Math 334 Homework 5

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Problem (1)**.** Find and classify all critical points of $f(x, y) = 3x - x^3 - 2y^2 + y^4$.

Proof. We will begin with $\nabla f(x, y) = (3 - 3x^2, -4y + 4y^3)$. Then, at $\nabla f = 0$, $3 - 3x^2 = 0 \implies x = \pm 1$ and $-4y + 4y^3 = 0 \implies y = 0, \pm 1$. So we will check the points $(\pm 1, \pm 1)$ and $(\pm 1, 0)$.

$$\begin{aligned} f(1, \pm 1) &= 3 - 1 - 2 + 1 = 1 \\ f(-1, \pm 1) &= -3 + 1 - 2 + 1 = -3 \\ f(1, 0) &= 3 - 1 = 2 \\ f(-1, 0) &= -3 + 1 = -2. \end{aligned}$$

So, (1,0) is a local max while $(-1, \pm 1)$ is a local min. The other two points must be saddle points.

We can further verify this using the determinant of the Hessian matrix. We expect the saddle points to produce a negative value D, where $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = -6x(12y^2 - 4)$. So $D(1, \pm 1) = -6(8) < 0$, and $D(-1, 0) = 6(-4) < 0$. So, both of these points are indeed saddle points. Checking for our local extrema, we expect to find positive D, and, indeed, $D(-1, \pm 1) = 6(8) > 0$ and $D(1, 0) = -6(-4) > 0$. □

Problem (2)**.** Let $V \geq 0$. Let $\{S_V\}$ be the set of rectangular prisms with volume $\leq V$. Find the minimum surface area of a prism in $\{S_V\}$. Is there a maximum possible surface area?

Proof. Let the volume function be $V(x, y, z) = xyz$. Let the surface area function be $A(x, y, z) = 2(xy + xz + yz)$.

First, we see that for a prism with zero volume, $x = y = z = 0$, then the area will also be zero. So, this is the minimum surface area for $V \geq 0$.

Next, we will use Lagrange multipliers, optimizing A with respect to V. So, $\nabla A = 2(y + z, x + z, x + y)$ and $\nabla V = (yz, xz, xy)$. With $\nabla A = \lambda \nabla V$ and $x, y, z \neq 0$, we are given the following equations,

$$\begin{aligned} yz &= 2\lambda(y + z) \\ xz &= 2\lambda(x + z) \\ xy &= 2\lambda(x + y). \end{aligned}$$

Considering the first and last equations, we have

$$\begin{aligned}\frac{yz}{y+z} &= \frac{xy}{x+y}\\ (x+z)y &= (y+z)x\\ xy+yz &= xy+xz\\ yz &= xz\\ y &= x.\end{aligned}$$

We can perform a similar computation so see that a critical point occurs at $x = y = z$. For the minimum, have already seen that these values must all be zero. However, for the maximum, since V can be any number, so too can A. Thus, there is no maximum possible surface area. $\square$

Problem (3)**.** Find the absolute max and min of $f(x, y) = 2x + 3y$ on $\sqrt{x} + \sqrt{y} = 5$, $x, y \geq 0$.

Proof. Let $g(x, y) = \sqrt{x} + \sqrt{y}$. We wish to optimize f on g. Then, with $\nabla f = (2, 3)$ and $\nabla g = \frac{1}{2}(\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{y}})$,

$$\nabla f = \lambda \nabla g \implies 2 = \frac{\lambda}{2\sqrt{x}}, 3 = \frac{\lambda}{2\sqrt{y}} \implies 2\sqrt{x} = 3\sqrt{y}.$$

Assume $x, y \neq 0$ and recall the constraint $\sqrt{x} + \sqrt{y} = 5$. The above becomes,

$$\begin{aligned}\frac{2}{3}\sqrt{x} &= 5 - \sqrt{x}\\ \frac{5}{3}\sqrt{x} &= 5\\ \sqrt{x} &= 3\\ x &= 9.\end{aligned}$$

Then, $y = 4$. So, there exists a critical point on the boundary at (9,4).We must also check the boundary points (25,0) and (0,25).

$$\begin{aligned}f(25, 0) &= 50\\ f(0, 25) &= 75\\ f(9, 4) &= 30.\end{aligned}$$

So, the absolute minimum of f on g is 30 at (9,4) and the absolute maximum is 75 at (0,25). $\square$

Problem (4)**.** Let $S \subset \mathbb{R}^3$ be defined by $x^2 - 4y^2 + z^2 = 1$. Let $p = (0, 0, 5)$.

(a) Find $x \in S$ closest to p, or prove there is no such x.

(b) Find $x \in S$ furthest from p, or prove there is no such x.

(c) Sketch S supporting answers to parts a and b.

Proof of a. We will use $f(x, y, z) = x^2 + y^2 + (z - 5)^2$ to give the distance squared from p. We will minimize f with respect to $g(x, y, z) = x^2 - 4y^2 + z^2 - 1$.

So, $\nabla f = (2x, 2y, 2z - 10)$ and $\nabla g = (2x, -8y, 2z)$. Then, $\nabla f = \lambda \nabla g \implies$

$$\begin{aligned} 2x = 2\lambda x &\implies x = 0 \vee \lambda = 1 \\ 2y = -8\lambda y &\implies y = 0 \vee \lambda = -1/4 \\ 2z - 10 = 2\lambda z &\implies z = \frac{5}{1-\lambda}. \end{aligned}$$

If $\lambda = 1$, then z is of indeterminate form. So we must have $x = 0$. For the first case, we will consider $\lambda = -\frac{1}{4}$, which gives $z = 4$. But, our constraint g and $x = 0$ provide that $-4y^2 + 4^2 = 1$, so $y^2 = \frac{15}{4} \implies y = \frac{\pm\sqrt{15}}{2}$. Thus, we have a critical point at $\left(0, \frac{\pm\sqrt{15}}{2}, 4\right)$.

For the second case, we will use $y = 0$, yet, with our constraint g and $x = 0$, then $z^2 = 1 \implies z = \pm 1$. So we also have critical points $(0, 0, \pm 1)$. Evaluating f at these critical points yields,

$$\begin{aligned} f(0,0,1) &= 16 \\ f(0,0,-1) &= 36 \\ f\left(0, \frac{\pm\sqrt{15}}{2}, 4\right) &= \frac{19}{4}. \end{aligned}$$

So, the minimum distance squared is $\frac{19}{4}$. Thus, the point of S closest to p is $\left(0, \frac{\pm\sqrt{15}}{2}\right)$, at $\frac{\sqrt{19}}{2}$ units away. $\square$

Proof of b. Since S is a hyperboloid of one sheet, then there is no point on S furthest from p. $\square$

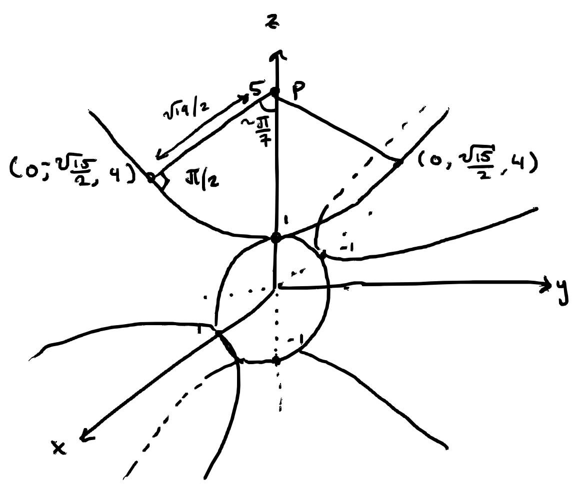

Figure 1: Sketch of $S \subset \mathbb{R}^3$

Problem (5)**.** (AMGM inequality) $\forall x_i \geq 0$,

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Prove AMGM inequality using Lagrange multipliers subject to the constraint $x_1 + \cdots + x_n = 1$.

Proof. Let $g = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$. We will optimize $\log g$ which increases as g increases. Let

$$G = \log g = \frac{1}{n} \log \left(\prod_{i=1}^{n} x_i\right) = \frac{1}{n} \sum_{i=1}^{n} \log x_i.$$

So, $\frac{\partial G}{\partial x_i} = \frac{1}{n x_i}$. Then, with the constraint $a = \sum_{i=1}^{n} x_i$, $\frac{\partial a}{\partial x_i} = 1$. With $\nabla G = \lambda \nabla a$, we have that $\frac{1}{n x_i} = \lambda$, which implies that all x_i must be equal. So, with the constraint a, $\forall i$, $x_i = \frac{1}{n}$ provides the critical point.

Next, we will use the Hessian matrix H. If we look at the pure second partials, we see that $\frac{\partial^2 G}{\partial x_i^2} = -\frac{1}{n x_i^2}$, while the mixed partials are zero. Thus, H_G contains only negative diagonal entries, since $\forall x_i \geq 0$, $-\frac{1}{n x_i^2} < 0$. Since H_G is a diagonal matrix with all negative eigenvalues, it is negative definite. Since H_G is negative definite, then the critical point given by $\forall i$, $x_i = \frac{1}{n}$ is a maximum.

Then, indeed,

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} = \left(\left(\frac{1}{n}\right)^{n}\right)^{\frac{1}{n}} = \frac{1}{n} \leq \frac{1}{n}.$$

So, the geometric and arithmetic means will be equal when all compared values are equal, but the geometric mean will be less than the arithmetic if the compared values are distinct. $\square$