Math 334 Midterm Extra Credit

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Problem (1)**.** Let $K \subset \mathbb{R}^n$ be compact. Let $f : K \longrightarrow K$ be a *shrinking map.* $\forall x, y \in K$, $x \neq y \implies |f(x) - f(y)| < |x - y|$.

Prove that f has a unique fixed point $x \in K : x = f(x)$.

If K compact with $K \supset C_1 \supset C_2 \supset \cdots$ nested sequence of non-empty closed subsets C_i, then

$$\bigcap_{i=1}^{\infty} C_i \neq \emptyset. \tag{*}$$

Proposition (Continuity of the shrinking map)**.** f is uniformly continuous.

Proof of Continuity Proposition. Choose $\delta = \epsilon$. Then, $\forall \epsilon > 0$, $\exists \delta > 0$,

$$|f(x) - f(y)| < |x - y| < \delta = \epsilon.$$

So, f is uniformly continuous as δ depends solely on ϵ. □

Proposition (Successive applications of the map form a nested sequence)**.** Let $C_0 = K$, and $C_{i+1} = f(C_i)$.

(a) C_i closed.

(b) C_i non-empty.

(c) C_i nested such that $K \supset C_1 \supset C_2 \supset \cdots$

Proof of Sequence Proposition (a). By induction, since f continuous and C_0 compact, then $\forall i$, C_i is compact as well. □

Proof of Sequence Proposition (b). Since f maps from K to K and $C_0 = K$ is non-empty, then each C_i, as the image of f, must be non-empty. □

Proof of Sequence Proposition (c). By induction, since $f(K) \subset K$ and $C_{i+1} = f(C_i)$, then $C_{i+1} \subset C_i$. □

Proof of Problem 1. Let $S \subset K$. So,

$$|f(x) - f(y)| < |x - y| \implies \sup_{x,y \in S} |f(x) - f(y)| < \sup_{x,y \in S} |x - y| \implies \operatorname{diam} f(S) < \operatorname{diam} S.$$

Since f maps from K to K, and $S, C_i \subset K$, then $\operatorname{diam} f(S) < \operatorname{diam} S \implies \operatorname{diam} C_{i+1} < \operatorname{diam} C_i$.

Let $(x_n) \subset \mathbb{R}$ be the sequence defined by $x_i = \operatorname{diam} C_i$. So, $\lim_{n\to\infty} x_n = \operatorname{diam} S$.

Since $\operatorname{diam} C_{i+1} < \operatorname{diam} C_i$, then (x_n) is decreasing. Since $\operatorname{diam} C_i \geq 0$, then x_n is bounded below by zero. Since $(x_n) \subset \mathbb{R}$ is decreasing and bounded below, it must converge.

Suppose, that (x_n) converges to $m > 0$.

Then, $S = \bigcap C_i$ must contain at least two points x, y such that $|x - y| = m$ But, $\forall i$, $x, y \in C_i$ means that we could not have $\operatorname{diam} C_{i+1} < \operatorname{diam} C_i$, which is a contraction. Thus, (x_n) must converge to zero.

Since $(x_n) \to 0$, then $\operatorname{diam} S = 0$.

By (*), $\forall i$, $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.

Since $\operatorname{diam} S = 0$ and $S \neq \emptyset$, then $S = \{x\}$. Since $x \in C_i \implies f(x) \in C_{i+1}$, then $f(x) \in S$. So, $f(x) = x$.

$\square$

Problem (2)**.** Give an example of a shrinking map that is not a contraction map.

Proof of Problem 2. Since a contraction map requires, for some fixed $\alpha \in (0, 1)$, that $\forall x, y \in K$, $x \neq y$,

$$|f(x) - f(y)| < \alpha|x - y|,$$

then we wish to find a map such that this relationship will not hold for any fixed choice of α.

$f : [0, 1] \longrightarrow \mathbb{R}$ defined as $f(x) = x - \frac{x^2}{2}$ is a shrinking map that is not a contraction map.

As x approaches zero, $f'(x) = 1 - x$ will get arbitrarily close to 1, so no fixed α will work. $\square$