Math 336 Homework 1

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Problem 1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.

(b) $1/z = \overline{z}$.

(c) $\mathrm{Re}(z) = 3$.

(d) $\mathrm{Re}(z) > c$, (resp., $\geq c$) where $c \in \mathbb{R}$.

(e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.

(f) $|z| = \mathrm{Re}(z) + 1$.

(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Proof. □

Problem 2. With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

Proof. □

Problem 3. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

[Hint: Why can one assume that z is real? It then suffices to prove that

$$(r - w)(r - \overline{w}) \leq (1 - rw)(1 - r\overline{w})$$

with equality appropriate for r and $|w|$.]

Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w - z}{1 - \overline{w}z}$$

satisfies the following conditions:

(i) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$]

Proof. □

Problem 4. Consider the function defined by

$$f(x + iy) = \sqrt{|x||y|}, \quad \text{whenever } x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Reimann equations at the origin, yet f is not holomorphic at 0.

Proof. □

Problem 5. In this problem we will go through a proof of the Fundamental Theorem of Algebra, that is: If

$$p(z) = a_n z^n + \cdots + a_0$$

is a polynomial with an $a_n \neq 0$, then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

(i) Suppose for the sake of contradiction that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Show that the function $g(z) = |p(z)|$ has a minimum at some point $z_0 \in \mathbb{C}$. (Hint: Remember that $\mathbb{C}$ is definitely not compact!)

(ii) Consider the function $q(z) = \frac{1}{|p(z)|} p(z - z_0)$. Show that q is a polynomial with $q(0) = 1$ and that $|q(z)|$ has its minimum at $z = 0$.

(iii) Show that for any sufficiently small $\epsilon > 0$, there is some θ for which $|q(\epsilon e^{i\theta})| < 1$, which provides the desired contradiction.

Proof. □

Problem 6. Consider the function f defined on R by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that f is indefinitely differentiable on R, and that $f^{(n)}(0) = 0$ for all $n \geq 1$. Conclude that f does not have a converging power series expansion $\sum_{n=0}^{\infty} a_n x^n$ for x near the origin.

Proof. $\square$

Problem 7. Show that if $|a| < r|b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Proof. $\square$