Math 334 Homework 9

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December 4, 2024

Problem (1)**.** The *Laplacian* of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by $\Delta f = \operatorname{div}(\nabla f)$. We will work in $\mathbb{R}^3$. Let $\mathbf{r} = (x, y, z)$ and $g(\mathbf{r}) = \frac{1}{|\mathbf{r}|}$.

(a) Compute ∇g at $\mathbf{r} \neq 0$.

(b) Show that $\Delta g = 0$ at $\mathbf{r} \neq 0$.

(c) Consider any sphere S with outward orientation centered at the origin.
Show directly that $\iint_S \nabla g \cdot d\mathbf{S} = -4\pi$.

(d) Explain why (b) and (c) do not contradict the divergence theorem.

(e) Let $E \subset \mathbb{R}^3$ be a regular domain with piecewise smooth boundary $R = \partial E$. Suppose that E contains the origin. Show that $\iint_R \nabla g \cdot d\mathbf{S} = -4\pi$.

Proof. (a) With $\frac{1}{|\mathbf{r}|} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, we have that $\nabla g = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

(b) For $\div(\nabla g)$, we will first consider the x partial,

$$\begin{aligned}
\frac{\partial}{\partial x}\left[-\frac{\mathbf{r}}{|\mathbf{r}|^3}\right] &= \frac{|\mathbf{r}|^3 - x\left(2x\left(\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\right)\right)}{|\mathbf{r}|^6} \\
&= \frac{|\mathbf{r}|^3 - 3x^2|\mathbf{r}|}{|\mathbf{r}|^6} \\
&= \frac{|\mathbf{r}|^2 - 3x^2}{|\mathbf{r}|^5}.
\end{aligned}$$

Then, repeating for the y and z partials, we have

$$\Delta g = \frac{1}{|\mathbf{r}|^5}(3|\mathbf{r}|^2 - 3(x^2+y^2+z^2)) = 0.$$

(c) Let the sphere S with radius R be parametrized by

$$\mathbf{r}(u, v) = (R\cos u \sin v, R\sin u \sin v, R\cos v),\ u \in [0, 2\pi],\ v \in [0, \pi].$$

So,

$$\begin{aligned}\mathbf{r}_u &= (-R\sin u\sin v, R\cos u\sin v, 0)\\ \mathbf{r}_v &= (R\cos u\sin v, R\sin u\cos v, -R\sin v)\\ \mathbf{r}_u\times\mathbf{r}_v &= (-R^2\cos u\sin^2 v, -R^2\sin u\sin^2 v, -R^2(\cos v\sin v))\\ &.\end{aligned}$$

But, we must flip the direction of the normal vector to ensure outward-facing normals,

$$\mathbf{r}_u\times\mathbf{r}_v = R^2(\cos u\sin^2 v, \sin u\sin^2 v, \cos v\sin v).$$

Since the magnitude of all $\mathbf{r}$ on S is the radius R, then $\nabla g(\mathbf{r}(u,v)) = -\frac{\mathbf{r}}{R^3}$.

Then,

$$\begin{aligned}\nabla g\cdot(\mathbf{r}_u\times\mathbf{r}_v) &= -(\cos^2 u\sin^3 v + \sin^2 u\sin^3 v + \cos^2 v\sin v)\\ &= -\sin v(\sin^2 v(\cos^2 u + \sin^2 u) + \cos^2 v)\\ &= -\sin v.\end{aligned}$$

So, we can compute,

$$-\int_0^\pi\int_0^{2\pi}\sin v\,du\,dv = -2\pi\int_0^\pi \sin v\,dv = -4\pi.$$

(d) The divergence theorem requires the vector field $\mathbf{F}$ to be defined everywhere on the domain E bounded by S, however, $\mathbf{F}$ is not defined at the origin.

(e) Since E is a regular domain that contains the origin, we can split E up into subdomains, one of which can be a small sphere around the origin, while the others can be simply connected. Since $\Delta g = 0$ where $\mathbf{r}\neq 0$, then the flux integral on the domains that do not contain the origin will vanish, and we will be left with the sole contribution of the domain containing the origin, which is -4π. ☐

Problem (2)**.** Fix $0 < a < b$ and consider the torus $T\subset\mathbb{R}^3$ formed by revolving the circle $(x-b)^2 + z^2 = a^2$ in the xz-plane about the z-axis.

(a) Find the surface area of T.

(b) Find the volume enclosed by T.

Proof of a. First, we will parametrize the cross sections of the torus, a circle in the xz-plane given by

$$(b + a\cos\theta, 0, a\sin\theta),\ \theta\in[0,2\pi].$$

Then, we can repeat for the y-coordinate as well,

$$(b + a\cos\theta, b + a\cos\theta, a\sin\theta),\ \theta\in[0,2\pi].$$

Next, we can rotate x and y components about the z-axis,

$$(\cos u(b + a\cos v), \sin u(b + a\cos v), a\sin v),\ u, v\in[0,2\pi].$$

So,

$$\begin{aligned}
\mathbf{r}_u &= (-\sin u(b + a\cos v), \cos u(b + a\cos v), 0) \\
\mathbf{r}_v &= (\cos u(-a\sin v), \sin u(-a\sin v), a\cos v) \\
\mathbf{r}_u \times \mathbf{r}_v &= (a\cos u\cos v(b + a\cos v), a\sin u\cos v(b + a\cos v), a\sin v(b + a\cos v)) \\
&= a(b + a\cos v)(\cos u\cos v, \sin u\cos v, \sin v) \\
|\mathbf{r}_u \times \mathbf{r}_v| &= a(b + a\cos v)\sqrt{\cos^2 v(\cos^2 u + \sin^2 u) + \sin^2 v} \\
&= a(b + a\cos v).
\end{aligned}$$

We can now compute the surface area integral,

$$\begin{aligned}
\iint_S dS &= \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos v)\, du\, dv \\
&= 4\pi^2 ab + 2\pi a^2 \int_0^{2\pi} \cos v\, dv \\
&= 4\pi^2 ab.
\end{aligned}$$

□

Proof of b. For the volume enclosed by T, by the divergence theorem, we wish to find an $\mathbf{F}$ such that $\operatorname{div}\mathbf{F} = 1$. We will use $\mathbf{F} = (0, 0, z)$.

Then, $\mathbf{F}(\mathbf{r}(u, v)) = a\sin v$.

So, with T enclosing the region E,

$$\begin{aligned}
\operatorname{vol} E = \iiint_E \operatorname{div}\mathbf{F}\, dV &= \oint\!\!\!\!\iint_T \mathbf{F}\cdot d\mathbf{S} \\
&= \int_0^{2\pi}\int_0^{2\pi} a\sin v(a\sin v(b + a\cos v))\, du\, dv \\
&= 2\pi a^2 \int_0^{2\pi} \sin^2 v(b + a\cos v)\, dv \\
&= 2\pi a^2 (b\pi + a\int_0^{2\pi} \cos v\sin^2 v\, dv) \\
&= 2\pi a^2 (b\pi + a\int_0^{0} x^2\, dx), \quad x = \sin v \\
&= 2\pi^2 a^2 b.
\end{aligned}$$

□

Problem (3)**.** Let C_r be the circle of radius r centered about the origin in the xz-plane and oriented anticlockwise when viewed from the positive y-axis. Suppose $\mathbf{F} \subset \mathbb{R}^3$ is a C^1 vector field on the compliment of the y-axis such that $\oint_{C_1} \mathbf{F}\cdot d\mathbf{x} = 5$ and $\operatorname{curl}\mathbf{F} = \left(\frac{z}{(x^2+z^2)^2}, 3, \frac{-x}{(x^2+z^2)^2}\right)$.

Compute $\forall r$, $\oint_{C_r} \mathbf{F}\cdot d\mathbf{x}$.

Proof. We already have $\oint_{C_1} \mathbf{F}\cdot d\mathbf{x}$, so we will consider the annular region parametrized by $\mathbf{s}(u, v) = (u\cos v, 0, u\sin v)$ where $u \in [1, r]$ and $v \in [0, 2\pi]$. With anticlockwise orientation of the boundary, the normals of this surface must point in positive y, so $\mathbf{s}_u \times \mathbf{s}_v = (0, u, 0)$.

So, using Stokes with $\operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 3u$, we will take the existing integral for C_1 and combine with our integral for radius 1 to r,

$$5 + \int_0^{2\pi} \int_1^r 3u \, du \, dv = 5 + 6\pi \left[\frac{u^2}{2} \right]_1^r = 5 + 3\pi(r^2 - 1).$$

Note that, if $r < 1$, then we will have $5 - 3\pi(1 - r^2)$ because we take away from the unit disk, with surface integral bounded from $u : r \to 1$. But, this is actually the same as the above. □

Problem (4)**.** Let $S \subset \mathbb{R}^3$ be the surface defined by $(x^2 + y^2)z = 1$, $z \in [1, 334]$ and oriented with normals pointing away from the z-axis. Let $E \subset \mathbb{R}^3$ be a regular domain bounded by S and $z \in [1, 334]$.

(a) Find a parametrization for S, and write an iterated integral which represents the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (0, 0, z)$.

(b) Find the volume of the region E.

(c) Let $\mathbf{G} = (\exp(y^2), \sin(x^2), z)$. Find the vector surface integral $\iint_S \mathbf{G} \cdot d\mathbf{S}$.

Proof. (a) We will parametrize S with $\mathbf{r}(u, v) = (u \cos v, u \sin v, \frac{1}{u^2})$ where $u \in [\frac{1}{\sqrt{334}}, 1]$ and $v \in [0, 2\pi]$.

So,

$$\begin{aligned} \mathbf{r}_u &= (\cos v, \sin v, \frac{-2}{u^3}) \\ \mathbf{r}_v &= (-u \sin v, u \cos v, 0) \\ \mathbf{r}_u \times \mathbf{r}_v &= \left(\frac{-2}{u^2} \cos v, \frac{-2}{u^2} \sin v, u \right). \end{aligned}$$

Since this normal vector has a positive z component, it agrees with the given normals of S.

Then, $\mathbf{F}(\mathbf{r}(u, v)) = (0, 0, \frac{1}{u^2})$, so $\mathbf{F} \cdot d\mathbf{S} = \frac{1}{u} \, du \, dv$.

So,

$$\int_0^{2\pi} \int_{\frac{1}{\sqrt{334}}}^1 \frac{1}{u} \, du \, dv = 2\pi \log u \Big|_{(334)^{-\frac{1}{2}}}^1 = \pi \log 334.$$

(b) We see that $\operatorname{div} \mathbf{F} = 1$. So, by the divergence theorem, the volume of the region E is given by the closed surface S', which is S combined with a top disk at $z = 334$ with radius $\frac{1}{\sqrt{334}}$, S_t with upward normals, and bottom disk at $z = 1$ with radius 1, S_b with downward normals.

The parametrization of S_t will have a fixed z component of 334 and, as a disk with upward normals, a normal vector given by $(0, 0, u)$. So, our vector surface integral in the field $\mathbf{F}$ will have an integrand $334u$. With a typical parametrization of a disk, our integral might look like $\int_0^{2\pi} \int_0^{\frac{1}{\sqrt{334}}} 334u \, du \, dv = \pi$.

Similarly, S_b will have an integrand of $-u$, due to having downward normals. Then, $\int_0^{2\pi} \int_0^1 -u \, du \, dv = -\pi$.

So, the flux contributed by S_t and S_b cancel, and we are left with the surface integral from (a), $\pi \log 334$ as the volume of E.

(c) Since $\operatorname{div} \mathbf{G} = 1$, and $\mathbf{G}$ is defined everywhere on E, then, by the divergence theorem, the flux thru S on $\mathbf{G}$ is the same as in (a). $\square$

Problem (5)**.** Let $\mathbf{G} = (-yz^2, xz^2, \exp(-z^2))$ and $\mathbf{F} = \operatorname{curl} \mathbf{G}$. Let S be the surface defined in cylindrical coordinates by $r = 1 + z^2, r \leq 3$ oriented with normals pointing towards the z-axis. Divide S into the portion S_+ in positive z, and S_- in negative z.

Determine whether the flux of $\mathbf{F}$ through S_+ or S_- is larger.

Proof. By stoke theorem, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{G} \cdot d\mathbf{S} = \oint_C \mathbf{G} \cdot d\mathbf{x}$$

where $\partial S = C$ and C is positively oriented.

So, we will consider the boundary curves of S_+ and S_-, which we will call C_+ and C_- respectively.

We will set up a difference equation for the upper and lower surfaces,

$$\iint_{S_+} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_-} \mathbf{F} \cdot d\mathbf{S} = \oint_{C_+} \mathbf{G} \cdot d\mathbf{x} - \oint_{C_-} \mathbf{G} \cdot d\mathbf{x}.$$

Since the image of S in the xy-plane is a unit circle ($r = 1$ where $z = 0$), then the part of the boundaries C_+ and C_- on the unit circle will be traversed in opposite directions. Therefore the line integrals over these paths will cancel each other in the difference equation.

So, we are left with the parts of C_+ and C_- that bound $r = 3$ at $z = \pm\sqrt{2}$ respectively. We will call these curves γ_+ and γ_-.

We will begin by parametrizing C_+ with $\gamma_+(t) = (3\cos t, 3\sin t, \sqrt{2})$. Since the normals of S point towards the z-axis, to maintain continuity of the normals, we must have an anticlockwise orientation for γ_+. So, $t \in [0, 2\pi]$. Then, $\gamma_+'(t) = (-3\sin t, 3\cos t, 0)$ and $\mathbf{G}(\gamma_+(t)) = (-6\sin t, 6\cos t, e^{-2})$.

So,

$$\oint_{\gamma_+} \mathbf{G} \cdot d\mathbf{x} = \int_0^{2\pi} \mathbf{G}(\gamma_+(t)) \cdot \gamma_+'(t)\, dt = \int_0^{2\pi} 18\, dt = 36\pi.$$

For γ_-, we will perform the same, except the z coordinate of the parametrization will be $-\sqrt{z}$, and the direction will be reversed. This will not impact the integrand, but rather the bounds for the integral, which will be reversed. So, the line integral around γ_- will be -36π.

Thus, the difference equation is will be positive, indicating that the flux on $\mathbf{F}$ through S_+ is larger than that over S_-. $\square$