Math 134 Homework 10

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10.5

Project

Consider a projectile fired at an angle θ, $\theta \in (0, \pi/2)$, from a point (x_0, y_0) with initial velocity v_0. The horizontal component of v_0 is $v_0 \cos\theta$, and the vertical component is $v_0 \sin\theta$. Let $x = x(t), \quad y = y(t)$ be parametric equations for the path of the projectile.

We neglect air resistance and the curvature of the earth. Under these circumstances there is no horizontal acceleration and,

$$x''(t) = 0.$$

The only vertical acceleration is due to gravity; therefore

$$y''(t) = -g.$$

1

Show that the path of the projectile (the trajectory) is given parametrically by the functions

$$x(t) = (v_0 \cos\theta)t + x_0, \quad y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0.$$

We begin with $x(t)$. We know that there is no horizontal acceleration. Given that acceleration is the second derivative of position, then $x''(t) = 0$.

We integrate to reveal that $x'(t)$, the velocity, is some constant. This constant is the initial horizontal velocity $v_0 \cos\theta$.

We then integrate, $\int x'(t)\, dt = \int v_0 \cos\theta\, dt$, which is $x(t) = (v_0 \cos\theta)t$ with a constant. This constant is the initial position x_0.

So, $x(t) = (v_0 \cos\theta)t + x_0$, as desired.

We repeat this process for $y(t)$, noting that there is vertical acceleration due to gravity. So $y''(t) = -g$.

Then, $\int y''(t)\, dt = \int -g\, dt$ where the constant term of the resulting integral is the vertical velocity $v_0 \sin\theta$.

So $y'(t) = -gt + v_0 \sin\theta$. We then integrate once more, noting that the final constant of integration will be the initial vertical position y_0.

Thus, $y(t) = \int y'(t)\,dt = \int -gt + v_0 \sin\theta\,dt = -\frac{gt^2}{2} + (v_0 \sin\theta)t + y_0$ as was given above.

2

Show that in rectangular coordinates the equation of the trajectory can be written

$$y = -\frac{g}{2{v_0}^2}\sec^2\theta(x - x_0)^2 + \tan\theta(x - x_0) + y_0.$$

First, we isolate t in the equation for x,

$$\begin{aligned} x &= (v_0\cos\theta)t + x_0 \\ \frac{x - x_0}{v_0\cos\theta} &= t. \end{aligned}$$

Then, we use this t in $y(t)$,

$$\begin{aligned} y(t) &= -\frac{1}{2}gt^2 + (v_0\sin\theta)t + y_0 \\ y\left(\frac{x - x_0}{v_0\cos\theta}\right) &= -\frac{g}{2}\left(\frac{x - x_0}{v_0\cos\theta}\right)^2 + (v_0\sin\theta)\left(\frac{x - x_0}{v_0\cos\theta}\right) + y_0 \\ &= -\frac{g}{2{v_0}^2}\left(\frac{1}{\cos^2\theta}\right)(x - x_0)^2 + v_0\left(\frac{\sin\theta}{\cos\theta}\right)(x - x_0) + y_0 \\ &= -\frac{g}{2{v_0}^2}\sec^2\theta(x - x_0)^2 + \tan\theta(x - x_0) + y_0. \end{aligned}$$

3

Measure distances in feet, time in seconds, and set $g = 32\text{ft/sec}^2$. Take (x_0, y_0) as the origin and the x-axis as ground level. Consider a projectile fired at an angle θ with initial velocity v_0.

a. Give parametric equations for the trajectory; give an equation in x and y for the trajectory.

b. Find the range of the projectile, which in this case is the x-coordinate of teh point of impact.

c. How many seconds after firing does the impact take place?

d. Choose θ so at to maximize the range.

e. Choose θ so that the projectile lands at $x = b$.

For (a), from (1) along with $(x_0, y_0) = (0, 0)$, we get

$$x(t) = (v_0\cos\theta)t, \quad y(t) = \frac{-gt^2}{2} + (v_0\sin\theta)t.$$

For (b), we wish to find t such that $y = 0$. With $y(0) = 0$ given, we proceed with $t \neq 0$. So,

$$
\begin{aligned}
y(t) = 0 &= \frac{-gt^2}{2} + (v_0 \sin\theta)t \\
0 &= t(v_0 \sin\theta - \frac{gt^2}{2}), \quad t \neq 0 \\
0 &= v_0 \sin\theta - \frac{gt^2}{2} \\
t &= \frac{2v_0 \sin\theta}{g}.
\end{aligned}
$$

If the projectile is fired at $t_i = 0$, then $t_f = \frac{2v_0 \sin\theta}{g}$ is the time at which the projectile impacts the ground.

The range of the projectile x, since $x(0) = 0$, is then $x(t)$ at the impact time t_f. We compute,

$$
\begin{aligned}
x(t) &= (v_0 \cos\theta)t \\
x(t_f) &= (v_0 \cos\theta)\left(\frac{2v_0 \sin\theta}{g}\right) \\
&= \frac{2{v_0}^2}{g} \cos\theta \sin\theta.
\end{aligned}
$$

For (c), we already computed that the impact time occurs at $t = \frac{2v_0}{g} \sin\theta$ seconds in (b).

For (d), we note that the range of the projectile at the impact time t_f is $\frac{2{v_0}^2}{g} \cos\theta \sin\theta$.

We will optimize this value for theta. We define the range as a function of theta,

$$
l(\theta) = \frac{2{v_0}^2}{g} \cos\theta \sin\theta.
$$

We compute $l'(\theta)$ and equate this value to zero to find the critical points of l.

$$
\begin{aligned}
l &= \frac{2{v_0}^2}{g} \cos\theta \sin\theta \\
l' &= \frac{2{v_0}^2}{g} \left(\cos^2\theta - \sin^2\theta\right) \\
0 &= \frac{2{v_0}^2}{g} \left(\cos^2\theta - \sin^2\theta\right) \\
0 &= \cos^2\theta - \sin^2\theta \\
\sin^2\theta &= \cos^2\theta \\
\tan\theta &= 1 \\
\theta &= \tan^{-1} 1 \\
\theta &= \frac{\pi}{4}.
\end{aligned}
$$

Thus, $\theta = \frac{\pi}{4}$ optimizes the distance x that the projectile travels.

For (e), we desire θ such that $l(\theta) = b$. So, we equate,

$$\begin{aligned} b &= \frac{2{v_0}^2}{g} \cos\theta \sin\theta \\ \frac{bg}{2{v_0}^2} &= \cos\theta \sin\theta; \end{aligned}$$

We use the double angle identity,

$$\frac{1}{2} \sin 2x = \cos x \sin x.$$

Then,

$$\begin{aligned} \frac{bg}{2{v_0}^2} &= \frac{\sin 2\theta}{2} \\ \frac{bg}{{v_0}^2} &= \sin 2\theta \\ \sin^{-1} \frac{bg}{{v_0}^2} &= 2\theta \\ \frac{1}{2} \sin^{-1} \frac{bg}{{v_0}^2} &= \theta. \end{aligned}$$

So, the angle required to impact the projectile at $x = b$ is $\theta = \frac{1}{2} \sin^{-1} \frac{bg}{{v_0}^2}$.

4

a. Graph the path of the projectile fired at an angle of $\frac{\pi}{6}$ with initial velocity $v_0 = 1500\text{ft/sec}$. Determine the range of the projectile and the heigh reached.

b. Keeping $v_0 = 1500\text{ft/sec}$, experiment with several values of θ. Confirm that $\theta = \pi/4$ maximizes the range. What angle maximizes the height reached?

For (a), we will first determine the maximum height and range of the projectile. From 3(b), we know that the range of the projectile is given by $\frac{2{v_0}^2}{g} \cos\theta \sin\theta$. So, with the given launch angle and velocity, we see that the range l is,

$$\begin{aligned} l &= \frac{2 \cdot 1500^2}{32} \cos \frac{\pi}{6} \sin \frac{\pi}{6} \\ &= \frac{2250000}{166} \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right) \\ &= \frac{140625\sqrt{3}}{4} \\ &\approx 60892 \end{aligned}$$

So the projectile travels about 60892 feet.

For the height of the projectile, we note that the projectile will be at its maximum height halfway through its trajectory path. We can also compute this $t_{\max}$ by finding the critical points of $y(t)$.

We use $y'(t)$ from (1), equating the expression to zero,

$$\begin{aligned} y'(t) &= -gt + v_0 \sin\theta \\ 0 &= -gt_{\max} + v_0 \sin\theta \\ t_{\max} &= \frac{v_0 \sin\theta}{g}. \end{aligned}$$

We note that $t_{\max}$ is exactly half of t_f.

We continue to find $y\,(t_{\max})$ to determine the maximum height of the projectile.

$$\begin{aligned} y(t) &= \frac{-gt^2}{2} + (v_0 \sin\theta)t \\ y(t_{\max}) &= \frac{-g}{2}\left(\frac{v_0 \sin\theta}{g}\right)^2 + v_0 \sin\theta \left(\frac{v_0 \sin\theta}{g}\right) \\ &= -\frac{1}{2}\frac{(v_0 \sin\theta)^2}{g} + \frac{(v_0 \sin\theta)^2}{g} \\ &= \frac{(v_0 \sin\theta)^2}{2g}. \end{aligned}$$

We then use the given conditions to finish solving for the height h,

$$\begin{aligned} h &= \frac{(1500 \sin\frac{\pi}{6})^2}{2 \cdot 32} \\ &= \frac{(750)^2}{64} \\ &= \frac{562500}{64} \\ &= \frac{140625}{16} \\ &\approx 8489. \end{aligned}$$

Therefore, the maximum height of the projectile is about 8489 feet.

We then graph the motion of the projectile.

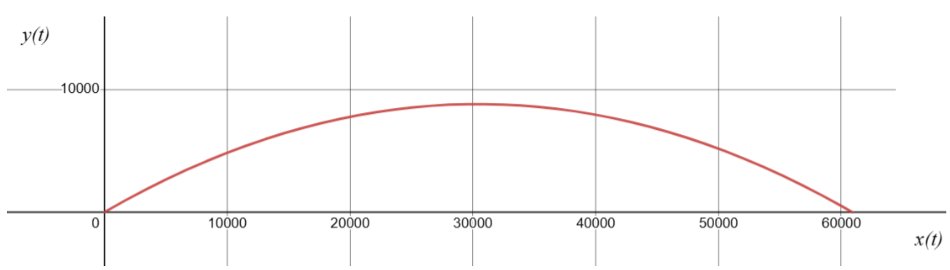

Figure 1: Trajectory of projectile with launch angle $\pi/6$ and initial velocity 1500 ft/sec.

For (b), we have already analytically confirmed that $\theta = \frac{\pi}{4}$ maximizes the range of the projectile for any initial velocity.

We might expect that a different launch angle would maximize the height reached by the projectile, however. We predict that shooting the projectile directly upwards, at an angle of $\theta = \frac{\pi}{2}$, will result in the maximum height attained.

We then optimize the maximum height h from (a), noting that the launch angle is bounded between 0 and $\pi/2$.

$$\begin{aligned} h &= \frac{(v_0 \sin\theta)^2}{2g} \\ h' &= \frac{{v_0}^2}{2g}(2\sin\theta\cos\theta) \\ 0 &= \frac{{v_0}^2}{2g}(2\sin\theta\cos\theta) \\ 0 &= \cos\theta\sin\theta \\ 0, \frac{\pi}{2} &= \theta. \end{aligned}$$

We get a critical point for h when θ is either 0 or $\frac{\pi}{2}$.

We first examine $\theta = 0$, noting that this will result in a maximum height of zero given that $\sin 0 = 0$ for the sine term in h.

Thus, it is our angle of $\frac{\pi}{2}$ that produces a maximum height.

Then, the maximum height for a given initial velocity v_0 is given by,

$$\begin{aligned} h_{\max} &= \frac{(v_0 \sin\frac{\pi}{2})^2}{2g} \\ &= \frac{{v_0}^2}{2g}. \end{aligned}$$

10.7

27

(a) Let $a > 0$. Find the length of the path traced out by

$$\begin{aligned} x(\theta) &= 3a\cos\theta + a\cos 3\theta, \\ y(\theta) &= 3a\sin\theta - a\sin 3\theta, \quad \text{as } \theta \in [0, 2\pi]. \end{aligned}$$

(b) Show that this path can also be parametrized by

$$x(\theta) = 4a\cos^3\theta, \quad y(\theta) = 4a\sin^3\theta \quad \theta \in [0, 2\pi].$$

We will start with (b). We will use the triple angle identities,

$$\begin{aligned} \sin 3\theta &= 3\sin\theta - 4\sin^3\theta \\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

We will start with finding an alternate parametrization for $x(\theta)$,

$$\begin{aligned}
a\cos 3\theta &= a4\cos^3\theta - a3\cos\theta \\
4a\cos^3\theta &= a\cos 3\theta + 3a\cos\theta \\
4a\cos^3\theta &= x(\theta).
\end{aligned}$$

Then, similarly, for $y(\theta)$,

$$\begin{aligned}
a\sin 3\theta &= a3\sin\theta - a4\sin^3\theta \\
4a\sin^3\theta &= 3a\sin\theta - a\sin 3\theta. \\
4a\sin^3\theta &= y(\theta).
\end{aligned}$$

So, we have shown that the alternate parametrizations hold,

$$\begin{aligned}
x(\theta) &= 4a\cos^3(\theta), \\
y(\theta) &= 4a\sin^3\theta.
\end{aligned}$$

For (a), we use these parametric equations from (b), to find the length of the curve from $\theta = 0$ to $\theta = 2\pi$, noting that $a > 0$.

We note that the arclength is given by,

$$\int_0^{2\pi} \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}.$$

We first find $x'(\theta)$ and $y'(\theta)$.

$$\begin{aligned}
\frac{d}{d\theta}x(\theta) &= \frac{d}{d\theta}4a\cos^3\theta \\
x'(\theta) &= 4a\left[3\cos^2\theta(-\sin\theta)\right] \\
&= -12a\cos^2\theta\sin\theta.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\frac{d}{d\theta}y(\theta) &= \frac{d}{d\theta}4a\sin^3\theta \\
y'(\theta) &= 4a\left[3\sin^2\theta(\cos\theta)\right] \\
&= 12a\sin^2\theta\cos\theta.
\end{aligned}$$

We then construct the integrand,

$$\begin{aligned}
\sqrt{[x'(\theta)]^+[y'(\theta)]^2} &= \sqrt{(-12a\cos^2\theta\sin\theta)^2 + \left(12a\sin^2\theta\cos\theta\right)^2} \\
&= \sqrt{144a^2\cos^4\theta\sin^2\theta + 144a^2\sin^4\theta\cos^2\theta} \\
&= 12a\sqrt{\cos^2\theta\sin^2\theta}\sqrt{\cos^2\theta + \sin^2\theta} \\
&= 12a|\cos\theta\sin\theta|.
\end{aligned}$$

Finally, we evaluate the integral,

$$\int_0^{2\pi} 12a|\cos\theta\sin\theta|\,d\theta$$
$$12a\int_0^{2\pi} |\cos\theta\sin\theta|\,d\theta$$

We note that that the magnitude of the area under each period of the function $\cos\theta\sin\theta$ is the same.

$$\int_0^{\pi/2} \cos\theta\sin\theta\,d\theta = -\int_{\pi/2}^{\pi} \cos\theta\sin\theta\,d\theta.$$

The same holds for the third and fourth periods.

So,

$$\int_0^{2\pi} |\cos\theta\sin\theta|\,d\theta = 4\int_0^{\pi/2} \cos\theta\sin\theta\,d\theta.$$

We continue from before,

$$\begin{aligned}
&12a\cdot 4\int_0^{\pi/2} \cos\theta\sin\theta\,d\theta \\
=&48a\int_{\sin 0}^{\sin\pi/2} u\,du \\
=&48a\left[\frac{u^2}{2}\right]_0^1 \\
=&48a\left(\frac{1}{2}\right) \\
=&24a.
\end{aligned}$$

So, the arclength of the curve is 24a.

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Show that the curve $y = \cosh x$ has the property that, for every interval $[a, b]$, the length of the curve from $x = a$ to $x = b$ is equal to the area under the curve from $x = b$ to $x = b$.

Proposition. *For $f(x) = \cosh^2 x$,*

$$\sqrt{1 + [f'(x)]^2} = f(x).$$

Proof. First, we start with two properties of the hyperbolic cosine,

$$1 + \sinh^2 x = \cosh^2 x, \quad \frac{d}{dx}\cosh x = \sinh x.$$

So, with $f(x) = \cosh x$,

$$\begin{aligned} f'(x) &= \sinh x \\ [f'(x)]^2 &= \sinh^2 x \\ [f'(x)]^2 + 1 &= \sinh^2 x + 1 \\ \sqrt{[f'(x)]^2 + 1} &= \sqrt{\cosh^2 x} \\ \sqrt{[f'(x)]^2 + 1} &= \cosh^2 x. \end{aligned}$$

Therefore the proposition holds. □

We begin by setting up the length and area integrals for a function f on the integral $[a, b]$.

The length of the curve is given by,

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

The length area under the curve is given by,

$$\int_a^b f(x)\, dx.$$

But, when $f(x) = \cosh x$, we notice that, by the proposition, the integrands are equivalent, $f(x) = \sqrt{1 + [f(x)]^2}$.

So, the area under the curve is equivalent to the length of the curve on a given interval.

10.8

27

Slice a sphere along two parallel planes which are a fixed distance apart. Show that the surface area of the band that is obtained depends only on the distance between the planes, not on their locations.

We will consider a sphere of radius R.

We will produce the surface area of this sphere via the shells produced by the rotation of horizontal sections of a circle.

Let this circle be parametrized by the equations

$$x(\theta) = R\cos\theta, \quad y(\theta) = R\sin\theta.$$

We will define the two parallel planes by the lines $x = a$ and $x = b$ which intersect the circle.

We note that the angle at at which the planes intersect the circle is given by the triangle formed between the radius of the sphere and the y-coordinate of the intersecting plane.

So, we let the parallel planes be defined by the two angles, α and β, which form the bounds of the inner region of the sphere.

$$\sin\alpha = \frac{a}{R}, \quad \alpha = \sin^{-1}\frac{a}{R};$$
$$\sin\beta = \frac{b}{R}, \quad \beta = \sin^{-1}\frac{b}{R}.$$

We then consider each shell piece of area to be the circumference of the shell, which 2π times the radius $x(\theta) = R\cos\theta$, times the length of the outside of the shell piece. The length of of the outside s is given by $s(\theta) = R\theta$

So, each piece of area is given by,

$$\Delta A \approx 2\pi R\cos\theta\Delta s.$$

We approximate $\Delta A \approx dA = 2\pi R\cos\theta ds$ and integrate between the bounds α and β.

$$\begin{aligned}\int_\alpha^\beta dA &= \int_\alpha^\beta 2\pi R\cos\theta\, ds\\ &= 2\pi R\int_\alpha^\beta \cos\theta R\, d\theta\\ &= 2\pi R^2\int_\alpha^\beta \cos\theta\, d\theta\\ &= 2\pi R^2[\sin\theta]_\alpha^\beta\\ &= 2\pi R^2(\sin\beta - \sin\alpha)\\ &= 2\pi R^2\left(\frac{b}{R} - \frac{a}{R}\right)\\ &= 2\pi R(b-a).\end{aligned}$$

So, the surface area of the region in between the planes is given by

$$2\pi R(b-a).$$

However, this values depend only on the distance between the two planes $(b-a)$, in addition to the radius of the sphere, but not the absolute position of the planes.

Project

Take a wheel and mark a point on the rim. Call that point P. Now roll the wheel, keeping your eyes on P. The jumping path described by P is called a *cycloid*. To obtain a mathematical characterization of the cycloid, let the radius of the wheel be R and set the wheel on the x-axis so that the point P starts at the origin.

The cycloid can be parametrized by the functions

$$x(\theta) = R(\theta - \sin\theta), \quad y(\theta) = R(1-\cos\theta).$$

2

a. At the end of each arch, the cycloid comes to a cusp. Show that x' and y' are both 0 at the end of each arch.

b. Show that the area under an arch of the cycloid is three times the area of the rolling circle.

c. Find the length of an arch of the cycloid.

For (a) we will demonstrate that x' and y' are zero at the each cusp. Each cusp occurs at the end of an arch, where $\theta = 2\pi k, k \in \mathbb{Z}$ which satisfies the sine term in y such that $y(\theta) = 0$.

We compute the first derivative of the parametrization,

$$x' = R(1 - \cos\theta), \quad y' = R\sin\theta.$$

We note that at integer multiples of 2π, $1 - \cos\theta = 0$ and $\sin\theta = 0$. So, it becomes clear that x' and y' are both zero at integer multiples of 2π.

For (b), we know that the area of a circle of radius R is πR^2.

We will show that the area under each arch of a cycloid is then $3\pi R^2$.

We will integrate under the first arch, where $\theta \in [0, 2\pi]$, recalling that $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$.

$$\begin{aligned}
\int_0^{2\pi} y(\theta)x'(\theta)\,d\theta &= \int_0^{2\pi} (R(1-\cos\theta))^2\,d\theta \\
&= R^2 \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta \\
&= R^2\left[\theta - 2\sin\theta + \int \cos^2\theta\,d\theta\right]_0^{2\pi} \\
&= R^2\left[\theta - 2\sin\theta + \int \frac{\cos 2\theta + 1}{2}\,d\theta\right]_0^{2\pi} \\
&= R^2\left[\theta - 2\sin\theta + \frac{1}{2}\left(\frac{\sin 2\theta}{2} + \theta\right)\right]_0^{2\pi} \\
&= R^2\left[\theta - 2\sin\theta + \frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_0^{2\pi} \\
&= R^2\left[\frac{3\theta}{2} - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi} \\
&= R^2\left(\frac{3(2\pi)}{2}\right) \\
&= 3\pi R^2.
\end{aligned}$$

So, the area under the cycloid is three times the area of the circle which traces it.

For (c) we determine the arclength directly, employing the half angle identity $\sin\frac{x}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ for

$x \in [0, 2\pi]$.

$$
\begin{aligned}
\int_0^{2\pi} \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\, d\theta &= \int_0^{2\pi} \sqrt{(R(1-\cos\theta))^2 + R\sin\theta^2}\, d\theta \\
&= R\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\
&= R\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\
&= 2R\int_0^{2\pi} \sqrt{\frac{1-\cos\theta}{2}}\, d\theta \\
&= 2R\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta \\
&= -4R[\cos\frac{\theta}{2}]_0^{2\pi} \\
&= -4R(\cos\pi - \cos 0) \\
&= -4R(-2) \\
&= 8R.
\end{aligned}
$$

So, the arclength of one arch of the cycloid is eight times the radius of the circle that describes it.

3

a. Locate the centroid of the region under the first arch of the cycloid.

b. Find the volume of the solid generate by revolving the region under an arch of the cycloid about the x-axis.

c. Find the volume of the solid generate by revolving the region under an arch of the cycloid about the y-axis.

For (a), by symmetry of each arch region, we expect that the x-coordinate of the centroid to lie in the middle of that region. This occurs at half of the width of each region, or πR.

Then, we compute the y-coordinate, $\overline{y}$, using the first region between $\theta = 0$ and $\theta = 2\pi$. We have already reduced the integrand of the arclength integral, noting that for the length s, the sum of the pieces, $\int ds = \int 2R\sin\theta/2$ for $\theta \in [0, 2\pi]$.

So, with the fact that $\cos 2x = 2\cos^2 x - 1$,[1]

$$\begin{aligned}
\overline{y}L &= \int_0^{2\pi} y(\theta)\sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\, d\theta \\
&= \int_0^{2\pi} R(1-\cos\theta)\left(2R\sin\frac{\theta}{2}\right)\, d\theta \\
&= 2R^2\left[\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta - \int_0^{2\pi} \cos\theta\sin\frac{\theta}{2}\, d\theta\right] \\
&= 2R^2\left[2[-\cos u]_0^{\pi} - \int_0^{\pi} \cos 2u \sin u\, du\right] \\
&= 2R^2\left[2(-\cos\pi + \cos 0) - \int_0^{\pi} \left(2\cos^2 u - 1\right)\sin u\, du\right] \\
&= 2R^2\left[4 + \int_{\cos 0}^{\cos\pi} \left(2v^2 - 1\right)\, dv\right] \\
&= 2R^2\left[4 + \left[\frac{2v^3}{3} - v\right]_1^{-1}\right] \\
&= 2R^2\left(4 + \left(\left(-\frac{2}{3}+1\right) - \left(\frac{2}{3}-1\right)\right)\right) \\
&= 2R^2\left(4 + \left(2 - \frac{4}{3}\right)\right) \\
&= 2R^2\left(\frac{16}{3}\right) \\
&= \frac{32}{3}R^2.
\end{aligned}$$

Since $L = 8R$, then $\overline{y} = \frac{\frac{32}{3}R^2}{8R} = \frac{4}{3}R$.

For (b) and (c), we will use Pappus theorem of a planar region rotated about an axis.

$$V = 2\pi\overline{R}A,$$

where $\overline{R}$ is the centroid coordinate against the axis of rotation, and A is the area of the region, $3\pi R^3$.

For (b), the rotation about the x-axis we use the $\overline{y}$, so $V = 2\pi\left(\frac{4}{3}R\right)(3\pi R^2) = \frac{8\pi^2}{3}R^2$

For (c), we use $\overline{x}$ to rotate about the y-axis. So, $V = 2\pi(\pi R)(3\pi R^2) = 6\pi^3 R^3$.

[1] This fact is clearly demonstrated once we take the identity $\cos 2x = \cos^2 x - \sin^2 x$ along with $\sin^2 x = 1 - \cos^2 x$.