Math 462 Homework 8

a lipson

June 06, 2025

Problem 1. Let G be a simple graph, and let v be a vertex of G. Let G' be the graph obtained from G by adding a new vertex v' and drawing an edge from v' to all the neighbors of v. Without using the strong perfect graph theorem, prove that G is perfect iff G' is perfect.

Proof. Recall that a graph G is perfect iff for all induced subgraphs H of G, $\chi(H) = \omega(H)$.

($\Longrightarrow$) Assume G is perfect. Let H' be any induced subgraph of G'. We will consider the following three cases for H':
1. $v' \notin H'$,
2. $v' \in H', v \notin H'$, and
3. $v', v \in H'$.

First, if H' does not contain v', then H' is also an induced subgraph of G which is perfect, so $\chi(H') = \omega(H')$.

Second, where H' contains v' but on v, let $H = H' - v' + v$. Since v and v' share the same neighborhood in G' but are not adjacent, then any clique in H' not containing v corresponds to the same clique in H. Also, any clique in H' containing v' corresponds to a clique in H where v' is replaced by v. Therefore $\omega(H') = \omega(H)$.

For coloring, any proper coloring of H corresponds to a proper coloring of H' by giving v the same color as v'. Hence $\chi(H') = \chi(H)$.

Since G is perfect, then

$$\chi(H') = \chi(H) = \omega(H) = \omega(H'),$$

so G' is perfect.

Finally, where both v' and v are in H, we have that v and v' have an identical neighborhood $N(v)$ in G'. So, removing either v or v' gives the same clique number, that is

$$\omega(H') = \omega(H' - v') = \omega(H' - v).$$

For coloring, we have that $\chi(H')$ must be the maximum chromatic number of $H' - v$ and $H' - v'$, but since v and v' can take on the same color, then we have

$$\chi(H') = \chi(H' - v') = \chi(H' - v).$$

Since both $H' - v'$ and $H' - v$ are induced subgraphs of G, then by the perfectness of G, we must have

$$\chi(H' - v') = \omega(H' - v') \text{ and } \chi(H' - v) = \omega(H' - v).$$

Since the above two equalities are themselves equal, then we must have $\chi(H') = \omega(H')$, which implies that G' is perfect.

($\Longleftarrow$) Assume G' is perfect. Let H be any induced subgraph of G. Since G is a subgraph of G', then H is also an induced subgraph of G'. Since G' is perfect, then $\chi(H) = \omega(H)$, so G is perfect. □

Problem 2. Let P be a poset such that the maximum size of an antichain is a and the maximum size of a chain is c. Let a' be the maximum size of an antichain of $P \times P$. Determine, as a function of a and c, the maximum possible value of a'.

Proof. First, consider the maximal case where $c = 1$, then P itself must be an antichain. So, all elements in $P \times P$ must be mutually incomparable. Therefore, $a' = |P \times P| = a^2$.

Next, consider the minimal case where $a = 1$, then P is a chain. So, all elements in $P \times P$ are comparable. WLOG let $P = [n]$ with partial ordering $\leq$. Then, we can construct a rectangular diagram with $P \times P$ which has a symmetric chain decomposition with an antichain along the diagonal, that is

$$(1, c), (2, c - 1), ..., (c - 1, 2), (c, 1),$$

which is an antichain of length c. Therefore, $a' = ac = c$.

Now, consider the case with $a, c > 1$. By Dilworth, we can partition P into a disjoint chains, call them $C_1 \sqcup \cdots \sqcup C_a = P$, where C_i has at most length c.

If all chains have exactly length c. Then, since antichains have size a, we have $a' = ac$.

Then, define the antichain along the antidiagonal similar to the construction in the minimal case,

$$A = \left\{\left(c_{i,j}, c_{i,c+1-j}\right) : 1 \leq i \leq a, 1 \leq j \leq c\right\},$$

which has size $|A| = ac$.

Thus, $a' = ac$. □