Math 335 Homework 5

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For Problems 1, 2, and 3, find the Fourier series of the 2π-periodic function $f(x)$ on $(-\pi, \pi)$.

Problem 1. (i) The square wave $f(x) = \begin{cases} -1 & (-\pi, 0), \\ 1 & (0, \pi). \end{cases}$

(ii) $f(x) = \sin^2 x$.

Proof of (i). We have that f is an odd function, so $a_n = 0$ for all n.

We will proceed to find the b_n Fourier coefficients,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx - \frac{1}{\pi} \int_{-\pi}^{0} \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx \\ &= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left(\frac{1 - \cos n\pi}{n} \right), \end{aligned}$$

which vanishes for all even n, and is $\frac{4}{n\pi}$ for all odd n.

Thus, we have that, for n odd,

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin nx}{n},$$

which is

$$\sum_{k=0}^{\infty} \frac{4 \sin (2k+1)x}{\pi(2k+1)}.$$

□

Proof of (ii). We have that

$$f(x) = \sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2} \cos 2x.$$

□

Problem 2. (i) $f(x) = e^{bx}$, $b > 0$.

(ii) $f(x) = x(\pi - |x|)$.

Proof of (i). We will consider the complex Fourier coefficient,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(b-in)x}\,dx \\ &= \frac{1}{\pi}\left[\frac{e^{(b-in)x}}{b-in}\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi(b-in)}\left(e^{(b-in)\pi} - e^{-(b-in)\pi}\right) \\ ; \end{aligned}$$

by Euler's identity, at $x = \pi$, we have that $e^{\pm in\pi} = (-1)^n$. So, with the hyperbolic sine identity $2\sinh b\pi = e^{b\pi} - e^{-b\pi}$, the above becomes,

$$\frac{(-1)^n \sinh b\pi}{\pi(b-in)}.$$

Thus, we have

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{\sinh b\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{b-in}e^{inx}.$$

□

Proof of (ii). Since f is an odd function, then $a_n = 0$ for all n.

So,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x(\pi - |x|)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} x(\pi - (-x))\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x(\pi - x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{0} (-u)(\pi - u)\sin(-nu)(-1)\,du + \frac{1}{\pi}\int_{0}^{\pi} u(\pi - u)\sin nu\,du \\
&= \frac{1}{\pi}\int_{0}^{\pi} u(\pi - u)(\sin nu - \sin(-nu))\,du \\
&= \frac{2}{\pi}\int_{0}^{\pi} x(\pi - x)\sin nx\,dx \\
&= 2\int_{0}^{\pi} x\sin nx\,dx - \frac{2}{\pi}\int_{0}^{\pi} x^2\sin nx\,dx \\
&= 2\left(\frac{-x\cos nx}{n}\bigg|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right) - \frac{2}{\pi}\left(\frac{-x^2\cos nx}{n}\bigg|_0^{\pi} + \int_0^{\pi}\frac{2x\cos nx}{n}\,dx\right) \\
&= 2\left(\frac{-\pi\cos n\pi}{n} + \left[\frac{\sin nx}{n}\right]_0^{\pi}\right) - \frac{2}{\pi}\left(\frac{-\pi^2\cos n\pi}{n} + 2\left(\frac{x\sin nx}{n^2}\bigg|_0^{\pi} + \int_0^{\pi}\frac{\sin nx}{n^2}\,dx\right)\right) \\
&= 2\left(\frac{\sin n\pi}{n^2} - \frac{\pi\cos n\pi}{n}\right) + \left(\frac{2\pi\cos n\pi}{n} - \frac{4}{\pi}\left(\frac{\pi\sin n\pi}{n^2} + \left[\frac{\cos nx}{n^3}\right]_0^{\pi}\right)\right) \\
&= \frac{2\sin n\pi}{n^2} - \frac{2\pi\cos n\pi}{n} + \frac{2\pi\cos n\pi}{n} - \frac{4\sin n\pi}{n^2} - \frac{4}{\pi}\left(\frac{\cos n\pi - 1}{n^3}\right) \\
&= -\frac{2\sin n\pi}{n^2} - \frac{4}{\pi}\left(\frac{\cos n\pi - 1}{n^3}\right).
\end{aligned}$$

This quantity vanishes for even n, and $b_n = \frac{8}{\pi n^3}$ for odd n.

So, for odd n,

$$f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n^3},$$

which is

$$\frac{8}{\pi}\sum_{0}^{\infty}\frac{\sin(2k+1)x}{(2k+1)^3}.$$

□

Problem 3. (i) $f(x) = \begin{cases}\frac{1}{a} & |x| < a, \\ -\frac{1}{\pi - a} & a < |x| < \pi,\end{cases}$ for $a \in (0, \pi)$.

The values of f are chosen to make the area under the curve of f on $[0, a]$ and $[a, \pi]$ both equal to one.

(ii) $f(x) = \begin{cases}\frac{a-|x|}{a^2} & |x| < a, \\ 0 & a < |x| < \pi,\end{cases}$ for $a \in (0, \pi)$.

The constraints on f are chosen such that the area under the triangle in the graph of f is equal to one.

Proof of (i). Since f is an even function, then $b_n = 0$ for all n.

Since the graph of f has equal positive and negative areas between the x-axis (both one) between 0 and π, and f is an even function, then

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

Next,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{-a} -\frac{1}{\pi-a}\cos nx\,dx + \frac{1}{\pi}\int_{a}^{\pi} -\frac{1}{\pi-a}\cos nx\,dx + \frac{1}{\pi}\int_{-a}^{a}\frac{1}{a}\cos nx\,dx \\
&= \frac{1}{\pi}\int_{\pi}^{a} -\frac{1}{\pi-a}\cos(-nx)(-1)\,dx + \frac{1}{\pi}\int_{a}^{\pi} -\frac{1}{\pi-a}\cos nx\,dx \ + \frac{1}{a\pi}\left[\frac{\sin nx}{n}\right]_{-a}^{a} \\
&= \frac{1}{\pi}\int_{a}^{\pi} -\frac{1}{\pi-a}(\cos nx + \cos nx)\,dx + \frac{1}{a\pi n}(\sin na - \sin(-na)) \\
&= -\frac{2}{\pi(\pi-a)}\left[\frac{\sin nx}{n}\right]_{a}^{\pi} + \frac{2\sin na}{a\pi n} \\
&= -\frac{2}{\pi(\pi-a)}\left(\frac{\sin n\pi - \sin na}{n}\right) + \frac{2\sin na}{a\pi n} \\
&= \frac{2\sin na}{\pi(\pi-a)n} + \frac{2\sin na}{a\pi n} \\
&= \frac{2a\sin na + 2(\pi-a)\sin na}{a\pi(\pi-a)n} \\
&= \frac{(2a+2\pi-2a)\sin na}{a\pi(\pi-a)n} \\
&= \frac{2\sin na}{a(\pi-a)n}.
\end{aligned}$$

Thus,

$$f(x) = \frac{2}{a(\pi-a)}\sum_{n=1}^{\infty}\frac{\sin na}{n}\cos nx.$$

□

Proof of (ii). Since f is even, then $b_n = 0$ for all n.

Since the area under the curve of the graph of f on $(-\pi,\pi)$ is one, then

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}.$$

Next,

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-a}^{a} \frac{a-|x|}{a^2}\cos nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{a} \frac{a-x}{a^2}\cos nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{a} \left(\frac{\cos nx}{a} - \frac{x\cos nx}{a^2}\right) dx \\
&= \frac{2}{\pi}\left[\frac{\sin nx}{an}\bigg|_0^a - \left(\frac{x\sin nx}{a^2 n}\bigg|_0^a - \int_0^a \frac{\sin nx}{a^2 n}\,dx\right)\right] \\
&= \frac{2}{\pi}\left[\frac{\sin na}{an} - \frac{a\sin na}{a^2 n} - \left[\frac{\cos nx}{a^2n^2}\right]_0^a\right] \\
&= \frac{2}{a^2\pi}\left(\frac{1-\cos na}{n^2}\right).
\end{aligned}
$$

Thus,

$$
f(x) = \frac{1}{2\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos na}{a^2n^2}\cos nx.
$$

□

Problem 4. Let the Fourier coefficients of f be a_n and b_n. Find the Fourier coefficients A_n and B_n of $g(x) = f(x)\sin x$.

Proof. We have that,

$$
f(x)\sin x = \left(\frac{1}{a}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)\sin x = \frac{\sin x}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx\sin x + b_n\sin nx\sin x).
$$

By the product identities, the above series becomes

$$
\frac{1}{2}\sum_{n=1}^{\infty}(a_n(\sin{(n+1)x} - \sin{(n-1)x}) + b_n(\cos{(n-1)x} - \cos{(n+1)x})).
$$

Considering the Fourier series of $g(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos nx + B_n\sin nx)$ and collecting the sine and cosine terms, we have that

$$
A_n = \frac{1}{2}(b_{n-1} - b_{n+1}) \text{ and } B_n = \frac{1}{2}(a_{n+1} - a_{n-1}),
$$

which are the Fourier coefficients of $f(x)\sin x$. □

Problem 5. Let f be 2π-periodic and monotonously decreasing on $(0, 2\pi)$. Prove that the Fourier coefficient $b_n \geq 0$ for all n.

Proof. Note that $\sin n(2\pi - x) = -\sin nx$. We will change the region of integration according to this identity.

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(\int_0^\pi f(x)\sin nx\,dx + \int_\pi^{2\pi} f(x)\sin nx\,dx\right) \\ &= \frac{1}{\pi}\left(\int_0^\pi f(x)\sin nx\,dx + \int_0^\pi f(2\pi - x)\sin n(2\pi - x)\,dx\right) \\ &= b_n\frac{1}{\pi}\int_0^\pi (f(x) - f(2\pi - x))\sin nx\,dx \end{aligned}$$

.

Since f is monotonously decreasing on $(0, 2\pi)$, then, for all $x \in (0, \pi)$, we have that $x \leq 2\pi - x \implies f(x) \geq f(2\pi - x)$, or $f(x) - f(2\pi - x) \geq 0$.

Then, with $\sin nx \geq 0$ for all $x \in (0, \pi)$ and $n \geq 0$, we have that

$$b_n = \frac{1}{\pi}\int_0^\pi (f(x) - f(2\pi - x))\sin nx\,dx \geq 0 \implies b_n \geq 0.$$

$\square$

Problem 6. Let $f \in C^2([-\pi, \pi])$, $f(-\pi) = f(\pi)$, and $f'(-\pi) = f'(\pi)$. Prove c_n $O(n^{-2})$ as $n \to \infty$.

Proof. We can perform IBP twice on c_n because $f \in C^2$.

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^\pi f(x)e^{-inx}\,dx \\ &= \frac{1}{2\pi}\left(f(x)\left(\frac{-e^{-inx}}{in}\right)\Bigg|_{-\pi}^{\pi} + \int_{-\pi}^\pi f'(x)\frac{e^{-inx}}{in}\,dx\right) \\ &\text{the left summand vanishes because } f(\pi) - f(-\pi) = 0 \\ &= \frac{1}{2\pi}\left(f'(x)\left(\frac{e^{-inx}}{n^2}\right)\Bigg|_{-\pi}^{\pi} - \int_{-\pi}^\pi f''(x)\frac{e^{-inx}}{n^2}\,dx\right) \\ &\text{the left summand vanishes because } f'(\pi) - f'(-\pi) = 0 \\ &= \frac{1}{2\pi n^2}\int_{-\pi}^\pi f''(x)e^{-inx}\,dx \sim O(n^{-2}). \end{aligned}$$

Thus, $c_n \sim O(n^{-2})$. $\square$

Problem 7. Find the following limits.

(i) $\lim\limits_{\lambda\to\infty}\int_0^\infty \frac{\cos^2\lambda x}{1+x^2}\,dx$.

(ii) $\lim\limits_{\lambda\to\infty}\int_{-\pi}^\pi \sin^2\lambda x\,dx$.

Proof of (i). We have that $\cos^2\lambda x = \frac{1+\cos 2\lambda x}{2}$. So,

$$\lim_{\lambda\to\infty}\int_0^\infty \frac{1+\cos 2\lambda x}{2(1+x^2)}\,dx = \frac{1}{2}\lim_{\lambda\to\infty}\int_0^\infty \left(\frac{1}{1+x^2} + \frac{\cos 2\lambda x}{1+x^2}\right)dx.$$

since $\frac{1}{1+x^2}$ is integrable over $(0,\infty)$, then the right summand vanishes by the Reimann-Lebesgue Lemma.

So, we have that the above limit becomes

$$\frac{1}{2}\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\arctan x\Big|_0^\infty = \frac{\pi}{4}.$$

□

Proof of (ii). We have that $\sin^2\lambda x = \frac{1-\cos 2\lambda x}{2}$. So,

$$\frac{1}{2}\lim_{\lambda\to\infty}\int_{-\pi}^{\pi}(1-\cos 2\lambda x)\,dx = \pi - \frac{1}{2}\lim_{\lambda\to\infty}\int_{-\pi}^{\pi}\cos 2\lambda x\,dx.$$

Since the unit function 1 is also integrable on $[-\pi,\pi]$, then the right summand vanishes by the Reimann-Lebesgue Lemma.

Hence, the limit in (ii) is π. □

Problem 8. (i) Prove

$$\sum_{k=0}^{n-1}\sin\left(k+\frac{1}{2}\right)x = \frac{\sin^2\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}, \quad \forall n\geq 1.$$

(ii) Use (i) to show

$$\int_0^\pi \frac{\sin^2\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\,dx = \pi, \quad \forall n\geq 1.$$

Proof of (i). We will induct on n.

For $n=1$,

$$\sum_{k=0}^{1-1=0}\sin\left(\frac{2k+1}{2}x\right) = \sin\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\sin\frac{x}{2}}.$$

Assume that (i) holds for n. Now, for $n+1$,

$$\sum_{k=0}^{n}\sin\frac{2k+1}{2}x = \frac{\sin^2\frac{nx}{2}}{\sin\frac{x}{2}} + \sin\frac{2n+1}{2}x.$$

Consider the right summand; using the product of sines,

$$\begin{aligned}\frac{\sin\left(\frac{2n+1}{2}\right)x\cdot\sin\frac{x}{2}}{\sin\frac{x}{2}} &= \frac{1}{2\sin\frac{x}{2}}\left(\cos\left(\frac{2n+1}{2}-\frac{1}{2}\right)x - \cos\left(\frac{2n+1}{2}+\frac{1}{2}\right)x\right)\\ &= \frac{1}{2\sin\frac{x}{2}}\left(\cos nx - \cos(n+1)x\right).\end{aligned}$$

Then, for left summand, using square sine identity,

$$\frac{\sin^2\frac{nx}{2}}{\sin\frac{x}{2}} = \frac{1-\cos nx}{2\sin\frac{x}{2}}.$$

So, the above sum in the inductive step simplifies to,

$$\frac{1-\cos{(n+1)x}}{2\sin\frac{x}{2}} = \frac{\sin^2\left(\frac{n+1}{2}\right)x}{\sin\frac{x}{2}},$$

which is our desired result with $n+1$. □

Proof of (ii). We will induct on n.

For $n = 1$,

$$\int_0^\pi \frac{\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}}\,dx = \int_0^\pi dx = 1\cdot\pi.$$

Assume (ii) holds for n. Now, for $n+1$, by part (i),

$$\int_0^\pi \frac{\sin^2\left(\frac{n+1}{2}\right)x}{\sin^2\frac{x}{2}}\,dx = \int_0^\pi \sum_{k=0}^{n} \frac{\sin\left(k+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx = \sum_{k=0}^{n}\int_0^\pi \frac{\sin\left(k+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx.$$

Here, we have the k^{th} Dirichlet kernel $D_k(x)$, where

$$D_k(x) = \frac{\sin\left(k+\frac{1}{2}\right)x}{2\pi\sin\frac{x}{2}}, \qquad \int_0^\pi D_k(x)\,dx = \frac{1}{2}.$$

So, the above is

$$\sum_{k=0}^{n}\int_0^\pi 2\pi D_k(x)\,dx = \sum_{0}^{n} 2\pi\cdot\frac{1}{2} = n\pi.$$

□

Problem 9. Let f be Reimann integrable on $[a,b]$. Let g be a continuous T-periodic function on $\mathbb{R}$. Prove

$$\lim_{\lambda\to\infty}\int_a^b f(x)g(\lambda x)\,dx = \frac{1}{T}\int_0^T g(x)\,dx\int_a^b f(x)\,dx.$$

Proof. Since f is Reimann integrable on $[a,b]$, then, given any $\epsilon > 0$, there exists a step function h such that

$$\int_a^b |f(x)-h(x)|\,dx < \epsilon.$$

So, we have that

$$\int_a^b f(x)g(\lambda x)\,dx = \int_a^b h(x)g(\lambda x)\,dx + \int_a^b (f(x)-h(x))g(\lambda x)\,dx. \tag{*}$$

Since h is a step function, then we can write it as

$$h(x) = \sum_{i=0}^{n} c_i\chi_{[x_i,x_{i+1}]}(x),$$

where $\chi_{[x_i,x_{i+1}]}(x)$ is the indicator function of the interval $[x_i,x_{i+1}]$.
Note that $\int_a^b h(x)\,dx = \sum_{i=0}^n c_i(x_{i+1}-x_i)$.

Then, the left summand of (*) becomes,

$$\int_a^b h(x)g(\lambda x)\,dx = \sum_{i=0}^{n} c_i \int_{x_i}^{x_{i+1}} g(\lambda x)\,dx.$$

Since g is continuous and T-periodic, then the integral of g over large intervals approaches the average over T as λ grows large.

$$\int_{x_i}^{x_{i+1}} g(\lambda x)\,dx = \frac{1}{\lambda}\int_{\lambda x_i}^{\lambda x_{i+1}} g(u)\,du \approx \frac{x_{i+1}-x_i}{T}\int_0^T g(u)\,du.$$

Thus, summing over i and for λ large,

$$\int_a^b h(x)g(\lambda x)\,dx \approx \frac{1}{T}\int_0^T g(x)\,dx \sum_{i=0}^{n} c_i(x_{i+1}-x_i) \to \frac{1}{T}\int_0^T g(x)\,dx \int_a^b h(x)\,dx.$$

Since g is continuous and T-periodic, then it is bounded, and $|g(\lambda x)| \leq M$. Hence,

$$\left|\int_a^b (f(x)-h(x))g(\lambda x)\,dx\right| \leq M\int_a^b |f(x)-h(x)|\,dx < M\epsilon.$$

Since ϵ was arbitrary, then the right summand of (*) vanishes.

Thus,

$$\lim_{\lambda\to\infty}\int_a^b f(x)g(\lambda x)\,dx = \frac{1}{T}\int_0^T g(x)\,dx \int_a^b h(x)\,dx.$$

□