Math 136 Homework 9

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1.

Problem. Find the minimum and maximum value of the function $f(x, y) = x^4 + 4y^3 + 5$ on the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$.

First, we will find critical points on the interior of the unit disk.

$$\{\vec{x} : \nabla f(\vec{x}) = \vec{0} = (4x^3, 12y^2)\} = \{\vec{0}\} = \{(0, 0)\}$$

So, at the origin, $f(0, 0) = 5$.

We will parametrize the boundary with a single variable by removing x from f according to the boundary condition $x^2 + y^2 = 1 \implies x^2 = 1 - y^2$. We will consider only $y \in [-1, 1]$.

$$\begin{aligned} f(y) &= (1 - y^2)^2 + 4y^3 + 5 \\ &= 1 - 2y^2 + y^4 + 4y^3 + 5 \\ &= y^4 + 4y^3 - 2y^2 + 6. \end{aligned}$$

Then we find where the derivative of $f(y)$ is zero,

$$\begin{aligned} f'(y) &= 4y^3 + 12y^2 - 4y \\ &= 4y(y^2 + 3y - 1). \end{aligned}$$

We see that this function is zero at $y = 0$. We will now consider the roots of the quadratic,

$$\begin{aligned} \left(y + \frac{3}{2}\right)^2 - \frac{13}{4} &= 0 \\ y + \frac{3}{2} &= \frac{\sqrt{13}}{2} \\ y &= \frac{\sqrt{13} - 3}{2}. \end{aligned}$$

We will evaluate the f at these critical points.

$$\begin{aligned} f(0) &= 6; \\ f\left(\frac{\sqrt{13} - 3}{2}\right) &\approx 5.9. \end{aligned}$$

We will also evaluate at the endpoints.

$$\begin{aligned} f(-1) &= 1; \\ f(1) &= 9. \end{aligned}$$

So, we see that, considering the critical points inside the region, the critical points on the boundary of the region, and the endpoints of the region,

$$\begin{aligned} \min f &= 1 \text{ at } (0,-1); \\ \max f &= 9 \text{ at } (0,1). \end{aligned}$$

2.

Problem. Show that a surface of the form $z = xf(\frac{x}{y})$ with f continuously differentiable, has the property that all tangent have a common point.

Let $g(x,y) = z = xf(\frac{x}{y})$.

The tangent plane $T(\vec{x})$ of the function g at a point $\vec{x_0}$ is given by,

$$T(\vec{x}) = g(\vec{x_0}) + \nabla g(\vec{x_0}) \cdot (\vec{x} - \vec{x_0}).$$

Let $\vec{x} = (x,y)$ and fix $\vec{x_0} = (a,b)$.

Then,

$$\nabla g = \left(f\left(\frac{x}{y}\right) + \frac{x}{y} f'\left(\frac{x}{y}\right), -\frac{x^2}{y^2} f'\left(\frac{x}{y}\right) \right).$$

So,

$$\begin{aligned} T(x,y) &= \left(f\left(\frac{a}{b}\right) + \frac{a}{b} f'\left(\frac{a}{b}\right) \right)(x-a) - \frac{a^2}{b^2} f'\left(\frac{a}{b}\right)(y-b) + af\left(\frac{a}{b}\right) \\ &= x\left(f\left(\frac{a}{b}\right) + \frac{a}{b} f'\left(\frac{a}{b}\right) \right) - \frac{a^2}{b} f'\left(\frac{a}{b}\right) - af\left(\frac{a}{b}\right) - y\frac{a^2}{b^2} f'\left(\frac{a}{b}\right) + \frac{a^2}{b} f'\left(\frac{a}{b}\right) + af\left(\frac{a}{b}\right) \\ &= x\left(f\left(\frac{a}{b}\right) + \frac{a}{b} f'\left(\frac{a}{b}\right) \right) - y\frac{a^2}{b^2} f'\left(\frac{a}{b}\right). \end{aligned}$$

Since a and b were fixed from $\vec{x_0}$, then we see that this function is indeed a plane as it is linear in x and y.

More importantly, all of the T planes share a common point of the origin $(x,y) = (0,0)$.

3.

Problem. Let $w = f(x,y), x = g(u,v)$, and set $k(u,v) = f(g(u,v),v)$. Assume that f and g are C^2. Find $\frac{\partial}{\partial v} k(u,v)$ and $\frac{\partial^2}{\partial v^2} k(u,v)$ in terms of partial derivatives of f and g.

First, for the first order differential,

$$
\begin{aligned}
\frac{\partial k}{\partial v} &= \frac{\partial f}{\partial g}\frac{\partial g}{\partial v}\frac{\partial v}{\partial v} + \frac{\partial f}{\partial v} + \frac{\partial v}{\partial v} \\
&= \frac{\partial f}{\partial g}\frac{\partial g}{\partial v} + \frac{\partial f}{\partial v}.
\end{aligned}
$$

Then, for the second order,

$$
\begin{aligned}
\frac{\partial^2 k}{\partial v^2} &= \frac{\partial^2 f}{\partial v \partial g}\frac{\partial g}{\partial v} + \frac{\partial f}{\partial g}\frac{\partial^2 g}{\partial v^2} + \frac{\partial^2 f}{\partial v^2} \\
\frac{\partial^2 k}{\partial v^2} &= \left(\frac{\partial^2 f}{\partial g^2}\frac{\partial g}{\partial v} + \frac{\partial^2 f}{\partial v^2}\right)\frac{\partial g}{\partial v} + \frac{\partial f}{\partial g}\frac{\partial^2 g}{\partial v^2} + \frac{\partial^2 f}{\partial v^2} \\
&= \frac{\partial^2 f}{\partial g^2}\left(\frac{\partial g}{\partial v}\right)^2 + \frac{\partial^2 f}{\partial v^2}\frac{\partial g}{\partial v} + \frac{\partial f}{\partial g}\frac{\partial^2 g}{\partial v^2} + \frac{\partial^2 f}{\partial v^2} \\
&= \frac{\partial^2 f}{\partial g^2}\left(\frac{\partial g}{\partial v}\right)^2 + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial g}{\partial v} + 1\right) + \frac{\partial f}{\partial g}\frac{\partial^2 g}{\partial v^2}.
\end{aligned}
$$