336 Final Project Draft

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Contents

Abstract

In this project, we plan to introduce the curious prospective mathematician to some introductory ideas in combinatorics. In particular, we will discuss recurrence relations, generating functions, and integer partitions.

For our peers, we will expand on these ideas and wield some of our analysis tools developed in 336 to introduce the circle method, a tool employed in proof of the Hardy-Ramanujan estimation formula for partitions.

The paper is organized into three sections:

1. The first section describes the student activities associated with the project, including the materials required for the physical interactive activity and guiding questions for students to ponder when working with the materials.
2. The second section acts as a primer for prospective students on combinatorics.
3. The third section provides a detailed explanation of the ideas at a deeper level for peers in 336.

1. Showcase Activity

We will present some problems for the reader can begin to ponder to build some intuition behind recurrence relations. These problems are intended to be accompanied by physical artifacts with the showcase.

We will also provide the solutions, but we encouraged the reader to first turn to the Discussion Section 2 to obtain the tools necessary for tackling the problems.

1.1. Domino Tiling

How many ways we can tile a $2 \times n$ space with 1×2 tiles?

Materials & Setup: We can use dominoes as our tiles and a narrow box for our space.

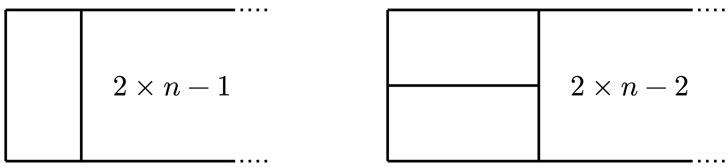

Figure 1: Domino tiling cases.

We see that there are two cases:

-
-

Guiding Questions

- Sketch a table of the space size n and the corresponding number of tilings which fill that space. Do you recognize this sequence?[1]
- Can you write down a relationship that describes how to get the next number of tilings in the sequence?

Extension Questions

- What are some other ways that we can build this sequence?

Proposition 1. The number of tiles follows the Fibonacci sequence.

1.2. Color Block Tiling

Let $h(n)$ be the number of ways to tile a $1 \times n$ space with

- 1×1 red and blue tiles; and
- 1×2 green, yellow, and black tiles.

Find the number of ways to tile the space with the given tiles.

Proof. We will use the sequence rule to find the generating function for $h(n)$.

Note that the empty space when $n = 0$ has one tiling where we use no tiles, i.e., $h(0) = 1$.

We can establish the following recurrence relation for $n \geq 2$,

$$h(n) = 2h(n-1) + 3h(n-2).$$

[1] If you are ever working with a sequence in the wild, try seeing if it is documented in The On-Line Encyclopedia of Integer Sequences (OEIS). https://oeis.org/

$\square$

Guiding Questions

Extension Questions

- How can we work with more complex shapes and arrangements? Perhaps natural curiosity might lead us to next explore the combinatorial possibilities of a Tetris game.

1.3. Parentheses Puzzle

How many ways are there to arrange sequences of nested and matched parentheses?

For any prefix of the string of parentheses, the number of left opening parentheses is at least the number of closing right parentheses. In the entire string, the number of left parentheses must be equal to the number of right parentheses. These are sequences are known as Dyck words.

((()))(()())

Figure 2: Example Dyck word.

Materials & Setup: Note cards with parentheses, colored red and blue. Set one color as the left parentheses, and one color as the right. Arrange the cards to form Dyck words.

Proposition 2. The number of Dyck words is the n-th Catalan number.

Proof of Proposition. Each Dyck word belongs to exactly one of the two following cases:
- it is empty, or
- it contains a matched pair of parentheses enclosing one Dyck word, and is followed by another, that is $(D_1)D_2$.

Let D be the set of all Dyck words, weighted by half the length of the string. Then, we have have weight-preserving bijection

$$D \to \{\} \sqcup \{()\} \times D \times D.$$

So,

$$F_D(x) = 1 + xF_D(x)^2 \Longrightarrow F_D(x) = \frac{1 - \sqrt{1-4x}}{2x} = \sum_0^\infty c_n x^n,$$

where c_n is the n-th Catalan number. $\square$

1.4. Polygon Triangulation

How many ways are there to cut a polygon along its diagonals into triangles where rotations are distinct?

Materials & Setup: Wooden block with labeled pegs stuck in to represent polygon vertices. Stretch rubber bands around the pegs to create triangulations.

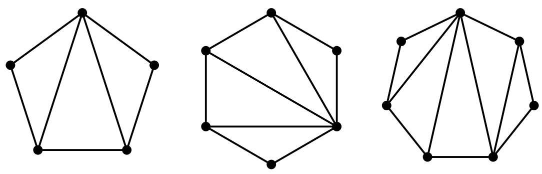

Figure 3: Example polygonal triangulations.

Guiding Questions

How many triangles are in the triangulation of an polygon with n sides (n-gon)?

Extension Questions

- What if we do not label the pegs, that is consider rotations to be indistinguishable?
- How many ways can we loop rubber bands around the pegs such that rubber bands do not cross over one another? Label the pegs so as to consider rotations to be distinct from one another. These are called non-crossing partitions.
- Can we connect these points to form a binary tree?
  Hint: what happens if we draw points (vertices of a tree) on each of the triangles in the triangulation.
- Can we do this for any triangulation?

Proposition 3. The number of ways to triangulate an $(n+2)$-sided polygon is the n-th Catalan number.

Proof of Proposition. Let T_n be the number of triangulations of an $n+2$-gon.

Each triangulation is in exactly one of the following cases:
- It is empty.
- The triangulation contains an outside edge of the polygon, and has two triangulations to the left and right.

Hence, there exists a weight-preserving bijection:

$$T \to \{\emptyset\} \sqcup \{\triangle\} \times T \times T.$$

Therefore

$$F_T(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n.$$

Thus, $|T_n| = c_n$, the n-th Catalan number. $\square$

2. Discussion for Students

We will take a first leap into the realm of enumerative combinatorics. Enumeration means finding the size of finite sets. In particular, we want to know how the size of sets change according to certain parameters.

Let A_n be a set parametrized by some natural number n. We write $f(n) = |A_n|$, assigning some function f which capture the size of the set.

Now, we can understand how the size of the set changes by understanding the behavior of the function f.

Note we have defined f to agree with the size of A_n only for integer values of n. What values might f take on for non-integer arguments?

There are several main cases for how we work with f:
- We have an explicit formula for $f(n)$.
- We have an approximation or asymptotic estimate for $f(n)$ as $n \to \infty$.
- We compare f to another function which we understand.
- We build a recurrence relation on the output of f.

[TODO: ensure intro covers all topics which we decide to retain.]

2.1. Asymptotic Behavior & Approximation

Asymptotic analysis allows us to simplify complex problems by focusing on how functions behave toward infinity.

This is a form of abstraction[2] where we strip away less significant terms to reveal the behavior of the function.

Oftentimes, we make comparisons with benchmark functions, such as logarithmic, linear, quadratic, and exponential.

We write $f \sim g$ when $\lim_{x\to\infty} f(x)/g(x) = 1$.

We use big and little "O" notations to relate the behavior of a given function f to these known functions.

2.1.1. Stirling's Formula

Theorem (*Stirling's Formula*).

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Proposition 4. Factorial grows faster than any exponential function.

Proof of Proposition 2. Let ab^n be an exponential function.

$$\lim_{n\to\infty} \frac{n!}{ab^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{ab^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{a}\left(\frac{n}{be}\right)^n = \infty.$$

Since this limit diverges, then $n!$ grows faster than any ab^n. □

[TODO: populate the definitions of the following sections.]

2.2. Generating Functions

Definition 1. recurrence relations

[NOTE: it would be beneficial to cut down on extra topics, so maybe we won't retain the following definition.]

Definition 2. homogeneous linear recurrence characteristic polynomial

[2] Abstraction is one of our most powerful tools in mathematics.

Definition 3. Suppose $f : \mathbb{N} \to \mathbb{C}$. The generating function of f is defined as follows:

$$F(x) = \sum_{n=0}^{\infty} f(n)x^n = \underbrace{f(0) + f(1)x + f(2)x^2 + \cdots}_{\text{contains all information from } f}$$

Generating functions are objects known as formal power series, they are a formal sum–which means we are not actually performing any addition–that does not have to converge for all x.[3]

We use generating functions to encode combinatorial rules as algebraic relations.

If $\{A_n\}$ is a sequence of finite sets, then we can define a generating function on the size of these sets

$$F_A(x) = \sum_{n=0}^{\infty} |A_n| x^n.$$

We can view the size of a set as a function that takes a set and returns the number of elements inside of the set. More generally, we can consider a weight function $w : A \to \mathbb{N}$ that takes a set and produces a given value n for each element in A.

With a set A, possibly infinite, and a weight function w on A, we can construct the generating function

$$F_A(x) = \sum_{a \in A} x^{w(a)}.$$

Example 1. Let A_n be the set of all binary strings of length n where A_0 is the empty string. Then $|A_n| = 2^n$ gives us that

$$F_A(x) = \sum_{n=0}^{\infty} (2x)^n = \frac{1}{1 - 2x}$$

using the geometric series $\sum x^n = \frac{1}{1-x}$.

[TODO: answer the following questions; they are prompts for writing.]

What are generating functions used for? What can we do with them?

How can we produce generating functions?

2.2.1. Constructing Generating Functions from Recurrence Relations

Theorem (*Sequence Rule*). Let A be a set with a weight function and no elements of weight 0. Let A^* be the set of all finite sequences of elements of A, including the empty sequence, where the weight of a sequence is given by the sum of the weights of its elements.

$$F_{A^*}(x) = \frac{1}{1 - F_A(x)}.$$

Proof. Every set in A^* belongs to exactly one of the following cases:
- it is empty, or
- its first element is in A, followed by an elements of A^*.

[3] Generating functions are neither generating, nor functions.

So, we can construct a weight-preserving bijection,

$$A^* \to \{(\,)\} \sqcup A \times A^*.$$

Hence, we have

$$\begin{aligned} F_{A^*}(x) &= 1 + F_A(x) F_{A^*}(x) \\ (1 - F_A(x)) F_{A^*}(x) &= 1 \\ F_{A^*}(x) &= \frac{1}{1 - F_A(x)}. \end{aligned}$$

Note that we can only divide formal power series with no constant term. Since we had that A had no elements with weight 0, then its power series □

Remark. If there were elements with weight zero, then we could create sequences with infinitely many zero-weighted elements.

2.3. Some Sequences

Definition 4. Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, ...$[4]

- recurrence relation $f(n+2) = f(n+1) + f(n)$.
- generating function $\frac{1}{1-x-x^2}$.
- closed form $\frac{\varphi^n - \psi^n}{\varphi - \psi}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\psi = \frac{1-\sqrt{5}}{2}$.
- asymptotics $f(n) \sim \varphi^n / \sqrt{5}$.

Definition 5. Catalan sequence $1, 1, 2, 5, 14, 42, ...$[5]

- recurrence relation $c(0) = 1$ and $c(n) = \frac{2(2n-1)}{n+1} c(n-1) = \sum_{i=1}^{n} c(i-1) c(n-i)$
- generating function $\frac{1-\sqrt{1-4x}}{2x}$
- closed form $\frac{1}{n+1}\binom{2n}{n}$
- asymptotics $c_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$

Proof of Catalan closed form. We will provide a combinatorial argument for $c_n = \frac{1}{n+1}\binom{2n}{n}$.

We have that c_n counts the number of lattice paths from $(0,0)$ to $(2n, 0)$ where each step is of the form $(1,1)$ or $(1,-1)$, and the path never crosses the $y = 0$ line.

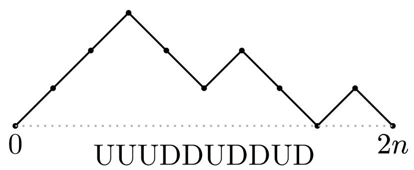

Figure 4: Example lattice path.

There are $\binom{2n}{n}$ lattice paths that never cross the $y = 0$ line without any other restriction, that is $2n$ total steps, n of which are up (or down).

[4] https://oeis.org/A000045
[5] https://oeis.org/A000108

Let b_n be the number of "bad" paths which cross $y = 0$. Given a bad path, consider the first time it touches $y = -1$ and reflect all points to the right across $y = -1$. The resulting lattice line is a path from $(0, 0)$ to $(2n, -2)$.

Given a path from $(0, 0)$ to $(2n, -2)$, we can go back to a bad path using the same process in reverse. Therefore, there exists a bijection between the set of bad paths to the set of paths from $(0, 0)$ to $(2n, -2)$; there are $\binom{2n}{n-1}$ such paths, for all $2n$ steps, $n - 1$ of these must be up steps.

Therefore,

$$\begin{aligned} c_n &= \binom{2n}{n} - \binom{2n}{n-1} \\ &= \binom{2n}{n} - \frac{(2n)!}{(n-1)!(n+1)!} \\ &= \binom{2n}{n} - \frac{n}{n+1}\frac{(2n)!}{n!n!} \\ &= \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} \\ &= \frac{1}{n+1}\binom{2n}{n}, \end{aligned}$$

which is what we wanted to show. □

So, both the parentheses puzzle and polygonal triangulation activities produce the same count.[6]

2.4. Integer Partitions

[TODO: expand the section on integer partitions, and perhaps give some applications of the object.]

Definition 6. For $n \in \mathbb{N}$, a partition of n is a way to write n as the sum of positive integers where the order of summation does not matter.[7]

Definition 7. We can represent partitions with Young and Ferrers diagrams.

Proposition 5. The number of partitions of n with at most k parts equals the number of partitions with largest part at most k.

Proof of Proposition 1. Consider the Ferrers diagram of the partition.

If a partition has k parts, then there are k rows in its diagram.

If the largest part of a partitions is k, then there are k columns in its partition.

Transpose the Ferrers diagram by flipping it across its central diagonal, swapping the number of rows and columns while maintaining the number of dots.

[6]In fact, the number of binary trees of n vertices is also the Catalan number.

[7]If the order of summation matters, then we have a strong composition.

This flipped partition is known as the conjugate partition.

Thus, we have a bijection between the set of partitions with k parts and the set of partitions with largest part k. □

Theorem. Let $P_{\leq k}$ be the set of all partitions with all parts at most k, weighted by sum.

$$F_{P_{\leq k}}(x) = \prod_{j=1}^{k} \frac{1}{1-x^j}.$$

Proof of Theorem.

□

If we let the maximum size of each part k exceed any number, i.e., $k \to \infty$, then we obtain the following theorem.

Theorem (*Partitions Generating Function*). Let P be the set of all partitions weighted by sum.

$$F_P(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

3. Explanation for Peers

We will build on material from previous section, so the reader is recommended to ensure familiarity with the ideas presented there. We will begin with a brief exploration of a kind of problems which, at first glance, does not appear to be related to complex analysis which we have been studying this quarter. Then, we will draw a connection between Number Theory problems and analysis techniques, in particular with demonstrating the Circle Method.

3.1. Additive Number Theory

Additive Number Theory is a branch concerned with the behavior of subsets of integers under addition. We have already seen an object of interest, integer partitions.

More generally, we consider k subsets of the nonnegative integers $\{A_i\}_{i=1}^{k}$ where $A_i \subset \mathbb{N}$. We are interested in the number of solutions $r_k(n)$ to the following equation with $n \in \mathbb{N}$:

$$n = \sum_{i=1}^{k} a_i, \quad a_i \in A_i.$$

We use $r_k(n)$ as a function between naturals to capture information about the solutions. Later, we will expand this function to a function on complex numbers using generating functions. This is the key step which allows us to bring in tools from analysis.

3.1.1. Additive Problem Examples

Now, we will see a few examples of additive problems.

Example 2. Weak[8] compositions of n into k parts with summands in A.[9]

[8]Strong compositions have all parts positive integers.

Let $A \subset \mathbb{N}$. Note that $\forall a \in A,\ 1 \leq a \leq n$, i.e., we cannot have a part of the partition greater than the sum. Expressing the number of solutions as a set, we have

$$r_k(n) = \#\{(a_1, ..., a_k) \in A^k \mid n = a_1 + \dots + a_k,\ a_i \in A\}.^{10}$$

When $A = \mathbb{N}_{\leq n}$, using a "stars" and "bars" argument with n stars and $k-1$ bars, we can show

$$r_k(n) = \binom{n+k-1}{k-1}.$$

Example 3. Goldbach's Conjecture, one of the oldest unsolved problems in number theory: any even natural number greater than 2 can be written as the sum of two primes.[11]

So, expressing this as a set in the above form, we write

$$r_2(n) = \#\{(p, q) \mid n = p + q, \quad p, q \text{ prime}\},$$

and the conjecture says that $\forall n > 2,\ r_2(n) \geq 1$.[12]

Example 4. Waring's problem: Let $g(k)$ be the minimum number such that for all positive integers n, the equation

$$n = \sum_{i=1}^{g(k)} a_i^k, \quad a_i \in \mathbb{N}$$

has at least one solution, i.e., $r_{g(k)}(n) \geq 1$. Here, we are considering both exponentiation and addition.[13]

Consider $7 = 1^2 + 1^2 + 1^2 + 2^2$. So, we have have that $g(2) \geq 4$. One can check that with 23 we have $g(3) \geq 9$ and with 79 we have $g(4) \geq 19$.

Remark. Lagrange's four-square theorem proves that exactly $g(2) = 4$.

3.2. Circle Method

The Hardy Ramanujan Littlewood Circle method is a technique in additive number theory. Our goal is to transform additive and combinatorics problems into complex analysis problems to use the tools of analysis. [1] Quoting Hardy and Ramanujan's original paper, "This idea [studying integrals from generating functions] has dominated nine-tenths of modern research in analytic theory of numbers." [2]

[9] Compositions are not the same as integer partitions; $r_k(n)$ counts ordered tuples, while partitions count unordered multisets.

[10] The # notation returns the size of the given set.

[11] The conjecture has been shown to hold for all integers less than $4 \cdot 10^{18}$ as of 2025 according to Wikipedia.

[12] The Weak Goldbach Conjecture posits that every odd number greater than 5 can be expressed as the sum of three primes, where a prime may be used more than once in the same sum. This conjecture was proven using the circle method by Harald Helfgott in 2013.

[13] Waring's problem is related to Fermat's polygonal number theorem.

[14] Particularly in the form of Theorem 4.4 in Stein & Shakarchi

From a high level, the method uses the Residue Theorem[14] to represent the coefficients of a generating function series as integrals around closed circular paths inside the unit circle. So, given a generating function $f(z) = \sum a_n (z - z_0)^n$, we have

$$a_n = \frac{1}{2\pi i} \oint_{\partial D(z_0)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

where f is holomorphic in an open set Ω with a disk D centered at z_0 such that $\overline{D} \subset \Omega$.

For the Circle Method, we will consider generating functions centered at $z_0 = 0$ with $\Omega = \mathbb{D}$. We will split up the circular paths of integration into "major" and "minor" arcs, where we get the main, often integrable, terms from the major arcs, and bounded error terms arise from minor arcs.

3.2.1. Example Application to Weak Compositions

We will use the circle method to find the number of weak compositions of n into k parts.

Setup

Begin with the case where we have $k = 2$ parts,

$$r_2(n) = \#\{(a_1, a_2) \mid n = a_1 + a_2,\ a_1, a_2 \in A\}.$$

For each $a_i \in A$, we can construct the generating function f using the indicator function for A:

$$f(z) = \sum_{n=0}^{\infty} \mathbf{1}_A(n) z^n, \qquad \text{where } \mathbf{1}_A(n) = \begin{cases} 1 & n \in A, \\ 0 & n \notin A. \end{cases}$$

Now, for $k = 2$, using the Cauchy series product, we have

$$f^2(z) = \left(\sum_{n=0}^{\infty} \mathbf{1}_A(n) z^n \right) \left(\sum_{m=0}^{\infty} \mathbf{1}_A(m) z^m \right) = \sum_{n=0}^{\infty} c(n) z^n$$

where $c(n) = \sum_{k=0}^{n} \mathbf{1}_A(k) \mathbf{1}_A(n-k)$, which we can rewrite as $\sum_{h+k=n} \mathbf{1}_A(h) \mathbf{1}_A(k)$.

Since $\mathbf{1}_A(h)\mathbf{1}_A(k) = 1$ iff both $h, k \in A$, then this expression of $c(n)$ is exactly the number of pairs of $(h, k) \in A^2$ which satisfy $h + k = n$.

Therefore, we have $f^2(x) = \sum_{n=0}^{\infty} r_2(n) z^n$.

Now, consider a composition of n into k parts with summands in A,

$$r_k(n) = \#\{(a_1, ..., a_k) \in A^k \mid n = a_1 + \cdots + a_k,\ a_i \in A\}.$$

After repeated applications of Cauchy product, we arrive at the generating function

$$f^k(z) = \sum_{n=0}^{\infty} r_k(n) z^n.$$

Since f^k is analytic, then with Residue Theorem applied to series coefficients, we have

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f^k(z)}{z^{n+1}}\, dz \tag{1}$$

for a circular closed loop C_ρ of radius ρ centered at the origin.

Now, we can express the number of solutions $r_k(n)$ in terms of the residues of this integral.

From Example 2 above with $A = \mathbb{N}$, we already know that $r_k(n) = \binom{n+k-1}{k-1}$. We will verify this using the Circle method.

Method Application

Since $A = \mathbb{N}$, We have

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

which converges for $|z| < 1$. So, by equation (1), we have

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{dz}{(1-z)^k z^{n+1}},$$

where the integral converges on the closed circular loop C_ρ of radius $\rho < 1$. Since we can evaluate this integral directly, then we can consider the entirety of C_ρ as our major arc, and we do not need to estimate any minor arcs.

Now, we will extract the nonzero residues, which come from the simple poles of the integral. Using the General Binomial Expansion, we have

$$\frac{1}{(1-z)^k} = \sum_{m=0}^{\infty} \binom{-k}{m} (-z)^m.$$

Hence, the integral becomes

$$\oint_{C_\rho} \sum_{m=0}^{\infty} \binom{-k}{m} (-z)^m z^{-(n+1)}\, dz.$$

So, for the simple poles, we will take the terms for which $m - n - 1 = -1 \Longrightarrow m = n$.

Thus,

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \binom{-k}{n} (-1)^n z^{-1}\, dz = (-1)^n \binom{-k}{n} = \binom{n+k-1}{k-1},$$

where the last equality is given by a combinatorial identity.[15]

3.3. Hardy-Ramanujan Estimation Formula

Theorem (*Hardy-Ramanujan*). Let $p(n)$ be the number of partitions of n.

$$p(n) \sim \frac{\exp \pi \sqrt{2n/3}}{4\sqrt{3}\, n}.$$

Sketch of Hardy-Ramanujan. We will follow a simplified version of the original proof which performs analysis on an asymptotically similar function which is analytic inside the unit

[15]This identity can be quickly shown by expressing both binomial coefficients in terms of falling factorials (Pochhammer symbols) and counting the appearances of -1.

circle. [3] The simplified proof uses merely one major arc and more elementary estimates as opposed to modular forms and theta functions.

The proof follows by careful analysis of the generating function given by the Partitions Generation Function Theorem,

$$f(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} \frac{1}{1 - z^m}, \quad \text{for } |z| < 1 \text{ where } p(0) = 1,$$

around the singularity $z = 1$. Note that f has poles at the roots of unity $z^n = 1$.

The lower order roots will correspond to the more dominant poles, that is the pole at $z = 1$ as the solution to $z - 1$, followed by the pole at $z = -1$ for the additional solution to $z^2 - 1$, and then the two additional cube roots of unity from $z^3 - 1$, and so on. These poles correspond to black regions inside the unit circle in the following figure. Note that while the figure is colored for the reciprocal of f, we can still observe the relative impact of the poles.

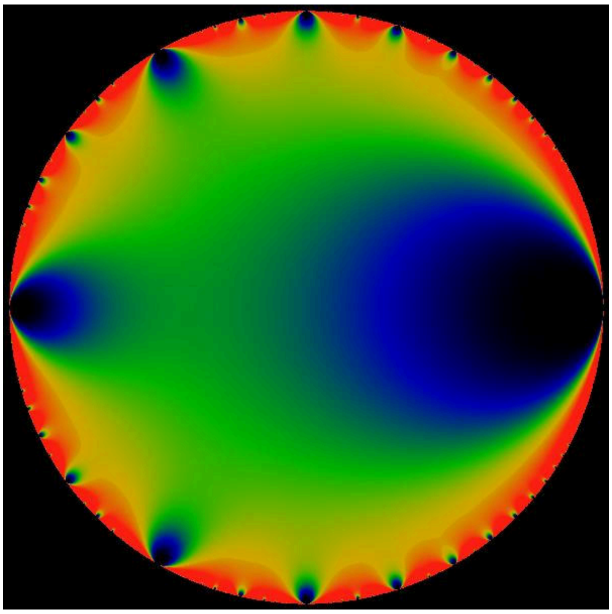

Figure 5: Modulus of Euler's q-series for integer partitions $\prod_{k=1}^{\infty}(1 - q^k)$ on the unit disk in the complex plane.[16]

We define a simpler auxiliary function which captures the asymptotic behavior of f:

$$\varphi(z) = \sum q(n) z^n = \left(\frac{1 - z}{2\pi}\right)^{1/2} \exp \frac{\pi^2}{12}\left(\frac{2}{1 + z} - 1\right).$$

We will show $p(n) \approx q(n)$ and compute the latter directly. We approximate f by φ with

$$f(z) = \varphi(z)(1 + O(1 - z))$$

for $|z| < 1$ near 1, in particular $|1 - z| \leq 2(1 - |z|)$. This is proven using variation.

We can show the bound

$$|f(z)| < \exp\left(\frac{1}{1 - |z|} + \frac{1}{|1 - z|}\right), \tag{2}$$

using the logarithm on the product definition of f.

[16] https://commons.wikimedia.org/wiki/File:Q-Eulero.jpeg

We will then employ the Circle Method on the following integral:

$$p(n) - q(n) = \frac{1}{2\pi i} \oint_C \frac{f(z) - \varphi(z)}{z^{n+1}}\, dz.$$

We choose the circular path of integration C with radius $1 - \pi/\sqrt{6n}$, which approaches 1 as $n \to \infty$. Since this path lies inside the unit circle, then $|z| < 1$ and the generating function f will converge.

We split the circle into two parts:
- the major arc $A = \left\{ z \in C \mid |1 - z| < \pi\sqrt{2/3n} \right\}$, which is the part of the circle C near the dominant singularity of f at $z = 1$, and
- the minor arc $B = C - A$, which is the rest of circle away form $z = 1$.

Asymptotically, the contribution of the minor arc will vanish exponentially fast, so we will be left with an approximation given by the integral around just the major arc.

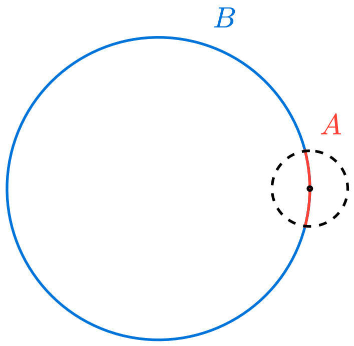

Figure 6: Major A and minor B arcs of C.

For the minor arc B, using the exponential bound on f from equation 2, we have the following estimate of asymptotic equivalence:

$$\begin{aligned}
\int_B \frac{f(z) - \varphi(z)}{z^{n+1}}\, dz &\sim \int_B |z|^{-n} \left(\exp \frac{\pi^2}{6|1-z|} + \exp\left(\frac{1}{1-|z|} + \frac{1}{|1-z|} \right) \right) dz \\
&\sim \int_B \exp \pi\sqrt{n/6} \left(\exp \frac{\pi}{6}\sqrt{3n/2} + \exp \frac{1}{\pi}\left(\sqrt{3n/2} + \sqrt{6n} \right) \right) dz \\
&= O\left(\exp a\sqrt{n}\right) \text{ where } a < \pi\sqrt{2/3}.
\end{aligned}$$

For the major arc, with the length of A as $O\left(n^{-1/2}\right)$,

$$\begin{aligned}
\int_A \frac{f(z) - \varphi(z)}{z^{n+1}}\, dz &\sim \int_A |z|^{-n}\ |1-z|^{3/2} \exp \frac{\pi}{6(1-|z|)}\, dz \\
&\sim n^{-3/4} \exp\left(\pi\sqrt{n/6} + \pi\sqrt{n/6} \right) n^{-1/2} \\
&= O\left(n^{-5/4} \exp \pi\sqrt{2n/3} \right).
\end{aligned}$$

Therefore

$$p(n) = q(n) + O\left(n^{-5/4} \exp \pi\sqrt{2n/3} \right).$$

Since this error is much smaller than the main asymptotic term of q, then we have $p \sim q$.

Using the method of steepest descent, we have

$$\pi\sqrt{2}\exp\frac{\pi^2}{12}\varphi(z) = (1-z)\int_{\mathbb{R}} \exp\left(\pi t\sqrt{2/3} - (1-z)t^2\right) dt$$

The remainder of the proof is (even more so) heavily abridged. Comparing the power series in z on both sides, we obtain

$$\begin{aligned}\pi\sqrt{2}\exp(\pi^2/12)q(n) &= \int_{\mathbb{R}} \exp\left(\pi t\sqrt{2/3} - t^2\right)\left(\frac{t^{2n}}{n!} - \frac{t^{2n-2}}{(n-1)!}\right) dt \\ &\sim \frac{\exp\pi\sqrt{2/3n}}{\sqrt{2\pi n}}\int_{\mathbb{R}} s\exp\left(\pi\sqrt{2/3}s - s^2 - 2\sqrt{n}s\right)\left(1+\frac{s}{\sqrt{n}}\right)^{2n-2}\left(2+\frac{s}{\sqrt{n}}\right) ds\end{aligned}$$

using the substitution $t = s + \sqrt{n}$ and Stirling's formula for $n!$.

Then,

$$\lim_{n\to\infty} e^{-2\sqrt{n}s}\left(1+\frac{s}{\sqrt{n}}\right)^{2n-2}\left(2+\frac{s}{\sqrt{n}}\right) = 2e^{-s^2}.$$

The integral is dominated by a function F whose integral converges over the reals, which enables the use of the Dominated Converge Theorem to take the above limit under the integral,

$$\begin{aligned}\pi\sqrt{2}\exp(\pi^2/12)q(n) &\sim \frac{\exp\pi\sqrt{2/3n}}{\sqrt{2\pi n}}\int_{\mathbb{R}} 2s\exp\left(\pi\sqrt{2/3}s - s^2\right) ds. \\ &= \frac{\pi}{2\sqrt{6n}}\exp\left(\pi^2/12 + \pi\sqrt{2n/3}\right) \\ q(n) &\sim \frac{\exp\pi\sqrt{2n/3}}{4\sqrt{3}n},\end{aligned}$$

which in turn yields the desired result for p. □

Remark. The full proof relies on several more advanced analysis concepts:

- Lebesgue's Dominated Convergence Theorem, which allows for the interchange of limits and integrals;
- the method of steepest descent, which enables us to analyze complex functions by evaluating integrals instead;
- as well as total variation of functions, which is useful in bounding terms.

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