Homotopy Type Theory

identifications are paths

a lipson

Types vs. Sets

Sets

Sets contain elements.

Elements belong to sets via membership (∈).

Sets may mix different kinds of elements.

Sets are built from other sets.

Types vs. Sets

Sets	Types
Sets contain elements.	Types contain terms.
Elements belong to sets via membership (∈).	Terms inhabit types via typing (:).
Sets may mix different kinds of elements.	Terms have exactly one type.
Sets are built from other sets.	Terms are built with constructors.

Common Types

Name	Symbol	Terms
empty	$\mathbb{0}$	
unit	$\mathbb{1}$	$\star$
naturals	$\mathbb{N}$	$0, 1, 2, \ldots$

Common Types

Name	Symbol	Terms
empty	$\mathbb{0}$	
unit	$\mathbb{1}$	$\star$
naturals	$\mathbb{N}$	$0, 1, 2, \ldots$
identity	$x =_A y$	depends

Common Types

Name	Symbol	Terms
empty	$\mathbb{0}$	
unit	$\mathbb{1}$	$\star$
naturals	$\mathbb{N}$	$0, 1, 2, \ldots$
identity	$x \underset{A}{=} y$	depends

Identity types may have no terms, one term, or even many!

Curry Howard

$$\text{Types} \longleftrightarrow \text{Propositions}$$

$$\text{Terms} \longleftrightarrow \text{Proofs}$$

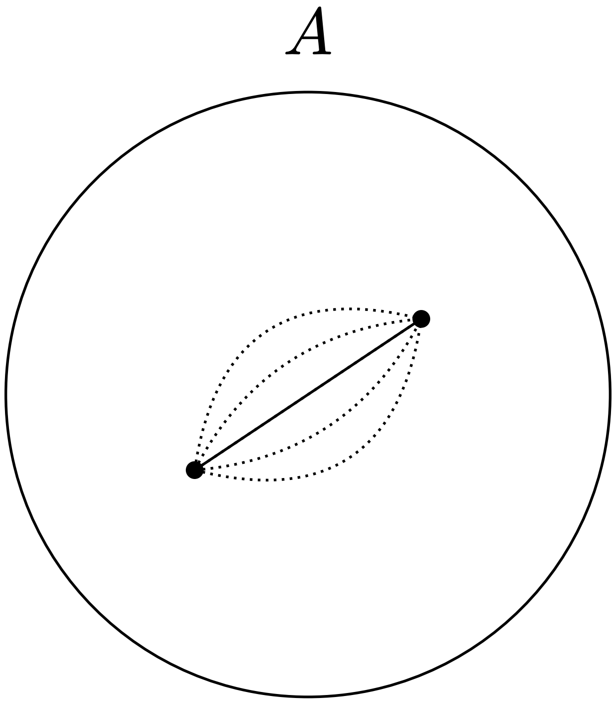
Curry Howard

$$\text{Types} \longleftrightarrow \text{Propositions}$$

$$\text{Terms} \longleftrightarrow \text{Proofs}$$

$\therefore$ Constructing a term of a type is the same as proving a proposition.

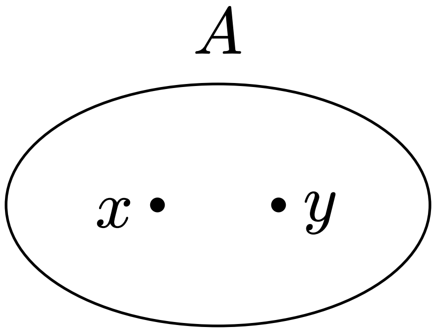
Homotopy



Homotopy Type Theory

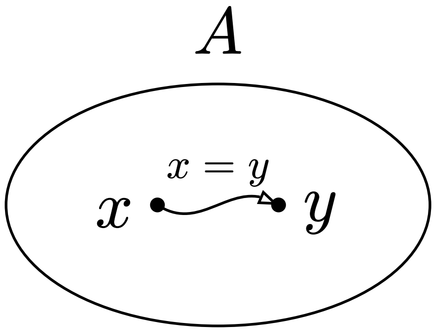
$$\text{Types} \sim \text{Spaces}$$

$$\text{Terms} \sim \text{Points in space}$$



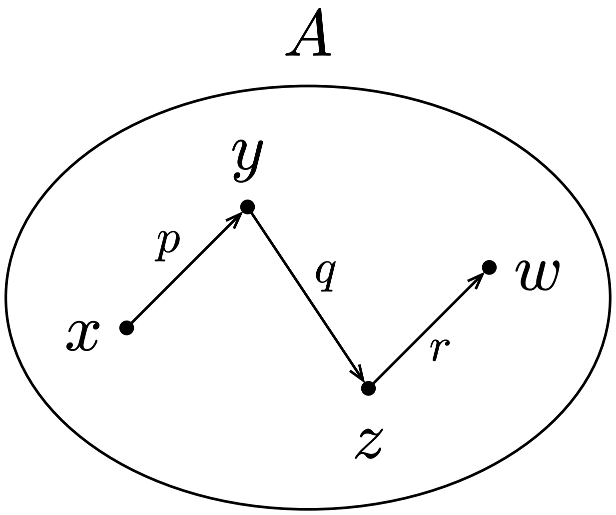
Homotopy Type Theory

$$\text{Types} \sim \text{Spaces}$$
$$\text{Terms} \sim \text{Points in space}$$



Terms of the identity type $x = y$ are paths from x to y.

Concatenate Paths

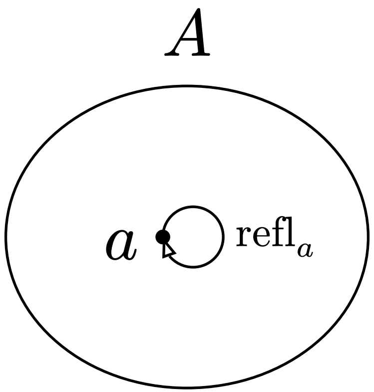


$$x \overset{p}{=\!=\!=} y \overset{q}{=\!=\!=} z \overset{r}{=\!=\!=} w$$

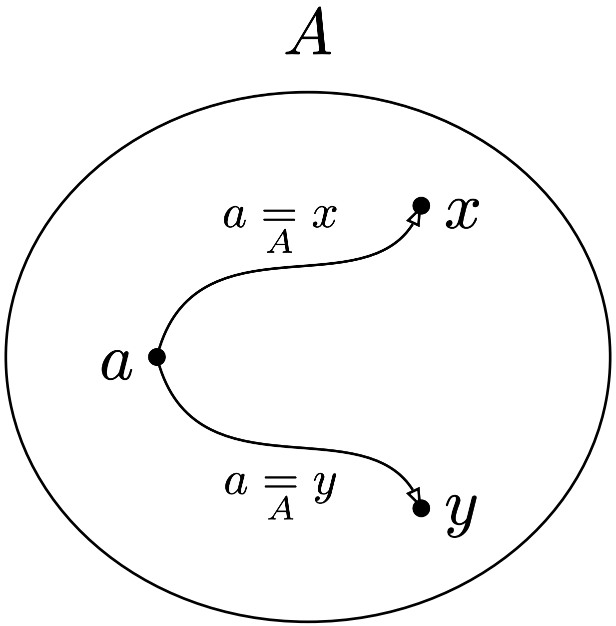
Identity Type

The identity type at a fixed point $a : A$ has one constructor:

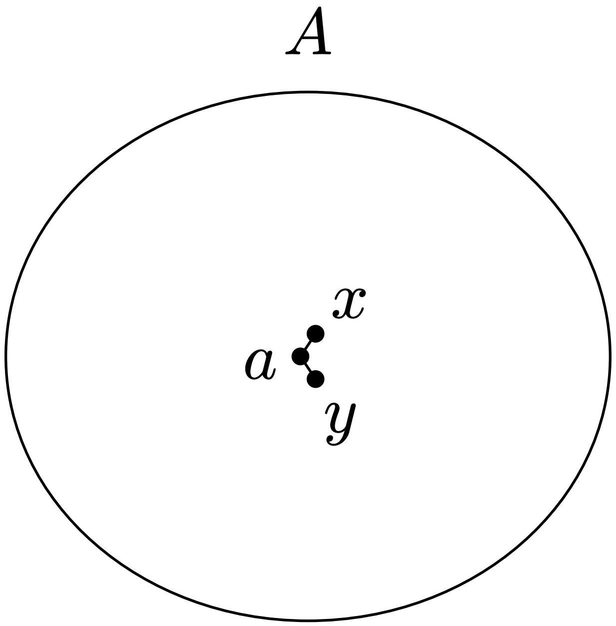
$$\mathrm{refl}_a : a \underset{A}{=} a.$$



Path Induction

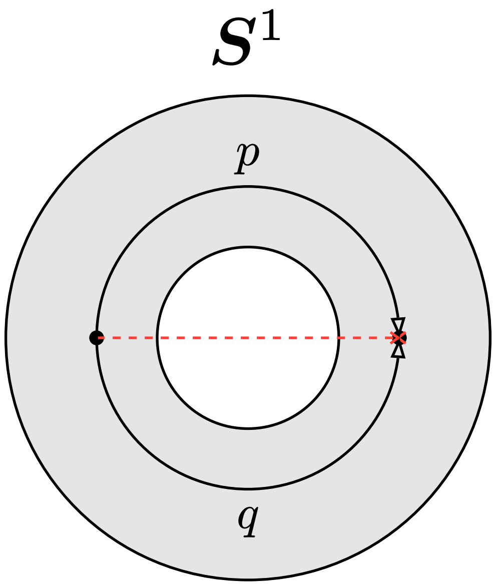


Path Induction



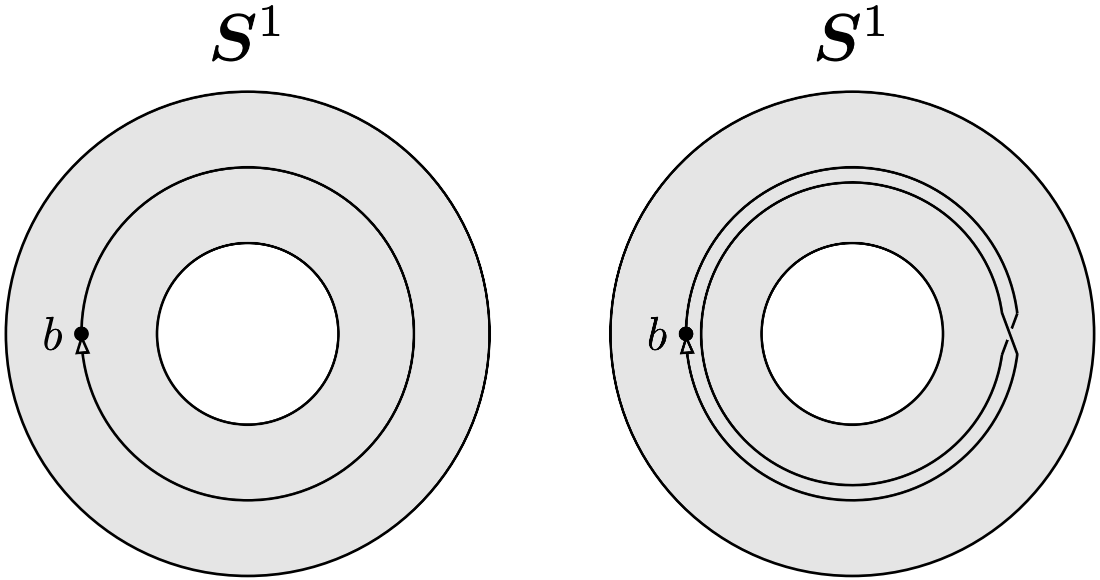
Any type that looks like this is contractible.

Circle Type



$$p \neq q$$

Circle Type



Distinct identifications between base b.

Truncation Levels

We separate types by their equalities:

−2. Contractible	$\mathbb{1}$	has identity type
−1. Propositions	$\mathrm{Eq}_{\mathbb{N}}$	
0. Sets	$\mathbb{N}$	
1. 1-types	$\boldsymbol{S}^1$	
$\vdots$		