Math 336 Homework 3

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Problem 1. f is holo on the disk D_{R_0} centered at the origin with radius R_0.

(a) Prove, for $0 < R < R_0$, $|z| < R$,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \operatorname{Re} \left[\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z} \right] d\varphi$$

(b) Show

$$\operatorname{Re} \left[\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r} \right] = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}$$

Proof of (a).

By CIF, we have that for all $|z| < R$,

$$f(z) = \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{f(w)}{w - z} \, dz.$$

Let $\zeta = \frac{R^2}{\bar{z}}$. Then $|z| < R$, we have $|\zeta| > R$. Since f is holomorphic, then

$$\int\limits_{|w|=R} \frac{f(w)}{w - \zeta} \, dw = 0.$$

So, we will consider sum of the above with this vanishing integral,

$$f(z) = \frac{1}{2\pi i} \int\limits_{|w|=R} f(w) \left(\frac{1}{w - z} - \frac{1}{w - \zeta} \right) dw.$$

With $w = Re^{i\varphi}$ and $dw = iRe^{i\varphi} \, d\varphi = iw \, d\varphi$ we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) w \left(\frac{1}{w - z} - \frac{1}{w - \zeta} \right) d\varphi.$$

But, we have that

$$
\begin{aligned}
\operatorname{Re}\left[\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right] &= \operatorname{Re}\left[\frac{w+z}{w-z}\right] \\
&= \frac{1}{2}\left(\frac{w+z}{w-z}+\frac{\overline{w}+\overline{z}}{\overline{w}-\overline{z}}\right) \\
&= \frac{1}{2}\left(\frac{w+z}{w-z}+\frac{\frac{R^2}{\overline{w}}+\overline{z}}{\frac{R^2}{\overline{w}}-\overline{z}}\right) \\
&= \frac{1}{2}\left(\frac{w+z}{w-z}-\frac{\frac{R^2}{\overline{z}}+w}{w-\frac{R^2}{\overline{z}}}\right) \\
&= \frac{1}{2}\left(\left(-1+\frac{2w}{w-z}\right)-\left(-1+\frac{2w}{w-\frac{R^2}{\overline{z}}}\right)\right) \\
&= w\left(\frac{1}{w-z}-\frac{1}{w-\zeta}\right).
\end{aligned}
$$

Thus,

$$
f(z)=\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\operatorname{Re}\left[\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right]d\varphi,
$$

which is what we wanted to show.

□

Proof of (b). We have, for $r \in \mathbb{R}$,

$$
\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}=\frac{(Re^{i\gamma}+r)\overline{(Re^{i\gamma}-r)}}{|re^{i\gamma}-r\ |^2}.
$$

For the numerator,

$$
\begin{aligned}
(Re^{i\gamma}+r)(Re^{-i\gamma}-r) &= R^2rRe^{i\gamma}+rRe^{-i\gamma}-r^2 \\
&= R^2-r^2-rR(e^{i\gamma}-e^{-i\gamma}) \\
&= R^2-r^2-rR(2i\sin\gamma).
\end{aligned}
$$

For the denominator,

$$
\begin{aligned}
(Re^{i\gamma}-r)(Re^{-i\gamma}-r) &= R^2-rR(e^{i\gamma}-e^{-i\gamma})+r^2 \\
&= R^2-rR2\cos\gamma+r^2.
\end{aligned}
$$

Thus,

$$
\operatorname{Re}\left[\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right]=\frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}
$$

as desired.

□

Problem 2. f non-vanishing and continuous on $\overline{\mathbb{D}}$, holomorphic on $\mathbb{D}$. Prove that if $|f(z)|=1$ when $|z|=1$, then f is constant.

Proof. Since f is holomorphic on $\overline{\mathbb{D}}$, then $\oint_\gamma f = 0$ for all closed curves γ contained in $\overline{\mathbb{D}}$.

Let $g = 1/\overline{f(1/\overline{z})})$. We will show that g is holomorphic as well.

Since $|z| \leq 1 \Rightarrow |\ 1/\overline{z}\ | \geq 1$ are both continuous, then any closed contour C contained in $\mathbb{D}$ will map to a closed counter C' outside of $\mathbb{D}$.

Therefore, we have that

$$\oint_{C'} g = \oint_{C} f = 0.$$

If C' covers part of $\mathbb{D}$, we can split it into two contours along the boundary of $\mathbb{D}$, and adjust each contour by some small about ε is in Schwarz reflection. Then, we have two separate contours: one of which is on the inside of $\mathbb{D}$, and one of which is on the outside. The contour on the inside must integrate

If C' covers $\mathbb{D}$ entirely, then we can split it up into multiple contours which only cover part of $\mathbb{D}$ and proceed as in the above case.

We can extend Morera to say that for all closed loops γ,

$$\oint_\gamma g = 0 \Rightarrow g \text{ holomorphic.}$$

Since g is holomorphic and agrees with f on the unit circle, then, by analytic continuation, we can construct the entire function

$$F(z) = \begin{cases} f(z), \ |z| \leq 1 \\ \\ g(z), \ |z| > 1. \end{cases}$$

We will show that F is bounded. Since F is continuous on the compact set $\mathbb{D}$, then it is bounded there by EVT.

Since $|z| \geq 1 \Rightarrow |\ 1/\overline{z}\ | \leq 1$ and f is non-vanishing and continuous on $\mathbb{D}$, then $g = 1/\overline{f(1/\overline{z})}$ continuous on a compact set and hence bounded.

Thus, since F is entire and bounded, then it is constant by Liouville.

□

Problem 3. Prove the converse to Runge's Theorem. If K compact, and K^c not connected, then there exists f holomorphic in a neighborhood of K which cannot be approximated uniformly by a polynomial on K.

Proof. Since K^c is not connected, then there must be an open component of K^c K, call it Ω.

For a contradiction, assume that f can be uniformly approximated by a polynomial on K. Then, there exists a polynomial p such that

$$|p(z) - f(z)| < \varepsilon$$
$$|(z - z_0)p(z) - 1| < \varepsilon(z - z_0)$$

where $z_0 \in \Omega$.

Let $g(z) = (z - z_0)p(z) - 1$.

Choose $\varepsilon = \max\limits_K \frac{1}{z - z_0}$ such that we have $|g| < 1$.

But, by MMP, since Ω open, then g cannot achieve a maximum in Ω unless g is constant.

Notice that $g(z_0) = 1$, which is a contradiction unless g is constant. But g cannot be constant because it has a linear term.

So, f cannot be uniformly approximated where K^c is not connected.

$\square$

Problem 4. Evaluate the following integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}$.

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}$.

Proof of (a). By the FTA, $\frac{1}{1+x^4}$ has four poles, one in each quadrant of the complex plane.

We will consider the positive imaginary semicircle contour C_R with radius R.

The desired integral is found on the bottom of the contour as $R \to \infty$.

Since $f = O(z^{-4})$, then the integral around the upper part of the contour with radius R will vanish as $R \to \infty$.

So, we are left with

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \oint\limits_{C_R} \frac{dz}{1 + z^4}.$$

By the residue theorem, the contour integral will evaluate to $2\pi i$ times the sum of the residues contained in the contour.

For the first pole at $z_1 = \frac{1}{\sqrt{2}}(1 + i)$, we have

$$\operatorname{res}_{z_1} f = \lim_{z \to z_1} \frac{z - z_1}{1 + z^4} = \frac{1}{\sqrt{8}(i - 1)}.$$

For the second pole at $z_2 = \frac{1}{\sqrt{2}}(-1 + i)$ we have

$$\operatorname{res}_{z_2} f = \lim_{z \to z_2} \frac{z - z_2}{1 + z^4} = \frac{1}{\sqrt{8}(i + 1)}.$$

So,

$$\oint\limits_{C_R} f\, dz = 2\pi i \Big(\operatorname{res}_{z_1} f + \operatorname{res}_{z_2} f\Big) = \frac{2\pi i}{\sqrt{8}} \left(\frac{1}{i + 1} + \frac{1}{i - 1}\right) = \frac{\pi}{\sqrt{2}}.$$

Thus,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{1+x^4}=\frac{\pi}{\sqrt{2}}$$

□

Proof of (b). We will consider the same contour C_R as in the previous problem.

For the integral on the top curve of C_R, we have that

$$f=\mathrm{Re}\left[\frac{e^{iz}}{z^2+a^2}\right].$$

Note that for $z=Re^{it}$ and $t\in[0,\pi]$,

$$\left[\frac{e^{iRe^{it}}}{R^2e^{2it}}+a^2\right]\leq\frac{e^{-R\sin t}}{R^2-a^2},$$

where $\left|e^{iRe^{it}}\right|=\left|e^{iR\cos t-R\sin t}\right|=e^{-R\sin t}$. Since $\sin t\geq 0$ for all $t\in[0,\pi]$, then this goes to zero as $R\to\infty$.

Hence, the integral of this term will vanish on the upper part of the contour with radius R.

We are left with the integral on the real line equal to the integral over the entire contour C_R.

We have that the poles of the integrand f occur at $\pm ai$. Since we have the semicircle contour in the positive imaginary axis, then we will find the residue at ai,

$$\mathrm{res}_{ai}f=\lim_{z\to ai}\frac{\cos z}{z+ai}=\frac{\cos ai}{2ai}=\frac{e^{-a}}{2ai}$$

where $e^{i(ai)}=e^{-a}=\cos(ai)+i\sin(ai)$.

So,

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}=\oint_{C_R}f=2\pi i\ \mathrm{res}_{ai}f=\frac{\pi e^{-a}}{a}$$

as desired.

□

Problem 5. Show for $|a|<1|$,

$$\int_0^{2\pi}\log\left|1-ae^{i\theta}\right|d\theta=0.$$

Show that the above also holds for $|a|=1$.

Proof of (a). Since $\log|1-ae^{i\theta}|$ is holomorphic for all $a<1$, then, by Goursat, the integral around the closed circular loop of radius a is zero.

Alternatively, we can use Corollary 7.3 with the fact that $\mathrm{Re}[\log z]=\log|z|$ and evaluate $\log|1-z|$ at $z=0$.

□

Proof of (b). Let $z = e^{i\theta}$ and C_1 be the unit circle contour.

We have that

$$\oint_{C_1} \frac{\log|1-z|}{iz}\, dz.$$

Note that the integrand f has poles at $z = 0$ and $z = 1$.

First, we will consider the pole at $z = 0$,

$$\operatorname{res}_0 f = \lim_{z \to 0} (z) \frac{\log|1-z|}{iz} = 0.$$

Since this residue is zero, then $z = 0$ was not a simple pole and instead was a removable singularity.

Thus, we will consider f to be holomorphic over the origin.

We will construct a new contour γ similar to C_1 which avoids the pole at $z = 1$ by an epsilon bubble of radius ε, given by the curve C_ε, as an indent to our circle contour.

Since f is holomorphic on the region enclosed by γ, then

$$0 = \oint_\gamma f\, dz = \int_{C_1} f\, dz + \int_{C_\varepsilon} f\, dz.$$

We will now show that the integral over C_ε will vanish as $\varepsilon \to 0$, so the integral over C_1 must vanish as well.

Let $z = 1 + \varepsilon e^{i\theta}$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, $dz = i\varepsilon e^{i\theta} d\theta = iz d\theta$ and $z - 1 = \varepsilon e^{i\theta}$.

So,

$$\int_{C_\varepsilon} \frac{\log|1-z|}{iz}\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log|\varepsilon e^{i\theta}|\, d\theta = \pi \log \varepsilon \to 0.$$

Thus, we have that the integral vanishes for all $a \leq 1$.

□

Problem 6. For even $n \in \mathbb{N}$, find

$$\int_0^\pi \sin^n(\theta) d\theta.$$

Proof. With Euler's identity and the binomial expansion, we have that

$$\int_0^\pi \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^n dx = \frac{1}{(2i)^n} \int_0^\pi \sum_{k=0}^n \binom{n}{k} e^{ix(n-k)} \left(-e^{ix}\right)^k dx.$$

Since we have a finite integral and series, we can interchange to achieve,

$$\frac{1}{(2i)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k \int_0^\pi e^{ix(n-2k)}\, dx.$$

Since $e^{ix(n-2k)}$ is even on $x \in [0, \pi]$ for all $n \neq 2k$, then the integral over this interval vanishes. So, for $k = \frac{n}{2}$, we have

$$\int_0^{\pi} e^{ix(n-2k)}\, dx = \pi.$$

Thus, we are left with

$$\frac{1}{(2i)^n}\binom{n}{\frac{n}{2}}(-1)^{\frac{n}{2}}\pi = \frac{\pi}{2^n}\binom{n}{\frac{n}{2}}.$$

$\square$