Math 334 Homework 5

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We will prove

Theorem. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ bounded, continuous on $[a, b] \setminus D$, where D has (Lebesgue) measure zero, then f is (Reimann) itegrable.

Definition. The *oscillation* of f at x is

$$\operatorname{osc}(f, x) = \lim_{\delta \to 0^+} \sup\{|f(y) - f(z)| \mid y, z \in B_\delta(x)\}.$$

Let $D_s = \{x \in [a, b] \mid \operatorname{osc}(f, x) \geq s\}$.
Let $D = \{x \in [a, b] \mid f \text{ discontinuous at } x\}$.
Let $m = \inf f$ and $M = \sup f$.

Problem (1)**.** Show that if S has measure zero, then any subset of S also has measure zero.

Proof. Since S has measure zero, then $\forall \delta > 0$, S has a cover $C \subset \bigcup\limits_{i=1}^{\infty} B_{r_i}(x_i)$ with $\sum\limits_{i=1}^{\infty} r_i < \delta$. Then, any subset can use the same cover, giving such a subset measure zero as well. $\square$

Problem (2)**.**

(a) Show $D = \bigcup_{s>0} D_s$

(b) Prove D_s closed and bounded for $s > 0$, and therefore compact.

Proof of a. First, we will show that $\operatorname{osc}(f, x) = 0 \implies f$ is continuous. So, with the given definition of oscillation, with $0 < \delta \to 0$ and $\forall \epsilon > 0$, then $|y - z| < \delta \implies |f(y) - f(z)| = 0 < \epsilon$. So f is uniformly continuous and therefore continuous on D_0.

However, for $s > 0$ and some $\epsilon > 0$, we have that $\operatorname{osc}(f, x) \geq s > \epsilon$, so f is discontinuous on D_s. Since D contains all x for which f is discontinuous, it must contain D_s for all s, which is $\bigcup\limits_{s>0} D_s$. $\square$

Proof of b. Since D_s is a subset of the bounded set $[a, b]$, then D_s is bounded as well.

We will now consider the sets $T_\delta = \{x \mid \exists y, z \in B_\delta(x) \wedge |f(y) - f(z)| \geq s\}$ such that $\bigcap\limits_{\delta>0} T_\delta = D_s$. Then, for any convergent sequence $\{x_n\} \subset \bigcap\limits_{\delta>0} T_\delta$ with $x_n \to x$, we will show that x belongs in the intersection as well.

Suppose $\forall\delta > 0$ and $\{x_n\} \subset T_\delta$, then, even if $x \notin K_\delta$, we will have $x \in T_{2\delta}$. But, $T_{2\delta} \subset \bigcap\limits_{\delta>0} T_\delta$. So, x must belong to the intersection $\bigcap\limits_{\delta>0} T_\delta$ whenever $x_n \to x$ and $\{x_n\}$ is a subset of the intersection as well. So, $D_s = \bigcap\limits_{\delta>0} T_\delta$ contains all of its limit points and is therefore closed.

Since D_s closed and bounded, it is also compact. □

Problem (3)**.** Let $\epsilon > 0$ and assume D has zero content.

(a) Prove that there is a finite set of open intervals $\{I_i\}_{i=1}^L$ which satisfy $D_\epsilon \subset \bigcup\limits_{i=1}^L I_i$ and $\sum\limits_{i=1}^L \text{len}(I_i) < \epsilon$.

(b) Show that, for any partition P of I_i, then $\sum\limits_{i=1}^L (U_P^{I_i} f - L_P^{I_i} f) < (M-m)\epsilon$.

Proof of a. Since D_ϵ compact by (2b), then it admits a finite subcover $\bigcup\limits_{i=1}^L I_i$. Since D has zero content, then it has a finite cover C with $C \subset \bigcup\limits_{i=1}^\infty B_{r_i}(x_i) \wedge \sum\limits_{i=1}^\infty r_i < \epsilon$. Since these balls are in $\mathbb{R}$, then they are open intervals. So, set $\text{len}(I_i) = 2r_i$. Then, $\sum\limits_{i=1}^L \text{len}(I_i) < \frac{\epsilon}{2} < \epsilon$. □

Proof of b. By definition, we have

$$L_P^{I_i} f = \sum_j m_j(x_j - x_{j-1}) \text{ and } U_P^{I_i} f = \sum_j M_j(x_j - x_{j-1}).$$

where $m_i = \inf\limits_{[x_{j-1},x_j]} f(x)$ and $M_i = \sup\limits_{[x_{j-1},x_j]} f(x)$.

Since P partitioned I_i, then by (3a),

$$\sum_{i=1}^L \sum_j (x_j - x_{j-1}) = \sum_{i=1}^L \text{len}(I_i) < \epsilon.$$

Since P partitions I_i which covers $[a,b]$, then the upper and lower sums will be bounded by the infimum and supremum of f,

$$m \sum_{i=1}^L \text{len}(I_i) \le \sum_{i=1}^L L_P^{I_i} f \le \sum_{i=1}^L U_P^{I_i} f \le M \sum_{i=1}^L \text{len}(I_i).$$

So

$$\sum_{i=1}^L (U_P^{I_i} f - L_P^{I_i} f) = (M-m)\sum_{i=1}^L \text{len}(I_i) < (M-m)\epsilon.$$

□

Problem (4)**.** Let $\epsilon > 0$. Let $I = \bigcup_{i=1}^L I_i$. Let $K = [a,b] \setminus I$.

(a) Show K closed and bounded, and therefore compact.

(b) Show $\forall x \in K$, $\exists \delta_x > 0$, $y, z \in B_{\delta_x}(x) \implies |f(y) - f(z)| < 2\epsilon$.

(c) The intervals $J_x = (x - \delta_x, x + \delta_x)$ form an open cover of K.

  (i) Show $\exists x_i$, $\forall i \in [1, N]$ with $J_i = J_{x_i}$, then $K \subset \bigcup_{i=1}^{N} J_i$.

  (ii) Show $\forall P$ partition of $J \subset J_i \implies U_P^J f - L_P^J f < 2\epsilon \operatorname{len}(J)$

Proof of a. Since $K \subset [a, b]$ bounded, then K bounded. Since each I_i open, then the finite union I is also open. Since a closed set minus an open set is open, and $[a, b]$ closed with I open, then K is closed. Since K is closed and bounded, then it is also compact. □

Proof of b. Since I covered D_ϵ by (3a), then $K \cap D_\epsilon = \emptyset$. So $K \subset D_0$. Since $\forall x \in D_0$, f is uniformly continuous by (2a), then $\forall x \in K$, f must be uniformly continuous as well. Since f uniformly continuous on a, then the statement holds. □

Proof of c. Since K compact, then it admits a finite subcover $\bigcup_{i=1}^{N} J_i$.

First, we have that $U_P^J f - L_P^J f = \sum_j (M_j - m_j)(x_j - x_{j-1})$ with m_j, M_j the infimum and supremum of the partitioned intervals respectively. So, $\sum_j (x_j - x_{j-1}) = \operatorname{len}(J)$

Then, by (4b), $\forall \epsilon > 0$, $\exists \delta_x > 0$, $\forall x \in K$, $\forall y, z \in B_{\delta_x}(x) \implies |f(y) - f(z)| < \epsilon$. So

$$\sup_{x \in J} f(x) - \inf_{x \in J} f(x) < \epsilon.$$

So, $U_P^J f - L_P^J f < \epsilon \operatorname{len}(J)$. □

Problem (5)**.** Let $\epsilon > 0$. Note that I_i and J_i form a finite open cover of $[a, b]$. Let E be the set of all endpoints of I_i and J_i.

(a) Show $\exists P$ partition of $[a, b]$ such that $\forall [x_{j-1}, x_j]$ in P is completely contained in some I_i or J_i.

(b) Using (3) and (4), show $U_P f - L_P f \leq C\epsilon$ where $C = (b - a)(2 + (M - m))$.

(c) Conclude f Reimann integrable.

Proof of a. First, we will consider E. But, we cannot use E alone to form P because the intervals between the endpoints of E may not be contained by the open sets I_i or J_i. So, for $x \in E$, $a, b \neq x$, we can construct a closed interval around $x \in [a_x, b_x]$, such that two closed intervals $[a_0, x], [x, b_0]$ which shared the endpoint x now become three closed intervals with a_x, b_x as shared boundary points.

Since x was an endpoint of either I_i or J_i, then $[a_0, a_x]$ and $[b_x, b_0]$ must be fully contained by I_i or J_i. Then, $[a_x, b_x]$ will be contained in both, thus satisfying the containment condition.

Perform this procedure for all such $x \in E$ to arrive at P. □

Proof of b. From (3b) we have that $\sum_{i=1}^{L} (U_P^{I_i} f - L_P^{I_i} f) < (M - m)\epsilon$, and, from (4c), $U_P^J f - L_P^J f <$

$2\epsilon\,\mathrm{len}(J)$. But, J could not be longer than $b-a$, so (4c) becomes $U_P^J f - L_P^J f < 2(b-a)\epsilon$. Then, with the partition P from (5a), all subintervals belong in either the partitions for I_i or J. So, P must be bounded above by the sum of the two other partition bounds,

$$U_P f - L_P f \leq (2(b-a) + (M-m))\epsilon = C\epsilon.$$

$\square$

Proof of c. Since $\exists P$, $\forall \epsilon > 0$, $U_P f - L_P f < C\epsilon$, then f is Reimann integrable by Lemma 4.5. $\square$