Math 135 Homework 3

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1. Let the n^{th} Taylor Polynomial for $f(x)$ at $x = a$ be given by,

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

(a) Show that

$$f^{(i)}(a) = T_{n,a}^{(i)}(a), \qquad i \in [0, n].$$

We begin by pulling out the first term of $T_{n,a}(x)$ and taking the i^{th} derivative of $T_{n,a}(x)$. We can differentiate a sum because of the linearity of the derivative.

$$\begin{aligned}
\frac{d^i}{dx^i}[T_{n,a}(x)] &= \frac{d^i}{dx^i}\left[f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] \\
&= \frac{d^i}{dx^i}[f(a)] + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}\left(\frac{d^i}{dx^i}(x-a)^k\right)
\end{aligned}$$

We notice that at $x = a$, the sum will reduce to zero given that $x - a = 0$. So, we are left with the i^{th} derivative of the series T,

$$T_{n,a}^{(i)}(a) = f^{(i)}(a),$$

which is what we wished to show.

(b) Show that

$$\lim_{x \to a} \frac{f(x) - T_{n,a}(x)}{(x-a)^n} = 0$$

using L'Hôpital's Rule.

First, we note that the n^{th} derivative of the function $(x-a)^n$ is $n!$,

$$\frac{d^n}{dx^n}[(x-a)^n] = n(n-1)(n-2)\cdots(n-n+1)(x-a)^{n-n} = n!.$$

We have shown in (a) that the following relationship holds for the i^{th} derivative up to $i = n$,

$$f^{(i)}(a) = T_{n,a}^{(i)}(a).$$

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In order to construct the n^{th} term of the series T, we need $f(x)$ to be at least n times differentiate. So, the n^{th} derivative of f, being differentiable itself, is also continuous. Then, since $T_{n,a}^{(i)}(x)$ is polynomial, it is continuous.

So, we can rewrite the above identity as a limit expression,

$$\lim_{x\to a} f^{(i)}(x) = \lim_{x\to a} T_{n,a}^{(i)}(x),$$

for all $i \in [0, n]$.

Next, we notice that the original limit is of indeterminate form. We have already seen that the numberator will equate to zero; the denominator $(x-a)^n$ will tend toward zero as $x \to a$.

We then proceed with L'Hôpital's Rule, and differentiate n times, such that the denominator becomes non-zero and the limit expression becomes of determine form.

Thus,

$$\lim_{x\to a} \frac{f(x) - T_{n,a}(x)}{(x-a)^n} \overset{\text{LH}}{=} \lim_{x\to a} \frac{f^{(n)}(x) - T_{n,a}^{(n)}(x)}{n!} = 0$$

(c) *Second Derivative Test Generalized* Assume that

$$f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$$

but $f^{(n)}(a) \neq 0$.

(i) If n is even and $f^{(n)}(a)$ is positive, then f has a local minimum at $x = a$.

(ii) If n is even and $f^{(n)}(a)$ is negative, then f has a local maximum at $x = a$.

(iii) If n is odd, then f has neither a minimum or a maximum at $x = a$.

Assume that $f(a) = 0$.

Notice that $f(x) = f(x) - f(a)$. This equation does not alter f or its derivatives.

From the definition,

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}.$$

So,

$$\begin{aligned} \frac{T_{n,a}(x)}{(x-a)^n} &= \frac{1}{(x-a)^n} \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^{k-n}}{k!}. \end{aligned}$$

Recall that $f^{(k)}(a) = 0$ for all $k < n$. So, our series reduces to,

$$\frac{f^{(n)}(a)}{n!}(x-a)^{n-n} = \frac{f^{(n)}(a)}{n!} = \frac{T_{n,a}(x)}{(x-a)^n}.$$

Then, we see that,

$$\frac{f(x)-T_{n,a}(x)}{(x-a)^n}=\frac{f(x)}{(x-a)^n}-\frac{f^{(n)}(a)}{n!}.$$

But, we have already seen that the left term goes to zero as x approaches a. This informs us that the difference between the right terms must also go to zero. In other words, the term must be very close together.

We showed earier that

$$\lim_{x\to a}\frac{f(x)-T_{n,a}(x)}{(x-a)^n}=0$$

So, for all $\epsilon>0$, there is a δ such that for all $|x-a|<\delta$, then $\left|\frac{f(x)-T_{n,a}(x)}{(x-a)^n}\right|<\epsilon$.

Then, we also know that

$$\left|\frac{f(x)}{(x-a)^n}-\frac{f^{(n)}(a)}{n!}\right|<\epsilon,$$

for all $|x-a|<\delta$.

Since $\frac{f(x)}{(x-a)^n}$ is ϵ close to the n^{th} derivative of f at a when x is δ close to a, we can conclude that this value shares the same sign as the n^{th} derivative of f at a.

We then proceed with (i) and (ii), where n is even.

For even n, $(x-a)^n$ will be positive for all $x\neq a$.

Also, for all positive n, $n!$ will be positive.

So, the ratio of these two terms, $\frac{(x-a)^n}{n!}$, will also be positive when $x\neq a$.

We will choose an ϵ such that $\epsilon<\left|f^{(n)}\frac{(x-a)^n}{n!}\right|$. For convenience, we will write $\epsilon_0=\epsilon(x-a)^n$.

We see that,

$$\begin{aligned}\frac{f^{(n)}(a)}{n!}-\epsilon<&\frac{f(x)}{(x-a)^n}<\frac{f^{(n)}(a)}{n!}+\epsilon\\ f^{(n)}(a)\frac{(x-a)^n}{n!}-\epsilon_0<&f(x)<f^{(n)}(a)\frac{(x-a)^n}{n!}+\epsilon_0,\end{aligned}$$

for all $|x-a|<\delta$

By our definition of epsilon, we see that in case (i), when $f^{(n)}(a)>0$ and $x\neq a$,

$$0<f^{(n)}(a)\frac{(x-a)^n}{n!}-\epsilon_0<f(x),$$

so,

$$f(x)>0.$$

Since $f(a)=0$ by our assumption, then with $f(x)>0$ for all $x\neq a$, we see that f attains a local minimum at $x=a$.

In case (ii), with $f^{(n)}(a) < 0$ and $x \neq a$,

$$f(x) < f^{(n)}(a)\frac{(x-a)^n}{n!} + \epsilon_0 < 0,$$

so

$$f(x) < 0.$$

Since $f(x) < 0$ for all $x \neq a$, and $f(a) = 0$, we see that $x = a$ is a local maximum for f.

For case (iii), when n is odd, the sign of the $(x-a)^n$ term depends on x. For $x < a$, the term is negative; for $x > a$, the term is positive.

We return to our description of the proximity of $\frac{f(x)}{(x-a)^n}$ and $\frac{f^{(n)}(a)}{n!}$. We will first consider $x > a$ such that $(x-a)^n$ is positive. The steps follow from above,

$$\begin{aligned} \frac{f^{(n)}(a)}{n!} - \epsilon <& \frac{f(x)}{(x-a)^n} < \frac{f^{(n)}(a)}{n!} + \epsilon \\ f^{(n)}(a)\frac{(x-a)^n}{n!} - \epsilon_0 <& f(x) < f^{(n)}(a)\frac{(x-a)^n}{n!} + \epsilon_0, \end{aligned}$$

and we again see that the sign of values of $f(x)$ follows the sign of the n^{th} derivative of f at a.

However, when $x < a$ and $(x-a)^n$ is negative, the identity flips,

$$\begin{aligned} \frac{f^{(n)}(a)}{n!} - \epsilon <& \frac{f(x)}{(x-a)^n} < \frac{f^{(n)}(a)}{n!} + \epsilon \\ f^{(n)}(a)\frac{(x-a)^n}{n!} - \epsilon_0 >& f(x) > f^{(n)}(a)\frac{(x-a)^n}{n!} + \epsilon_0. \end{aligned}$$

Similarly, we see that the sign of f depends on $f^{(n)}(a)$, but now it does so inversely.

For $f^{(n)}(a) > 0$ and $x \neq a$, again recalling that $(x-a)^n < 0$,

$$0 > f^{(n)}(a)\frac{(x-a)^n}{n!} > f(x)$$

implies that $f(x) < 0$.

The same can be repeated for $f^{(n)}(a)$ negative.

Since the sign of $f^{(n)}(a)$ is fixed, and the sign of f depends inversely on this term when $x < a$, but then directly on this term after $x > a$, we see that the sign of f changes about $x = a$, which makes it neither a maximum nor a minimum.

2. **12.5.45**

Show that

(a) If $\sum a_k$ converges absolutely, then $\sum a_k^2$ converges.

Since $\sum a_k$ converges absolutely, then $\sum |a_k|$ converges.

So, by (12.2.5), as $k \to \infty$, then $|a_k| \to 0$.

Thus, there exists an K, such that, for all $k \geq K$, $|a_k| < 1$.

So,

$$\begin{aligned}|a_k| &< 1\\ |a_k| \cdot |a_k| &< |a_k|\\ a_k^2 &< |a_k| \quad \forall k \geq K.\end{aligned}$$

Thus,

$$\sum_{k=K}^{\infty} a_k^2 < \sum_{k=K}^{\infty} |a_k|.$$

Then, by the series comparison theorem (12.3.6), since $\sum |a_k|$ converges, $\sum a_k^2$ also converges.

(b) Show by example that the converse of (a) is false.

Let $a_k = \frac{1}{k}$.

Then, with $a_k^2 = \frac{1}{k^2}$, $\sum a_k^2$ converges as the p-series with $p = 2 > 1$.

But, with $|a_k| = \left|\frac{1}{k}\right|$, which is the same as $\frac{1}{k}$ for positive k, we get $\sum a_k$, which is the harmonic series, which diverges. Since $\sum a_k$ diverges, it is not absolutely convergent. Therefore, the converse of (a) is false.

3. **12.5.42**

Alternating Series Test without decreasing condition. Find an example where $\lim\limits_{n\to\infty} a_n = 0$, $a_n \geq 0$ but the alternating series $\sum (-1)^k a_k$ diverges.

If we alternate between a divergent series of positive terms with a convergent series of positive terms, we will wind up with a diverging alternating series whose terms are all positive.

For simplicity, we will choose the constant zero series as our convergent series, and the harmonic series as our divergent series.

We define our new series a_k as the combination of the two aforementioned series, where the odd terms are from the zero series and the even terms come from the harmonic series.

$$a_k = 1 - 0 + \frac{1}{2} - 0 + \frac{1}{3} - 0 + \cdots.$$

We can more clearly represent this series by its odd and even components, as we did when constructing the series in the first place, being careful of division by zero with the first-indexed term,

$$a_0 = 1, \quad a_{2k} = \frac{2}{k}, \quad a_{2k+1} = 0, \quad \forall k > 0.$$

If we consider the convergence of sequence, a_k, we note that all the nonzero terms have a corresponding term in the sequence $\frac{1}{k}$, which converges to zero as $k \to \infty$. Additionally, all the these terms are positive. This satisfies the conditions of a_k.

Then, for $\sum (-1)^k a_k$, any odd k will provide a negative coefficient, yet the value of a_k will be zero. With even k, the positive values of the harmonic sequence will be added to the series with positive coefficients, which will cause the alternating series with a_k to diverge, just as the harmonic series does.[1]

If we desire a more explicit formula, we can consider the square of the sine or cosine functions, which have ranges between zero and one.

Take $f(k) = \cos^2\left(\frac{\pi}{2}k\right)$.

We see that,

$$f(k) = \begin{cases} 1, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

We can then combine this function with the "harmonic sequence"[2] $1/k$ to produce a new sequence with only odd or even terms.

We will consider the sequence,

$$b_k = \frac{\cos^2\left(\frac{\pi}{2}k\right)}{k}.$$

This sequence contains only positive terms and will tend toward zero as $k \to \infty$ by the comparison test (12.3.6) with the harmonic sequence $1/k$. As such, it is a valid candidate for this example.

Then, we note, for all odd k, where the coefficient will be negative, the value of b_k is zero. This is the harmonic series with only even denominators Thus, the sequence of partial sums of the series $\sum (-1)^k b_k$ will always grow.

In fact, the harmonic series with only even denominators, $\sum \frac{1}{2k}$, is clearly related to the harmonic series itself, in that the above is,

$$\frac{1}{2}\sum \frac{1}{k}.$$

Since the harmonic series diverges, so too does any constant multiple of the series (12.3.5).

A similar procedure can be performed for all even k, noting that we could consider instead $\sin^2\left(\frac{\pi}{2}k\right)$.

[1] But, perhaps only half as fast? Can we speak about the speed of divergence?

[2] I will call the sequence of the harmonic series' terms the harmonic sequence for the remainder of this excessively long example.