Math 134 Homework 7

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5.9

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For a rod that extends from $x = a$ to $x = b$ and has mass density $\lambda(x)$, the integral

$$\int_a^b (x - c)\lambda(x)\,dx$$

gives what is called the mass moment of the rod about the point $x = c$. Show that the mass moment about the center of mass is zero. (The center of mass can be defined as the point about which the mass moment is zero.)

Definition (5.9.4)**.** *The mass of a rod with density $\lambda(x)$ is*

$$M = \int_a^b \lambda(x)\,dx.$$

Definition (5.9.5)**.** *The center of mass x_M of a rod of variable density $\lambda(x)$ is*

$$x_M M = \int_a^b x\lambda(x)\,dx.$$

Proof. First, we evaluate the mass moment at the center of mass x_M,

$$\begin{aligned} &\int_a^b (x - x_m)\lambda(x)\,dx \\ =&\int_a^b x\lambda(x)\,dx - \int_a^b x_M\lambda(x)\,dx \end{aligned}$$

We note that x_M is a constant. Then, using Definitions 5.9.4 and 5.9.5, we see that,

$$\begin{aligned} &\int_a^b x\lambda(x)\,dx - \int_a^b x_M\lambda(x)\,dx \\ =&x_M \cdot M - x_M \cdot M \\ =&0. \end{aligned}$$

So, the mass moment about the center of mass is zero.

$\square$

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Prove the proposition.

Proposition. *Two distinct continuous functions cannot have the same average on every interval.*

Proof 1. Let $h(x) = f(x) - g(x)$. Since f and g are continuous, then their difference h is also continuous.

Since $f \neq g$, there exists a c such that $h(c) = f(c) - g(c) \neq 0$.

Without loss of generality, suppose that $f(c) > g(c)$.[1] So, $h(c) > 0$.

Since h continuous, then there is an interval I that is $\delta > 0$ close to c, defined as $(c - \delta, c + \delta)$, such that $h(c) > 0$.

So, by the definition of continuity, $|x - c| < \delta$ implies $|h(x) - h(c)| < h(c)$.

But, $|h(x) - h(c)| < h(c)$ is equivalent to $h(c) - h(c) < h(x) < h(c) + h(c)$.

So, $0 < h(x)$.

Since $h > 0$ on I, then $f > g$ on I.

Then, by 5.8.2,

$$\int_{c-\delta}^{c+\delta} f(x) - g(x)\, dx > 0.$$

So,

$$\int_{c-\delta}^{c+\delta} f(x)\, dx > \int_{c-\delta}^{c+\delta} g(x)\, dx.$$

Or,

$$\frac{1}{2\delta} \int_{c-\delta}^{c+\delta} f(x)\, dx > \frac{1}{2\delta} \int_{c-\delta}^{c+\delta} g(x)\, dx.$$

Therefore, there exists an interval where the averages of f and g are no the same, so the proposition holds.

□

Proof 2. Define $h(x) = f(x) - g(x)$.

Assume that $h(x) \neq 0$.

For a contradiction, assume that f and g have the same average on any arbitrary interval $[a, b]$.

Then,

$$\frac{1}{b-a} \int_a^b h(x) dx = 0.$$

[1] If g is greater than f, repeat the process with the labels swapped.

So, for all $a < b$,

$$\int_a^b h(x) = 0.$$

Let

$$H(x) = \int_a^x h(t)dt = 0.$$

Then,

$$H'(x) = h(x) = 0.$$

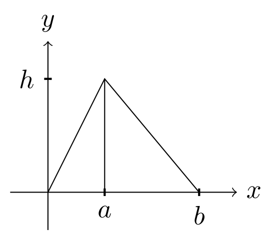
But, we assumed $h(x) \neq 0$. So the assumption was false and the statement holds.

$\square$

6.4

30

Find the volume of the solid generated by revolving the triangular region in the figure.



(a) about the x-axis.

(b) about the y-axis.

For (a), we start by finding two function to represent the hypotenuses of the triangles.

Let the hypotenuse segment with positive slope be defined by the function f, while the segment with negative slope be defined by g.

The slope of f with respect to the vertical axis is $\frac{a}{h}$. Since f intersects the origin, then $f(y) = \frac{a}{h}y$.

The slope of g with respect to y is $\frac{a-b}{h}$. It intercepts the x-axis at $x = b$. So, $g(y) = \frac{a-b}{h}y + b$.

We integrate around the x-axis with cylindrical shells.

The circumference of these shells is given by $2\pi y$, where y, the distance from the x-axis, is the shell's radius.

The height of each shell is given by the difference between f and g. Since, $g > f$ on $[0, h]$, then we will take g to define the upper part of the cylindrical shell to retain a positive area.

So, the height is given by,

$$\begin{aligned} g(y) - f(y) &= \frac{a-b}{h}y + b - \frac{a}{h}y \\ &= b - \frac{b}{h}y \\ &= b\left(1 - \frac{y}{h}\right). \end{aligned}$$

The thickness of each cylindrical shell is the differential Δy.

We then integrate the volume of shells over $[0, h]$,

$$\begin{aligned} &\int_0^h 2\pi y \cdot b\left(1 - \frac{y}{h}\right)\, dy \\ &= 2\pi b \int_0^h y - \frac{y^2}{h}\, dy \\ &= 2\pi b \left[\frac{y^2}{2} - \frac{y^3}{3h}\right]\Bigg|_0^h \\ &= 2\pi b \left(\frac{h^2}{2} - \frac{h^3}{3h}\right) \\ &= 2\pi b \left(\frac{h^2}{6}\right) \\ &= \frac{\pi b h^2}{3}. \end{aligned}$$

So, the solid of revolution about the x-axis has an area of $\frac{\pi b h^2}{3}$.

For (b), we will integrate about the y-axis with washers, where $g(y)$ is further from the axis of rotation. So, we take $g(y)$ to be the outer function.

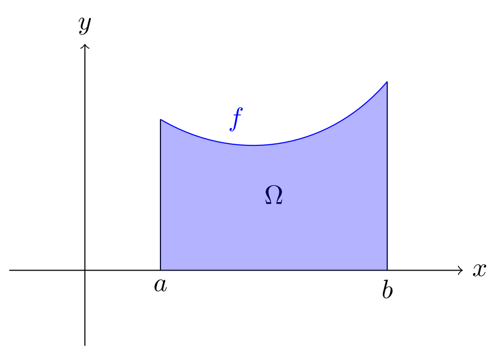
We integrate the washers over the same interval, $[0, h]$,

$$
\begin{aligned}
&\pi \int_0^h [g(y)]^2 - [f(y)]^2 dy \\
&= \pi \int_0^h \left(\frac{a-b}{h}y + b\right)^2 - \left(\frac{a}{h}y\right)^2 dy \\
&= \pi \int_0^h \frac{(a-b)^2}{h^2}y^2 + \frac{2b(a-b)}{h}y + b^2 - \frac{a^2}{h^2}y^2 \, dy \\
&= \pi \int_0^h \frac{a^2 - 2ab + b^2}{h^2}y^2 + \frac{2b(a-b)}{h}y + b^2 - \frac{a^2}{h^2}y^2 \, dy \\
&= \pi \int_0^h \frac{b^2 - 2ab}{h^2}y^2 + \frac{2b(a-b)}{h}y + b^2 \, dy \\
&= \pi b \left[\frac{b-2a}{3h^2}y^3 + \frac{a-b}{h}y^2 + by\right] \Bigg|_0^h \\
&= \pi b \left(\frac{b-2a}{3h^2}h^3 + \frac{a-b}{h}h^2 + bh\right) \\
&= \pi b \left(\frac{(b-2a)h}{3} + (a-b)h + bh\right) \\
&= \pi bh \left(\frac{b-2a}{3} + a - b + b\right) \\
&= \pi bh \left(\frac{b+a}{3}\right) \\
&= \frac{\pi}{3} bh(a+b).
\end{aligned}
$$

So, the volume of the solid of revolution about the y-axis is $\frac{\pi}{3}bh(a+b)$.

Project

If a solid is *homogeneous* (constant mass density), then the center of mass depends only on the shape of the solid and is called the *centroid*. In general, determination of the centroid of a solid requires triple integration. However, if the solid is a solid of revolution, then the centroid can be found by one-variable integration.



Explain Problems 1 and 2 briefly with a picture.

1

Let Ω be the region shown in the figure and let T be the solid generated by revolving Ω around the x-axis. By symmetry, the centroid of T is on the x-axis. Thus the centroid of T is determined solely by its x-coordinate $\overline{x}$.

Show that $\overline{x}V = \int_a^b \pi x[f(x)]^2 dx$ where V is the volume of T.

Use the following principle: if a solid of volume V consists of a finite number of pieces with volumes $V_1, V_2, \ldots, V_n$ and the pieces have centroids $x_1, x_2, \ldots, x_n$, then $\overline{x}V = x_1V_1 + x_2V_2 + \cdots + x_nV_n$.

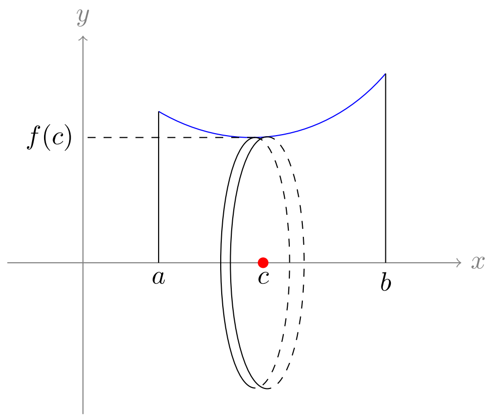


Figure 1: A disk volume piece of radius $f(c)$ and width Δx. By radial symmetry, the center of mass is in the middle of the disk, at point $(c, 0)$ as Δx tends toward zero.

Using the disk method, we sum the pieces of volume $\pi[f(x)]^2\Delta x$.

For each disk, the centroid simplify lies at the x coordinate, as Δx tends to zero, and the disk is symmetrical by rotation.

So, each centroid volume product is given as,

$$\pi x[f(x)]^2\Delta x.$$

Then, we partition the interval $[a, b]$ and sum of all of the pieces from x_0 to x_n, in order to reveal the integral

$$\sum_{i=0}^{n} \pi x_i[f(x_i)]^2\Delta x_i = \int_a^b \pi x[f(x)]^2\,dx.$$

2

Now revolve Ω around the y-axis and let S be the resulting solid. By symmetry, the centroid of S lies on the y-axis and is determined solely by its y-coordinate $\overline{y}$.

Show that $\overline{y}V = \int_a^b \pi x[f(x)]^2 dx$ where V is the volume of S.

Using the shell method, each cylindrical shell contributes a volume of the difference between two cylinders,

$$f(x)\pi(x + \Delta x)^2 - f(x)\pi x^2.$$

We combine this with the fact that the center of mass of each shell occurs at the average value of the functions $f(x)$ and 0, so $\frac{f(x)}{2}$.

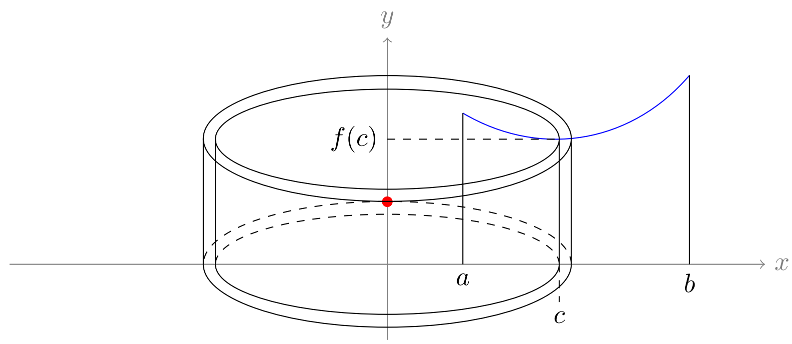


Figure 2: A cylindrical shell volume piece of width Δx and height $f(c)$. By radial symmetry, the center of mass occurs in the middle of the shell, at a height of $\frac{f(c)}{2}$.

Then, we simplify the center of mass and volume for each piece, noting that a second degree differential is approximated as zero,

$$\begin{aligned}
&\frac{f(x)}{2}\left[f(x)\pi(x+\Delta x)^2 - f(x)\pi x^2\right] \\
=&\frac{f(x)}{2}\cdot \pi f(x)\left[x^2 + 2x\Delta x + (\Delta x)^2 - x^2\right] \\
=&\frac{\pi}{2}[f(x)]^2(2x\Delta x) \\
=&\pi x[f(x)]^2\Delta x
\end{aligned}$$

As in 1, we recognize this to be the integral,

$$\sum_{i=0}^{n} \pi x_i[f(x_i)]^2\Delta x_i = \int_a^b \pi x[f(x)]^2\,dx.$$

3(d)

Use the results in Problems 1 and 2 to locate the centroid of the solid generated by revolving the region below the graph of $f(x) = \sqrt{x}$, $x \in [0,1]$,

(i) about the x-axis;

(ii) about the y-axis.

For (i), by Problem 1, the centroid about the x-axis is given by,

$$\overline{x} = \frac{1}{V}\int_0^1 \pi x^2\,dx.$$

First, we compute the volume of the solid of rotation about the x-axis using the disk method,

$$\begin{aligned} V &= \int_0^1 \pi(\sqrt{x})^2\,dx \\ &= \pi\left[\frac{x^2}{2}\right]\Big|_0^1 \\ &= \frac{\pi}{2}. \end{aligned}$$

So,

$$\begin{aligned} \overline{x} &= \frac{2}{\pi}\int_0^1 \pi x^2\,dx \\ &= 2\left[\frac{x^3}{3}\right]\Big|_0^1 \\ &= \frac{2}{3}. \end{aligned}$$

Thus,

$$\overline{x} = \frac{2}{3}.$$

For (ii), by Problem 2, the centroid about the y-axis is given by,

$$\overline{y} = \frac{1}{V}\int_0^1 \pi x^2\,dx.$$

We compute the volume of the solid of revolution about the y-axis using the shell method,

$$\begin{aligned} V &= \int_0^1 2\pi x \cdot \sqrt{x}\,dx \\ &= 2\pi\int_0^1 x^{\frac{3}{2}}\,dx \\ &= 2\pi\left[\frac{2}{5}x^{\frac{5}{2}}\right]\Big|_0^1 \\ &= 2\pi\left(\frac{2}{5}\right) \\ &= \frac{4\pi}{5}. \end{aligned}$$

Then,

$$\begin{aligned} \overline{y} &= \frac{5}{4\pi}\int_0^1 \pi x^2\,dx \\ &= \frac{5}{4}\left[\frac{x^3}{3}\right]\Big|_0^1 \\ &= \frac{5}{12}. \end{aligned}$$

So,

$$\overline{y} = \frac{5}{12}.$$

6.5

40

A storage tank in the form of a hemisphere topped by a cylinder is filled with oil that weighs 60 pounds per cubic foot. The hemisphere has a 4-foot radius; the height of the cylinder is 8 feet.

(a) How much work is required to pump the oil to the top of the tank?

(b) How long would it take a $\frac{1}{2}$-horsepower motor to empty out the tank?

For (a), we integrate the work done by considering small sections of oil traveling up to the top of the tank.

Each little bit of force is given by the horizontal cross sectional area of the tank times the density of oil that fills it times gravity and, finally, times the small change in height of the cross section.

The cross sectional area of a hemisphere is given by πr^2, where the radius is a function of the vertical variable y and is given by the equation of a circle with radius 4, $\sqrt{4^2 - y^2}$. So, the area at height y in the hemisphere is

$$\pi \left(16 - y^2\right).$$

Then, given the oil density as above, each small piece of force is given by,

$$60\pi \left(16 - y^2\right) \Delta y.$$

For the area of the cylindrical section, the cross sectional area is a constant $4^2\pi = 16\pi$. Similarly, each bit of force is given by,

$$60 \cdot 16\pi = 960\pi.$$

The distance that each slice of area travels is given by its current depth below the top of the tank. We will consider center of the hemisphere to lie at the origin. Thus, height is modelled as the difference between the current height of the oil and the top of the tank,

$$8 - y.$$

We Integrate the work for all the pieces of the hemisphere lying from the base at $y = -4$ to the top at

$y = 0$, and the pieces of the cylinder from $y = 0$ to $y = 8$,

$$\begin{aligned}
&60\pi \int_{-4}^{0} \left(16 - y^2\right)(8 - y)dy + \int_{0}^{8} 960\pi(8 - y)dy \\
=&60\pi \int_{-4}^{0} 128 - 16y - 8y^2 + y^3\, dy + 960\pi \left[8y - \frac{1}{2}y^2\right]\Bigg|_0^8 \\
=&60\pi \left[128y - 8y^2 - \frac{8}{3}y^3 + \frac{1}{4}y^4\right]\Bigg|_{-4}^{0} + 960\pi\left(64 - \frac{64}{2}\right) \\
=&60\pi \left(0 - \left(-512 - 128 + \frac{512}{3} + \frac{256}{4}\right)\right) + 960\pi(32) \\
=&60\pi \left(576 - \frac{512}{3}\right) + 30720\pi \\
=&20\pi(1728 - 512) + 30720\pi \\
=&24320\pi + 30720\pi \\
=&55040\pi \\
\approx&172913.
\end{aligned}$$

So, the total work done to move the oil to the top of the tank is 172913 foot pounds.

For (b), we note that one horsepower (HP) is 550 foot pounds per second.

So, half of one horsepower is 275 foot pounds per second.

Thus, cancelling foot pounds by a rate of foot pounds per second, we are left with seconds,

$$\frac{172913}{275} \approx 629.$$

Finally, 629 seconds is about $\frac{629}{60} \approx 10.5$ minutes.

Therefore, it takes about ten and a half minutes to pump the oil out of the top of the tank.

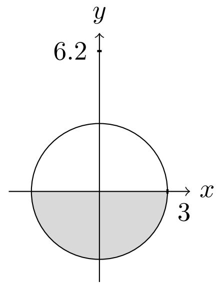
Worksheet

3

(a) Gasoline is stored in a cylindrical tank of radius 3 meters and length 12 meters lying under ground in a gas station. The tank is buried on its side with the highest part of the tank 2.5 meters below ground. The tank is initially half-full. Suppose that the filler cap of each car is 0.7 meters above the ground. Express the work done in pumping all the gasoline as an integral. The density of gasoline is 750 kilograms per cubic meter.

(b) Show that the work done is the weight of the gasoline times the distance its center of mass travels vertically. Note that you only need the y-coordinate of its center of mass. What are the x and z coordinates of the center of mass of the gasoline?

For (a), we start with the cross sectional area of each part of gas in the tank.



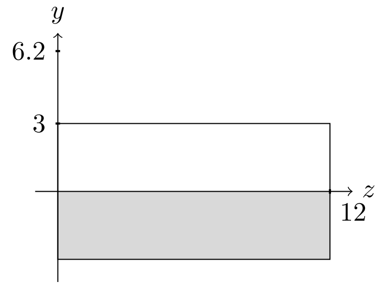


Figure 3: Note that from the origin, the height of the spout is $3 + 2.5 + 0.7 = 6.2$ feet.

We model the rectangular areas by their depth of 12 meters times the width of the circular side of the tank at height y.

So, the minute change is volume ΔV is given by the area at y times Δy, or $12 \cdot 2\sqrt{9 - y^2}\Delta y$.

Then, the force is given by the volume times the density of 750 kilos per cubic meter times the gravitational constant g, $750 \cdot g \cdot 24\sqrt{9 - y^2}\Delta y = 18000g\sqrt{9 - y^2}\Delta y$.

Next, the height that each slice travel is $6.2 - y$.

So, we express the work done to pump all the gasoline from the tank as the integral from the base of the tank at $y = -3$ to the middle of the tank at $y = 0$,

$$\int_{-3}^{0} 18000g\sqrt{9 - y^2}(6.2 - y)dy.$$

Then, we integrate,

$$\begin{aligned}
&\int_{-3}^{0} 18000g\sqrt{9 - y^2}(6.2 - y)dy \\
=&18000g\left[6.2\int_{-3}^{0}\sqrt{9 - y^2} - \int_{-3}^{0} y\sqrt{9 - y^2}\right]
\end{aligned}$$

We note that the second integral is just the area of a circle with radius 3 in the second quadrant, equivalent to $\frac{9\pi}{4}$.

For the latter integral, we make the substitution $u(y) = 9 - y^2$, so that $dy = -\frac{1}{2y}du$.

$$\begin{aligned}
=&18000g\left[6.2 \cdot \frac{9\pi}{4} - \int_{u(-3)}^{u(0)} \frac{-1}{2}\sqrt{u}\, du\right] \\
=&18000g\left(6.2 \cdot \frac{9\pi}{4} + \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]\Bigg|_{0}^{9}\right) \\
=&18000g\left(6.2 \cdot 9 \cdot \frac{\pi}{4} + 9\right) \\
=&18000g\left(55.8\left(\frac{\pi}{4} + 1\right)\right) \\
=&1004400g\left(1 + \frac{\pi}{4}\right).
\end{aligned}$$

With $g = 9.8$, then $1004400g\left(1 + \frac{\pi}{4}\right) \approx 1.76 \cdot 10^7$ joules.

For (b), we are only concerned with the distance that the gasoline moves vertically, as we are computing work done against gravity. The center of mass of the gasoline will lie in the middle of the tank due to symmetry, and may move laterally to the pump. We are only concerned with vertical movement.

The total mass of the gasoline is given by the integral of the density times each bit of volume,

$$\int \rho\, dV.$$

The total weight of the gasoline is given by the integral of the mass times the gravitational constant g,

$$\int \rho g\, dV.$$

The mass moment of the gasoline about the vertical axis y is given by the integral of the density ρ times the bits of volume,

$$\int y\rho g\, dV.$$

We define $\overline{y}$ as center of mass of the gasoline, which is given by the moment about vertical axis y divided by the total mass of the gasoline,

$$\frac{\int y\rho g\, dV}{\int \rho\, dV}.$$

We define the distance that a portion of gas travels as the vertical linear difference between the destination height a and the current height y, such that $h(y) = a - y$.

The total work done is the integral of all the pieces of weight times the distance that the pieces moved.

We claim that the total work is equivalent to the distance that the center of mass $\overline{y}$ traveled times the total weight of the gasoline.

So,

$$\begin{aligned} h(\overline{y}) \cdot \int \rho g\, dV &= \int h(y)\rho g\, dV \\ \left(a - \frac{\int y\rho\, dV}{\int \rho\, dV}\right) \int \rho g\, dV &= \int (a - y)\rho g\, dV \\ \int a\rho g\, dV - \int \rho g y\, dV &= \int a\rho g\, dV - \int \rho g y\, dV. \end{aligned}$$

Thus, we see that the statement holds in general. In our case, $\rho = 750$ kilograms per cubic meter, $a = 6.2$ meters, and $g = 9.8$ meters per second per second.

7.1

52

Set

$$f(x) = \int_1^{2x} \sqrt{16 + t^4}\, dt$$

(a) Show that f has an inverse.

The statement is equivalent to showing that f is injective.

We will show that f is increasing for all x in its domain.

We differentiate $f(x)$ with respect to x by 5.8.7,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\int_1^{2x} \sqrt{16 + t^4}\, dt\right] \\ &= 2\sqrt{16 + (2x)^4} \\ &= 2\sqrt{16 + 16x^4} \\ &= 8\sqrt{1 + x^4}. \end{aligned}$$

We note that the range of $\sqrt{x} > 0$ for all $x \in \mathbb{R}$.

So, $f'(x) > 0$ for all x in the domain of f.

Since, $f'(x) > 0$ for all $x \in \operatorname{dom}(f)$, then f is increasing.

Since f is increasing for all x in its domain, then f is injective.

(b) Find $\left(f^{-1}\right)'(0)$.

Theorem 7.1.8 states that, when f is injective and differentiable, then $\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$ where $f(a) = b$.

First, we solve for a according to Theorem 7.1.8 from $b = 0$ by noting that for any function $h(x)$ and number k, $\int_k^k h(x)dx = 0$,

$$0 = f(a) = \int_1^{2a} \sqrt{16 + t^4}\, dt$$

So, $2a = 1$, or $a = \frac{1}{2}$.

Next, with $f'(x)$ from (a),

$$\begin{aligned}
\left(f^{-1}\right)'(0) &= \frac{1}{f'\left(\frac{1}{2}\right)} \\
&= \frac{1}{8\sqrt{1+\left(\frac{1}{2}\right)^2}} \\
&= \frac{1}{8\sqrt{1+\frac{1}{16}}} \\
&= \frac{1}{8\frac{\sqrt{17}}{4}} \\
&= \frac{1}{2\sqrt{17}}.
\end{aligned}$$

Therefore, we see that $\left(f^{-1}\right)'(0) = \frac{1}{2\sqrt{17}}$.