Math 462 Homework 5

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Proposition. $X(G) \geq n/\alpha(G)$ where $n = |V(G)|$ and $\alpha(G)$ gives the size of the largest independent set of G.

Proof of Proposition. Create a coloring of G such that all the vertices in the largest independent set $\alpha(G)$ have the same color.

By the definition of $\alpha(G)$, all color classes must be assigned to at most $\alpha(G)$ vertices.

Since there are $X(G)$ color classes, then n vertices can be partitioned into at most $X(G)$ parts of size $\alpha(G)$. Hence $n \leq X(G)\alpha(G)$, which is what we wanted to show. □

Problem 1. Let $G = (V, E)$ be a simple graph, $|V| = n$, and $k \in \mathbb{N}$. Prove that if $|E| \leq nk/2$, then $\alpha(G) \geq n/(k+1)$.

Proof. By the handshake lemma, $2|E| \leq nk$ implies that the average degree of the vertices in G is at most k.

So, in any such graph G, there exists a vertex with degree at most the average degree k, call it v.

Consider the extreme case where v has k neighbors, all of which are assigned different colors. Then, v must be colored with a new and different color, so we must have $k+1$ total colors. Hence $X(G) \leq k+1$.

By the Proposition, we have $\alpha(G) \geq n/X(G)$.

Thus, we have

$$\frac{n}{k+1} \leq \frac{n}{X(G)} \leq \alpha(G),$$

as desired. □

Problem 2. Let T be a tree with $t \geq 2$ vertices.

(a) Prove that if G is a simple graph with all vertices of degree at least $t-1$, then G is not T-free.

(b) Prove that if $G = (V, E)$ is simple, with $|V| = n$ and $|E| \geq (t-2)n + 1$, then G is not T-free.

(c) Conclude that

$$\text{ex}(n, T) \leq (t-2)n.$$

Proof of (a). We will embed T into G using induction on t, the number of vertices of T.

For the base case $t = 2$, T is a single edge. Since all vertices in G have degree at least $t - 1 = 1$, then G contain at least one edge, so G contains as subgraph isomorphic to T.

Assume the statement holds for $t > 2$, and consider a tree T with $t + 1$ vertices.

Choose any leaf ℓ in T and let p be its parent vertex. Such a leaf always exists any any tree with more than two vertices.

Let $T' = T - \ell$. Note that T' has t vertices.

By the inductive hypothesis, G contains a subgraph G' which is isomorphic to T'.

Let the vertex v in G' correspond to the vertex p in T.

Since v is in G, then it has degree at least $(t + 1) - 1 = t$.

Since T' is a tree with t vertices, then it has $t - 1$ edges.

So, v has at most $t - 1$ incident edges in G'.

Therefore, v must have at least one neighbor w in G that is not in G'.

Now, map ℓ in T to w in G, which gives us an embedded a subgraph of G isomorphic to T'.

Thus, by induction, if every vertex in G has degree at least $t - 1$, then G contains a subgraph isomorphic to T. $\square$

Proof of (b). By part (a), if every vertex of G has degree at least $t - 1$, then G contains a subgraph isomorphic to T.

So, assume G contains a vertex v of degree less than $t - 1$.

Consider $G' = G - v$. Then, G' has $n - 1$ vertices and at least

$$(t - 2)n + 1 - (t - 2) = (t - 2)(n - 1) + 1$$

edges because v was incident to at most $t - 2$ edges.

If we have removed all vertices of degree less than $t - 1$, then we have achieved a subgraph isomorphic to T by part (a).

Otherwise, we can repeat this process up to $n - t$ times to a graph with only t vertices remaining.

After removing $n - t$ vertices, each with degree at most $t - 2$, then we have at least

$$(t - 2)n + 1 - (n - t)(t - 2) = (t - 2)t + 1$$

edges remaining.

But, a simple graph of t vertices can have at most $\frac{t(t-1)}{2}$ edges.

However,

$$(t - 2)t + 1 - \frac{t(t - 1)}{2} = t^2 - 2t + 1 - \frac{t^2}{2} + \frac{t}{2} = \frac{t^2 - 3t + 2}{2},$$

which is indeed positive for all $t > 2$, so such a graph would have more than the maximum possible number of edges, which is a contradiction.

Hence G must not have had any vertices with degree less than $t - 1$, which again by part (a), means that G is not T-free.

For $t = 2$, we have a single edge, so G is directly isomorphic to T.

Thus, for a graph G with n vertices and at least $(t-2)n+1$ edges, G must have a subgraph isomorphic to T. □

Proof of (c). Recall that $\mathrm{ex}(n, T)$ is the maximum number of edges in a T-free n-vertex graph.

From part (b), we have that any such graph with n vertices and at least $(t-2)n+1$ edges contains a subgraph isomorphic to T.

Therefore, a T-free graph with n vertices can have at most $(t-2)n$ edges. Thus,

$$\mathrm{ex}(n, T) \leq (t-2)n.$$

□