Math 462 Homework 7

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Problem 1. For each of the following graphs, determine the vertex-connectivity and edge-connectivity of the graph.

(a) Let n be an odd positive integer. The directed graph with vertices $\{0, 1, ..., n-1\}$, with a directed edge ij if $j - i \bmod n \in \{1, 2, ..., (n-1)/2\}$.

(b) Let n be an odd positive integer. The directed graph with vertices $\{0, 1, ..., n-1\}$, with
- a directed edge ij if $1 \leq i < j \leq n-1$,
- a directed edge $0i$ if i is odd, and
- a directed edge $i0$ if i is even.

Proof of (a). We will first determine the vertex connectivity κ and edge connectivity κ' of the underlying graph G.

We have edges between vertices if there is a directed edge ij or ji in the digraph, described by the following two cases:
- ij exists if $j - i \equiv k \pmod n$ where $k \in \{1, 2, ..., (n-1)/2\}$, and
- ji exists if $i - j \equiv \ell \pmod n$ where $\ell \in \{1, 2, ..., (n-1)/2\}$.

Note that the second condition in equivalent to $-(j-i) \equiv -k \equiv n - k \pmod n$.

Since $k \in \{1, 2, ..., (n-1)/2\}$, then $n - k \in \{n-1, n-2, ..., (n-1)/2\}$.

Since n is odd, then $\{1, 2, ..., (n-1)/2\} \cup \{n-1, n-2, ..., (n-1)/2\} = \{1, 2, ..., n-1\}$.

Therefore, vertices i and j are connected iff $j - i \not\equiv 0 \pmod n \implies i \neq j$, which means that the underlying graph is the complete graph $K_n = G$.

Thus, we have $\kappa(G) = \kappa'(G) = n - 1$.

Now, we will find the strong vertex and edge connectivity of the digraph. Since each vertex has an in-degree and an out-degree of exactly $(n-1)/2$, then we can disconnect a vertex by removing $(n-1)/2$ incident edges or adjacent vertices.

So, we must have that $\kappa(G) \leq (n-1)/2$ and $\kappa'(G) \leq (n-1)/2$.

From the directed version of Menger's theorems, we have that strong connectivity equals the maximum k such that every ordered pair of vertices (x, y) has an xy-connector of size k, and similarly for strong edge connectivity with an xy-edge-connector of size k.

So, it suffices to show that there are xy vertex and edge connectors each of size k.

We will construct connectors which are simultaneously edge and vertex connectors of size $(n-1)/2$.

Since the digraph has rotational symmetry, we will fix the first vertex at 0 and consider the pair $(0, v)$ for some vertex $v \in \{1, 2, ..., n-1\}$.

We will construct paths to any vertex v backwards to 0.

From 0, we have directed edges to all vertices in $S = \left\{1, 2, ..., \frac{n-1}{2}\right\}$.

Begin with the $\frac{n-1}{2}$ inward edges to v, which are incident to the $\frac{n-1}{2}$ vertices $v-1, v-2, ..., v-\frac{n-1}{2}$ all $\operatorname{mod} n$. If any of these vertices are in S, we have a unique path to zero, and hence we are done.

Otherwise, we can connect the vertices adjacent to 0, $v-i$ for $i \in \left[1, \frac{n-1}{2}\right]$, to the next unconnected vertex $u = v - \frac{n-1}{2} - i \operatorname{mod} n$ through an inward edge. Since $v \leq n-1$, then u must belong to S because $u \leq n-1-\frac{n-1}{2}-i \leq \frac{n-3}{2} \in S$.

Since each of these $\frac{n-1}{2}$ paths go through unique edges and vertices, then they are $\frac{n-1}{2}$ strong edge and vertex connectors.

Thus the digraph is $\frac{n-1}{2}$ strong vertex and edge connected. □

Proof of (b). Let G be the underlying graph of the digraph. We will first determine $\kappa(G)$ and $\kappa'(G)$.

As in part (a), the underlying graph is the complete graph K_n, so $\kappa(G) = \kappa'(G) = n-1$.

Now, we will determine the strong vertex and edge connectivity of the graph.

For strong vertex connectivity, If we delete vertex 1, then vertex 0 cannot reach vertex 2, so the digraph is 1 strong vertex connected.

For strong edge connectivity, if we delete the directed edge 01, then the vertex 1 becomes disconnected because 01 is the only the inward directed edge to 1. So the digraph is 1 strongly edge connected. □

Problem 2. Given a simple graph G, let $\overline{G}$ denote the *complement* of G. That is, $\overline{G}$ is a simple graph with the same vertex set as G, and for any two distinct vertices x and y, x and y are adjacent in G iff they are not adjacent in G.

(a) Let G be a simple graph with n vertices. Prove that $\chi(G) + \chi\left(\overline{G}\right) \leq n+1$. Hint: Use induction.

(b) Use part (a) to prove that $\chi(G)\chi\left(\overline{G}\right) \leq (n+1)^2/4$.

(c) Conclude that $\chi(G) \leq (n+1)^2/(4\alpha(G))$, where $\alpha(G)$ is the size of the largest independent set of G.

Proof of (a). Induction on the number of vertices n.

For the base case when $n = 1$, both G and $\overline{G}$ contain one vertex, so they each have a chromatic number of 1. Hence, $\chi(G) + \chi\left(\overline{G}\right) = 1 + 1 \leq 1 + 1$, as desired.

Assume the result for $n-1$ vertices,

$$\chi(G) + \chi\left(\overline{G}\right) \leq (n-1) + 1 = n,$$

and consider G with n vertices.

Choose any vertex $v \in V(G)$, and let $G' = G - v$, $\overline{G'} = \overline{G} - v$. By the inductive hypothesis,

$$\chi(G') + \chi\left(\overline{G'}\right) \leq n.$$

Now, adding back the vertex v can increase the chromatic number by at most one in either G' or $\overline{G'}$ but not both.

If v is connected to all color classes in G', then it needs a new color in G. But, this means that v could not have been connected to all color classes in $\overline{G'}$.

Similarly, the argument holds for v connected to all color classes in $\overline{G'}$.

So, either $\chi(G) \leq \chi(G')$ or $\chi\left(\overline{G}\right) \leq \chi\left(\overline{G'}\right)$.

So, we must have that

$$\chi(G) + \chi(G') \leq \chi(G') + \chi\left(\overline{G'}\right) + 1 \leq n + 1,$$

which is what we wanted to show. □

Proof of (b). From part (a), we have

$$\chi(G) + \chi(G') + 1 \leq n + 1,$$

Then, with the AM-GM inequality, we have

$$\chi(G)\chi(G') \leq \left(\frac{\chi(G) + \chi(G')}{2}\right)^2 \leq \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)^2}{4}.$$

□

Proof of (c). Since any proper coloring of $\overline{G}$ requires at least $\omega\left(\overline{G}\right)$ colors, then $\chi\left(\overline{G}\right) \geq \omega\left(\overline{G}\right) = \alpha(G)$.

Combining the above with part (b) yields the desired result. □