

Math 135 Homework 9

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1. Assume that the triangle identity holds,

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

Note that $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$.

Then,

$$\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|.$$

So,

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|.$$

Then,

$$\|\mathbf{y}\| = \|\mathbf{y} - \mathbf{x} + \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\|.$$

So,

$$\|\mathbf{x}\| - \|\mathbf{y}\| \geq -\|\mathbf{y} - \mathbf{x}\| = -\|\mathbf{x} - \mathbf{y}\|.$$

Thus,

$$\begin{aligned} -\|\mathbf{x} - \mathbf{y}\| &\leq \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| \\ \left| \|\mathbf{x}\| - \|\mathbf{y}\| \right| &\leq \|\mathbf{x} - \mathbf{y}\|. \end{aligned}$$

2. Note that

$$\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}.$$

So,

$$\begin{aligned} &\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 \\ &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + 2(\mathbf{a} \cdot \mathbf{b}) + \mathbf{b} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} - 2(\mathbf{a} \cdot \mathbf{b}) + \mathbf{b} \cdot \mathbf{b} \\ &= 2(\mathbf{a} \cdot \mathbf{a}) + 2(\mathbf{b} \cdot \mathbf{b}) \\ &= 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2). \end{aligned}$$

3. The three points P_1 , P_2 , and Q form a plane that intersects the sphere on which they lie to form a circle.

Given that P_1 and P_2 are antipodal, we can instead consider a semicircle with P_1 and P_2 at the endpoints of the arc.

With these conditions, we will show that $\overrightarrow{P_1Q}$ is perpendicular to $\overrightarrow{P_2Q}$ in problem 5.

4. If \mathbf{a} , \mathbf{b} , and \mathbf{c} are linearly independent, then, for any vector \mathbf{d} ,

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}.$$

Since \mathbf{b} , \mathbf{c} are linearly independent, they are not parallel. So, $\mathbf{b} \times \mathbf{c} \neq \mathbf{0}$.

Additionally, since \mathbf{a} , \mathbf{b} , \mathbf{c} are linearly independent, then \mathbf{a} is not a linear combination of \mathbf{b} and \mathbf{c} . So, \mathbf{a} does not lie in the plane formed by \mathbf{b} and \mathbf{c} .

Thus, by 13.6.4,

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq 0. \quad (*)$$

To define α , we note that either \mathbf{b} or \mathbf{c} crossed with the vector $\mathbf{b} \times \mathbf{c}$ will always be zero since $\mathbf{b} \times \mathbf{c}$ is perpendicular to both \mathbf{b} and \mathbf{c} . Thus, we consider the following,

$$\begin{aligned} \mathbf{d} \cdot (\mathbf{b} \times \mathbf{c}) &= (\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \alpha\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ \frac{\mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} &= \alpha. \end{aligned}$$

The denominator, being the triple product of \mathbf{a} , \mathbf{b} , and \mathbf{c} , is not zero by (*), and is therefore non-zero for any arrangement of the vectors in the triple product by 13.4.7.

We repeat the computation above using $\mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ to retain only \mathbf{b} or \mathbf{c} respectively.

This yields the following definitions for α , β , and γ ,

$$\alpha = \frac{\mathbf{d} \cdot (\mathbf{b} \times \mathbf{c})}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}, \quad \beta = \frac{\mathbf{d} \cdot (\mathbf{a} \times \mathbf{c})}{\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})}, \quad \gamma = \frac{\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})}{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}.$$

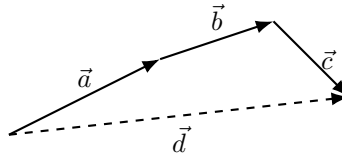
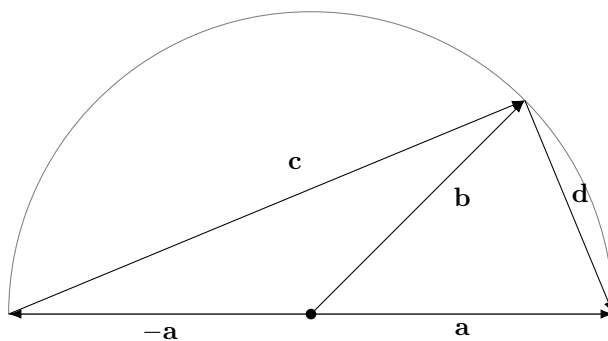


Figure 1: Linear combination of \mathbf{a} , \mathbf{b} , and \mathbf{c} to form \mathbf{d} .

5.

Theorem. *Every angle inscribed in a semicircle is a right angle*



Proof. Let θ be the angle between \mathbf{c} and \mathbf{d} .

We will prove that $\mathbf{c} \cdot \mathbf{d} = 0$. This is an equivalent statement to the Theorem.

Since $\mathbf{c} - \mathbf{a} = \mathbf{b}$, then $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

We see that $\mathbf{d} = \mathbf{a} - \mathbf{b}$.

We also notice that $\|\mathbf{a}\| = \|\mathbf{b}\|$, which is the radius of the semicircle.

Then,

$$\begin{aligned}
 \mathbf{c} \cdot \mathbf{d} &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
 &= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\
 &= \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\
 &= \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 \\
 &= 0.
 \end{aligned}$$

So $\mathbf{c} \cdot \mathbf{d} = 0$.

□