Math 336 Homework 6

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Problem 1. Prove that $f \in \mathcal{F}_a$ and $0 \le b < a$ implies $\forall n \in \mathbb{N}, \ f^{(n)} \in \mathcal{F}_b$.

Proof. We wish to show that for $f^{(n)} \in \mathcal{F}_b$,

$$\exists B : \left| f^{(n)}(x+iy) \right| \le \frac{B}{1+x^2} \ \text{ and } \ f^{(n)}(z) \text{ is holorphic for all } \ z : |\mathrm{Im}(z)| < b < a.$$

First, since f is holomorphic on $|\mathrm{Im}(z)| < a$ and $b < a$, then $f^{(n)}$ is holomorphic on $|\mathrm{Im}(z)| < b$.

Next, using the Cauchy inequality and considering disks of radius a centered on the real axis, we have that where $z = x + iy$,

$$\forall x \in \mathbb{R}, \ \left| f^{(n)}(x) \right| \le \frac{n!}{a^n} \sup_{z \in C_{R(x)}} |f(z)| \le \frac{n!A}{a^n(1+x^2)}.$$

Let $B = \frac{An!}{a^n}$, which is constant for all a and n fixed. Hence $\left| f^{(n)}(x+iy) \right| \le \frac{B}{1+x^2}$.

Thus, $f^{(n)} \in \mathcal{F}_b$ as desired. $\square$

Problem 2.
(a) Show that for $a > 0$ and $\xi \in \mathbb{R}$,

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{a}{a^2+x^2} e^{-2\pi i x \xi} \, dx = e^{-2\pi a|\xi|}.$$

(a) Check

$$\int_{\mathbb{R}} e^{-2\pi a|\xi|} e^{2\pi x \xi} \, d\xi = \frac{1}{\pi} \frac{a}{a^2+x^2}.$$

Proof of (a). Let $f(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$. Let $g(x) = f(x)e^{-2\pi x \xi}$.

Since f has moderate decay and is holomorphic at poles $\pm ia$, then $f \in \mathcal{F}_c, a < c$, so we can write $|f(x+iy)| \le \frac{C}{1+x^2}$.

WLOG let $\xi > 0$. Otherwise, use $x \mapsto -x$ where $f(x) = f(-x)$ with $\xi \mapsto -\xi$ to yield,

$$g(x) = f(x)e^{-2\pi i x \xi} = f(-x)e^{-2\pi i(-x)(-\xi)}.$$

Now, we will construct a semicircular contour γ in the negative imaginary half-plane which picks up a residue from the simple pole of f at $-ia$.

On the semicircular curve C_R^- of the contour, we have

$$\left| \int_{C_{\bar{R}}} g(z)\, dz \right| \le \int_{C_{\bar{R}}} |f(z)| \left| e^{-2\pi i z \xi} \right| dz.$$

Write $z = x - iy$. Since we are in the negative imaginary half-plane $y > 0$, then

$$\left| e^{-2\pi i (x - iy)\xi} \right| = e^{-2\pi y \xi} \le 1,$$

with equality when $\operatorname{Im} z = -y = 0$.

But, on $|z| = R$, we have $f(z) = O(R^{-2})$, which dominates over the length of the integration path , so the integral over $C_{\bar{R}}^-$ vanishes as $R \to \infty$.

So, we recover

$$\int_{\mathbb{R}} g(x)\, dx = -\lim_{R \to \infty} \int_{L} g(z)\, dz,$$

where the negation is obtained because the positively oriented contour traverses the real axis the in the opposing direction.

Hence we have,

$$\oint g\, dz = -\int_{\mathbb{R}} g\, dz = 2\pi i \operatorname{res}_{-ia} d.$$

Then,

$$\begin{aligned}
\operatorname{res}_{-ia} g &= \lim_{z \to -ia} (z + ia) \frac{1}{\pi} \frac{a}{a^2 + x^2} e^{-2\pi i z \xi} \\
&= \lim_{z \to -ia} \frac{1}{\pi} \frac{a}{z - ia} e^{-2\pi i z \xi} \\
&= \frac{1}{\pi} \frac{a}{-2ia} e^{-2\pi i (-ia)\xi} \\
&= -\frac{1}{2\pi i} e^{-2\pi a \xi}.
\end{aligned}$$

So,

$$-\int_{\mathbb{R}} g\, dz = -\frac{2\pi i}{2\pi i} e^{-2\pi a \xi} = -e^{-2\pi a \xi}.$$

But, the above holds for both ξ and $-\xi$, so

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{a}{a^2 + x^2} e^{-2\pi i x \xi}\, dx = e^{-2\pi a |\xi|},$$

as desired. $\qquad\square$

Proof of (b). We will split the integral into two parts to separate $|\xi|$,

$$\int_{-\infty}^{0} e^{2\pi a \xi} e^{2\pi i x \xi}\, d\xi + \int_{0}^{\infty} e^{-2\pi a \xi} e^{2\pi i x \xi}\, d\xi.$$

For the first integral, let $u = -\xi$. Then,

$$\int_0^\infty e^{-2\pi u(a+ix)} \, du = \frac{1}{2\pi(a+ix)}.$$

Similarly, for the second integral,

$$\int_0^\infty e^{-2\pi\xi(a-ix)} \, d\xi = \frac{1}{2\pi(a-ix)}.$$

Recombining the two integrals,

$$\int_{-\infty}^\infty e^{-2\pi a|\xi|} e^{2\pi i x\xi} \, d\xi = \frac{1}{2\pi}\left(\frac{1}{a-ix} + \frac{1}{a+ix}\right) = \frac{1}{\pi}\frac{a}{a^2+x^2},$$

which is what we wanted to show. $\qquad\square$

Problem 3. Let Q be a polynomial of degree at least 2 with distinct roots, not of which on the real axis. Calculate

$$\int_{\mathbb{R}} \frac{e^{-2\pi i x\xi}}{Q(x)} \, dx, \quad \xi \in \mathbb{R}$$

in terms of the roots of Q. What happens when several roots coincide?

Proof. Let z_i be the roots of Q. For now, assume that these are only simple zeros.

We will consider the cases when $\xi > 0$, $\xi < 0$, and $\xi = 0$.

First, for $\xi > 0$, the exponential term $e^{-2\pi z\xi}$ decays in the upper half-plane, so we can close a semicircular contour there. Then, by the Residue theorem, we have

$$I = \int_{\mathbb{R}} \frac{e^{-2\pi i x\xi}}{Q(x)} \, dx = 2\pi i \sum_{z_i:\ \text{Im } z_i > 0} \text{res}_{z_i} \frac{e^{-2\pi i z\xi}}{Q(z)}.$$

Since z_i are all simple roots, then

$$\text{res}_{z_i} \frac{e^{-2\pi i z\xi}}{Q(z)} = \frac{e^{-2\pi i z_i\xi}}{Q'(z_i)}.$$

Thus,

$$I = 2\pi i \sum_{z_i:\ \text{Im } z_i > 0} \frac{e^{-2\pi i z_i\xi}}{Q'(z_i)}.$$

Second, for $\xi < 0$, the exponential term decays in the lower half-plane, but the positive orientation of a semicircular contour in the negative imaginary half-plane gives a backwards traversal of the real axis. Hence, by the Residue theorem and as is the first case, we have

$$I = -2\pi i \sum_{z_i:\ \text{Im } z_i < 0} \frac{e^{-2\pi i z_i\xi}}{Q'(z_i)}.$$

Third, when $\xi = 0$, the integral becomes $\int_{\mathbb{R}} \frac{dx}{Q(x)}$.

We will consider the contour in the upper half-plane. Since $\deg Q \geq 2$, then $1/Q(z) = O(z^{-2})$, which will vanish for $|z|$ sufficiently large. For $|z| = R$,

$$\left| \int_{C_R^+} \frac{dz}{Q(z)} \right| \leq \pi R \frac{1}{R^2} \to 0 \quad \text{as} \quad R \to \infty.$$

Thus, as before,

$$\int_{\mathbb{R}} \frac{dx}{Q(x)} = 2\pi i \sum_{z_i:\, \text{Im } z_i > 0} \frac{1}{Q(z_i)}.$$

Now, when roots of Q coincide, we have for a root z_j with multiplicity $m_j > 1$,

$$Q(z) = \left(z - z_j\right)^{m_j} \widetilde{Q}(z)(z) \quad \text{where} \quad \widetilde{Q}(z)(z_j) \neq 0.$$

The residue at z_j is given by

$$\operatorname{res}_{z_j} \frac{e^{-2\pi i z \xi}}{Q(z)} = \frac{1}{(m_j - 1)!} \lim_{z \to z_j} \left(\frac{d}{dz}\right)^{m_j - 1} \frac{e^{-2\pi i z \xi}}{\widetilde{Q}(z)}.$$

$\square$

Problem 4. Solve the differential equation

$$\sum_{j=0}^n a_j \left(\frac{d}{dt}\right)^j u(t) = f(t),$$

where $a_k \in \mathbb{C}$ and $f \in C^2$ has bounded support.

(a) Let

$$\hat{f}(z) = \int_{\mathbb{R}} f(t) e^{-2\pi i z t} \, dt.$$

Note that $\hat{f}$ is entire. Show using IBP that, for fixed $0 \leq a$, if $|y| \leq a$, then

$$\left| \hat{f}(x + iy) \right| \leq \frac{A}{1 + x^2}.$$

(a) Write

$$P(z) = \sum_{j=0}^n a_j (2\pi i z)^j.$$

Find $c \in \mathbb{R}$ such that P does not vanish on the line

$$L = \{z \mid z = x + ic, x \in \mathbb{R}\}.$$

(a) Set

$$u(t) = \int_L \frac{e^{2\pi i x z}}{P(z)} \hat{f}(z) \, dz.$$

Check that

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) = \int_L e^{2\pi i z t} \hat{f}(z) \, dz$$

and

$$\int_L e^{2\pi i z t} \hat{f}(z) \, dz = \int_{\mathbb{R}} e^{2\pi i x t} \hat{f}(x) \, dx.$$

Conclude by the Fourier inversion theorem that

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) = f(t).$$

Note that the solution u depends on the choice of c.

Proof of (a). Since f has bounded support, say on $[-M, M]$, then for all $|t| > M$, $f(t) = 0$ and $f'(t) = 0$.

So,

$$\int_{\mathbb{R}} f(t) e^{-2\pi i z t} \, dt = \int_{-M}^{M} f(t) e^{-2\pi i z t} \, dt.$$

Since $f \in C^2$, then we can perform IBP twice. First, we have

$$\int_{[-M,M]} f(t) e^{-2\pi i z t} \, dt = f(t) \frac{e^{-2\pi i z t}}{-2\pi i z} \Big|_{-M}^{M} - \int_{-M}^{M} \frac{e^{-2\pi i z t}}{-2\pi i z} f'(t) \, dt,$$

but the first term vanishes by the bounded support of f. Second, we have

$$\int_{-M}^{M} \frac{e^{2\pi i z t}}{-2\pi i z} f'(t) \, dt = f'(t) \frac{e^{-2\pi i z t}}{4\pi^2 z^2} \Big|_{-M}^{M} - \frac{1}{4\pi^2 z^2} \int_{-M}^{M} e^{-2\pi i z t} f''(t) \, dt,$$

but the first term vanishes by the bounded support of f for f'.

So, we have that

$$\left| \hat{f}(x + iy) \right| = \left| \frac{1}{4\pi^2 (x + iy)^2} \right| \left| \int_{-M}^{M} \left| e^{-2\pi i (x+iy)t} \right| |f''(t)| \, dt \le \frac{1}{4\pi^2} \frac{1}{x^2 + y^2} 2MB e^{2\pi a M}$$

where $f \in C^2$ implies that f'' is continuous and bounded by some B on $[-M, M]$ by EVT.

Hence, exchanging constants and using $|y| \le a$, we have

$$\left| \hat{f}(x + iy) \right| \le \frac{A}{1 + x^2},$$

as desired. $\qquad\square$

Proof of (b). Since P is a polynomial of degree n, then it has n roots by FTA. Let these roots be $z_j = x_j + iy_j$.

Then, we can choose any $c \neq y_i$ for any j. In particular, let $|c| > \max\limits_{1 \leq j \leq n} |y_j|$.

Thus, $P(x + ic)$ will not vanish for all x in $\mathbb{R}$. $\qquad\square$

Proof of (c). We will first find the derivatives of u,

$$\left(\frac{d}{dt}\right)^j u(t) = \left(\frac{d}{dt}\right)^j \int_L \frac{e^{2\pi izt}}{P(z)} \hat{f}(z) \, dz$$

$$= \int_L \left(\frac{d}{dt}\right)^j \frac{e^{2\pi izt}}{P(z)} \hat{f}(z) \, dz$$

$$= \int_L (2\pi iz)^j \frac{e^{2\pi izt}}{P(z)} \hat{f}(z) \, dz,$$

where the interchange of the derivative and the integral is justified with Leibniz rule given that the path L does not depend on t, and the integrand has continuous partials in both t and z.

So,

$$\sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t) = \sum_{j=0}^{n} a_j \int_L (2\pi iz)^j \frac{e^{2\pi izt}}{P(z)} \hat{f}(z) \, dz$$

$$= \int_L \left(\sum_{j=0}^{n} a_j (2\pi iz)^j\right) \frac{e^{2\pi izt}}{P(z)} \hat{f}(z) \, dz$$

$$= \int_L e^{2\pi izt} \hat{f}(z) \, dz.$$

by considering the definition of P and where a finite sum can be interchanged without consequence.

Now, create a closed rectangular contour joining the real axis and R with sides at $\pm R$.

Since $\hat{f}(z)$ and $e^{2\pi izt}$ are entire, then the integral over the closed loop contour will vanish.

Furthermore, at $x = \pm R$,

$$\left|\hat{f}(x + iy)\right| \leq \frac{A}{1 + R^2} \to 0 \ \text{ as } \ R \to \infty.$$

So, the integral will vanish on the vertical sides of the contour.

Therefore, we are left with

$$\int_L e^{2\pi izt} \hat{f}(z) \, dz = \int_{\mathbb{R}} e^{2\pi ixt} \hat{f}(x) \, dx.$$

Thus, by the Fourier inversion theorem, we have

$$f(t) = \int_{\mathbb{R}} e^{2\pi ixt} \hat{f}(x) \, dx = \sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^j u(t),$$

which is what we wanted to show. $\qquad\square$