Math 462 Homework 3

a lipson

April 25, 2025

Problem 1. Draw n points on a circle labeled $1, ..., n$. Draw zero or more line segments between points so that no two line segments intersect.

Let D be the set of all such drawings over all n, where the weight of a drawing is n, the point of points on the circle.

(a) Find a closed form for the generating function $F_D(x)$ of D.

(b) A full drawing is a drawing where every point on the circle is on a line segment. Prove that, if n is even, then the number of full drawings of weight n is $c_{n/2}$, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

(c) Let d_n be the number of drawings (not necessarily full) of we weight n. Prove that

$$d_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k.$$

Proof of (a). We will consider a circle of $n+1$ points. The $(n+1)$-th points fits in exactly one of the following two cases:

- it is not connected to any other point in the circle, or
- it is connected to exactly one other point in the circle, call this point i.

In the second case, then the segment between $n+1$ and i bisects the circle into two circles with $i-1$ and $n-i$ points respectively.

So, we can construct the following recurrence relationship:

$$d_{n+1} = d_n + \sum_{i=1}^{n} d_{i-1} d_{n-i},$$

where we consider the cases where the point $n+1$ is be connected to each one of the other points 1 through n in the circle.

With the relation, we can construct the generation function $F_D(x)$,

$$
\begin{aligned}
\sum_{0}^{\infty} d_{n+1}x^n &= \sum_{0}^{\infty} d_n x^n + \sum_{0}^{\infty}\sum_{i=1}^{n} d_{i-1}d_{n-i}x^n \\
\sum_{0}^{\infty} d_{n+1}x^n &= \sum_{0}^{\infty} d_n x^n + x\sum_{0}^{\infty}\sum_{i=0}^{n-1} d_i d_{(n-1)-i}x^{n-1} \\
\sum_{0}^{\infty} d_{n+1}x^n &= \sum_{0}^{\infty} d_n x^n + x\left(\sum_{n=0}^{\infty} d_n x^n\right)^2 \\
\frac{1}{x}(F_D(x) - d_0) &= F_D(x) + x(F_D(x))^2 \\
F_D(x) - 1 &= xF_D(x) + x^2F_D(x)^2 \\
0 &= 1 + (x-1)F_D(x) + x^2F_D(x)^2 \\
F_D(x) &= \frac{1 - x \pm \sqrt{(x-1)^2 - 4x^2}}{2x^2} \\
F_D(x) &= \frac{1 - x \pm \sqrt{1 - 2x - 3x^2}}{2x^2},
\end{aligned}
$$

where the reduction from the double sum is given by the Cauchy product for series.

Since we have that $1 = d_0 = F(0)$, then we can consider $\lim_{x\to 0} F(x)$ to determine the proper branch of the above function. Applying L'Hôpital twice in the limit yields

$$F_D(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

$\square$

Proof of (b). Let f_n be the number of full drawings with n points and let F be the set of all full drawings, weighted by number of points.

Then, the elements of F belong to one of the following two cases:
- an empty drawing f_0 with weight zero, or
- a drawing with a segment chord that splits the circle into two more full drawings.

So, we have a weight preserving bijection given by

$$F \leftrightarrow \{\bigcirc\} \sqcup \left\{\ominus\right\} \times F \times F,$$

where circle the circle with zero points has weight zero and the circle with two points has weight two.

This gives,

$$F_F(x) = 1 + x^2F_F(x)^2 = \frac{1 \pm \sqrt{1 - 4x^2}}{2x^2}.$$

As before, with $f_0 = 1 \Rightarrow F(0) = 1$, we can consider the limit as $x \to 0$ to see that

$$F_F(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x^2}.$$

By the general binomial theorem, we have that

$$\left(1-4x^2\right)^{1/2} = \sum_0^\infty \binom{\frac{1}{2}}{n}\left(-4x^2\right)^n = 1 + \sum_1^\infty (-1)^{n-1}\frac{2}{4^n n}\binom{2n-2}{n-1}x^{2n}.$$

So,

$$\begin{aligned} F &= \frac{1-\left(1-4x^2\right)^{1/2}}{2x^2} \\ &= -\frac{1}{2x^2}\sum_1^\infty (-1)^{n-1}\frac{2}{4^n n}\binom{2n-2}{n-1}(-4)^n x^{2n} \\ &= \sum_1^\infty \binom{2(n-1)}{n-1}\frac{x^{2(n-1)}}{n} \\ &= \sum_0^\infty \binom{2n}{n}\frac{x^{2n}}{n+1} \\ &= \sum_0^\infty \binom{n}{n/2}\frac{x^n}{n/2+1}, \end{aligned}$$

which implies that

$$f_n = \binom{n}{n/2}\frac{1}{n/2+1} = c_{n/2}.$$

□

Proof of (c). Consider the provided definition of d_n, we have that k is the number of segments in the circle.

Then, we must use $2k$ points to create the k segments.

By part (b), we have that there are $c_{2k/k} = c_k$ ways to create k non-intersecting segments on a circle with $2k$ points.

For any circle with n points, we can have up to $\lfloor n/2 \rfloor$ segments, because we need two points per segment.

For each of the $k \in [0, \lfloor n/2 \rfloor]$ segments, we can pick $2k$ points of n total points as segment endpoint; there are $\binom{n}{2k}$ ways to do this.

Hence, considering all possible number of segments, we have

$$d_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k,$$

as desired □

Problem 2. Let $k \in \mathbb{Z}^+$. A k-ary tree is defined recursively as follows:
- The empty set a k-ary treee.
- If $T_1, ..., T_k$ are k-ary trees, then we can form a k-ary tree by drawing a root vertex r, drawing $T_1, ..., T_k$ below r in that order, and rdawing an edge from r to each of the roots of $T_1, ..., T_k$.

Prove that the number of k-ary trees with n vertices is

$$\frac{1}{(k-1)n+1}\binom{kn}{n}.$$

Proof. By the definition of k-ary trees, we have the following generating function F defined by $F = 1 + xF^k$.

Let $H = F - 1 = xF^k$. Note that $H(0) = 0$ and $H'(0) \neq 0$.

Then, for the generating function $G = \sum g_n x^n$, using the Lagrange inversion formula with H, we have that

$$g_n = \frac{1}{n}[x^{n-1}]x^n\left(xF^k\right)^{-n} = \frac{1}{n}[x^{n-1}]F^{-kn}.$$

So, we must find the coefficient of x^{n-1} is the power series F^{-kn}.

Using the generalized binomial expansion, we have that,

$$\begin{aligned} F^{-kn} &= (H+1)^{-kn} \\ &= \sum_{j=0}^{\infty}\binom{-kn}{j}H^j \\ &= \sum_{j=0}^{\infty}(-1)^j\binom{kn+j-1}{j}H^j. \end{aligned}$$

So, consider $j = n - 1$, which gives

$$(-1)^{n-1}\binom{kn+n-2}{n-1}[x^{n-1}]H^{n-1}.$$

Since $H = xF^k$ and $F(0) = 1$, then the coefficient of x^{n-1} is H^{n-1} is one.

Therefore,

$$g_n = \frac{1}{n}(-1)^{n-1}\binom{kn+n-2}{n-1}.$$

Thus, the number of k-ary trees with n vertices is

$$\frac{1}{(k-1)n+1}\binom{kn}{n}.$$

□