Math 462 Homework 7

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Problem 1. Modify Euler's formula so that it holds for any planar graph, not just connected planar graphs. Prove your formula.

Proof. Since G is planar and not connected, then G has k connected components.

We can connect align these components in a row and connect them with $k-1$ edges.

Now, the new graph is connected and planar, so $V-(E+(k-1))+F=2$ holds.

Hence, we have $V-E+F=3-k$ for a disconnected planar graph of k connected components. □

Problem 2. (a) Prove that if G is a simple planar graph with no triangles and at least three vertices, then $E \leq 2V-4$.

(b) Use this to prove that $K_{3,3}$ is not planar.

Proof of (a). Since G is simple and planar with no triangles, then every face must have at least length 4.

So,

$$4F \leq \sum_{f \in F} \text{length}(f) = 2E \implies F \leq \frac{1}{2}E.$$

Then, substituting the above into Euler's formula we have,

$$2 = V-E+F \leq V - \frac{1}{2}E \implies E \leq 2V-4.$$

□

Proof of (b). We have that, for $K_{3,3}$, $V=6$ and $E=9$.

But, $V > 2V-4$ as $9 > 2 \cdot 6 - 4 = 8$.

Therefore, $K_{3,3}$ is not planar. □

Problem 3.

Let G be a simple graph (not necessarily planar). Let d be the maximum degree of a vertex in G.

(a) Prove that $\chi(G) \leq d+1$. *Hint:* Use induction.

(b) Find an infinite family of non-isomorphic graphs for which equality holds in the above inequality.

Proof of (a). We will prove the statement by induction on d.

For the base case, when $d = 0$, the all vertices must have degree zero and must therefore be not connected. Hence $\chi(G) = 0 + 1 = 1$ color for each vertex.

Assume, that $\chi(G) = d$ for the maximum vertex degree in G of $d - 1$.

Now, for the maximum degree of d, we have by the inductive hypothesis that the chromatic number must be at least d.

Then, the vertex with degree d, call it v is adjacent to d other vertices, each of which can be given one of d colors.

But, v must have a different color from the other d vertices; hence the graph must be $(d+1)$-colorable.

Therefore $\chi(G) \leq d+1$. □

Proof of (b). Consider the complete graphs K_n. The degree of each vertex in K_n is $n-1$, so $\chi(K_n) = n$.

Each K_n is not isomorphic to K_{n-1} as these graphs have a different number of vertices.

So, for all n, we have that the infinite family of complete graphs has their chromatic number equal to one more than their maximum degree. □

Problem 4.

A polyhedron has 26 faces, 20 of which are triangles and 6 of which are quadrilaterals. Find the number of vertices and edges of this polyhedron.

Proof. We have that,

$$2E = \sum_{f \in F} \text{length}(f) = 20 \cdot 3 + 6 \cdot 4 = 84 \implies E = 42.$$

Then, by Euler's formula,

$$V - E + F = V - 42 + 26 = 2 \implies V = 18.$$

□

Problem 5.

An *n-gonal pyramid* is a polyhedron formed by connecting each vertex of an n-sided polygon with one additional vertex.

(a) What is the dual polyhedron of an n-gonal pyramid?

(b) What is the chromatic number of an n-gonal pyramid?

Proposition 1. An n-gonal pyramid is its own dual.

Proof of Proposition and (a). Let N be the n-gonal pyramid and N^* be its dual.

Then, for N, $V = n + 1$ for the vertices in the base n-gon and the apex.

$F = n + 1$ for the n-gon base and the n triangular faces.

$E = 2n$ for the n edges in the base n-gon and the n edges connecting the base vertices to the apex.

Next, for N^*, there is a vertex at each face of N, so $V^* = F = n + 1$.

Similarly, there is a face on each vertex of N, so $F^* = V = n + 1$.

Each edge in N corresponds to an edge in the dual N^*, so $E^* = N = 2n$.

In fact, N^* has the same structure as N.

The vertex in the face of the base n-gon in N is the apex in N^*.

The n vertices on triangular faces of N form the vertices of the n-gon base in N^*.

Thus, $N \cong N^*$ implies that an n-gonal pyramid is its own dual. □

Proof of (b). By Problem 3, since the maximum degree in N belongs to the apex vertex with degree n, then $\chi(N) = n + 1$. □

Problem 6.

Let G_n be the graph with vertex set $\{0, 1\}^n$ and an edge between x, $y \in \{0, 1\}^n$ if x and y differ at exactly 2 positions.

(a) Prove that $\chi(G_3) = 4$ and $\chi(G_4) = 4$.

(b) Prove that $\chi(G_n) \geq n$ for all positive integers n.

We have that the vertices of G_n are n-bit binary strings.

Define the parity of a binary string to be the number of ones modulo 2; i.e., strings with an even parity have an even number of ones, and those with an odd parity have an odd number of ones.

Note that, if two binary strings differ by exactly one position, then the must have opposite parities. If two binary strings differ by exactly two positions, then they must have the same parity.

Proof of (a). G_3 contains two connected components, each of which are K_4. Since K_4 is 4-colorable, then $\chi(G_3) = 4$ as well.

For G_4, we will split the 16 vertices into two sets, S_0 and S_1, by whether the first bit in the vertex binary string is 0 or 1. Each set contains an isomorphic copy of G_3 where each vertex binary string is prefixed by either 0 or 1.

Since we already know G_3 to be 4-colorable, then G_4 must be at least 4 colorable as well. Color the subgraph formed by S_0 with the same coloring c_4 that works for G_3.

Note that this coloring ensures that adjacent vertices with the same parity get different colors.

Now, we must connect the vertices between the subsets S_0 and S_1 of G_4.

Since the first vertex of each subset already differs by one position, then we must place edges between $v_0 \in S_0$ and $v_1 \in S_1$ where the last 3-bits of the binary strings differ by exactly one position.

So, we will connect vertices whose last 3-bits differ in parity. This connects vertices in S_0 with vertices in S_1 with opposing parity.

Hence, each vertex of S_0 will be connected to a corresponding vertex in S_1 which it was not connected to in S_0, since vertices in S_0 are connected to those of the same parity in S_0.

Thus, we can take the coloring c_4 from S_0 and swap the colors used for even and odd parity vertices (we have two choices for each parity because there are four colors). This will ensure that the vertices connected between S_0 and S_1 have different colors.

Hence we still have a 4-coloring in G_4 and $\chi(G_4) = 4$. □

Proof of (b). We will induct on n. For the base case, we have that $\chi(G_1) = 1$ as there are two disconnected vertices.

We have that G_2 is a 4-cycle, which is bipartite and hence 2-colorable; i.e., $\chi(G_2) = 2$.

We also have that $\chi(G_3) = \chi(G_4) = 4$ by part (a).

Now, assume that the statement holds for $n \geq 5$.

Consider a partition $\{S_0, S_1\}$ of the vertices of G_{n+1} based on the first bit of the vertex binary string as before in part (a).

Now, G_{n+1} contains two subgraphs, each of which are n-colorable by the inductive hypothesis.

Then, vertices are connected between the subgraphs if the tail n-bit binary strings differ by exactly one bit since these vertices already differ by the first bit.

So, the vertices in S_0 must be connected to vertices of the opposite tail parity in S_1, and they were not connected to the corresponding vertex in S_0.

Suppose that G_{n+1} was only n-colorable, with n colors for S_0 and S_1 each, possible permuting colors to avoid conflicts.

Suppose we assign a coloring c_n to S_0 such that that all even-parity vertices use a subset of the n colors and all odd-parity vertices use a different subset of the n colors.

However, when we attempt to copy this coloring c_n to S_1, we see that an even-parity vertex in S_0 must now be adjacent to an odd-parity vertex in S_1.

Since these two vertices were not adjacent in S_0, they may have been given the same color originally.

But now, they are forced to be adjacent in G_{n+1}, so they must have different colors.

Since this occurs for all pairs, we need an extra color.

Therefore, G_{n+1} must require at least $n + 1$ colors.

Thus, $\chi(G_{n+1}) \geq n + 1$. $\square$