Math 462 Homework 2

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Problem 1. Let $f : \mathbb{N} \to \mathbb{R}$ such that

$$f(n) = f(n-1) + 2f(n-2) + 2^n$$

for $n \geq 2$, $f(0) = 1$, and $f(1) = 2$.

(a) Find a closed form expression for the generating function $F(x)$ of $f(n)$.

(b) Prove that $f(n) \sim cn2^n$ for some constant c.

Proof of (a). We will form a generating function using the recurrence relation,

$$\begin{aligned}
\sum_0^\infty f(n+2)x^n &= \sum_0^\infty f(n+1)x^n + 2\sum_0^\infty f(n)x^n + \sum_0^\infty 2^{n+2}x^n \\
\frac{1}{x^2}\left(F(x) - xf(1) - f(0)\right) &= \frac{1}{x}(F(x) - f(0)) + 2F(x) + \frac{4}{1-2x} \\
F(x) - 2x - 1 &= xF(x) - x + 2x^2F(x) + \frac{4x^2}{1-2x} \\
F(x)(1 - x - 2x^2) &= 1 + x + \frac{4x^2}{1-2x} \\
F(x)(1+x)(1-2x) &= \frac{(1-2x)(1+x) + 4x^2}{1-2x} \\
F(x) &= \frac{1 + x - 2x + 2x^2}{(1-2x)^2(1+x)},
\end{aligned}$$

which is our closed form for $F(x)$. $\square$

Proof of (b). Since the degree of the numerator of the closed form of F is less than that of the denominator, then the quotient remainder of F can be expressed using partial fractions with constant numerators.

So, we have

$$\frac{c_1}{1-2x} + \frac{c_2}{(1-2x)^2} + \frac{c_3}{1+x}.$$

Note that

$$\frac{1}{(1-y)^2} = \frac{d}{dx}\left(\frac{1}{1-y}\right) = \frac{d}{dx}\sum_{n\geq 0} x^n = \sum_{n\geq 0}(n+1)x^n.$$

Therefore, using the geometric power series expansion, we have

$$\begin{aligned} F(x) &= c_1 \sum_{n\geq 0}(2x)^n + c_2 \sum_{n\geq 0}(n+1)(2x)^n + c_3 \sum_{n\geq 0} c_3 \sum_{n\geq 0}(-x)^n \\ &= \sum_{n\geq 0}\left(c_1 2^n + c_2(n+1)2^n + c_3(-1)^n\right) x^n \\ &= \sum_{n\geq 0}(c_2 n + (c_1+c_2))2^n + c_3(-1)^n x^n. \end{aligned}$$

Thus,

$$f(n) = (c_2 n + (c_1+c_2))2^n + c_3(-1)^n \implies f(n) \sim cn2^n,$$

as desired. □

Problem 2. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series such that $F(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials, and $Q(0) \neq 0$. Let $r_1, \ldots r_k \in \mathbb{C}$ be the distinct roots of $Q(x)$, and let m_i be the multiplicity of the root r_i. Assume $|r_1| \leq \ldots \leq |r_k|$, and also if $|r_1| = |r_i|$ for some $i \neq 1$, then $m_1 > m_i$.

(a) Prove that $a_n = O\left(\frac{n^{m_1-1}}{|r_1|^n}\right)$, where a_n is the coefficient of x^n in $F(x)$.

(b) Let $c = \lim_{x\to r_1}\left(1 - \frac{x}{r_1}\right)^{m_1} F(x)$. Prove that if $c \neq 0$, then

$$a_n \sim \frac{c}{(m_1-1)!}\frac{n^{m_1-1}}{r_1^n}.$$

Proof of (a). We will consider the partial fraction decomposition with k roots of m_i multiplicity of $F = \frac{P}{Q}$,

$$F(x) = \frac{P(x)}{Q(x)} = h(x) + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{c_{ij}}{(r_i - x)^j},$$

where $h(x)$ is a polynomial, zero if $\deg P < \deg Q$, and each c_{ij} is constant.

By the general binomial expansion, we have

$$\frac{1}{(x-r_i)^j} = \frac{1}{r_i^j(1 - x/r_i^j)} = \frac{1}{r_i^j}\sum_{0}^{\infty}\binom{n+j-1}{n}\left(\frac{x}{r_i}\right)^n.$$

For n large,

$$\binom{n+j-1}{n} = \frac{(n+j-1)}{n!(j-1)!} = \frac{(n+j-1)\cdots(n+1)}{(j-1)!} \sim \frac{n^{j-1}}{(j-1)!}.$$

Since $h(x)$ has finite order, then for large $d = \deg h < n$, we have

$$F(x) = \sum_{d}^{\infty} a_n x^n = \sum_{i=1}^{k}\sum_{j=1}^{m_i} c_{ij}\sum_{0}^{\infty}\binom{n+j-1}{n}\frac{x^n}{r_i^{n+j}}.$$

Since we only have one infinite sum, we can rearrange with the finite sums to extract the relationship with the a_n coefficient. So, for n large,

$$a_n \sim c_{ij}\frac{n^{j-1}}{(j-1)!r_i^{n+j}}.$$

This term is largest when $i = 1$ because, for all i, $|r_1| \leq |r_i|$, which is in the denominator. Also, since we have n^{j-1} in the numerator, then we must have the largest $j = m_1$ because for all $i \neq 1$, $m_1 > m_i$. Thus,

$$a_n \sim c_{1m_1} \frac{n^{m_1-1}}{(m_1-1)! r_1^{n+m_1}} = O\left(\frac{n^{m_1-1}}{r_1^n}\right),$$

which is what we wanted to show. □

Proof of (b). We have, from part (a),

$$\left(1 - \frac{x}{r_1}\right)^{m_1} F(x) = \tilde{h}(x) + \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \frac{\left(1 - \frac{x}{r_1}\right)^{m_1}}{(r_i - x)^j},$$

where $\tilde{h}$ is a polynomial of degree $\tilde{d}$.

When $i = 1$ and $j = m_1$, we have

$$c_{1m_1} \frac{\left(1 - \frac{x}{r_1}\right)^{m_1}}{(r_1 - x)^{m_1}} = c_{1m_1} \frac{\left(1 - \frac{x}{r_1}\right)^{m_1}}{r_1^{m_1}\left(1 - \frac{x}{r_1}\right)^{m_1}} = \frac{c_{1m_1}}{r_1^{m_1}}.$$

As $x \to r_1$, all other terms in the sum will approach 0.

For $1 < i$ and $j < m_1$, we have that $|r_1| \leq |r_i|$, so the numerator $(1 - x/r_1)^{m_1}$ will go to zero while the denominator $(r_i - x)^j$ will remain bounded if we have strict inequality between $|r_1|$ and $|r_i|$. Otherwise, if the two terms are equal, then we have that $m_1 > m_i$, which will give cancellation, and therefore the term will tend to zero as well.

Therefore

$$\lim_{x \to r_1} \left(1 - \frac{x}{r_1}\right)^{m_1} F(x) = \frac{c_{1m_1}}{r_1^{m_1}} = c.$$

Substituting $c_{1m_1} = c r_1^{m_1}$ into the result from part (a),

$$a_n \sim c r_1^{m_1} \frac{n^{m_1-1}}{(m_1-1)! r_1^{n+m_1}} = \frac{c}{(m_1-1)!} \frac{n^{m_1-1}}{r_1^n},$$

as desired. □