Math 334 Homework 8

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Problem (1)**.** Let $I_n = \int_{\mathbb{R}^n} \exp\left(-|x|^2\right) dx$ converge, such that, $\forall K_1 \subset K_2 \subset \cdots$ compact with $\bigcup K_i = \mathbb{R}^n$,

$$\lim_{i\to\infty} \int_{K_i} \exp\left(-|x|^2\right) dx = I_n.$$

(a) With a sequence of compact sets, show

$$\int_{\mathbb{R}^2} \exp\left(-|x|^2\right) dx = \left(\int_{\mathbb{R}} \exp\left(-x_1^2\right) dx_1\right)\left(\int_{\mathbb{R}} \exp\left(-x_2^2\right) dx_2\right).$$

(b) With a different sequence of compact sets, and using a change of variables, evaluate $\int_{\mathbb{R}^2} \exp\left(-|x|^2\right) dx$.

(c) Conclude that $\int_{\mathbb{R}} \exp\left(-x^2\right) dx = \sqrt{\pi}$.

Proof of a. We will show that the equivalent statement, $I_2 = I_1^2$ holds.

Let $K_m = [-m, m]^2$. K_m is clearly nested, closed, and bounded. So, K_m is compact and $\bigcup K_m = \mathbb{R}^2$.

Then, $\forall m$,

$$\int_{K_m} \exp\left(-|x|^2\right) dx = \iint_{K_m} \exp\left(-(x_1^2 + x_2^2)\right) dx_1\, dx_2 = \int_{-m}^{m}\int_{-m}^{m} \exp(-x_1^2)\exp(-x_2^2)\, dx_1\, dx_2.$$

Since x_1 is constant in x_2 and vise versa, then the above is

$$\int_{-m}^{m} \exp(-x_1^2)\, dx_1 \int_{-m}^{m} \exp(-x_2^2)\, dx_2.$$

By assumption, $\lim\limits_{m\to\infty} \int_{K_m} \exp(-|x|^2)\, dx = I_2$ and $\lim\limits_{m\to\infty} \int_{-m}^{m} \exp(-x^2)\, dx = I_1$. So,

$$I_2 = \lim_{m\to\infty} \int_{K_m} \exp(-|x|^2)\, dx = \left(\lim_{m\to\infty} \int_{-m}^{m} \exp(-x_1^2)\, dx_1\right)\left(\lim_{m\to\infty} \int_{-m}^{m} \exp(-x_2^2)\, dx_2\right) = I_1^2.$$

□

Proof of b. Let $K_r = \{x_1, x_2 \in \mathbb{R}^2 \ \mid \ x_1^2 + x_2^2 \leq r^2\}$. Clearly, K_r is nested and compact. Also, $\bigcup K_r = \mathbb{R}^2$.

We will evaluate the integral in $\mathbb{R}^2$ using polar coordinates with radius ρ and angle θ. The Jacobian determinant for this transformation is then ρ. Since $|x|^2 = \rho^2$, then

$$\int_{K_r} \exp(-|x|^2)\,dx = \int_0^{2\pi}\int_0^r \exp(-\rho^2)\rho\,d\rho\,d\theta = \int_0^{2\pi} d\theta \int_0^r \exp(-\rho^2)\,d\rho.$$

Let $u = -\rho^2$, $du = -2\rho d\rho$ so the above becomes,

$$2\pi\left(\frac{-1}{2}\right)\int_{u(0)}^{u(r)} \exp(u)\,du = -\pi(\exp(-r^2) - 1) = \pi(1 - \exp(-r^2)).$$

Then, by assumption,

$$I_2 = \lim_{r\to\infty}\int_{K_r} \exp(-|x|^2)\,dx = \lim_{r\to\infty} \pi(-\exp(-r^2)) = \pi.$$

So, $I_2 = \pi$. $\square$

Proof of c. By (a), $I_2 = I_1^2$. Since $\forall x$, $\exp(x) > 0$, the integral of $\exp(x)$ must have positive signed area above the x-axis. Thus, $\int_{\mathbb{R}} \exp(-x^2)\,dx = I_1 = \sqrt{\pi}$ (and not $-\sqrt{\pi}$). $\square$

Problem (2)**.** Let $E \subset \mathbb{R}^3$ be the region inside the surface $x^2 + y^2 + z^2 = 4$ and outside the surface $4x^2 + 4y^2 - z^2 = 1$. Let the density of E be given by $f(x, y, z) = z^2$. Write an iterated integral for the mass of E using spherical coordinates.

Proof. The outside surface is a sphere of radius 2. The inside surface is a hyperboloid of one sheet with a minimum radius of $\frac{1}{2}$.

We will proceed with spherical coordinates where the angle ϕ gives altitude and θ gives azimuth.

So, with $r^2 = x^2 + y^2$ as in cylindrical coordinates, we can form the spherical radius ρ from the right triangle with the radius r of any point projected onto the xy-plane, and the height z. With z adjacent to the azimuth ϕ and r opposite, we see that $z = \rho\cos\phi$ and $r = \rho\sin\phi$. But, with $x = r\cos\theta$ and $y = r\sin\theta$, we have $x = \rho\sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$.

Now, we will parametrize the bounding surfaces as $\rho(\theta, \phi)$. The sphere has a constant radius of 2 and is given by $\rho = 2$. The hyperboloid can be parametrized as follows:

$$\begin{aligned}
4(x^2 + y^2) - z^2 &= 1\\
4(\rho\sin\phi)^2 + (\rho\cos\phi)^2 &= 1\\
\rho^2(4\sin^2\phi - \cos^2\phi) &= 1\\
\rho^2(5\sin^2\phi - 1) &= 1\\
\rho &= \frac{1}{\sqrt{5\sin^2\phi - 1}}.
\end{aligned}$$

Since we wish to parametrize all around the sphere and hyperboloid, we will let $\theta \in [0, 2\pi]$. Next, we will find the intersection points of the bounding surfaces which will give us bounds for ϕ.

$$\begin{aligned} 2 &= (5\sin^2\phi - 1)^{-\frac{1}{2}} \\ 5\sin^2\phi - 1 &= \frac{1}{4} \\ \sin^2\phi &= \frac{1}{4} \\ \sin\phi &= \pm\frac{1}{2},\ \phi \in [0,\pi] \\ \phi &= -\frac{\pi}{6}, \frac{\pi}{6}. \end{aligned}$$

So, $\phi \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$.

Then, the integrand z^2 in polar coordinates becomes $\rho\cos^2\phi$. With the Jacobian determinant for spherical coordinates, $\rho^2\sin\phi$, the mass of E is given by

$$\int_0^{2\pi}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{(5\sin^2\phi-1)^{-\frac{1}{2}}}^{2} \rho^4\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta.$$

□

Problem (3)**.** Let $S \subset \mathbb{R}^2$ in the first quadrant be the region bounded by $xy = 1, 3$ and $x^2 - y^2 = 1, 4$. Compute $\iint_S (x^2+y^2)\, dA$ with the map $(x,y) \mapsto (xy, x^2-y^2)$.

Proof. Let $u(x,y) = xy$ such that $u \in [1,3]$. Let $v(x,y) = x^2 - y^2$ such that $v \in [1,4]$.

We will find the Jacobian determinant of the transformation from xy-space to uv-space. Since we have u and v as functions of x and y, then we will use the fact that $\det(DG) = \left(\det(DG)^{-1}\right)^{-1}$. Then,

$$\begin{aligned} |\det DG| &= \left|\det(DG)^{-1}\right|^{-1} \\ &= \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} \\ &= \left|\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\right|^{-1} \\ &= \left|\begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix}\right|^{-1} \\ &= \left|-2y^2 - 2x^2\right|^{-1} \\ &= \frac{1}{2(x^2+y^2)}. \end{aligned}$$

So,

$$\iint_S (x^2+y^2)\, dA = \int_1^4\int_1^3 \frac{x^2+y^2}{2(x^2+y^2)}\, du\, dv = \frac{1}{2}\int_1^4\int_1^3 du\, dv = 3.$$

□