Math 334 Homework 1

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1.

Problem. Let $x, y \in \mathbb{R}^n$ and $x, y \neq 0$. Prove $\langle x, y \rangle = |x||y| \implies \exists \lambda \in \mathbb{R} : x = \lambda y$.

2.

Problem. Let $x, y \in \mathbb{R}^n$.

a) Prove $2\left(|x|^2 + |y|^2\right) = |x+y|^2 + |x-y|^2$.

Proof. First, we will expand $|x+y|^2$,

$$\begin{aligned}
|x+y|^2 &= \langle x+y, x+y \rangle \\
&= \langle x, x+y \rangle + \langle y, x+y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&= |x|^2 + 2\langle x, y \rangle + |y|^2.
\end{aligned}$$

We will do the same for $|x-y|^2$,

$$\begin{aligned}
|x-y|^2 &= \langle x-y, x-y \rangle \\
&= \langle x, x-y \rangle - \langle y, x-y \rangle \\
&= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\
&= |x|^2 - 2\langle x, y \rangle + |y|^2.
\end{aligned}$$

So, combining these,

$$|x+y|^2 + |x-y|^2 = 2\left(|x|^2 + |y|^2\right),$$

as desired. □

b) Prove the polarization identity, $\langle x, y \rangle = \frac{|x+y|^2 - |x-y|^2}{4}$.

3.

Problem. Show $x_1, \ldots, x_m \in \mathbb{R}^n$ and $\forall i \neq j.\ \langle x_i, x_j \rangle = 0 \implies |x_1 + \cdots + x_m|^2 = |x_1|^2 + \cdots + |x_m|^2$.

4.

Problem. Let

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx, \quad f, g : [0,1] \longrightarrow \mathbb{R}.$$

Prove

$$\left| \int_0^1 f(x)g(x)\,dx \right| \le \left(\int_0^1 f(x)^2\,dx \right)^{1/2} \left(\int_0^1 g(x)^2\,dx \right)^{1/2}.$$

Proof. If $f(x)$ or $g(x)$ are zero $\forall x \in [0,1]$, then the Cauchy-Schwarz inequality holds as both sides are zero.

So, we now assume f and g are nonzero somewhere in the unit interval. We assemble a non-negative function $\forall t \in \mathbb{R}$. $h(t) \ge 0$,

$$\begin{aligned} h(t) &= |f(x) - tg(x)|^2 \\ &= \langle f - tg, f - tg \rangle \\ &= \int_0^1 (f(x) - tg(x))(f(x) - tg(x))\,dx \\ &= \int_0^1 f(x)^2 - 2tf(x)g(x) + t^2 g(x)^2\,dx \end{aligned}$$

We see that this function is a quadratic in t. We now wish to find the minimum value of this function, which occurs at

$$t_0 = \frac{\int_0^1 f(x)g(x)\,dx}{\int_0^1 g(x)^2\,dx}.$$

□