Math 334 Homework 4

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Problem (1)**.** Prove $f : \mathbb{R} \longrightarrow \mathbb{R}$, $\exists c \in \mathbb{R}$, $\forall x \in \mathbb{R}$, $|f'(x)| \leq c \implies f$ uniformly continuous.

Proof. Let $\epsilon = c\delta$. For $|x - y| < \delta$, we must also have $y \to x$. So, the limit definition of the derivative can be expressed as $\left|\frac{f(y)-f(x)}{y-x}\right| = |f'(x)| \leq c$. Then,

$$\begin{aligned} c|x-y| &< c\delta = \epsilon \\ \left|\frac{f(y)-f(x)}{y-x}\right| |x-y| &< \epsilon \\ |f(x)-f(y)| &< \epsilon. \end{aligned}$$

Thus, $|x - y| < \delta \implies |f(y) - f(x)| < \epsilon$, so f is uniformly continuous. □

Problem (2)**.** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ continuous at (0,0),

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

i) Show $\forall v \in \mathbb{S}$, $\exists \lim\limits_{t \to 0} \frac{f(tv)-f(0)}{t}$.

ii) Prove f not differentiable at (0,0).

Proof of i. Let $v = (x, y)$. Since $f(0) = 0$ and $x^2 + y^2 = |v|^2 = 1$,

$$\begin{aligned} \lim_{t\to 0} \frac{f(tv)}{t} &= \lim_{t\to 0} \frac{t^3x^2y}{t|v|^2} \\ &= \lim_{t\to 0} t^2x^2y \\ &= 0. \end{aligned}$$

□

Proof of ii. First, we find $\nabla f(x,y) = \frac{1}{(x^2+y^2)^2}\left(2xy^3, x^2\left(x^2-y^2\right)\right)$. In order for f to be differentiable at $(0,0)$, then the following must hold,

$$\lim_{h\to 0} \frac{f(h) - f(0) + \nabla f(0) \cdot h}{|h|} = 0.$$

But, clearly, $\nabla f(0)$ is indeterminate at (0,0). For a contradiction, assume that $\nabla f = (a, b)$.

Then,

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^2y}{x^2+y^2} - ax - by}{\sqrt{x^2+y^2}} = 0.$$

Now, if we approach on the path $y = x^2$, then

$$\begin{aligned}\lim_{x\to 0} \frac{\frac{x^4}{x^2+x^4} - ax - bx^2}{\sqrt{x^2+x^4}} &= \lim_{x\to 0} \frac{\frac{x^2}{1+x^2} - ax - bx^2}{x\sqrt{1+x^2}} \\ &= \lim_{x\to 0} \frac{\frac{x}{1+x^2} - a - bx}{\sqrt{1+x^2}} \\ &= a.\end{aligned}$$

So, along $y = x^2$, if the first coordinate of $\nabla f(0)$ is not zero, then the derivative at the origin does not exist.

If $\nabla f(0) = (0,0)$, then we must have that $\lim\limits_{h\to\infty} \frac{f(h)}{|h|} = 0$.

But, along the path $y = ax$,

$$\lim_{x\to 0} \frac{\frac{a^2x^3}{x^2+a^2x^2}}{\sqrt{x^2+a^2x^2}} = \frac{a^2}{(1+a^2)^{\frac{3}{2}}},$$

which is depends on some a as well. So, the derivative does not exist at the origin. $\square$

Problem (3)**.** Let $f : U \longrightarrow \mathbb{R}$, $U \subset \mathbb{R}^n$ open. Show $\forall i$, $\exists \frac{\partial f}{\partial x_i} : U \longrightarrow \mathbb{R}$ bounded $\implies$ f continuous on U.

Proof. Since $U \subset \mathbb{R}^n$ and $\left|\frac{\partial f}{\partial x_i}\right| \leq c$, then we can apply problem (1) to each partial of f. Since f is continuous in all component directions, it is also everywhere continuous. $\square$

Problem (4)**.** Use the linear approximation of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ to approximate the distance between $p = (0.99, -0.97, 2.02)$ and $q = (4.02, 0.98, 8.01)$.

Proof. The distance between p and q is given by $f(q-p)$. So, we will approximate f near $q-p$ for some nice integer-valued vector $r = (3,2,6) \approx q - p$.

Then, $f(r) = \sqrt{3^2+2^2+6^2} = \sqrt{9+4+36} = \sqrt{49} = 7$. Note that our approximation of the distance between p and q should be close to this value.

Then, $\nabla f(x) = \frac{1}{f(x)}(x,y,z)$, so $\nabla f(r) = \frac{1}{7}(3,2,6)$.

Our linear approximation L at r is given by,

$$L_r(x) = f(r) + \nabla f(r) \cdot (x - r).$$

So, at the point $x = q - p = (3.03, 1.95, 5.99)$,

$$\begin{aligned} L_r(x) &= 7 + \frac{1}{7}(3,2,6) \cdot (0.03, -0.05, -0.01) \\ &= 7 + \frac{1}{7}(0.09 - 0.1 - 0.04) \\ &= 7 - \frac{0.05}{7} \\ &= 7 - \frac{1}{140} \\ &= \frac{979}{140}. \end{aligned}$$

So, the approximate distance between p and q is $\frac{979}{140}$, which is indeed close to 7. □

Problem (5)**.** Let $S_1 = \{(x,y,z) \mid y + z^3 = 2\}$ and $S_2 = \{(x,y,z) \mid x^2 + xy + y^4 = 21\}$. Let $C = S_1 \cap S_2$, C smooth.

(a) Sketch S_1, S_2, and C on the same diagram.

(b) Find a parametric equation for the tangent line to C at $p = (4,1,1)$.

For (a), we note that S_1 is a cylinder in x, and S_2 is a cylinder in z. So, we can construct level set diagrams for S_1 and S_2. The diagram that best visualizes their intersection C is level sets of the z-axis. Let $f(x,y) = x^2 + xy + x^4 - 21$ and $g(y) = (y-2)^{\frac{1}{3}}$. Then, we will consider the preimages $f^{-1}(c)$ and $g^{-1}(c)$ for several c. The z-axis best illustrates C.

Proposition (b)**.** The tangent line to C at p is given by $\ell(t) = (4 - 24t, 1 + 27t, 1 - 9t)$.

Proof. Let $f(x,y,z) = x^2 + xy + x^4 - 20$ and $g(x,y,z) = y + z^3 - 2$.

We will find ∇f and ∇g which are both perpendicular to f and g respectively. Thus, the cross product of these gradient vectors will produce a vector tangent to both f and g, which will allow us to construct a tangent line.

First, $\nabla f = (2x + y, x + 4y^3, 0)$ and $\nabla g(0, 1, 3z^2)$.

Then, at p, $\nabla f(p) = (9, 8, 0)$ and $\nabla g(p) = (0, 1, 3)$.

So, $\nabla f(p) \times \nabla g(p) = (9,8,0) \times (0,1,3) = (24, -27, 9)$.

Thus, the tangent line at p is given by $\ell(t) = (4,1,1) - (24, -27, 9)t$. □