# Math 135 Homework 2

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## 11.7.69

Let $f > 0$ be continuous and decreasing on $[1, \infty)$.

We will show that,

$$\int_1^\infty f(x)\,dx \text{ converges iff } a_n = \int_1^n f(x)\,dx \text{ converges.}$$

$(\implies)$ We define $\int_1^\infty f(x)\,dx$ converging to $L$ as,

$$\lim_{n\to\infty} \int_1^n f(x)\,dx = L.$$

We also see that,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \int_1^n f(x)\,dx = L.$$

So,

$$\lim_{n\to\infty} a_n = L.$$

Therefore, the sequence $(a_n)$ converges to $L$ when $\int_1^\infty f(x)\,dx$ converges to $L$.

$(\impliedby)$ We follow the same definitions in the other direction to arrive at the fact that the converge of $(a_n)$ implies the convergence of $\int_1^\infty f(x)\,dx$.

## 12.2.31

Let $\sum a_k$ converge.

Define the remainder as $R_n = \sum_{k=n+1}^\infty a_k$.

Let $S_n$ be the $n^{\text{th}}$ partial sum of $\sum a_k$.

*Proof.* We will prove that, as $n \to \infty$, $R_n \to 0$.

We begin by splitting the series into the two parts of the partial sum and the remainder at the index $n$,

$$\sum_{k=0}^\infty a_k = \sum_{k=0}^n a_k + \sum_{k=n+1}^\infty a_k = S_n + R_n.$$

$$R_n = \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{n} a_k.$$

We then consider the behavior of $R_n$ as $n$ tends toward infinity.

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \left[ \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{n} a_k \right] \\
&= \sum_{k=0}^{\infty} a_k - \lim_{n\to\infty} \sum_{k=0}^{n} a_k \\
&= \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} a_k \\
&= 0.
\end{aligned}$$

So, the remainder of a convergent series tends toward zero. □

## 11.Review.28

Suppose that $(a_n)$ converges to $L$.

Define $m_n = \frac{1}{n}\sum_{k=1}^{n} a_k$.

Prove that $m_n$ converges to $L$.

*Proof.* Since $(a_n)$ converges to $L$, then, for all $\epsilon > 0$, there exists a $K$ such that for all $k \geq K$,

$$|a_k - L| < \epsilon.$$

So,

$$\begin{gathered}
L - \epsilon < a_k < L + \epsilon \\
\sum_{k=1}^{n} L - \epsilon < \sum_{k=1}^{n} a_k < \sum_{k=1}^{n} L + \epsilon \\
\frac{1}{n}\sum_{k=1}^{n} L - \epsilon < \frac{1}{n}\sum_{k=1}^{n} a_k < \frac{1}{n}\sum_{k=1}^{n} L + \epsilon \\
\frac{1}{n} \cdot n(L - \epsilon) < \frac{1}{n}\sum_{k=1}^{n} a_k < \frac{1}{n} \cdot n(L + \epsilon) \\
L - \epsilon < \frac{1}{n}\sum_{k=1}^{n} a_k < L + \epsilon.
\end{gathered}$$

Thus,

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| &< \epsilon \\ |m_n - L| &< \epsilon.\end{aligned}$$

So, for all $k$ sufficiently large, $m_n$ is $\epsilon$-close to $L$.

□

## 12.2.42

Prove that $\sum_{k=1}^{\infty}(a_{k+1} - a_k)$ converges iff $a_n$ converges.

*Proof.* ( $\Longrightarrow$ ) Let $\sum(a_{k+1} - a_k)$ converge to $L$.

So,

$$\lim_{n\to\infty}\sum_{k=1}^{n}(a_{k+1} - a_k) = L.$$

Then,

$$\begin{aligned} L &= \lim_{n\to\infty}\left[\sum_{k=1}^{n} a_{k+1} - \sum_{k=1}^{n} a_k\right] \\ &= \lim_{n\to\infty}\left[\sum_{k=2}^{n+1} a_k - \sum_{k=1}^{n} a_k\right] \\ &= \lim_{n\to\infty}\left[a_{n+1} + \sum_{k=2}^{n} a_k - \left(\sum_{k=2}^{n} a_k + a_1\right)\right] \\ &= \lim_{n\to\infty}[a_{n+1} - a_1] \\ L + a_1 &= \lim_{n\to\infty} a_{n+1} \\ L + a_1 &= \lim_{n\to\infty} a_n. \end{aligned}$$

So, $a_n$ converges.

( $\Longleftarrow$ ) Let $(a_n)$ converge to $L$; Since $a_n \to L$, then $a_{n+1} \to L$ as well.

We compute the sum,

$$\begin{aligned}
\sum_{k=1}^{\infty}(a_{k+1}-a_k) &= \sum_{k=1}^{\infty} a_{k+1} - \sum_{k=1}^{\infty} a_k \\
&= \lim_{n\to\infty}\left[\sum_{k=1}^{n} a_{k+1} - \sum_{k=1}^{n} a_k\right] \\
&= \lim_{n\to\infty}\left[\sum_{k=2}^{n+1} a_k - \sum_{k=1}^{n} a_k\right] \\
&= \lim_{n\to\infty}\left[a_{n+1}\sum_{k=2}^{n} a_k - \left(\sum_{k=2}^{n} a_k + a_1\right)\right] \\
&= \lim_{n\to\infty}[a_{n+1}-a_1] \\
&= \lim_{n\to\infty} a_{n+1} - a_1 \\
&= L - a_1.
\end{aligned}$$

So, the sum also converges. $\square$

## 12.3.48

Let the series $\sum a_k$ and $\sum b_k$ be defined such that, for all $k$, $a_k, b_k > 0$.

Suppose that $\frac{a_k}{b_k} \to \infty$ as $k \to \infty$.

So, $a_k > b_k$ for all sufficiently large $k$.

(a) Since $\sum b_k$ diverges and $a_k > b_k$ for all sufficiently large $k$, then, by the Basic Comparison Theorem, $\sum a_k$ diverges as well.

(b) Since $\sum a_k$ converges and $a_k > b_k$ for all sufficiently large $k$, then, by the Basic Comparison Theorem, $\sum b_k$ must converge as well.

(c) Let $a_k = \frac{1}{k}$, the harmonic series. This series diverges.

First, Let $b_k = \frac{1}{k^2}$, the $p$-series with $p = 2 > 1$, which converges.

Then, $\frac{a_k}{b_k} = \frac{1/k}{1/k^2} = k$.

So, $\lim\limits_{k\to\infty} \frac{a_k}{b_k} = \lim\limits_{k\to\infty} k = \infty$.

Next, let $a_k = \frac{1}{\sqrt{k}}$, the $p$-series with $p = \frac{1}{2} < 1$, which diverges, and $b_k$ be the harmonic series.

Then, $\frac{a_k}{b_k} = \frac{1/\sqrt{k}}{1/k} = \sqrt{k}$, which tends to infinity as $k$ tends to infinity.

So, if $\sum a_k$ diverges and $\frac{a_k}{b_k} \to \infty$ as $k \to \infty$, then $b_k$ can either converge or diverge.

(d) First, note that $a_k$ can diverge when $b_k$ converges as in the first example in (c).

So, we will consider an $a_k$ and $b_k$ which converge. We will choose the 2 and 3 $p$-series respectively,

$$a_k = \frac{1}{k^2}, \quad b_k = \frac{1}{k^3}.$$

Then,

$$\frac{a_k}{b_k} = \frac{1/k^2}{1/k^3} = k,$$

which we have already seen tends toward infinity as $k$ tends toward infinity.

Thus, $a_k$ can either diverge or converge when $b_k$ converges.

## Problem Six

(a) We will show that $\int_2^\infty \frac{dx}{x(\ln x)^p}$ converges iff $p > 1$.

We make the substitution $u = \ln x$, which produces, $\int_{\ln 2}^\infty \frac{du}{u^p}$.

We note that this converges where $p > 1$ by the $p$-series convergence theorem (11.7.1).

(b) By the integral test theorem (12.3.2), $\sum_{k=2}^\infty \frac{1}{k(\ln k)^p}$ converges when $\int_2^\infty \frac{dx}{x(\ln x)^p}$ converges.

With $p = 2$, the series $\sum_{k=2}^\infty \frac{1}{k \ln^2 k}$ will converge by part (a).

We note that a series can be represented as the sum of its $n^{\text{th}}$ partial sum and the remainder at the $n^{\text{th}}$ index.

So, we will write,

$$\sum_{k=2}^\infty \frac{1}{k \ln^2 k} = \sum_{k=2}^n \frac{1}{k \ln^2 k} + \sum_{k=n+1}^\infty \frac{1}{k \ln^2 k}.$$

We note that $\frac{1}{x \ln^2 x}$ is a positive decreasing function, so the series, which models right-hand Reimann sums given by the indexing $n+1$ of the regular partition of width one, will always be less than or equal to the area of the integral.

Thus,

$$R_n = \sum_{n+1}^\infty \frac{1}{k \ln^2 x} \leq \int_n^\infty \frac{dx}{x \ln^2 x}.$$

We desire that the maximum error $R_n$ should be no greater than $0.05 = \frac{1}{20}$; we will find an index $n$ that serves this purpose.

$$R_n \leq \int_n^\infty \frac{dx}{x \ln^2 x} \leq \frac{1}{20}.$$

We evaluate the integral, considering that the multiplicative inverse of the natural log tends toward zero, to see that,

$$\begin{aligned} \frac{1}{\ln n} &\leq \frac{1}{20} \\ \ln n &\geq 20 \\ n &\geq e^{20}. \end{aligned}$$

We verify using a calculator that with a series index of $e^{20}$, the partial sum will be within 0.05 of 2.1,

$$\sum_{k=2}^{e^{20}} \frac{1}{k \ln^2 k} \approx 0.06.$$

(c)

We demonstrated the following chain of inequalities in class as a result of the integral test *ibid*,

$$\int_m^n f(x)\,dx \leq \sum_{k=m}^{n} f(k) \leq f(m) + \int_m^n f(x)\,dx.$$

So, with $f(x) = \frac{1}{x \ln x}$, $m = 2$, and $n = 10^{1000}$, the above becomes,

$$\int_2^{10^{1000}} \frac{dx}{x \ln x} \leq \sum_{k=2}^{10^{1000}} \frac{1}{k \ln k} \leq \frac{1}{2 \ln 2} + \int_2^{10^{1000}} \frac{dx}{x \ln x}.$$

We then compute the integral $\int \frac{dx}{x \ln x}$ with the substitution $u = \ln x$.

We see that the indefinite integral equates to $\ln(\ln x)$. We compute via calculator the definite integral on the interval $[2, 10^{1000}]$, $\ln\left(\ln 10^{1000}\right) - \ln(\ln 2) \approx 8.1$.

Additionally, we calculate that $\frac{1}{2 \ln 2} \approx 0.72$.

Thus,

$$8.11 \leq \sum_{k=2}^{10^{1000}} \frac{1}{k \ln k} \leq 8.83.$$