Math 334 Homework 2

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Problem (1)**.**

a. Prove that an infinite union of open sets is open. Where U_i is an open subset of $\mathbb{R}^n$, $\cup_{i=1}^{\infty} U_i$ is open.

   Is the countable size of the collection of sets important?

b. Give an example of an infinite collection of closed sets S_i whose union $\cup_{i=1}^{\infty} S_i$ is not closed.

Proposition. The union of any two open sets $S_1, S_2 \in \mathbb{R}^n$ is open.

Proof of Proposition. We wish to show that, for S_1, S_2 open, $\partial(S_1 \cup S_2) \cap (S_1 \cup S_2) = \emptyset$.

Since S_1 open, $\forall x_1 \in S_1$. $\exists r > 0$. $B_r(x) \subset S_1 \implies B_r(x) \subset S_1 \cup S_2$.

Since S_2 open, $\forall x_2 \in S_2$. $\exists r > 0$. $B_r(x) \subset S_2 \implies B_r(x) \subset S_1 \cup S_2$.

Therefore, $\forall x \in S_1 \cup S_2$. $\exists r > 0$. $B_r(x) \subset S_1 \cup S_2$, which means that $S_1 \cup S_2$ is open. $\square$

Proof of a. We will prove the statement by induction on $m \in \mathbb{Z}^+$.

For the base case, if we choose m as one, we see that the single union of an open set will produce itself, an already open set. Thus, we choose $m = 2$, $\cup_{i=1}^{2} U_i = U_1 \cup U_2$, which is open by the proposition.

Assume the $m = k$ case holds, that is, $\cup_{i=1}^{k} U_i$ in open.

Then, for the $m = k + 1$ case,

$$\cup_{i=1}^{k+1} U_i = \left(\cup_{i=1}^{k} U_i\right) \cup U_{k+1}.$$

But, we see that the left hand side is open by the I.H., and the right hand side is open by the statement. So, $\cup_{i=1}^{k+1} U_i$ is open by the proposition and the $k + 1$ case holds. $\square$

If we had an uncountable infinity, we could not have performed induction. Is it possible that a sufficiently large infinite union of open sets is no longer open?

Proof for b. Consider last week's problem using a set of rationals.

Let S_i for some index i be a set with a single vector with rational components, $x \in \mathbb{Q}^n$. The singleton S_i is closed because, for the only value $x \in S_i$,

$$\forall r > 0.\ B_r(x) \cap S_i = \{x\} \neq \emptyset \wedge B_r(x) \cap S_i^c = \mathbb{R}^n \setminus \{x\} \neq \emptyset.$$

But, the infinite (or even finite) union of such closed singletons produces a set whole boundary contains irrationals, as we have seen that the interior of such a union is $\emptyset$.

Such irrationals are not contained within any S_i as they contain only rational-valued components.

Thus, we have that $\partial S \not\subset S$ where $S = \cup_{i=1}^{\infty} S_i$, which means that such a union is not closed. □

Problem (2)**.** Let $f(x) = \frac{1}{q}$ where $\forall p, q \in \mathbb{Z}.\ x = \frac{p}{q},\ q > 0$ such that p, q coprime, and $f(x) = 0$ where $x \in \mathbb{R} \setminus \mathbb{Q}$.

Determine all x for which $f(x)$ is continuous.

Proof. First, we will find an upper bound for f □

Problem (3)**.** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$f(x, y) = \begin{cases} \frac{y(y-x^2)}{x^4} & 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Determine all points s.t. f not continuous.

Proposition. f is continuous everywhere but at $(0, 0)$.

Proof. First, we will show that $\lim\limits_{(x,y)\to(0,0)} \neq 0 = f(0)$.

Consider approaching the origin along the path $y = x$,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{y(y-x^2)}{x^4} &= \lim_{x\to 0} \frac{x(x-x^2)}{x^4} \\ &= \lim_{x\to 0} \frac{x^2 - x^3}{x^4} \\ &= \lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x} \right). \end{aligned}$$

We see that this quantity will exceed any number and is not equal to zero.

Since $f(0) = 0$ but the limit approaching zero along the path $y = x$ is not zero, then f is not continuous at zero.

We will now show that f is continuous along the boundary path $y = x^2$ except at zero.

$$\lim_{x\to 0} \frac{x^2(x^2 - x^2)}{x^4} = \lim_{x\to 0} 0 = 0 = f(S),$$

where $S = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$.

We will now show that f is continuous for all x, y such that $0 < y < x^2$.

Let $\epsilon > 0$ be given and $\delta =$. $\square$