Math 462 Homework 6

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Problem 1. Let $G = (V, E)$ be a bipartite graph with maximum degree d. Prove that G has a matching of size at least $|E|/d$.

Proof. Each vertex in any vertex cover can cover at most d edges due to the degree constraint, so we need at least $|E|/d$ vertices to cover all the edges.

Since G is bipartite, then by Kőnig, the size of a maximum matching equals the size of a minimum vertex cover.

Let C be a minimum vertex cover. We will show that $|C| \geq |E|/d$.

Since C is a cover, then every edge in E is incident to some vertex in C.

For each vertex in C, we can assign up to $\deg(v) \leq d$ edges from E.

So,

$$|E| \leq \sum_{v \in C} \deg(v) \leq \sum_{v \in C} d = |C|\ d.$$

Therefore the size of the minimum vertex cover is at least $|E|/d$. By Kőnig, this equals the size of the maximum matching in G.

Thus, there exists a matching of size at least $|E|/d$ in G. □

Problem 2. The *distance* $d(x, y)$ between two vertices x, y of a graph is the number of edges in the shortest path between the two vertices.

The *diameter* of a graph is the maximum of $d(x, y)$ over all pairs of vertices x, y.

Let $G = (V, E)$ be a graph with $\kappa(G) = k > 0$ and diameter D.

(a) Prove that $|V| \geq k(D - 1) + 2$.

(b) Prove that the largest independent set of G has size at least $\lceil (D + 1)/2 \rceil$.

(c) For each $k \geq 1$ and $D \geq 2$, construct a graph with connectivity k and diameter D for which equality holds simultaneously in (a) and (b).

Proof of (a). Since the diameter of G is D, then the distance between any two vertices is at most D. A path of distance D contains $D - 1$ internal vertices.

Since G is k-connected, then by Menger, there exists a k-connector between any two vertices of G.

Since a k-connector in G consists of k pairwise internally disjoint paths of distance at most D, then it contains $k(D - 1)$ internal vertices.

Now, adding the two endpoints of this k-connector, we must have $|V| \geq k(D + 1) + 2$, as desired. □

Proof of (b). Since the diameter is D, then there exists a path connecting $D+1$ vertices.

We can construct an independent set by taking every other vertex of this path.

If these vertices were adjacent, then we could have constructed a path between the endpoints of distance less than D, a contradiction. So, these vertices form an independent set.

If D is even, then we have $\frac{D}{2}+1$ independent vertices.
If D is odd, then we have $\frac{D+1}{2}$ independent vertices.

In both cases, we have $\lceil \frac{D+1}{2} \rceil$ vertices in the independent set. □

Proof of (c). To start, consider a graph constructed by k parallel internally disjoint paths:

$$v_{i,1}, v_{i,2}, ..., v_{i,D-1} \text{ where } i \in [1,k],$$

of length $D-1$ each joined at both ends by two vertices.

This graph has exactly $k(D-1)+2$ vertices.

However, for k connectivity, each vertex must have degree at least k.

So, for each j in $v_{i,j}$, we can connect the vertices across all $i \in [1,k]$ to construct a copies of the complete graph K_k.

Since we still have k pairwise internally disjoint paths, then by Menger, this graph is k-connected.

As before, we can take alternating vertices in any of the k-connector paths to construct an independent set of size $\lceil (D+1)/2 \rceil$. Each vertex $v_{i,j}$ is connected to all other vertices $v_{\tilde{\imath},j}$ at the same j level $\forall i, \tilde{\imath} \in [1,k]$, so we cannot obtain a larger independent set.

Therefore, a graph with $D-1$ copies of K_k all vertices connected in a line to their corresponding vertex in the adjacent copy with two additional vertices connected to the complete graph copies on ends will satisfy the requirements. □