Math 336 Homework 4

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Problem 1. Prove that for $u \notin \mathbb{Z}$,

$$\sum_{-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}.$$

by integrating $f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$ on the circle $|z| = R_N = N + \frac{1}{2}$ for $N \in \mathbb{Z}$ and $N \geq |w|$, and adding residues of f on the inside of the circle C_{R_N}, letting $N \to \infty$.

Proof. The simple poles of f occur at $z \in [-N, N] \subset \mathbb{Z}$ and there is a second order pole at $z = -u$.

We have that

$$\oint_{C_{R_N}} f\, dz = 2\pi i \left(\sum_{-N}^{N} \mathrm{res}_k f + \mathrm{res}_{-u} f \right).$$

For the integer residues $[-N, N]$,

$$\begin{aligned}
\mathrm{res}_k f &= \lim_{z \to k} (z-k) \frac{\pi \cos \pi z}{(u+z)^2 \sin \pi z} \\
&\overset{\text{LH}}{=} \lim_{z \to k} \frac{\pi \cos \pi k - (z-k)\pi^2 \sin \pi z}{2(u+z)\sin \pi z + (u+z)^2 \pi \cos \pi z} \\
&= \frac{\pi(-1)^k}{(u+z)^2 \pi (-1)^k} \\
&= \frac{1}{(u+k)^2}.
\end{aligned}$$

For the second order pole,

$$\begin{aligned}
\mathrm{res}_{-u} f &= \lim_{z \to -u} \frac{d}{dz} \left((z+u)^2 \frac{\pi \cot \pi z}{(u+z)^2} \right) \\
&= \lim_{z \to -u} \frac{d}{dz} (\pi \cot \pi z) \\
&= \lim_{z \to -u} -\pi^2 \csc^2 \pi z \\
&= -\frac{\pi^2}{(\sin \pi u)^2},
\end{aligned}$$

by the oddness of sine.

So, we have

$$\oint_{C_{R_N}} f\, dz = 2\pi i \left(\sum_{-N}^{N} \frac{1}{(u+n)^2} - \frac{\pi^2}{(\sin \pi u)^2} \right).$$

We will show that the contour integral vanishes as $N \to \infty$. We begin by splitting the circle contour into parts and estimating $\cot \pi z$ on each part. We will write $z = x + iy$.

For the first part, we will consider the pieces of the circle with a modulus of real part between N and $N + 1$. Since $\cot \pi z$ has a period of 1, with singularities at 0 and 1 but is bounded between, then it is also bounded when $|\mathrm{Re}(z)| = |x|$ is on the open interval $(N, N + 1)$.

For the second part, we will consider the pieces of the contour with a modulus of imaginary part greater than the value of the contour for which the real part is above N is magnitude. Since the contour is a circle, then we can find the height of the contour that is achieved prior to $|\mathrm{Re}(z)| = |x| = N$. So,

$$\begin{aligned} x^2 + y^2 &= R_N^2 \\ y &= \sqrt{\left(N + \frac{1}{2}\right)^2 - N^2} \\ &= \sqrt{N + \frac{1}{4}} \approx \sqrt{N}. \end{aligned}$$

Next, we will show that $\cot \pi z$ is bounded for $|\mathrm{Im}(z)| = |y| \geq \sqrt{N}$.

We begin with the identifying cotangent with exponential functions using Euler's formula,

$$\cot \pi z = i\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1}.$$

So, as $|y| = \sqrt{N} \to \infty$, we have

$$i\frac{e^{2\pi i(x+iy)} + 1}{e^{2\pi i(x+iy)} - 1} = i\frac{e^{-2\pi y}e^{2\pi i x} + 1}{e^{-2\pi y}e^{2\pi i x} - 1} \to -i.$$

Hence, cotangent is bounded on the two parts of the contour, while the denominator of the integrand grows without bound.

So, we have

$$\int\limits_{|\mathrm{Re}(z)|\, > N} \frac{\pi \cot \pi z}{(u+z)^2}\, dz \leq \left|\frac{C}{N^2}\right| \to 0$$

$$\int\limits_{|\mathrm{Im}(z)|\, > \sqrt{N}} \frac{\pi \cot \pi z}{(u+z)^2}\, dz \leq \left|\frac{1}{N^2}\right| \to 0$$

Therefore the contour integral vanishes as $N \to \infty$. Thus we are left with,

$$\lim_{N\to\infty} \sum_{-N}^{N} \frac{1}{(u+k)^2} = \frac{\pi^2}{(\sin \pi u)^2},$$

as desired. $\square$

Problem 2. Prove that all entire and injective functions are linear.

Proof. Let f be an entire injective function.

We will consider the behavior of f at infinity using $g(z) = f(1/z)$, which is holomorphic on the punctured plane $\mathbb{C} \setminus \{0\}$ and has a singularity at $z = 0$.

The singularity of g at $z = 0$ must be one of three types: removable, a pole, or essential.

If $z = 0$ is an essential singularity, then by Casorati-Weierstrass, the image of g is locally dense in $\mathbb{C}$ in a deleted neighborhood of zero. Consider a region $\Omega \subset D_r$ such that $0 \notin \Omega$. Since the inversion map is holomorphic and nonconstant on Ω, then we have an open mapping from Ω to Ω' outside $\mathbb{D}$.

But then $f(z) = g(1/z)$ would be locally dense in Ω', contradicting the injectivity of f.

Hence $z = 0$ is not an essential singularity of g.

If $z = 0$ is a removable singularity, then g can be extended to a holomorphic entire function on $\hat{\mathbb{C}}$ with no poles, which makes g a constant function on $\mathbb{C}$, contradicting injectivity.

So $z = 0$ is not a removable singularity of g.

Therefore, $z = 0$ is a pole of g.

Since f is injective, then it can have at most one root with a multiplicity of one. So, by FTA, f must be linear.

Since $f(z)$ is linear and $g(z) = f(1/z)$ has a pole at $z = 0$, we must have that $f(z) = az + b$, for some $a \neq 0$. $\square$

Problem 3. Suppose f and g are holomorphic in a region containing the closed unit disk $\overline{\mathbb{D}}$. Suppose f has a simple zero at $z = 0$, and vanishes nowhere else in the $\overline{\mathbb{D}}$. Let

$$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

Show that for ε sufficiently small, then

(a) $f_\varepsilon(z)$ has a unique zero z_ε in D, and

(b) the map $\varepsilon \mapsto z_\varepsilon$ is continuous.

Proof of (a). Since f and g are holomorphic on the $\overline{\mathbb{D}}$, which is compact, then they are continuous and bounded there as well.

Since $|f| > 0$ on the $\partial\mathbb{D}$, then, for all z on the $\partial\mathbb{D}$ there exists an ε such that $|f(z)| > \varepsilon|g(z)| \geq 0$. So, by Rouché, f and $f_\varepsilon = f + \varepsilon g$ have the same number of zeros in $\mathbb{D}$.

Since f has one zero in $\mathbb{D}$, then so does f_ε, call it z_ε. $\square$

Proof of (b). Consider the argument principle for f_ε, which has one zero and no poles in $\mathbb{D}$,

$$1 = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{f_\varepsilon'(z)}{f_\varepsilon(z)} \, dz.$$

Since f has a single simple zero inside $\mathbb{D}$, then, counting the multiplicity of 1, f_ε must have a simple pole at z_ε. So, we can write $f_\varepsilon(z) = (z - z_\varepsilon)p(z)$ for some holomorphic function p nonvanishing on D.

Consider the following function,

$$z\frac{f_\varepsilon'(z)}{f_\varepsilon(z)} = z\frac{p(z) + (z - z_\varepsilon)p'(z)}{(z - z_\varepsilon p(z))} = \frac{z}{z - z_\varepsilon} + z\frac{p'(z)}{p(z)}$$

Now, if we take the contour integral on the unit circle $\partial\mathbb{D}$ of the above, then the left hand term will yield $2\pi i z_\varepsilon$ by the residue theorem.

But, since $p(z)$ was nonvanishing in the unit disk $\overline{\mathbb{D}}$, then the integrand is holomorphic, and so the contour integral will vanish.

Therefore, we have that

$$z_\varepsilon = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{z f_\varepsilon'(z)}{f_\varepsilon(z)}\,dz.$$

Since f_ε is nonvanishing on the unit circle, holomorphic, and continuous in ε, then the integrand is continuous in ε. Hence z_ε is continuous in ε as well. $\square$

Problem 4. Let f be nonconstant and holomorphic on an open set Ω containing the closed unit disk.

(a) Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains $\mathbb{D}$.

(b) Show that if $|f(z)| \leq 1$ whenever $|z| = 1$ and there exists $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains D.

Proof of (a). By MMP, if f is holomorphic on $\mathbb{D}$ with $|f| = 1$ on $\partial\mathbb{D}$, then, for all z in $\mathbb{D}$, $|f| < 1$.

Suppose, for a contradiction, that f has no zeros in $\mathbb{D}$, then $1/f$ is holomorphic.

So, applying MMP again, on $\partial\mathbb{D}$, we have that $|1/f| = 1$, so $|1/f| < 1 \Rightarrow |f| > 1$ on the interior of D, which is a contradiction.

Therefore, f must have at least one zero in $\mathbb{D}$.

Then, for all $|w_0| < 1$, we have that $|f(z)| = 1 > |w_0|$ on the unit circle $\partial\mathbb{D}$.

So, by Rouché, $f(z)$ and $f(z) - w_0$ both have the same number of zeros in $\mathbb{D}$.

Therefore there exists a z_0 such that $f(z_0) = w_0$.

Since w_0 was arbitrary inside $\mathbb{D}$, then we have that the image of f contains $\mathbb{D}$. $\square$

Proof of (b). Proceeding as in part (a), we will show that f still has a zero in $\mathbb{D}$.

Since $\overline{\mathbb{D}}$ is compact, then f attains a minimum there, say at z_0 as given above.

If $|f(z_0)| = 0 < 1$, then we have a zero.

Otherwise, if $|f(z)| > 0$ for all $z \in \mathbb{D}$, then $1/f$ is holomorphic on $\mathbb{D}$, and attains a max at z_0 inside $\mathbb{D}$, which contradicts MMP.

Hence, f must have a zero inside $\mathbb{D}$, and we can finish the proof as in part (a) using Rouché with the fact that $|f(z)| \geq 1 > |w_0|$ on ∂D. $\square$

Problem 5. Prove for f holomorphic in an annulus $A = \{z \mid r_1 \leq |z - z_0| \leq r_2\}$, with $0 < r_1 < r_2$, then

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n,$$

where the series converges absolutely on the interior of the annulus.

Proof. Consider the two contour integrals

$$f(z) = \frac{1}{2\pi i} \left[\oint\limits_{C_{r_2}} \frac{f(w)}{w - z}\, dw - \oint\limits_{C_{r_1}} \frac{f(w)}{w - z}\, dw \right]$$

where the circles C_{r_2} and C_{r_1} bound the annulus A with $z \in A$.

For the outer circle contour integral on C_{r_2}, we have

$$\frac{1}{w - z} = \frac{1}{z - z_0 - (z - z_0)} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}}.$$

Note that, for w on C_{r_2}, we have that $|w - z_0| > |z - z_0|$, hence $\left| \frac{z - z_0}{w - z_0} \right| < 1$.

So, with the geometric series, the above becomes

$$\frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n,$$

which converges uniformly for all $w \in C_{r_2}$.

For the inner circle contour integral on C_{r_1}, we have

$$\frac{1}{w - z} = -\frac{1}{z - w} = \frac{1}{(z - z_0) - (w - z_0)} = -\frac{1}{z - z_0} \frac{1}{1 - \frac{w - z_0}{z - z_0}},$$

because $|w - z_0| < |z - z_0|$ on C_{r_1}.

Then, with the geometric series, we have,

$$-\frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^n,$$

which converges for all $w \in C_{r_1}$.

Substituting back into the integrals, we have

$$f(z) = \frac{1}{2\pi i} \left[\oint\limits_{C_{r_2}} \frac{f(w)}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n dw - \oint\limits_{C_{r_1}} \frac{f(w)}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^n dw \right].$$

Since both integrals converge uniformly, then we can interchange the sums and integrals. So, we have

$$f(z) = \frac{1}{2\pi i}\left[\sum_{n=0}^{\infty}\left(\oint_{C_{r_2}} \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)(z-z_0)^n - \sum_{n=0}^{\infty}\left(\oint_{C_{r_1}} f(w)(w-z_0)^n\,dw\right)(z-z_0)^{-(n+1)}\right].$$

Each integral can be evaluated with CIF for the nonnegative and negative coefficients respectively.

Note that the right hand series begins at a pole of order 1, and has no constant term. Thus we have

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n,$$

which is what we wanted to show. □

Problem 6. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a continuous map where $\hat{\mathbb{C}}$ is the Reimann sphere.

We say that f is holomorphic at a if either:

- $a \neq \infty, f(a) \neq \infty$, and f is holomorphic at a in the usual sense;
- $a \neq \infty, f(a) = \infty$, and $z \mapsto 1/(f(z))$ is holomorphic at a;
- $a = \infty, f(a) \neq \infty$, and $z \mapsto f(1/z)$ is holomorphic at 0;
- $a = \infty, f(a) = \infty$, and $z \mapsto 1/f(1/z)$ is holomorphic at 0.

We say that f is biholomorphic if it is holomorphic and has a holomorphic inverse.

(a) Check that nonconstant holomorphic functions $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ may be identified with nonconstant meromorphic functions $\hat{\mathbb{C}} \to \mathbb{C}$ and are hence rational functions.

(b) Prove that any biholomorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has the form $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$, with $ad - bc \neq 0$.

Proof of (a). Let f be a nonconstant holomorphic function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

We will proceed by cases.

If $f(\infty) \neq \infty$, then, if f holomorphic is in the usual sense, then it is also meromorphic. If instead we have that $f(1/z)$ is holomorphic at zero, then it has a finite analytic expansion in $1/z$ which yields a finite number of poles; hence $f(1/z)$ is meromorphic.

If $f(\infty) = \infty$, then, if f is a meromorphic function, then $1/f(z)$ is also meromorphic. Lastly, if $1/f(1/z)$ is holomorphic at 0, consider $g(z) = \frac{1}{f(z)}$. By the first case, g is meromorphic, so the inverse of $f = \frac{1}{g}$ is also meromorphic. □

Proof of (b). Since f is holomorphic on $\hat{\mathbb{C}}$, then, by part (a), f is rational.

Since f is biholomorphic, then f is bijective and f^{-1} is holomorphic on $\hat{\mathbb{C}}$ and hence rational as well.

We have that a poly in $\mathbb{C}$ is injective iff it is linear, hence f and f^{-1} must be a ratio of linear polynomials. So we have that $f(z) = \frac{az+b}{cz+d}$.

But, f is nonconstant, so the numerator $az + b$ must not be proportional to the denominator $cz + d$.

Suppose, for a contradiction, that $az + b = \lambda(cz + d)$. Then we would have $\binom{a}{b} = \lambda\binom{c}{d}$, meaning that the vectors of coefficients are linearly dependent. This gives us

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \Longrightarrow ad - bc = 0.$$

Thus, we must have $ad - bc \neq 0$ for f nonconstant.

We will now verify that f is invertible and f^{-1} is holomorphic on $\hat{\mathbb{C}}$.

Consider,

$$\begin{aligned} w &= \frac{az + b}{cz + d} \\ cwz + dw &= az + b \\ (cw - a)z &= b - dw \\ z &= \frac{b - dw}{cw - a} \\ &= \frac{dw - b}{-cw + a}, \end{aligned}$$

which implies that $f^{-1}(z) = \frac{dz-b}{-cz+a}$.

Then, we have that $da - (-b)(-c) \neq 0$, so f^{-1} is nonconstant and rational, and hence holomorphic on $\hat{\mathbb{C}}$.

Thus, all biholomorphic functions f take the form $\frac{az+b}{cz+d}$ where $ad - bc \neq 0$. $\square$