336 Final Project Draft

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Contents

Abstract

In this project, we plan to introduce the curious prospective mathematician to some introductory ideas in combinatorics. In particular, we will discuss recurrence relations, generating functions, and integer partitions.

For our peers, we will expand on these ideas and wield some of our analysis tools developed in 336 to demonstrate a proof of the Hardy-Ramanujan estimation formula for partitions.

1. Showcase Activity

We will present some problems which the reader can begin to ponder. The solutions will also be provided, but the reader is encouraged to first turn to the Discussion Section 2 in order to obtain the tools necessary for talking the provided problems.

[*note that some of these are intended to be presented as physical artifacts with the showcase*]

1.1. Domino Tiling

We want to know how many ways we can tile a $2 \times n$ space with 1×2 tiles. We can use dominoes as our tiles.

[insert diagrams of both cases]

We see that there are two cases:

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-

1.2. Color Block Tiling

Let $h(n)$ be the number of ways to tile a $1 \times n$ space with
- 1×1 red and blue tiles; and
- 1×2 green, yellow, and black tiles.

Find the number of ways to tile the space with the given tiles.

Proof. We will use the sequence rule to find the generating function for $h(n)$.

Note that the empty space when $n = 0$ has one tiling where we use no tiles, i.e., $h(0) = 1$.

We can establish the following recurrence relation for $n \geq 2$,

$$h(n) = 2h(n-1) + 3h(n-2).$$

□

Perhaps the next natural curiosity might be to explore the combinatorial possibilities of Tetris games.

2. Discussion for Students

recurrence relations

2.1. Generating Functions

Definition 1. recurrence relation

Definition 2. generating function

$$\sum_{n=0}^{\infty} f(n) x^n$$

Theorem (*Sequence Rule*). Let A be a set with a weight function and no elements of weight 0. Let A^* be the set of all finite sequences of elements of A, including the empty sequence, where the weight of a sequence is given by the sum of the weights of its elements.

$$F_{A^*}(x) = \frac{1}{1 - F_A(x)}.$$

Proof. Every set in A^* belongs to exactly one of the following cases:
- it is empty, or
- its first element is in A, followed by an elements of A^*.

So, we can construct a weight-preserving bijection,

$$A^* \to \{(\,)\} \sqcup A \times A^*.$$

Hence, we have

$$\begin{aligned} F_{A^*}(x) &= 1 + F_A(x) F_{A^*}(x) \\ (1 - F_A(x)) F_{A^*}(x) &= 1 \\ F_{A^*}(x) &= \frac{1}{1 - F_A(x)}. \end{aligned}$$

Note that we can only divide formal power series with no constant term. Since we had that A had no elements with weight 0, then its power series □

Remark. If there were elements with weight zero, then we could create sequences with infinitely many zero-weighted elements.

2.2. Integer Partitions

Definition 3. partition

Definition 4. Young and Ferrers diagrams

3. Project for Peers

3.1. Additive Number Theory

Additive number theory is a branch concerned with the behavior of subsets of integers under addition. We have already seen an object of interest, integer partitions.

Consider k subsets of the nonnegative integers $\{A_i\}_{i=1}^{k}$ where $A_i \subset \mathbb{N}$. Then, we are interested in the number of solutions $r_k(n)$ to the following equation with $n \in \mathbb{N}$:

$$n = \sum_{i=1}^{k} a_i, \quad a_i \in A_i.$$

3.1.1. Additive Problem Examples

Now, we will see a few examples of additive problems.

Example 1. Weak[1] compositions of n into k parts with summands in A.[2]

Let $A \subset \mathbb{N}$. Note that $\forall a \in A,\ 1 \leq a \leq n$, i.e., we cannot have a part of the partition greater than the sum.

$$r_k(n) = \#\{(a_1, ..., a_k) \in A^k \mid n = a_1 + \cdots + a_k,\ a_i \in A\}.$$

When $A = \mathbb{N}$, using a "stars" and "bars" argument with n stars and $k - 1$ bars, we can show

$$r_k(n) = \binom{n + k - 1}{k - 1}.$$

[1]Strong compositions have all parts positive integers.

[2]Compositions are not the same as integer partitions; $r_k(n)$ counts ordered tuples, while partitions count unordered multisets.

Example 2. Goldbach's Conjecture, one of the oldest unsolved problems in number theory:[3] any even natural number greater than 2 can be written as the sum of two primes.

So, expressing this as in the above form, we write

$$r_2(n) = \#\{(p, q) \mid n = p + q, \quad p, q \text{ prime}\},$$

and the conjecture says that $\forall n > 2,\ r_2(n) \geq 1$.[4]

Example 3. Waring's problem:[5] Let $g(k)$ be the minimum number such that for all positive integers n, the equation

$$n = \sum_{i=1}^{g(k)} a_i^k, \quad a_i \in \mathbb{N}$$

has at least one solution, i.e., $r_{g(k)}(n) \geq 1$. Here, we are considering both exponentiation and addition.

Consider $7 = 1^2 + 1^2 + 1^2 + 2^2$. So, we have have that $g(2) \geq 4$. One can check that with 23 we have $g(3) \geq 9$ and with 79 we have $g(4) \geq 19$.

Remark. Lagrange's four-square theorem proves that exactly $g(2) = 4$.

3.2. Circle Method

Cited as the Hardy Ramanujan Littlewood Circle method technique in additive number theory Our goal is to transform additive and combinatorics problems into complex analysis problems to use the tools of analysis. [1] Indeed, quoting Hardy and Ramanujan's original paper, "This idea [studying integrals from generating functions] has dominated nine-tenths of modern research in analytic theory of numbers." [2]

The circle method is aptly named by using Cauchy's theorem for series coefficients, Theorem 4.4 in Stein and Shakarchi, to represent the coefficients of generating function series as integrals around closed circular paths which package the information in their residues.

3.2.1. Example Application to Weak Compositions

We will use the circle method to find the number of weak compositions of n into k parts.

Begin with the case when $k = 2$,

$$r_2(n) = \#\{(a_1, a_2) \mid n = a_1 + a_2,\ a_1, a_2 \in A\}.$$[6]

Construct the generating function using the indicator function for A:

$$f(z) = \sum_{n=0}^{\infty} \mathbf{1}_A(n) z^n, \qquad \text{where } \mathbf{1}_A(n) = \begin{cases} 1 & n \in A, \\ 0 & n \notin A. \end{cases}$$

Now, using the Cauchy series product, we have

[3] The conjecture has been shown to hold for all integers less than $4 \cdot 10^{18}$ as of 2025 according to Wikipedia.

[4] The Weak Goldbach Conjecture posits that every odd number greater than 5 can be expressed as the sum of three primes, where a prime may be used more than once in the same sum. This conjecture was proven using the circle method by Harald Helfgott in 2013.

[5] Waring's problem is related to Fermat's polygonal number theorem.

[6] The # notation returns the size of the given set.

$$f^2(z) = \left(\sum_{n=0}^{\infty} \mathbf{1}_A(n) z^n\right)\left(\sum_{m=0}^{\infty} \mathbf{1}_A(m) z^m\right) = \sum_{n=0}^{\infty} c(n) z^n$$

where $c(n) = \sum_{k=0}^{n} \mathbf{1}_A(k)\mathbf{1}_A(n-k)$, which we can rewrite as $\sum_{h+k=n} \mathbf{1}_A(h)\mathbf{1}_A(k)$.

Since $\mathbf{1}_A(h)\mathbf{1}_A(k) = 1$ iff both $h, k \in A$, then this expression of $c(n)$ is exactly the number of pairs of $(h, k) \in A^2$ which satisfy $h + k = n$.

Therefore, we have $f^2(x) = \sum_{n=0}^{\infty} r_2(n) z^n$.

Now, consider a composition of n into k parts with summands in A,

$$r_k(n) = \#\{(a_1, ..., a_k) \in A^k \mid n = a_1 + \cdots + a_k,\ a_i \in A\}.$$

After repeated applications of Cauchy product, we can arrive at

$$f^k(z) = \sum_{n=0}^{\infty} r_k(n) z^n.$$

Since f^k is holomorphic in z, then, with Cauchy's theorem for series coefficients, we have

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f^k(z)}{z^{n+1}}\, dz \tag{1}$$

for a circular closed loop C_ρ centered at the origin of radius ρ.

Now, we can express the number of solutions $r_k(n)$ in terms of the residues of this integral.

From Example 1 above with $A = \mathbb{N}$, we already know that $r_k(n) = \binom{n+k-1}{k-1}$.

Since $A = \mathbb{N}$, We have

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

which converges for $|z| < 1$. So, by equation (1), we have

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{dz}{(1-z)^k z^{n+1}},$$

where the integral converges on the closed circular loop C_ρ of radius $\rho < 1$.

We will extract the simple poles of the integral.

Using the general binomial expansion, we have

$$\frac{1}{(1-z)^k} = \sum_{m=0}^{\infty} \binom{-k}{m} (-z)^m.$$

Hence, the integral becomes

$$\oint_{C_\rho} \sum_{m=0}^{\infty} \binom{-k}{m} (-z)^m z^{-(n+1)}\, dz.$$

So, we will take the terms for which $m - n - 1 = -1 \implies m = n$.

Thus,

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \binom{-k}{n} (-1)^n z^{-1}\, dz = (-1)^n \binom{-k}{n} = \binom{n+k-1}{k-1},$$

where the last equality is given by a combinatorial identity.[7]

3.3. Hardy-Ramanujan Estimation Formula

Theorem (*Hardy-Ramanujan*). Let $p(n)$ be the number of partitions of n.

$$p(n) \sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4\sqrt{3}\, n}.$$

Proof of Hardy-Ramanujan. We have adapted a simplified version of the proof. [3]

$$f(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} \frac{1}{1 - z^m}, \quad |z| < 1.$$

$\square$

Proposition 1.

Bibliography

[1] "Introducing Hardy littlewood Ramanujan Circle Method." [Online]. Available: https://www.youtube.com/watch?v=QqjyAQNMZck

[2] G. H. Hardy and S. Ramanujan, "Asymptotic Formulæ in Combinatory Analysis," *Proceedings of the London Mathematical Society*, no. 1, pp. 75–115, 1918, doi: https://doi.org/10.1112/plms/s2-17.1.75.

[3] D. J. Newman, "A simplified proof of the partition formula.," *Michigan Mathematical Journal*, vol. 9, no. 3, pp. 283–287, 1962, doi: 10.1307/mmj/1028998729.

[4] A. Karatsuba, "Circle method," *Encyclopedia of Mathematics*, [Online]. Available: http://encyclopediaofmath.org/index.php?title=Circle_method&oldid=39763

[7] This identity can be quickly shown by expressing both binomial coefficients in terms of falling factorials (Pochhammer symbols) and counting the appearances of -1.