336 Final Project Draft

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[TODO: make small subsections not appear in TOC]

Contents

Abstract

In this project, we plan to introduce the curious prospective mathematician to some introductory ideas in combinatorics. In particular, we will discuss recurrence relations, generating functions, and integer partitions.

For our peers, we will expand on these ideas and wield some of our analysis tools developed in 336 to demonstrate a proof of the Hardy-Ramanujan estimation formula for partitions.

1. Showcase Activity

We will present some problems with which the reader can begin to ponder. The solutions will also be provided, but the reader is encouraged to first turn to the Discussion Section 2 in order to obtain the tools necessary for tackling the provided problems.

[NOTE: these are intended to be presented as physical artifacts with the showcase; we wish to help students build some intuition behind recurrence relations and recursion.]

1.1. Domino Tiling

How many ways we can tile a $2 \times n$ space with 1×2 tiles.

Materials & Setup: We can use dominoes as our tiles.

[insert diagrams of both cases]

We see that there are two cases:

-
-

Guiding Questions

Sketch a table of the space size n and the corresponding number of tilings which fill that space. Do you recognize this sequence?[1]

Can you write down a relationship that describes how to get the next number of tilings in the sequence?

Extension Questions

What are some other ways that we can build this sequence?

Proposition 1. The number of tiles follows the Fibonacci sequence.

[NOTE: we might want to remove this example because that will allow us to get rid of the sequence rule part; i think this example was mostly fun because we would get to play with colored tiles.]

1.2. Color Block Tiling

Let $h(n)$ be the number of ways to tile a $1 \times n$ space with

- 1×1 red and blue tiles; and
- 1×2 green, yellow, and black tiles.

Find the number of ways to tile the space with the given tiles.

Proof. We will use the sequence rule to find the generating function for $h(n)$.

Note that the empty space when $n = 0$ has one tiling where we use no tiles, i.e., $h(0) = 1$.

We can establish the following recurrence relation for $n \geq 2$,

$$h(n) = 2h(n-1) + 3h(n-2).$$

□

Perhaps natural curiosity might lead us to next explore the combinatorial possibilities of a Tetris game.

How can we work with more complex shapes and arrangements?

[1] If you are ever working with a sequence in the wild, try seeing if it is documented in The On-Line Encyclopedia of Integer Sequences (OEIS). https://oeis.org/

[NOTE: the next examples are exploring the Catalan sequence]

1.3. Parenthesis Puzzle

How many ways are there to arrange sequences of nested and matched parentheses?[2]

Materials & Setup: Note cards with left and right parenthesis. Or, some 3D-printed plastic parentheses that clip together.

[TODO: expand + provide examples]

1.4. Binary Trees

How many binary trees are there with n vertices?

Materials & Setup:

[TODO: define rooted binary trees]

1.5. Polygon Triangulation

How many ways are there to cut a polygon along its diagonals into triangles.

Materials & Setup: Wooden block some with pegs as polygon vertices. Stretch rubber bands between pegs to create triangulations.

Guiding Questions

How many triangles are in the triangulation of an polygon with n sides (n-gon)?

Proposition 2. The number of ways to triangulate an $(n+2)$-sided polygon is the n-th Catalan number.

Proof. Let T_n be the number of triangulations of an $n+2$-gon.

Each triangulation is in exactly one of the following cases:
- It is empty.
- The triangulation contains an outside edge of the polygon, and has two triangulations to the left and right.

Hence, there exists a weight-preserving bijection:

$$T \to \{\emptyset\} \sqcup \{\triangle\} \times T \times T.$$

Therefore

$$F_T(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n.$$

Thus, $|T_n| = c_n$, the n-th Catalan number. □

Remark. We can also construct a proof with a bijection to binary trees.

[2] These are known as Dyck words.

Extension Questions

Can you connect this problem to the previous one? Hint: what happens if we draw points (vertices of a tree) on each of the triangles in the triangulation. Can we connect these points to form a binary tree? Can we do this for any triangulation?

2. Discussion for Students

We will take a first leap into the realm of enumerative combinatorics. Enumeration means finding the size of finite sets. In particular, we want to know how the size of sets change according to certain parameters.

Let A_n be a set parametrized by some natural number n. We write $f(n) = |A_n|$, assigning some function f which capture the size of the set.

Now, we can understand how the size of the set changes by understanding the behavior of the function f.

Note we have defined f to agree with the size of A_n only for integer values of n. What values might f take on for non-integer arguments?

There are several main cases for how we work with f:

- We have an explicit formula for $f(n)$.
- We have an approximation or asymptotic estimate for $f(n)$ as $n \to \infty$.
- We compare f to another function which we understand.
- We build a recurrence relation on the output of f.

[TODO: ensure intro covers all topics which we decide to retain.]

2.1. Asymptotic Behavior & Approximation

Asymptotic analysis allows us to simplify complex problems by focusing on how functions behave toward infinity.

This is a form of abstraction[3] where we strip away less significant terms to reveal the behavior of the function.

Oftentimes, we make comparisons with benchmark functions, such as logarithmic, linear, quadratic, and exponential.

We write $f \sim g$ when $\lim_{x \to \infty} f(x)/g(x) = 1$.

We use big and little "O" notations to relate the behavior of a given function f to these known functions.

2.1.1. Stirling's Formula

Theorem (*Stirling's Formula*).

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Proposition 3. Factorial grows faster than any exponential function.

[3]Abstraction is one of our most powerful tools in mathematics.

Proof of Proposition 2. Let ab^n be an exponential function.

$$\lim_{n\to\infty}\frac{n!}{ab^n}=\lim_{n\to\infty}\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{ab^n}=\lim_{n\to\infty}\frac{\sqrt{2\pi n}}{a}\left(\frac{n}{be}\right)^n=\infty.$$

Since this limit diverges, then $n!$ grows faster than any ab^n. $\square$

[TODO: populate the definitions of the following sections.]

2.2. Generating Functions

Definition 1. recurrence relations

[NOTE: it would be beneficial to cut down on extra topics, so maybe we won't retain the following definition.]

Definition 2. homogeneous linear recurrence characteristic polynomial

Definition 3. Suppose $f:\mathbb{N}\to\mathbb{C}$. The generating function of f is defined as follows:

$$F(x)=\sum_{n=0}^{\infty}f(n)x^n=\underbrace{f(0)+f(1)x+f(2)x^2+\cdots}_{\text{contains all information from } f}$$

Generating functions are objects known as formal power series, they are a formal sum–which means we are not actually performing any addition–that does not have to converge for all x.[4]

We use generating functions to encode combinatorial rules as algebraic relations.

If $\{A_n\}$ is a sequence of finite sets, then we can define a generating function on the size of these sets

$$F_A(x)=\sum_{n=0}^{\infty}|A_n|x^n.$$

We can view the size of a set as a function that takes a set and returns the number of elements inside of the set. More generally, we can consider a weight function $w:A\to\mathbb{N}$ that takes a set and produces a given value n for each element in A.

With a set A, possibly infinite, and a weight function w on A, we can construct the generating function

$$F_A(x)=\sum_{a\in A}x^{w(a)}.$$

Example 1. Let A_n be the set of all binary strings of length n where A_0 is the empty string. Then $|A_n|=2^n$ gives us that

$$F_A(x)=\sum_{n=0}^{\infty}(2x)^n=\frac{1}{1-2x}$$

using the geometric series $\sum x^n=\frac{1}{1-x}$.

[4]Generating functions are neither generating, nor functions.

[TODO: answer the following questions; they are prompts for writing.]

What are generating functions used for? What can we do with them?

How can we produce generating functions?

2.2.1. Constructing Generating Functions from Recurrence Relations

[NOTE: do we wish to remain this level of detail? i think it might be excessive; the sequence rule is put in place for the color block tiling problem.]

Theorem (*Sequence Rule*). Let A be a set with a weight function and no elements of weight 0. Let A^* be the set of all finite sequences of elements of A, including the empty sequence, where the weight of a sequence is given by the sum of the weights of its elements.

$$F_{A^*}(x) = \frac{1}{1 - F_A(x)}.$$

Proof. Every set in A^* belongs to exactly one of the following cases:
- it is empty, or
- its first element is in A, followed by an elements of A^*.

So, we can construct a weight-preserving bijection,

$$A^* \to \{(\,)\} \sqcup A \times A^*.$$

Hence, we have

$$\begin{aligned} F_{A^*}(x) &= 1 + F_A(x)F_{A^*}(x) \\ (1 - F_A(x))F_{A^*}(x) &= 1 \\ F_{A^*}(x) &= \frac{1}{1 - F_A(x)}. \end{aligned}$$

Note that we can only divide formal power series with no constant term. Since we had that A had no elements with weight 0, then its power series □

Remark. If there were elements with weight zero, then we could create sequences with infinitely many zero-weighted elements.

2.3. Some Sequences

Definition 4. Fibonacci sequence
- recurrence relation $f(n+2) = f(n+1) + f(n)$.
- generating function $\frac{1}{1-x-x^2}$.
- closed form
- asymptotics

[TODO: expand upon the Catalan numbers.]

Definition 5. Catalan sequence,
- recurrence relation
- generating function
- closed form $\frac{1}{n+1}\binom{2n}{n}$.

- asymptotics

2.4. Integer Partitions

[TODO: expand the section on integer partitions, and perhaps give some applications of the object.]

Definition 6. For $n \in \mathbb{N}$, a partition of n is a way to write n as the sum of positive integers where the order of summation does not matter.[5]

Definition 7. We can represent partitions with Young and Ferrers diagrams.

Theorem. Let $P_{\leq k}$ be the set of all partitions with all parts at most k, weighted by sum.

$$F_{P_{\leq k}}(x) = \prod_{j=1}^{k} \frac{1}{1-x^j}.$$

Proposition 4. The number of partitions of n with at most k parts equals the number of partitions with largest part at most k.

Proof of Proposition 1. Consider the Ferrers diagram of the partition.

If a partition has k parts, then there are k rows in its diagram.

If the largest part of a partitions is k, then there are k columns in its partition.

Transpose the Ferrers diagram by flipping it across its central diagonal, swapping the number of rows and columns while maintaining the number of dots.

This flipped partition is known as the conjugate partition.

Thus, we have a bijection between the set of partitions with k parts and the set of partitions with largest part k. □

If we let the maximum size of each part k exceed any number, i.e., $k \to \infty$, then we obtain the following theorem.

Theorem. Let P be the set of all partitions weighted by sum.

$$F_P(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$ [6]

[5] If the order of summation matters, then we have a strong composition.

[6] Be careful with infinite products; these objects can be very unwieldy.

3. Project for Peers

3.1. Additive Number Theory

Additive number theory is a branch concerned with the behavior of subsets of integers under addition. We have already seen an object of interest, integer partitions.

Consider k subsets of the nonnegative integers $\{A_i\}_{i=1}^{k}$ where $A_i \subset \mathbb{N}$. Then, we are interested in the number of solutions $r_k(n)$ to the following equation with $n \in \mathbb{N}$:

$$n = \sum_{i=1}^{k} a_i, \quad a_i \in A_i.$$

3.1.1. Additive Problem Examples

Now, we will see a few examples of additive problems.

Example 2. Weak[7] compositions of n into k parts with summands in A.[8]

Let $A \subset \mathbb{N}$. Note that $\forall a \in A,\ 1 \leq a \leq n$, i.e., we cannot have a part of the partition greater than the sum.

$$r_k(n) = \#\{(a_1, ..., a_k) \in A^k \mid n = a_1 + \cdots + a_k,\ a_i \in A\}.$$

When $A = \mathbb{N}$, using a "stars" and "bars" argument with n stars and $k - 1$ bars, we can show

$$r_k(n) = \binom{n+k-1}{k-1}.$$

Example 3. Goldbach's Conjecture, one of the oldest unsolved problems in number theory:[9] any even natural number greater than 2 can be written as the sum of two primes.

So, expressing this as in the above form, we write

$$r_2(n) = \#\{(p, q) \mid n = p + q, \quad p, q \text{ prime}\},$$

and the conjecture says that $\forall n > 2,\ r_2(n) \geq 1$.[10]

Example 4. Waring's problem:[11] Let $g(k)$ be the minimum number such that for all positive integers n, the equation

$$n = \sum_{i=1}^{g(k)} a_i^k, \quad a_i \in \mathbb{N}$$

has at least one solution, i.e., $r_{g(k)}(n) \geq 1$. Here, we are considering both exponentiation and addition.

[7] Strong compositions have all parts positive integers.

[8] Compositions are not the same as integer partitions; $r_k(n)$ counts ordered tuples, while partitions count unordered multisets.

[9] The conjecture has been shown to hold for all integers less than $4 \cdot 10^{18}$ as of 2025 according to Wikipedia.

[10] The Weak Goldbach Conjecture posits that every odd number greater than 5 can be expressed as the sum of three primes, where a prime may be used more than once in the same sum. This conjecture was proven using the circle method by Harald Helfgott in 2013.

[11] Waring's problem is related to Fermat's polygonal number theorem.

Consider $7 = 1^2 + 1^2 + 1^2 + 2^2$. So, we have have that $g(2) \geq 4$. One can check that with 23 we have $g(3) \geq 9$ and with 79 we have $g(4) \geq 19$.

Remark. Lagrange's four-square theorem proves that exactly $g(2) = 4$.

3.2. Circle Method

Cited as the Hardy Ramanujan Littlewood Circle method technique in additive number theory Our goal is to transform additive and combinatorics problems into complex analysis problems to use the tools of analysis. [1] Indeed, quoting Hardy and Ramanujan's original paper, "This idea [studying integrals from generating functions] has dominated nine-tenths of modern research in analytic theory of numbers." [2]

The circle method is aptly named by using Cauchy's theorem for series coefficients, Theorem 4.4 in Stein and Shakarchi, to represent the coefficients of generating function series as integrals around closed circular paths which package the information in their residues.

3.2.1. Example Application to Weak Compositions

We will use the circle method to find the number of weak compositions of n into k parts.

Begin with the case where we have $k = 2$ parts,

$$r_2(n) = \#\{(a_1, a_2) \mid n = a_1 + a_2,\ a_1, a_2 \in A\}.^{12}$$

Construct the generating function using the indicator function for A:

$$f(z) = \sum_{n=0}^{\infty} \mathbf{1}_A(n) z^n, \qquad \text{where } \mathbf{1}_A(n) = \begin{cases} 1 & n \in A, \\ 0 & n \notin A. \end{cases}$$

Now, using the Cauchy series product, we have

$$f^2(z) = \left(\sum_{n=0}^{\infty} \mathbf{1}_A(n) z^n\right)\left(\sum_{m=0}^{\infty} \mathbf{1}_A(m) z^m\right) = \sum_{n=0}^{\infty} c(n) z^n$$

where $c(n) = \sum_{k=0}^{n} \mathbf{1}_A(k)\mathbf{1}_A(n-k)$, which we can rewrite as $\sum_{h+k=n} \mathbf{1}_A(h)\mathbf{1}_A(k)$.

Since $\mathbf{1}_A(h)\mathbf{1}_A(k) = 1$ iff both $h, k \in A$, then this expression of $c(n)$ is exactly the number of pairs of $(h, k) \in A^2$ which satisfy $h + k = n$.

Therefore, we have $f^2(x) = \sum_{n=0}^{\infty} r_2(n) z^n$.

Now, consider a composition of n into k parts with summands in A,

$$r_k(n) = \#\{(a_1, ..., a_k) \in A^k \mid n = a_1 + \cdots + a_k,\ a_i \in A\}.$$

After repeated applications of Cauchy product, we can arrive at

$$f^k(z) = \sum_{n=0}^{\infty} r_k(n) z^n.$$

Since f^k is holomorphic in z, then, with Cauchy's theorem for series coefficients, we have

[12] The # notation returns the size of the given set.

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f^k(z)}{z^{n+1}}\, dz \tag{1}$$

for a circular closed loop C_ρ centered at the origin of radius ρ.

Now, we can express the number of solutions $r_k(n)$ in terms of the residues of this integral.

From Example 1 above with $A = \mathbb{N}$, we already know that $r_k(n) = \binom{n+k-1}{k-1}$.

Since $A = \mathbb{N}$, We have

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

which converges for $|z| < 1$. So, by equation (1), we have

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{dz}{(1-z)^k z^{n+1}},$$

where the integral converges on the closed circular loop C_ρ of radius $\rho < 1$.

We will extract the simple poles of the integral.

Using the general binomial expansion, we have

$$\frac{1}{(1-z)^k} = \sum_{m=0}^{\infty} \binom{-k}{m} (-z)^m.$$

Hence, the integral becomes

$$\oint_{C_\rho} \sum_{m=0}^{\infty} \binom{-k}{m} (-z)^m z^{-(n+1)}\, dz.$$

So, we will take the terms for which $m - n - 1 = -1 \Longrightarrow m = n$.

Thus,

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \binom{-k}{n} (-1)^n z^{-1}\, dz = (-1)^n \binom{-k}{n} = \binom{n+k-1}{k-1},$$

where the last equality is given by a combinatorial identity.[13]

3.3. Hardy-Ramanujan Estimation Formula

Theorem (*Hardy-Ramanujan*). Let $p(n)$ be the number of partitions of n.

$$p(n) \sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4\sqrt{3}\, n}.$$

Proof of Hardy-Ramanujan. We have adapted a simplified version of the proof. [3]

[13] This identity can be quickly shown by expressing both binomial coefficients in terms of falling factorials (Pochhammer symbols) and counting the appearances of -1.

$$f(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} \frac{1}{1 - z^m}, \quad |z| < 1.$$

Proposition 5.

[TODO: complete the proof] □

Bibliography

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