Math 335 Homework 5

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For Problems 1, 2, and 3, find the Fourier series of the 2π-periodic function $f(x)$ on $(-\pi, \pi)$.

Problem 1. (i) The square wave $f(x) = \begin{cases} -1 & (-\pi, 0), \\ 1 & (0, \pi). \end{cases}$

(ii) $f(x) = \sin^2 x$.

Proof of (i). We have that f is an odd function, so $a_n = 0$ for all n.

We will proceed to find the b_n Fourier coefficients,

$$
\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx && = \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx - \frac{1}{\pi} \int_{-\pi}^{0} \sin nx \, dx \\
&= \frac{2}{\pi} \int_{0}^{\pi} \sin nx \, dx \\
&= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_{0}^{\pi} \\
&= \frac{2}{\pi} \left(\frac{1 - \cos n\pi}{n} \right),
\end{aligned}
$$

which vanishes for all even n, and is $\frac{4}{n\pi}$ for all odd n.

Thus, we have that, for n odd,
$$
f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin nx}{n},
$$
which is
$$
\sum_{k=0}^{\infty} \frac{4 \sin (2k+1)x}{\pi(2k+1)}.
$$
$\square$

Proof of (ii). We have that
$$
f(x) = \sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2} \cos 2x.
$$
$\square$

Problem 2. (i) $f(x) = e^{bx}$, $b > 0$.

(ii) $f(x) = x(\pi - |x|)$.

Proof of (i). We will consider the complex Fourier coefficient,

$$
\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(b-in)x}\, dx \\
&= \frac{1}{\pi} \left[\frac{e^{(b-in)x}}{b-in} \right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi(b-in)} \left(e^{(b-in)\pi} - e^{-(b-in)\pi} \right)
\end{aligned}
$$

;

by Euler's identity, at $x = \pi$, we have that $e^{\pm in\pi} = (-1)^n$. So, with the hyperbolic sine identity $2\sinh b\pi = e^{b\pi} - e^{-b\pi}$, the above becomes,

$$
\frac{(-1)^n \sinh b\pi}{\pi(b-in)}.
$$

Thus, we have

$$
f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{\sinh b\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n}{b-in} e^{inx}.
$$

$\square$

Proof of (ii). Since f is an odd function, then $a_n = 0$ for all n.

So,

$$
\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x(\pi - |x|) \sin nx\, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{0} x(\pi - (-x)) \sin nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} x(\pi - x) \sin nx\, dx \\
&= \frac{1}{\pi} \int_{\pi}^{0} (-u)(\pi - u) \sin(-nu)(-1)\, du + \frac{1}{\pi} \int_{0}^{\pi} u(\pi - u) \sin nu\, du \\
&= \frac{1}{\pi} \int_{0}^{\pi} u(\pi - u)(\sin nu - \sin(-nu))\, du \\
&= \frac{2}{\pi} \int_{0}^{\pi} x(\pi - x) \sin nx\, dx \\
&= 2 \int_{0}^{\pi} x \sin nx\, dx - \frac{2}{\pi} \int_{0}^{\pi} x^2 \sin nx\, dx \\
&= 2 \left(\frac{-x \cos nx}{n} \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n}\, dx \right) - \frac{2}{\pi} \left(\frac{-x^2 \cos nx}{n} \Big|_0^{\pi} + \int_0^{\pi} \frac{2x \cos nx}{n}\, dx \right) \\
&= 2 \left(\frac{-\pi \cos n\pi}{n} + \left[\frac{\sin nx}{n} \right]_0^{\pi} \right) - \frac{2}{\pi} \left(\frac{-\pi^2 \cos n\pi}{n} + 2 \left(\frac{x \sin nx}{n^2} \Big|_0^{\pi} + \int_0^{\pi} \frac{\sin nx}{n^2}\, dx \right) \right) \\
&= 2 \left(\frac{\sin n\pi}{n^2} - \frac{\pi \cos n\pi}{n} \right) + \left(\frac{2\pi \cos n\pi}{n} - \frac{4}{\pi} \left(\frac{\pi \sin n\pi}{n^2} + \left[\frac{\cos nx}{n^3} \right]_0^{\pi} \right) \right) \\
&= \frac{2 \sin n\pi}{n^2} - \frac{2\pi \cos n\pi}{n} + \frac{2\pi \cos n\pi}{n} - \frac{4 \sin n\pi}{n^2} - \frac{4}{\pi} \left(\frac{\cos n\pi - 1}{n^3} \right) \\
&= -\frac{2 \sin n\pi}{n^2} - \frac{4}{\pi} \left(\frac{\cos n\pi - 1}{n^3} \right).
\end{aligned}
$$

This quantity vanishes for even n, and $b_n = \frac{8}{\pi n^3}$ for odd n.

So, for odd n,

$$f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^3},$$

which is

$$\frac{8}{\pi} \sum_{0}^{\infty} \frac{\sin(2k+1)x}{(2k+1)^3}.$$

$\square$

Problem 3. (i) $f(x) = \begin{cases} \frac{1}{a} & |x| < a, \\ -\frac{1}{\pi - a} & a < |x| < \pi, \end{cases}$ for $a \in (0, \pi)$.

The values of f are chosen to make the area under the curve of f on $[0, a]$ and $[a, \pi]$ both equal to one.

(ii) $f(x) = \begin{cases} \frac{a - |x|}{a^2} & |x| < a, \\ 0 & a < |x| < \pi, \end{cases}$ for $a \in (0, \pi)$.

The constraints on f are chosen such that the area under the triangle in the graph of f is equal to one.

Proof of (i). Since f is an even function, then $b_n = 0$ for all n.

Since the graph of f has equal positive and negative areas between the x-axis (both one) between 0 and π, and f is an even function, then

$$\frac{1}{2} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0.$$

Next,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{-a} -\frac{1}{\pi - a} \cos nx \, dx + \frac{1}{\pi} \int_{a}^{\pi} -\frac{1}{\pi - a} \cos nx \, dx + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{a} \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{\pi}^{a} -\frac{1}{\pi - a} \cos(-nx)(-1) \, dx + \frac{1}{\pi} \int_{a}^{\pi} -\frac{1}{\pi - a} \cos nx \, dx + \frac{1}{a\pi} \left[\frac{\sin nx}{n} \right]_{-a}^{a}$$

$$= \frac{1}{\pi} \int_{a}^{\pi} -\frac{1}{\pi - a} (\cos nx + \cos nx) \, dx + \frac{1}{a\pi n} (\sin na - \sin(-na))$$

$$= -\frac{2}{\pi(\pi - a)} \left[\frac{\sin nx}{n} \right]_{a}^{\pi} + \frac{2 \sin na}{a\pi n}$$

$$= -\frac{2}{\pi(\pi - a)} \left(\frac{\sin n\pi - \sin na}{n} \right) + \frac{2 \sin na}{a\pi n}$$

$$= \frac{2 \sin na}{\pi(\pi - a)n} + \frac{2 \sin na}{a\pi n}$$

$$= \frac{2a \sin na + 2(\pi - a) \sin na}{a\pi(\pi - a)n}$$

$$= \frac{(2a + 2\pi - 2a) \sin na}{a\pi(\pi - a)n}$$

$$= \frac{2 \sin na}{a(\pi - a)n}.$$

Thus,

$$f(x) = \frac{2}{a(\pi - a)} \sum_{n=1}^{\infty} \frac{\sin na}{n} \cos nx.$$

$\square$

Proof of (ii). Since f is even, then $b_n = 0$ for all n.

Since the area under the curve of the graph of f on $(-\pi, \pi)$ is one, then

$$\frac{1}{2} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi}.$$

Next,

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
&= \frac{1}{\pi} \int_{-a}^{a} \frac{a - |x|}{a^2} \cos nx \, dx \\
&= \frac{2}{\pi} \int_{0}^{a} \frac{a - x}{a^2} \cos nx \, dx \\
&= \frac{2}{\pi} \int_{0}^{a} \left(\frac{\cos nx}{a} - \frac{x \cos nx}{a^2} \right) dx \\
&= \frac{2}{\pi} \left[\frac{\sin nx}{an} \Big|_0^a - \left(\frac{x \sin nx}{a^2 n} \Big|_0^a - \int_0^a \frac{\sin nx}{a^2 n} \, dx \right) \right] \\
&= \frac{2}{\pi} \left[\frac{\sin na}{an} - \frac{a \sin na}{a^2 n} - \left[\frac{\cos nx}{a^2 n^2} \right]_0^a \right] \\
&= \frac{2}{a^2 \pi} \left(\frac{1 - \cos na}{n^2} \right).
\end{aligned}$$

Thus,

$$f(x) = \frac{1}{2\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos na}{a^2 n^2} \cos nx.$$

$\square$

Problem 4. Let the Fourier coefficients of f be a_n and b_n. Find the Fourier coefficients A_n and B_n of $g(x) = f(x) \sin x$.

Proof. We have that,

$$f(x) \sin x = \left(\frac{1}{a} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right) \sin x = \frac{\sin x}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx \sin x + b_n \sin nx \sin x).$$

By the product identities, the above series becomes

$$\frac{1}{2} \sum_{n=1}^{\infty} (a_n (\sin (n+1)x - \sin (n-1)x) + b_n (\cos (n-1)x - \cos (n+1)x)).$$

Considering the Fourier series of $g(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$ and collecting the sine and cosine terms, we have that

$$A_n = \frac{1}{2}(b_{n-1} - b_{n+1}) \text{ and } B_n = \frac{1}{2}(a_{n+1} - a_{n-1}),$$

which are the Fourier coefficients of $f(x)\sin x$. $\qquad\square$

Problem 5. Let f be 2π-periodic and monotonously decreasing on $(0, 2\pi)$. Prove that the Fourier coefficient $b_n \geq 0$ for all n.

Proof. Note that $\sin n(2\pi - x) = -\sin nx$. We will change the region of integration according to this identity.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\left(\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right)\\
&= \frac{1}{\pi}\left(\int_0^{\pi} f(x)\sin nx\,dx + \int_0^{\pi} f(2\pi - x)\sin n(2\pi - x)\,dx\right)\\
&= b_n\frac{1}{\pi}\int_0^{\pi}(f(x) - f(2\pi - x))\sin nx\,dx
\end{aligned}$$

.

Since f is monotonously decreasing on $(0, 2\pi)$, then, for all $x \in (0, \pi)$, we have that $x \leq 2\pi - x \implies f(x) \geq f(2\pi - x)$, or $f(x) - f(2\pi - x) \geq 0$.

Then, with $\sin nx \geq 0$ for all $x \in (0, \pi)$ and $n \geq 0$, we have that

$$b_n = \frac{1}{\pi}\int_0^{\pi}(f(x) - f(2\pi - x))\sin nx\,dx \geq 0 \implies b_n \geq 0.$$

$\qquad\square$

Problem 6. Let $f \in C^2([-\pi, \pi])$, $f(-\pi) = f(\pi)$, and $f'(-\pi) = f'(\pi)$. Prove $c_n\ O(n^{-2})$ as $n \to \infty$.

Proof. We can perform IBP twice on c_n because $f \in C^2$.

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\\
&= \frac{1}{2\pi}\left(f(x)\left(\frac{-e^{-inx}}{in}\right)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{in}\,dx\right)
\end{aligned}$$

the left summand vanishes because $f(\pi) - f(-\pi) = 0$

$$= \frac{1}{2\pi}\left(f'(x)\left(\frac{e^{-inx}}{n^2}\right)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f''(x)\frac{e^{-inx}}{n^2}\,dx\right)$$

the left summand vanishes because $f'(\pi) - f'(-\pi) = 0$

$$= \frac{1}{2\pi n^2}\int_{-\pi}^{\pi} f''(x)e^{-inx}\,dx \sim O(n^{-2}).$$

Thus, $c_n \sim O(n^{-2})$. $\qquad\square$

Problem 7. Find the following limits.

(i) $\lim_{\lambda \to \infty} \int_0^\infty \frac{\cos^2 \lambda x}{1+x^2}\, dx.$

(ii) $\lim_{\lambda \to \infty} \int_{-\pi}^\pi \sin^2 \lambda x\, dx.$

Proof of (i). We have that $\cos^2 \lambda x = \frac{1+\cos 2\lambda x}{2}$. So,

$$\lim_{\lambda \to \infty} \int_0^\infty \frac{1+\cos 2\lambda x}{2(1+x^2)}\, dx = \frac{1}{2} \lim_{\lambda \to \infty} \int_0^\infty \left(\frac{1}{1+x^2} + \frac{\cos 2\lambda x}{1+x^2} \right) dx.$$

since $\frac{1}{1+x^2}$ is integrable over $(0, \infty)$, then the right summand vanishes by the Reimann-Lebesgue Lemma.

So, we have that the above limit becomes

$$\frac{1}{2} \int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2} \arctan x \Big|_0^\infty = \frac{\pi}{4}.$$

$\square$

Proof of (ii). We have that $\sin^2 \lambda x = \frac{1-\cos 2\lambda x}{2}$. So,

$$\frac{1}{2} \lim_{\lambda \to \infty} \int_{-\pi}^\pi (1 - \cos 2\lambda x)\, dx = \pi - \frac{1}{2} \lim_{\lambda \to \infty} \int_{-\pi}^\pi \cos 2\lambda x\, dx.$$

Since the unit function 1 is also integrable on $[-\pi, \pi]$, then the right summand vanishes by the Reimann-Lebesgue Lemma.

Hence, the limit in (ii) is π.

$\square$

Problem 8.   (i) Prove
$$\sum_{k=0}^{n-1} \sin \left(k + \frac{1}{2} \right) x = \frac{\sin^2 \left(\frac{nx}{2} \right)}{\sin \left(\frac{x}{2} \right)}, \quad \forall n \geq 1.$$

(ii) Use (i) to show
$$\int_0^\pi \frac{\sin^2 \left(\frac{nx}{2} \right)}{\sin \left(\frac{x}{2} \right)}\, dx = \pi, \quad \forall n \geq 1.$$

Proof of (i). We will induct on n.

For $n = 1$,
$$\sum_{k=0}^{1-1=0} \sin \left(\frac{2k+1}{2} x \right) = \sin \frac{x}{2} = \frac{\sin^2 \frac{x}{2}}{\sin \frac{x}{2}}.$$

Assume that (i) holds for n. Now, for $n + 1$,
$$\sum_{k=0}^n \sin \frac{2k+1}{2} x = \frac{\sin^2 \frac{nx}{2}}{\sin \frac{x}{2}} + \sin \frac{2n+1}{2} x.$$

Consider the right summand; using the product of sines,

$$\frac{\sin\left(\frac{2n+1}{2}\right)x \cdot \sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{1}{2\sin\frac{x}{2}}\left(\cos\left(\frac{2n+1}{2}-\frac{1}{2}\right)x - \cos\left(\frac{2n+1}{2}+\frac{1}{2}\right)x\right)$$

$$= \frac{1}{2\sin\frac{x}{2}}\left(\cos nx - \cos(n+1)x\right).$$

Then, for left summand, using square sine identity,

$$\frac{\sin^2\frac{nx}{2}}{\sin\frac{x}{2}} = \frac{1-\cos nx}{2\sin\frac{x}{2}}.$$

So, the above sum in the inductive step simplifies to,

$$\frac{1-\cos(n+1)x}{2\sin\frac{x}{2}} = \frac{\sin^2\left(\frac{n+1}{2}\right)x}{\sin\frac{x}{2}},$$

which is our desired result with $n+1$. $\qquad\square$

Proof of (ii). We will induct on n.

For $n=1$,

$$\int_0^\pi \frac{\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx = \int_0^\pi dx = 1\cdot\pi.$$

Assume (ii) holds for n. Now, for $n+1$, by part (i),

$$\int_0^\pi \frac{\sin^2\left(\frac{n+1}{2}\right)x}{\sin^2\frac{x}{2}}\,dx = \int_0^\pi \sum_{k=0}^n \frac{\sin\left(k+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx = \sum_{k=0}^n \int_0^\pi \frac{\sin\left(k+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx.$$

Here, we have the kth Dirichlet kernel $D_k(x)$, where

$$D_k(x) = \frac{\sin\left(k+\frac{1}{2}\right)x}{2\pi\sin\frac{x}{2}}, \qquad \int_0^\pi D_k(x)\,dx = \frac{1}{2}.$$

So, the above is

$$\sum_{k=0}^n \int_0^\pi 2\pi D_k(x)\,dx = \sum_0^n 2\pi\cdot\frac{1}{2} = n\pi.$$

$\qquad\square$

Problem 9. Let f be Reimann integrable on $[a,b]$. Let g be a continuous T-periodic function on $\mathbb{R}$. Prove

$$\lim_{\lambda\to\infty}\int_a^b f(x)g(\lambda x)\,dx = \frac{1}{T}\int_0^T g(x)\,dx\int_a^b f(x)\,dx.$$

Proof. Since f is Reimann integrable on $[a,b]$, then, given any $\epsilon>0$, there exists a step function h such that

$$\int_a^b |f(x)-h(x)|\,dx < \epsilon.$$

So, we have that

$$\int_a^b f(x)g(\lambda x)\,dx = \int_a^b h(x)g(\lambda x)\,dx + \int_a^b (f(x) - h(x))g(\lambda x)\,dx. \qquad (*)$$

Since h is a step function, then we can write it as

$$h(x) = \sum_{i=0}^n c_i \chi_{[x_i, x_{i+1}]}(x),$$

where $\chi_{[x_i, x_{i+1}]}(x)$ is the indicator function of the interval $[x_i, x_{i+1}]$.
Note that $\int_a^b h(x)\,dx = \sum_{i=0}^n c_i(x_{i+1} - x_i)$.

Then, the left summand of $(*)$ becomes,

$$\int_a^b h(x)g(\lambda x)\,dx = \sum_{i=0}^n c_i \int_{x_i}^{x_{i+1}} g(\lambda x)\,dx.$$

Since g is continuous and T-periodic, then the integral of g over large intervals approaches the average over T as λ grows large.

$$\int_{x_i}^{x_{i+1}} g(\lambda x)\,dx = \frac{1}{\lambda} \int_{\lambda x_i}^{\lambda x_{i+1}} g(u)\,du \approx \frac{x_{i+1} - x_i}{T} \int_0^T g(u)\,du.$$

Thus, summing over i and for λ large,

$$\int_a^b h(x)g(\lambda x)\,dx \approx \frac{1}{T} \int_0^T g(x)\,dx \sum_{i=0}^n c_i(x_{i+1} - x_i) \to \frac{1}{T} \int_0^T g(x)\,dx \int_a^b h(x)\,dx.$$

Since g is continuous and T-periodic, then it is bounded, and M. $|g(\lambda x)| \leq$. Hence,

$$\left| \int_a^b (f(x) - h(x))g(\lambda x)\,dx \right| \leq M \int_a^b |f(x) - h(x)|\,dx < M\epsilon.$$

Since ϵ was arbitrary, then the right summand of $(*)$ vanishes.

Thus,

$$\lim_{\lambda \to \infty} \int_a^b f(x)g(\lambda x)\,dx = \frac{1}{T} \int_0^T g(x)\,dx \int_a^b h(x)\,dx.$$

$\square$