Math 134 Homework 10

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9.1

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Find the general solution of $y' + ry = 0$, r constant.

(a) Show that if y is a solution and $y(a) = 0$ at some number $a \geq 0$, then $y(x) = 0$ for all x. (Thus a solution y is either identically zero or never zero.)

(b) Show that if $r < 0$, then all nonzero solutions are unbounded.

(c) Show that if $r > 0$, then all solutions tend to 0 as $x \to \infty$.

(d) What are the solutions if $r = 0$?

First, we compute the general solution to the first order linear homogeneous differential equation by separation of variables.

$$\begin{aligned} \frac{dy}{dx} + ry &= 0 \\ \int \frac{dy}{y} &= \int -r\,dx \\ \ln|y| &= c_0 - rx \\ |y| &= e^{c_0 - rx} \\ |y| &= e^{c_0}e^{-rx} \\ y &= \pm e^{c_0}e^{-rx} \\ y &= c_1 e^{-rx}. \end{aligned}$$

The general solution is

$$y = ce^{-rx}.$$

For (a), we will show that y is either the trivial solution $y(x) = 0$, or y is never zero.

If, for some $a \geq 0$, $y(a) = 0$, then, $0 = ce^{-rx}$.

But, $e^b > 0$ for all $-rx = b \in \mathbb{R}$.

Then $ce^{-rx} = 0$ implies that $c = 0$.

So $y(x) = 0 \cdot e^{-rx} = 0$ for all x.

If $c \neq 0$ and $e^{-rx} \neq 0$ for all x, then their product $f(x) \neq 0$ either.

For (b), if $r < 0$, then $-r > 0$.

If $c = 0$, then, by (a), the solution is identically zero, so it does grow without bound. So, we continue with $c \neq 0$.

So, when x exceeds any number, with $0 < b = -r \lim_{x\to\infty} ce^{bx}$ grows without bound.

Thus, when $r < 0$, the nontrivial solutions y are unbounded.

For (c), we note that the trivial solution is zero everywhere, so it tends toward zero as x exceeds any number.

As x exceeds any number, with $r > 0$, $\lim_{x\to\infty} ce^{rx}$ will become unbounded by (b).

We note that $y = ce^{-rx} = \frac{1}{ce^{rx}}$.

So, $\lim_{x\to\infty} \frac{1}{ce^{rx}}$ will have is denominator grow without bound and thus will tend toward zero as x exceeds any number.

Therefore for all solutions with $r > 0$, y tends toward zero.

For (d), when $r = 0$, we see that $y = ce^{-rx} = c$.

So, the solution is uniquely $y = c$ when $r = 0$.

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Consider the differential equation

$$y' + p(x)y = 0$$

with p continuous on an interval I.

(a) Show that if y_1 and y_2 are solutions, then $u = y_1 + y_2$ is also a solution.

(b) Show that if y is a solution and C is a constant, then $u = Cy$ is also a solution.

For (a), since y_1 and y_2 are both solutions, we are given that

$$\begin{aligned} {y_1}' + p(x)y_1 &= 0, \\ {y_2}' + p(x)y_2 &= 0. \end{aligned}$$

We take their sum,

$$({y_1}' + {y_2}') + p(x)(y_1 + y_2) = 0.$$

Given that the sum of a derivative is the same is the derivative of a sum, we get

$$(y_1 + y_2)' + p(x)(y_1 + y_2) = 0.$$

When we let $u = y_1 + y_2$, we see that u is also a solution to the differential equation,

$$u' + p(x)u = 0.$$

For (b), we begin with the solution y where

$$y' + p(x)y = 0.$$

We multiply by the arbitrary constant C,

$$\begin{aligned} C(y' + p(x)y) &= C \cdot 0 \\ Cy' + p(x)Cy &= 0. \end{aligned}$$

We note that the derivative of a constant times a function is the constant times the derivative of the function. So, we get

$$(Cy)' + p(x)Cy = 0.$$

We define the function $u = Cy$ where y is the given solution and see that,

$$u' + p(x)u = 0.$$

So, u is also a solution to the differential equation.

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Show that if y_1 and y_2 are solutions to

$$y' + p(x)y = q(x),$$

then $y = y_1 - y_2$ is a solution to $y' + p(x)y = 0$.

Since y_1 and y_2 are both solutions, we see that

$$\begin{aligned} {y_1}' + p(x)y_1 &= q(x), \\ {y_2}' + p(x)y_2 &= q(x). \end{aligned}$$

We take their difference,

$${y_1}' - {y_2}' + p(x)(y_1 - y_2) = 0.$$

We note, by linearity of the derivative, the above becomes

$$(y_1 - y_2)' + p(x)(y_1 - y_2) = 0.$$

We define $y = y_1 - y_2$ and see that,

$$y' + p(x)y = 0.$$

So, y is a solution to the equation $y' + p(x)y = 0$.

9.2

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When an object of mass m moves through air or a viscous medium, it is acted on by a frictional force that acts in the direction opposite to its motion. This frictional force depends on the velocity of the object and (within close approximation) is given by

$$F(v) = -\alpha v - \beta v^2,$$

where α and β are positive constants.

(a) From Newton's second law, $F = ma$, we have

$$m\frac{dv}{dv} = -\alpha v - \beta v^2.$$

Solve this differential equation to find $v(t)$.

(b) Find v if the object has initial velocity $v(0) = v_0$.

(c) What happens to $v(t)$ as $t \to \infty$?

For (a), we begin by solving the differential equation.

First, we will compute the partial fraction decomposition of the structure $\frac{1}{x(1+ax)}$,

$$\begin{aligned}
\frac{1}{x(1+ax)} &= \frac{A}{x} + \frac{B}{1+ax} \\
1 &= A(1+ax) + B(x) \\
x = 0 &\implies A = 1 \\
x = -\frac{1}{a} &\implies B = -a \\
\frac{1}{x(1+ax)} &= \frac{1}{x} - \frac{a}{1+ax}.
\end{aligned}$$

We will also isolate x in $A = \frac{x}{1+ax}$,

$$\begin{aligned}
\frac{x}{1+ax} &= A \\
x &= A(1+ax) \\
x &= A + axA \\
x - axA &= A \\
x(1-aA) &= A \\
x &= \frac{A}{1-aA}.
\end{aligned}$$

These two algebraic reductions will aid us in solving the differential equation.

We proceed using the partial fraction decomposition for with the substitution $a = \frac{\beta}{\alpha}$ and the simplification where $A = ce^{-\frac{\alpha}{m}t}$.

$$\begin{aligned}
m\frac{dv}{dt} &= -\alpha - \beta v^2 \\
m\frac{dv}{dt} &= -\alpha \cdot v \left(1 + \frac{\beta}{\alpha}v\right) \\
\int \frac{dv}{v\left(1+\frac{\beta}{\alpha}v\right)} &= \int \frac{-\alpha}{m}\, dt \\
\int \frac{1}{v} + \frac{-\frac{\beta}{\alpha}}{1+\frac{\beta}{\alpha}v}\, dv &= -\frac{\alpha}{m}\int dt \\
\ln|v| - \ln\left|1+\frac{\beta}{\alpha}v\right| &= c_0 - \frac{\alpha}{m}t \\
\ln\left|\frac{v}{1+\frac{\beta}{\alpha}v}\right| &= c_0 - \frac{\alpha}{m}t \\
\left|\frac{v}{1+\frac{\beta}{\alpha}v}\right| &= e^{c_0-\frac{\alpha}{m}t} \\
\frac{v}{1+\frac{\beta}{\alpha}v} &= ce^{-\frac{\alpha}{m}t} \\
v &= \frac{ce^{-\frac{\alpha}{m}t}}{1-\frac{\beta}{\alpha}ce^{-\frac{\alpha}{m}t}} \\
&= \frac{1}{ce^{\frac{\alpha}{m}t} - \frac{\beta}{\alpha}} \\
v(t) &= \frac{\alpha}{ce^{\frac{\alpha}{m}t} - \beta}.
\end{aligned}$$

So, the solution to the differential equation is

$$v(t) = \frac{\alpha}{ce^{\frac{\alpha}{m}t} - \beta}.$$

For (b), we consider the initial condition $v(0) = v_0$.

$$\begin{aligned}
v(0) &= \frac{\alpha}{ce^{\frac{\alpha}{m}(0)} - \beta} \\
v_0 &= \frac{\alpha}{c-\beta} \\
c - \beta &= \frac{\alpha}{v_0} \\
c &= \frac{\alpha}{v_0} + \beta.
\end{aligned}$$

Thus the solution becomes,

$$\begin{aligned}
v(t) &= \frac{\alpha}{\left(\frac{a}{v_0}+\beta\right)e^{\frac{\alpha}{m}t} - \beta} \\
&= \frac{\alpha v_0}{(\alpha+\beta v_0)e^{\frac{\alpha}{m}t} - \beta v_0}.
\end{aligned}$$

For (c), we consider the terminal behavior of $v(t)$ by taking the limit as t exceeds any number. We will show that,

$$\lim_{t\to\infty} v(t) = 0.$$

We will consider the limit,

$$\lim_{t\to\infty} \frac{\alpha v_0}{(\alpha + \beta v_0)e^{\frac{\alpha}{m}t} - \beta v_0}.$$

We notice that the mass of an object m, and the given constant α are both positive. So, $\frac{\alpha}{m}$, the coefficient of the exponent term in the denominator is positive. Thus, as t grows without bound, $e^{\frac{\alpha}{m}t}$ will grow as well. As the numerator is constant, an increasing denominator suggest that v will decrease to zero as t grows large.

Thus the terminal behavior of $v(t)$ is that it tends to zero; the object will come to a stop in the viscous medium.

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A descending parachutist is acted on by two forces: a constant downward force mg and the upward force of air resistance, which (within close approximation) is of the form $-\beta v^2$ where β is a positive constant. We will take the downward direction as positive.

(a) Express t in terms of the velocity v, the initial velocity v_0, and the *terminal velocity* $v_c = \sqrt{\frac{mg}{\beta}}$.

(b) Express v as a function of t.

(c) Express the acceleration a as a function of t. Verify that the acceleration never changes sign and in time tends to zero.

(d) Show that, in time, v tends to v_c.

First we set up the equation to model the force on the parachutist,

$$F(v) = mg - \beta v^2.$$

For (a), we note that the force $F(v)$ is the mass m times the acceleration, which is the change in velocity $\frac{dv}{dt}$.

So, we begin by setting up the differential equation and isolating $\int dt = t$.

$$\begin{aligned} m\frac{dv}{dt} &= mg - \beta v^2 \\ \frac{dv}{dt} &= g - \frac{\beta}{m}v^2 \\ \int \frac{dv}{g - \frac{\beta}{m}v^2} &= \int dt \\ \frac{1}{g}\int \frac{dv}{1 - \frac{\beta}{mg}v^2} &= t. \end{aligned}$$

We notice that $\frac{\beta}{mg} = \frac{1}{{v_c}^2}$. So, the equation becomes,

$$\begin{aligned} t &= \frac{1}{g} \int \frac{dv}{1 - \frac{v^2}{{v_c}^2}} \\ &= \frac{v_c^2}{g} \int \frac{dv}{{v_c}^2 - v^2}. \end{aligned}$$

Thus, we have expressed t in terms of v and v_c. We note that the initial velocity v_0 will arise as the constant resulting from the evaluation of the integral $\int dv$.

For (b), we first reduce the partial fraction of a difference of squares to later simplify the computations,

$$\begin{aligned} \frac{1}{a^2 - x^2} &= \frac{1}{(a+x)(a-x)} \\ &= \frac{A}{a+x} + \frac{B}{a-x} \\ 1 &= A(a-x) + B(a+x) \\ x = -a \implies A &= \frac{1}{2a} \\ x = a \implies B &= \frac{1}{2a} \\ \frac{1}{a^2 - x^2} &= \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right) \end{aligned}$$

We will also perform an algebraic simplification of $\frac{a+x}{a-x} = A$.

$$\begin{aligned} \frac{a+x}{a-x} &= A \\ a + x &= A(a-x) \\ a + x &= aA - Ax \\ Ax + x &= aA - a \\ x(A+1) &= a(A-1) \\ x = a\frac{A+1}{A-1}. \end{aligned}$$

We will use the equation given in (a) and employ the partial fraction decomposition and simplification above where $a = v_c$, $x = v$, and $A = ce^{\frac{2g}{v_c}t}$.

$$\begin{aligned}
\frac{v_c^2}{g}\int \frac{dv}{{v_c}^2 - v^2} &= t \\
\int \frac{dv}{{v_c}^2 - v^2} &= \frac{g}{{v_c}^2}t \\
\frac{1}{2v_c}\int \left(\frac{1}{v_c + v} + \frac{1}{v_c - v}\right) dv &= \frac{g}{{v_c}^2}t \\
\ln|v_c + v| - \ln|v_c - v| + c_0 &= \frac{2g}{v_c}t \\
\ln\left|\frac{v_c + v}{v_c - v}\right| &= c_1 + \frac{2g}{v_c}t \\
\left|\frac{v_c + v}{v_c - v}\right| &= e^{c_1}e^{\frac{2g}{v_c}t} \\
\frac{v_c + v}{v_c - v} &= c_2e^{\frac{2g}{v_c}t} \\
v(t) &= v_c \cdot \frac{c_2e^{\frac{2g}{v_c}t} - 1}{c_2e^{\frac{2g}{v_c}t} + 1}.
\end{aligned}$$

We can further simplify this solution with the hyperbolic tangent identity,[1]

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

We rewrite $v(t)$ as

$$v(t) = v_c \cdot \frac{e^{\frac{2g}{v_c}t + c} - 1}{e^{\frac{2g}{v_c}t + c} + 1}.$$

We then make the substitution $2x = \frac{2g}{v_c}t + c$, such that

$$v(t) = v_c \tanh \frac{\frac{2g}{v_c}t + c}{2} = v_c \tanh\left(\frac{g}{v_c}t + \frac{c}{2}\right).$$

For (c), we note that acceleration $a = \frac{dv}{dt}$. We replace $v(t)$ in the original differential equation, recalling

[1] From `https://en.wikipedia.org/wiki/Hyperbolic_functions`.

that $v_c = \sqrt{\frac{mg}{\beta}}$.

$$\begin{aligned}
a(t) &= \frac{dv}{dt} \\
&= g - \frac{\beta}{m} v^2 \\
&= g - \frac{\beta}{m} \left(v_c \tanh\left(\frac{g}{v_c} t + \frac{c}{2} \right) \right)^2 \\
&= g - \frac{\beta}{m} \left(\frac{mg}{\beta} \right) \tanh^2 \left(\frac{g}{v_c} t + \frac{c}{2} \right) \\
&= g \left(1 - \tanh^2 \left(\frac{g}{v_c} t + \frac{c}{2} \right) \right) \\
&= g \operatorname{sech}^2 \left(\frac{g}{v_c} t + \frac{c}{2} \right).
\end{aligned}$$

We can verify this result by differentiating $v(t)$ as well.

$$\begin{aligned}
a(t) &= \frac{dv}{dt} \\
&= \frac{d}{dt} \left[v_c \tanh \left(\frac{g}{v_c} t + \frac{c}{2} \right) \right] \\
&= v_c \operatorname{sech}^2 \left(\frac{g}{v_c} t + \frac{c}{2} \right) \left(\frac{g}{v_c} \right) \\
&= g \operatorname{sech}^2 \left(\frac{g}{v_c} t + \frac{c}{2} \right).
\end{aligned}$$

We note that the range of the hyperbolic secant is the interval is bounded below by a horizontal asymptote at zero. Thus, the acceleration $a(t)$, being the product of the gravitational constant g and the square of the hyperbolic secant function, is always positive.

Lastly, given that

$$\lim_{x \to \infty} \operatorname{sech} x = 0,$$

we see that as t exceeds any given number, then

$$\lim_{t \to \infty} a(t) = 0.$$

So, the acceleration tends toward zero in time.

For (d), we observe the limit as the time t exceeds any number of the velocity function $v(t)$,

$$\lim_{t \to \infty} v_c \tanh \left(\frac{g}{v_c} t + \frac{c}{2} \right).$$

We know that the end behavior of the hyperbolic tangent is that it approaches its vertical bound of one. Thus,

$$\lim_{t \to \infty} v_c \tanh \left(\frac{g}{v_c} t + \frac{c}{2} \right) = v_c.$$

So, the velocity function approaches the terminal velocity in time.

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A rescue package of mass 100 kilograms is dropped from a plane flying at a height of 4000 meters. As the object falls, the air resistance is equal to twice its velocity. After 10 seconds, the package's parachute opens and the air resistance is now four times the square of its velocity.

(a) What is the velocity of the package the instant the parachute opens?

(b) What is the velocity of the package t seconds after the parachute opens?

(c) What is the terminal velocity of the package?

We note that it is possible to set up a single differential equation and thereby single solution function for the behavior of the package's fall using the Heaviside step function.

For (a), we begin with the differential equation of the package in free fall to determine the package's velocity at the ten second mark.

The air resistance in free fall is given by the term $-2v$. The term is negated as it opposes the positive downwards direction of motion.

Thus, we model the force being applied to the package as $F = mg - 2v$, and arrive at the differential equation for velocity as a function of time as,

$$m\frac{dv}{dt} = mg - 2v.$$

We find the general solution to this equation by rearranging the terms to the form $v' + \frac{2}{m}v = g$ and then using the method of an integrating factor as in 9.1.2.

$$\begin{aligned}
v &= e^{-\int \frac{2}{m}\,dt}\left[\int e^{\int \frac{2}{m}\,dt} g\,dt\right] \\
v &= e^{-\frac{2}{m}t}\left[\int ge^{\frac{2}{m}t}\right] \\
v &= e^{-\frac{2}{m}t}\left(\frac{mg}{2}e^{\frac{2}{m}t} + c\right) \\
v &= \frac{mg}{2} + ce^{-\frac{2}{m}t}.
\end{aligned}$$

We know that the package started with no vertical velocity. Thus, we consider,

$$\begin{aligned}
v(0) &= 0 \\
&= \frac{mg}{2} + ce^{-\frac{2}{m}(0)} \\
c &= -\frac{mg}{2}.
\end{aligned}$$

So, the equation for the velocity of the package becomes,

$$v(t) = \frac{mg}{2}\left(1 - e^{-\frac{2}{m}t}\right).$$

We then solve for the velocity at the time $t = 10$ seconds given that the mass $m = 100$kg and the gravitational constant g in standard units is 9.8m/s^2.

$$
\begin{aligned}
v(10) &= \frac{100 \cdot 9.8}{2}\left(1 - e^{-\frac{2}{100}(10)}\right) \\
&= 490\left(1 - e^{-\frac{1}{5}}\right) \\
&\approx 88.8.
\end{aligned}
$$

So, the velocity of the package after ten seconds when the parachute opens is about 88.8 meters per seconds.

For (b), we consider the differential equation where the upwards negative air resistance force is $-4v^2$.

We will solve the equation $m\frac{dv}{dt} = mg - 4v^2$ using separation of variables, noting that the initial velocity is given by the solution to (a) when the parachute first opens.

So,

$$
\frac{dv}{dt} = g - \frac{4}{g}v^2 \tag{1}
$$

$$
\frac{dv}{dv} = \frac{4}{m}\left(\frac{mg}{4} - v^2\right). \tag{2}
$$

$$
\tag{3}
$$

We will let $k^2 = \frac{mg}{4}$, so $\frac{dv}{dt} = \frac{4}{m}\left(k^2 - v^2\right)$.

We proceed in a similar manner to the previous problem, following the given partial fraction decomposition and algebraic simplifications,

$$
\begin{aligned}
\int \frac{dv}{k^2 - v^2} &= \int \frac{4}{m}\, dt \\
\frac{1}{2k}\int\left(\frac{1}{k+v} + \frac{1}{k-v}\right) dv &= \frac{4}{m}t + c_0 \\
\ln\left|\frac{k+v}{k-v}\right| &= \frac{8k}{m}t + c_1 \\
\left|\frac{k+v}{k-v}\right| &= e^{\frac{8k}{m}t + c_1} \\
\frac{k+v}{k-v} &= c_2 e^{\frac{8k}{m}t} \\
v &= k\left(\frac{c_2 e^{\frac{8k}{m}t} - 1}{c_2 e^{\frac{8k}{m}t} + 1}\right) \\
v &= \frac{\sqrt{mg}}{2}\left(\frac{c_2 e^{4\sqrt{\frac{m}{g}}t} - 1}{c_2 e^{4\sqrt{\frac{m}{g}}t} + 1}\right)
\end{aligned}
$$

We will then solve for c_2 with the initial velocity v_0 of the second equation, $v(0) = v_0 \approx 88.8$ meters per second, recalling the values of the constants m and g.

$$\begin{aligned}
v(0) &= v_0 \\
&= \frac{\sqrt{mg}}{2}\left(\frac{c_2 e^{4\sqrt{\frac{m}{g}}(0)}-1}{c_2 e^{4\sqrt{\frac{m}{g}}(0)}+1}\right) \\
v_0 &= \frac{\sqrt{mg}}{2}\left(\frac{c_2-1}{c_2+1}\right) \\
2v_0(c_2+1) &= \sqrt{mg}(c_2-1) \\
2c_2v_0+2v_0 &= c_2\sqrt{mg}-\sqrt{mg} \\
2c_2v_0-c_2\sqrt{mg} &= -2v_0-\sqrt{mg} \\
c_2(2v_0-\sqrt{mg}) &= -(2v_0+\sqrt{mg}) \\
c_2 &= \frac{2v_0+\sqrt{mg}}{2v_0-\sqrt{mg}} \\
&\approx \frac{2\cdot 88.8+\sqrt{100\cdot 9.8}}{2\cdot 88.8+\sqrt{100\cdot 9.8}} \\
&\approx \frac{177.9+99.0}{177.9-99} \\
&\approx \frac{276.9}{78.9} \\
&\approx 3.51.
\end{aligned}$$

So,

$$\begin{aligned}
v(t) &= \frac{\sqrt{mg}}{2}\left(\frac{3.51e^{4\sqrt{\frac{m}{g}}t}-1}{3.51e^{4\sqrt{\frac{m}{g}}t}+1}\right) \\
&= \frac{\sqrt{100\cdot 9.8}}{2}\left(\frac{3.51e^{4\sqrt{\frac{100}{9.8}}t}-1}{3.51e^{4\sqrt{\frac{100}{9.8}}t}+1}\right) \\
&= 49.5\left(\frac{3.51e^{12.78t}-1}{3.51e^{12.78t}+1}\right).
\end{aligned}$$

For (c), we note that the terminal velocity can be found as t gets very large.

We can express $v(t)$ in terms of the hyperbolic tangent as before, noting that it approaches one as t gets very large.

$$v(t) = 49.5\tanh\left(\frac{12.78t+\ln 3.51}{2}\right).$$

So,

$$\lim_{t\to\infty} v(t) = 49.5.$$

Thus, the terminal velocity of the package is 49.5 meters per second.

Coffee

Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings. Experimenting, I have found that 12 ounces of 180F coffee in my favorite cup will take 20 minutes to cool to a drinking temperature of 110F in a 70F room. Assume that when I add cream to the coffee, the two liquids are mixed instantly, and the temperature of the mixture instantly becomes the weighted average of the temperature of the coffee and of the cream (weighted by the number of ounces of each fluid). Also, assume that the cooling constant of the liquid (the proportionality constant in the differential equation) does not change when I add the cream.

I take my coffee with cream. I am going to add 2 ounces of cream at 40F to my coffee. In order to reach drinking temperature as quickly as possible, should I

(a) Add the cream immediately to my 12 ounces of 180F coffee and wait for it to cool down to 110F?

(b) Wait 5 minutes before adding the cream?

To begin, we will model the cooling of plain coffee.

We will model the temperature T over time t with the differential equation

$$\begin{aligned}\frac{dT}{dt} &\propto T - T_s \\ \frac{dT}{dt} &= k(T - T_s),\end{aligned}$$

where T_s is the ambient temperature of the room.

We solve using separation of variables,

$$\begin{aligned}\int \frac{dT}{T - T_s} &= \int k\,dt \\ \ln|T - T_s| &= kt + c_0 \\ T - T_s &= ce^{kt} \\ T = ce^{kt} + T_s.\end{aligned}$$

With the given ambient temperature of 70F and an initial temperature for the coffee of 180F, we solve for the constant c,

$$\begin{aligned}T(0) &= 180 \\ &= ce^{k(0)} + 70 \\ 110 &= c.\end{aligned}$$

We also know that, when this coffee is set out for twenty minutes, it cools to a temperature of 110 degrees. So, given $T(20) = 110$, we solve for the cooling rate k,

$$\begin{aligned}110 &= 110e^{20k} + 70 \\ \frac{40}{110} &= e^{20k} \\ \ln\frac{4}{11} &= 20k \\ \frac{\ln\frac{4}{11}}{20} &= k.\end{aligned}$$

So, the equation for the temperature of the plain coffee at time t is then

$$T(t) = 70 + 110\left(\frac{4}{11}\right)^{\frac{t}{20}}.$$

We then compute the temperature of the plain coffee after it is let to sit for five minutes.

$$\begin{aligned} T(5) &= 70 + 110\left(\frac{4}{11}\right)^{\frac{5}{20}} \\ &= 70 + 110\sqrt[4]{\left(\frac{4}{11}\right)} \\ &= 155.42. \end{aligned}$$

We define this value as $T_5 = 155.42$.

We then compute the temperature of the mixture of 12 ounces of plain coffee at this temperature T_5 with 2 ounces of 40F cream, given by the volume weighted average of their temperatures.

$$\begin{aligned} T_0 &= \frac{12 \cdot T_5 + 2 \cdot 40}{12 + 2} \\ &= \frac{1865.04 + 80}{14} \\ &\approx 138.93. \end{aligned}$$

So, the temperature for coffee that has sat for five minutes and then has had cream added is 138.93F.

We model the temperature from the solution to the differential equation from Newton's law of cooling using the same cooling rate as before, $k = \frac{\ln \frac{4}{11}}{20}$.[2]

Then, we solve for the new arbitrary constant at $t = 0$,

$$\begin{aligned} 138.93 &= 70 + c \\ c &= 68.93. \end{aligned}$$

We proceed to determine the time t at which the coffee and cream mixture reaches the 110F drinking

[2] We imagine that this is realistic, and that the surface area of the beverage that is exposed to air remains roughly the same. This would be the case in a cylindrical mug.

temperature.

$$\begin{aligned}
T(t) &= 110 \\
110 &= 70 + 68.93\left(\frac{4}{11}\right)^{\frac{t}{20}} \\
\frac{40}{68.93} &= \left(\frac{4}{11}\right)^{\frac{t}{20}} \\
\ln\frac{40}{68.93} &= \frac{t}{20}\ln\left(\frac{4}{11}\right) \\
20\frac{\ln\frac{40}{68.93}}{\ln\frac{4}{11}} &= t \\
10.76 &\approx t.
\end{aligned}$$

So, it takes about 10.76 minutes for the mixed beverage to cool to drinking temperature.

Thus, a total of 15.76 minutes elapsed prior to being able to drink the coffee for choice (b).

For choice (a), we consider the initial temperature of the mixture of 12 ounces of coffee with 2 ounces of cream when the coffee is at 180F.

So,

$$\begin{aligned}
T_0 &= \frac{12\cdot 180 + 2\cdot 40}{12+2} \\
&= \frac{2160+80}{14} \\
&= \frac{2240}{14} \\
&= 160.
\end{aligned}$$

With $T(0) = 160$, we can compute the constant for the general solution $T(t) = 70 + c\left(\frac{4}{11}\right)^{\frac{t}{20}}$,

$$\begin{aligned}
T(0) &= 160 \\
&= 70 + c \\
90 &= c.
\end{aligned}$$

Finally, we find the time t at which the equation for the instantly-mixed coffee and cream comes to drinking temperature.

$$\begin{aligned}
T(t) &= 110 \\
110 &= 70 + 90\left(\frac{4}{11}\right)^{\frac{t}{20}} \\
\ln\frac{4}{9} &= \frac{t}{20}\ln\frac{4}{11} \\
20\frac{\ln\frac{4}{9}}{\ln\frac{4}{11}} &= t \\
16 &\approx t.
\end{aligned}$$

So, when the cream is added as soon as the coffee is poured, it takes 16 minutes for the beverage to arrive at the drinking temperature.

Thus, choice (b) is the superior choice; Wait some time to add the cream as the coffee will cool quicker.

As it turns out, waiting as long as possible before adding cream will mean the total time to wait before the beverage arrives at the drinking temperature will be reduced.

Also, I don't drink coffee.