Math 135 Homework 6

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1. **19.1**

Prove that, if $p \neq q$, then e^{px} and e^{qx} are linearly independent.

Proof. For a contradiction, assume that $p \neq q$ and e^{px} and e^{qx} are linearly dependent and therefore satisfy

$$c_1 e^{px} + c_2 e^{qx} = 0 \tag{*}$$

for all x and c_1 and c_2 not both zero.

Assume that $c_2 \neq 0$.

Then,

$$\begin{aligned} c_1 e^{px} &= -c_2 e^{qx} \\ -\frac{c_1}{c_2} &= e^{(q-p)x}. \end{aligned}$$

But $q - p \neq 0$ so $e^{(q-p)x}$ cannot be constant for all x.

Therefore the assumption that e^{px} and e^{qx} were linearly dependent when $p \neq q$ was false.

Thus, by contraction, e^{px} and e^{qx} are linearly independent when $p \neq q$.

Then, $e^{(q-p)} \neq 1$.

So, $e^{(q-p)x} \neq 1^x = 1$ cannot be constant for all x.

So, $e^{(q-p)x} \neq c$ where $c = -\frac{c_1}{c_2}$ such that $c_2 \neq 0$.

Then,

$$\begin{aligned} -\frac{c_1}{c_2} &\neq e^{(q-p)x} \\ -c_1 e^{px} &\neq c_2 e^{qx} \\ 0 &\neq c_1 e^{px} + c_2 e^{qx}. \end{aligned}$$

$\square$

19.2

Prove that e^{ax} and xe^{ax} are linearly independent.

Proof. For a contradiction, assume that e^{xa} and xe^{ax} are linearly dependent.

Assume that $c_2 \neq 0$.

Then

$$c_1 e^{ax} + c_2 x e^{ax} = 0$$

for all x.

Since $e^{ax} \neq 0$ for all x, then

$$c_1 + c_2 x = 0.$$

So

$$x = -\frac{c_1}{c_2}.$$

But, this statement cannot hold for all x.

So, the assumption that e^{ax} and xe^{ax} was false, therefore the two equations are linearly independent. □

2. **20.24**

Find the general solution $y'' - y' + y = 0$.

The characteristic equation of this differential equation is

$$r^2 - r + 1 = 0.$$

This equation as roots

$$r = \frac{1 \pm \sqrt{3}i}{2}.$$

So, the general solution to the homogeneous equation is

$$y = e^{t/2}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \sin\frac{\sqrt{3}}{2}t\right).$$

3. **21.30**

Find the general solution to $y''' - 2y'' + y' = 2e^x + 2x$ with the initial conditions $y(0) = y'(0) = y''(0) = 0$.

Then, via the characteristic equation, the homogeneous solution is

$$y_h = c_1 e^x + c_2 x e^x + c_3$$

Since the characteristic equation contains a root of degree two and the non-homogeneous equation contains an e^x term, the form of the particular solution is given by two degrees above the highest degree present in the homogeneous solution,

$$y_p = A + Bx + Cx^2 + Ex^2e^x + Fx^3e^x.$$

Then, we also have

$$u_p' = B + 2Cx + (Ex^2 + 2Ex)e^x + (Fx^3 + 3Ex^2)e^x,$$

and

$$u_p'' = 2C + (Ex^2 + 4Ex + 2x)e^x + (Fx^3 + 6Fx^2 + 6Fx)e^x.$$

Then,

$$\begin{aligned} u_p'' - 2u_p' + u_p &= 2e^x + 2x \\ A + Bx + Cx^2 - 2B - 4Cx + 2C + 2Ee^x + 5Fxe^x &= 2e^x + 2x \\ (A - 2B + 2C) + (B - 4C)x + Cx^2 + 2Ee^x + 5Fxe^x &= 2e^x + 2x. \end{aligned}$$

The system of equations yields[1]...

The initial conditions reveal the final solution[2]...

So

$$y = x^2 + 4x + 4 + (x^2 - 4)e^x.$$

4. **22.18**

Use variation of parameters to find the general solution to

$$x^2y'' + xy' - 4y = x^3$$

given that $y_1 = x^2$ and $y_2 = x^{-2}$.

The homogeneous solution is

$$y_h = c_1x^2 + c_2x^{-2}.$$

We will place the differential equation into standard form,

$$y'' + x^{-1}y' - 4x^{-2}y = x.$$

Let $g(x) = x$.

Then, $y_p = u_1y_1 + u_2y_2$ where

$$\begin{aligned} u_1 &= \int \frac{-u_2 g(x)}{W(y_1, y_2)}\,dx, \\ u_2 &= \int \frac{u_1 g(x)}{W(y_1, y_2)}\,dx. \end{aligned}$$

[1] Did not have time to finish typing up the solution.
[2] Same as above.

First, we compute the Wronskian of the two solutions,

$$\begin{aligned} W(x^2, x^{-2}) &= \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} \\ &= -2x^{-1} - 2x^{-1} \\ &= -4x^1. \end{aligned}$$

So,

$$\begin{aligned} u_1 &= \int \frac{-x^{-2} \cdot x}{-4x^{-1}}\, dx = \frac{1}{4} \int dx = \frac{x}{4}, \\ u_2 &= \int \frac{x^2 \cdot x}{-4x^{-1}}\, dx = -\frac{1}{4} \int x^4\, dx = -\frac{x^5}{20}. \end{aligned}$$

Assembling y_p, we obtain,

$$\begin{aligned} y_p &= \frac{x}{4}x^2 - \frac{x^5}{20}x^{-2} \\ &= \frac{x^3}{4} - \frac{x^3}{20} \\ &= \frac{5x^3 - x^3}{20} \\ &= \frac{x^3}{5}. \end{aligned}$$

Thus,

$$y = c_1x^2 + c_2x^{-2} + \frac{x^3}{5}.$$

5. **23.16**

Find the general solution to $x^2y'' - 2y = 2x^2$ given $y_1 = x^2$.

We proceed using the method of reduction of order.

With the form $f_2(x)y'' + f_1(x)y' + f_0(x)y = 0$ from TP 23.14, we see that $f_2(x) = x^2$, $f_1(x) = 0$ and $f_0(x) = -2$.

Then, by TP 23.28,

$$y_2 = y_1 \int \frac{e^{-\int \frac{f_1(x)}{f_2(x)}\, dx}}{y_1^2}\, dx.$$

So, with $y_1 = x^2$,

$$\begin{aligned} y_2 &= x^2 \int \frac{e^{-\int \frac{0}{x^2}\,dx}}{x^4}\,dx \\ &= x^2 \int \frac{dx}{x^4} \\ &= x^2 \frac{1}{-3x^3} \\ &= \frac{1}{-3x}. \end{aligned}$$

We construct the homogeneous solution with this y_2,

$$y_h = c_1 x^2 + c_2 x^{-1}.$$

Then, we place the differential equation into standard form,

$$y'' - 2x^{-2}y = 2$$

and proceed with the technique of variation of parameters.

Let $y_1 = x^2$, $y_2 = x^{-1}$, and $g(x) = 2$.

Then,

$$y_p = u_1 y_1 + u_2 y_2$$

where

$$\begin{aligned} u_1 &= \int \frac{-u_2 g(x)}{W(y_1, y_2)}\,dx, \\ u_2 &= \int \frac{u_1 g(x)}{W(y_1, y_2)}\,dx. \end{aligned}$$

First, we compute the Wronskian of the two solutions found in the homogeneous solution,

$$\begin{aligned} W(x^2, x^{-1}) &= \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} \\ &= -x^2 \cdot x^{-2} - 2x \cdot x^{-1} \\ &= -1 - 2 \\ &= -3. \end{aligned}$$

Then, we compute u_1,

$$\begin{aligned} u_1 &= \int \frac{-x^{-1} \cdot 2}{-3}\,dx \\ &= \frac{2}{3} \int \frac{dx}{x} \\ &= \frac{2}{3} \ln x. \end{aligned}$$

Similarly, we compute u_2

$$u_2 = \int \frac{x^2 \cdot 2}{-3}\, dx$$
$$= -\frac{2}{9}x^3.$$

So, we combine to get,

$$y_p = \frac{2}{3}x^2 \ln x - \frac{2}{9}x^2.$$

When combining y_h and y_p, we see that y_h already contains an arbitrary constant for an x^2 term, so we only need to consider the $\frac{2}{3}x^2 \ln x$ part of the particular solution.

Thus, our general solution is,

$$y = c_1x^2 + c_2x^{-1} + \frac{2}{3}x^2 \ln x.$$

6. Suppose that y_1 and y_2 form the fundamental set of solutions to

$$y'' + p(t)y' + q(t)y = 0 \tag{*}$$

on $t \in \mathbb{R}$ such that p, q continuous for all t.

Prove that there is only one zero of y_1 between two consecutive zeros of y_2.

Proof. Assume y_1 and y_2 form the fundamental set of solutions to (*).

Since y_1 and y_2 form the fundamental set of solutions to the differential equation (*), then they are twice differentiable, linearly independent, and their Wronskian is non-zero for all t.

$$0 \neq W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

For a contraction, assume that there is no zero of y_1 between two consecutive zeros of y_2.

Define c_1, c_2 as consecutive roots of y_2 such that $y_2(c_1) = y_2(c_2) = 0$.

Let $I = [c_1, c_2]$.

Then, by the assumption, $y_1(t) \neq 0$ for all $t \in I$.

Let $f(t) = \frac{y_2(t)}{y_1(t)}$.

Since $y_2(c_1) = y_2(c_2) = 0$, then $f(c_1) = f(c_2) = 0$.

Since $y_1(t) \neq 0, \quad \forall t \in I$, then $f(t)$ is continuous for all $t \in I$.

So, there exists an $a \in (c_1, c_2)$ such that $f'(a) = 0$ by Rolle's Theorem.

Then, with

$$f'(t) = \frac{d}{dt}\left[\frac{y_2(t)}{y_1(t)}\right] = \frac{y_2'y_1 - y_2y_1'}{y_1^2},$$

we see that

$$0 = f'(a) = \frac{y_2'(a)y_1(a) - y_2(a)y_1'(a)}{y_1^2(a)}.$$

But $y_1 \neq 0$; so,

$$0 = y_2'(a)y_1(a) - y_2(a)y_1'(a),$$

which is the Wronskian of y_1 and y_2 evaluated at $t = a$.

But y_1 and y_2 form the fundamental set of solutions to (*) and therefore their Wronskian is never zero by the assumption.

So, the statement that y_1 has no zero between c_1 and c_2 is false.

So, by contradiction, y_1 must have at least one zero between the two consecutive zeros of y_2 at c_1 and c_2.

Then, for a contraction, assume that y_1 has more than one zero between two consecutive zeros of y_2. This statement is equivalent to stating that y_2 has no zeros between two consecutive zeros of y_2.

We will repeat the above part of the proof by defining the utility function $g(t) = \frac{y_1(t)}{y_2(t)}$.

Let $J = [b_1, b_2]$

So, by the assumption, there exists b_1, b_2 such that $y_1(b_1) = y_1(b_2) = 0$.

Then, by the assumption, $y_2(t) \neq 0$ for all $t \in J$.

So, g, the quotient of two continuous functions, whose denominator is not zero, is continuous.

Since $g(b_1) = g(b_2) = 0$ and g continuous, then there exists a $k \in J$ such that $g'(k) = 0$ by Rolle.

Then, with

$$g'(t) = \frac{y_1'y_2 - y_1y_2'}{y_2^2},$$

$$0 = g'(k) = \frac{y_1'(k)y_2(k) - y_1(k)y_2'(k)}{y_2^2(k)}.$$

But $y_2 \neq 0$; so,

$$0 = y_1'(k)y_2(k) - y_1(k)y_2'(k),$$

which is the Wronskian of y_1 and y_2 evaluated at $t = k$.

But, the Wronskian of y_1 and y_2 is never zero by the assumption that they form the fundamental set of solutions.

So $W(y_1(k), y_2(k)) = 0$ is a contradiction.

Thus y_1 has no more than one zero between two consecutive zeros of y_2.

Since y_1 has at least one zero and no more than one zero between two consecutive zeros of y_2, then y_1 has exactly one zero between two consecutive zeros of y_2.

$\square$

7. Let y_1 and y_2 be solutions to
$$y'' + py' + qy = 0 \tag{*}$$
where p, q continuous on $I = (a, b)$.

Show that if there as a point in I where y_1 and y_2 are both zero or where both have maxima or minima, then y_1 and y_2 are linearly dependent.

We will show that there exists a $c \in I$ such that, if either
$$y_1(c) = y_2(c) = 0, \tag{a}$$
or
$$y'_1(c) = y'_2(c) = 0 \tag{a}$$
and $y''_1(c) \neq 0$ and $y''_2(c) \neq 0$, then y_1 and y_2 are linearly dependent.

For a contradiction, assume that y_1 and y_2 are linearly independent solutions to (*).

So, the Wronskian of y_1 and y_2 is non zero for all $t \in I$.
$$0 \neq W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

Then, for case (a),
$$y_1(c) = y_2(c) = 0$$
implies that
$$y_1(c)y_2'(c) - y_1'(c)y_2(c) = 0.$$

But,
$$y_1 y_2' - y_1' y_2 \neq 0$$
for all $t \in I$ by the assumption that the solutions y_1 and y_2 were linearly independent.

Therefore y_1 and y_2 are linearly dependent and also constant multiples of each other.

Next, for case (b),
$$y_1'(c) = y_2'(c) = 0$$
implies that
$$y_1(c)y_2'(c) - y_1'(c)y_2(c) = 0.$$

But, this contracts the assumption of the linear independence of the solutions.

The non-zero second derivative condition was not necessary.

So, we have shown that if there exists a c such that the solutions are both zero at c, or both attain a critical point at c, then the two solutions are linearly dependent and therefore constant multiples of one another.