Math 334 Homework 1

Alexandre Lipson

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1.

Problem. Let $x, y \in \mathbb{R}^n$ and $x, y \neq 0$. Prove $\langle x, y \rangle = |x||y| \implies \exists \lambda \in \mathbb{R} : x = \lambda y$.

Proof. Note that $\langle x, y \rangle = |x||y| \implies$.

$$0 = |x|^2 - \frac{\langle x, y \rangle}{|y|^2} \tag{*}$$

Let $f(t) = |x - ty|^2$, which is zero when $x = ty$. We will expand f to observe its minimum,

$$\begin{aligned} |x - ty|^2 &= \langle x - ty, x - ty \rangle \\ &= |x|^2 - 2t\langle x, y \rangle + t^2|y|^2. \end{aligned}$$

Then, for the minimum at $t = \lambda$,

$$\begin{aligned} 0 &= f'(\lambda) \\ &= 2\lambda|y|^2 - 2\langle x, y \rangle \\ \lambda &= \frac{\langle x, y \rangle}{|y|^2}. \end{aligned}$$

So,

$$\begin{aligned} f(\lambda) &= |x|^2 - 2\frac{\langle x, y \rangle^2}{|y|^2} + \left(\frac{\langle x, y \rangle}{|y|^2}\right)^2 |y|^2 \\ &= |x|^2 - \frac{\langle x, y \rangle^2}{|y|^2} \end{aligned}$$

But, this quantity is zero by (*).

So, $f(\lambda) = |x - \lambda y|^2 = 0$, which occurs when

$$x = \lambda y, \quad \lambda = \frac{\langle x, y \rangle}{|y|^2} = \frac{|x|}{|y|}.$$

□

2.

Problem. Let $x, y \in \mathbb{R}^n$.

a) Prove $2\left(|x|^2 + |y|^2\right) = |x+y|^2 + |x-y|^2$.

Proof. First, we will expand $|x+y|^2$,

$$\begin{aligned} |x+y|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x+y\rangle + \langle y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= |x|^2 + 2\langle x, y\rangle + |y|^2. \end{aligned}$$

We will do the same for $|x-y|^2$,

$$\begin{aligned} |x-y|^2 &= \langle x-y, x-y\rangle \\ &= \langle x, x-y\rangle - \langle y, x-y\rangle \\ &= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= |x|^2 - 2\langle x, y\rangle + |y|^2. \end{aligned}$$

So, combining these,

$$|x+y|^2 + |x-y|^2 = 2\left(|x|^2 + |y|^2\right),$$

as desired. ∎

b) Prove the polarization identity, $\langle x, y\rangle = \frac{|x+y|^2 - |x-y|^2}{4}$.

Proof. We note that, with the commutativity of the inner product,

$$\begin{aligned} \langle a+b, a+b\rangle &= \langle a, a+b\rangle + \langle b, a+b\rangle \\ &= \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle \\ &= \langle a, a\rangle + 2\langle a, b\rangle + \langle b, b\rangle. \end{aligned}$$

Furthermore, scalars distribute over the inner product. For $x, y \in \mathbb{R}^n,\ a \in \mathbb{R}$,

$$\langle x, ay\rangle = a\langle x, y\rangle = \langle ax, y\rangle.$$

So, $\langle x-y, x-y\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle$.

So, the identity can expand,

$$\begin{aligned} \langle x, y\rangle &= \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle\right) \\ &= \frac{1}{4}\left(\langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle\right) - \left(\langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle\right) \\ &= \frac{1}{4}\left(4\langle x, y\rangle\right) \\ &= \langle x, y\rangle, \end{aligned}$$

as desired. $\square$

3.

Problem. Show $x_1, \ldots, x_m \in \mathbb{R}^n$ and $\forall i \neq j.\ \langle x_i, x_j \rangle = 0 \implies |x_1 + \cdots + x_m|^2 = |x_1|^2 + \cdots + |x_m|^2$.

Proof. We will prove the statement by induction on $m \in \mathbb{Z}^+$.

For the base case, when $m = 2,\ \langle x_1, x_2 \rangle = 0$,

$$\begin{aligned} x_1 + x_2{}^2 &= \langle x_1 + x_2, x_1 + x_2 \rangle \\ &= |x_1|^2 + 2\langle x_1, x_2 \rangle + |x_2|^2 \\ &= |x_1|^2 + |x_2|^2. \end{aligned}$$

We will assume that the $m = k$ case holds by the Inductive Hypothesis,

$$|x_1 + \cdots + x_m|^2 = |x_1|^2 + \cdots + |x_m|^2.$$

Then, for the following $m = k + 1$ case,

$$|x_1 + \cdots + x_m + x_{m+1}|^2 = \langle x_1 + \cdots + x_m + x_{m+1}, x_1 + \cdots + x_m + x_{m+1} \rangle.$$

We distribute only the $x_{m+1}{}^{\text{th}}$ term,

$$|x_1 + \cdots + x_m + x_{m+1}|^2 = \langle x_1 + \cdots + x_m, x_1 + \cdots + x_m \rangle + 2\langle x_{m+1}, x_1 + \cdots + x_m \rangle + \langle x_{m+1}, x_{m+1} \rangle.$$

We see that the first term becomes $|x_1|^2 + \cdots + |x_m|^2$ by the I.H., and that the last term is $|x_{m+1}|^2$. Now, for the middle term, if we distribute x_{m+1} across the right hand side of the inner product, we will be left with inner products of the form $\langle x_{m+1}, x_i \rangle$ where $1 \leq i \leq m$. However, $m + 1 \neq i$ for all such i. Therefore, by the problem statement, the middle term must be zero.

Thus,

$$|x_1 + \cdots + x_{m+1}|^2 = |x_1|^2 + \cdots + |x_m + 1|^2.$$

Since the base case $m = 2$ holds, and the inductive case $m = k + 1$ holds when $m = k$ holds, then the statement is true for all $m \in \mathbb{Z}^+$.

$\square$

4.

Problem. Let

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx, \quad f, g : [0, 1] \longrightarrow \mathbb{R}.$$

Prove

$$\left| \int_0^1 f(x)g(x)\,dx \right| \leq \left(\int_0^1 f(x)^2\,dx \right)^{1/2} \left(\int_0^1 g(x)^2\,dx \right)^{1/2}.$$

Proof. If $f(x)$ or $g(x)$ are zero $\forall x \in [0, 1]$, then the Cauchy-Schwarz inequality holds as both sides are zero.

So, we now assume f and g are nonzero somewhere in the unit interval. We assemble a non-negative function $\forall t \in \mathbb{R}$. $h(t) \geq 0$, which gives a notion of closeness to f and g being scalar multiples of one another as in problem 1.

$$\begin{aligned} h(t) &= |f(x) - tg(x)|^2 \\ &= \langle f - tg, f - tg \rangle \\ &= \int_0^1 (f(x) - tg(x))\,(f(x) - tg(x))\; dx \\ &= \int_0^1 f(x)^2 - 2tf(x)g(x) + t^2 g(x)^2 \, dx \end{aligned}$$

We see that this function is a quadratic in t. We now wish to find the minimum value of this function, which occurs at

$$t_0 = \frac{\int_0^1 f(x)g(x)\,dx}{\int_0^1 g(x)^2\,dx}.$$

So, at this t_0 and $\forall t \in \mathbb{R}$,

$$\begin{aligned} 0 \leq h(t_0) &= \int_0^1 f(x)^2\,dx - 2\frac{\left(\int_0^1 f(x)g(x)\,dx\right)^2}{\int_0^1 g(x)^2\,dx} + \left(\frac{\int_0^1 f(x)g(x)\,dx}{\int_0^1 g(x)^2\,dx}\right)^2 \int_0^1 g(x)^2\,dx \\ &= \int_0^1 f(x)^2\,dx - \frac{\left(\int_0^1 f(x)g(x)\,dx\right)^2}{\int_0^1 g(x)^2\,dx} \end{aligned}$$

$$\frac{\left(\int_0^1 f(x)g(x)\,dx\right)^2}{\int_0^1 g(x)^2\,dx} \leq \int_0^1 f(x)^2\,dx$$

$$\begin{aligned} \left(\int_0^1 f(x)g(x)\,dx\right)^2 &\leq \int_0^1 f(x)^2\,dx \int_0^1 g(x)^2\,dx \\ \left|\int_0^1 f(x)g(x)\,dx\right| &\leq \left(\int_0^1 f(x)^2\,dx\right)^{1/2}\left(\int_0^1 g(x)^2\,dx\right)^{1/2}. \end{aligned}$$

So, original Schwarz form of the inequality holds. $\square$

5.

Problem. Let $\mathbb{Q}$ be the set of all rationals.

Let $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid \forall i = 1, 2,\ x_i \in \mathbb{Q},\ 0 \leq x_i \leq 1\}$.

a) Describe $\mathring{S}, \partial S$

Proposition. $\mathring{S} = \emptyset$.

Proof. For $x \in S$ to be an interior point, it must satisfy the property $\exists r > 0.\ B_r(x) \subset S$. However, since there is an irrational between any two rational numbers, if this ball contains another point $x_0 \in S$, then it must also contain an irrational number a between x and x_0, so $a \in B_r(x)$. But, $a \notin S$ by definition of S. Thus, $B_r(x) \subset S$ no longer holds.

If we restrict the radius of the ball around x to not include any irrationals, we are forced to set $r = 0$, which no longer constitutes a neighborhood around x

Thus S contains no interior points. $\square$

Proposition. $\partial S = \{(x_1, x_2) \in \mathbb{R}^2 \mid \forall i = 1, 2,\ 0 \leq x_i \leq 1\}$, *the unit square.*

Proof. ($\supset$) Since S is a subset of all rational numbers, S^c contains all irrational numbers.

By the reasoning in the previous proposition, all balls around $x \in S$ must contain both $x_0 \in S$ and $a \in \mathbb{R} \setminus \mathbb{Q}$ as well. Thus,

$$\forall r > 0.\ B_r(x) \cap S \neq \emptyset \ \wedge\ B_r(x) \cup S^c \neq \emptyset.$$

So, $S \subset \partial S$.

Then, at the same time, we can choose any irrational $y \in R^2$ in the unit square such that a ball around x contains rationals. So, for all such irrational y,

$$\forall r > 0.\ B_r(y) \cap S \neq \emptyset \ \wedge\ B_r(y) \cup S^c \neq \emptyset.$$

So, for $T = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_1, y_2 \in \mathbb{R} \setminus \mathbb{Q},\ 0 \leq y_1, y_2 \leq 1\}$, $T \subset \partial S$.

Since $S \subset \partial S$ and $T \subset \partial S$, then $\{(a, b) \in \mathbb{R}^2 \mid 0 \leq a, b \leq 1\} = S \cup T \subset \partial S$.

($\subset$) If $x \in \partial S$ is rational and in the unit square, then $x \in S$ as well. If x is instead irrational, then it belongs in T. Thus, $\partial S \subset S \cup T$. $\square$

b) Determine if S in open, and if S is closed.

Proposition. *S is not open.*

Proof. We will consider the point (1,1). We see that $(1, 1) \in \partial S$ and $(1, 1) \in S$. So, $\partial S \cap \neq \emptyset$. Therefore, S is not open. $\square$

Proposition. *S is not closed.*

Proof. Let $a \in [0, 1]$ be irrational. Then, $(0, a) \in \partial S$, but $(0, a) \notin S$. So, $\exists x.\ x \in \partial S \wedge x \notin S$. Thus, $\partial S \not\subset$, which implies that S is not closed. $\square$

c) Describe $\overline{S}, \overset{\circ}{\overline{S}}$.

Proposition. $\overline{S} = \partial S$.

Proof. $\overline{S} = S \cup \partial S$, but $S \subset \partial S$. So, $\overline{S} = \partial S$. $\square$

Proposition. $\overset{\circ}{\overline{S}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1, x_2 < 1\}$.

Proof. First, we note that $\overset{\circ}{\overline{S}} \subset \overline{S}$. If $x \in \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 = 0 \vee x_1, x_2 = 1\}$, then $\exists r > 0,\ B_r(x)$ such that $\exists (y_1, y_2) \in B_r(x)$ where $y_1 < 0$, $y_2 < 0$, $1 < y_1$, or $1 < y_2$. But then $B_r(x)$ contains elements outside of $\overline{S}$. Thus, $B_r(x) \cap \overline{S}^c \neq \emptyset$, which means that x must have been a boundary point, and not an interior point. So, we restrict $\mathring{\overline{S}}$ to strict inequalities between 0 and 1. $\square$

6.

Problem. Let A be a finite set with an odd number of elements.

Let $f : A \longrightarrow A,\ \forall x \in A,\ f(f(x)) = x$. Show that f has a fixed point $\exists x.\ f(x) = x$.

What does this mean in the context of a ballroom dance?

Proof. Suppose, for a contradictions, that $\forall x \in A,\ f(x) \neq x$. Let $|A| = 2k + 1$ for some $k \in \underset{\geq 0}{\mathbb{Z}}$.

We will construct pairings of type $A \times A$ where the left hand side contains objects from A in the preimage of f, and the right hand side contains objects in the image. Since $f(x) \neq x$, then each element of A must appear only once per pairing.

If $x \in A$ appears in a pair, it must be as x in the preimage, or as $f(y) = x$ in the image for some $y \neq x$ in the preimage. If x appears as x in the preimage, its pair must be distinct value $f(x) \in A$. If x appears as $f(y)$ in the image, its preimage pair must by y as $f(f(y)) = y$.

Since A has an odd number of elements $2k+1$, we cannot form k mutually distinct pairs with all of its elements without leaving one element unpaired. Therefore, to map all elements of A with f, we are forced to pair some x with itself, contradicting our assumption that f had to fixed points. $\square$

In the context of a ballroom dance, this means that there must be an even number of people in order for all persons to have a dance partner.