Math 462 Homework 1

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Problem 1. Solving systems of linear recurrences for single homogeneous linear recurrences.

Let $\mathbf{f}(n) = [f_1(n), \ldots, f_d(n)]^T$. Let A be the $d \times d$ matrix $(a_{ij})_{i,j=1}^{d}$ such tha;t,

$$\mathbf{f}(n+1) = A\,\mathbf{f}(n).$$

(a) Assume A diagonalizable over $\mathbb{C}$ with d eigenvalues $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$, and $\mathbb{C}^d$ has eigenbasis $\mathbf{v}_1, \ldots, \mathbf{v}_d$ where $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for all i.

Let $c_1, \ldots, c_d \in \mathbb{C}$ such that

$$\mathbf{f}(0) = \sum_{i=1}^{d} c_i \mathbf{v}_i.$$

Prove

$$\mathbf{f}(n) = \sum_{i=1}^{d} c_i \lambda_i^n \mathbf{v}_i.$$

(b) Suppose $|\lambda_i| \leq |\lambda_1|$ for all i. Assume $c_1 \neq 0$ and that every coordinate of $\mathbf{v}_1$ is nonzero. Prove that, as $n \to \infty$,

$$\mathbf{f}(n) \sim c_1 \lambda_1^n \mathbf{v}_1.$$

(c) Suppose that $f : \mathbb{N} \to \mathbb{R}$ satisfies a homogeneous linear recurrence

$$f(n+d) = \sum_{i=1}^{d} a_i f(n+d-i)$$

for constants a_i. Explain how to use the above method with the functions $f_i(n) = f(n+d-i)$ to find a formula for $f(n)$. Assume that the matrices we obtain are diagonalizable.

Proof of (a). Let P be the eigenbasis matrix with eigenvector columns. Let $\mathbf{c} = [c_1, \ldots, c_d]^T$.

Since A is diagonalizable, then we have that $A = P\Lambda P^{-1}$ where Λ is the eigenvalue diagonal matrix. So,

$$A^n = P\Lambda^n P^{-1} \implies A^n P = P\Lambda^n.$$

We will show that $\mathbf{f}(n) = P\Lambda^n \mathbf{c}$.

By induction, we have

$$\mathbf{f}(n+1) = A\,\mathbf{f}(n) \implies \mathbf{f}(n) = A^n \mathbf{f}(0).$$

But, we have $A^n P = P\Lambda^n$, and we are given that $\mathbf{f}(0) = P\mathbf{c}$, so $\mathbf{f}(n) = A^n P\mathbf{c} = P\Lambda^n\mathbf{c}$. $\square$

Proof of (b). Since $|\lambda_i| < |\lambda_1|$ for all $i \neq 1$, then $\left|\frac{\lambda_i}{\lambda_1}\right| < 1$.

We will factor out $c_1\lambda_1^n$ from $\mathbf{f}(n)$,

$$\mathbf{f}(n) = c_1\lambda_1^n \left(\mathbf{v}_1 + \sum_{i=2}^{d} \frac{c_i}{c_1}\left(\frac{\lambda_i}{\lambda_1}\right)^n\right).$$

Since $\left|\frac{\lambda_i}{\lambda_1}\right| < 1$, then the terms of the finite sum will vanish as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{f}(n) &= \lim_{n\to\infty} c_1\lambda_1^n \left(\mathbf{v}_1 + \sum_{i=2}^{d} \frac{c_i}{c_1}\left(\frac{\lambda_i}{\lambda_1}\right)^n\right) \\
&= \lim_{n\to\infty} (c_1\lambda_1^n) \lim_{n\to\infty} \left(\mathbf{v}_1 + \sum_{i=2}^{d} \frac{c_i}{c_1}\left(\frac{\lambda_i}{\lambda_1}\right)^n\right) \\
&= \lim_{n\to\infty} c_1\lambda_1^n\mathbf{v}_1 + \lim_{n\to\infty} \sum_{i=2}^{d} \frac{c_i}{c_1}\left(\frac{\lambda_i}{\lambda_1}\right)^n \\
&= \lim_{n\to\infty} c_1\lambda_1^n\mathbf{v}_1 + \sum_{i=2}^{d} \frac{c_i}{c_1} \lim_{n\to\infty}\left(\frac{\lambda_i}{\lambda_1}\right)^n \\
&= \lim_{n\to\infty} c_1\lambda_1^n\mathbf{v}_1.
\end{aligned}$$

Hence, $f(n) \sim c_1\lambda_1^n\mathbf{v}_1$. $\square$

Proof of (c). Let $f_i(n) = f(n+d-i)$. In particular, $f_d(n) = f(n+d-d) = f(n)$.

Note that we have a relation between subsequent $f_i(n)$:

$$f_{i+1}(n+1) = f((n+1)+d-(i+1)) = f(n+d-i) = f_i(n),$$

where $i \in [1, d-1]$.

So, $f_i(n+1) = f_{i-1}(n)$ for $i \in [2, d]$.

Consider the case when $i = 1$,

$$f_1(n+1) = f(n+d) = \sum_{i=1}^{d} a_i f(n+d-i) = \sum_{i=1}^{d} a_i f_i(n).$$

Therefore we can write $\mathbf{f}(n+1) = A\,\mathbf{f}(n)$ using the diagonal matrix A given by,

$$A_{d\times d} = \begin{bmatrix} a_1 & \cdots & a_d \\ I_{d-1} & & \vec{0} \end{bmatrix}.$$

So, by part (a), we have that $\mathbf{f}(n) = A^n\mathbf{f}(0)$ where $\mathbf{f}(0) = [f(d-1), \ldots, f(0)]^T$.

Since the last component of $\mathbf{f}(n)$, $f_d(n) = f(n)$, then, to recover $f(n)$, we must find A^n.

Since A is diagonalizable by assumsption, then we diagonalize A and exponentiation its diagonal matrix.

Then, we can multiply $A^n\mathbf{f}(0)$ and consider the last component of the result, which yields $f(n)$. $\square$

Problem 2. Let $f : \mathbb{N} \to \mathbb{R}$ satisfy a linear recurrence of the form

$$f(n+d) = \sum_{i=1}^{d} a_i f(n+d-i)$$

where each a_i is either 0 or 1. Prove that $f(n) = o(2^n)$.

Proof. By Problem 1 part (b), we have that

$$\mathbf{f}(n) \sim c_1 \lambda_1^n \mathbf{v}_1,$$

where λ_1 is the eigenvalue with largest magnitude, c_1 is a complex constant, and $\mathbf{v}_1$ is an eigenvector, which is also constant.

Each λ_i, not necessarily distinct, is root of the characteristic polynomial.

The characteristic polynomial of the homogeneous linear occurrence gives

$$\lambda^d = \sum_{k=1}^{d-1} a_{d-k} \lambda^k.$$

With all a_i either 0 or 1, then we must have,

$$\lambda^d \leq \sum_{k=1}^{d-1} \lambda^k.$$

By the triangle inequality, we have that,

$$|\lambda|^d \leq \left| \sum_{k=1}^{d-1} \lambda^k \right| \leq \sum_{k=1}^{d-1} |\lambda|^k = \frac{1-|\lambda|^d}{1-|\lambda|}.$$

Suppose, that $|\lambda| \geq 2 \implies 2|\lambda|^d \leq |\lambda|^{d+1}$. Then,

$$|\lambda|^{d+1} - |\lambda|^d \leq |\lambda|^d - 1 \implies |\lambda|^{d+1} \leq 2|\lambda|^d - 1 \leq |\lambda|^{d+1} - 1,$$

a contradiction.

Therefore, for all solutions λ to the characteristic polynomial, we must have that $|\lambda| < 2$.

Therefore,

$$\mathbf{f}(n) \sim c_1 \lambda_1^n \mathbf{v}_1 \leq c_1 2^n \mathbf{v}_1.$$

Since $f(n)$ is the last component of this vector function, then $f(n) = o(2^n)$. $\square$