Math 136 Homework 2

Alexandre Lipson

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1. Find the matrix of the rotation in $\mathbb{R}^3$ through the angle α around the vector $(1,2,3)^T$. We assume that the rotation is counterclockwise if we sit at the tip of the vector and looking at the origin.

   Present the answer as a product of several matrices: you don't have to perform the multiplication.

   We already have the counterclockwise rotation matrix for $\mathbb{R}^2$. We can expand this matrix to operate on the xy-plane in $\mathbb{R}^3$ by preserving $\hat{k}$.

   Thus, a counterclockwise rotation of α degrees about the z-axis is given by,

$$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   First, we must make a change of basis to put the vector $\langle 1,2,3\rangle$ at $\langle 0,0,1\rangle$.

   We will construct an orthonormal set of vectors that includes $\vec{v_3} = \langle 1,2,3\rangle$. We pick the vector $\vec{v_2} = \langle 2,-1,0\rangle$ which is normal to the first since their dot product is zero.

   We then take the cross product of these two vectors to build a third that is normal to the other two,

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{vmatrix} = \langle -5,4,-1\rangle = \vec{v_1}.$$

   We then normalize each of these vectors to establish the orthonormal set,

$$\begin{aligned} \|\, \langle 1,2,3\rangle \,\| &= \sqrt{14} \\ \|\, \langle 1,1,-1\rangle \,\| &= \sqrt{3} \\ \|\, \langle -5,4,-1\rangle \,\| &= \sqrt{42}. \end{aligned}$$

   So,

$$\begin{aligned} \hat{v_3} &= \frac{1}{\sqrt{14}} \langle 1,2,3\rangle \\ \hat{v_2} &= \frac{1}{\sqrt{3}} \langle 1,1,-1\rangle \\ \hat{v_1} &= \frac{1}{\sqrt{42}} \langle -5,4,-1\rangle . \end{aligned}$$

However, we notice that the orientation of this basis is different than the orientation of the standard basis. In the future, this can be checked with the sign of the determinant. For now, we notice that this basis is left-handed while the standard basis is right-handed by attempting to place our fingers in in the appropriate octants.

In order to preserve the direction of rotation, we will reverse $\hat{v_1}$.

We assemble the transformation matrix A by set the image of each of the standard basis vectors in each of columns.

We will write the new basis vectors in terms of the common denominator $\frac{1}{\sqrt{42}}$.

$$A = \frac{1}{\sqrt{42}} \begin{bmatrix} 5 & \sqrt{13} & \sqrt{3} \\ -4 & \sqrt{14} & 2\sqrt{3} \\ 1 & -\sqrt{14} & 3\sqrt{3} \end{bmatrix}.$$

In order to perform the total transformation, we first transform the skewed orthonormal basis to the position the standard basis–this is the inverse transform of A–in order to rotate about $\hat{k}$, then we do the transformation A to place $\hat{k}$ back at $\vec{v_3}$.

So, the final transformation is given by the composition,

$$T(\vec{x}) = AR_\alpha A^{-1}\vec{x}.$$

2. Write the systems of equations below in matrix form and as vector equations

$$\begin{cases} x_1 + 2x_2 \quad\quad + 2x_4 = 6 \\ 3x_1 + 5x_2 - x_3 + 6x_4 = 17 \\ 2x_1 + 4x_2 + x_3 + 2x_4 = 12 \\ 2x_1 \quad\quad - 7x_3 + 11x_4 = 7 \end{cases}$$

We can write the system as a vector equation,

$$\vec{x_1}\begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix} + \vec{x_2}\begin{pmatrix} 2 \\ 5 \\ 4 \\ 0 \end{pmatrix} + \vec{x_3}\begin{pmatrix} 0 \\ -1 \\ 1 \\ -7 \end{pmatrix} + \vec{x_4}\begin{pmatrix} 2 \\ 6 \\ 2 \\ 11 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 12 \\ 7 \end{pmatrix}.$$

We can also write the system as a matrix,

$$\begin{pmatrix} 1 & 2 & 0 & 2 & 6 \\ 3 & 5 & -1 & 6 & 17 \\ 2 & 4 & 1 & 2 & 12 \\ 2 & 0 & -7 & 11 & 7 \end{pmatrix}.$$

We convert the matrix to an augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 3 & 5 & -1 & 6 & 17 \\ 2 & 4 & 1 & 2 & 12 \\ 2 & 0 & -7 & 11 & 7 \end{array}\right).$$

We then perform row reduction,

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 3 & 5 & -1 & 6 & 17 \\ 2 & 4 & 1 & 2 & 12 \\ 2 & 0 & -7 & 11 & 7 \end{array}\right) \qquad \begin{array}{l} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 2R_1 \\ R_4 = R_4 - 2R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & -4 & -7 & 7 & -5 \end{array}\right) \qquad \begin{array}{l} R_2 = -R_2 \\ R_1 = R_1 - 2R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & -4 & -7 & 7 & -5 \end{array}\right) \qquad R_4 = R_4 + R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -3 & 7 & -1 \end{array}\right) \qquad \begin{array}{l} R_2 = R_2 - R_3 \\ R_4 = R_4 + 3R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \qquad \begin{array}{l} R_2 = R_2 - 2R_4 \\ R_3 = R_3 + 2R_4 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \qquad R_1 = R_1 + 2R_3 - 2R_4$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

Thus, the vector form solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \\ -1 \end{pmatrix}.$$

3. Given a linear transformation $T : V \longmapsto W$, the kernel of V is $\ker T = \left\{\vec{v} \in V : T(\vec{v}) = \vec{0}\right\}$ and the image of V is $\operatorname{Im} T = \{T(\vec{v}) : \vec{v} \in V\}$. The kernel is a subset of V and the image is a subset of W.

   (a) Show that the kernel and the image of a transformation are vector subspaces of the domain and target set, respectively

   Let $\vec{v_1}, \vec{v_2} \in \ker T$. Then,

$$T(\vec{v_1}) = T(\vec{v_2}) = \vec{0}.$$

   So,

$$T(\alpha_1\vec{v_1} + \alpha_2\vec{v_2}) = \alpha_1 T(\vec{v_1}) + \alpha_2 T(\vec{v_2}) = \vec{0}.$$

Since the kernel of T is closed under addition and scalar multiplication, it is a vector space.

Since the kernel is a subset of V and it is also a vector space, it is a subspace of V.

Let $\vec{w_1} = T(\vec{v_1}), \vec{w_2} = T(\vec{v_2})$ be vectors in the image of T.

Then,

$$\alpha_1 \vec{w_1} + \alpha_2 \vec{w_2} = \alpha_1 T(\vec{v_1}) + \alpha_2 T(\vec{v_2}) = T(\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2}).$$

Since the sum of two arbitrary scalar multiples of vectors in the image of T are also in the image of T, then $\operatorname{Im} T$ is a vector space.

Since $\operatorname{Im} T$ is a vector space and a subset of W, then it is a subspace of W.

(b) Recall that a function is injective if $f(x) = f(y) \implies x = y$. Show that a linear transformation T is injective if and only if $\ker T = \vec{0}$

$(\Longleftarrow)$

Suppose that $\ker T = 0$. If $T(x) = T(y)$, then, $T(x) - T(y) = 0$ implies that $T(x - y) = 0$ by the linearity of T.

So, $x - y$ must belong to the kernel of T contains only the zero function.

Thus $x - y = 0$ so $x = y$. Therefore T is injective.

$(\Longrightarrow)$

Suppose that T is injective.

For a contradiction, suppose that the kernel of T contains zero and also $x \neq 0$.

Since T is injective, then $0 = T(x) = T(0)$ implies that $x = 0$, contradicting the assumption that $\ker T$ contained $x \neq 0$.

Therefore $\ker T = 0$.

(c) Let $T : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ defined by $(x\ y\ z)^T \longmapsto (x\ 0\ y)$. Describe the image and kernel of this transformation, both geometrically and using a basis for each, and find their dimensions.

Since the transformation drops all information in z, the kernel of T is the z-axis where x and y are zero. It can be described with the basis vector $\langle 0, 0, 1 \rangle$ and has dimension one.

The image of T is spanned by the basis $\{\langle 1, 0, 0 \rangle, \langle 0, 0, 1 \rangle\}$ which produces the xz-plane and has dimension two.

(d) Let W be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ with continuous derivatives. Let D be the differentiation map. Then, D is a linear transformation, why?. What are the image and kernel of D? Same question with $D_1 : V_{P,n} \longmapsto V_{P,n}$.

D is a linear transformation since differentiation is a linear operator.

The kernel of D is the set of all constant functions.

Take any continuous function f. Let $g(x) = \int_0^x f(t)\,dt$.

So g has continuous derivatives and $D(g) = f$.

So, the image of D is the set of all continuous functions.

With the transformation D_1 that maps from $V_{p,n}$ as the domain and codomain, the kernel is the same as D above, $V_{p,0}$.

The image of D_1 is one $V_{p,n-1}$, or the polynomials of a degree one less than n.