Math 136 Homework 3

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1. True or false:

   (a) Any system of linear equations has at least one solution;

   False. Represent the system of linear as an augmented matrix that has a row without pivots but with a nonzero entry in the augment column of that row. This indicates that the system has no solutions.

   (b) Any system of linear equations has at most one solution;

   False. Use any augmented matrix that has at least on column without a pivot. This indicates that the solution to the system has a free variable and therefore infinitely many solutions. For example, a matrix of all zeros has infinitely many solutions.

   (c) Any homogeneous system of linear equations has at least one solution;

   True. The solution set to a homogeneous system is the same as the kernel of the coefficient matrix. The kernel will include at least the zero vector. Thus, the any homogeneous system has at least one solution.

   (d) Any system of n linear equations in n unknowns has at least one solution;

   False. This is a more specific case of (a), where the coefficient matrix is square. We can create an augmented matrix with a nonzero particular solution entry next to a row of zeros in the coefficient matrix.

   (e) Any system of n linear equations in n unknowns has at most one solution;

   False. With the same argument as (b), a column without a pivot yields a free variable which produces infinitely many solutions.

   (f) If the homogeneous system corresponding to a given system of a linear has a solution, then the given system has a solution;

   False. Take any system without a solution. We can find such a system because (a) does not hold. Add the homogeneous augment column of all zeros. Since this system is now homogeneous, it has a solution by (c). But, the original system did not have a solution. Therefore a solution to a homogeneous system does not imply that the linear system given by coefficient matrix also has a solution. Therefore the statement is false.

   (g) If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no non-zero solution;

True. Let the coefficient matrix by A.

Since A has n equations and n unknowns, then it is a square $n \times n$ matrix.

Since A is invertible and square, then A has pivots in every row and column. So, $\text{rank}A = n$ and, more importantly, the nullity of A is zero by the Rank Nullity Theorem (FTLA).

So, $\ker A = \vec{0}$. Then, since $\ker A = \left\{\vec{v} \in \text{dom}A : A(\vec{v}) = \vec{0}\right\}$, we also see that the zero vector is a solution to the homogeneous system. Thus, the statement holds.

(h) The solution set of any system of m equations in n unknowns is a subspace in $\mathbb{R}^n$;

False. By (a), the solution set $\mathcal{S}$ of a system of equations can be empty. But, if $\vec{0} \notin \mathcal{S}$, then $\mathcal{S}$ cannot be a vector space.[1] If the solution set of a system of equations is not a vector space, then it cannot be a subspace.

(i) The solution set of any homogeneous system of m equations in n unknowns is a subspace in $\mathbb{R}^n$.

False. The solution set of a homogeneous system is equivalent to the kernel of its coefficient matrix. Since the kernel is a subspace of the domain, which is $\mathbb{R}^n$ given that there are n unknowns and therefore n input variables, then the solution set is a subspace of $\mathbb{R}^n$.

2. True or false:

(a) The rank of a matrix is equal to the number of its non-zero columns;

False. A row reduced matrix may have a nonzero column with no pivots. This column would then be a linear combination of the other columns. Therefore, this column would not contribute to the dimension of the image of the transformation, which is the rank. So, the rank is not equal to the number of nonzero columns and the statement is false.

(b) The $m \times n$ zero matrix is the only $m \times n$ matrix with rank 0;

True. Since the rank is the dimension of the image of of the matrix, which are both zero, then the image of the matrix transformation must be zero. Thus, the matrix can have no nonzero columns and must thereby be the zero matrix.

(c) Elementary row operations preserve rank;

True; this was proved in lecture. By 7.2.3, all row operations preserve row space and therefore the rank of the row space, or just the rank.

(d) Elementary column operations do not necessarily preserve rank;

False. By the same reasoning as (c), column operations preserve column space. So, the dimension of the column space is preserved, which is the rank.

[1] Instead, it is an affine space that is translated from the origin.

(e) The rank of a matrix is equal to the maximum number of linearly independent columns in the matrix;

True. All of the linearly independent columns form a basis for the column space. Since the number of columns in a basis for a space is dimension of a given space, then the maximum number of linearly independent columns are the dimension of the column space which is the rank.

(f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix;

True. The linearly independent rows form a basis for the row space. By the same argument as (e), the number of linearly independent rows is the dimension of the row space, which is again the rank.

Alternatively, using (e), we see that the size of the largest set of linearly independent rows is the rank of the transpose of the give matrix. This is equal to the rank of the matrix itself by the Rank theorem.

(g) The rank of an $n \times n$ matrix A is at most n;

True. Suppose, for a contradiction, that $\operatorname{rank} A = \dim \operatorname{Im} A = m > n$.

But A has a codomain of dimension n and an $n \times n$ matrix cannot produce vectors of a higher dimension that the target set. So, by contradiction, the rank of A must be at most n.

(h) An $n \times n$ matrix A with rank n is invertible.

True. A has a domain dimension of n. Since it has rank n, then it has a nullity of zero by the FTLA. Therefore it is invertible by the identities given in lecture.

3. Define $T : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 3 & -2 \\ 1 & -2 & 2 \\ 2 & -6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let $\mathcal{S}$ be the set of all vectors that are fixed by T, which means that

$$\mathcal{S} = \left\{\vec{v} \in \mathbb{R}^3 : T(\vec{v}) = \vec{v}\right\}.$$

(a) Show, using the definition of a subspace, that $\mathcal{S}$ is a subspace of $\mathbb{R}^3$.

Let $\vec{v_1}, \vec{v_2} \in \mathbb{R}^3$ such that $\vec{v_1} = T(\vec{v_1})$ and $\vec{v_2} = T(\vec{v_2})$.

Then,

$$\begin{aligned} T(\alpha_1\vec{v_1} + \alpha_2\vec{v_2}) &= \alpha_1 T(\vec{v_1}) + \alpha_2 T(\vec{v_2}) \\ &= \alpha_1\vec{v_1} + \alpha_2\vec{v_2} \in \mathbb{R}^3 \end{aligned}.$$

Therefore $\mathcal{S}$ is a vector space. Since $\mathcal{S}$ contains only elements from $\mathbb{R}^3$, $\mathcal{S}$ is a subspace of $\mathbb{R}^3$.

(b) Find a linear equation $ax + by + cz = d$ such that $\mathcal{S}$ is the set of solutions to that equation.

Let the transformation matrix of the transformation T be given by A. We wish to find $\vec{x}$ such that $A\vec{x} = \vec{x}$.

Note that $\vec{x} = I\vec{x}$. So, $A\vec{x} = \vec{x}$ becomes $A\vec{x} - I\vec{x} = \vec{0} = (A - I)\vec{x}$.

So, we now need the kernel of $A - I$.

We reduce the matrix,

$$
\begin{aligned}
&= \begin{pmatrix} 0 & 3 & -2 & \\ 1 & -2 & 2 & \\ & 2 & -6 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 2 & -2 \\ 1 & -3 & 2 \\ 2 & -6 & 4 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

When we consider the homogeneous system of these equations, we see that the solution in the given form is

$$x - 3y + 2z = 0.$$

(c) Geometrically, what king of object is $\mathcal{S}$.

$\mathcal{S}$ is a plane.

(d) Find a basis for $\mathcal{S}$.

We will choose two linearly independent vectors that are orthogonal to the plane's normal vector $\langle 1, -3, 2 \rangle$.

We see that the vectors $\langle 1, 1, 1 \rangle$ and $\langle -2, 0, 1 \rangle$ will work.

So, a basis for $\mathcal{S}$ is given by the set of the two,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

4. (a) Show that the complex numbers $\mathbb{C}$ may be viewed as a 2-dimensional real vector space.

$\mathcal{C}$ is defined as $\mathbb{R}^2$ paired with the maps $+, -, \cdot, /$. Since the vector space of $\mathbb{R}^2$ is $\mathbb{R}^2$ with the addition (and substraction maps), then we can reductively view $\mathbb{C}$ as $\mathbb{R}^2$.

(b) Let $L : \mathbb{C} \longmapsto M_{2\times 2}$ be the map defined by

$$L(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Verify that L is a linear map. What is the rank of L?

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then,

$$\begin{aligned} L(\alpha_1 z_1 + \alpha_2 z_2) &= L(\alpha_1 x_1 + \alpha_2 x_2 + i(\alpha_1 y_1 + \alpha_2 y_2)) \\ &= \begin{pmatrix} \alpha_1 x_1 + \alpha_2 x_2 & \alpha_1 y_1 + \alpha_2 y_2 \\ -(\alpha_1 y_1 + \alpha_2 y_2) & \alpha_1 x_1 + \alpha_2 x_2 \end{pmatrix} \\ &= \alpha_1 \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} \\ &= \alpha_1 L(z_1) + \alpha_2 L(z_2). \end{aligned}$$

Since L is closed under addition and scalar multiplication, it is a linear map.

We can see that the kernel of L will be solely $0 \in \mathbb{C}$, so the nullity of L is zero.

Since the dimension of the domain of L is 2 and the zero nullity of L, by the Rank Nullity Theorem, the rank of L is 2.

(c) Show that L satisfies the identity $L(z_1 z_2) = L(z_1)L(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

We start from the product of the two matrices, using the above definitions for z_1 and z_2,

$$\begin{aligned} L(z_1)L(z_2) &= \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 x_2 - y_1 y_2 & x_1 y_2 + x_2 y_1 \\ -(x_2 y_1 + x_1 y_2) & x_1 x_2 - y_1 y_2 \end{pmatrix} \\ &= L\left(x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)\right) \\ &= L\left((x_1 + iy_1)(x_2 + iy_2)\right) \\ &= L(z_1 z_2). \end{aligned}$$