Math 462 Homework 6

Alexandre Lipson

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Problem 1. Let G be a graph with bipartition $\{X, Y\}$. Suppose that every vertex in X has degree a, and every vertex in Y has degree b, and $a \geq b \geq 1$. Prove there is a matching of G such that every element of X is covered.

Proof. We wish to show that $\forall S \subset X$, $|S| \leq |N(S)|$.

Consider any $S \subset X$. The number of edges from S is $a|S|$.

These edges go into the vertices $N(S) \subset Y$, which have at most $b|N(S)|$ edges. So, $a|S| \leq b|N(S)|$.

Since $1 \leq b \leq a$, then we have $b/a \leq 1$ and $|S|/|N(S)| \leq b/a \leq 1$, which means that $|S| \leq |N(S)|$.

Since this holds for any $S \subset X$, then, by Hall's Theorem, we must have a matching covering X. $\square$

Problem 2. Let $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$. Using Problem 1, prove that if $k < n/2$, then there is an injective map $f : \binom{[n]}{k} \to \binom{[n]}{k+1}$ such that $S \subset f(S)$ for all $S \in \binom{[n]}{k}$.

Proof. We will construct a graph with the bipartition $\left\{X = \binom{[n]}{k}, Y = \binom{[n]}{k+1}\right\}$.

For every k element subset of $[n]$, there are $n - k$ ways to choose the $k + 1^{\text{th}}$ element.
So, the degree of each vertex in X is $n - k$.

For every $k + 1$ element subset of $[n]$, there are $k + 1$ ways to remove an element down to k elements.
So, the degree of each vertex in Y is $k + 1$.

Since $\frac{n}{2} > k$, then we have $\frac{n}{2} + 1 > k + 1$ and $n - k > n - \frac{n}{2} = \frac{n}{2}$.

So, $n - k + 1 > \frac{n}{2} + 1 > k + 1$. Since n and k are integers, then $n - k \geq k + 1$.

By Problem 1, $n - k \geq k + 1 \geq 1$ implies that we must have a matching covering X, which means that every vertex in Y is matched to at most one vertex in X.

Thus, we can define an injective mapping f given by the edges from $X = \binom{[n]}{k}$ to $Y = \binom{[n]}{k+1}$.

$\square$

Problem 3. Players 1 and 2 are playing a game on a graph G. Player 1 starts by choosing a vertex v_1. Player 2 then chooses a vertex v_2 different from v_1 and adjacent to v_1. Player 1 then chooses a vertex v_3 different from v_1 and v_2, and adjacent to v_2. Player 2 then chooses a vertex v_4 different from v_1, v_2, and v_3, and adjacent to v_3. And so on. If a player has no vertices they can choose on their turn, they lose and the other player wins.

1. Prove that if G has a perfect matching, then Player 2 has a strategy to always win.

2. Prove that if G does not have a perfect matching, then Player 1 has a strategy to always win.

   Hint: Consider a matching with a maximum number of edges, and have Player 1 start at an unmatched vertex.

Proof of a. Let M be the perfect matching of G. For whichever vertex that P1 chooses, P2 can choose the matched vertex from M.

If there is no vertex adjacent to P2's choice, then P2 wins.

Otherwise, P1 will pick an adjacent vertex, call it v, which cannot be matched to any previously chosen vertices as P2 only chooses vertices that complete the perfect matching according to M.

Since G has a perfect matching, then v will also have a matched vertex v', which P2 can choose.

Thus, there are either zero or at least two vertices remaining.

Repeat the above steps until there are zero vertices remaining, at which point P2 will have chosen the last vertex and thereby win as P1 will have no unmatched choices remaining. □

Proof of b. Consider the maximum matching M of G. Since G does not have a perfect matching, then there must be at least one unmatched vertex in G.

P1 should start by choosing such an unmatched vertex u.

If there are no edges incident to u, then P2 cannot make any moves, so P1 wins.

Otherwise, there must be another vertex v adjacent to u which must be matched to a vertex w in M.

If v was unmatched, then we could have constructed larger matching than M by adding an edge between the u, which was unmatched as well, and its neighbor v.

After P2 chooses their only option v, then P1 must choose its partner w in M.

P2 must then choose an adjacent unused vertex, which cannot be matched to any previously chosen vertex since u was unmatched and v and w were matched.

P1 can continue to follow the strategy given in part (a): whenever P2 chooses a vertex x, P1 responds by choosing the vertex y matched with x in M. This creates an M-alternating path P in G which starts at the unmatched vertex u.

Since M is a maximum matching, then there cannot be an M-augmenting path in G. Therefore, P cannot end at another unmatched vertex.

Eventually, P2 will be forced into a position where they have no valid move and lose, which means P1 wins. □

Problem 4. Determine all pairs (m, n) of positive integers such that $K_{m,n}$ has an Eulerian walk.

Proof. If m and n are even, then all vertices of $K_{m,n}$ must have an even degree, which means that we have an Eulerian circuit and thus and Eulerian walk as well.

If we have $K_{r,2}$ or $K_{2,r}$ such that r is odd, then we will have exactly two vertices of odd degree, which means that we also have an Eulerian walk.

For $K_{1,1}$, we also have an Eulerian walk.

If $m > 1$ and $n > 1$ are both odd, then we must have $m + n > 2$ odd vertices, which prevents us from having an Eulerian walk.

Thus, the valid pairings, which we can write irrespective of order, are: (m, n) for m and n even, $(2, r)$ for r odd, and $(1, 1)$.

All other pairings will have more than 2 odd vertices, and therefore will not have an Eulerian walk. □