

Math 336 Homework 6

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Problem 1. Poisson summation formula.

(a) Fix τ with $\text{Im } \tau > 0$. Apply Poisson summation formula to

$$f(z) = (\tau + z)^{-k}$$

for $2 \leq k \in \mathbb{N}$ to obtain

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m \tau}.$$

(b) Set $k = 2$ in the above identity and use $\text{Im } \tau > 0$ to show

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^2} = \frac{\pi^2}{\sin^2(\pi \tau)}.$$

(c) Can we conclude the above identity holds when τ is any complex number and not just an integer?

Proof of (a). Using residues, we will show that when $\xi < 0$, we have $\hat{f}(\xi) = 0$, and for $\xi > 0$, we have

$$\hat{f}(\xi) = \frac{(-2\pi i)^k}{(k-1)!} \xi^{k-1} e^{2\pi i \xi \tau}.$$

Since $\text{Im } \tau > 0$ and f has a pole of order k at $z = -\tau$, then this pole is not on the real axis, and f is holomorphic in some strip of width $a < \tau$.

Since f satisfies sufficient decay conditions for $k \geq 2$, and is holomorphic within a strip, then we can apply the Fourier transform. So, $\forall x \in \mathbb{R}$,

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$$

Let

$$g(z) = f(z) e^{-2\pi i z \xi} = \frac{e^{2\pi i z \xi}}{(\tau + z)^k}.$$

Since $e^{-2\pi i z \xi}$ is holomorphic, then g also has only a pole of order k at $z = -\tau$.

Since $\text{Im } \tau > 0$, then $\text{Im } (-\tau) < 0$, so the pole of g is in the lower half-plane.

We will consider the three cases where $\xi < 0$, $\xi = 0$, and $\xi > 0$.

For $\xi < 0$, the exponential term decays in the upper half-plane, so a semicircular contour there will enclose no poles, contributing no residues to the integral $\oint g dz$, thus $\hat{f}(\xi) = 0$ for $\xi < 0$.

For $\xi = 0$,

$$\int_{\mathbb{R}} \frac{dx}{(\tau + x)^k}$$

, since the integrand decays at sufficient magnitude of z , then can use the upper semicircular contour, again with no residues, or the lower semicircular contour, where the residue will vanish. So, the integral vanishes when $\xi = 0$.

For $\xi > 0$, the exponential term will vanish in the lower half-plane, so we will construct a semicircular contour there, picking up the residue from the pole at $z = -\tau$. Note that the positively oriented contour traverses the real axis in the opposite direction, so we pick up a negative sign. So, we have

$$-\hat{f}(\xi) = \int_{\mathbb{R}} g(x) dx = 2\pi i \operatorname{res}_{-\tau} g(z).$$

For the residue, we have

$$\begin{aligned} \operatorname{res}_{-\tau} g(z) &= \lim_{z \rightarrow -\tau} \frac{1}{(k-1)!} \left(\frac{d}{dx} \right)^{k-1} e^{-2\pi i z \xi} \\ &= \lim_{z \rightarrow -\tau} \frac{(-2\pi i \xi)^{k-1}}{(k-1)!} e^{-2\pi i z \xi} \\ &= \frac{(-2\pi i)^{k-1}}{(k-1)!} \xi^{k-1} e^{2\pi i \tau \xi}. \end{aligned}$$

Therefore, $\forall \xi > 0$,

$$\hat{f}(\xi) = \frac{(-2\pi i)^k}{(k-1)!} \xi^{k-1} e^{2\pi i \tau \xi},$$

and $\hat{f}(\xi)$ vanishes otherwise.

Now, with the Poisson summation formula, we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

Since $\forall \xi \leq 0$, $\hat{f}(\xi) = 0$, then the above becomes

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \in \mathbb{Z}^+} n^{k-1} e^{2\pi i n \tau},$$

as desired. □

Proof of (b). For $k = 2$, we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^2} = \frac{(-2\pi i)^2}{(2-1)!} \sum_{n \in \mathbb{Z}^+} n^{2-1} e^{2\pi i n \tau} = -4\pi^2 \sum_{n \in \mathbb{Z}^+} n e^{2\pi i n \tau}.$$

Note that

$$\sum_{n \in \mathbb{Z}_{\geq 0}} z^n = \frac{1}{1-z} \Rightarrow \sum_{n \in \mathbb{Z}^+} n z^n = \frac{z}{(1-z)^2}.$$

Let $w = e^{\pi i \tau}$,

$$\sin \pi \tau = \frac{e^{\pi i \tau} - e^{-\pi i \tau}}{2i} = \frac{w - w^{-1}}{2i}.$$

Let $w^2 = z = e^{2\pi i \tau}$,

$$\sin^2 \pi \tau = -\frac{1}{4}(w - w^{-1})^2 = -\frac{1}{4}(w^2 - 2 + w^{-2}) = -\frac{z^2 - 2z + 1}{4z} = -\frac{(z-1)^2}{4z}.$$

Therefore we have $\sum_{n \in \mathbb{Z}^+} n z^n = -\frac{1}{4 \sin^2 \pi \tau}$, which implies

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^2} = -4\pi^2 \sum_{n \in \mathbb{Z}^+} n e^{2\pi i n \tau} = \frac{\pi^2}{\sin^2 \pi \tau},$$

which is what we wanted to show. □

Proof of (c). Both functions in the identity of part (b) are meromorphic functions of τ which agree on the open half-plane and have identical poles at integer values.

We will show that the principle part of both functions matches at the poles.

The principle part of the series near integer τ poles is 1.

We will take the Taylor expansion of $\frac{\pi^2}{\sin^2 \pi \tau}$ near for τ near integers m ,

$$\sin \pi \tau = \sin(\pi m + \pi(\tau - m)) \approx \pi(\tau - m)$$

where the approximation holds by the Fundamental Theorem of Engineering.

Therefore we have $\sin^2 \pi \tau \rightarrow \pi^2(\tau - m)^2$ near integers m , so $\frac{\pi^2}{\sin^2 \pi \tau} \rightarrow \frac{1}{(\tau - m)^2}$ there as well, which has principle part 1 as well.

Since the principle parts agree near poles, then we have matching analytic continuations both functions on all $\tau \in \mathbb{C}$.

Thus, the identity holds for all complex τ . □

Problem 2. Suppose \hat{f} has compact support in $[-M, M]$ and $f(z) = \sum_{n \in \mathbb{N}} a_n z^n$.

(a) Show

$$a_n = \frac{(2\pi i)^n}{n!} \int_{-M}^M \hat{f}(\xi) \xi^n d\xi.$$

(b) Show

$$\limsup_{n \rightarrow \infty} (n! |a_n|)^{\frac{1}{n}} \leq 2\pi M.$$

(c) In the converse direction, show that if f is analytic with the limit supremum condition, then f is entire and

$$\forall \varepsilon > 0, \exists A_\varepsilon > 0 : |f(z)| \leq A_\varepsilon e^{2\pi(M+\varepsilon)|z|}.$$

Proof of (a). Since f has compact support, then f and \hat{f} have moderate decay.

Since f is entire, then $f \in \mathcal{F}_a$, so Fourier inversion holds.

Therefore, by the compact support of \hat{f}

$$f(z) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i z \xi} d\xi = \int_{-M}^M \hat{f}(\xi) e^{2\pi i z \xi} d\xi.$$

By the Cauchy integral formula for series coefficients,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz.$$

Since f is entire, then the integrand has a pole of order $n+1$ at $z=0$.

Using residues,

$$\begin{aligned} a_n &= \lim_{z \rightarrow 0} \frac{1}{n!} \left(\frac{d}{dz} \right)^n f(z) \\ &= \lim_{z \rightarrow 0} \frac{1}{n!} \left(\frac{d}{dz} \right)^n \int_{-M}^M \hat{f}(\xi) e^{2\pi i z \xi} d\xi \\ &= \int_{-M}^M \frac{1}{n!} \hat{f}(\xi) \lim_{z \rightarrow 0} \left(\frac{d}{dz} \right)^n e^{2\pi i z \xi} d\xi \\ &= \int_{-M}^M \lim_{z \rightarrow 0} \frac{1}{n!} \hat{f}(\xi) (2\pi i \xi)^n e^{2\pi i z \xi} d\xi \\ &= \frac{(2\pi i)^n}{n!} \int_{-M}^M \hat{f}(\xi) \xi^n d\xi, \end{aligned}$$

where the interchange of limit processes is justified by the finite integral and continuity of the integrand in both ξ and z .

Note that we can also arrive at this conclusion much faster by using the series expansion of $e^{2\pi i z \xi}$ inside the Fourier inversion integral for f . \square

Proof of (b). With $|\xi| \leq M$ by the bounds of the integral and $\hat{f}(\xi)$ bounded by some constant C from compact support, we have

$$\begin{aligned} n! |a_n| &= (2\pi)^n \left| \int_{-M}^M \hat{f}(\xi) \xi^n d\xi \right| \\ &\leq (2\pi)^n \int_{-M}^M |\hat{f}(\xi)| |\xi|^n d\xi \\ &\leq (2\pi)^n M^n \int_{-M}^M |\hat{f}(\xi)| d\xi \\ &\leq C(2\pi M)^n. \end{aligned}$$

So,

$$\limsup_{n \rightarrow \infty} (n! |a_n|)^{1/n} \leq \limsup_{n \rightarrow \infty} C^{1/n} 2\pi M = 2\pi M.$$

□

Proof of (c). For all $\varepsilon > 0$, there exists an N_ε such that for all $n > N_\varepsilon$, we have

$$(n! |a_n|)^{1/n} < 2\pi(M + \varepsilon) \implies |a_n| < \frac{(2\pi(M + \varepsilon))^n}{n!}.$$

We can split the series of f at N_ε ,

$$|f(z)| \leq \sum_{n=0}^{N_\varepsilon} |a_n| |z|^n + \sum_{n=N_\varepsilon+1}^{\infty} |a_n| |z|^n.$$

The first sum is bounded by $C_1 |z|^{N_\varepsilon}$.

For second sum with $n > N_\varepsilon$, we have the bound

$$\sum_{n=N_\varepsilon+1}^{\infty} \frac{(2\pi(M + \varepsilon))^n}{n!},$$

which is part of the series expansion for the exponential function, hence we also have the bound $C_2 e^{2\pi(M+\varepsilon)|z|}$.

Combining the bounds,

$$|f(z)| < C_1 |z|^{N_\varepsilon} + C_2 e^{2\pi(M+\varepsilon)|z|}.$$

Since exponential functions grow faster than polynomials, then, for some A_ε , we have

$$|f(z)| \leq A_\varepsilon e^{2\pi(M+\varepsilon)|z|},$$

which is what we wanted to show. □

Problem 3. We will show results similar to Phragmén-Lindelöf.

- (a) Suppose F is holomorphic in the right half-plane and extends continuously to the imaginary axis boundary. Given the boundary condition $\forall y \in \mathbb{R}, |F(iy)| \leq 1$ and the growth condition $|F(z)| \leq C \exp(c|z|^\gamma)$ for $c, C > 0$ and $\gamma > 1$, prove $|F(z)| \leq 1$ for all z in the right half-plane.
- (b) Let S be the sector with vertex at the origin, forming an angle of $\frac{\pi}{\beta}$. Suppose F is holomorphic in S and continuous on the boundary, $|F(z)| \leq 1$ on ∂S , and $|F(z)| \leq C \exp(c|z|^\alpha)$ for all z in S , with $c, C > 0$ and $0 < \alpha < \beta$. Prove $\forall z \in S, |F(z)| \leq 1$.

Proof of (b). Note that we will first prove part (b) as a more general case of part (a).

We have that $|\arg z| < \frac{\pi}{2\beta}$.

Assume that $\beta > 1$, i.e. the sector remains in the right half-plane.¹

We will take the principal log branch cut on \mathbb{R}^- .

Consider the function $\exp(-\varepsilon z^\beta)$ where $z = \operatorname{Re} e^{i\theta}$,

$$|\exp(-\varepsilon z^\beta)| = \exp(-\varepsilon r^\beta \cos \beta\theta).$$

¹We need $\cos(\arg z) > 0$ for decay.

But, $|\beta\theta| < \frac{\pi}{2\beta}$, so $|\theta| < \frac{\pi}{2}$ and $\cos \theta > 1$.

Let $F_\varepsilon(z) = F(z) \exp(-\varepsilon z^\beta)$. So,

$$|F_\varepsilon(z)| \leq |F(z)| |\exp(-\varepsilon z^\beta)| \leq C \exp(c|z|^\alpha - \varepsilon z^\beta).$$

Since $\alpha < \beta$, then $|F_\varepsilon| \rightarrow 0$ as $|z| \rightarrow \infty$.

Therefore

$$\sup_{\partial S \cap D_R} |F_\varepsilon| \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Consider the compact region $\bar{S} \cap D_R$, the intersection between the closure of S and the closed disk of radius R .

On ∂S , since $|F| \leq 1$, then $|F_\varepsilon| \leq \exp(-\varepsilon z^\beta) \leq 1$.

On the outer arc, with $\alpha < \beta$ and for sufficiently large R , $|F_\varepsilon| \leq C \exp(cR^\alpha - \varepsilon R^\beta) < 1$.

By MMP, as $R \rightarrow \infty$,

$$\sup_{\bar{S} \cap D_R} |F_\varepsilon| \leq \sup_{\partial(\bar{S} \cap D_R)} |F_\varepsilon| \leq 1.$$

Since F_ε is continuous in ε , then as $\varepsilon \rightarrow 0$,

$$\sup_{\bar{S} \cap D_R} |F_\varepsilon| \rightarrow \sup_{\bar{S} \cap D_R} |F| \leq 1,$$

which is what we wanted to show. □

Proof of (a). Let S be the right half-plane sector. On S , we have $\arg z \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so we will use the principal log branch.

For $\varepsilon > 0$, let $F_\varepsilon(z) = F(z) \exp(-\varepsilon z)$.

On S , the real part of z is be positive. So, $|F_\varepsilon| \leq C \exp(c|z|^\gamma - \varepsilon z)$ will vanish as $|z| \rightarrow \infty$ because $\gamma < 1$, and F_ε is bounded.

On ∂S , $|F_\varepsilon| = |F| |e^{-\varepsilon z}| \leq 1$.

On the outer arc $|F_\varepsilon| \leq 1$ from the decay demonstrated above.

So, as $R \rightarrow \infty$, we have $|F_\varepsilon| \leq 1$ on the boundary $\partial(\bar{S} \cap D_R)$, which bounds F_ε on the interior by MMP.

Then, as $\varepsilon \rightarrow 0$, we have that $|F| \leq 1$ on S , which is what we wanted to show. □

Problem 4.

Note that there is likely a typo in the suggested solution, the original erroneous statement was taking $\beta \rightarrow \pi$.

Phragmén-Lindelöf