Math 134 Homework 4

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1

Prove that a non-constant linear function is uniformly continuous on the real line.

Proposition. *The function* $f(x) = ax + b, \quad x \neq 0$ *is uniformly continuous on* $x \in \mathbb{R}$.

Proof. By the definition on uniform continuity, for every $\epsilon > 0$, there is a $\delta > 0$ such that, for all x, y, $|x - y| < \delta$ implies $\Big|(ax + b) - (ay + b)\Big| < \epsilon$.

Let $\epsilon > 0$ be given.

Define $\delta = \frac{\epsilon}{|a|}$.

Assume $|x - y| < \delta = \frac{\epsilon}{|a|}$.

Then,

$$\begin{aligned} \Big|(ax + b) - (ay + b)\Big| &= |ax - ay| \\ &= |a||x - y| \\ &< |a|\frac{\epsilon}{|a|} = \epsilon. \end{aligned}$$

Since, $\Big|(ax + b) - (ay + b)\Big| < \epsilon$, the proposition holds. □

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Show that the function $f(x) = x^2$ is not uniformly continuous on the real line.

Proposition. *For every* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that, for all* x *and* y*,* $|x - y| < \delta$ *implies* $|f(x) - f(y)| < \epsilon$.

Proof. For a proof by contradiction, we will negate the statement.

We will that that, there is an $\epsilon > 0$, with all $\delta > 0$ such that for some x and y, $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$ are true.

Let $\epsilon = 1$. Choose any $\delta > 0$.

Let $x = \frac{1}{\delta}, \quad y = \frac{1}{\delta} + \frac{\delta}{2}$.

Notice that,

$$|x - y| = |\frac{1}{\delta} + \frac{\delta}{2} - \frac{1}{\delta}| = \frac{\delta}{2} < \delta.$$

We have that,

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |x - y||x + y| \\ &= \frac{\delta}{2}|x + y| \\ &= \frac{\delta}{2}\left|\frac{1}{\delta} + \frac{\delta}{2} + \frac{1}{\delta}\right| \\ &= \frac{\delta}{2}\left(\frac{2}{\delta} + \frac{\delta}{2}\right) \\ &= 1 + \frac{\delta^2}{4} > 1 \geq \epsilon. \end{aligned}$$

So, the negation of the statement holds.

Thus, by contradiction, the proposition is false. Therefore $f(x)$ is not uniformly continuous on the real line.

$\square$

5.2

12

(a) Given that $P = x_0, x_1, \ldots, x_n$ is an arbitrary partition of $[a, b]$, find $L_f(P)$ and $U_f(P)$ for $f(x) = x + 3$.

(b) Evaluate $\int_a^b f(x)dx$.

Since $f'(x) = 1 > 0$ for all $x \in \mathbb{R}$, then f is strictly increasing.

Then the maximum of f on any part of P is $M_i = x_i + 3$ and the minimum is $m_i = x_{i-1} + 3$.

Note that $\Delta x_i = x_i - x_{i-1}$ and $\sum_{i=1}^{n} \Delta x_i = \frac{b-a}{n}$.

The upper bound $U_f(P)$ of f on P can be found with M_i,

$$
\begin{aligned}
U_f(P) &= \sum_{i=1}^{n} M_i \Delta x_i \\
&= \sum_{i=1}^{n} (x_i + 3)\, \Delta x_i \\
&= \sum_{i=1}^{n} x_i \Delta x_i + 3 \sum_{i=1}^{n} \Delta x_i \\
&= \sum_{i=1}^{n} x_i \Delta x_i + 3(b-a).
\end{aligned}
$$

Similarly, the lower bound $L_f(P)$ of f on P can be found with m_i,

$$
\begin{aligned}
L_f(P) &= \sum_{i=1}^{n} m_i \Delta x_i \\
&= \sum_{i=1}^{n} (x_{i-1} + 3)\, \Delta x_i \\
&= \sum_{i=1}^{n} x_{i-1} \Delta x_i + 3(b-a).
\end{aligned}
$$

So,

$$
L_f(P) = \sum_{i=1}^{n} x_{i-1} \Delta x_i + 3(b-a), \quad U_f(P) = \sum_{i=1}^{n} x_i \Delta x_i + 3(b-a).
$$

For (b), we note that the mean of $L_f(P)$ and $U_f(P)$ is between these two values. So, provided that, $\Delta x > 0$,

$$
L_f(P) < \sum_{i=1}^{n} \frac{x_i + x_{i-1}}{2} \Delta x_i + 3(b-a) < U_f(P).
$$

Since $\Delta x_i = x_i - x_{i-1}$, the middle term becomes

$$
\begin{aligned}
&\sum_{i=1}^{n} \frac{x_i + x_{i-1}}{2} (x_i - x_{i-1}) + 3(b-a) \\
&= \frac{1}{2} \sum_{i=1}^{n} x_i^2 - x_{i-1}^2 + 3(b-a).
\end{aligned}
$$

We notice that the sum of the alternating terms, after cancellation, ultimately yields $x_n^2 - x_0^2$, where $x_n = b, x_0 = a$.

So,

$$
\frac{1}{2} \sum_{i=1}^{n} x_i^2 - x_{i-1}^2 = \frac{b^2 - a^2}{2}.
$$

Then,

$$L_f(P) < \frac{b^2 - a^2}{2} + 3(b - a) < U_f(P).$$

By uniqueness of the integral,

$$\int_a^b (x + 3)dx = \frac{b^2 - a^2}{2} + 3(b - a).$$

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Show that the proposition holds.

Proposition. *If f is continuous and decreasing on $[a, b]$ and P is a regular partition on $[a, b]$, then $U_f(P) - L_f(P) = [f(a) - f(b)]\Delta x$.*

Since f is decreasing, we define the minimum m_i and the maximum M_i on the i^{th} part of P as,

$$m_i = f(x_i), \quad M_i = f(x_{i-1}).$$

Since P is regular, Δx is constant.

Then,

$$L_f(P) = \sum_{i=1}^{n} f(x_i)\Delta x, \quad U_f(P) = \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

So,

$$U_f(P) - L_f(P) = \sum_{i=1}^{n} \left[f(x_{i-1}) - f(x_i)\right] \Delta x.$$

Note that the alternating sum, $f(x_0) - f(x_1) + f(x_1) - f(x_2) + \cdots + f(x_{n-1}) - f(x_n)$ will reduce to $f(x_0) - f(x_n)$, where $x_0 = a$, and $x_n = b$.

So,

$$\sum_{i=1}^{n} \left[f(x_{i-1}) - f(x_i)\right] \Delta x = \left[f(a) - f(b)\right] \Delta x$$

Therefore, the proposition holds.

25–30

Assume that f and g are continuous, that $a < b$, and that

$$\int_a^b f(x)dx > \int_a^b g(x)dx.$$

Which of the statements necessarily holds for all partitions P of $[a, b]$? Justify your answer.

Let $I_f = \int_a^b f(x)dx, \quad I_g = \int_a^b g(x)dx$. So $I_g < I_f$.

From the definition of the integral in 5.2.6, we construct the following relationship,

$$L_g(P) \leq I_g < I_f \leq U_f(P). \tag{1}$$

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$L_g(P) < U_f(P)$

Clearly holds by equation 1.

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$L_g(P) < L_f(P)$

By counterexample, we will provide an f and g such that $L_g(P) \geq L_f(P)$.

Use the course partition P on the interval $I = [0, c]$ of $\{0, c\}$ such that $\Delta x = c$.

Let $f(x) = x, \quad g(x) = 0$.

Since f is increasing on I, its lower bound on the coarse partition will occur at the beginning of P, at $x = 0$.

Since g is constant, its lower and upper bounds will be the same for all x in P.

So, $f(0) = 0 = g(0)$ and therefore $L_f(P) = L_g(P)$.

We also see that, while f has some positively signed area under its curve between the x-axis, g has no area under its curve.

So, $I_f > I_g$ still holds.

Therefore, by counterexample, the statement does not always hold.

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$L_g(P) < \int_a^b f(x)dx$

Follows from equation 1 and the definition of I_f.

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$U_g(P) < U_f(P)$

Let $f(x) = 1, \quad g(x) = x$.

Use the course partition P on $[0, 1]$ of $\{0, 1\}$.

Since f is constant, then $U_f(P) = 1$.

Since g is strictly increasing throughout P, then $U_g(P)$ will occur on the end of P, where $x = 1$. So, $U_g(P) = g(1) = 1$.

So,

$$U_f(P) = U_g(P).$$

By examples (4) and (5) in the textbook, $I_f = 1$ and $I_g = 1/2$.

So the initial condition $I_f > I_g$ is satisfied.

Therefore, the statement does not always hold.

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$U_f(P) > \int_a^b g(x)dx$

Also holds by I_g and equation 1.

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$U_g(P) < \int_a^b f(x)dx$

Use the coarse partition P on $[0, c]$ of $\{0, c\}$ such that $x_0 = 0$, $x_1 = c$, and $\Delta x = c$.

Define $k > 0$ such that $\frac{c}{2} < k \leq c$.

Let $f(x) = k, \quad g(x) = x$.

Then, by example (4) and (5),

$$I_g = \int_0^c xdx = \frac{c^2}{2}, \quad I_f = \int_0^c kdx = ck$$

Since $k > \frac{c}{2}$, then $ck > \frac{c^2}{2}$ So, the initial condition $I_f > I_g$ is satisfied.

Since g is increasing, it's upper bound on P occurs at the right endpoint of P, at $x = c$.

So,

$$U_g(P) = \sum_{i=0}^{1} g(x_i)\Delta x = (g(0) + g(c))\, c = c^2.$$

Since $0 < k \leq c$, then $ck \leq c^2$.

So, $U_g(P) \geq I_f$.

Therefore, the statement does not always hold.

5.3

21

Suppose that f is differentiable with $f'(x) > 0$ for all x, and suppose that $f(1) = 0$. Set

$$F(x) = \int_0^x f(t)dt.$$

Justify each statement and make a rough sketch of the graph of F.

(a) F is continuous.

Since f is differentiable, it is continuous. Then, by 5.3.5, F is also continuous.

(b) F is twice differentiable.

Since $F'(x) = f(x)$ by 5.3.5 and f is differentiable, then $F''(x) = f'(x)$. So, F is twice differentiable.

(c) $x = 1$ is a critical point for F.

Since $F' = f$ and $f(1) = 0$, then $x = 1$ is a critical number of F.

(d) F takes on a local minimum at $x = 1$.

Since $F''(x) = f'(x)$ and $f'(x) > 0$ for all $x > 0$, then all critical points on F will be local minima by the second derivate test.

(e) $F(1) < 0$.

Since $f'(x) > 0$ for all $x > 0$, then f is strictly increasing.

Since $f(1) = 0$, then $f(x) < 0$ for all $x < 1$.

Since $F'(x) = f(x)$ and $f(x) < 0$ for all $x < 1$, then $F(x)$ is strictly decreasing when $x < 1$.

Since F is strictly decreasing when $x < 1$ and $F(0) = \int_0^0 f(t)dt = 0$, then $F(1) < 0$.

We can produce a simple sketch around the interval $(0, 1)$ using the data that $F(0) = 0$; and that F is strictly decreasing until $F(1)$, which is a local minima.

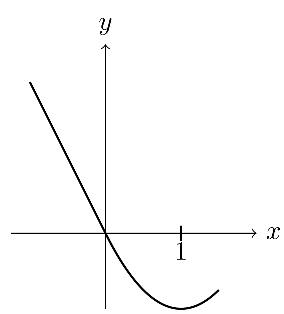

Figure 1: Sketch of F.

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Let f be everywhere continuous and set

$$F(x) = \int_0^x t \left[\int_1^t f(u)du \right] dt.$$

Find each of the following,

(a) $F'(x)$

By 5.3.5, $F'(x) = x \int_1^x f(u)du$.

(b) $F'(1)$

Since $F'(1) = x\int_1^1 f(u)du = x \cdot 0$ by 5.3.4, then $F'(1) = 0$.

(c) $F''(x)$

Again, by 5.3.5,

$$\begin{aligned} F''(x) &= \int_1^x f(u)du + x\frac{d}{dx}\left[\int_1^x f(u)du\right] \\ &= \int_1^x f(u)du + xf(x). \end{aligned}$$

(d) $F''(1)$

$F''(1) = 0 + 1 \cdot f(1)$ by 5.3.4. So, $F''(1) = f(1)$.