Math 336 Homework 4

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Problem 1. Prove that for $u \notin \mathbb{Z}$,

$$\sum_{-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}.$$

by integrating $f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$ on the circle $|z| = R_N = N + \frac{1}{2}$ for $N \in \mathbb{Z}$ and $N \geq |w|$, and adding residues of f on the inside of the circle C_{R_N}, letting $N \to \infty$.

Proof. The simple poles of f occur at $z \in [-N, N] \subset \mathbb{Z}$ and there is a second order pole at $z = -u$.

We have that

$$\oint_{C_{R_N}} f\, dz = 2\pi i \left(\sum_{-N}^{N} \operatorname{res}_k f + \operatorname{res}_{-u} f \right).$$

For the integer residues $[-N, N]$,

$$\begin{aligned}
\operatorname{res}_k f &= \lim_{z \to k} (z-k) \frac{\pi \cos \pi z}{(u+z)^2 \sin \pi z} \\
&\overset{\text{LH}}{=} \lim_{z \to k} \frac{\pi \cos \pi k - (z-k)\pi^2 \sin \pi z}{2(u+z)\sin \pi z + (u+z)^2 \pi \cos \pi z} \\
&= \frac{\pi(-1)^k}{(u+z)^2 \pi (-1)^k} \\
&= \frac{1}{(u+k)^2}.
\end{aligned}$$

For the second order pole,

$$\begin{aligned}
\operatorname{res}_{-u} f &= \lim_{z \to -u} \frac{d}{dz} \left((z+u)^2 \frac{\pi \cot \pi z}{(u+z)^2} \right) \\
&= \lim_{z \to -u} \frac{d}{dz} (\pi \cot \pi z) \\
&= \lim_{z \to -u} -\pi^2 \csc^2 \pi z \\
&= -\frac{\pi^2}{(\sin \pi u)^2},
\end{aligned}$$

by the oddness of sine.

So, we have

$$\oint_{C_{R_N}} f\, dz = 2\pi i \left(\sum_{-N}^{N} \frac{1}{(u+n)^2} - \frac{\pi^2}{(\sin \pi u)^2} \right).$$

We will show that the contour integral vanishes as $N \to \infty$. We begin by splitting the circle contour into parts and estimating $\cot \pi z$ on each part. We will write $z = x + iy$.

For the first part, we will consider the pieces of the circle with a modulus of real part between N and $N+1$. Since $\cot \pi z$ has a period of 1, with singularities at 0 and 1 but is bounded between, then it is also bounded when $|\mathrm{Re}(z)| = |x|$ is on the open interval $(N, N+1)$.

For the second part, we will consider the pieces of the contour with a modulus of imaginary part greater than the value of the contour for which the real part is above N is magnitude. Since the contour is a circle, then we can find the height of the contour that is achieved prior to $|\mathrm{Re}(z)| = |x| = N$. So,

$$
\begin{aligned}
x^2 + y^2 &= R_N^2 \\
y &= \sqrt{\left(N + \frac{1}{2}\right)^2 - N^2} \\
&= \sqrt{N + \frac{1}{4}} \approx \sqrt{N}.
\end{aligned}
$$

Next, we will show that $\cot \pi z$ is bounded for $|\mathrm{Im}(z)| = |y| \geq \sqrt{N}$.

We begin with the identifying cotangent with exponential functions using Euler's formula,

$$\cot \pi z = i \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1}.$$

So, as $|y| = \sqrt{N} \to \infty$, we have

$$i \frac{e^{2\pi i(x+iy)} + 1}{e^{2\pi i(x+iy)} - 1} = i \frac{e^{-2\pi y} e^{2\pi i x} + 1}{e^{-2\pi y} e^{2\pi i x} - 1} \to -i.$$

Hence, cotangent is bounded on the two parts of the contour, while the denominator of the integrand grows without bound.

So, we have

$$
\begin{aligned}
\int\limits_{|\mathrm{Re}(z)| > N} \frac{\pi \cot \pi z}{(u+z)^2}\, dz &\leq \left| \frac{C}{N^2} \right| \to 0 \\
\int\limits_{|\mathrm{Im}(z)| > \sqrt{N}} \frac{\pi \cot \pi z}{(u+z)^2}\, dz &\leq \left| \frac{1}{N^2} \right| \to 0
\end{aligned}
$$

Therefore the contour integral vanishes as $N \to \infty$. Thus we are left with,

$$\lim_{N \to \infty} \sum_{-N}^{N} \frac{1}{(u+k)^2} = \frac{\pi^2}{(\sin \pi u)^2},$$

as desired. $\square$

Problem 2. Prove that all entire and entire functions are linear.

Proof. First, consider the case where f is a polynomial. By FTA, f must have $\deg f$ roots in $\mathbb{C}$.

If f is injective, then f must have at most one root, hence f must be linear; i.e., f has a simple pole at ∞.

Now, for the case when f is not a polynomial, we have that $f(z)$ holomorphic on $\mathbb{C}$ implies that $f\left(\frac{1}{z}\right)$ is holomorphic on the punctured plane $\mathbb{C} \setminus \{0\}$.

If $z = 0$ is an essential singularity, then by Casorati-Weierstrass, in a deleted neighborhood of zero, the image of g is locally dense in $\mathbb{C}$, i.e., we get arbitrarily close to any value. □

Problem 3.

Proof. □

Problem 4.

Proof. □

Problem 5.

Proof. □

Problem 6.

Proof. □