Math 336 Homework 1

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Problem 1. Prove that
$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{\sqrt{2\pi}}{4}.$$
Integrate e^{-z^2} over the path given in the figure. Recall that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

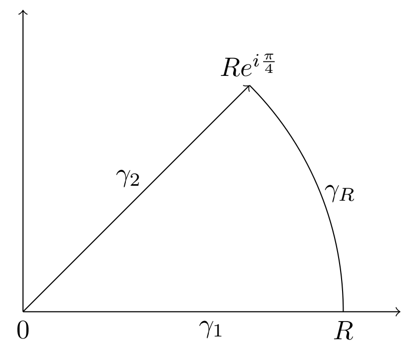

Figure 1. Section contour; consider $\int_0^\infty$ as $\lim_{R\to\infty} \int_R^\infty$.

Proof. Since $f(z) = e^{-z^2}$ is a composition of holomorphic functions on $\mathbb{C}$, then f is holomorphic on $\mathbb{C}$.

Let $\gamma = \gamma_1 \cup \gamma_R \cup \gamma_2$ be the closed loop section contour in the figure. Since f is holomorphic, then it has the closed loop property and $\oint_\gamma f\,dz = 0$.

So, we have that
$$0 = \oint_\gamma f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \int_{\gamma_R} f\,dz.$$

But, as $R \to \infty$, γ_1 parametrizes the positive real line, so
$$\int_{\gamma_1} e^{-z^2}\,dz = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$
because e^{-x^2} is an even function.

We will show that the integral of f over the arc γ_R goes to zero. Let $z = re^{i\theta}$. On this curve, r is fixed at R and θ varies from 0 to $\frac{\pi}{4}$.

So, $-z^2 = -R^2 e^{2i\theta}$ and $dz = iRe^{i\theta}$. Then,

$$\int_0^{\frac{\pi}{4}} iRe^{i\theta} e^{-R^2 e^{2i\theta}}\, d\theta = \int_0^{\frac{\pi}{4}} iRe^{i\theta - R^2 e^{2i\theta}}\, d\theta.$$

For $\theta \in \left[0, \frac{\pi}{4}\right]$, we have that $e^{2i\theta} \in \left[e^0, e^{i\frac{\pi}{2}}\right] = [1, i]$.

So, as $R \to \infty$, we have,

$$\forall \theta \in \left[0, \frac{\pi}{4}\right), \quad \left| Re^{i\theta - R^2 e^{2i\theta}} \right| \to 0 \quad \text{and} \quad \theta = \frac{\pi}{4}, \quad \left| Re^{i(\theta - R^2)} \right| \to \infty.$$

However, since the integrand only blows up at a simple point, which has measure zero, then the contribution of this point to the overall integral is zero. Hence

$$\lim_{R\to\infty} \int_{\gamma_R} f\, dz = 0.$$

So, maintaining the orientation of each part of the curve γ, we have that

$$\int_{\gamma_1} f\, dz = -\int_{\gamma_2} f\, dz.$$

Now, for the integral on the path γ_2, we fix $\theta = \frac{\pi}{4}$ and vary r from 0 to R. Note that this parameterization traverses γ_2 in the opposite direction of the positively oriented curve γ, so we will consider the opposite of the resulting integral.

So, $-z^2 = -r^2 e^{2i\theta} = -r^2 e^{i\frac{\pi}{2}} = -ir^2$ and $dz = e^{i\frac{\pi}{4}}\, dr = \frac{1}{\sqrt{2}}(1+i)dr$.

Then, with the Euler identity, we have that

$$-\int_{\gamma_2} f\, dz = \frac{1}{\sqrt{2}}(1+i) \int_0^\infty e^{-ir^2}\, dr = \frac{1}{\sqrt{2}}(1+i) \int_0^\infty (\cos r^2 - i \sin r^2)\, dr.$$

With the above equality between the integrals of γ_1 and γ_2, we have

$$\begin{aligned}\frac{\sqrt{\pi}}{2} &= \frac{1}{\sqrt{2}}(1+i) \int_0^\infty (\cos r^2 - i \sin r^2)\, dr \\ \sqrt{\frac{\pi}{2}} &= \int_0^\infty \cos x^2\, dx + \int_0^\infty \sin x^2\, dx + i\left(\int_0^\infty \cos x^2\, dx - \int_0^\infty \sin x^2\, dx \right) \\ .\end{aligned}$$

Considering the real and imaginary parts separately, we have

$$\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx$$

and therefore also

$$\frac{\sqrt{2\pi}}{4} = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx$$

as desired. □

Problem 2. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter of the image of f, $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ satisfies

$$2|f'(0)| \le d,$$

and that equality holds when f is linear.

Proof. By the Cauchy integral formula, we have

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - z)^2} \, d\omega$$

for some circle $C \subset \mathbb{D}$ centered at z.

Then, at $z = 0$, we have

$$f'(0) = \frac{1}{2\pi i} \int_{|\omega|=r} \frac{f(\omega)}{\omega^2} \, d\omega, \quad \forall r \in (0,1),$$

which we can also substitute $\omega \mapsto -\omega$ to achieve the opposite of the above.

Note that $|f(\omega) - f(-\omega)| \leq d$.

Hence, $\forall r \in (0,1)$,

$$\begin{aligned}
2f'(0) &= \frac{1}{2\pi i} \int_{|\omega|=r} \frac{f(\omega) - f(-\omega)}{\omega^2} \, d\omega \\
2|f'(0)| &= \left| \frac{1}{2\pi i} \int_{|\omega|=r} \frac{f(\omega) - f(-\omega)}{\omega^2} \, d\omega \right| \\
&\leq \frac{1}{2\pi} \int_{|\omega|=r} \frac{d}{|\omega|^2} d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{r^2} r \, d\theta \\
&= \frac{d}{2\pi} \int_0^{2\pi} \frac{1}{r} \, d\theta \\
&= \frac{d}{r}.
\end{aligned}$$

Since the above holds for all $r \in (0,1)$, then, considering the limit,

$$\lim_{r \to 1^-} \frac{d}{r} = d \implies 2|f'(0)| \leq d,$$

as desired.

If f is linear, then we can write $f = a + bz$.

Then, $|f(\omega) - f(-\omega)| = 2|b|$.

But, we also have that the diameter of the image d is given by

$$|f(z) - f(\omega)| = |(a + bz) - (a + b\omega)| = |b||z - \omega|$$

So, we have that

$$\sup_{z,\omega \in \mathbb{D}} |f(z) - f(\omega)| = |b| \sup_{z,\omega \in \mathbb{D}} |z - \omega| = 2|b|.$$

Thus, we have equality when proceeding as above. $\square$

Problem 3. Let $\Omega \subset \mathbb{C}$ be bounded and open, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$, then φ is linear.

Proof. Consider $f(z) = \varphi(z + z_0) - z_0$. If $z_0 = 0$, then $f(z) = \varphi(z)$. Otherwise, $f(0) = \varphi(z_0) - z_0 = 0$. So, WLOG, assume $z_0 = 0$.

For a contradiction, assume φ is not linear, so we can write it as

$$\varphi(z) = z + a_n z^n + O(z^{n+1})$$

for $n \geq 2$ and $a_n \neq 0$.

Let φ_k be the composition of φ with itself k times. We will show that $\varphi_k(z) = z + ka_n z^n + O(z^{n+1})$ by induction.

For the base case with $k = 1$, we have that

$$\varphi(z) = z + a_n z^n + O(z^{n+1}) = \varphi_1(z).$$

Assume the result holds for k and consider $k + 1$,

$$\begin{aligned}\varphi_{k+1}(z) &= \varphi(\varphi_k(z)) \\ &= \varphi_k(z) + ka_n(\varphi_k(z))^n + O((\varphi_k(z))^{n+1}) \\ &= z + a_n z^n + O(z^{n+1}) + ka_n(z + a_n z^n + O(z^{n+1})) + O((z + a_n z^n + O(z^{n+1}))^{n+1}).\end{aligned}$$

By the binomial expansion, we have that

$$(z + a_n z^n + O(z^{n+1}))^n = z^n + O(z^{n+1}).$$

Furthermore, we also can simplify the following,

$$O((z + a_n z^n + O(z^{n+1}))^{n+1}) = O(z^{n+1}).$$

Therefore,

$$\varphi_{k+1}(z) = z + a_n z^n + ka_n(z^n + O(z^{n+1})) + O(z^{n+1}) = z + (k+1)a_n z^n + O(z^{n+1}),$$

as desired.

Since Ω is bounded, then there exists an $R > 0$ such that, for all $z \in \Omega$, $|z| < R$. Since $\varphi_k(z)$ maps between Ω, then for all k and $z \in \Omega$,

$$\sup_{C_R} |\varphi_k(z)| \leq R.$$

Then, with Cauchy's inequality,

$$|\varphi_k^{(n)}(z)| = |kn! a_n + O(z)| \leq \frac{n!}{R^n} \sup_{C_R} |\varphi_k(z)| \leq \frac{n!}{R^{n-1}}.$$

Next, for z near 0,

$$\lim_{k\to\infty} |a_n| \leq \lim_{k\to\infty} \frac{1}{kR^{n-1}} = 0.$$

Therefore $a_n = 0$ for all $n \geq 2$. So, we are left with $\varphi(z) = z$, which is indeed linear.

Lastly, we also must have $\varphi'(z) = 1$ as any non-unit coefficient on z in φ would also result in an unbounded φ_k as $k \to \infty$.

$$\varphi(z) = az \implies \varphi_k(z) = a^k z,$$

but

$$\lim_{k\to\infty} |\varphi_k'(z)| = \lim_{k\to\infty} a^k \leq \frac{1}{R} \sup_{C_R} |\varphi_k(z)| \leq 1.$$

So we must have $a = 1$. $\square$

Problem 4. Let $u : \mathbb{D} \to \mathbb{R}$. Suppose $u \in \mathbb{C}$ and $\Delta u(x, y) = 0$ for all $(x, y) \in \mathbb{D}$.

(a) Prove that there exists a holomorphic function on the unit disk such that $\mathrm{Re}(f) = u$. Show that the imaginary part of f is defined up to an additive real constant.

(b) Deduce the Poisson integral representation formula from the Cauchy integral formula. If u is harmonic in the unit disk and continuous on its closure, then, with $z = re^{i\theta}$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)u(\varphi)\,d\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disk given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Proof of (a). Let $f' = g = 2\partial_z u$. We will show that $\partial_{\overline{z}} g = 0$, which implies that g is holomorphic.

We have the following relation between the mixed Wirtinger derivatives and the Laplacian,

$$\partial_{\overline{z}}\partial_z = \frac{1}{4}\Delta,$$

note that the order of the derivatives may be switched. So,

$$\partial_{\overline{z}} g = \partial_{\overline{z}}(2\partial_z u) = \frac{1}{2}\Delta u = 0.$$

Hence g is holomorphic on $\mathbb{D}$, and it has a primitive G on $\mathbb{D}$ such that $G' = g$.

Then, with $G' = g = 2u'$, integrating both sides gives,

$$G = 2\int \frac{d}{dz} u\,dz = 2u + (a + bi)$$

where $a + bi$ is a complex constant of integration. So,

$$\mathrm{Re}(G) = 2u + a \text{ and } \mathrm{Im}(G) = b.$$

Thus, let $f = \frac{1}{2}G$ and we have found f which satisfies the required conditions. □

Proof of (b). Note that we will show the result up to $u(e^{i\varphi})$ in place of $u(\varphi)$, as the latter representation is an equivalent notation.

We are given that, on the unit disk,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})P_z(\varphi)\,d\varphi.$$

Since $\cos\gamma$ is even, then with the other given definition of $P_r(\gamma)$,

$$P_r(-\gamma) = P_r(\gamma) = \mathrm{Re}\left(\frac{e^{i\gamma} + r}{e^{i\gamma} - r}\right).$$

Then, substituting $z = re^{i\theta}$,

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})\,\mathrm{Re}\left(\frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} - re^{i\theta}}\right)d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})\,\mathrm{Re}\left(\frac{e^{i(\varphi-\theta)} + r}{e^{i(\varphi-\theta)} - r}\right)d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})P_r(\varphi - \theta)\,d\varphi. \end{aligned}$$

Finally, taking the real part of f,

$$u(z) = u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})P_r(\theta - \varphi)\,d\varphi.$$

□

Problem 5. Analytic functions on the unit disk that cannot be extended analytically past the unit circle.

Definition 1 (regular)**.** Let f be defined on the unit disk $\mathbb{D}$ with boundary circle C. A point w on C is regular for f if there is an open neighborhood U of w and an analytic function g on U such that $f = g$ on $\mathbb{D} \cap U$.

Lemma 1. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of C is regular for f.

Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad \forall |z| < 1.$$

Note that the radius of convergence is 1. Show that f cannot be analytically continued past the unit disk.

Proof. □

Problem 6. Consider $f : \mathbb{C} \to \mathbb{C}$ of the form $f = u + iv$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions. Assume that f is a conformal map, i.e. the Jacobian is an orthogonal matrix. Prove that either f or its conjugate $\overline{f}$ satisfies the Cauchy-Riemann equations.

Proof. Since f is conformal and has an orthogonal Jacobian matrix J, then

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \implies \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \pm \begin{pmatrix} v_y \\ -u_y \end{pmatrix}.$$

So, we have that either

$$u_x = v_y \text{ and } v_x = -u_y, \quad \text{or} \quad u_x = -v_y \text{ and } v_x = u_y.$$

We see that the first set of equations are the Cauchy-Riemann equations for $f = u + iv$.

Instead, if we consider $\overline{f} = u - iv$, then the second set of equations holds using substitution by $v(x, y) \mapsto -v(x, y)$. □