Math 334 Midterm Extra Credit

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Problem (1)**.** Let $K \subset \mathbb{R}^n$ be compact. Let $f : K \longrightarrow K$ be a *shrinking map.* $\forall x, y \in K$, $x \neq y \implies |f(x) - f(y)| < |x - y|$.

Prove that f has a unique fixed point $x \in K : x = f(x)$.

If K compact with $K \supset C_1 \supset C_2 \supset \cdots$ nested sequence of non-empty closed subsets C_i, then

$$\bigcap_{i=1}^{\infty} C_i \neq \emptyset. \tag{*}$$

Proposition (Continuity of the shrinking map)**.** f is uniformly continuous.

Proof of Continuity Proposition. Choose $\delta = \epsilon$. Then, $\forall \epsilon > 0$, $\exists \delta > 0$,

$$|f(x) - f(y)| < |x - y| < \delta = \epsilon.$$

So, f is uniformly continuous as δ depends solely on ϵ. □

Proposition (Uniqueness of the fixed point)**.** The fixed point of f is unique.

Proof of Uniqueness Proposition. Suppose, for a contradiction, that there were x, y fixed points with $x \neq y$ and $f(x) = x$, $f(y) = y$.

Then, $|f(x) - f(y)| = |x - y|$. But we had, by the definition of f, $|f(x) - f(y)| < |x - y|$, which is a contradiction.

Thus, $x = y$ and the fixed point is unique. □

Proof of Problem. Let $g : K \longrightarrow \mathbb{R}$ be the distance between x and its image under f,

$$g(x) = |f(x) - x|.$$

Since f is continuous, then g, the difference between two continuous functions, is also continuous.

Since K is compact and g is continuous and positive, then, by EVT, there is a minimum of g at $x \in K$,

$$0 \leq \min_{x \in K} g = g(a).$$

We wish to show that $g(a)$ must be zero. This a is the fixed point of f.

First, if $g(a) = 0$, then we're done.

Next, if $g(a) > 0$, we will consider the image of $f(a)$ under g. But, the definition of f bounds this value,

$$g(f(a)) = |f(a) - f(f(a))| < |a - f(a)| = g(a).$$

So, we have that $g(f(a)) < g(a)$, contracting the fact that $g(a)$ was the minimum of g.

Thus, $g(a) = 0$ and the fixed point of f occurs at $x = a$. □

Problem (2)**.** Give an example of a shrinking map that is not a contraction map.

Proposition (2)**.** $\forall \epsilon : 0 < \epsilon < 1$, the map $f : [0, 1] \longrightarrow \mathbb{R}$ defined as $f(x) = (1 - \epsilon)x - \frac{x^2}{2}$ is a shrinking map that is not a contraction map.

Proof of 2. Since a contraction map requires, for some fixed $\alpha \in (0, 1)$, that $\forall x, y \in K$, $x \neq y$,

$$|f(x) - f(y)| < \alpha|x - y|,$$

then we wish to find a map such that this relationship will not hold for any fixed choice of α.

As x approaches zero, $f'(x)$ can get arbitrarily close to 1, but no fixed α will work as we can always choose a smaller ϵ. □