Math 134 Homework 6

Alexandre Lipson

November 6, 2023

1

Find the slope of the line through the origin which divides the area under the parabola $y = 4x - x^2$ and above the x-axis into two equal parts.

First, we find the total area A_T under the curve.

The zeros of $y = 4x - x^2$ can be found by factoring; we see that $y = x(4 - x)$ indicates that the parabola has zeros at $x = 0$, and $x = 4$. These will become the bounds of integration.

$$\begin{aligned} A_T &= \int_0^4 4x - x^2 dx \\ &= 2x^2 - \frac{x^3}{3}\bigg|_0^4 \\ &= 32 - \frac{64}{3} \\ &= \frac{32}{3}. \end{aligned}$$

Then, half of the total area, $\frac{A_T}{2} = \frac{16}{3}$, will be the area between the line $y = ax$ of slope a and the parabola.

We find the bounds of integration for this region by first noticing that the origin is a root of both the parabola and the line. So, we will now consider only $x \neq 0$,

$$\begin{aligned} 4x - x^2 &= ax \\ 4 - x &= a \\ x &= 4 - a. \end{aligned}$$

So, we integrate by x between 0 and $4 - a$.

The line must lie underneath the parabola in order to divide the parabola's area in twain.[1] So, the upper function is the parabola and the lower one is the line.

[1] The phrase "in twain" is an archaic way of saying, "in two." It is intended to provide joy to the reader when used here. Its inclusion was not designed to detract from the clarity of the proof.

We integrate to find the slope a.

$$\begin{aligned}
\frac{16}{3} &= \int_0^{4-a} 4x - x^2 - ax \\
&= \int_0^{4-a} (4-a)x - x^2 dx \\
&= \frac{(4-a)x^2}{2} - \frac{x^3}{3}\bigg|_0^{4-a} \\
&= \frac{(4-a)^3}{2} - \frac{(4-a)^3}{3} \\
\frac{16}{3} &= \frac{(4-a)^3}{6} \\
32 &= (4-a)^3 \\
\sqrt[3]{32} &= 4-a \\
a &= 4 - 2\sqrt[3]{4} \approx 0.8252.
\end{aligned}$$

So, the slope of the line that divides the total area under the parabola in half is $4 - 2\sqrt[3]{4}$, or about 0.8252.

5.9

16

Determine whether the assertion is true or false on an arbitrary interval $[a, b]$ on which f and g are continuous.

We choose to denote the average of a function f as $\overline{f}$ instead of f_{avg} as is used in the textbook.

Definition. *By 5.9.1, the average of a function h is defined as,*

$$\int_a^b h(x)dx = \overline{h}(b-a)$$

where $\overline{h} = h(c)$ for some $c \in [a, b]$.

We see that,

$$\frac{1}{b-a}\int_a^b h(x)dx.$$

(a) $\overline{f+g} = \overline{f} + \overline{g}$

Using the definition with $h(x) = f(x) + g(x)$ and 5.4.4, we see that,

$$\frac{1}{b-a}\int_a^b f(x) + g(x)dx = \frac{1}{b-a}\int_a^b f(x)dx + \frac{1}{b-a}\int_a^b g(x)dx,$$

which is $\overline{f} + \overline{g}$ by the definition.

(b) $\overline{\alpha f} = \alpha\overline{f}$

Using the definition and $h(x) = \alpha f(x)$ and 5.4.3, we get that

$$\frac{1}{b-a}\int_a^b \alpha f(x)dx = \alpha\frac{1}{b-a}\int_a^b f(x)dx,$$

which is $\alpha\overline{f}$ by the definition.

(c) $\overline{fg} = \overline{f}\cdot\overline{g}$

With $h(x) = f(x)g(x)$, by the definition,

$$\overline{fg} = \frac{1}{b-a}\int_a^b f(x)g(x)dx.$$

We will show that the statement does not always hold by providing a counterexample.

Let $f(x) = g(x) = x$.

Let interval of integration $[a, b]$ be $[0, 1]$.

So, $\frac{1}{b-a} = 1$.

Then,

$$\overline{fg} = 1\cdot\int_0^1 x\cdot x\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

But,

$$\overline{f}\cdot\overline{g} = \left[\int_0^1 x\,dx\right]^2 = \left[\left.\frac{x^2}{2}\right|_0^1\right]^2 = \left[\frac{1}{2}\right]^2 = \frac{1}{4}.$$

Since, $\frac{1}{3}\neq\frac{1}{4}$, then, for some continuous functions f and g and a closed and bounded interval, $\overline{fg}\neq\overline{f}\cdot\overline{g}$.

So, the assertion is not always true.

(d) $\overline{fg} = \overline{f}/\overline{g}$

Again, we will show that the statement does not always hold by providing a counterexample.

Using the same f and g as in the previous statement, we have that $\overline{fg} = \frac{1}{3}$.

But, with $\overline{f} = \overline{g} = \frac{1}{2}$,

$$\overline{f}/\overline{g} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

Since $\frac{1}{3}\neq 1$, then $\overline{fg}\neq\overline{f}/\overline{g}$ for some continuous functions f and g on a closed and bounded interval.

So, the assertion does not always hold for all continuous functions f and g on an arbitrary closed and bounded interval.

6.1

40

(a) Calculate the area of the region in the first quadrant bounded by the coordinate axes and the parabola $y = 1 + a - ax^2, \quad a > 0$.

(b) Determine the value of a that minimizes this area.

For (a), we start by finding the bounds of integration, noting that $x > 0$ in the first quadrant.

$$
\begin{aligned}
0 &= 1 + a - ax^2 \\
\frac{-(1+a)}{-a} &= x^2 \\
x &= \sqrt{\frac{a+1}{a}}.
\end{aligned}
$$

Since we are in the first quadrant, the lower bound of integration will be 0.

Since the leading coefficient is negative, we know that this is a downward opening parabola. So, the signed area will be positive. We integrate to find the area function $A(a)$.

$$
\begin{aligned}
A(a) &= \int_0^{\sqrt{\frac{a+1}{a}}} 1 + a - ax^2 dx \\
&= \left[(a+1)x - \frac{a}{3}x^3 \right] \Big|_0^{\sqrt{\frac{a+1}{a}}} \\
&= (a+1)\sqrt{\frac{a+1}{a}} - \frac{a}{3}\left(\sqrt{\frac{a+1}{a}}\right)^3 \\
&= \frac{(a+1)\sqrt{a+1}}{\sqrt{a}} - \frac{a}{3}\left(\frac{(a+1)\sqrt{a+1}}{a\sqrt{a}}\right) \\
&= \frac{(a+1)\sqrt{a+1}}{\sqrt{a}} - \frac{1}{3}\left(\frac{(a+1)\sqrt{a+1}}{\sqrt{a}}\right) \\
&= \frac{2}{3}\left(\frac{(a+1)\sqrt{a+1}}{\sqrt{a}}\right) \\
&= \frac{2}{3}(a+1)\sqrt{1 + \frac{1}{a}}.
\end{aligned}
$$

For (b), we optimize $A(a)$.

We differentiate to find the critical number of A.

$$\begin{aligned}
A' &= \frac{d}{da}\left[\frac{2}{3}(a+1)\sqrt{1+\frac{1}{a}}\right] \\
&= \frac{2}{3}\left(\sqrt{1+\frac{1}{a}} + (a+1)\frac{1}{2\sqrt{1+\frac{1}{a}}}\left(\frac{-1}{a^2}\right)\right) \\
&= \frac{2}{3}\left(\sqrt{1+\frac{1}{a}} - \frac{a+1}{2a^2\sqrt{1+\frac{1}{a}}}\right) \\
&= \frac{2}{3}\left(\sqrt{1+\frac{1}{a}} - \frac{1+\frac{1}{a}}{2a\sqrt{1+\frac{1}{a}}}\right) \\
&= \frac{2}{3}\left(\sqrt{1+\frac{1}{a}} - \frac{\sqrt{1+\frac{1}{a}}}{2a}\right) \\
&= \frac{2}{3}\left(\frac{2a-1}{2a}\sqrt{1+\frac{1}{a}}\right) \\
&= \left(\frac{2a-1}{3a}\right)\sqrt{1+\frac{1}{a}}
\end{aligned}$$

Then, we find the critical points of A by setting $A' = 0$.

Since $a > 0$, then $\frac{\sqrt{1+\frac{1}{a}}}{3a} \neq 0$.

So,

$$\begin{aligned}
0 = A' &= (2a-1)\frac{\sqrt{1+\frac{1}{a}}}{3a} \\
0 &= 2a-1 \\
a &= \frac{1}{2}.
\end{aligned}$$

We take the second derivative of A to determine concavity and to classify the critical number $a = \frac{1}{2}$.

$$\begin{aligned}
A'' &= \frac{d}{da}\left[\left(\frac{2a-1}{3a}\right)\sqrt{1+\frac{1}{a}}\right] \\
&= \left[2\left(\frac{1}{3a}\right) + (2a-1)\left(\frac{-1}{6a^2}\right)\right]\sqrt{1+\frac{1}{a}} + \left(\frac{2a-1}{3a}\right)\left[\left(\frac{-1}{a^2}\right)\frac{1}{2\sqrt{1+\frac{1}{a}}}\right] \\
&= \left(\frac{2}{3a} + \frac{1-2a}{6a^2}\right)\sqrt{1+\frac{1}{a}} + \frac{1-2a}{6a^3\sqrt{1+\frac{1}{a}}}
\end{aligned}$$

We evaluate A'' at the critical point $a = \frac{1}{2}$. We note that the second term is immediately reduced since its numerator, $1-2a$ is zero when $a=\frac{1}{2}$. The same is true true for $1-2a$ in the first term.

$$\begin{aligned} A''\left(\frac{1}{2}\right) &= \left(\frac{2}{3\left(\frac{1}{2}\right)}\right)\sqrt{1+\frac{1}{\frac{1}{2}}} \\ &= \frac{4}{3}\sqrt{3} \\ &= \frac{4}{\sqrt{3}}. \end{aligned}$$

Since $\frac{4}{\sqrt{3}}>0$, then $a=\frac{1}{2}$ is a local minimum which minimizes the area under the parabola $y=1+a-ax^2$ in the first quadrant.

6.2

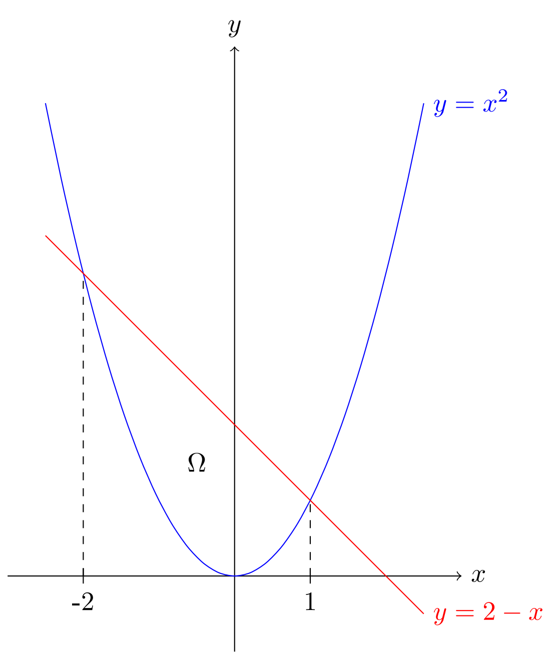
10

Sketch the region Ω bounded by the curves $y=x^2$ and $y=2-x$ and find the volume of the solid generated by revolving this region about the x-axis.

Also, rotate about $y=-1$, $y=4$, and $y=\frac{1}{4}$.

In order sketch the region Ω, we first need to find the intersection points.

$$\begin{aligned} x^2 &= 2-x \\ x^2+x-2 &= 0 \\ (x+2)(x-1) &= 0 \\ x &= -2, 1. \end{aligned}$$



Then, to find the volume of the solid of revolution by the x-axis, we note that the outer function is $y=2-x$ and the inner is $y=x^2$. We integrate with the bounds given by the intersection of the curves above.

$$\begin{aligned}
V_0 &= \pi \int_{-2}^{1} (2-x)^2 - (x^2)^2 dx \\
&= \pi \int_{-2}^{1} 4 - 4x + x^2 - x^4 dx \\
&= \pi \left[4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5} \right] \Bigg|_{-2}^{1} \\
&= \frac{\pi}{15} \left[60x - 30x^2 + 5x^3 - 3x^5 \right] \Big|_{-2}^{1} \\
&= \frac{\pi}{15} \left[(60 - 30 + 5 - 3) - (-120 - 120 - 40 + 96) \right] \\
&= \frac{\pi}{15} \left[32 - (-184) \right] \\
&= \frac{216\pi}{15} \\
&= \frac{72\pi}{5}.
\end{aligned}$$

For rotating about $y = -1$, we take each of the functions shifted up by one unit, $y = 2 - x + 1$ and $y = x^2 + 1$. The line is the outer function and the parabola is the inner function.

$$\begin{aligned}
V_{-1} &= \pi \int_{-2}^{1} (2 - x + 1)^2 - (x^2 + 1)^2 dx \\
&= \pi \int_{-2}^{1} 9 - 6x + x^2 - x^4 - 2x^2 - 1 dx \\
&= \pi \int_{-2}^{1} -x^4 - x^2 - 6x + 8 dx \\
&= \pi \left[\frac{-x^5}{5} - \frac{x^3}{3} - 3x^2 + 5x \right] \Bigg|_{-2}^{1} \\
&= \pi \left(\left(\frac{-1}{5} - \frac{1}{3} - 3 + 8 \right) - \left(\frac{32}{5} + \frac{8}{3} - 12 - 16 \right) \right) \\
&= \pi \left(\frac{-33}{5} - \frac{9}{3} + 33 \right) \\
&= \pi \left(\frac{150}{5} - \frac{33}{5} \right) \\
&= \frac{117\pi}{5}.
\end{aligned}$$

When rotating about $y = 4$, we note that the functions are shifted downward by four, and that the parabola is the outer function while the line is the inner function.

$$
\begin{aligned}
V_4 &= \pi \int_{-2}^{1} \left(x^2-4\right)^2 - (-x-2)^2 dx \\
&= \pi \int_{-2}^{1} \left(x^4-8x^2+16\right) - \left(x^2+4x+4\right) dx \\
&= \pi \int_{-2}^{1} x^4-9x^2-4x+12 dx \\
&= \pi \left[\frac{x^5}{5} - 3x^3 - 2x^2 + 12x\right]\Bigg|_{-2}^{1} \\
&= \pi \left(\left(\frac{1}{5} - 3 - 2 + 12\right) - \left(\frac{-32}{5} + 24 - 8 - 24\right)\right) \\
&= \pi \left(\frac{33}{5} + 15\right) \\
&= \frac{108\pi}{5}.
\end{aligned}
$$

Now, when rotating about $y = \frac{1}{4}$, we notice that the axis of rotation intersects the region. Thus, the part of the region underneath the axis will have a negative signed area. If we integrate the region as a solid of rotation, we will obtain a cavity in the final volume. To remedy this, we simply add back a positive portion of the solid of rotation obtained by the part of the region that is below the axis.

The axis $y = \frac{1}{4}$ will only intersect region when it is defined by the parabola. The roots of the parabola are found by $x^2 - \frac{1}{4} = 0$, which are $x = \pm\frac{1}{2}$.

The root of the line with the axis is $2 - x - \frac{1}{4} = 0$, which is $x = \frac{9}{4}$. This x value lies outside the x bounds of the region Ω, so we know that $x = -\frac{1}{2}$, and $x = \frac{1}{2}$ are the bounds of the cavity.

So, $y = 2 - x - \frac{1}{4} = \frac{7}{4} - x$ is the outer function and $y = x^2 - \frac{1}{4}$ is the inner function.

Thus, our final volume integral is given by the solid of revolution with a cavity combined with the volume to fill the cavity,

$$
V_{\frac{1}{4}} = \pi \int_{-2}^{1} \left(\frac{7}{4} - x\right)^2 - \left(x^2 - \frac{1}{4}\right)^2 dx + \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(x^2 - \frac{1}{4}\right)^2 dx.
$$

44

A hemispherical punch bowl 2 feet in diameter is filled to within 1 inch of the top. Thirty minutes after the party starts, there are only 2 inches of punch left at the bottom of the bowl.

(a) How much punch was there at the beginning?

(b) How much punch was consumed?

To find the volume of a sphere of radius 12, we will consider the solid of rotation of a circle.

We center the circle at the origin, and thus our punch bowl is defined by the curve, $y = -\sqrt{12^2 - y^2} = -\sqrt{144 - y^2}$, where y is below the x-axis to represent the bowl.

For (a), we integrate up to one inch below the x-axis, which is the top of the bowl.

With the interval $y = -12$, the bottom of the bowl, given by the radius, to $y = -1$, we integrate a solid of rotation to determine the initial volume of punch.

$$\begin{aligned}
V_i &= \pi \int_{-12}^{-1} \sqrt{144 - y^2}^2 dy \\
&= \pi \int_{-12}^{-1} 144 - y^2 dy \\
&= \pi \left[144y - \frac{y^3}{3} \right] \Bigg|_{-12}^{-1} \\
&= \pi \left(\left(-144 + \frac{1}{3} \right) - \left(-1728 - \frac{-1728}{3} \right) \right) \\
&= \pi \left(-\frac{431}{3} - \left(-\frac{3456}{3} \right) \right) \\
&= \frac{3025\pi}{3}.
\end{aligned}$$

Similarly, for the final volume of punch, the interval is $y \in [-12, -10]$, where only two inches of punch remain in the bowl, so $-12 + 2 = -10$.

$$\begin{aligned}
V_f &= \pi \int_{-12}^{-10} \sqrt{144 - y^2}^2 dy \\
&= \pi \int_{-12}^{-10} 144 - y^2 dy \\
&= \pi \left[144y - \frac{y^3}{3} \Bigg|_{-12}^{-10} \right] \\
&= \pi \left(\left(-1440 + \frac{1000}{3} \right) - \left(-1728 - \frac{-1728}{3} \right) \right) \\
&= \pi \left(\left(-\frac{3320}{3} \right) - \left(-\frac{3456}{3} \right) \right) \\
&= \pi \left(\frac{136}{3} \right) \\
&= \frac{136\pi}{3}.
\end{aligned}$$

So, for (b), the volume of punch consumed, being the difference between the initial volume and the final volume, is,

$$\frac{3025\pi}{3} - \frac{136\pi}{3} = \frac{2889\pi}{3} = 963\pi,$$

or about 3025 cubic inches of punch.

This volume, is roughly equivalent to 13.1 gallons or about 1676 fluid ounces.

Considering that one drink is about 12 ounces, then a total of $\frac{1676}{12} \approx 140$ drinks were consumed in 30 minutes.

Assuming that the number of persons who didn't drink at all is equivalent to the number of persons who had two drinks within 30 minutes, we guess that the attendance of this party was roughly 140 persons.

6.3

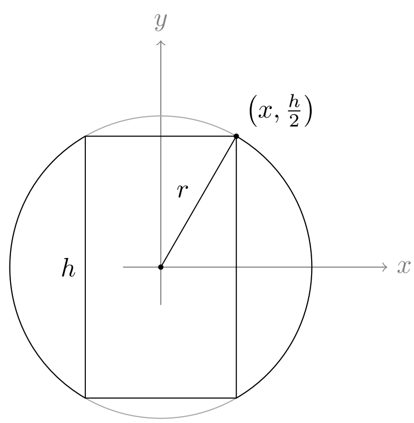
46

A hole is drilled through the center of a ball of radius r, leaving a solid with a hollow cylindrical core of height h. Show that the volume of this solid is independent of the radius of the ball.

This volume is known as a napkin ring.

We will consider both r, h to be non-negative.

We take a cross section of the ball to examine the relationship between r and h.



We see that,

$$r^2 = x^2 + \left(\frac{h}{2}\right)^2.$$

From this relationship, we can see that the height of the cylinder h is,

$$h = 2\sqrt{r^2 - x^2}.$$

Likewise, we also see that the radius of the cylinder, equal to x, which we will call R, is

$$R = \sqrt{r^2 - \frac{h^2}{4}}.$$

We will form the volume of the napkin ring by a solid of revolution from the portion of the ball to the right of the y-axis on positive x. Thus the bounds of integration are $x \in [R, r]$.

We integrate using the cylindrical shells, noting that the height of each shell is the total height of the ball, or $h = 2\sqrt{r^2 - x^2}$, and the circumference is $2\pi x$.

We will also use a substitution with the function $u(x) = r^2 - x^2$. We see that $du = -2xdx$, $u(r) =$

$r^2 - r^2 = 0$, and $u(R) = r^2 - \sqrt{r^2 - \frac{h^2}{4}}^2 = \frac{h^2}{4}$.

$$\begin{aligned}
V &= \int_R^r 2\pi x \left(2\sqrt{r^2 - x^2}\right) dx \\
&= 4\pi \int_R^r x\sqrt{r^2 - x^2} dx \\
&= \frac{4\pi}{-2} \int_R^r -2x\sqrt{r^2 - x^2} dx \\
&= -2\pi \int_{u(R)}^{u(r)} \sqrt{u}\, du \\
&= -2\pi \left[\frac{2}{3} u^{\frac{3}{2}}\right]\Bigg|_{\frac{h^2}{4}}^{0}.
\end{aligned}$$

We can immediately see that the volume will not depend on r without further computation.

Still,

$$V = -2\pi \left(-\frac{2}{3}\left(\frac{h^2}{4}\right)^{\frac{3}{2}}\right) = \frac{4\pi}{3}\left(\frac{h^3}{8}\right) = \frac{h^3\pi}{6}.$$

Thus, the volume of the napkin ring only depends on the height of the cylindrical core.