Math 135 Homework 4

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1. Show that $\sum_{k=0}^{\infty} \frac{\sin k}{2^k}$ converges.

   We will demonstrate convergence of the given series via the stronger property of absolute convergence given by a comparison test with a geometric series.

$$\sum \frac{\sin k}{2^k} \leq \sum \left| \frac{\sin k}{2^k} \right| \leq \sum \frac{1}{2^k} = \sum \left(\frac{1}{2} \right)^k,$$

   which converges since $\left|\frac{1}{2}\right| < 1$.

   Since $\sum \left| \frac{\sin k}{2^k} \right|$ converges, $\sum \frac{\sin k}{2^k}$ converges absolutely and therefore also converges.

   We will use Euler's formula,

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

   The equality provides that,

$$\sum \frac{\cos k}{2^k} + i \sum \frac{\sin k}{2^k} = \sum \frac{e^{ik}}{2^k}.$$

   We note that the value of series series at hand is the imaginary part of $\sum \frac{e^{ik}}{2^k}$.

   The latter term can be recognized as a particular form of the geometric series,

$$\sum \frac{e^{ik}}{2^k} = \sum \left(\frac{e^i}{2} \right)^k.$$

   Since e^i is a complex number of the form $re^{i\theta}$ with $r = 1$, we know that the norm of e^i is one. Thus,

$$\left| \frac{e^i}{2} \right| = \frac{1}{2} < 1.$$

   So, the series converges.

   We will now consider the value of the geometric series with $\frac{e^i}{2}$ given by the identity $\sum x^k = \frac{1}{1-x}$.

Using Euler's identity,

$$\begin{aligned}\frac{1}{1-e^i/2} &= \frac{2}{2-e^i}\\ &= \frac{2}{2-\cos 1 - i\sin 1}\\ &= \frac{2(2-\cos 1 + i\sin 1)}{(2-\cos 1)^2 - (i\sin 1)^2}\\ &= \frac{4 - 2\cos 1 + 2i\sin 1}{(2-\cos 1)^2 + \sin 1}.\end{aligned}$$

Since we are concerned with $\sum \frac{\sin k}{2^k}$, we will consider the imaginary part of this value as noted above.

$$\operatorname{Im}\left[\frac{4-2\cos 1 + 2i\sin 1}{(2-\cos 1)^2 + \sin 1}\right] = \frac{2\sin 1}{(2-\cos 1)^2+\sin 1} \qquad \approx 0.566.$$

2. Prove by induction that

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty}\binom{n+k}{k}x^n, \quad \forall |x| < 1.$$

Proof. First, with given by the identity,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

we simplify the binomial coefficient to

$$\binom{n+k}{k} = \frac{(n+k)!}{k!n!}.$$

We will rewrite the relationship we wish to demonstrate given the above; we will also move the $k!$ term from the denominator of the sum to the left as it does not depend on the summation index n and reindex the sum starting at $n = 1$.

$$\frac{k!}{(1-x)^{k+1}} = 1 + \sum_{n=1}^{\infty}\frac{(n+k)!}{n!}x^n, \quad \forall |x| < 1.$$

We will perform induction on $k \geq 0$. We start with the base case $k = 0$.

$$\begin{aligned}\frac{0!}{(1-x)^{0+1}} &= 1 + \sum_{n=1}^{\infty}\frac{(n+0)!}{n!}x^n\\ \frac{1}{1-x} &= 1 + \sum_{n=1}^{\infty} x^n = \sum_{k=0}^{\infty} x^n.\end{aligned}$$

We recognize this equality to hold as the identity of the geometric series. So, the base case holds as well.

We claim that the k^{th} case holds by the inductive hypothesis,

$$\frac{k!}{(1-x)^{k+1}} = 1 + \sum_{n=1}^{\infty} \frac{(n+k)!}{n!} x^n, \quad \forall |x| < 1.$$

Then, for the inductive step, we differentiate the relationship given by the inductive hypothesis.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{k!}{(1-x)^{k+1}}\right] &= \frac{d}{dx}\left[1 + \sum_{n=1}^{\infty} \frac{(n+k)!}{n!} x^n\right] \\
\frac{(k+1)!}{(1-x)^{k+2}} &= \sum_{n=1}^{\infty} \frac{(n+k)!}{n!} n x^{n-1} \\
\frac{(k+1)!}{(1-x)^{(k+1)+1}} &= \sum_{n=1}^{\infty} \frac{(n-k)!}{(n-1)!} x^{n-1} \\
&= \sum_{j=0}^{\infty} \frac{(j+1+k)!}{(j+1-1)!} x^{j+1-1} \\
&= \sum_{j=0}^{\infty} \frac{(j+(k+1))!}{j!} x^{j} \\
&= 1 + \sum_{j=1}^{\infty} \frac{(j+(k+1))!}{j!} x^{j},
\end{aligned}$$

which is the same as the formula with an index of $k+1$. So, by induction, the statement holds.

□

3. (a) Prove the Cauchy Mean Value Theorem 11.5.2

Theorem (Cauchy Mean Value Theorem 11.5.2)**.** *For f, g differentiable on (a, b) and continuous on $[a, b]$, with $g' \neq 0$ on (a, b), there exists and r in (a, b) such that*

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Define $h(x) = (g(b) - g(a))(f(x) - f(a)) - (f(b) - f(a))(g(x) - g(a))$ such that $h(a) = h(b) = 0$.

Let $\underset{[a,b]}{\Delta} f = f(b) - f(a)$ and $\underset{[a,b]}{\Delta} g = g(b) - g(a)$.

Since $g' \neq 0$ and the fact that g is continuous on $[a, b]$, then $\underset{[a,b]}{\Delta} g$ cannot be zero by the conditions of Rolle's Theorem that if $g(b) = g(a) \implies \underset{[a,b]}{\Delta} g = 0$, then there would be an r_0 such that $g'(r_0) = 0$. But, this is not the case, so we continue with $\underset{[a,b]}{\Delta} \neq 0$.

Since h is a combination of continuous functions on $[a, b]$, h is continuous on $[a, b]$.

Since $h(a) = h(b) = 0$ and h is continuous on $[a, b]$, then by the Mean Value Theorem, there exists an r in (a, b) such that $h'(r) = 0$.

Then, with $h'(x) = \underset{[a,b]}{\Delta} g \cdot f'(x) - \underset{[a,b]}{\Delta} f \cdot g'(x)$, $h'(r) = 0$ implies that

$$\underset{[a,b]}{\Delta} f \cdot g'(r) = \underset{[a,b]}{\Delta} g \cdot f'(r).$$

Then, recalling that $g' \neq 0$ on $[a, b]$ and $\underset{[a,b]}{\Delta} g \neq 0$,

$$\frac{f'(r)}{g'(r)} = \frac{\underset{[a,b]}{\Delta} f}{\underset{[a,b]}{\Delta} g} = \frac{f(b) - f(a)}{g(b) - g(a)}, \quad r \in (a, b).$$

So, the statement holds for some r. □

(b) Prove that, for f, g contiunous on $[a, b]$, there exists a $c \in [a, b]$ such that,

$$g(c) \int_a^b f(t)\, dt = f(c) \int_a^b g(t)\, dt.$$

Proof. Let $F'(x) = f(x)$ and $G'(x) = g(x)$.

By the Cauchy Mean Value Theorem, $\exists c \in (a, b)$ such that,

$$\begin{gathered} G'(c)[F(b) - F(a)] = F'(c)[G(b) - G(a)] \\ g(c)[F(b) - F(a)] = f(c)[G(b) - G(a)]. \end{gathered}$$

By the Fundamental Theorem of Calculus, for $F'(x) = f(x)$, $F(b) - F(a) = \int_a^b f(t)\, dt$. The same holds for $G'(x)$.

So, for some c in (a, b),

$$g(c) \int_a^b f(t)\, dt = f(c) \int_a^b g(t)\, dt.$$

□

(c) Prove that, for ϕ, h continuous on $[a, b]$, with $h(t) \neq 0$ for all $t \in [a, b]$, then,

$$\int_a^b \phi(t)h(t)\, dt = \phi(c) \int_a^b h(t)\, dt.$$

If $h(t) \neq 0$ for all $t \in (a, b)$, then $h(t) > 0$ or $h(t) < 0$ for all $t \in (a, b)$.

With $h(t) \geq 0$, the Second Mean Value Theorem for Integrals 5.9.3 applies to demonstrate that there is a c in (a, b) such that the statement holds.

Similarly, for $h(t) \leq 0$, the proof of 5.9.3 can be altered with the use of $-h$, so that the minimum and maximum values attained by h on $[a, b]$ are flipped.

(d) Prove that, for some c between a and x,

$$\frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n\, dt = \frac{f^{(n+1)}(c)(x - a)^{n+1}}{(n + 1)!}, \qquad (x - t)^n \neq 0, \quad \forall t \in (a, x),$$

which is the remainder in Taylor's theorem.

Proof. By (c), there exists an $c \in (a, x)$ such that,

$$\begin{aligned}\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt &= \frac{f^{(n+1)}(c)}{n!}\int_a^x (x-t)^n\,dt \\ &= \frac{f^{(n+1)}(c)}{n!}\left[\frac{-(x-t)^{n+1}}{n+1}\bigg|_a^x\right] \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}\left((x-a)^{n+1} - (x-x)^{n+1}\right) \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.\end{aligned}$$

□

(e) Show that,

$$\frac{1}{10\sqrt{2}} < \int_0^1 \frac{x^9}{\sqrt{1+x}}\,dx < \frac{1}{10}.$$

Let $f(x) = \frac{1}{\sqrt{1+x}}$; f is decreasing on the interval $[0, 1]$.

Then, the local extrema can be determined by the endpoints, $\min_{[0,1]} f = \frac{1}{\sqrt{2}}$ and $\max_{[0,1]} f = 1$.

So,

$$\frac{1}{\sqrt{2}} < \frac{1}{\sqrt{1+x}} < 1, \quad \forall x \in [0, 1].$$

Then, with $x^9 > 0$ for all x in closed positive unit interval $[0, 1]$,

$$\frac{x^9}{\sqrt{2}} < \frac{x^9}{\sqrt{1+x}} < x^9.$$

We then integrate on the interval $[0, 1]$,

$$\begin{gathered}\int_0^1 \frac{x^9}{\sqrt{2}}\,dx < \int_0^1 \frac{x^9}{\sqrt{1+x}}\,dx < \int_0^1 x^9\,dx \\ \frac{1}{10\sqrt{2}} < \int_0^1 \frac{x^9}{\sqrt{1+x}}\,dx < \frac{1}{10}.\end{gathered}$$

4. (a) Prove the triangle identity in $\mathbb{C}$,

$$|z + \omega| \leq |z| + |\omega|.$$

Proposition (*)**.** *The square of the norm of a number is the product of a number and its conjugate,*

$$|z|^2 = z\bar{z}.$$

Proof of (*). Let $z = \alpha + \beta i$.

So, $|z| = \sqrt{\alpha^2 + \beta^2}$ and $|z|^2 = \alpha^2 + \beta^2$.

Then, $z\bar{z} = (\alpha + \beta i)(\alpha - \beta i) = \alpha^2 - \beta^2 i^2 = \alpha^2 + \beta^2$.

It is now clear that $|z|^2 = z\bar{z}$. ☐

Proposition (**)**.** *The conjugate of a sum of two numbers is equal to the sum of their conjugates,*

$$\overline{z + \omega} = \bar{z} + \bar{\omega}.$$

Proposition (***)**.** *The conjugate of a product of two numbers is equal to the product of their conjugates,*

$$\overline{z\omega} = \bar{z}\bar{\omega}.$$

Proposition (****)**.** *The sum of a number and its conjugate is equal to twice the real part of the number,*

$$z + \bar{z} = 2\,Re[z].$$

Proof of (****). Let $z = \alpha + \beta i$. Then $\mathrm{Re}[z] = \alpha$.

With $\bar{z} = \alpha - \beta i$, $z + \bar{z} = \alpha + \beta i + \alpha - \beta i = 2\alpha = 2\,\mathrm{Re}[z]$. ☐

Proposition (*****)**.** *The norm of a number is equal to the norm of its conjugate,*

$$|z| = |\bar{z}|.$$

Proposition (******)**.** *The norm of a product of two numbers is equal to the product of their norms,*

$$|z\omega| = |z||\omega|.$$

Proof of (******). Let $z = a + bi$ and $\omega = x + yi$.

Then,

$$\begin{aligned} z\omega &= ax + ayi + bxi + byi^2 \\ &= (ax - by) + (ay + bx)i. \end{aligned}$$

So,

$$\begin{aligned} |z\omega| &= \sqrt{ax - by^2 + ay + bx^2} \\ &= \sqrt{a^2x^2 - 2abxy + b^2y^2 + a^2y^2 + 2abxy + b^2x^2} \\ &= \sqrt{a^2x^2 + b^2y^2 + a^2y^2 + b^2x^2} \\ &= \sqrt{(a^2 + b^2)(x^2 + y^2)}. \end{aligned}$$

With $|z| = \sqrt{a^2 + b^2}$ and $|\omega| = \sqrt{x^2 + y^2}$, then

$$|z||\omega| = \sqrt{(a^2 + b^2)(x^2 + y^2)}.$$

So, the proposition holds. ☐

Lemma.

$$z\bar{\omega} + \bar{z}\omega \leq 2|z||\omega|.$$

Proof of Lemma. Since $\overline{z\bar{\omega}} = \bar{z}\omega$ by (***), then $z\bar{\omega} + \bar{z}\omega = 2\,\mathrm{Re}[z\bar{\omega}]$ by (****).

Since $\mathrm{Re}[z] \leq z$, then $2\,\mathrm{Re}[z\bar{\omega}] \leq 2|z\bar{\omega}|$.

But, $2|z\bar{\omega}| = 2|z||\omega|$ by (*****) and (******).

So, $z\bar{\omega} + \bar{z}\omega \leq 2|z||\omega|$ and the Lemma holds. □

Proof of statement. We begin with

$$\begin{aligned} |z+\omega|^2 &= (z+\omega)(\bar{z}+\bar{\omega}) \quad \text{by (*) and (**)} \\ &= z\bar{z} + z\bar{\omega} + \bar{z}\omega + \omega\bar{\omega} \\ &= |z|^2 + z\bar{\omega} + \bar{z}\omega + |\omega|^2 \quad \text{by (*)} \\ &\leq |z|^2 + 2|z||\omega| + |\omega|^2 \quad \text{by the Lemma} \\ &= (|z| + |\omega|)^2. \end{aligned}$$

Since

$$|z+\omega|^2 \leq (|z|+|\omega|)^2,$$

then,

$$|z+\omega| \leq |z| + |\omega|$$

holds as well. □

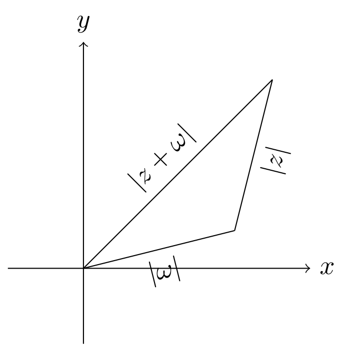

Figure 1: Proof of triangle inequality in the complex plane.

(b) Prove that,

$$|z+\omega|^2 + |z-\omega|^2 = 2\left(|z|^2 + |\omega|^2\right).$$

Proof. We proceed with the propositions (*) and (**) above,

$$\begin{aligned} |z+\omega|^2 + |z-\omega|^2 &= (z+\omega)(\bar{z}+\bar{\omega}) + (z-\omega)(\bar{z}-\bar{\omega}) \\ &= z\bar{z} + \bar{z}\omega + z\bar{\omega} + \omega\bar{\omega} + z\bar{z} - \bar{z}\omega - \bar{\omega}z + \omega\bar{\omega} \\ &= 2z\bar{z} + 2\omega\bar{\omega} \\ &= 2\left(|z|^2 + |\omega|^2\right). \end{aligned}$$

$\square$