.

First, we differentiate the parabola equation with respect to x.

$$\begin{aligned} \frac{d}{dx}[y] &= \frac{d}{dx}\left[2x^2\right] \\ \frac{dy}{dx} &= 4x \end{aligned}$$

Next, we also differentiate the circle equation.

$$\begin{aligned} \frac{d}{dx}\left[x^2 + (y-a)^2\right] &= \frac{d}{dx}[1] \\ 2x + 2(y-a)\frac{d}{dx}[y-a] &= 0 \\ 2x + 2(y-a)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-x}{y-a} \\ \frac{dy}{dx} &= \frac{x}{a-y} \end{aligned}$$

We then find all y where these two derivates are equal,

$$\begin{aligned} 4x &= \frac{x}{a-y} \\ a-y &= \frac{1}{4} \\ y &= a - \frac{1}{4} \end{aligned}$$

We use this y to solve for x using the parabola equation $y = 2x^2$.

$$\begin{aligned} a - \frac{1}{4} &= 2x^2 \\ \frac{a}{2} - \frac{1}{8} &= x^2 \\ x &= \pm\sqrt{\frac{a}{2} - \frac{1}{8}} \end{aligned}$$

We also know that all contacting points between the parabola and the circle require that they intersect. We find the intersections by substituting the relationship $x^2 = \frac{y}{2}$ from the parabola into the circle equation in addition to our relationship between y and a derived from the parallel tangents, $y = a - \frac{1}{4}$.

$$\begin{aligned} x^2 + (y-a)^2 &= 1 \\ \frac{y}{2} + (y-a)^2 &= 1 \\ \frac{a - \frac{1}{4}}{2} + \left(a - \frac{1}{4} - a\right)^2 &= 1 \\ \frac{a}{2} - \frac{1}{8} + \left(-\frac{1}{4}\right)^2 &= 1 \\ \frac{a}{2} - \frac{1}{8} + \frac{1}{16} &= 1 \\ a &= 2 + \frac{1}{4} - \frac{1}{8} \\ a &= \frac{17}{8} \end{aligned}$$

We then use a to solve for x and y to find the two points of intersection with parallel tangent lines.

First, $y = a - \frac{1}{4} = \frac{17}{8} - \frac{2}{8} = \frac{15}{8}$.

Next, $x = \pm\sqrt{\frac{a}{2} - \frac{1}{8}} = \pm\sqrt{\frac{17}{16} - \frac{2}{16}} = \pm\sqrt{\frac{15}{16}} = \pm\frac{\sqrt{15}}{4}$.

So, the two contacting points of the parabola and circle are $\left(\frac{\sqrt{15}}{4}, \frac{15}{8}\right)$ and $\left(-\frac{\sqrt{15}}{4}, \frac{15}{8}\right)$.