336 Final Project

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June 01, 2025

Contents

Abstract

In this project, we plan to introduce the curious prospective mathematician to some introductory ideas in combinatorics. In particular, we will discuss recurrence relations, generating functions, and integer partitions.

For our peers, we will expand on these ideas and wield some of our analysis tools developed in 336 to introduce the circle method, a tool employed in proof of the Hardy-Ramanujan estimation formula for partitions.

The paper is organized into three sections:

1. The first section describes the student activities associated with the project, including the materials required for the physical interactive activity and guiding questions for students to ponder when working with the materials.
2. The second section acts as a primer for prospective students on combinatorics.

3. The third section provides a detailed explanation of the ideas at a deeper level for peers in 336.

1. Showcase Activity

We will present some problems for the reader to ponder and build some intuition behind recurrence relations. These problems are intended to be accompanied by physical artifacts with the showcase.

We will also provide the solutions, but the reader in encouraged to first turn to the Discussion Section 2 to obtain the tools necessary for tackling the problems.

1.1. Domino Tiling

How many ways we can tile a $2 \times n$ rectangular space with 1×2 tiles?

Materials & Setup: We can use dominoes as our tiles.

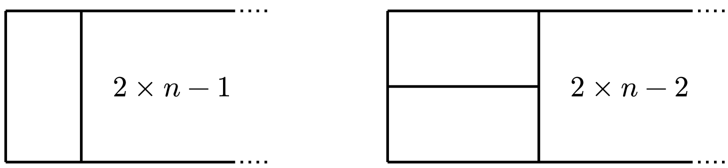

Figure 1: Domino tiling cases.

Guiding Questions

- Sketch a table of the space size n and the corresponding number of tilings which fill that space. Do we recognize this sequence?[1]
- Can we write down a relationship that describes how to get the next number of tilings in the sequence? Does this relationship match the possible cases in the above Figure 1?
- If we have a tiling of a $2 \times (n-1)$ space, in how many ways can you extend it to tile a $2 \times n$ space? What about starting from a $2 \times (n-2)$ space? Could we have written down this relationship by building a recursive definition of the tilings?

Extension Questions

- What are some other ways that we can build this sequence?
- If the dominoes come in k different colors, how many ways can we tile a $2 \times n$ space?
- What happens with a $3 \times n$ space using the same 1×2 dominoes?
  (This is tricky; explore small cases.)
- More generally, what happens if we change the shape of our dominoes?
  What about if we change the size of our tiling space?

Proposition 1. The number of tiles follows the Fibonacci sequence.

1.2. Parentheses Puzzle

How many ways are there to arrange sequences of nested and matched parentheses?

[1] If you are ever working with a sequence in the wild, try seeing if it is documented in The On-Line Encyclopedia of Integer Sequences (OEIS). https://oeis.org/

For any prefix of the string of parentheses, the number of left parentheses is at least the number of right parentheses. In the entire string, the number of left parentheses must be equal to the number of right parentheses. These are sequences are known as Dyck words.

$$((()))(()())$$

Figure 2: Example Dyck word.

Materials & Setup: Notecards with parentheses, colored with two colors, e.g. red and blue. Set one color as the left parentheses, and one color as the right. Arrange the cards to form Dyck words.

Guiding Questions

- Let's start small: how many valid arrangements are there with 1, 2, or 3 pairs of parentheses? Create a systematic list for each case using your colored cards.
- Use the cards to build all possible arrangements for 3 pairs. Can we group them by some common structure?
- What makes a sequence "invalid"? Try to construct some invalid sequences and identify exactly where they break the rules.
- For any valid sequence, what happens if we remove the first left parenthesis and its matching right parenthesis?
- Can we find a pattern: if we know the number of valid sequences for smaller cases, how might we build up to larger cases? (This is tricky.)

Extension Questions

- **Mountain ranges**: Draw each parenthesis sequence as a path where "(" means "go up" and ")" means "go down." What do valid sequences look like as paths?
- **Ballot problem**: In an election, candidate A receives n votes and candidate B receives n votes. If votes are counted one by one, in how many ways can A always be ahead or tied throughout the counting?
- **Binary trees**: Can we connect Dyck words to binary trees? How might parentheses represent the structure of a tree?
- **Stack operations**: If "(" represents "push" and ")" represents "pop" on a stack, what do valid sequences represent in terms of stack behavior?
- **Generalizations**: What if we had more types of brackets, e.g. (), [], {}?
  What if we required that () must be nested within [] which must be nested within {}?

Proposition 2. The number of Dyck words is the n-th Catalan number.

Proof of Proposition. Each Dyck word belongs to exactly one of the two following cases:
- it is empty, or
- it contains a matched pair of parentheses enclosing one Dyck word, and is followed by another, that is $(D_1)D_2$.

Let D be the set of all Dyck words, weighted by half the length of the string. Then, we have have weight-preserving bijection

$$D \to \{\} \sqcup \{()\} \times D \times D,$$

and we can conclude by the proof of Proposition 5. □

1.3. Polygon Triangulation

How many ways are there to cut a polygon along its diagonals into triangles where rotations are distinct?

Materials & Setup: Wooden block with labeled pegs stuck in to represent polygon vertices. Stretch rubber bands around the pegs to create triangulations.

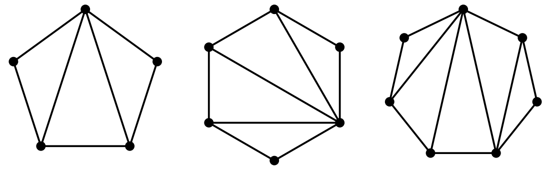

Figure 3: Example polygonal triangulations.

Guiding Questions

- How many triangles are in the triangulation of an polygon with n sides (n-gon)?
- Pick one vertex of your polygon. In any triangulation, how many triangles must include this vertex? What does this tell us about the structure?

Extension Questions

- What if we do not label the pegs, that is consider rotations to be indistinguishable?
- How many ways can we loop rubber bands around the pegs such that rubber bands do not cross over one another? Label the pegs so as to consider rotations to be distinct from one another. These are called non-crossing partitions.
- Can we connect these points to form a binary tree?
  Hint: what happens if we draw points (vertices of a tree) on each of the triangles in the triangulation. Can we do this for any triangulation?
- **Associativity**: Can you connect polygon triangulations to different ways of associating the product $a_1 \cdot a_2 \cdot a_3 \cdots a_n$, that is placing parentheses around the terms?

Proposition 3. The number of ways to triangulate an $(n+2)$-sided polygon is the n-th Catalan number.

Proof of Proposition. Let T_n be the number of triangulations of an $n+2$-gon.

Each triangulation is in exactly one of the following cases:
- It is empty.
- The triangulation contains an outside edge of the polygon, and has two triangulations to the left and right.

Hence, there exists a weight-preserving bijection:

$$T \to \emptyset \sqcup \{\triangle\} \times T \times T,$$

and we can conclude by the proof of Proposition 5. □

1.4. Young Diagrams

In how many ways can we arrange n identical squares into left-justified rows, where each row has no more squares than the row above it? These arrangements are known as Young diagrams.[2]

Materials & Setup: We can use small square tiles (like Scrabble tiles, square counters, or cut paper squares), to arrange into Young diagrams.

```
□ □ □ □    □ □ □    □ □    □ □    □
           □        □ □    □      □
                           □      □
                                  □
```

Listing 1: Young diagrams for $n = 4$.

Guiding Questions

- Start with small numbers ($n = 1, 2, 3, 4, 5$) and systematically build all possible arrangements with the tiles; can we reuse previous diagrams for when creating larger diagrams?
- What's the connection between these diagrams and writing n as a sum? (e.g., $4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$)
- If we flip a diagram along its diagonal (transpose), what do we get? How does this relate to partitions into distinct vs. repeated parts?

Extension Questions

- **Restricted partitions**: What if we limit the maximum row length to k squares?
- **Odd partitions**: What if we only allow rows with odd lengths?
- **Self-conjugate**: Which arrangements look the same when flipped along the diagonal?
- **Hook lengths**: Can we find a pattern in the "hook" shapes (the squares extending below and to the right) extending from each square?

Remark. The number of partitions of size n is given by the partitions function $p(n)$, the approximation of which is given by the Theorem in the last section. There is no known closed form for $p(n)$.

2. Discussion for Students

We will take a first leap into the realm of enumerative combinatorics. Enumeration means finding the size of finite sets. In particular, we want to know how the size of sets change according to certain parameters.

Let A_n be a set parametrized by some natural number n. We will assign a function f to capture the size of the set and write $f(n) = |A_n|$. Now, we can understand how the size of the set changes by understanding the behavior of the function f.

Note that we have defined f to agree with the size of A_n only for integers n. However, if f is a nice function, we can ask the question: what values might f take on for non-integer arguments?

There are several main cases for how we work with f:

- we have an explicit formula for $f(n)$,
- we have an approximation or asymptotic estimate for $f(n)$ as $n \to \infty$,
- we compare f to another function which we already understand well, or
- we build a recurrence relation on the output of f.

[2]If we did this with points instead of boxes, we would have a Ferrers diagram.

2.1. Asymptotic Behavior & Approximation

Asymptotic analysis allows us to simplify complex problems by focusing on how functions behave for large inputs toward infinity. This is a form of abstraction[3] where we strip away less significant terms to expose the dominant behavior of the function. Oftentimes, we make comparisons with benchmarks, such as the logarithmic, linear, quadratic, and exponential functions.

We write $f \sim g$ for asymptotic equivalence, that is when

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

meaning the functions grow at the same rate.

A polynomial function is asymptotically equivalent to its highest degree term:

$$ax^2 + bx + c \sim ax^2.$$

Exponential functions grow faster than linear functions:

$$a^n + b^n + n^c \sim b^n \quad \text{where} \quad a < b \quad \text{and} \quad 1 < b,$$

which grow faster than logarithmic functions:

$$ax^b + c\log x \sim ax^b \quad \text{with} \quad c > 0.$$

Note that asymptotic equivalence requires constant.

2.1.1. Stirling's Formula

In fact, we have the following proposition that factorials grow faster than any of the above functions.

Proposition 4. Factorial grows faster than any exponential function.

We will need the following Theorem to prove the result.

Theorem (*Stirling's Formula*).

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Proof of Proposition. Let ab^n be an exponential function. With Stirling's Formula, we have:

$$\lim_{n\to\infty} \frac{n!}{ab^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{ab^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{a}\left(\frac{n}{be}\right)^n = \infty.$$

Since this limit diverges, then $n!$ grows faster than any ab^n. □

2.2. Generating Functions

Generating functions are objects known as formal power series, they are a formal sum–which means we are not actually performing any addition–that does not have to converge for all x.[4]

[3] Abstraction is one of our most powerful tools in mathematics.

[4] Generating functions are neither generating, nor functions.

We have already seen how to capture combinatorial information using functions; now we will extend these functions to build generating functions. We use generating functions to encode combinatorial rules as algebraic relations.

2.2.1. Uses for Generating Functions

Generating functions serve as a "mathematical machine" that transforms counting problems into algebraic problems. Instead of working directly with sequences like $1, 1, 2, 5, 14, 42, ...$, we encode them as coefficients in a power series and use algebra to find patterns.

Some key uses are as follows:
- **Finding closed forms**: Turn recurrence relations into algebraic equations we can solve.
- **Proving identities**: Show that two different counting problems give the same answer.
- **Asymptotic analysis**: Extract growth rates of sequences.
- **Solving complex recurrences**: Handle cases where direct methods fail.

2.2.2. How weights become coefficients

This is the core idea: if an object has weight w, it contributes x^w to the generating function.

Method:
1. Each object gets a numerical weight (often size, but could be any property)
2. An object of weight w becomes the monomial x^w
3. Objects with the same weight are collected; if there are k objects of weight n, they contribute $k \cdot x^n$ total
4. The generating function is then

$$\sum_{\text{all objects}} x^{\text{weight of object}} = \sum_{n=0}^{\infty} \text{count of weight-}n\text{ objecst} \cdot x^n$$

2.2.3. Producing Generating Functions

- **Direct counting**: Count objects by weight and write down the series.
- **Combinatorial constructions**: Use union and product rules (like the $S \to \emptyset \sqcup \{\star\} \times S \times S$ pattern).
- **Recurrence relations**: Transform recursive formulas into generating function equations.
- **Known series**: Recognize patterns that match geometric series, binomial series, etc.

We will now transition into the technical construction of generation functions.

Definition 1. A recurrence relation is an equation that expresses each element of a sequence as a function of the preceding ones. For our purposes, a function f satisfies a recurrence relation if we can expression f on some input n in terms of the value of f on "previous" inputs. For example,

$$\forall n, f(n + 2) = f(n + 1) + f(n)$$

is the recurrence relation for the Fibonacci sequence.

Definition 2. Suppose f is a function on the natural numbers. The generating function of f is defined as follows:

$$F(x) = \sum_{n=0}^{\infty} f(n)x^n = \underbrace{f(0) + f(1)x + f(2)x^2 + \cdots}_{\text{contains all information from } f}$$

where the values of f at n are paired with the coefficients of x^n.

If $\{A_n\}$ is a sequence of finite sets, then we can define a generating function on the size of these sets

$$F_A(x) = \sum_{n=0}^{\infty} |A_n| x^n.$$

We can view the size of a set as a function that takes a set and returns the number of elements inside of the set. More generally, we can consider a weight function $w : A \to \mathbb{N}$ that takes a set A and produces a given natural value for each element in A.

With a set A, possibly infinite, and a weight function w on A, we can construct the generating function

$$F_A(x) = \sum_{a \in A} x^{w(a)}.$$

Example 1. Let A_n be the set of all binary strings of length n where A_0 is the empty string. Then $|A_n| = 2^n$ gives us that

$$F_A(x) = \sum_{n=0}^{\infty} (2x)^n = \frac{1}{1-2x}$$

using the geometric series $\sum x^n = \frac{1}{1-x}$.

2.3. Some Sequences

We will now define in detail the sequences we have previously seen in Activity Section 1.

Definition 3. Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$[5] The sequence is defined by the recurrence relation:

$$f(n+2) = f(n+1) + f(n)$$

with closed form:

$$\frac{\varphi^n - \psi^n}{\varphi - \psi} \sim \varphi^n / \sqrt{5}$$

, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\psi = \frac{1-\sqrt{5}}{2}$ is its conjugate. The above is produced by the generating function: $\left(1 - x - x^2\right)^{-1}$.

Definition 4. Catalan sequence $1, 1, 2, 5, 14, 42, \ldots$[6] The sequence is defined by the recurrence relation:

[5] https://oeis.org/A000045
[6] https://oeis.org/A000108

$$c(0) = 1 \;\text{ and }\; c(n) = \frac{2(2n-1)}{n+1}c(n-1) = \sum_{i=1}^{n} c(i-1)c(n-i)$$

with closed form:

$$\frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

produced by the generating function;

$$\frac{1-\sqrt{1-4x}}{2x}.$$

Proof of Catalan closed form. We will provide a combinatorial argument for $c_n = \frac{1}{n+1}\binom{2n}{n}$. We have that c_n counts the number of lattice paths from $(0,0)$ to $(2n,0)$ where each step is of the form $(1,1)$ or $(1,-1)$, and the path never crosses the $y=0$ line.

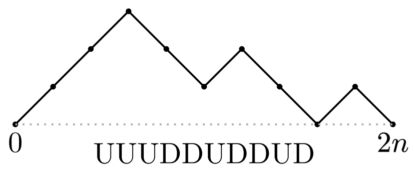

Figure 4: Example lattice path.

There are $\binom{2n}{n}$ lattice paths that never cross the $y=0$ line without any other restriction, that is $2n$ total steps, n of which are up (or down).

Let b_n be the number of "bad" paths which cross $y=0$. Given a bad path, consider the first time it touches $y=-1$ and reflect all points to the right across $y=-1$. The resulting lattice line is a path from $(0,0)$ to $(2n,-2)$.

Given a path from $(0,0)$ to $(2n,-2)$, we can go back to a bad path using the same process in reverse. Therefore, there exists a bijection between the set of bad paths to the set of paths from $(0,0)$ to $(2n,-2)$; there are $\binom{2n}{n-1}$ such paths, for all $2n$ steps, $n-1$ of these must be up steps.

Therefore,

$$\begin{aligned}
c_n &= \binom{2n}{n} - \binom{2n}{n-1} \\
&= \binom{2n}{n} - \frac{(2n)!}{(n-1)!(n+1)!} \\
&= \binom{2n}{n} - \frac{n}{n+1}\frac{(2n)!}{n!n!} \\
&= \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} \\
&= \frac{1}{n+1}\binom{2n}{n},
\end{aligned}$$

which is what we wanted to show. $\square$

2.3.1. Constructing Generating Functions from Recurrence Relations

Given a functional recurrence relation on some function $f(n)$, we can perform the generating function transform to collapse the recursive equation into a direct relationship for a generating function $F(x)$. If we are lucky, we can then invert the transform to acquire a closed form for $f(n)$.

$$\begin{tikzcd} f(n) \arrow[r, bend left, "\mathcal{G}"] & F(x) \arrow[l, bend left, "\mathcal{G}^{-1}"] \end{tikzcd}$$

Figure 5: Generation function transform ($\mathcal{G}$) diagram.

2.3.1.1. Binary Trees

We can also acquire the Catalan numbers using binary trees and their generating function.

Definition 5. (Rooted) Binary trees are defined recursively as:
- The empty set is a binary tree.
- If T_1 and T_2 are binary trees, then $T_1 \bullet T_2$ is binary tree.

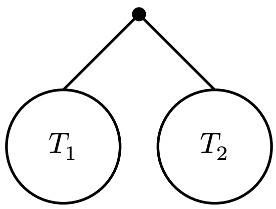

Figure 6: Binary tree.

Each vertex of a binary tree can have a left child, a right child, neither, or both.

Proposition 5. The number of binary trees with n vertices is c_n.

Proof of Proposition. Let B be the set of all binary trees weighted by the number of vertices.

Each binary tree is in exactly one of the following two cases:
- it is empty, or
- it has two children, $T_1, T_2 \in B$.

So, there exists a weight-preserving bijection:

$$B \to \emptyset \sqcup \{\bullet\} \times B \times B.$$

Therefore,

$$\begin{aligned} F_B(x) &= x^0 + x^1 F_B(x)^2 \\ 0 &= 1 - F_B(x) + x F_B(x)^2 \\ F_B(x) &= \frac{1 \pm \sqrt{1-4x}}{2x}, \end{aligned}$$

where we use the Quadratic Formula to solve for $F_B(x)$.

But, we need $F_B(x)$ to be a function with $F_B(0) = 1$ because there is one binary tree with zero vertices.

So, we will take the following limit:

$$\lim_{x\to 0} F_B(x) = 1 \implies \begin{cases} \lim\limits_{x\to 0} \frac{1+\sqrt{1-4x}}{2x} \text{ DNE}, \\ \lim\limits_{x\to 0} \frac{1-\sqrt{1-4x}}{2x} = \lim_{x\to 0} \frac{(-\frac{1}{2})(-4)(1-4x)^{-1/2}}{2} = 1. \end{cases}$$

Therefore $F_B(x) = \frac{1-\sqrt{1-4x}}{2x}$

By the General Binomial Theorem, we have

$$(1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4)^n x^n \quad \text{where} \quad \binom{1/2}{n} = \begin{cases} (-1)^{n-1} \frac{2}{4^n n} \binom{2n-2}{n-1} & , \; n > 0 \\ 1, & n = 0. \end{cases}$$

Hence, initially extracting the $n = 0$ term from the sum, we have

$$\begin{aligned} F_B(x) &= \frac{1}{2x} \left(1 - \left(1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{4^n n} \binom{2n-2}{n-1} (-4)^n x^n \right) \right) \\ &= \frac{1}{2x} \sum_{n=1}^{\infty} \frac{2}{n} \binom{2(n-1)}{n-1} x^n \\ &= \frac{1}{x} \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n+1} \binom{2n}{n} \implies |B_n| = \frac{1}{n+1} \binom{2n}{n}, \end{aligned}$$

which is what we wanted to show. □

Remark. This proof uses several techniques. The key ideas are:

1. Every binary tree is either empty or has exactly two subtrees.
2. This recursive structure translates to the equation $F(x) = 1 + xF(x)^2$.
3. Solving this quadratic gives us the Catalan numbers.

Don't worry if the algebraic details are challenging; focus on understanding the recursive structure!

Example 2. There are 14 binary trees with 4 vertices: $B_4 = \frac{1}{5}\binom{8}{4} = 14$, find them all.

Remark. Once we recognize the recurrence relationship $S \to \emptyset \sqcup \{\star\} \times S \times S$, then we can immediately conclude that we have the Catalan numbers. This fact is used to condense the proofs in the first section.

2.4. Integer Partitions

We will now turn to another

Definition 6. For $n \in \mathbb{N}$, a partition of n is a way to write n as the sum of positive integers where the order of summation does not matter.[7] As seen in Activity Section 1, partitions can be represented using Young diagrams.

[7] If the order of summation matters, then we have a strong composition.

Proposition 6. The number of partitions of n with at most k parts equals the number of partitions with largest part at most k.

Proof of Proposition. Consider the Young diagram of the partition. If a partition has k parts, then there are k rows in its diagram. If the largest part of a partitions is k, then there are k columns in its partition.

Transpose the Young diagram by flipping it across its central diagonal, swapping the number of rows and columns while maintaining the number of dots. This flipped partition is known as the conjugate partition.

Thus, we have a bijection between the set of partitions with k parts and the set of partitions with largest part k. □

Theorem. Let $P_{\leq k}$ be the set of all partitions with all parts at most k, weighted by sum.

$$F_{P_{\leq k}}(x) = \prod_{j=1}^{k} \frac{1}{1-x^j}.$$

Proof of Theorem. We can specify any partition in $P_{\leq k}$ uniquely by the number of k's, $k-1$'s, … , and $1'$s.

Therefore, we have a weight-preserving bijection:

$$P_{\leq k} \to \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}},$$

where the weight of each partition $(a_1, ..., a_k) = \sum_{n=1}^{k} n a_n$ with the a_n representing the number of n's in the partition.

Equivalently, we have a weight-preserving bijection:

$$P_{\leq k} \to \mathbb{N} \times 2\mathbb{N} \times 3\mathbb{N} \times \cdots \times k\mathbb{N}$$

where $i\mathbb{N} = \{0, i, 2i, 3i, ...\}$ and the weight of $x \in i\mathbb{N}$ is itself.

So,

$$F_{P_{\leq k}}(x) = F_{\mathbb{N}}(x) F_{2\mathbb{N}}(x) \cdots F_{k\mathbb{N}}(x).$$

Then, we can obtain the generating function for each $F_{i\mathbb{N}}(x)$ using the geometric series:

$$F_{\mathbb{N}}(x) = \sum_{0}^{\infty} x^n = \frac{1}{1-x}$$
$$F_{2\mathbb{N}}(x) = \sum_{0}^{\infty} x^{2n} = \frac{1}{1-x^2}$$
$$\vdots$$
$$F_{k\mathbb{N}}(x) = \sum_{0}^{\infty} x^{kn} = \frac{1}{1-x^k}.$$

Thus,

$$F_{P_{\leq k}}(x) = \prod_{j=1}^{k} \frac{1}{1 - x^j},$$

which is what we wanted to show. □

If we let the maximum size of each part k exceed any number, i.e., $k \to \infty$, then we obtain the following theorem.

Theorem (*Partitions Generating Function*). Let P be the set of all partitions weighted by sum.

$$F_P(x) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}.$$

3. Explanation for Peers

We will build on material from previous section, so the reader is recommended to ensure familiarity with the ideas presented there. We will begin with a brief exploration of a kind of problem which, at first glance, does not appear to be related to the complex analysis that we have been studying this quarter. Then, we will draw a connection between Number Theory problems and analysis techniques, in particular by demonstrating the Circle Method.

3.1. Additive Number Theory

Additive Number Theory is a branch concerned with the behavior of subsets of integers under addition. We have already seen one such object of interest, integer partitions.

More generally, we consider k subsets of the nonnegative integers $\{A_i\}_{i=1}^{k}$ where $A_i \subset \mathbb{N}$. We are interested in the number of solutions $r_k(n)$ to the following equation with $n \in \mathbb{N}$:

$$n = \sum_{i=1}^{k} a_i, \quad a_i \in A_i.$$

We use $r_k(n)$ as a function between naturals to capture information about the solutions. Later, we will expand this function to a function on complex numbers using generating functions. This is the key step which allows us to bring in tools from analysis.

3.1.1. Additive Problem Examples

Now, we will see a few examples of additive problems.

Example 3. Weak[8] compositions of n into k parts with summands in A.[9]

Let $A \subset \mathbb{N}$. Note that $\forall a \in A,\ 1 \leq a \leq n$, i.e., we cannot have a part of the partition greater than the sum. Expressing the number of solutions as a set, we have

$$r_k(n) = \#\{(a_1, ..., a_k) \in A^k \mid n = a_1 + \cdots + a_k,\ a_i \in A\}.^{10}$$

[8]Strong compositions have all parts positive integers.

[9]Compositions are not the same as integer partitions; $r_k(n)$ counts ordered tuples, while partitions count unordered multisets.

[10]The # notation returns the size of the given set.

When $A = \mathbb{N}_{\leq n}$, using a "stars" and "bars" argument with n stars and $k-1$ bars, we can show

$$r_k(n) = \binom{n+k-1}{k-1}.$$

Example 4. Goldbach's Conjecture, one of the oldest unsolved problems in number theory: any even natural number greater than 2 can be written as the sum of two primes.[11]

So, expressing this as a set in the above form, we write

$$r_2(n) = \#\{(p,q) \mid n = p+q, \quad p,q \text{ prime}\},$$

and the conjecture says that $\forall n > 2,\ r_2(n) \geq 1$.[12]

Example 5. Waring's problem: Let $g(k)$ be the minimum number such that for all positive integers n, the equation

$$n = \sum_{i=1}^{g(k)} a_i^k, \quad a_i \in \mathbb{N}$$

has at least one solution, i.e., $r_{g(k)}(n) \geq 1$. Here, we are considering both exponentiation and addition.[13]

Consider $7 = 1^2 + 1^2 + 1^2 + 2^2$. So, we have have that $g(2) \geq 4$. One can check that with 23 we have $g(3) \geq 9$ and with 79 we have $g(4) \geq 19$.

Remark. Lagrange's four-square theorem proves that exactly $g(2) = 4$.

3.2. Circle Method

The Hardy-Ramanujan-Littlewood Circle Method is a technique in additive number theory. Our goal is to transform additive and combinatorics problems into complex analysis problems to use the tools of analysis. [1] Quoting Hardy and Ramanujan's original paper, "This idea [studying integrals from generating functions] has dominated nine-tenths of modern research in analytic theory of numbers." [2]

From a high level, the method uses the Residue Theorem[14] to represent the coefficients of a generating function series as integrals around closed circular paths inside the unit circle. So, given a generating function $f(z) = \sum a_n (z - z_0)^n$, we have

$$a_n = \frac{1}{2\pi i} \oint_{\partial D(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

where f is holomorphic in an open set Ω with a disk D centered at z_0 such that $\overline{D} \subset \Omega$.

[11] The conjecture has been shown to hold for all integers less than $4 \cdot 10^{18}$ as of 2025 according to Wikipedia.

[12] The Weak Goldbach Conjecture posits that every odd number greater than 5 can be expressed as the sum of three primes, where a prime may be used more than once in the same sum. This conjecture was proven using the circle method by Harald Helfgott in 2013.

[13] Waring's problem is related to Fermat's polygonal number theorem.

[14] Particularly in the form of Theorem 4.4 in Stein & Shakarchi

For the Circle Method, we will consider generating functions centered at $z_0 = 0$ with $\Omega = \mathbb{D}$. We will split up the circular paths of integration into "major" and "minor" arcs, where we get the main, often integrable, terms from the major arcs, and bounded error terms arise from minor arcs.

3.2.1. Example Application to Weak Compositions

We will use the circle method to find the number of weak compositions of n into k parts.

Setup

Begin with the case where we have $k = 2$ parts,

$$r_2(n) = \#\{(a_1, a_2) \mid n = a_1 + a_2,\ a_1, a_2 \in A\}.$$

For each $a_i \in A$, we can construct the generating function f using the indicator function for A:

$$f(z) = \sum_{n=0}^{\infty} \mathbf{1}_A(n) z^n, \qquad \text{where } \mathbf{1}_A(n) = \begin{cases} 1 & n \in A, \\ 0 & n \notin A. \end{cases}$$

Now, for $k = 2$, using the Cauchy series product, we have

$$f^2(z) = \left(\sum_{n=0}^{\infty} \mathbf{1}_A(n) z^n\right)\left(\sum_{m=0}^{\infty} \mathbf{1}_A(m) z^m\right) = \sum_{n=0}^{\infty} c(n) z^n$$

where $c(n) = \sum_{k=0}^{n} \mathbf{1}_A(k)\mathbf{1}_A(n-k)$, which we can rewrite as $\sum_{h+k=n} \mathbf{1}_A(h)\mathbf{1}_A(k)$.

Since $\mathbf{1}_A(h)\mathbf{1}_A(k) = 1$ iff both $h, k \in A$, then this expression of $c(n)$ is exactly the number of pairs of $(h, k) \in A^2$ which satisfy $h + k = n$.

Therefore, we have $f^2(x) = \sum_{n=0}^{\infty} r_2(n) z^n$.

Now, consider a composition of n into k parts with summands in A,

$$r_k(n) = \#\{(a_1, ..., a_k) \in A^k \mid n = a_1 + \cdots + a_k,\ a_i \in A\}.$$

After repeated applications of Cauchy product, we arrive at the generating function

$$f^k(z) = \sum_{n=0}^{\infty} r_k(n) z^n.$$

Since f^k is analytic, then with Residue Theorem applied to series coefficients, we have

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f^k(z)}{z^{n+1}}\, dz \tag{1}$$

for a circular closed loop C_ρ of radius ρ centered at the origin.

Now, we can express the number of solutions $r_k(n)$ in terms of the residues of this integral.

From Example 2 above with $A = \mathbb{N}$, we already know that $r_k(n) = \binom{n+k-1}{k-1}$. We will verify this using the Circle method.

Method Application

Since $A = \mathbb{N}$, We have

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

which converges for $|z| < 1$. So, by equation (1), we have

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{dz}{(1-z)^k z^{n+1}},$$

where the integral converges on the closed circular loop C_ρ of radius $\rho < 1$. Since we can evaluate this integral directly, then we can consider the entirety of C_ρ as our major arc, and we do not need to estimate any minor arcs.

Now, we will extract the nonzero residues, which come from the simple poles of the integral. Using the General Binomial Expansion, we have

$$\frac{1}{(1-z)^k} = \sum_{m=0}^{\infty} \binom{-k}{m} (-z)^m.$$

Hence, the integral becomes

$$\oint_{C_\rho} \sum_{m=0}^{\infty} \binom{-k}{m} (-z)^m z^{-(n+1)}\, dz.$$

So, for the simple poles, we will take the terms for which $m - n - 1 = -1 \implies m = n$.

Thus,

$$r_k(n) = \frac{1}{2\pi i} \oint_{C_\rho} \binom{-k}{n} (-1)^n z^{-1}\, dz = (-1)^n \binom{-k}{n} = \binom{n+k-1}{k-1},$$

where the last equality is given by a combinatorial identity.[15]

3.3. Hardy-Ramanujan Estimation Formula

Theorem (*Hardy-Ramanujan*). Let $p(n)$ be the number of partitions of n.

$$p(n) \sim \frac{\exp \pi \sqrt{2n/3}}{4\sqrt{3}\, n}.$$

Sketch of Hardy-Ramanujan. We will follow a simplified version of the original proof which performs analysis on an asymptotically similar function which is analytic inside the unit circle. [3] The simplified proof uses merely one major arc and more elementary estimates as opposed to modular forms and theta functions.

The proof follows by careful analysis of the generating function given by the Partitions Generation Function Theorem,

$$f(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} \frac{1}{1-z^m}, \quad \text{for } |z| < 1 \text{ where } p(0) = 1,$$

[15] This identity can be quickly shown by expressing both binomial coefficients in terms of falling factorials (Pochhammer symbols) and counting the appearances of -1.

around the singularity $z = 1$. Note that f has poles at the roots of unity $z^n = 1$.

The lower order roots will correspond to the more dominant poles, that is the pole at $z = 1$ as the solution to $z - 1$, followed by the pole at $z = -1$ for the additional solution to $z^2 - 1$, and then the two additional cube roots of unity from $z^3 - 1$, and so on. These poles correspond to black regions inside the unit circle in the following figure. Note that while the figure is colored for the reciprocal of f, we can still observe the relative impact of the poles.

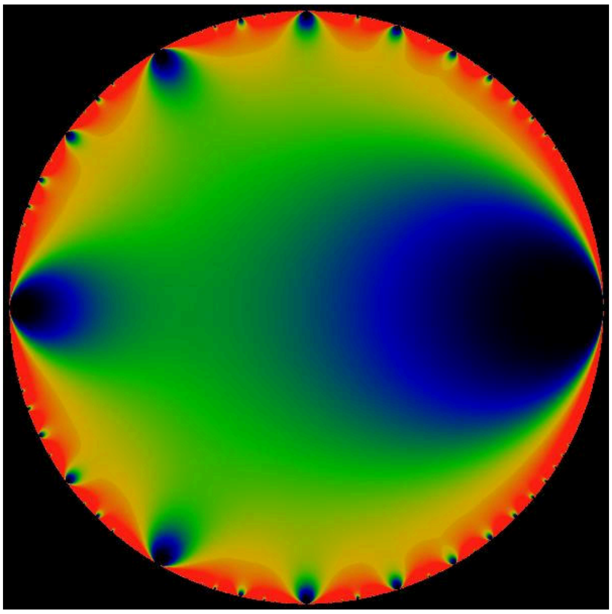

Figure 7: Modulus of Euler's q-series for integer partitions $\prod_{k=1}^{\infty}(1 - q^k)$ on the unit disk in the complex plane.[16]

We define a simpler auxiliary function which captures the asymptotic behavior of f:

$$\varphi(z) = \sum q(n) z^n = \left(\frac{1-z}{2\pi}\right)^{1/2} \exp \frac{\pi^2}{12}\left(\frac{2}{1+z} - 1\right).$$

We will show $p(n) \approx q(n)$ and compute the latter directly. We approximate f by φ with

$$f(z) = \varphi(z)(1 + O(1-z))$$

for $|z| < 1$ near 1, in particular $|1 - z| \leq 2(1 - |z|)$. This is proven using variation.

We can show the bound

$$|f(z)| < \exp\left(\frac{1}{1-|z|} + \frac{1}{|1-z|}\right), \tag{2}$$

using the logarithm on the product definition of f.

We will then employ the Circle Method on the following integral:

$$p(n) - q(n) = \frac{1}{2\pi i} \oint_C \frac{f(z) - \varphi(z)}{z^{n+1}}\, dz.$$

[16] https://commons.wikimedia.org/wiki/File:Q-Eulero.jpeg

We choose the circular path of integration C with radius $1-\pi/\sqrt{6n}$, which approaches 1 as $n\to\infty$. Since this path lies inside the unit circle, then $|z|<1$ and the generating function f will converge.

We split the circle into two parts:
- the major arc $A=\left\{z\in C \mid |1-z|<\pi\sqrt{2/3n}\right\}$, which is the part of the circle C near the dominant singularity of f at $z=1$, and
- the minor arc $B=C-A$, which is the rest of circle away form $z=1$.

Asymptotically, the contribution of the minor arc will vanish exponentially fast, so we will be left with an approximation given by the integral around just the major arc.

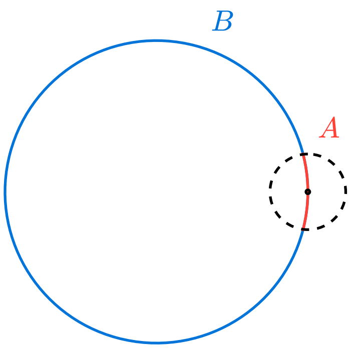

Figure 8: Major A and minor B arcs of C.

For the minor arc B, using the exponential bound on f from equation 2, we have the following estimate of asymptotic equivalence:

$$\begin{aligned}\int_B \frac{f(z)-\varphi(z)}{z^{n+1}}\,dz &\sim \int_B |z|^{-n}\left(\exp\frac{\pi^2}{6|1-z|}+\exp\left(\frac{1}{1-|z|}+\frac{1}{|1-z|}\right)\right)dz\\ &\sim \int_B \exp\pi\sqrt{n/6}\left(\exp\frac{\pi}{6}\sqrt{3n/2}+\exp\frac{1}{\pi}\left(\sqrt{3n/2}+\sqrt{6n}\right)\right)dz\\ &= O\left(\exp a\sqrt{n}\right)\text{ where } a<\pi\sqrt{2/3}.\end{aligned}$$

For the major arc, with the length of A as $O\left(n^{-1/2}\right)$,

$$\begin{aligned}\int_A \frac{f(z)-\varphi(z)}{z^{n+1}}\,dz &\sim \int_A |z|^{-n}\ |1-z|^{3/2}\exp\frac{\pi}{6(1-|z|)}\,dz\\ &\sim n^{-3/4}\exp\left(\pi\sqrt{n/6}+\pi\sqrt{n/6}\right)n^{-1/2}\\ &= O\left(n^{-5/4}\exp\pi\sqrt{2n/3}\right).\end{aligned}$$

Therefore

$$p(n)=q(n)+O\left(n^{-5/4}\exp\pi\sqrt{2n/3}\right).$$

Since this error is much smaller than the main asymptotic term of q, then we have $p\sim q$.

Using the method of steepest descent, we have

$$\pi\sqrt{2}\exp\frac{\pi^2}{12}\varphi(z)=(1-z)\int_{\mathbb{R}}\exp\left(\pi t\sqrt{2/3}-(1-z)t^2\right)dt$$

The remainder of the proof is (even more so) heavily abridged. Comparing the power series in z on both sides, we obtain

$$\pi\sqrt{2}\exp(\pi^2/12)q(n) = \int_{\mathbb{R}} \exp\left(\pi t\sqrt{2/3} - t^2\right)\left(\frac{t^{2n}}{n!} - \frac{t^{2n-2}}{(n-1)!}\right) dt$$
$$\sim \frac{\exp \pi\sqrt{2/3n}}{\sqrt{2\pi n}} \int_{\mathbb{R}} s\exp\left(\pi\sqrt{2/3}s - s^2 - 2\sqrt{n}s\right)\left(1 + \frac{s}{\sqrt{n}}\right)^{2n-2}\left(2 + \frac{s}{\sqrt{n}}\right) ds$$

using the substitution $t = s + \sqrt{n}$ and Stirling's formula for $n!$.

Then,

$$\lim_{n\to\infty} e^{-2\sqrt{n}s}\left(1 + \frac{s}{\sqrt{n}}\right)^{2n-2}\left(2 + \frac{s}{\sqrt{n}}\right) = 2e^{-s^2}.$$

The integral is dominated by a function F whose integral converges over the reals, which enables the use of the Dominated Converge Theorem to take the above limit under the integral,

$$\pi\sqrt{2}\exp(\pi^2/12)q(n) \sim \frac{\exp \pi\sqrt{2/3n}}{\sqrt{2\pi n}} \int_{\mathbb{R}} 2s\exp\left(\pi\sqrt{2/3}s - s^2\right) ds.$$
$$= \frac{\pi}{2\sqrt{6}n}\exp\left(\pi^2/12 + \pi\sqrt{2n/3}\right)$$
$$q(n) \sim \frac{\exp \pi\sqrt{2n/3}}{4\sqrt{3}n},$$

which in turn yields the desired result for p. □

Remark. The full proof relies on several more advanced analysis concepts:
- Lebesgue's Dominated Convergence Theorem, which allows for the interchange of limits and integrals;
- the method of steepest descent, which enables us to analyze complex functions by evaluating integrals instead;
- as well as total variation of functions, which is useful in bounding terms.

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