Math 135 Homework 5

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Probation Practice

1. Solve the differential equations and check the solution.

   (a) $y' = x - 4xy$

   We begin by reorganizing the equation to the form $y' + p(x)y = q(x)$ in order to arrive at the solution via the method of integrating factors. S.H.E. 9.1.2 Provides that, with $H(x) = \int p(x)\,dx$,

   $$y(x) = e^{-H(x)} \left[\int e^{H(x)} q(x)\,dx \right],$$

   with a constant of integration after the evaluation of the indefinate integral.

   So, $y' + 4xy = x$ gives

   $$H(x) = \int 4x\,dx = 2x^2.$$

   Then,

   $$\begin{aligned} y(x) &= e^{-2x^2} \left[\int x e^{2x^2}\,dx \right] \\ &= e^{-2x^2} \left[\frac{1}{4} \int e^u\,du \right] \\ &= e^{-2x^2} \left(\frac{e^{2x^2}}{4} + c \right) \\ &= ce^{-2x^2} + \frac{1}{4}. \end{aligned}$$

   We verify this result by taking the derivative of our solution,

   $$y' = \frac{d}{dx} \left[ce^{-2x^2} + \frac{1}{4} \right] = -4cxe^{-2x^2},$$

and then matching it with the equation for y above,

$$\begin{aligned} y' &= x - 4xy \\ -4cxe^{-2x^2} &= x - 4x\left(ce^{-2x^2} + \frac{1}{4}\right) \\ &= x - 4cxe^{-2x^2} - \frac{4x}{4} \\ &= -4cxe^{-2x^2}. \end{aligned}$$

This statement is true, and therefore we have arrived at a valid general solution.

(b) $y' = \csc x + y \cot x$

Again, we will put the first order linear differential equaition into standard form such that we can use the form provided by S.H.E. 9.1.2.

With $y' - y\cot x = \csc x$, we will need $H(x) = \int -\cot x \, dx = -\ln \sin x$.

Then,

$$\begin{aligned} y(x) &= e^{\ln \sin x}\left[\int \csc x e^{-\ln \sin x}\, dx\right] \\ &= \sin x \left[\int -\frac{dx}{\sin x e^{\ln \sin x}}\right] \\ &= \sin x \left[\int -\csc^2 x \, dx\right] \\ &= \sin x\,(-\cot x + c) \\ &= c\sin x - \cos x. \end{aligned}$$

We diffentiate our solution,

$$y'(x) = c\cos x + \sin x.$$

We compare this to the given form for $y'(x)$,

$$\begin{aligned} y'(x) &= \csc x + y\cot x \\ c\cos x + \sin x &= \csc x + (c\sin x - \cos x)\cot x \\ &= \frac{1}{\sin x} + c\cos x - \frac{\cos^2 x}{\sin x} \\ &= \frac{1 - \cos^2 x}{\sin x} + c\cos x \\ &= \frac{\sin^2 x}{\sin x} + c\cos x \\ &= \sin x + c\cos x. \end{aligned}$$

We see that the statement is true and so the general solution is valid.

(c) $x^2y' + 2xy = 8x^3$

We put this equation into standard form and proceed with the methods used in the previous problems,

$$y' + \frac{2}{x}y = 8x.$$

With, $\int p(x)\,dx = \int \frac{2}{x}\,dx = 2\ln x$,

$$\begin{aligned} y(x) &= e^{-2\ln x}\left[\int 8xe^{2\ln x}\,dx\right] \\ &= \frac{1}{x^2}\left[\int 8x^3\,dx\right] \\ &= \frac{1}{x^2}\left(2x^4 + c\right) \\ &= 2x^2 + \frac{c}{x^2}. \end{aligned}$$

To check this y, we differentiate,

$$y'(x) = 4x - \frac{2c}{x^3}.$$

Then we compare, with the given differential equation where y' is isolated,

$$\begin{aligned} 4x - \frac{2c}{x^3} &= 8x - \frac{2}{x}\left(2x^2 + \frac{c}{x^2}\right) \\ &= 8x - 4x - \frac{2c}{x^3} \\ &= 4x - \frac{2c}{x^3}, \end{aligned}$$

which is true, so the general solution holds as well.

(d) $y' = xe^{y-x^2}, \quad y(0) = 0$

This is a separable differential equation. We also have an initial condition that will pin the single constant at a fixed number.

We rewrite the equation with regards to differentials $\frac{dy}{dx}$,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{xe^y}{e^{x^2}} \\
\int \frac{dy}{e^y} &= \int \frac{x\,dx}{e^{x^2}} \\
-e^{-y} &= -\frac{1}{2}\int e^u\,du \\
e^{-y} &= \frac{1}{2}e^u + c \\
e^{-y} &= \frac{1}{2}e^{-x^2} + c \\
-y &= \ln\left(\frac{1}{2}e^{-x^2} + c\right) \\
y &= -\ln\left(\frac{1}{2}e^{-x^2} + c\right).
\end{aligned}$$

We then consider the initial condition $y(0) = 0$,

$$\begin{aligned}
0 &= -\ln\left(\frac{1}{2}e^{-0^2} + c\right) \\
0 &= -\ln\left(\frac{1}{2} + c\right) \\
1 &= \frac{1}{2} + c \\
\frac{1}{2} &= c.
\end{aligned}$$

So, we have the solution,

$$\begin{aligned}
y(x) &= -\ln\left(\frac{1}{2}(1 + e^{-x^2})\right) \\
&= \ln 2 - \ln\left(1 + e^{-x^2}\right).
\end{aligned}$$

We can verify this solution in the same manner as previous problems.

First, we differentiate our solution,

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\left[\ln 2 - \ln\left(1 + e^{-x^2}\right)\right] \\
&= -\frac{-2xe^{-x^2}}{1 + e^{-x^2}} \\
&= \frac{2x}{1 + e^{x^2}}.
\end{aligned}$$

Then we compare it to the given y' with our derived solution y,

$$\begin{aligned} y'(x) &= \frac{xe^{\left(\ln 2 - \ln\left(1+e^{-x^2}\right)\right)}}{e^{x^2}} \\ &= \frac{xe^{\ln 2}}{e^{x^2} e^{\ln\left(1+e^{-x^2}\right)}} \\ &= \frac{2x}{e^{x^2}\left(1+e^{-x^2}\right)} \\ &= \frac{2x}{1+e^{x^2}}. \end{aligned}$$

Since both y' match, we confirm that we have arrived at a valid general solution.

(e) $(x + yx)\,dx = (x^2y^2 + x^2 + y^2 + 1)\,dy$

We rearrange the terms to reveal that this equation is separable.

$$\begin{aligned} (x + yx)\,dx &= (x^2y^2 + x^2 + y^2 + 1)\,dy \\ x(y+1)\,dx &= (x^2+1)(y^2+1)\,dy \\ \int \frac{x\,dx}{x^2+1} &= \int \frac{y^2+1}{y+1}\,dy \\ \frac{1}{2}\int \frac{du}{u} &= \int \frac{(y+1)^2 - 2y}{y+1}\,dy \\ \frac{\ln u}{2} + c/2 &= \int \left(y + 1 - \frac{2(y+1)-2}{y+1}\right)\,dy \\ \frac{\ln(x^2+1)}{2} + c/2 &= \int \left(y + 1 - 2 + \frac{2}{y+1}\right)\,dy \\ &= \int \left(y - 1 + \frac{2}{y+1}\right)\,dy \\ &= \frac{y^2}{2} - y + 2\ln(y+1) \\ \ln(x^2+1) + c &= y^2 - 2y + 4\ln(y+1) \end{aligned}$$

2. Find the first four Picard approximations of $y' = e^x + y$ with $y(0) = 0$.

Since $y(x_0) = y_0$, then $(x_0, y_0) = (0, 0)$.

So, $y_0(x) = y_0 = 0$.

The n^{th} Picard approximation is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f[x, y_{n-1}(x)]\,dx,$$

where $y' = f(x, y(x))$.

So, for y_1,

$$\begin{aligned} y_1(x) &= 0 + \int_0^x f(x, y_0(x))\,dx \\ &= \int_0^x (e^x + 0)\,dx \\ &= e^x\big|_0^x \\ &= e^x - 1. \end{aligned}$$

Next, for y_2,

$$\begin{aligned} y_2(x) &= 0 + \int_0^x \left(e^x + (e^x - 1)\right)\,dx \\ &= \int_0^x (2e^x - 1)\,dx \\ &= [2e^x - x]_0^x \\ &= 2e^x - x - 2. \end{aligned}$$

For y_3,

$$\begin{aligned} y_3(x) &= 0 + \int_0^x \left(e^x + (2e^x - x - 2)\right)\,dx \\ &= \int_0^x (3e^x - x - 2)\,dx \\ &= \left[3e^x - \frac{x^2}{2} - 2x\right]_0^x \\ &= 3e^x - \frac{x^2}{2} - 2x - 3. \end{aligned}$$

Lastly, for y_4,

$$\begin{aligned} y_4(x) &= 0 + \int_0^x \left(e^x + \left(3e^x - \frac{x^2}{2} - 2x - 3\right)\right)\,dx \\ &= \int_0^x \left(4e^x - \frac{x^2}{2} - 2x - 3\right)\,dx \\ &= \left[4e^x - \frac{x^3}{6} - x^2 - 3x\right]_0^x \\ &= 4e^x - \frac{x^3}{6} - x^2 - 3x - 4. \end{aligned}$$

3. First the first three Picard approximations of the system of differential equations,

$$\begin{aligned} x'(t) &= t + y^2 & \qquad x(0) &= 0, \\ y'(t) &= x - t & \qquad y(0) &= 1. \end{aligned}$$

We first notice that $t_0 = 0$, $x_0 = 0$, and $y_0 = 1$.

For a parametric system of differential equations, the Picard approximations are given by

$$x_n = x_0 + \int_{t_0}^{t} f_1[t, x_{n-1}(t) y_{n-1}(t)]\, dt,$$
$$y_n = y_0 + \int_{t_0}^{t} f_2[t, x_{n-1}(t) y_{n-1}(t)]\, dt,$$

where $x'(t) = f_1(t, x(t), y(t))$ and $y'(t) = f(t, x(t), y(t))$.

We start with the first pair of approximations,

$$\begin{aligned} x_1(t) &= 0 + \int_0^t (t + 1^2)\, dt \\ &= \left[t + \frac{t^2}{2} \right]_0^t \\ &= t + \frac{t^2}{2} + t, \end{aligned}$$

and

$$\begin{aligned} y_1(t) &= 1 + \int_0^t (0 - t)\, dt \\ &= 1 - \frac{t^2}{2}{}_0^t \\ &= 1 - \frac{t^2}{2}. \end{aligned}$$

We continue recursively as before,

$$\begin{aligned} x_2(t) &= 0 + \int_0^t \left(t + \left(1 - \frac{t^2}{2}\right)^2 \right)\, dt \\ &= \int_0^t \left(1 + t - t^2 + \frac{t^4}{4} \right)\, dt \\ &= \left[t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20} \right]_0^t \\ &= t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20}, \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= 10y \int_0^t \left(t + \frac{t^2}{2} - t \right)\, dt \\ &= 1 + \left. \frac{t^3}{6} \right|_0^t \\ &= 1 + \frac{t^3}{6}. \end{aligned}$$

Finally,

$$\begin{aligned}
x_3(t) &= 0 + \int_0^t \left(t + \left(1 + \frac{t^3}{6} \right)^2 \right) dt \\
&= \int_0^t \left(1 + t + \frac{t^3}{3} + \frac{t^6}{36} \right) dt \\
&= t + \frac{t^2}{2} + \frac{t^4}{12} + \frac{t^7}{252},
\end{aligned}$$

and

$$\begin{aligned}
y_3(t) &= 1 + \int_0^t \left(t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{20} - t \right) dt \\
&= 1 + \left[\frac{t^3}{6} - \frac{t^4}{12} + \frac{t^6}{120} \right]_0^x.
\end{aligned}$$

Homework

1. The equation $y' + P(x)y = Q(x)y^n$ is called the Bernoulli equation. It becomes a linear equation after the change of variable $y^{1-n} = z$. Solve the equation $y(6y^2 - x - 1)\,dx + 2x\,dy = 0$ using this idea.

First, we rearrange the equation to the given form,

$$\begin{aligned}
y(6y^2 - x - 1)\,dx + 2x\,dy &= 0 \\
-2x\frac{dy}{dx} &= 6y^3 - y(x+1) \\
y' - \frac{x+1}{2x}y &= -\frac{3}{x}y^3.
\end{aligned}$$

We see that this is a Bernoulli differential equation of degree three.

We will perform the substitution with $y^{1-n} = y^{1-3} = y^{-2} = z$ to find y and y' in z,

$$\begin{aligned}
y^{-2} &= z \\
y &= z^{-1/2} \\
y' &= -\frac{1}{2}z^{-3/2}z'.
\end{aligned}$$

We then rewrite the DE in z,

$$\begin{aligned}
-\frac{1}{2}z^{-3/2}z' - \frac{x+1}{2x}\left(z^{-1/2}\right) &= -\frac{3}{x}\left(z^{-1/2}\right)^3 \\
-\frac{z'}{2} - \frac{x+1}{2x}z &= -\frac{3}{x} \\
z' + \frac{x+1}{x}z &= \frac{6}{x}.
\end{aligned}$$

We can use the method of an integrating factor to solve this first order linear differential equation.

Let

$$\mu = e^{\int \frac{x+1}{x}\,dx} = e^{x+\ln x} = xe^x.$$

Then,

$$\begin{aligned}
\mu z' + \mu \frac{x+1}{x} z &= \mu \frac{6}{x} \\
xe^x z' + e^x(x+1)z &= 6e^x \\
\frac{d}{dx}\left[xe^x z\right] &= 6e^x \\
xe^x z &= \int 6e^x \, dx \\
xe^x z &= 6e^x + c \\
z &= \frac{6e^x + c}{xe^x}.
\end{aligned}$$

But recall that $y^{-2} = z$. So,

$$\begin{aligned}
y^{-2} &= \frac{6e^x + c}{xe^x} \\
y^2 &= \frac{xe^x}{6e^x + c} \\
y &= \pm\sqrt{\frac{xe^x}{6e^x + c}} \\
y &= \pm\sqrt{\frac{x}{6 + ce^{-x}}},
\end{aligned}$$

which is the general solution to our differential equation.

2. Let $I = [0,1]$ and $Y_n(t) = t^n$. Show that the sequence (Y_n) is not Cauchy by computing $||Y_n - Y_m||$.

   Fix $n, m \in \mathbb{Z}^+$ such that $n > m$.

   Let k be the difference between the two indices, $k = n - m$ such that $k > 0$.

   Then,

$$\begin{aligned}
||Y_n - Y_m|| &= \max_{t\in I} |t^n - t^m| \\
&= \max_{t\in I} \left[t^m \left|t^k - 1\right|\right].
\end{aligned}$$

   But,

$$0 \le t^k \le 1, \quad \forall k \in \mathbb{Z}^+, \forall t \in [0,1] = I.$$

   So,

$$||Y_n - Y_m|| = \max_{t\in I} \left[t^m \left(1 - t^k\right)\right].$$

We then wish to find the extrema of this function $t^m\left(1-t^k\right)$.

$$\begin{aligned}0 &= \frac{d}{dt}\left[t^m\left(1-t^k\right)\right] \\ &= mt^{m-1}\left(1-t^k\right)+t^m\left(-kt^{k-1}\right) \\ &= t^{m-1}\left[m\left(1-t^k\right)+t\left(-kt^{k-1}\right)\right] \\ &= t^{m-1}\left[m-mt^k-kt^k\right] \\ &= t^{m-1}\left[m-(m+k)t^k\right],\end{aligned}$$

which occurs when either,

$$t^{m-1}=0 \implies t=0,$$

or,

$$\begin{aligned}m-(m+k)t^k &= 0 \\ t^k &= \frac{m}{m+k} \\ t &= \left(\frac{m}{m+k}\right)^{1/k}.\end{aligned}$$

We evaluate the function at the endpoints $0, 1$ and the extrema $0, \left(\frac{m}{m+k}\right)^{1/k}$. We see that, at 0 and 1, the function becomes zero. So, at the last critical point, when $t=\left(\frac{m}{m+k}\right)^{1/k}$,

$$\begin{aligned}&\left(\frac{m}{m+k}\right)^{\frac{m}{k}}\left(1-\left(\frac{m}{m+k}\right)^{\frac{k}{k}}\right) \\ =&\left(\frac{m}{m+k}\right)^{\frac{m}{k}}\left(\frac{k}{m+k}\right).\end{aligned}$$

Now, we will consider the value of the function as k, the gap between n and m, grows large.

$$\lim_{k\to\infty}\left[\left(\frac{m}{m+k}\right)^{\frac{m}{k}}\left(\frac{k}{m+k}\right)\right].$$

We will consider each term in the limit separately and show that they both exist and are finite.

With

$$\lim_{k\to\infty}\frac{k}{m+k},$$

we have a limit of indeterminate form. We apply L'Hôpital's Rule,

$$\lim_{k\to\infty}\frac{1}{1}=1.$$

Next, for

$$\begin{aligned}
&\lim_{k\to\infty}\left(\frac{m}{m+k}\right)^{\frac{m}{k}}\\
=&\left[\lim_{k\to\infty}\left(\frac{m}{m+k}\right)^{1/k}\right]^{m},
\end{aligned}$$

we will show that the numerator and denominator are both finite and nonzero so that the limit of the quotient is equal to the quotient of the limits.

The numerator, $\lim\limits_{k\to\infty} m^{1/k} = 1, \quad m > 0$ by S.H.E 11.4.1.

For the denominator, we know that $\lim\limits_{k\to\infty} \frac{\ln k}{k} = 0$ by S.H.E. 11.4.1.

So,

$$\begin{aligned}
\lim_{k\to\infty}\frac{\ln(m+k)}{k} &= \lim_{k\to\infty}\frac{\ln m \ln k}{k}\\
&= \ln m \lim_{k\to\infty}\frac{\ln k}{k}\\
&= \ln m \cdot 0\\
&= 0.
\end{aligned}$$

Since,

$$(m+k)^{1/k} = e^{\ln\left((m+k)^{1/k}\right)} = e^{\frac{\ln(m+k)}{k}}.$$

Then,

$$\begin{aligned}
\lim_{k\to\infty}(m+k)^{1/k} &= \lim_{k\to\infty} e^{\frac{\ln(m+k)}{k}}\\
&= e^{\lim\limits_{k\to\infty}\frac{\ln(m+k)}{k}}\\
&= e^{0}\\
&= 1.
\end{aligned}$$

Since the limit of the numerator and denominator are both one, then the limit of their quotient is also one.

Since the both limits in the original product exist and equate to one, then,

$$\left[\lim_{k\to\infty}\left(\frac{m}{m+k}\right)^{1/k}\right]^{m} = 1^{m} = 1.$$

We had that,

$$||Y_n - Y_m|| = \max_{t\in I}\left[t^m\left(1-t^k\right)\right],$$

So, for $k\to\infty$,

$$||Y_n - Y_m|| = \max_{t\in I}\left[t^m\left(1-t^k\right)\right] = \left(\frac{m}{m+k}\right)^{\frac{m}{k}}\left(\frac{k}{m+k}\right) \to 1.$$

Since $||Y_n - Y_m||$ goes to one as $n - m$ goes to infinity, then $||Y_n - Y_m||$ cannot be less than any number ϵ for some large number N with any $n, m \geq N$. So, the sequence (Y_n) does not meet the definition of Cauchy and is therefore not a Cauchy sequence.

3. Let $I = [-\pi, \pi]$, and consider the function $f_0 : I \to \mathbb{R}$ defined by $f_0(t) = e^t$. Let $f_n, n \in \mathbb{Z}^+$ be the sequence of functions on I defined inductively by the formula

$$f_{n+1}(t) = \cos(t) + \frac{1}{2}\sin(t) f_n(t).$$

So $f_1(t) = \cos(t) + \frac{1}{2}\sin(t)e^t, \quad f_2(t) = \cos(t) + \frac{1}{2}\sin(t)\left(\cos(t) + \frac{1}{2}\sin(t)e^t\right)$, etc.

Show that f_n converges uniformly to a continuous function and find the limit $\lim_{n\to\infty} f_n$.

We will show that f_n is a contraction map.

We consider, $||f_n(Y_1) - f_n(Y_2)||$, where Y_1 and Y_2 are two arbitrary functions in $\mathcal{F}$, defined as a subset of C, which is the set of all continuous functions on I, that meets the following conditions,

$$\mathcal{F} = \{Y \in C : Y(t_0) = y_0, \text{ and } |Y(t) - y_0| \leq b \quad \forall t \in I\}.$$

Then,

$$\begin{aligned}
||f_n(Y_1) - f_n(Y_2)|| &= \max_{t\in I}|f_n(Y_1) - f_n(Y_2)| \\
&= \max_{t\in I}\left|\cos(t) + \frac{1}{2}\sin(t)Y_1 - \left(\cos(t) + \frac{1}{2}\sin(t)Y_2\right)\right| \\
&= \max_{t\in I}\left|\frac{\sin(t)}{2}(Y_1 - Y_2)\right| \\
&= \frac{1}{2}\max_{t\in I}\big|Y_1 - Y_2\big| \\
&= \frac{1}{2}||Y_1 - Y_2||.
\end{aligned}$$

So, by the definition of the contraction map, if we choose $K = 1/2$, then $0 < K < 1$ and

$$||f_n(Y_1) - f_n(Y_2)|| \leq K||Y_1 - Y_2.$$

So, f_n is a contraction map.

Since f_n is a contraction map, then f_n is Cauchy. Since f_n is Cauchy, then f_n converges uniformly to a function f. Since f_n converges uniformly to f, then,

$$f(t) = \lim_{n\to\infty} f_n(t).$$

By Theorem 3 in the Cauchy Sequences of Functions handout, $f(t)$ is a fixed point of $f_n(t)$.

So,

$$\begin{aligned}
f(t) &= \cos t + \frac{1}{2}\sin t f(t) \\
f(t)\left(1 - \frac{1}{2}\sin t\right) &= \cos t \\
f(t) &= \frac{\cos t}{1 - \frac{1}{2}\sin t}.
\end{aligned}$$