Math 462 Homework 3

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Problem 1. How many ways are there to arrange the letters of the word ESTATE so that no two consecutive letters are the same?

Proof. There are $\frac{6!}{(2!)^2} = 180$ total arrangements of the letters in the word.

Using PIE, subtract the cases where two E's are consecutive as well as those where two T's are consecutive, then add back the cases where both two E's and two T's are consecutive.

First, with two consecutive E's or T's, we will consider the pair as a unit; we have $5!/2! = 60$ choices, where there are 5 options to arrange, and a 2! ways to order the possible non-consecutive other pair of letters.

Next, with two consecutive E's and T's we will consider both pairs each as a unit, so we have $4! = 24$ ways to arrange the remaining options.

Thus, we have $180 - (2 * 60 - 24) = 84$ ways to arrange the letters so that no two consecutive letters are the same. □

Problem 2. Let $n \geq 3$ be an integer.

1. How many functions $f : [n] \to [n]$ satisfy $f(1) = f(2)$ or $f(2) = f(3)$?

2. How many functions $f : [n] \to [n]$ satisfy at least one of $f(1) = f(2)$, $f(2) = f(3)$, $f(3) = f(4)$, $\dots$, $f(n-1) = f(n)$?

   Hint: It's easier not to use PIE.

Proof of a. First, note that in either case $f(1) = f(2)$ or $f(2) = f(3)$, we have n choices for all other mapping pairs except for the pair of outputs which are equal, so there are n^{n-1} such functions in either case.

However, we are over-counting functions where $f(1) = f(2) = f(3)$. In a similar fashion, we have n choices except for the three equal outputs, so there are n^{n-2} functions where this condition holds.

Thus, we have $2n^{n-1} - n^{n-2}$ total functions which satisfy $f(1) = f(2)$ or $f(2) = f(3)$. □

Proof of b. We will count the compliment of this set, where $\forall i$, $f(i) \neq f(i+1)$, and subtract it from the total number of functions from [n] to [n], n^n.

If we have n choices for $f(i)$, then we have $n-1$ choices for $f(i+1)$. Proceeding inductively, we will have $n-1$ choices for each other image of $i \in [n]$. So, the total number of such functions with non equal images of consecutive integers is $n(n-1)^{n-1}$.

Thus, we have $n^n - n(n-1)^{n-1}$ as the total number of functions where at least one pair of consecutive integers in the domain produce equal images in the codomain. □

Problem 3. How many 2×2 matrices are there with entries from $\{0, 1, \ldots, n\}$ and no zero rows and no zero columns? (A "zero row" is a row with all zeroes.)

Proof. We will count the total number of matrices and take away the number with zero rows as well as the number with zero columns, and finally add back the number with both zero rows and zero columns.

However, the last set has only one element, the zero matrix is the only matrix with both zero rows and zero columns.

For the total number of 2×2 matrices, we have 4 entries with $n+1$ possible entries each, for a total of $(n+1)^4$.

Since we can change a matrix with a zero column into one with a zero row using the transpose, which is invertible and therefore a bijection, the size of the sets of matrices with zero rows and zero columns is the same.

We will look at matrices with zero rows; either the top or the bottom row must be zero. Then, there are $(n+1)^2$ choices for the other 2 entries. So, we have $2(n+1)^2$ possible zero row matrices, and the same number for zero column matrices as well.

Thus, we have

$$(n+1)^4 - 2(2(n+1)^2) + 1 = ((n+1)^2 - 2)^2$$

matrices with no zero rows and no zero columns.

Note that it is interesting to see the squared term here, it seems to suggest some sort of symmetry with the matrix? □

Problem 4. Let k, n, and r be positive integers. Come up with a formula for the number of weak compositions of k into n parts with no part equal to r. Your formula can involve a sum with at most $n+1$ terms.

Proof. Let A be the total number of weak completions of k into n parts,

$$|A| = \binom{k+n-1}{n-1}.$$

Let A_i be the weak compositions with i parts equal to r. There can be at most $\left\lfloor \frac{k}{r} \right\rfloor = m$ such parts.

Then, for each A_i, having distributed $i \cdot r$ into i parts already, we are left with a weak completion of $k - ir$ into $n - i$ parts. So,

$$|A_i| = \binom{k - ir + n - i - 1}{n - i - 1}.$$

We have that

$$\bigcup_{j=1}^{m} A_j = \sum_{S \subset [m],\, S \neq \emptyset} (-1)^{|S|-1} \left| \bigcap_{i \in S} A_i \right|.$$

Suppose $|S| = i$. Then we have exactly the parts indexed by S equal to r; there are $\binom{n}{i}$ ways to choose those i parts. So,

$$\left| \bigcup_{j=1}^{m} A_j \right| = \sum_{i=1}^{\lfloor \frac{k}{r} \rfloor} (-1)^{i-1} \binom{n}{i} \binom{k - ir + n - i - 1}{n - i - 1}.$$

Thus, we have

$$\begin{aligned} |A| - \left| \bigcup_{j=1}^{m} A_j \right| &= \binom{k + n - 1}{n - 1} - \sum_{i=1}^{\lfloor \frac{k}{r} \rfloor} (-1)^{i-1} \binom{n}{i} \binom{k - ir + n - i - 1}{n - i - 1} \\ &= \sum_{i=0}^{\lfloor \frac{k}{r} \rfloor} (-1)^{i} \binom{n}{i} \binom{k - ir + n - i - 1}{n - i - 1}. \end{aligned}$$

Note that we are splitting k into n parts and i of which are equal to r. So we must have that $i \leq n$. Since i ranges from 0 to at most n, then the sum at has at most $n + 1$ terms. $\square$