Math 134 A Homework 2

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2.2

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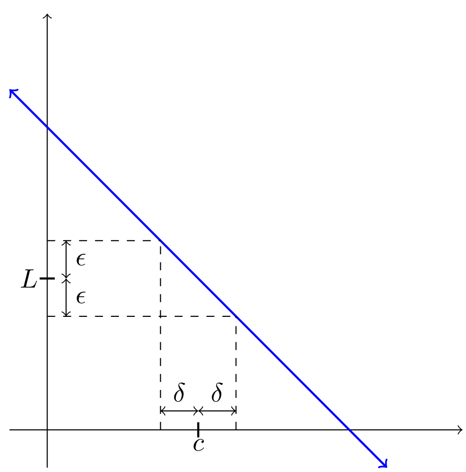
For each of the limits stated and the ϵ's given, use a graphing utility to find a $\delta > 0$ which is such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Draw the graph of f together with the vertical lines $x = c - \delta, x = c + \delta$ and the horizontal lines $y = L - \epsilon, y = L + \epsilon$

$\lim_{x \to 0}(2 - 5x) = 2.$



Proposition. *For every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - 0| < \delta$ implies $|2 - 5x - 2| < \epsilon$.*

Proof. Let $\epsilon > 0$ be given.

Define $\delta = \frac{\epsilon}{5}$

Assume $|x| < \delta$, then

$$\begin{aligned} |x| &< \frac{\epsilon}{5} \\ 5|x| &< \epsilon \\ |5x| &< \epsilon \\ |-5x| &< \epsilon \\ |-5x + 2 - 2| &< \epsilon \\ |2 - 5x - 2| &< \epsilon \end{aligned}$$

Hence, $\lim_{x \to 0} (2 - 5x) = 2$.

$\square$

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Give an $\epsilon - \delta$ proof for $\lim_{x \to 3} \sqrt{x+1} = 2$.

Theorem 2.3.2. *$\lim_{x \to c} f(x) = M$ and $\lim_{x \to M} g(x) = L$ implies $\lim_{x \to c} g(f(x)) = L$.*

Proposition. *For every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - 3| < \delta$ implies $|\sqrt{x+1} - 2| < \epsilon$.*

Proof. We will split this limit into a composition of two limits, where $f(x) = x + 1$ and $g(x) = \sqrt{x}$ such that $g(f(x)) = \sqrt{x+1}$ by Theorem 2.3.2.

First, since $x + 1$ is polynomial, then $x + 1$ is continuous. Therefore, $\lim_{x \to c} f(x) = f(c)$.

So, $\lim_{x \to 3} x + 1 = 4$.

For an $\epsilon - \delta$ proof, we make the assumption that, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - 3| < \delta$ implies $|x + 1 - 4| < \epsilon$.

Let $\epsilon > 0$ be given. Define $\delta = \epsilon$. Assume $|x - 3| < \delta$.

Then, $|x - 3| = |x + 1 - 4| < \epsilon$.

So, $\lim_{x \to 3} x + 1 = 4$.

Second, for $\lim_{x \to 4} \sqrt{x} = 2$, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - 4| < \delta$ implies $|\sqrt{x} - 2| < \epsilon$.

Let $\epsilon > 0$ be given.

Choose $\delta = \min\{4, \epsilon\}$.

Assume $0 < |x - 4| < \delta$.

Since $\delta \leq 4$, then $x \geq 0$. So, $\sqrt{x}$ is defined.

$$\begin{aligned} x - 4 &= \sqrt{x}^2 - 2^2 \\ x - 4 &= (\sqrt{x}+2)(\sqrt{x}-2) \\ |x - 4| &= \left|(\sqrt{x}+2)(\sqrt{x}-2)\right| \\ |x - 4| &= |\sqrt{x}+2||\sqrt{x}-2| \end{aligned}$$

Since $\sqrt{x} + 2 \geq 2 > 1$, then $|x-4| > |\sqrt{x} - 2|$.

Since $|x - 4| < \delta$ and $\delta \leq \epsilon$, then $|\sqrt{x} - 2| < \epsilon$.

Therefore, $\lim\limits_{x\to 4} = 2$.

Since $\lim\limits_{x\to 3} x + 1 = 4$ and $\lim\limits_{x\to 4} = 2$, then $\lim\limits_{x\to 3} \sqrt{x+1} = 2$ by Theorem 2.3.2.

□

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Prove that, for the function

$$g(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational,} \end{cases}$$

$\lim\limits_{x\to 0} g(x) = 0$.

Proposition. *For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - 0| = |x| < \delta$ implies $|g(x) - 0| = |g(x)| < \epsilon$.*

Proof. Let $\epsilon > 0$ be given. Define $\delta = \epsilon$. Assume $0 < |x| < \delta$.

Then, there are two cases, for $x \in \mathbb{R}$, x is either rational or irrational.

In the first case, where $x \in \mathbb{Q}$, $|g(x)| = |x| < \delta = \epsilon$.

So, $|g(x)| < \epsilon$ and the statement holds for the first case.

In the second case, when $x \in \mathbb{R} \setminus \mathbb{Q}$, $|g(x)| = |0| = 0$.

Since $0 < \epsilon$ was given, $|g(x)| < \epsilon$ also holds in the second case.

Therefore, $\lim\limits_{x\to 0} g(x) = 0$. □

2.3

61e

Calculate

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

for $f(x) = x^n : n \in \mathbb{Z}^+$

Proposition. $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = nx^{n-1}$.

Proof. Proof by induction.

For the base case $n = 1$, $f(x) = x$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1x^{1-1} = 1x^0 = 1.$$

So, the statement holds for the base case.

Next, we assume that the statement holds for $n = k$ such that $f(x) = x^k$ by the inductive hypothesis.

$$\lim_{h \to 0} \frac{(x+h)^k - x^k}{h} = kx^{k-1}.$$

Then, for the inductive step, when $n = k + 1$,

$$\begin{aligned}
\lim_{h \to 0} \frac{(x+h)^{k+1} - x^{k+1}}{h} &= \lim_{h \to 0} \frac{(x+h)(x+h)^k - x^{k+1}}{h} \\
&= \lim_{h \to 0} \frac{x(x+h)^k + h(x+h)^k - x^{k+1}}{h} \\
&= \lim_{h \to 0} \frac{x(x+h)^k - x^{k+1}}{h} + \lim_{h \to 0} \frac{h(x+h)^k}{h} \\
&= \lim_{h \to 0} x \frac{(x+h)^k - x^k}{h} + \lim_{h \to 0} (x+h)^k
\end{aligned}$$

If g is a polynomial in h, then x is constant. So, $\lim_{h \to 0} g(x+h) = g(x)$.

So, with $g(x) = x^k$, then $\lim_{h \to 0} (x+h)^k = \lim_{h \to 0} g(x+h) = g(x) = x^k$.

Additionally, we recognize that the first quantity is equal to $x \cdot kx^{k-1}$ by the induction hypothesis and the fact that x is constant when taking the limit with respect to h.

$$\begin{aligned}
\lim_{h \to 0} x \frac{(x+h)^k - x^k}{h} + \lim_{h \to 0} (x+h)^k &= x \cdot kx^{k-1} + x^k \\
&= kx^k + k^k \\
&= (k+1)x^k \\
&= (k+1)x^{(k+1)-1}
\end{aligned}$$

Which is the statement when $n = k + 1$. Therefore, by induction, the statement holds for all $n \in \mathbb{Z}^+$.

$\square$

2.4

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Let $f(x) = \begin{cases} A^2x^2, & x \leq 2 \\ (1-A)x, & x > 2. \end{cases}$ For what values of A is f continuous at 2?

The function f is continuous when $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.

Since $(1-A)x$ is polynomial, $\lim_{x\to 2^-}(1-A)x = 2(1-A)$.

Since A^2x^2 is polynomial, $\lim_{x\to 2^+} A^2x^2 = 4A^2$.

So, f will be continuous when,

$$\begin{aligned}
2(1-A) &= 4A^2 \\
-4A^2 - 2A + 2 &= 0 \\
2A^2 + A - 1 &= 0 \\
A^2 + \frac{A}{2} - \frac{1}{2} &= 0 \\
A^2 + \frac{2A}{4} + \frac{1}{16} &= \frac{1}{2} + \frac{1}{16} \\
\left(A + \frac{1}{4}\right)^2 &= \frac{9}{16} \\
A + \frac{1}{4} &= \pm\frac{3}{4} \\
A &= -\frac{1}{4} \pm \frac{3}{4} \\
A &= -1, \frac{1}{2}.
\end{aligned}$$

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Suppose that the function f has the property that there exists a number B such that

$$|f(x) - f(c)| \leq B|x - c|$$

for all x in the interval $(c-p, c+p)$. Prove that f is continuous at c.

Proof. We begin by noting that a function is continuous when $\lim_{x\to c} f(x) = f(c)$.

We will rewrite $x \in (c-p, c+p)$ as $|x-c| < p$.

For consistency, we will δ such that, $p = \delta > 0$.

For every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x-c| < p = \delta$ implies $|f(x) - f(c)| < \epsilon$.

Let $\epsilon > 0$ be given. Define $\delta = \frac{\epsilon}{B}$.

Assume $|x-c| < \delta$, then

$$\begin{aligned}
|x-c| &< \frac{\epsilon}{B} \\
B|x-c| &< \epsilon \\
|f(x) - f(c)| &< \epsilon \text{ by the property of } f.
\end{aligned}$$

Therefore, $\lim_{x \to c} f(x) = f(c)$.

So, the function f is continuous at c. $\square$

2.5

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Evaluate the limit, taking a and b as nonzero constants.

$\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

Proof. Assume that $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} g(x)}$ such that $\lim_{x \to 0} g(x) \neq 0$.

From 2.5.6, assume that $\lim_{x \to 0} \frac{\sin cx}{cx} = 1$.

So, $\lim_{x \to 0} \frac{\sin cx}{x} = c$.

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin ax}{\sin bx} &= \lim_{x \to 0} \frac{\sin ax}{\sin bx} \cdot \frac{1/x}{1/x} \\
&= \frac{\lim_{x \to 0} \frac{\sin ax}{x}}{\lim_{x \to 0} \frac{\sin bx}{x}} \text{ given that } \lim_{x \to 0} \frac{\sin bx}{x} \neq 0. \\
&= \frac{a}{b}
\end{aligned}
$$

Therefore, $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$. $\square$

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Show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

First, we visualize $x \sin \frac{1}{x}$ to see that it appears to be bounded by the functions $-|x|$ and x.

Proposition. *For all $x \in \mathbb{R}$, $-|x| \leq x \sin \frac{1}{x} \leq |x|$.*

Proof. For all $x \in \mathbb{R}$, $-|x| \leq 0 \leq |x|$.

For all $\theta \in \mathbb{R}$, the range of $\sin \theta$ is $[-1, 1]$.

Since $-1 \leq \sin \theta \leq 1$, then $-1 \leq \sin \frac{1}{x} \leq 1$.

Since $-|x| < x < |x|$, then $-|x| \leq x \sin \frac{1}{x} \leq |x|$.

Therefore the proposition is true. $\square$

We will now prove the statement.

Proposition. $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

Proof. By 2.2.4, assume $\lim_{x \to c} |x| = |c|$.

So, $\lim_{x \to 0} |x| = 0$ and $\lim_{x \to 0} -|x| = 0$ by the assumption.

Since $-|x| \leq x \sin \frac{1}{x} \leq |x|$ and $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$, then $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ by the Sandwich Theorem.

Therefore, the statement is true. □

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Let f be the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$$

Show that $\lim_{x \to 0} x f(x) = 0$.

Proof. First, we will show that $0 \leq x f(x) \leq x$ for $x \geq 0$ and $x \leq x f(x) \leq 0$ for $x < 0$.

The range of f is $[0, 1]$.

So, $0 \leq f(x) \leq 1$.

If $x \geq 0$, then $0 \leq x f(x) \leq x$ else $x \leq x f(x) \leq 0$.

Since $x \leq x f(x) \leq 0$ and $\lim_{x \to 0} x = 0$ and $\lim_{x \to 0} 0 = 0$, then, for either x negative or non-negative, $\lim_{x \to 0} x f(x) = 0$. □

Worksheet

5

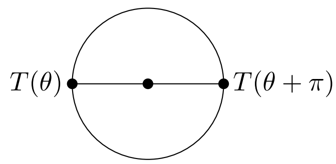
At any given time, prove that there are two points on the surface of the earth diametrically opposite from each other that have the exact same temperature.

Since two diametrically opposed points on a sphere lie on a single plane, we can take a slice of the spherical earth through the center to produce a circle with the two diametrically opposed points.

We will parametrize the temperature on the ring as $T(\theta)$ where we pick any point on the circle and assign its θ to zero. Then, θ represents a another point on the circle by the angle created from that point, the center of the ring, and the chosen θ_0.

So, given that the period of a circle is 2π, then $T(\theta) = T(\theta + 2\pi)$.

We can assume that this function will be continuous because, between any two points with different temperatures, there must exist a point in between with a temperature in between as well.



The two diametrically opposed points are $T(\theta)$ and $T(\theta + \pi)$.

We seek some θ such that the difference between between these two points is zero.

So, we will let f represent the different between the temperature at one point and its opposing point on the other hemisphere,

$$f(\theta) = T(\theta) - T(\theta + \pi).$$

The difference of the continuous function $T(\theta)$ with a horizontal offset of itself $T(\theta + a)$ is also continuous. Therefore f is continuous.

We will show that there exists a c such that $f(c) = 0$.

First, we note that going measuring the temperature in one direction is the same as measuring temperature in the opposite direction.

$$\begin{aligned}
f(\theta + \pi) &= T(\theta + \pi) - T(\theta + 2\pi) \\
f(\theta + \pi) &= T(\theta + \pi) - T(\theta) \\
f(\theta + \pi) &= -\left(T(\theta) - T(\theta + \pi)\right) \\
f(\theta + \pi) &= -f(\theta)
\end{aligned}$$

If $f(\theta) = 0$, then $f(\theta + \pi) = 0$.

This is analogous to a constant temperature at all points. So, for all pairs of points by a, $f(a) = 0$. In this case, the statement holds.

So, we can now assume $|f(\theta)| > 0$ which implies $|f(\theta + \pi)| > 0$.

Since $f(\theta)$ and $f(\theta + \pi)$ have the opposite signs, there must exist a $K = 0$ in between because f is continuous.

So, there exists a $c \in [0, \pi]$ such that $f(c) = K = 0$ by the IVT.

Therefore, since we need to trace over one half of the circle, $[0, \pi]$ to check for equal points on the other half, $[\pi, 2\pi]$, the statement is true.

6

A hiker begins a backpacking trip at 6am on Saturday morning, arriving at camp at 6pm that evening. The next day, the hiker returns on the same trail leaving at 6am in the morning and finishing at 6pm. Show that there is some place on the trail that the hiker visited at the same time of day both coming and going.

Let the interval $[0, 2a] = [0, a] \cup [a, 2a]$ define the total duration of the journey where a is the twelve-hour one-way duration.

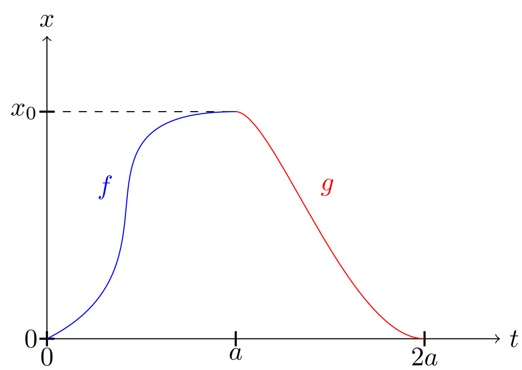
Assume that the journey ranges from a minimum starting position 0 to some maximum position $x_0 > 0$.

We will represent the first first trip by $f(t)$ and the second by $g(t)$. We assume that these functions are continuous because the hiker cannot teleport and therefore must pass through all intermediary positions between any two given positions.

It is given that the starting point of f by the same as the ending point of g where both are the starting position 0, $f(0) = 0 = g(2a)$.

Then, it is also given that the ending point of f is th same as the starting point of g, which is the maximum x_0, $f(a) = x_0 = g(a)$

We can then visualize the two functions f and g,



We can also expand the domain of f to include $[a, 2a]$ where its value remains constant at x_0. We can do the same for g such that its value is the minimum 0 on $[0, a]$. Therefore, both functions are defined and continuous on the total interval $[0, 2a]$.

Then, we define $h(t) = f(t) - g(t + a)$ which represents the difference in position between f and g after their starting time. Since both f and g are continuous on $[0, 2a]$, their difference is also continuous.

When h is zero, the hiker's position f and g will be the same at the same time in the journey.

So, we seek a c such that $h(c) = 0$.

For $t = 0$, we have, $h(0) = f(0) - g(a) = 0 - x_0 = -x_0$.

For $t = a$, we have, $h(a) = f(a) - g(2a) = x_0 - 0 = x_0$.

Since $x_0 > 0$, then $-x_0 < 0x_0$.

So, there exists a $c \in [0, a]$ such that $h(c) = 0$ by the IVT.

Therefore, there is some point c that the hiker will visit at the same time after starting the journey on each day.