

# Math 336 Homework 2

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**Problem 1.** Prove that

$$\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{\sqrt{2\pi}}{4}.$$

Integrate  $e^{-z^2}$  over the path given in the figure. Recall that  $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$ .

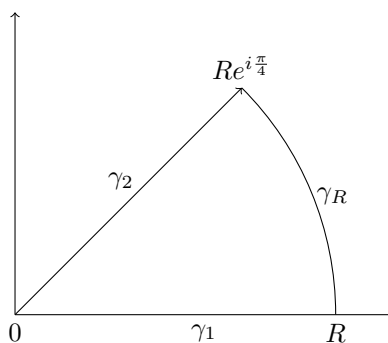


Figure 1. Section contour; consider  $\int_0^\infty$  as  $\lim_{R \rightarrow \infty} \int_R^\infty$ .

*Proof.* Since  $f(z) = e^{-z^2}$  is a composition of holomorphic functions on  $\mathbb{C}$ , then  $f$  is holomorphic on  $\mathbb{C}$ .

Let  $\gamma = \gamma_1 \cup \gamma_R \cup \gamma_2$  be the closed loop section contour in the figure. Since  $f$  is holomorphic, then it has the closed loop property and  $\oint_\gamma f dz = 0$ .

So, we have that

$$0 = \oint_\gamma f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz + \int_{\gamma_R} f dz.$$

But, as  $R \rightarrow \infty$ ,  $\gamma_1$  parametrizes the positive real line, so

$$\int_{\gamma_1} e^{-z^2} dz = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

because  $e^{-x^2}$  is an even function.

We will show that the integral of  $f$  over the arc  $\gamma_R$  goes to zero. Let  $z = re^{i\theta}$ . On this curve,  $r$  is fixed at  $R$  and  $\theta$  varies from 0 to  $\frac{\pi}{4}$ .

So,  $-z^2 = -R^2 e^{2i\theta}$  and  $dz = iRe^{i\theta}$ . Then,

$$\int_0^{\frac{\pi}{4}} iRe^{i\theta} e^{-R^2 e^{2i\theta}} d\theta = \int_0^{\frac{\pi}{4}} iRe^{i\theta - R^2 e^{2i\theta}} d\theta.$$

For  $\theta \in [0, \frac{\pi}{4}]$ , we have that  $e^{2i\theta} \in [e^0, e^{i\frac{\pi}{2}}] = [1, i]$ .

So, as  $R \rightarrow \infty$ , we have,

$$\forall \theta \in \left[0, \frac{\pi}{4}\right), \quad \left|Re^{i\theta - R^2 e^{2i\theta}}\right| \rightarrow 0 \quad \text{and} \quad \theta = \frac{\pi}{4}, \quad \left|Re^{i(\theta - R^2)}\right| \rightarrow \infty.$$

However, since the integrand only blows up at a simple point, which has measure zero, then the contribution of this point to the overall integral is zero. Hence

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} f dz = 0.$$

So, maintaining the orientation of each part of the curve  $\gamma$ , we have that

$$\int_{\gamma_1} f dz = - \int_{\gamma_2} f dz.$$

Now, for the integral on the path  $\gamma_2$ , we fix  $\theta = \frac{\pi}{4}$  and vary  $r$  from 0 to  $R$ . Note that this parameterization traverses  $\gamma_2$  in the opposite direction of the positively oriented curve  $\gamma$ , so we will consider the opposite of the resulting integral.

So,  $-z^2 = -r^2 e^{2i\theta} = -r^2 e^{i\frac{\pi}{2}} = -ir^2$  and  $dz = e^{i\frac{\pi}{4}} dr = \frac{1}{\sqrt{2}}(1+i)dr$ .

Then, with the Euler identity, we have that

$$- \int_{\gamma_2} f dz = \frac{1}{\sqrt{2}}(1+i) \int_0^\infty e^{-ir^2} dr = \frac{1}{\sqrt{2}}(1+i) \int_0^\infty (\cos r^2 - i \sin r^2) dr.$$

With the above equality between the integrals of  $\gamma_1$  and  $\gamma_2$ , we have

$$\begin{aligned} \frac{\sqrt{\pi}}{2} &= \frac{1}{\sqrt{2}}(1+i) \int_0^\infty (\cos r^2 - i \sin r^2) dr \\ \sqrt{\frac{\pi}{2}} &= \int_0^\infty \cos x^2 dx + \int_0^\infty \sin x^2 dx + i \left( \int_0^\infty \cos x^2 dx - \int_0^\infty \sin x^2 dx \right) \end{aligned}$$

Considering the real and imaginary parts separately, we have

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx$$

and therefore also

$$\frac{\sqrt{2\pi}}{4} = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx$$

as desired. □

**Problem 2.** Suppose  $f : \mathbb{D} \rightarrow \mathbb{C}$  is holomorphic. Show that the diameter of the image of  $f$ ,  $d = \sup_{z, w \in \mathbb{D}} |f(z) - f(w)|$  satisfies

$$2|f'(0)| \leq d,$$

and that equality holds when  $f$  is linear.

*Proof.* By the Cauchy integral formula, we have

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - z)^2} d\omega$$

for some circle  $C \subset \mathbb{D}$  centered at  $z$ .

Then, at  $z = 0$ , we have

$$f'(0) = \frac{1}{2\pi i} \int_{|\omega|=r} \frac{f(\omega)}{\omega^2} d\omega, \quad \forall r \in (0, 1),$$

which we can also substitute  $\omega \mapsto -\omega$  to achieve the opposite of the above.

Note that  $|f(\omega) - f(-\omega)| \leq d$ .

Hence,  $\forall r \in (0, 1)$ ,

$$\begin{aligned} 2f'(0) &= \frac{1}{2\pi i} \int_{|\omega|=r} \frac{f(\omega) - f(-\omega)}{\omega^2} d\omega \\ 2|f'(0)| &= \left| \frac{1}{2\pi i} \int_{|\omega|=r} \frac{f(\omega) - f(-\omega)}{\omega^2} d\omega \right| \\ &\leq \frac{1}{2\pi} \int_{|\omega|=r} \frac{d}{|\omega|^2} d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{r^2} r d\theta \\ &= \frac{d}{2\pi} \int_0^{2\pi} \frac{1}{r} d\theta \\ &= \frac{d}{r}. \end{aligned}$$

Since the above holds for all  $r \in (0, 1)$ , then, considering the limit,

$$\lim_{r \rightarrow 1^-} \frac{d}{r} = d \implies 2|f'(0)| \leq d,$$

as desired.

If  $f$  is linear, then we can write  $f = a + bz$ .

Then,  $|f(\omega) - f(-\omega)| = 2|b|$ .

But, we also have that the diameter of the image  $d$  is given by

$$|f(z) - f(\omega)| = |(a + bz) - (a + b\omega)| = |b||z - \omega|$$

So, we have that

$$\sup_{z, \omega \in \mathbb{D}} |f(z) - f(\omega)| = |b| \sup_{z, \omega \in \mathbb{D}} |z - \omega| = 2|b|.$$

Thus, we have equality when proceeding as above.  $\square$

**Problem 3.** Let  $\Omega \subset \mathbb{C}$  be bounded and open, and  $\varphi : \Omega \rightarrow \mathbb{C}$  a holomorphic function. Prove that if there exists  $z_0 \in \Omega$  such that  $\varphi(z_0) = z_0$  and  $\varphi'(z_0) = 1$ , then  $\varphi$  is linear.

*Proof.* Consider  $f(z) = \varphi(z + z_0) - z_0$ . If  $z_0 = 0$ , then  $f(z) = \varphi(z)$ . Otherwise,  $f(0) = \varphi(z_0) - z_0 = 0$ . So, WLOG, assume  $z_0 = 0$ .

For a contradiction, assume  $\varphi$  is not linear, so we can write it as

$$\varphi(z) = z + a_n z^n + O(z^{n+1})$$

for  $n \geq 2$  and  $a_n \neq 0$ .

Let  $\varphi_k$  be the composition of  $\varphi$  with itself  $k$  times. We will show that  $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$  by induction.

For the base case with  $k = 1$ , we have that

$$\varphi(z) = z + a_n z^n + O(z^{n+1}) = \varphi_1(z).$$

Assume the result holds for  $k$  and consider  $k + 1$ ,

$$\begin{aligned} \varphi_{k+1}(z) &= \varphi(\varphi_k(z)) \\ &= \varphi_k(z) + k a_n (\varphi_k(z))^n + O((\varphi_k(z))^{n+1}) \\ &= z + a_n z^n + O(z^{n+1}) + k a_n (z + a_n z^n + O(z^{n+1})) + O((z + a_n z^n + O(z^{n+1}))^{n+1}). \end{aligned}$$

By the binomial expansion, we have that

$$(z + a_n z^n + O(z^{n+1}))^n = z^n + O(z^{n+1}).$$

Furthermore, we also can simplify the following,

$$O((z + a_n z^n + O(z^{n+1}))^{n+1}) = O(z^{n+1}).$$

Therefore,

$$\varphi_{k+1}(z) = z + a_n z^n + k a_n (z^n + O(z^{n+1})) + O(z^{n+1}) = z + (k + 1) a_n z^n + O(z^{n+1}),$$

as desired.

Since  $\Omega$  is bounded, then there exists an  $R > 0$  such that, for all  $z \in \Omega$ ,  $|z| < R$ . Since  $\varphi_k(z)$  maps between  $\Omega$ , then for all  $k$  and  $z \in \Omega$ ,

$$\sup_{C_R} |\varphi_k(z)| \leq R.$$

Then, with Cauchy's inequality,

$$|\varphi_k^{(n)}(z)| = |k n! a_n + O(z)| \leq \frac{n!}{R^n} \sup_{C_R} |\varphi_k(z)| \leq \frac{n!}{R^{n-1}}.$$

Next, for  $z$  near 0,

$$\lim_{k \rightarrow \infty} |a_n| \leq \lim_{k \rightarrow \infty} \frac{1}{k R^{n-1}} = 0.$$

Therefore  $a_n = 0$  for all  $n \geq 2$ . So, we are left with  $\varphi(z) = z$ , which is indeed linear.

Lastly, we also must have  $\varphi'(z) = 1$  as any non-unit coefficient on  $z$  in  $\varphi$  would also result in an unbounded  $\varphi_k$  as  $k \rightarrow \infty$ .

$$\varphi(z) = az \implies \varphi_k(z) = a^k z,$$

but

$$\lim_{k \rightarrow \infty} |\varphi_k'(z)| = \lim_{k \rightarrow \infty} a^k \leq \frac{1}{R} \sup_{C_R} |\varphi_k(z)| \leq 1.$$

So we must have  $a = 1$ . □

**Problem 4.** Let  $u : \mathbb{D} \rightarrow \mathbb{R}$ . Suppose  $u \in \mathbb{C}$  and  $\Delta u(x, y) = 0$  for all  $(x, y) \in \mathbb{D}$ .

- (a) Prove that there exists a holomorphic function on the unit disk such that  $\operatorname{Re}(f) = u$ . Show that the imaginary part of  $f$  is defined up to an additive real constant.
- (b) Deduce the Poisson integral representation formula from the Cauchy integral formula. If  $u$  is harmonic in the unit disk and continuous on its closure, then, with  $z = re^{i\theta}$ ,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) u(e^{i\varphi}) d\varphi$$

where  $P_r(\gamma)$  is the Poisson kernel for the unit disk given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$

*Proof of (a).* Let  $f' = g = 2\partial_z u$ . We will show that  $\partial_{\bar{z}}g = 0$ , which implies that  $g$  is holomorphic.

We have the following relation between the mixed Wirtinger derivatives and the Laplacian:

$$\partial_{\bar{z}}\partial_z = \frac{1}{4}\Delta.$$

Note that the order of the derivatives may be switched. So,

$$\partial_{\bar{z}}g = \partial_{\bar{z}}(2\partial_z u) = \frac{1}{2}\Delta u = 0.$$

Hence  $g$  is holomorphic on  $\mathbb{D}$ , and it has a primitive  $G$  on  $\mathbb{D}$  such that  $G' = g$ .

Then, with  $G' = g = 2u'$ , integrating both sides gives,

$$G = 2 \int \frac{d}{dz} u dz = 2u + (a + bi)$$

where  $a + bi$  is a complex constant of integration. So,

$$\operatorname{Re}(G) = 2u + a \text{ and } \operatorname{Im}(G) = b.$$

Thus, let  $f = \frac{1}{2}G$  and we have found  $f$  which satisfies the required conditions. □

*Proof of (b).* We are given that, on the unit disk,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) P_z(\varphi) d\varphi.$$

Since  $\cos \gamma$  is even, then with the other given definition of  $P_\rho(\gamma)$ ,

$$P_\rho(-\gamma) = P_\rho(\gamma) = \operatorname{Re} \left( \frac{e^{i\gamma} + \rho}{e^{i\gamma} - \rho} \right).$$

Then, substituting  $z = re^{i\theta}$ ,

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) \operatorname{Re} \left( \frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} - re^{i\theta}} \right) d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) \operatorname{Re} \left( \frac{e^{i(\varphi-\theta)} + r}{e^{i(\varphi-\theta)} - r} \right) d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) P_r(\varphi - \theta) d\varphi. \end{aligned}$$

Finally, taking the real part of  $f$  and considering that  $P_r(\gamma)$  is even,

$$\operatorname{Re}(f(z)) = u(z) = u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\varphi}) P_r(\theta - \varphi) d\varphi,$$

as desired.  $\square$

**Problem 5.** Analytic functions on the unit disk that cannot be extended analytically past the unit circle.

**Definition 1** (regular). Let  $f$  be defined on the unit disk  $\mathbb{D}$  with boundary circle  $C$ . A point  $w$  on  $C$  is regular for  $f$  if there is an open neighborhood  $U$  of  $w$  and an analytic function  $g$  on  $U$  such that  $f = g$  on  $\mathbb{D} \cap U$ .

**Lemma 1.** A function  $f$  defined on  $\mathbb{D}$  cannot be continued analytically past the unit circle if no point of  $C$  is regular for  $f$ .

Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad \forall |z| < 1.$$

Note that the radius of convergence is 1. Show that  $f$  cannot be analytically continued past the unit disk.

*Proof.* Let  $\theta = \frac{2\pi p}{2^k}$  for  $p, k \in \mathbb{Z}^+$ . Let  $z = re^{i\theta}$  such that

$$z^{2^n} = (re^{i\theta})^{2^n} = r \exp 2\pi i p (2^{n-k}).$$

For all  $n \geq k$ , the above is  $\exp 2\pi i p (2^{n-k}) = 1$ . So, we can write

$$\sum_0^{\infty} z^{2^n} = \sum_0^{n-1} r \exp 2\pi i p (2^{n-k}) + \sum_n^{\infty} r.$$

But,

$$\lim_{r \rightarrow 1^-} \sum_n^{\infty} r = \infty \implies \lim_{|z| \rightarrow 1} \left| \sum_0^{\infty} z^{2^n} \right| = \infty.$$

Therefore  $f$  has a singularity on the unit circle where  $\theta = 2\pi \frac{p}{2^k}$ .

However, we can use  $\frac{p}{2^k}$  for positive integers  $p$  and  $k$  to produce the binary decimal representation of any number  $\frac{\varphi}{2\pi} \in [0, 1] \subset \mathbb{R}$ .

Hence,  $\forall \varepsilon > 0, \exists p, k$  such that

$$\left| \frac{\varphi}{2\pi} - \frac{p}{2^k} \right| < \frac{\varepsilon}{2\pi} \implies |\varphi - \theta| < \varepsilon.$$

Therefore  $e^{i\theta}$  with the above assignment of  $\theta$  will cover all angles in around the unit circle  $C$ .

Thus, there exist singularities at all points around the unit circle, so no points are regular on  $C$ , which means that  $f$  cannot be analytically continued past  $C$ .  $\square$

**Problem 6.** Consider  $f : \mathbb{C} \rightarrow \mathbb{C}$  of the form  $f = u + iv$ , where  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$  are smooth functions. Assume that  $f$  is a conformal map, i.e. the Jacobian is an orthogonal matrix. Prove that either  $f$  or its conjugate  $\bar{f}$  satisfies the Cauchy-Riemann equations.

*Proof.* Since  $f$  is conformal and has an orthogonal Jacobian matrix  $J$ , then

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \implies \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \pm \begin{pmatrix} v_y \\ -u_y \end{pmatrix}.$$

So, we have that either

$$u_x = v_y \text{ and } v_x = -u_y, \quad \text{or} \quad u_x = -v_y \text{ and } v_x = u_y.$$

We see that the first set of equations are the Cauchy-Riemann equations for  $f = u + iv$ .

Instead, if we consider  $\bar{f} = u - iv$ , then the second set of equations holds using substitution by  $v(x, y) \mapsto -v(x, y)$ .  $\square$