Math 136 Homework 5

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1. Prove that the geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.

Proof. Let $\vec{v_1}, \ldots, \vec{v_r}$ be a basis of the eigenspace of A for a corresponding eigenvalue λ_0. So, r is the geometric multiplicity of λ_0.

By 4.1.8, we can complete this to form a basis in $\mathbb{R}^n$,

$$A = \begin{pmatrix} \lambda_0 I_r & * \\ \mathbf{0} & B \end{pmatrix}$$

where I_r is the $r \times r$ identity matrix.

By 4.1.7, this characteristic polynomial of this matrix is given by its determinant,

$$\det(A - \lambda I) = \det(\lambda_0 I_g - \lambda I)\det(B - \lambda I).$$

This first determinant term will become the λ_0 roof of degree r and the second determinant will be a polynomial in λ which we will call $q(\lambda)$.

So,

$$\det(A - \lambda I) = (\lambda_0 - \lambda)^r q(\lambda).$$

Since the determinant does not depend on basis, we will also consider the characteristic equation for the operator A in the standard basis. This will be a polynomial with roots for each eigenvalue λ_i with degree m_i for $i = 1, 2, \ldots, k$. m_i represents the algebraic multiplicity of the root.

$$P(\lambda) = (\lambda_0 - \lambda)^{m_0} (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_k - \lambda)^{m_k}.$$

Then, we equate these two characteristic polynomials and divide by the r^{th} power of the λ_0 root.

$$(\lambda_0 - \lambda)^{m_0 - r} (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_k - \lambda)^{m_k} = q(\lambda).$$

But q is a polynomial and not a rational function, so the power of the λ_0 root must be positive.

Hence $m_0 \geq r$; the geometric multiplicity r cannot exceed the algebraic multiplicity m_0. □

2. Prove that the trace of a matrix is equal to the sum of its eigenvalues.

Proof. From 4.1.10, the determinant of a matrix is the product of its eigenvalues,

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda).$$

We expand and then isolate the terms of degree λ^{n-1} to obtain

$$(\lambda_1 + \lambda_2 + \cdots + \lambda_n)(-1)^{n-1}\lambda^{n-1}.$$

We will now show that $\det(A - \lambda I)$ can be represented as

$$(a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) + q(\lambda),$$

where $q(\lambda)$ is a polynomial of degree at most $n-2$.

First, we will compute the determinant through minor matrices. We will consider the minor formed by taking the top left element of A,

$$\det(A - \lambda I) = (a_{11} - \lambda)\det\begin{pmatrix} a_{22} - \lambda & & * \\ & \ddots & \\ * & & a_{nn} - \lambda \end{pmatrix} + q(\lambda)..$$

Then, q is the product of all the other minors which are obtained either the first row or the first column. Let us take from the first row. Then, for each a_{1j}, for $j = 2, 3, \ldots, n$, the corresponding minor matrix will not have the $a_{11} - \lambda$ term, nor the $a_{jj} - \lambda$ term. Therefore, the highest order that the roots of in this minor matrix could be is $n-2$.

Since the highest order of the terms in q is only $n-2$, all of the $n-1$ terms must be contained in the product given by the first minor.

By expanding and isolating λ^{n-1} terms, we see get

$$(a_{11} + a_{22} + \cdots + a_{nn})(-1)^{n-1}\lambda^{n-1}.$$

Then, by comparing coefficients we see that

$$\begin{aligned} \lambda_1 + \cdots + \lambda_n &= a_{11} + \cdots + a_{nn} \\ \sum_{i=1}^{n} \lambda_i &= \sum_{i=1}^{n} a_{ii} = \text{trace} A \end{aligned}$$

.

□

3.

Problem. *Find a closed form for the n^{th} Fibonacci number φ_n. The series is defined recursively as $\varphi_{n+2} = \varphi_{n+1} + \varphi_n$.*

First, we will find a matrix $A_{2\times 2}$ such that

$$\begin{pmatrix} \varphi_{n+2} \\ \varphi_{n+1} \end{pmatrix} = A \begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix}.$$

This holds for the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We will diagonalize A to find a form for A^n.

We compute the eigenvalues of the matrix,

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{pmatrix} \\ &= (1-\lambda)(-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$
.

We recognize the solutions to this equation to be the golden ratio φ and its conjugate $-\varphi^{-1}$; these are our two eigenvalues λ_1 and λ_2.

So, for diagonalizable $A = SDS^{-1}$, we now have that

$$D = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^{-1} \end{pmatrix}.$$

To determine the isomorphic matrix S, we compute the eigenvectors for A.

For $\lambda_1 = \varphi$, along with the fact that $1 - \varphi = -\varphi^{-1}$, $A - \lambda I$ gives,

$$\begin{pmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{pmatrix} \to \begin{pmatrix} -\varphi^{-1} & 1 \\ \varphi^{-1} & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -\varphi \\ 0 & 0 \end{pmatrix}.$$

which yields the eigenvector $\begin{pmatrix} \varphi \\ 1 \end{pmatrix}$.

Similarly, for $\lambda_2 = -\varphi^{-1}$,

$$\begin{pmatrix} 1-(-\varphi)^{-1} & 1 \\ 1-(-\varphi)^{-1} & \end{pmatrix} = \begin{pmatrix} \varphi & 1 \\ 1 & \varphi^{-1} \end{pmatrix} \to \begin{pmatrix} \varphi & 1 \\ 0 & 0 \end{pmatrix},$$

which provides the eigenvector $\begin{pmatrix} -\varphi^{-1} \\ 1 \end{pmatrix}$.

So,

$$S = \begin{pmatrix} \varphi & -\varphi^{-1} \\ 1 & 1 \end{pmatrix}.$$

Then, for S^{-1}, we use the form of the inverse of a 2×2 matrix with the determinant of S to see that

$$S^{-1} = \frac{1}{2\varphi - 1} \begin{pmatrix} 1 & \varphi^{-1} \\ -1 & \varphi \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \varphi^{-1} \\ -1 & \varphi \end{pmatrix}.$$

Then, with $A^n = SD^nS^{-1}$,

$$A^n = \begin{pmatrix} \varphi & -\varphi^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^n & 0 \\ 0 & (-\varphi^{-1})^n \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \varphi^{-1} \\ -1 & \varphi \end{pmatrix}.$$

Since

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix} = A^n \begin{pmatrix} \varphi_1 \\ \varphi_0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then,

$$\begin{aligned}
\begin{pmatrix}\varphi_{n+1}\\ \varphi_n\end{pmatrix} &= \begin{pmatrix}\varphi & -\varphi^{-1}\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^n & 0\\ 0 & (-\varphi^{-1})^n\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1 & \varphi^{-1}\\ -1 & \varphi\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\
&= \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi & -\varphi^{-1}\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^n & 0\\ 0 & (-\varphi)^{-n}\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}\\
&= \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi & -\varphi^{-1}\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^n\\ -(-\varphi)^{-n}\end{pmatrix}\\
&= \frac{1}{\sqrt{5}}\begin{pmatrix}\varphi^{n+1}-(-\varphi)^{-(n+1)}\\ \varphi^n-(-\varphi)^{-n}\end{pmatrix}.
\end{aligned}$$

So, we see that

$$\varphi_n = \frac{1}{\sqrt{5}}\left(\varphi^n - (-\varphi)^{-n}\right).$$

We will show that $\begin{pmatrix}\frac{\varphi_{n+1}}{\varphi_n}\\ 1\end{pmatrix}$ converges to an eigenvector of A.

4. (a) *Proof.* For a contradiction, suppose that $\vec{x_1}$ and $\vec{x_2}$ are linearly dependent.

So, $\vec{x_2} = a\vec{x_1}$ for some scalar $a \in \mathbb{R}$.

Then,

$$A\vec{z} = A(\vec{x_1} + ia\vec{x_1})..$$

For the case when $a = 0$ and $\vec{x_2} = \vec{0}$, then $A\vec{z} = A\vec{x_1}$, which has no imaginary part, contradicting the requirement that $\beta \neq 0$.

So, we continue with $a \neq 0$.

$$A(\vec{x_1} + ia\vec{x_1}) = (\alpha - \beta a)\vec{x_1} + i(\alpha a + \beta)\vec{x_1}.$$

By separating into real and imaginary components, we see that

$$\begin{aligned}
A\vec{x_1} &= (\alpha - \beta a)\vec{x_1}\\
aA\vec{x_1} &= (\alpha a + \beta)\\
A\vec{x_1} &= \left(\frac{\alpha a + \beta}{a}\right)\vec{x_1}.
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\alpha a + \beta}{a} &= \alpha - \beta a\\
\alpha a + \beta &= \alpha a - \beta a^2\\
\beta &= -\beta a^2\\
-1 &= a^2.
\end{aligned}$$

But, this implies that $a \in \mathbb{C}$, contradicting our assumption that it was real.

So, by contradiction, $\vec{x_1}$ and $\vec{x_2}$ are linearly independent. $\square$

(b) *Proof.* Let $\text{span}\{\vec{x_1}, \vec{x_2}\} = W$.

We see that L_W maps vectors from W onto linear combinations of vectors in W, so its codomain is also W,

$$A\vec{x_1} + iA\vec{x_2} = (\alpha\vec{x_1} - \beta\vec{x_2}) + i(\beta\vec{x_1} + \alpha\vec{x_2}).$$

Treating $\vec{x_1}$ and $\vec{x_2}$ as basis vectors $(1, 0)$ and $(0, 1)$ respectively, we see that they map to $(\alpha, -\beta)$ and (β, α) through L_W by looking at the corresponding real and imaginary parts.

So, writing L_W with a matrix,

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

With $\lambda = \alpha + i\beta$, using Euler's formula where $\lambda = \|\lambda\|e^{i\theta}$, we get that

$$\|\lambda\|e^{i\theta} = \|\lambda\|(\cos\theta + i\sin\theta).$$

So, it is clear that $\alpha = \cos\theta$ and $\beta = \sin\theta$.

Thus, we can rewrite our matrix as,

$$A = \|\lambda\| \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

$\square$

(c) Scaling by λ and rotating by θ in a plane W.