Math 336 Homework 6

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Problem 1. Prove that $f \in \mathcal{F}_a$ and $0 \leq b < a$ implies $\forall n \in \mathbb{N},\ f^{(n)} \in \mathcal{F}_b$.

Proof. We wish to show that for $f^{(n)} \in \mathcal{F}_b$,

$$\exists B : \left|f^{(n)}(x+iy)\right| \leq \frac{B}{1+x^2} \quad \text{and} \quad f^{(n)}(z) \text{ is holorphic for all } \ z : |\mathrm{Im}(z)| < b < a.$$

First, since f is holomorphic on $|\mathrm{Im}(z)| < a$ and $b < a$, then $f^{(n)}$ is holomorphic on $|\mathrm{Im}(z)| < b$.

Next, using the Cauchy inequality and considering disks of radius a centered on the real axis, we have that where $z = x + iy$,

$$\forall x \in \mathbb{R},\ \left|f^{(n)}(x)\right| \leq \frac{n!}{a^n} \sup_{z \in C_{R(x)}} |f(z)| \leq \frac{n!A}{a^n(1+x^2)}.$$

Let $B = \frac{An!}{a^n}$, which is constant for all a and n fixed. Hence $\left|f^{(n)}(x+iy)\right| \leq \frac{B}{1+x^2}$.

Thus, $f^{(n)} \in \mathcal{F}_b$ as desired. □

Problem 2.

(a) Show that for $a > 0$ and $\xi \in \mathbb{R}$,

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{a}{a^2+x^2} e^{-2\pi i x\xi}\, dx = e^{-2\pi a|\xi|}.$$

(a) Check

$$\int_{\mathbb{R}} e^{-2\pi a|\xi|} e^{2\pi x\xi}\, d\xi = \frac{1}{\pi} \frac{a}{a^2+x^2}.$$

Proof of (a). Let $f(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$. Let $g(x) = f(x)e^{-2\pi x\xi}$.

WLOG let $\xi > 0$. Otherwise, use $x \mapsto -x$ where $f(x) = f(-x)$ with $\xi \mapsto -\xi$ to yield,

$$g(x) = f(x)e^{-2\pi i x\xi} = f(-x)e^{-2\pi i(-x)(-\xi)}.$$

Now, we will construct a semicircular contour γ in the negative imaginary half-plane which picks up a residue from the simple pole of f at $-ia$.

On the semicircular curve C_R^- of the contour, we have

$$\left|\int_{C_R^-} g(z)\, dz\right| \leq \int_{C_R^-} |f(z)|\left|e^{-2\pi i z\xi}\right| dz.$$

Write $z = x - iy$. Since we are in the negative imaginary half-plane $y > 0$, then

$$\left|e^{-2\pi i(x-iy)\xi}\right| = e^{-2\pi y\xi} \leq 1,$$

with equality when $\operatorname{Im} z = -y = 0$.

But, on $|z| = R$, we have $f(z) = O(R^{-2})$, which dominates over the length of the integration path , so the integral over C_R^- vanishes as $R \to \infty$.

So, we recover

$$\int_{\mathbb{R}} g(x)\,dx = -\lim_{R\to\infty} \int_L g(z)\,dz,$$

where the negation is obtained because the positively oriented contour traverses the real axis the in the opposing direction.

Hence we have,

$$\oint g\,dz = -\int_{\mathbb{R}} g\,dz = 2\pi i \operatorname{res}_{-ia} d.$$

Then,

$$\begin{aligned}
\operatorname{res}_{-ia} g &= \lim_{z\to -ia} (z+ia)\frac{1}{\pi}\frac{a}{a^2+x^2}e^{-2\pi i z\xi} \\
&= \lim_{z\to -ia} \frac{1}{\pi}\frac{a}{z-ia}e^{-2\pi i z\xi} \\
&= \frac{1}{\pi}\frac{a}{-2ia}e^{-2\pi i(-ia)\xi} \\
&= -\frac{1}{2\pi i}e^{-2\pi a\xi}.
\end{aligned}$$

So,

$$-\int_{\mathbb{R}} g\,dz = -\frac{2\pi i}{2\pi i}e^{-2\pi a\xi} = -e^{-2\pi a\xi}.$$

But, the above holds for both ξ and $-\xi$, so

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{a}{a^2+x^2}e^{-2\pi i x\xi}\,dx = e^{-2\pi a|\xi|},$$

as desired. □

Proof of (b). We will split the integral into two parts to separate $|\xi|$,

$$\int_{-\infty}^0 e^{2\pi a\xi}e^{2\pi i x\xi}\,d\xi + \int_0^\infty e^{-2\pi a\xi}e^{2\pi i x\xi}\,d\xi.$$

For the first integral, let $u = -\xi$. Then,

$$\int_0^\infty e^{-2\pi u(a+ix)}\,du = \frac{1}{2\pi(a+ix)}.$$

Similarly, for the second integral,

$$\int_0^\infty e^{-2\pi\xi(a-ix)}\,d\xi = \frac{1}{2\pi(a-ix)}.$$

Recombining the two integrals,

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|}e^{2\pi ix\xi}\,d\xi = \frac{1}{2\pi}\left(\frac{1}{a-ix}+\frac{1}{a+ix}\right) = \frac{1}{\pi}\frac{a}{a^2+x^2},$$

which is what we wanted to show. □

Problem 3. Let Q be a polynomial of degree at least 2 with distinct roots, not of which on the real axis. Calculate

$$\int_{\mathbb{R}} \frac{e^{-2\pi ix\xi}}{Q(x)}\,dx, \quad \xi \in \mathbb{R}$$

in terms of the roots of Q. What happens when several roots coincide?

Proof. Let z_i be the roots of Q. For now, assume that these are only simple zeros.

We will consider the cases when $\xi > 0$, $\xi < 0$, and $\xi = 0$.

First, for $\xi > 0$, the exponential term $|e^{-2\pi zi\xi}| = e^{2\pi(\mathrm{Im}\ z)\xi}$ decays in the lower half-plane, so we can close a semicircular contour there.

Since a positively oriented semicircular contour traverses the real axis in the negative direction, which is backwards of our integral, then, by the Residue theorem, we have

$$I = \int_{\mathbb{R}} \frac{e^{-2\pi ix\xi}}{Q(x)}\,dx = -2\pi i \sum_{z_i:\ \mathrm{Im}\ z_i<0} \mathrm{res}_{z_i} \frac{e^{-2\pi iz\xi}}{Q(z)}.$$

Since z_i are all simple roots, then

$$\mathrm{res}_{z_i} \frac{e^{-2\pi iz\xi}}{Q(z)} = \frac{e^{-2\pi iz_i\xi}}{Q(z_i)}.$$

Thus,

$$I = -2\pi i \sum_{z_i:\ \mathrm{Im}\ z_i<0} \frac{e^{-2\pi iz_i\xi}}{Q(z_i)}.$$

Second, for $\xi < 0$, the exponential term decays in the upper half-plane. The corresponding contour traverses in the real line in the same direction as our integral. Hence, by the Residue theorem and as is the first case, we have

$$I = 2\pi i \sum_{z_i:\ \mathrm{Im}\ z_i>0} \frac{e^{-2\pi iz_i\xi}}{Q(z_i)}.$$

Third, when $\xi = 0$, the integral becomes $\int_{\mathbb{R}} \frac{dx}{Q(x)}$, which vanishes at a sufficient distance from the origin for all complex arguments.

We will choose the semicircular contour contour in the upper half-plane. Since $\deg Q \geq 2$, then $1/Q(z) = O(z^{-2})$, which will vanish for $|z|$ sufficiently large. For $|z| = R$,

$$\left| \int_{C_R^+} \frac{dz}{Q(z)} \right| \le \pi R \frac{1}{R^2} \to 0 \quad \text{as} \quad R \to \infty.$$

Thus, as before,

$$\int_{\mathbb{R}} \frac{dx}{Q(x)} = 2\pi i \sum_{z_i \colon \operatorname{Im} z_i > 0} \frac{1}{Q(z_i)}.$$

Now, when roots of Q coincide, we have for a root z_j with multiplicity $m_j > 1$,

$$Q(z) = \left(z - z_j\right)^{m_j} \widetilde{Q}(z)(z) \quad \text{where} \quad \widetilde{Q}(z)\left(z_j\right) \neq 0.$$

The residue at z_j is given by

$$\operatorname{res}_{z_j} \frac{e^{-2\pi i z \xi}}{Q(z)} = \frac{1}{\left(m_j - 1\right)!} \lim_{z \to z_j} \left(\frac{d}{dz}\right)^{m_j - 1} \frac{e^{-2\pi i z \xi}}{\widetilde{Q}(z)}.$$

$\square$

Problem 4. Solve the differential equation

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) = f(t),$$

where $a_k \in \mathbb{C}$ and $f \in C^2$ has bounded support.

(a) Let

$$\hat{f}(z) = \int_{\mathbb{R}} f(t) e^{-2\pi i z t} \, dt.$$

Note that $\hat{f}$ is entire. Show using IBP that, for fixed $0 \le a$, if $|y| \le a$, then

$$\left|\hat{f}(x + iy)\right| \le \frac{A}{1 + x^2}.$$

(a) Write

$$P(z) = \sum_{j=0}^{n} a_j (2\pi i z)^j.$$

Find $c \in \mathbb{R}$ such that P does not vanish on the line

$$L = \{z \mid z = x + ic, x \in \mathbb{R}\}.$$

(a) Set

$$u(t) = \int_L \frac{e^{2\pi i x z}}{P(z)} \hat{f}(z) \, dz.$$

Check that

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) = \int\limits_L e^{2\pi izt} \hat{f}(z)\, dz$$

and

$$\int_L e^{2\pi izt} \hat{f}(z)\, dz = \int\limits_{\mathbb{R}} e^{2\pi ixt} \hat{f}(x)\, dx.$$

Conclude by the Fourier inversion theorem that

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) = f(t).$$

Note that the solution u depends on the choice of c.

Proof of (a). Since f has bounded support, say on $[-M, M]$, then for all $|t| > M$, $f(t) = 0$ and $f'(t) = 0$.

So,

$$\int\limits_{\mathbb{R}} f(t)e^{-2\pi izt}\, dt = \int_{-M}^{M} f(t)e^{-2\pi izt}\, dt.$$

Since $f \in C^2$, then we can perform IBP twice. First, we have

$$\int\limits_{[-M,M]} f(t)e^{-2\pi izt}\, dt = f(t)\frac{e^{-2\pi izt}}{-2\pi iz}\bigg|_{-M}^{M} - \int_{-M}^{M} \frac{e^{-2\pi izt}}{-2\pi iz} f'(t)\, dt,$$

but the first term vanishes by the bounded support of f. Second, we have

$$\int_{-M}^{M} \frac{e^{2\pi izt}}{-2\pi iz} f'(t)\, dt = f'(t)\frac{e^{-2\pi izt}}{4\pi^2 z^2}\bigg|_{-M}^{M} - \frac{1}{4\pi^2 z^2}\int_{-M}^{M} e^{-2\pi izt} f''(t)\, dt,$$

but the first term vanishes by the bounded support of f for f'.

So, we have that

$$\left|\hat{f}(x+iy)\right| = \left|\frac{1}{4\pi^2(x+iy)^2}\right| \int_{-M}^{M} \left|e^{-2\pi i(x+iy)t}\right| \left|f''(t)\right| dt \leq \frac{1}{4\pi^2}\frac{1}{x^2+y^2} 2MBe^{2\pi aM}$$

where $f \in C^2$ implies that f'' is continuous and bounded by some B on $[-M, M]$ by EVT.

Hence, exchanging constants and using $|y| \leq a$, we have

$$\left|\hat{f}(x+iy)\right| \leq \frac{A}{1+x^2},$$

as desired. □

Proof of (b). Since P is a polynomial of degree n, then it has n roots by FTA. Let these roots be $z_j = x_j + iy_j$.

Then, we can choose any $c \neq y_i$ for any j. In particular, let $|c| > \max\limits_{1\leq j\leq n} |y_j|$.

Thus, $P(x+ic)$ will not vanish for all x in $\mathbb{R}$. $\square$

Proof of (c). We will first find the derivatives of u,

$$\begin{aligned}\left(\frac{d}{dt}\right)^j u(t) &= \left(\frac{d}{dt}\right)^j \int_L \frac{e^{2\pi i z t}}{P(z)}\hat{f}(z)\,dz \\ &= \int_L \left(\frac{d}{dt}\right)^j \frac{e^{2\pi i z t}}{P(z)}\hat{f}(z)\,dz \\ &= \int_L (2\pi i z)^j \frac{e^{2\pi i z t}}{P(z)}\hat{f}(z)\,dz,\end{aligned}$$

where the interchange of the derivative and the integral is justified with Leibniz rule given that the path L does not depend on t, and the integrand has continuous partials in both t and z.

So,

$$\begin{aligned}\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) &= \sum_{j=0}^{n} a_j \int_L (2\pi i z)^j \frac{e^{2\pi i z t}}{P(z)}\hat{f}(z)\,dz \\ &= \int_L \left(\sum_{j=0}^{n} a_j (2\pi i z)^j\right) \frac{e^{2\pi i z t}}{P(z)}\hat{f}(z)\,dz \\ &= \int_L e^{2\pi i z t}\hat{f}(z)\,dz.\end{aligned}$$

by considering the definition of P and where a finite sum can be interchanged without consequence.

Now, create a closed rectangular contour joining the real axis and R with sides at $\pm R$.

Since $\hat{f}(z)$ and $e^{2\pi i z t}$ are entire, then the integral over the closed loop contour will vanish.

Furthermore, at $x = \pm R$,

$$\left|\hat{f}(x+iy)\right| \leq \frac{A}{1+R^2} \to 0 \quad \text{as} \quad R \to \infty.$$

So, the integral will vanish on the vertical sides of the contour.

Therefore, we are left with

$$\int_L e^{2\pi i z t}\hat{f}(z)\,dz = \int_{\mathbb{R}} e^{2\pi i x t}\hat{f}(x)\,dx.$$

Thus, by the Fourier inversion theorem, we have

$$f(t) = \int_{\mathbb{R}} e^{2\pi i x t}\hat{f}(x)\,dx = \sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t),$$

which is what we wanted to show. $\square$