Math 334 Homework 9

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Problem (1)**.** Let $\mathbf{F} \subset \mathbb{R}^3$ be defined by $\mathbf{F} = (2y - z, y - z, -x)$. Let $C = \{(x, y, z) \mid x^2 + y^2 = 1\} \cap \{(x, y, z) \mid z = y\}$ positively oriented when viewed from the positive z-axis. Find $\int_C \mathbf{F} \cdot d\mathbf{x}$.

Proof. We will parametrize C by $\gamma(t) = (\cos t, \sin t, \sin t)$ with $t \in [0, 2\pi]$. Because of the x and y components, we see that C is traversed anticlockwise through t.

We will construct the line integral $\int_0^{2\pi} \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt$. So, $\gamma'(t) = (-\sin t, \cos t, \cos t)$.

Then, $\mathbf{F}(\gamma(t)) = (\sin t, 0, -\cos t)$. So, $\mathbf{F}(\gamma(t)) \cdot \gamma'(t) = -\sin^2 t - \cos^2 t = -1$.

Thus, $\int_0^{2\pi} -1\, dt = -2\pi$. $\square$

Problem (2)**.** Let $\mathbf{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an everywhere continuous, path-independent vector field. $\forall x \in \mathbb{R}^2$, set $f(\mathbf{x}) = \int_0^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{x}$ as the line integral on any piecewise C^1 path from 0 to x.

Prove $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is C^1 and satisfies $\nabla f = \mathbf{F}$.

Proof. Since $\mathbf{F}$ is path-independent, then f will be a well defined function regardless of the path chosen from 0 to x.

We will split the arbitrary C^1 path into vertical and horizontal components.

So, considering a horizontal then vertical decomposition of the path, we have

$$f(x, y) = \int_0^x \mathbf{F}_1(t, 0)\, dt + \int_0^y \mathbf{F}_2(x, t)\, dt.$$

Similarly, we can construct a vertical then horizontal decomposition,

$$f(x, y) = \int_0^y \mathbf{F}_2(0, t)\, dt + \int_0^x \mathbf{F}_1(t, y)\, dt.$$

We see that, if we take the $\frac{\partial f}{\partial y}$ of the first form of f, the integral over the horizontal interval $[0, x]$ will vanish, leaving,

$$\frac{\partial f}{\partial y} = \frac{d}{dy} \int_0^y \mathbf{F}_2(x, t)\, dt = \mathbf{F}_2(x, y).$$

Likewise, for the second form of f, the vertical integral will vanish, so

$$\frac{\partial f}{\partial x} = \mathbf{F}_1(x, y).$$

So, we have shown that, by each component, $\nabla f = \mathbf{F}$. Since each component of $\mathbf{F}$ was continuous, then f as the sum of integrals of continuous functions $\mathbf{F}_1$ and $\mathbf{F}_2$ is also continuous. $\square$

Problem (3)**.** $S \subset \mathbb{R}^n$ *star-shaped* $\iff \exists a \in S : \forall a \neq x \in S$, there is a straight line segment from a to x.

(a) Give an example of a star-shaped set that is not convex. A set $S \subset \mathbb{R}^n$ is convex $\iff$ $\forall a, x \in S$, the line segment from a to x is contained in S.

(b) Prove that any star-shaped set is simply connected.

Proof of a. Consider a pacman shaped set. Since this set is like a disk with a slice missing, then all points can be reached from the center of the disk which would cover this pacman set. However, between the top and bottom of the pacman mouth (the vertices between the arc of the circle and the straight line slices) no straight path can be drawn such that the line stays completely within the body of the pacman, which is the set. $\square$

Proof of b. We will show that all closed loops inside S are homotopic to the star point.

First, we translate S such that the star point lies on the origin.

Then Let the map $\gamma : [0, 1] \times \varphi(\mathbb{S}^1) \longrightarrow \mathbb{R}^2$ defined by $(t, x) \mapsto (1 - t)x$ be a continuous map from any point $x \in \varphi(\mathbb{S}^1)$ on the loop to the star point.

Since $(1 - t)x$ is a linear interpolation between the points x on the loop and the star point, then all points on this straight path must also belong in S since S is star shaped.

So, γ defines a homotopy between any closed loop $\varphi(\mathbb{S}^1)$ and the star point 0.

Since all closed loops in S are homotopic to a point by γ, then S is simply connected. $\square$

Problem (4)**.** Find a simple closed curve γ, positively oriented, that maximizes the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{x}$ where $\mathbf{F} = (y^3, 3x - x^3)$.

Proof. Since γ is a closed and positively oriented loop, using Green's theorem, we have that, with $\mathbf{F} = (P(x, y), Q(x, y))$,

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \iint_D (Q_x - P_y)\, dA = \iint_D (3 - 3x^2 - 3y^2)\, dA.$$

So, we wish to find a region D such that the integrand of 2D curl $Q_x - P_y$ is positive on the interior of D and negative on the exterior of D, and, therefore, zero on the boundary of D.

If the integrand were positive on the exterior of D, then we could have chosen a better region D to further increase the value of the integral.

Considering a region D such that $3 - 3x^2 - 3y^2 = 0$ yields $x^2 + y^2 = 1$, or the unit circle.

So, the curve which maximizes $\int_\gamma \mathbf{F} \cdot d\mathbf{x}$ is the positively oriented unit circle. □

Problem (5)**.** Let $\mathbf{F} = ((1 + 3x^3)\exp(x^3 + y^3) + x,\, 3xy^2 \exp(x^3 + y^3))$.

(a) What is the maximal domain of $\mathbf{F}$?

(b) Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{x}$ where C_1 is the vertical line segment from (1,1) to (1,3).

(c) Determine whether $\mathbf{F}$ is conservative on the domain from (a). If yes, find a potential function for $\mathbf{F}$. If not, prove that $\mathbf{F}$ is not conservative.

(d) Evaluate $\int_{C_2} \mathbf{F} \cdot d\mathbf{x}$ where C_2 is the left semicircle defined by $(x-1)^2 + (y-2)^2 = 1$, $x \le 1$ traversed from (1,1) to (1,3).

Proof of a. Since each component of $\mathbf{F}$ is the product of everywhere continuous functions, then $\mathbb{R}^2$ is the maximal domain for $\mathbf{F}$. □

Proof of b. Let $\gamma(t) = (1, 1 + 2t)$, $t \in [0, 1]$ parametrize C_1. Then $\gamma'(t) = (0, 2)$.

So,

$$\int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^1 2(3(1+2t)^2)\exp(1 + (1+2t)^3)\, dt.$$

Let $u = 1 + (1+2t)^3$ such that $du = 2(3(1+2t)^2)dt$. So, the above becomes

$$\int_{u(0)}^{u(1)} e^u\, du = e^u\Big|_2^{65} = e^{65} - e^2.$$

□

Proof of c. We will show that $\mathbf{F}$ is the gradient field of a potential function f.

We will integrate $\mathbf{F}_2$ with respect to y, as $f(x, y) = \int \mathbf{F}_2\, dy + h(x)$; we will recover $h(x)$ from the difference between $\int \mathbf{F}_1\, dx$ and $\int \mathbf{F}_2\, dy$.

So, with $u = x^3 + y^3$, and noting that x is constant when integrating in y,

$$\begin{aligned}\int \mathbf{F}_2\, dy &= \int 3xy^2 \exp(x^3 + y^3)\, dy \\ &= \int xe^u\, du \\ &= x\exp(x^3 + y^3).\end{aligned}$$

So, $f(x, y) = x\exp(x^3 + y^3) + h(x)$

Then, letting $h(x) = \frac{x^2}{2}$, we can verify our potential function f by recovering $\mathbf{F}_1$ from $\frac{\partial f}{\partial x}$.

Since $\mathbf{F}$ is the gradient of a potential function f, it is conservative. □

Proof of d. Since $\mathbf{F}$ is conservative, then $\mathbf{F}$ is path-independent.

Since C_1 and C_2 share respective start and end points, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int_{C_2} \mathbf{F} \cdot d\mathbf{x} = e^{65} - e^2$. $\square$