Math 336 Homework 1

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April 16, 2025

Problem 1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.

Each z is equidistant to both z_1 and z_2. So, these z describe the perpendicular bisector to the line segment joining z_1 and z_2.

(b) $1/z = \overline{z}$.

Since $|z|^2 = z\overline{z} = \frac{z}{z} = 1$, then $|z| = 1$. Then, $z = e^{it}$ and $\overline{z} = e^{-it}$ give $\left(e^{it}\right)^{-1} = e^{-it}$, which holds for all t. So, these z contain all points on the unit circle.

(c) $\text{Re}(z) = 3$.

These z form a line parallel to the imaginary axis, intersecting the real axis at 3.

(d) $\text{Re}(z) > c$, (resp., $\geq c$) where $c \in \mathbb{R}$.

These z describe the half-plane on $\mathbb{C}$ extending in the positive real direction with boundary parallel to the imaginary axis at c, either inclusive of z with real part equal to c, or exclusive, respectively.

(e) $\text{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.

We have that $\text{Re}(az) + \text{Re}(b) > 0$. Let $a = \alpha + i\beta$, $z = x + iy$, and $\text{Re}(b) = \gamma$.

Since $\text{Re}(az) = \alpha x - \beta y$, then $\alpha x - \beta y + \gamma > 0$, this gives the half plane with normal $\overline{a}$ pointing outwards of the region defined by $\{z\}$.

This is the same as the region above the line $\alpha x - \beta y + \gamma = 0$ in $\mathbb{R}^2$ superimposed on the complex plane.

(f) $|z| = \text{Re}(z) + 1$.

Let $z = x + iy$. Then, $x^2 + y^2 = (x+1)^2 \implies y^2 = 2x + 1 \implies x = \frac{y^2 - 1}{2}$, which is a parabola opening the direction of the positive real axis with apex at $-\frac{i}{2}$.

(g) $\text{Im}(z) = c$ with $c \in \mathbb{R}$.

These z describe a line parallel to the real axis, intersecting the imaginary axis ic.

Problem 2. With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

Proof. Let $z = re^{i\theta}$. Then, $z^n = r^n e^{in\theta} = se^{i\varphi}$. So, we have $r^n = s$ and $n\theta = \varphi + 2\pi k$, $k \in \mathbb{Z}$.

WLOG, let $\varphi = 0$ because we can rotate both vectors such that ω lies on the positive real axis. In any case, we are looking for solutions to $n\theta - \varphi = 0$ with the following constraint on θ:

$$\theta < 2\pi \implies \theta = \frac{\varphi + 2\pi k}{n} < 2\pi.$$

Since this holds for $k = 0, \ldots, n-1$ until $k = n \implies \theta = 2\pi \not< 2\pi$, then we have n solutions of θ.

For $s = 0$, then $\omega = 0 = z^n \implies z = 0$, which gives one more solution.

Thus, we have a total of $n + 1$ possible solutions. $\square$

Problem 3. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

[Hint: Why can one assume that z is real? It then suffices to prove that

$$(r - w)(r - \overline{w}) \leq (1 - rw)(1 - r\overline{w})$$

with equality appropriate for r and $|w|$.]

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w - z}{1 - \overline{w}z}$$

satisfies the following conditions:

(i) F maps the unit disk to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$]

Proof of (a). Since z and w are inside of the unit disk, then we may assume z is real by rotational symmetry.

Let $z' = |z|$ and $w' = we^{-i\operatorname{Arg}(z)}$. Then,

$$1 = \left|e^{-i\operatorname{Arg}(z)}\right| \implies \left|\frac{e^{-i\operatorname{Arg}(z)}(w-z)}{1-\overline{w}z}\right| < 1.$$

We have that

$$z = |z|e^{i\operatorname{Arg}(z)} \implies ze^{-i\operatorname{Arg}(z)} = |z| = z'.$$

So,

$$e^{-i\operatorname{Arg}(z)}(w-z) = w' - z'.$$

But,

$$\overline{w'}z' = \overline{w}e^{i\operatorname{Arg}(z)}|z| = \overline{w}z.$$

So, our substitution retains the original equality:

$$\left|\frac{w-z}{1-\overline{w}z}\right| = \left|\frac{w'-z'}{1-\overline{w'}z'}\right| < 1.$$

Thus, we will now let $z \subset \mathbb{C} = r \in \mathbb{R}$.

Then,

$$\begin{aligned}
\left|\frac{w-r}{1-\overline{w}r}\right| &\le 1 \\
|r-w| &\le |1-\overline{w}r| \\
|r-w|^2 &\le |1-\overline{w}r|^2 \\
(r-w)(\overline{r-w}) &\le (1-\overline{w}r)(\overline{1-\overline{w}r}) \\
(r-w)(r-\overline{w}) &\le (1-rw)(1-r\overline{w}).
\end{aligned}$$

If $|z| = r = 1$, then, clearly, equality holds.

Next, considering $r < 1 \implies r^2 - 1 \neq 0$, we will reduce further

$$\begin{aligned}
(r-w)(r-\overline{w}) &\le (1-rw)(1-r\overline{w}) \\
r^2 - r(w+\overline{w}) + w\overline{w} &\le 1 - r(w+\overline{w}) + r^2 w\overline{w} \\
r^2 + |w|^2 &\le 1 + r^2|w|^2 \\
r^2 - 1 &\le (r^2-1)|w|^2 \\
|w| &\le 1,
\end{aligned}$$

which is indeed what we wished to show. $\square$

Proof of b. (i) Since $\forall z \in \mathbb{D}$ and fixed w, $\left|\frac{w-z}{1-\overline{w}z}\right| < 1$ by part (a), then the image of F on $\mathbb{D}$ must be a subset of $\mathbb{D}$.

Since the F is a quotient of holomorphic functions, then F is holomorphic except where the denominator is zero, where $\overline{w}z = 1 \implies z = \frac{1}{\overline{w}}$.

But, we had that $|w| = |\overline{w}| \leq 1 \implies 1 \leq \left|\frac{1}{\overline{w}}\right| = |z|$, which means that the singularities occur only on the boundary of the unit disk $\mathbb{D}$.

Thus, F is holomorphic on $\mathbb{D}$, which is open.

(ii) We have that

$$F(0) = \frac{w - 0}{1 - \overline{w}(0)} = w,$$

and also

$$F(w) = \frac{w - w}{1 - \overline{w}w} = 0.$$

(iii) By part (a), if $r = |z| = 1$, then $|F(z)| = 1$.

(iv) We will show $(F \circ F)(z) = z$. We have that

$$F(F(z)) = \frac{w - \frac{w-z}{1-\overline{w}z}}{1 - \overline{w}\frac{w-z}{1-\overline{w}z}}.$$

We will consider the numerator and denominator separately. First, for the numerator,

$$\begin{aligned} w - \frac{w-z}{1-\overline{w}z} &= \frac{w(w - \overline{w}z) - (w - z)}{1 - \overline{w}z} \\ &= \frac{w - w\overline{w}z - w + z}{1 - \overline{w}z} \\ &= \frac{z - w\overline{w}z}{1 - \overline{w}z} \\ &= z\frac{1 - |w|^2}{1 - \overline{w}z}. \end{aligned}$$

Second, for the denominator,

$$\begin{aligned} 1 - \overline{w}\left(\frac{w-z}{1-\overline{w}z}\right) &= \frac{1 - \overline{w}z - w\overline{w} + \overline{w}z}{1 - \overline{w}z} \\ &= \frac{1 - |w|^2}{1 - \overline{w}z}. \end{aligned}$$

Hence, the quotient of the above is

$$z\left(\frac{1 - |w|^2}{1 - \overline{w}z}\right)\left(\frac{1 - \overline{w}z}{1 - |w|^2}\right) = z.$$

Thus, $F(F(z)) = z$, which implies that $F \circ F$ is the identity function and that F is bijective.

$\square$

Problem 4. Consider the function defined by

$$f(x + iy) = \sqrt{|x||y|}, \quad \text{whenever } x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Reimann equations at the origin, yet f is not holomorphic at 0.

Proof. Let $f = u + iv$. Since $f = \sqrt{|x||y|} \in \mathbb{R}$, then f has no imaginary component. So, $f = u$ and f vanishes at the origin.

Hence, $v = 0 \implies \partial_x v = \partial_y v = 0$. Then, for the real component, we have that,

$$\partial_x f = \frac{1}{2}\sqrt{\left|\frac{y}{x}\right|}, \qquad \partial_y f = \frac{1}{2}\sqrt{\left|\frac{x}{y}\right|}.$$

We will consider the limit definition of the derivative along the real and imaginary axes:

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{\sqrt{|x||0|}-0}{x} = 0, \qquad \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}\frac{\sqrt{|0||y|}-0}{y} = 0.$$

So, indeed, approaching the origin along the coordinate axes, the derivative of $f = u$ vanishes, so the Cauchy-Reimann conditions are trivially satisfied there.

But, for the path $x = y$, parametrized in $h > 0$,

$$\lim_{h\to 0}\frac{f(h,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{|h||h|}}{h} = \lim_{h\to 0}\frac{|h|}{h} = 1.$$

So, the derivative of f is not continuous at the origin, and hence $f \notin C^1$ there. □

Problem 5. In this problem we will go through a proof of the Fundamental Theorem of Algebra, that is: If

$$p(z) = a_n z^n + \cdots + a_0$$

is a polynomial with an $a_n \neq 0$, then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

(i) Suppose for the sake of contradiction that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Show that the function $g(z) = |p(z)|$ has a minimum at some point $z_0 \in \mathbb{C}$. (Hint: Remember that $\mathbb{C}$ is definitely not compact!)

(ii) Consider the function $q(z) = \frac{1}{|p(z_0)|}p(z + z_0)$. Show that q is a polynomial with $|q(0)| = 1$ and that $|q(z)|$ has its minimum at $z = 0$.

(iii) Show that for any sufficiently small $\varepsilon > 0$, there is some θ for which $|q(\varepsilon e^{i\theta})| < 1$, which provides the desired contradiction.

Proof of (a). By the triangle inequality, we have that

$$|p(z)| = \left|\sum_0^\infty a_n z^n\right| \leq \sum_0^\infty |a_n z^n|.$$

So $g(z) = |p(x)| = O(|z|^n)$.

Then, as $|z| \to \infty$, we have that $g \to \infty$; i.e., for sufficiently large R, we have

$$R < |z| \implies a_0 = |p(0)| < |p(z)| = g. \tag{*}$$

Consider the closed disk D of radius R centered at the origin. Since D is closed and bounded, then it is compact.

Since p is continuous, then $g = |p|$ is also continuous.

Then, by EVT, g attains a minimum value on D at some point $z_0 \in D$.

So, $|p(z_0)|$ is the minimum value of g on D.

For all z outside D, then $R < |z|$, so we have $|p(0)| < g$ by (*).

If $|p(z_0)| < |p(0)|$, then g attains a global min at z_0.
If $|p(x_0)| = |p(0)|$, then g attains a global min at either z_0 or 0.
In either case, g attains its min at some point in $\mathbb{C}$. $\square$

Proof of (b). Since $p(z)$ is a polynomial, then $p(z + z_0)$ is also a polynomial, just translated by z_0. Then, $\frac{1}{|p(z_0)|}$ is a constant. So, q, is a scaled and translated polynomial, which is still a polynomial.

Note that

$$|q(0)| = \frac{|p(0 + z_0)|}{|p(z_0)|} = 1.$$

Since $|p(z_0)|$ is the min of $|p(z)|$, then $|p(z_0)| \leq |p(z + z_0)|$ for all $z \in \mathbb{C}$.

Hence, $1 \leq |q(z)|$ for all $z \in \mathbb{C}$, with equality when $z = 0$. Thus, $|q(z)|$ has a min at $z = 0$. $\square$

Proof of (c). Since q is a polynomial with $q(0) = 1$, then it can be represented with a finite series,

$$q(x) = 1 + \sum_{1}^{n} c_k z^k, \qquad q(\varepsilon e^{i\theta}) = 1 + \sum_{1}^{n} c_k \varepsilon^k e^{ik\theta}.$$

Note that, for ε sufficiently small,

$$|c_k \varepsilon^k| > |c_{k+1} \varepsilon^{k+1}| \tag{*}$$

regardless of the constants c_k, c_{k+1}.

Assume that c_k is the lowest indexed nonzero coefficient. Then,

$$q(\varepsilon e^{i\theta}) = 1 + c_k \varepsilon^k e^{ik\theta} + \psi(\theta),$$

where $|\psi(\theta)| < |c_k \varepsilon^k e^{ik\theta}|$ by (*).

We wish to choose θ in the opposite direction of c_k. Since $c_k = |c_k| e^{i\varphi}$, then we will consider $\theta = \frac{\pi - \varphi}{k}$ so that

$$ik\theta = ik\left(\frac{\pi - \varphi}{k}\right) = i(\pi - \varphi).$$

Therefore,

$$c_k e^{ik\theta} = |c_k| e^{i\varphi} e^{i(\pi - \varphi)} = |c_k| e^{i\pi} = -|c_k|.$$

So, for this choice of θ,

$$q(\varepsilon e^{i\theta}) = 1 - |c_k| \varepsilon^k + \psi(\theta),$$

and $|\psi(\theta)| < |c_k| \varepsilon^k$.

Thus, $q(\varepsilon e^{i\theta}) < 1$, a contradiction.

So, there must exist z such that $p(z) = 0$, meaning that all polynomials in $\mathbb{C}$ must have at least one root. $\square$

Problem 6. Consider the function f defined on R by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that f is indefinitely differentiable on R, and that $f^{(n)}(0) = 0$ for all $n \geq 1$. Conclude that f does not have a converging power series expansion $\sum_{n=0}^{\infty} a_n x^n$ for x near the origin.

Proof. Since f is the composition of infinitely differentiable functions, then f is also infinitely differentiable for positive x.

We wish to find the derivatives $f^{(n)}(x)$ for $x > 0$.

By the chain rule, $f'(x)$ will the product of a polynomial in $\frac{1}{x}$ with e^{-1/x^2}. Indeed, we have $f'(x) = \frac{2}{x^3} e^{-1/x^2}$.

By the chain and product rule, $f''(x)$ will also the product of a polynomial in $\frac{1}{x}$ with e^{-1/x^2}. Specifically, $f''(x) = \left(\frac{4}{x^6} - \frac{6}{x^4}\right) e^{-1/x^2}$.

Thus, we can write

$$f^{(n)}(x) = q_d(1/x) e^{-1/x^2},$$

where q_d is a polynomial in $\frac{1}{x}$ of degree d; in fact, we have $d = 3n$ because the derivative will contribute a factor of $\frac{2}{x^3}$ to q_d.

Now, for the continuity of the derivative, we have that $\lim\limits_{x \to 0^-} f^{(n)}(x) = 0$, so we wish to show that $\lim\limits_{x \to 0^+} f^{(n)}(x) = 0$ as well.

For any polynomial p of degree d, we have that p is bounded: $|p| \leq N x^d$ for some N.

So, for the polynomial q_d in $\frac{1}{x}$, we have that $|q_d(1/x)| \leq \frac{M}{x^d}$ for some M.

Hence,

$$\lim_{x \to 0^+} |f^{(n)}(x)| = \lim_{x \to 0^+} |q_d(1/x) e^{-1/x^2}| \leq \lim_{x \to 0} |M x^{-d} e^{-1/x^2}|.$$

Then, with the substitution $t = x^{-2}$, we have $x \to 0 \implies t \to \infty$. Since exponential functions grow faster than any polynomial function, then $\lim\limits_{t \to \infty} M t^{\frac{d}{2}} e^{-t} = 0$.

Hence, $\lim\limits_{x \to 0^+} f^{(n)}(x) = 0$, and $f^{(n)}(x) = 0$ for all n at $x = 0$.

So, we can produce a Taylor series expansion centered at zero,

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0.$$

But, $e^{-1/x^2} \neq 0$ for all $x > 0$. Thus, f cannot be represented by a Taylor series around $x = 0$ despite being infinitely differentiable there.

So, f is smooth but not analytic at zero. $\square$

Problem 7. Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Proof. We will expand the integrand using partial fraction decomposition,

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right).$$

Then, we will represent each of the above functions by a geometric series which converges on in the given annulus of bounded by radii $r \in [|a|, |b|]$.

First,

$$\frac{1}{z-b} = -\frac{1}{b}\left(\frac{1}{1-z/b}\right) = -\frac{1}{b}\sum_0^\infty \left(\frac{z}{b}\right)^n = -\sum_0^\infty b^{-n} z^n,$$

which converges for $|z/b| < 1 \implies |z| < b$.

Second,

$$\frac{1}{z-a} = \frac{1}{z}\left(\frac{1}{1-a/z}\right) = \frac{1}{z}\sum_0^\infty \left(\frac{a}{z}\right)^n = \sum_0^\infty a^n z^{-(n+1)},$$

which congers for $|a/z| < 1 \implies |a| < |z|$.

So, our integral becomes,

$$\frac{1}{a-b}\int_\gamma \left(\sum_0^\infty a^n z^{-(n+1)} + \sum_0^\infty b^{-n} z^n\right) dz.$$

But, we have that, for all integers $k \neq -1$, $\int_\gamma z^k \, dz = 0$.

So all terms in the right series vanish, and all terms in the left series vanish except for $n = 0$. Hence, we are left with $a^0 z^{-(0+1)} = \frac{1}{z}$.

Thus,

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{1}{a-b}\int_\gamma \frac{1}{z}\, dz = \frac{2\pi i}{a-b}.$$

$\square$