Math 462 Homework 4

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Problem 1. Let F be the generating function for the set of all partitions, weighted by sum. Prove that

$$F(x) = \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{x^{k^2}}{(1-x^j)^2}.$$

Proof. Consider a partition λ. Let k be the size of the largest square that fits in the upper left corner of the Young diagram of λ.

Then, λ has at least k parts of size at least k. Now, we can decompose the Young diagram into three sub pieces:

- the $k \times k$ square;
- a partition α to the right of the square, with at most k parts; and
- a partition β below the square, with parts at most k.

The $k \times k$ square has weight k^2, which contributes x^{k^2} to the generating function F.

Both α and β are in bijection with $P_{\leq k}$, which counts the number of partitions weighted by sum with at most k parts, and, by conjugation, also the number of partitions with parts at most k.

We have that the generation function for $P_{\leq k}$ is $\prod_{j=1}^{k} \frac{1}{1-x^j}$. So, this gives the generating function for the partitions α and β as well.

Combining these and summing over all possible k yields

$$\sum_{k=0}^{\infty} x^{k^2} \left(\prod_{j=1}^{k} \frac{1}{1-x^j} \right)^2 = \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{x^{k^2}}{(1-x^j)^2},$$

as desired. □

Problem 2. Let $p(n)$ denote the number of partitions of n. Let $p_{\leq k}(n)$ denote the number of partitions of n where all parts have size at most k.

(a) We have that $p_{\leq k}(n) \sim Cn^{k-1}$. Find the constant C.

(b) Without citing an asymptotic formula for $p(n)$, prove that $p(n)$ grows faster than any polynomial function; i.e., prove that for any positive integer d,

$$\lim_{n \to \infty} \frac{p(n)}{n^d} = \infty.$$

Proposition 1. $1 - x^j \approx j(1-x)$ near $x = 1$.

Proof of Proposition 1. Consider the Taylor expansion of $f(x) = 1 - x^j$ centered at 1,

$$\begin{aligned}1 - x^j &= 1 - \sum_{n=0}^{\infty} f^{(n)} \frac{1}{n!}(x-1)^n \\ &= 1 - (f(1) + f'(1)(x-1) + \text{higher order terms in } (x-1)) \\ &\approx j(1-x),\end{aligned}$$

where the higher order terms will go vanish faster as $x \to 1$. $\square$

Proof of (a). We have that the generating function of $p_{\leq k}(n)$ is given by $F(x) = \prod_{j=1}^{k} \frac{1}{1-x^j}$, where 1 is a root of multiplicity k in the denominator. All other roots of the denominator have magnitude 1, with multiplicity less that k.

So, by Homework 2, we have

$$p_{\leq k}(n) \sim \frac{c}{(k-1)!} \frac{n^{k-1}}{1^n} = c\frac{n^{k-1}}{(k-1)!},$$

where

$$c = \lim_{x \to 1} (1-x)^k F(x) = \lim_{x \to 1} \prod_{j=1}^{k} \frac{(1-x)^k}{1-x^j} \approx \prod_{j=1}^{k} \frac{(1-x)^k}{j(1-x)^k} = \frac{1}{k!},$$

where the approximation is given by Proposition 1.

Thus, $C = \frac{c}{(k-1)!} = \frac{1}{k!(k-1)!}$. $\square$

Proof of (b). From part (a), we have that $p_{\leq k} \sim Cn^{k-1}$, so for any fixed d, set $k - 1 = d + 1$ such that

$$p_{\leq k} \frac{n}{n^d} \sim \frac{cd^{d+1}}{n^d} \to \infty.$$

But $p(n) \geq p_{\leq k}(n)$ for any k, because $p(n)$ contains all partitions, including those with above k parts.

Thus,

$$\lim_{n \to \infty} \frac{p(n)}{n^d} \geq \lim_{n \to \infty} \frac{p_{\leq k}(n)}{n^d} = \infty.$$

$\square$