Homework 1

1. You are trying to create a password using the twenty-six characters A–Z and ten characters 0–9. (Assume no other characters, such as lower-case letters or punctuation, are allowed.)

   a) How many n-character passwords can be created if you must use at least one letter and at least one number?

   The total number of n-character passwords is 36^n. The number of n-character passwords with no numbers is 26^n, and the number with no letters is 10^n. Note that the passwords with no numbers and the passwords with no letters have no overlap. So, the total number of n-character passwords with at least on letter and at least one number is $36^n - 26^n - 10^n$.

   b) How many n-character passwords can be created if you must use at least one letter, at least one number, and no character can be used more than once?

   Note that $n \leq 36$ in order to have no repeat characters. First, we will pick n characters from the 36 possible, which is $\binom{36}{n}$. But, we care about order, so we have $n!$ ways to arrange each selection of n characters. Hence, we have $\binom{36}{n}n! = \frac{36!}{(36-n)!}$ ways to create n-character passwords with no duplicate letters. However, we wish to remove those which contain no letters and no numbers as well. Similarly, these counts are given by $\frac{26!}{(26-n)!}$ and $\frac{10!}{(10-n)!}$ respectively.

   c) In terms of n, approximately how many digits do your answers to (a) and (b) have?

   For (a), the count is approximately 36^n. So, the number of digits of this number can be found with $\log_{10} 36^n = n \log_{10} 36 \approx 1.56n$.
   For (b), the dominant term of the count is $\frac{36!}{(36-n)!}$. We can use Stirling's approximation formula
   $$\log_{10} n! \approx \left(n + \frac{1}{2}\right) \log_{10} n - \frac{n}{\log 10} + \frac{1}{2} \log_{10} 2\pi$$
   to approximate the number of digits for this number in terms of n.
   Then $\log_{10} \frac{36!}{(36-n)!} = \log_{10}(36!) - \log_{10}((36-n)!)$. So,
   $$\begin{array}{rl} \log_{10}(36!) & \approx \left(36 + \frac{1}{2}\right) \log_{10} 36 - \frac{36}{\log 10} + \frac{1}{2} \log_{10} 2\pi \\ -\log_{10}((36-n)!) & \approx \left(36 - n + \frac{1}{2}\right) \log_{10}(36-n) - \frac{36-n}{\log 10} + \frac{1}{2} \log_{10} 2\pi \\ \hline & = \frac{73}{2} \log_{10} 36 - \left(\frac{73}{2} - n\right) \log_{10}(36-n) + \frac{n}{\log 10} \\ & \approx \left(n - \frac{73}{2}\right) \log_{10}(36-n) + \frac{n}{\log 10} + 57 \end{array}.$$

2. A class of 28 students is divided into seven teams of four for a group assignment. In addition, in each team one person is designated to be the leader. How many ways can the teams and leaders be assigned? (Assume all teams are doing the same assignment.)

   There are 28! arrangements of students Yet, we must also choose one leader from each of the groups, there are 4^7 ways to do this. But, we can swap students inside of each of the seven groups; there are $4!^7$ ways to do this. We can also swap the groups themselves; there are 7! ways to do this.

   So, multiplying by the first two independent choices and dividing by the ways we achieve equivalent choices through swaps, we find the total count is
   $$\frac{28!4^7}{4!^7 7!} = \frac{28!}{3!^7 7!} = \frac{28!}{6^7 7!} \approx 2.16 \cdot 10^{20}.$$

3. You are arranging a collection of n different books on a bookshelf. Among these books are The Lord of the Rings 1, 2, and 3.

   a) How many ways are there to arrange the books if The Lord of the Rings books must be next to each other, in the correct order?

      Consider the three adjacent ordered LOTR books to be one unit. Then, there are $(n-2)!$ was to arrange the books while preserving the LOTR unit.

   b) How many ways are there to arrange the books if The Lord of the Rings books must be next to each other, but not necessarily in the correct order?

      Consider the result from (a), then multiply by the $3! = 6$ permutations of the three LOTR books. This gives a $6(n-2)!$.

   c) How many ways are there to arrange the books if The Lord of the Rings books must be in the correct order, but not necessarily next to each other?

      First, we will choose 3 spots from the n possible to place the LOTR books in order. This is $\binom{n}{3}$. Then, we will arrange the $n-3$ remaining books. We have $(n-3)!$ ways to do so. Thus, the total permutation count is the product of these independent actions, $\binom{n}{3}(n-3)!$.

4. Consider an $n \times n$ checkerboard, which consists of n^2 small squares arranged in a square grid. Prove that the number of rectangles which are unions of small squares is $\binom{n+1}{2}^2$.

   Note that the $n \times n$ checkerboard is equivalent to a grid of $n+1$ lines. Consider forming a rectangle by choosing 2 of the $n+1$ grid lines for both the vertical and horizontal sides of the rectangle. These choices are independent. Furthermore, the order of the choices does not matter as a rectangle with a given top and bottom (or left and right) bound would be the same as one with the bottom and top (or right and left) bound.

   So, we will have $\binom{n+1}{2}$ choices for each pair of parallel sides of the rectangle. Since these choices are independent, we have a total of $\binom{n+1}{2}^2$ choices for all of the two sides.

5. Let C be a set with n elements. Prove that the number of ordered pairs (A, B) of sets such that $A \cup B = C$ is 3^n.

   Proof. $\forall x \in C$, there are three possibilities:

$$\begin{array}{ccc} x \in A & \wedge & x \in B, \\ x \in A & \wedge & x \notin B, \\ x \notin A & \wedge & x \in B. \end{array}$$

   Each choice of x from the n-element set C is independent. So, the total product of n independent choices of 3 is 3^n $\square$

6. For each nonnegative integer n, let $f(n)$ be the number of subsets of [100] which sum to n. For example, $f(6) = 4$ because there are 4 subsets of [100] which sum to 6, specifically $\{6\}$, $\{1,5\}$, $\{2,4\}$, and $\{1,2,3\}$. Prove that $f(2000) = f(3050)$.

   Hint: What is $1 + 2 + \cdots + 100$?

   Proof. Note that the 100th triangle number is $5050 = \frac{100(101)}{2}$.

   In order to show that $f(2000) = f(3050)$, we wish to find a bijection between subsets of [100] which sum to 2000 and 3050 respectively.

Consider the set $S : \{s \in [100] \ \mid \ \text{sum of elements of } \ s = 2000\}$. Then, the complement of S in $[100]$, S^c, must contain the rest of the numbers, which sum to $5050 - 2000 = 3050$. So, the sum of the elements of S^c is 3050.

Similarly, if we start with S^c whose sum is 3050, then $(S^c)^c = S$ will have a sum of 2000.

Hence, we can define a bijection $f : S \longmapsto S^c$ by the mapping $s \mapsto [100] \setminus s$.

This map is well defined because every $s \in S$ has exactly one complement.

This map is invertible, $[100] \setminus s \mapsto [100] \setminus ([100] \setminus s) = s \implies f \circ f = f$. So f is its own inverse.

Thus, f is a bijection.

Since f is a bijection, then $f(2000) = |S| = |S^c| = f(3050)$, which is what we wished to show. $\square$