Math 336 Homework 2

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Problem 1. Prove that
$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{\sqrt{2\pi}}{4}.$$
Integrate e^{-z^2} over the path given in the figure. Recall that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

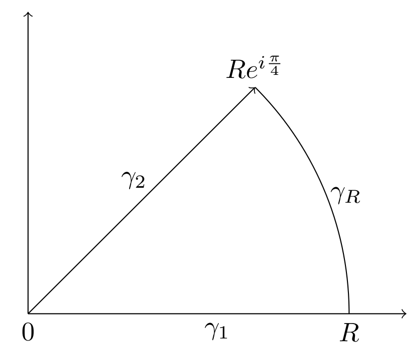

Figure 1. Section contour; consider $\int_0^\infty$ as $\lim\limits_{R\to\infty}\int_R^\infty$.

Proof. Since $f(z) = e^{-z^2}$ is a composition of holomorphic functions on $\mathbb{C}$, then f is holomorphic on $\mathbb{C}$.

Let $\gamma = \gamma_1 \cup \gamma_R \cup \gamma_2$ be the closed loop section contour in the figure. Since f is holomorphic, then it has the closed loop property and $\oint_\gamma f\,dz = 0$.

So, we have that
$$0 = \oint_\gamma f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \int_{\gamma_R} f\,dz.$$

But, as $R \to \infty$, γ_1 parametrizes the positive real line, so
$$\int_{\gamma_1} e^{-z^2}\,dz = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$
because e^{-x^2} is an even function.

We will show that the integral of f over the arc γ_R goes to zero. Let $z = re^{i\theta}$. On this curve, r is fixed at R and θ varies from 0 to $\frac{\pi}{4}$.

So, $-z^2 = -R^2e^{2i\theta}$ and $dz = iRe^{i\theta}$. Then,

$$\int_0^{\frac{\pi}{4}} iRe^{i\theta}e^{-R^2e^{2i\theta}}\,d\theta = \int_0^{\frac{\pi}{4}} iRe^{i\theta-R^2e^{2i\theta}}\,d\theta.$$

For $\theta \in \left[0, \frac{\pi}{4}\right]$, we have that $e^{2i\theta} \in \left[e^0, e^{i\frac{\pi}{2}}\right] = [1, i]$.

So, as $R \to \infty$, we have,

$$\forall\theta \in \left[0, \frac{\pi}{4}\right), \quad \left|Re^{i\theta-R^2e^{2i\theta}}\right| \to 0 \quad \text{and} \quad \theta = \frac{\pi}{4}, \quad \left|Re^{i(\theta-R^2)}\right| \to \infty.$$

However, since the integrand only blows up at a simple point, which has measure zero, then the contribution of this point to the overall integral is zero. Hence

$$\lim_{R\to\infty}\int_{\gamma_R} f\,dz = 0.$$

So, maintaining the orientation of each part of the curve γ, we have that

$$\int_{\gamma_1} f\,dz = -\int_{\gamma_2} f\,dz.$$

Now, for the integral on the path γ_2, we fix $\theta = \frac{\pi}{4}$ and vary r from 0 to R. Note that this parameterization traverses γ_2 in the opposite direction of the positively oriented curve γ, so we will consider the opposite of the resulting integral.

So, $-z^2 = -r^2e^{2i\theta} = -r^2e^{i\frac{\pi}{2}} = -ir^2$ and $dz = e^{i\frac{\pi}{4}}\,dr = \frac{1}{\sqrt{2}}(1+i)dr$.

Then, with the Euler identity, we have that

$$-\int_{\gamma_2} f\,dz = \frac{1}{\sqrt{2}}(1+i)\int_0^\infty e^{-ir^2}\,dr = \frac{1}{\sqrt{2}}(1+i)\int_0^\infty (\cos r^2 - i\sin r^2)\,dr.$$

With the above equality between the integrals of γ_1 and γ_2, we have

$$\begin{aligned}\frac{\sqrt{\pi}}{2} &= \frac{1}{\sqrt{2}}(1+i)\int_0^\infty (\cos r^2 - i\sin r^2)\,dr\\ \sqrt{\frac{\pi}{2}} &= \int_0^\infty \cos x^2\,dx + \int_0^\infty \sin x^2\,dx + i\left(\int_0^\infty \cos x^2\,dx - \int_0^\infty \sin x^2\,dx\right)\\ &.\end{aligned}$$

Considering the real and imaginary parts separately, we have

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx$$

and therefore also

$$\frac{\sqrt{2\pi}}{4} = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx$$

as desired. $\square$

Problem 2. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter of the image of f, $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ satisfies

$$2|f'(0)| \leq d,$$

and that equality holds when f is linear.

Proof. By the Cauchy integral formula, we have

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega - z)^2}\, d\omega$$

for some circle $C \subset \mathbb{D}$ centered at z.

Then, at $z = 0$, we have

$$f'(0) = \frac{1}{2\pi i}\int_{|\omega|=r} \frac{f(\omega)}{\omega^2}\, d\omega, \quad \forall r \in (0,1),$$

which we can also substitute $\omega \mapsto -\omega$ to achieve the opposite of the above.

Note that $|f(\omega) - f(-\omega)| \leq d$.

Hence, $\forall r \in (0,1)$,

$$\begin{aligned}
2f'(0) &= \frac{1}{2\pi i}\int_{|\omega|=r} \frac{f(\omega) - f(-\omega)}{\omega^2}\, d\omega \\
2|f'(0)| &= \left|\frac{1}{2\pi i}\int_{|\omega|=r} \frac{f(\omega) - f(-\omega)}{\omega^2}\, d\omega\right| \\
&\leq \frac{1}{2\pi}\int_{|\omega|=r} \frac{d}{|\omega|^2} d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{r^2} r\, d\theta \\
&= \frac{d}{2\pi}\int_0^{2\pi} \frac{1}{r}\, d\theta \\
&= \frac{d}{r}.
\end{aligned}$$

Since the above holds for all $r \in (0,1)$, then, considering the limit,

$$\lim_{r\to 1^-} \frac{d}{r} = d \implies 2|f'(0)| \leq d,$$

as desired.

If f is linear, then we can write $f = a + bz$.

Then, $|f(\omega) - f(-\omega)| = 2|b|$.

But, we also have that the diameter of the image d is given by

$$|f(z) - f(\omega)| = |(a + bz) - (a + b\omega)| = |b||z - \omega|$$

So, we have that

$$\sup_{z,\omega\in\mathbb{D}} |f(z) - f(\omega)| = |b| \sup_{z,\omega\in\mathbb{D}} |z - \omega| = 2|b|.$$

Thus, we have equality when proceeding as above. $\square$

Problem 3. Let $\Omega \subset \mathbb{C}$ be bounded and open, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

Proof. Consider $f(z) = \varphi(z + z_0) - z_0$. If $z_0 = 0$, then $f(z) = \varphi(z)$. Otherwise, $f(0) = \varphi(z_0) - z_0 = 0$. So, WLOG, assume $z_0 = 0$.

For a contradiction, assume φ is not linear, so we can write it as

$$\varphi(z) = z + a_n z^n + O(z^{n+1})$$

for $n \geq 2$ and $a_n \neq 0$.

Let φ_k be the composition of φ with itself k times. We will show that $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$ by induction.

For the base case with $k = 1$, we have that

$$\varphi(z) = z + a_n z^n + O(z^{n+1}) = \varphi_1(z).$$

Assume the result holds for k and consider $k + 1$,

$$\begin{aligned}
\varphi_{k+1}(z) &= \varphi(\varphi_k(z)) \\
&= \varphi_k(z) + ka_n(\varphi_k(z))^n + O((\varphi_k(z))^{n+1}) \\
&= z + a_n z^n + O(z^{n+1}) + ka_n(z + a_n z^n + O(z^{n+1})) + O((z + a_n z^n + O(z^{n+1}))^{n+1}).
\end{aligned}$$

By the binomial expansion, we have that

$$(z + a_n z^n + O(z^{n+1}))^n = z^n + O(z^{n+1}).$$

Furthermore, we also can simplify the following,

$$O((z + a_n z^n + O(z^{n+1}))^{n+1}) = O(z^{n+1}).$$

Therefore,

$$\varphi_{k+1}(z) = z + a_n z^n + ka_n(z^n + O(z^{n+1})) + O(z^{n+1}) = z + (k+1)a_n z^n + O(z^{n+1}),$$

as desired.

Since Ω is bounded, then there exists an $R > 0$ such that, for all $z \in \Omega$, $|z| < R$. Since $\varphi_k(z)$ maps between Ω, then for all k and $z \in \Omega$,

$$\sup_{C_R} |\varphi_k(z)| \leq R.$$

Then, with Cauchy's inequality,

$$|\varphi_k^{(n)}(z)| = |kn!a_n + O(z)| \leq \frac{n!}{R^n} \sup_{C_R} |\varphi_k(z)| \leq \frac{n!}{R^{n-1}}.$$

Next, for z near 0,

$$\lim_{k\to\infty} |a_n| \leq \lim_{k\to\infty} \frac{1}{kR^{n-1}} = 0.$$

Therefore $a_n = 0$ for all $n \geq 2$. So, we are left with $\varphi(z) = z$, which is indeed linear.

Lastly, we also must have $\varphi'(z) = 1$ as any non-unit coefficient on z in φ would also result in an unbounded φ_k as $k \to \infty$.

$$\varphi(z) = az \implies \varphi_k(z) = a^k z,$$

but

$$\lim_{k\to\infty} |\varphi_k'(z)| = \lim_{k\to\infty} a^k \leq \frac{1}{R} \sup_{C_R} |\varphi_k(z)| \leq 1.$$

So we must have $a = 1$. □

Problem 4. Let $u : \mathbb{D} \to \mathbb{R}$. Suppose $u \in \mathbb{C}$ and $\Delta u(x, y) = 0$ for all $(x, y) \in \mathbb{D}$.

(a) Prove that there exists a holomorphic function on the unit disk such that $\mathrm{Re}(f) = u$. Show that the imaginary part of f is defined up to an additive real constant.

(b) Deduce the Poisson integral representation formula from the Cauchy integral formula. If u is harmonic in the unit disk and continuous on its closure, then, with $z = re^{i\theta}$,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) u(e^{i\varphi})\, d\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disk given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}.$$

Proof of (a). Let $f' = g = 2\partial_z u$. We will show that $\partial_{\bar{z}} g = 0$, which implies that g is holomorphic.

We have the following relation between the mixed Wirtinger derivatives and the Laplacian:

$$\partial_{\bar{z}}\partial_z = \frac{1}{4}\Delta.$$

Note that the order of the derivatives may be switched. So,

$$\partial_{\bar{z}} g = \partial_{\bar{z}}(2\partial_z u) = \frac{1}{2}\Delta u = 0.$$

Hence g is holomorphic on $\mathbb{D}$, and it has a primitive G on $\mathbb{D}$ such that $G' = g$.

Then, with $G' = g = 2u'$, integrating both sides gives,

$$G = 2\int \frac{d}{dz} u\, dz = 2u + (a + bi)$$

where $a + bi$ is a complex constant of integration. So,

$$\mathrm{Re}(G) = 2u + a \text{ and } \mathrm{Im}(G) = b.$$

Thus, let $f = \frac{1}{2}G$ and we have found f which satisfies the required conditions. □

Proof of (b). We are given that, on the unit disk,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi}) P_z(\varphi)\, d\varphi.$$

Since $\cos\gamma$ is even, then with the other given definition of $P_r(\gamma)$,

$$P_r(-\gamma) = P_r(\gamma) = \mathrm{Re}\left(\frac{e^{i\gamma} + r}{e^{i\gamma} - r}\right).$$

Then, substituting $z = re^{i\theta}$,

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})\,\mathrm{Re}\left(\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\right)d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})\,\mathrm{Re}\left(\frac{e^{i(\varphi-\theta)}+r}{e^{i(\varphi-\theta)}-r}\right)d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})P_r(\varphi-\theta)\,d\varphi. \end{aligned}$$

Finally, taking the real part of f and considering that $P_r(\gamma)$ is even,

$$\mathrm{Re}(f(z)) = u(z) = u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})P_r(\theta-\varphi)\,d\varphi,$$

as desired. □

Problem 5. Analytic functions on the unit disk that cannot be extended analytically past the unit circle.

Definition 1 (regular)**.** Let f be defined on the unit disk $\mathbb{D}$ with boundary circle C. A point w on C is regular for f if there is an open neighborhood U of w and an analytic function g on U such that $f = g$ on $\mathbb{D}\cap U$.

Lemma 1. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f.

Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad \forall |z| < 1.$$

Note that the radius of convergence is 1. Show that f cannot be analytically continued past the unit disk.

Proof. Let $\theta = \frac{2\pi p}{2^k}$ for $p, k \in \mathbb{Z}^+$. Let $z = re^{i\theta}$ such that

$$z^{2^n} = \left(re^{i\theta}\right)^{2^n} = r\exp 2\pi ip(2^{n-k}).$$

For all $n \geq k$, the above is $\exp 2\pi ip(2^{n-k}) = 1$. So, we can write

$$\sum_0^\infty z^{2^n} = \sum_0^{n-1} r\exp 2\pi ip(2^{n-k}) + \sum_n^\infty r.$$

But,

$$\lim_{r\to 1^-}\sum_n^\infty r = \infty \implies \lim_{|z|\to 1}\left|\sum_0^\infty z^{2^n}\right| = \infty.$$

Therefore f has a singularity on the unit circle where $\theta = 2\pi\frac{p}{2^k}$.

However, we can use $\frac{p}{2^k}$ for positive integers p and k to produce the binary decimal representation of any number $\frac{\varphi}{2\pi} \in [0,1] \subset \mathbb{R}$.

Hence, $\forall\varepsilon > 0, \exists p, k$ such that

$$\left|\frac{\varphi}{2\pi} - \frac{p}{2^k}\right| < \frac{\varepsilon}{2\pi} \implies |\varphi - \theta| < \varepsilon.$$

Therefore $e^{i\theta}$ with the above assignment of θ will cover all angles in around the unit circle C.

Thus, there exist singularities at all points around the unit circle, so no points are regular on C, which means that f cannot be analytically continued past C. $\square$

Problem 6. Consider $f : \mathbb{C} \to \mathbb{C}$ of the form $f = u + iv$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions. Assume that f is a conformal map, i.e. the Jacobian is an orthogonal matrix. Prove that either f or its conjugate $\overline{f}$ satisfies the Cauchy-Riemann equations.

Proof. Since f is conformal and has an orthogonal Jacobian matrix J, then

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \implies \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \pm \begin{pmatrix} v_y \\ -u_y \end{pmatrix}.$$

So, we have that either

$$u_x = v_y \text{ and } v_x = -u_y, \quad \text{or} \quad u_x = -v_y \text{ and } v_x = u_y.$$

We see that the first set of equations are the Cauchy-Riemann equations for $f = u + iv$.

Instead, if we consider $\overline{f} = u - iv$, then the second set of equations holds using substitution by $v(x, y) \mapsto -v(x, y)$. $\square$