Math 336 Homework 6

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Definition 1. $f : U \to V$ is a local bijection on U for all $z \in U$ if there exists an open disk $D \subset U$ centered at z such that $f : D \to f(D)$ is a bijection.

Problem 1. Prove $f : U \to V$ is a holomorphic local bijection on U iff $\forall z \in U,\ f'(z) \neq 0$.

Proof. ($\Longrightarrow$) Since f is a local bijection on $D \to f(D)$, then f must be injective there as well. As seen in lecture[1], if $f : U \to V$ is holomorphic and injective, then $\forall z \in U, f'(z) \neq 0$.

Reconstructing the argument, suppose for a contradiction that $\exists z_0 \in U : f'(z_0) = 0$.

For z near z_0, since f is analytic, we have

$$f(z) = f(z_0) + a(z - z_0)^k + G(z)$$

where $a \neq 0$, $k \geq 2$, and $G = o\left(z^{k+1}\right)$ near z_0.

For small $w > 0$,

$$f(z) - f(z_0) - w = \underbrace{a(z - z_0)^k - w}_{F(z)} + G(z).$$

Since $G = o\left(z^{k+1}\right)$ near z_0, then for small r, $|G| < |F|$ on $\partial D_r(z_0)$.

So, by Rouché, F and G have the same number of zeros inside $D_r(z_0)$.

But, F has a k zeros, so $f(z) - f(z_0) - w$ must have k zeros in $D_r(z_0)$ as well, i.e., $f(z) = f(z_0) + w$ has multiple solutions there, contradicting the injectivity of f.

($\Longleftarrow$) Since f is holomorphic at z_0 with $f'(z_0) \neq 0$, then its analytic expansion contains a linear term,

$$f(z) = f(z_0) + a(z - z_0) + o\Big((z - z_0)^k\Big)$$

for $a \neq 0$, $k \geq 1$.

We can factor the above as

$$f(z) - f(z_0) = (z - z_0)h(z)$$

where h is holomorphic with $h(z_0) = a$.

Because h is continuous and nonvanishing at z_0, then we can find a small disk $D \subset U$ centered at z_0 where h is nonvanishing in U.

Hence, in D, $f(z) = f(z_0)$ only at z_0, so f is injective there. By the Open Mapping Theorem, the image $f(D)$ is open because f is nonconstant and holomorphic.

So $f : D \to f(D)$ is a bijection on open sets.

[1] Proposition 8.1.1 in Stein & Shakarchi

Since z_0 was arbitrary, we can perform this procedure for each $z_0 \in U$ to see that f is a local bijection. $\square$

Problem 2. Is there a surjective map from $\mathbb{D} \to \mathbb{C}$?

Proof. Yes. We already have the bijective inverse Cayley map $G : \mathbb{D} \to \mathbb{H}$, so we will construct a surjective map $\mathbb{H} \to \mathbb{C}$.

Now, consider the map $H : \mathbb{H} \to \mathbb{C}$ defined by the squaring function $z \mapsto z^2$; where $z = re^{i\theta}$, we have $r^2 e^{2i\theta}$.

For $z \in \mathbb{H}$, $\arg z \in (0, \pi)$. In order to cover the real axis, we shift the upper half-plane down to $-i$. Then, we have at least $\arg z \in [0, \pi]$.

So, for $w \in H(\mathbb{H})$, we have $\arg w \in [0, 2\pi]$, which covers the whole imaginary plane $\mathbb{C}$. Since r covers $\mathbb{R}_{\geq 0}$, then the squaring map H will have r^2 covering $\mathbb{R}_{\geq 0}$ as well.

So, the map H is surjective. Thus, the composition $H \circ G : \mathbb{D} \to \mathbb{C}$ is surjective. $\square$

Problem 3. Dirichlet problem in a strip. It suffices to compute solutions at points $z = iy$ for $y \in (0, 1)$.

(a) Show that if $re^{i\theta} = G(iy)$, then

$$re^{i\theta} = i\frac{\cos \pi y}{1 + \sin \pi y}.$$

leads to two cases: $y \in \left(0, \frac{1}{2}\right]$ and $\theta = \frac{\pi}{2}$, or $y \in \left[\frac{1}{2}, 1\right)$ and $\theta = -\frac{\pi}{2}$.

In either case, show

$$r^2 = \frac{1 - \sin \pi y}{1 + \sin \pi y} \quad \text{and} \quad P_r(\theta - \varphi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}.$$

(b) With the change of variables $t = F\left(e^{i\varphi}\right)$ in $\frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)\tilde{f}_0(\varphi)\,d\varphi$, show

$$e^{i\varphi} = \frac{i - e^{\pi t}}{i + e^{\pi t}}.$$

Take the imaginary part and differentiate to establish

$$\sin \varphi = \frac{1}{\cosh \pi t} \quad \text{and} \quad \frac{d\varphi}{dt} = \frac{\pi}{\cosh \pi t}.$$

Hence deduce

$$\begin{aligned} \frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)\tilde{f}_0(\varphi)\,d\varphi &= \frac{1}{2\pi}\int_0^\pi \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}\tilde{f}_0(\varphi)\,d\varphi \\ &= \frac{\sin \pi y}{2}\int\limits_{\mathbb{R}} \frac{f_0(t)}{\cos \pi t - \cos \pi y}\,dt. \end{aligned}$$

(c) Use a similar argument to prove the formula for the integral

$$\frac{1}{2\pi}\int_{-\pi}^0 P_r(\theta - \varphi)\tilde{f}_1(\varphi)\,d\varphi.$$

Proof of (a). From the definition of G, we have

$$re^{i\theta} = \frac{i - e^{i\pi y}}{i + e^{i\pi y}}.$$

Multiplying by the conjugate of the denominator, $\cos \pi y - i(1 + \sin \pi y)$, we have the following:

For the numerator, $(i(1 - \sin \pi y) - \cos \pi y)(\cos \pi y - i(1 + \sin \pi y)) = 2i \cos \pi y$.
For the denominator, $(i(1 + \sin \pi y) + \cos \pi y)(\cos \pi y - i(1 + \sin \pi y)) = 2(1 + \sin \pi y)$.

So, we have

$$re^{i\theta} = i\frac{\cos \pi y}{1 + \sin \pi y},$$

as desired.

In the first case, with $y \in \left(0, \frac{1}{2}\right] \implies \pi y \in \left(0, \frac{\pi}{2}\right]$, we have $\cos \pi y \geq 0$ and $\sin \pi y > 0$. Therefore the above expression is purely imaginary with positive imaginary part, so

$$\theta = \frac{\pi}{2} \quad \text{and} \quad r = \frac{\cos \pi y}{1 + \sin \pi y} \in [0, 1).$$

In the second case, with $y \in \left[\frac{1}{2}, 1\right)$, we have $\cos \pi y \leq 0$ and $\sin \pi y > 0$. Therefore

$$\theta = -\frac{\pi}{2} \quad \text{and} \quad 0 < r = \frac{-\cos \pi y}{1 + \sin \pi y} \in [0, 1).$$

We can now verify the following:

$$r^2 = \frac{\cos^2 \pi y}{(1 + \sin \pi y)^2} = \frac{1 - \sin^2 \pi y}{(1 + \sin \pi y)^2} = \frac{1 - \sin \pi y}{1 + \sin \pi y}.$$

For

$$P_r(\theta - \varphi) = \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2},$$

note that for $\theta = \frac{\pi}{2}, -\frac{\pi}{2}$, we have $\cos\left(\frac{\pi}{2} - \varphi\right) = \sin \varphi$ and $\cos\left(-\frac{\pi}{2} - \varphi\right) = -\sin \varphi$ respecively. But, in either case,

$$r\cos(\theta - \varphi) = \frac{\cos \pi y}{1 + \sin \pi y} \sin \varphi.$$

For the numerator, we have

$$1 - r^2 = \frac{2 \sin \pi y}{1 + \sin \pi y}.$$

For the denominator,

$$\begin{aligned} 1 - 2r\cos(\theta - \varphi) + r^2 &= 1 - 2\frac{\cos \pi y}{1 + \sin \pi y} \sin \varphi + \frac{1 - \sin \pi y}{1 + \sin \pi y} \\ &= \frac{2(1 - \cos \pi y \sin \varphi)}{1 + \sin \pi y}. \end{aligned}$$

Therefore,

$$P_r(\theta-\varphi)=\frac{\sin\pi y}{1-\cos\pi y\sin\varphi}.$$

□

Proof of (b). Multiplying by the conjugate of the denominator, we have

$$e^{i\varphi}=\frac{1+2ie^{\pi t}-e^{2\pi t}}{1+e^{2\pi t}},$$

which provides the following:

$$\begin{aligned}\cos\varphi&=\frac{1-e^{2\pi t}}{1+e^{2\pi t}}=-\tanh\pi t=-\frac{\sinh\pi t}{\cosh\pi t},\ \text{and}\\ \sin\varphi&=\frac{2e^{\pi t}}{1+e^{2\pi t}}=\frac{1}{\cosh\pi t}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\sin\varphi&=\frac{d}{dt}\frac{1}{\cosh\pi t}\\ \frac{d\varphi}{dt}\cos\varphi&=\frac{-\pi\sinh\pi t}{\cosh^2\pi t}\\ \frac{d\varphi}{dt}&=\frac{\pi}{\cosh\pi t}.\end{aligned}$$

With the change of variables $t=F\big(e^{i\varphi}\big)$ where $F:\mathbb{D}\to\mathbb{H}$ is the Cayley map, as φ traverses $(0,\pi)$, t will traverse the real axis. We have that $\tilde{f}_0(\varphi)=f_0\big(F\big(e^{i\varphi}\big)\big)$.

So,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{\pi}P_r(\theta-\varphi)\tilde{f}_0(\varphi)\,d\varphi&=\frac{1}{2\pi}\int_0^{\pi}\frac{\sin\pi y}{1-\cos\pi y\sin\varphi}f_0\big(F\big(e^{i\varphi}\big)\,d\varphi\\ &=\frac{\sin\pi y}{2\pi}\int\limits_{t(0)\to\infty}^{t(\pi)\to\infty}\frac{1}{1-\frac{\cos\pi y}{\cosh\pi t}}\frac{\pi}{\cosh\pi t}f_0(t)\,dt\\ &=\frac{\sin\pi y}{2\pi}\int\limits_{\mathbb{R}}\frac{f_0(t)}{\cosh\pi t-\cos\pi y}\,dt.\end{aligned}$$

□

Proof of (c). A similar argument follows for $\frac{1}{2\pi}\int_{-\pi}^{0}P_r(\theta-\varphi)\tilde{f}_1(\varphi)\,d\varphi$ using the case with $\theta=-\frac{\pi}{2}$ and $y\in\big[\frac{1}{2},1\big)$.

Thus,

$$u(0,y)=\frac{\sin\pi y}{2}\left(\int\limits_{\mathbb{R}}\frac{f_0(t)}{\cosh\pi t-\cos\pi y}\,dt+\int\limits_{\mathbb{R}}\frac{f_1(t)}{\cosh\pi t+\cos\pi y}\,dt\right)$$

□

Proof of bonus. Since "translation of the boundary condition by x results in a translation of the solution by x as well,"[2] then applying the same derivation to the boundary conditions $f_0(x+t)$ and $f_1(x+t)$ yields the same intermediate result,

$$u(x,y) = \frac{\sin \pi y}{2}\left(\int_{\mathbb{R}} \frac{f_0(x+t)}{\cosh \pi t - \cos \pi y}\,dt + \int_{\mathbb{R}} \frac{f_1(x+t)}{\cosh \pi t + \cos \pi y}\,dt\right).$$

Using the substitution $t \mapsto -t$ with the oddness of hyperbolic cosine and reversing the integration direction, we have

$$u(x,y) = \frac{\sin \pi y}{2}\left(\int_{\mathbb{R}} \frac{f_0(x-t)}{\cosh \pi t - \cos \pi y}\,dt + \int_{\mathbb{R}} \frac{f_1(x-t)}{\cosh \pi t + \cos \pi y}\,dt\right),$$

which is what we wanted to show. □

Problem 4.
(a) Prove if $f : \mathbb{D} \to \mathbb{D}$ is analytic with two fixed points, then f is the identity function.

(b) Must every holomorphic map $f : \mathbb{D} \to \mathbb{D}$ have a fixed point?

Proof of (a). Consider the biholomorphic automorphism $\psi_\alpha : \mathbb{D} \to \mathbb{D}$, which maps $0 \mapsto \alpha$ and $\alpha \mapsto 0$.

Let z and ω be fixed points of f.

Consider the map $\varphi = \psi_z \circ f \circ \psi_z$, which fixes the origin because $0 \overset{\psi_z}{\mapsto} z \overset{f}{\mapsto} z \overset{\psi_z}{\mapsto} 0$.

Let $\tilde{\omega}$ be the preimage of ω under ψ_z, that is $\psi_z(\tilde{\omega}) = \omega$. Since we have $\tilde{\omega} \overset{\psi_z}{\mapsto} \omega \overset{f}{\mapsto} \omega \overset{\psi_z}{\mapsto} \tilde{\omega}$, then $\tilde{\omega}$ is another fixed point of φ.

Note that $\tilde{\omega} \neq 0$, otherwise $\tilde{\omega}$ would map to z under ψ_z, which is a contradiction with the assumption that f had two distinct fixed points.

Since $\varphi(0) = 0$ and $\varphi(\tilde{\omega}) = \tilde{\omega} \implies |\varphi(\tilde{\omega})| = |\tilde{\omega}|$, then by Schwarz, φ is a rotation. But, the second fixed point $\tilde{\omega}$ means that we must have a rotation by zero, otherwise there would only be one fixed point. Hence, φ must be the identity function.

But, ψ_z is an involution, so with left and right composition, we have that

$$\psi_z \circ \varphi \circ \psi_z = f = \psi_z \circ \mathrm{id} \circ \psi_z = \psi_z \circ \varphi_z = \mathrm{id},$$

which is what we wanted to show. □

Proof of (b). Consider a translation map $H : \mathbb{H} \to \mathbb{H}$ which has no fixed points and the Cayley maps between $\mathbb{D}$ and $\mathbb{H}$, $F : \mathbb{D} \to \mathbb{H}$ and $G : \mathbb{H} \to \mathbb{D}$.

Assume for a contradiction that $G \circ H \circ F : \mathbb{D} \to \mathbb{D}$ has a fixed point z.

Then, $H(z) = (F \circ G)(z) = z$, a contradiction with the fact that H has no fixed points. □

[2]Stein & Shakarchi p. 216

Problem 5. Let $G : \mathbb{D} \to \mathbb{H}$ be the inverse Cayley map defined by $G(w) = i\frac{1-w}{1+w}$. Find the image of the following curves under G:

(a) Circles centered at the origin, and

(b) Straight line segments through the origin.

Sketch the original curves in $\mathbb{D}$ on one diagram, and the image curves in $\mathbb{H}$ on another.

Consider $z = re^{i\theta}$

(a) Here we fix the magnitude r and vary θ. We will obtain circles with the equation

$$x^2 + \left(y - \frac{1+r^2}{1-r^2}\right)^2 = \left(\frac{2r}{1-r^2}\right)^2,$$

which start centered at $(0,1) = i$ near $r = 0$, and expand to have infinite radius as $r \to 1$ and become tangent to the real axis at the origin.

(b) Here we fix the argument θ and vary r. We will obtain circles with the equation

$$\left(x - \frac{1}{2}\left(\tan\frac{\theta}{2} - \cot\frac{\theta}{2}\right)\right)^2 + y^2 = \left(\frac{1}{2}\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\right)^2,$$

which intersect the points $(0, \pm 1) = \pm i$, and expand to infinite radius as $\theta \to 0, \pi$. At $\theta = \frac{\pi}{2}$, the circle is centered at the origin with radius 1.

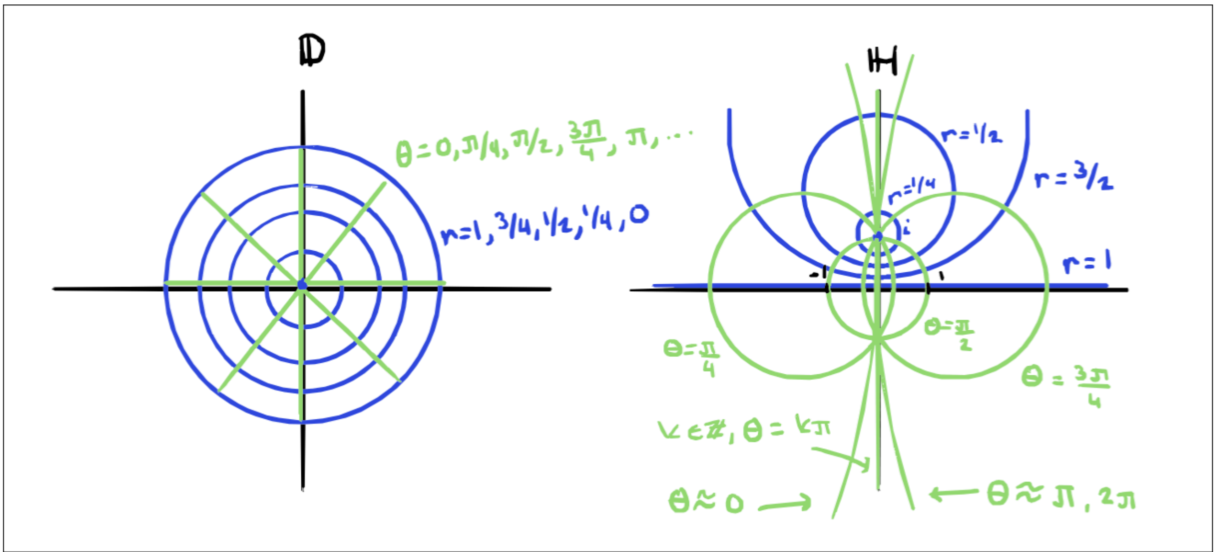

Figure 1: Sketch of curves in $\mathbb{D}$ and $\mathbb{H}$; (a) is blue and (b) is green.