Math 335 Extra Credit Problem

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Problem 1. $a_n > 0$, $\sum a_n$ converges $\implies \sum \frac{1}{n}\left(a_n + a_{n+1} + \cdots + a_{2n}\right)$ converges.

Proof. We will show that the series converges using the Dirichlet test.

Note that we can express the sum inside the series as a difference of partial sums of a_n,

$$a_n + a_{n+1} + \cdots + a_{2n} = S_{2n} - S_{n-1},$$

where $S_n = \sum_{k=1}^{n} a_n$.

So, the given series can be written as,

$$\sum \frac{1}{n}\left(S_{2n} - S_{n-1}\right).$$

We will now show that $S_{2n} - S_{n-1}$ is uniformly bounded for all n.

Since S_n converges, then it must be bounded. So, there exists some M such that, $|S_n| \leq M$ for all n.

Then, we have that $|S_{2n}| \leq M$ and $|S_{n-1}| \leq M$ as well.

Thus, by the triangle inequality,

$$|S_{2n} - S_{n-1}| \leq |S_{2n}| + |S_{n-1}| \leq 2M.$$

So, $S_{2n} - S_{n-1}$ is uniformly bounded for all n.

Since $\frac{1}{n} \to 0$ as $n \to \infty$ and $S_{2n} - S_{n-1}$ is uniformly bounded for all n, then

$$\sum \frac{1}{n}\left(S_{2n} - S_{n-1}\right) = \sum \frac{1}{n}\left(a_n + a_{n+1} + \cdots + a_{2n}\right)$$

converges by the Dirichlet test. $\square$