# Math 336 Homework 5

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May 06, 2025

**Problem** 1. Find

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta.$$

*Proof.* Let $t = 2\theta$. Then, with the period of cosine, we have,

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta &= \frac{1}{2}\int_0^{4\pi} \frac{\cos^2\left(\frac{3}{2}t\right)}{5 - 4\cos t}\, dt \\
&= \int_0^{2\pi} \frac{\cos^2\left(\frac{3}{2}t\right)}{5 - 4\cos t}\, dt \\
&= \frac{1}{2}\int_0^{2\pi} \frac{1 + \cos 3t}{5 - 4\cos t}\, dt \\
&= \frac{1}{2}\oint_{\partial\mathbb{D}} \frac{1 + \frac{z^3 + z^{-3}}{2}}{5 - 4\frac{z + z^{-1}}{2}} \frac{dz}{iz} \\
&= \frac{1}{4i}\oint_{\partial\mathbb{D}} \frac{z^3 + 2 + z^{-3}}{-2z^2 + 5z - 2}\, dz \\
&= -\frac{1}{8i}\oint_{\partial\mathbb{D}} \frac{z^6 + 2z^3 + 1}{z^3(z-2)\left(z - \frac{1}{2}\right)}\, dz \\
&= -\frac{1}{8i}\oint_{\partial\mathbb{D}} \left(\frac{z^3 + 2}{(z-2)\left(z - \frac{1}{2}\right)} + \frac{1}{z^3(z-2)\left(z - \frac{1}{2}\right)}\right) dz.
\end{aligned}$$

Call the functions in the sum $g$ and $h$ respectively, left to right. We must consider the residues of $g$ and $h$ inside the unit disk. We see that $g$ has a simple pole at $\frac{1}{2}$, and $h$ has a third order pole at 0 and a simple pole at $\frac{1}{2}$.

First, we have that

$$\mathrm{res}_{\frac{1}{2}} g = \lim_{z \to \frac{1}{2}} \frac{z^3 + 2}{z - 2} = \frac{\frac{1}{8} + 2}{-\frac{3}{2}} = \frac{\frac{17}{8}}{-\frac{3}{2}} = -\frac{17}{12}.$$

Next, by partial fractions, we have

$$\frac{1}{(z-2)\left(z - \frac{1}{2}\right)} = \frac{2}{3}\left(\frac{1}{z-2} - \frac{1}{z - \frac{1}{2}}\right).$$

So, for the third order pole at zero of $h$,

$$
\begin{aligned}
\operatorname{res}_0 h &= \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\frac{2}{3}\left(\frac{1}{z-2} - \frac{1}{z-\frac{1}{2}}\right) \\
&= \frac{1}{3}\lim_{z\to 0}\left(\frac{2}{(z-3)^3} - \frac{2}{\left(z-\frac{1}{2}\right)^3}\right) \\
&= \frac{2}{3}\left(-\frac{1}{8} - \frac{1}{-\frac{1}{8}}\right) \\
&= \frac{2}{3}\left(8 - \frac{1}{8}\right) \\
&= \frac{21}{4}.
\end{aligned}
$$

Then, for the simple pole at $\frac{1}{2}$,

$$
\operatorname{res}_{\frac{1}{2}} h = \lim_{z\to\frac{1}{2}} \frac{1}{z^3(z-2)} = \frac{1}{\frac{1}{8}\left(-\frac{3}{2}\right)} = -\frac{16}{3}.
$$

Thus, we have

$$
\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta &= -\frac{1}{8i}\oint_{\partial\mathbb{D}}\left(\frac{z^3+2}{(z-2)\left(z-\frac{1}{2}\right)} + \frac{1}{z^3(z-2)\left(z-\frac{1}{2}\right)}\right)dz \\
&= -\frac{2\pi i}{8i}\left(\operatorname{res}_0 g + \operatorname{res}_0 h + \operatorname{res}_{\frac{1}{2}} h\right) \\
&= -\frac{\pi}{4}\left(\frac{21}{4} - \frac{16}{3} - \frac{17}{12}\right) \\
&= -\frac{\pi}{4}\left(-\frac{3}{2}\right) \\
&= \frac{3\pi}{8}.
\end{aligned}
$$

□

**Problem** 2. Find the number of zeros, counting multiplicities, of

$$
p(z) = z^4 - 2x^3 + 9x^2 + z - 1
$$

inside $|z| = 2$.

*Proof.* Consider the behavior of the terms of $p$ on the boundary $|z| = 2$,

$$
\begin{aligned}
|9z^2| &= 36 \\
|z^4 - 2x^3 + z - 1| &\le |z^4| + |2z^3| + |z| + 1 = 16 + 16 + 2 + 1 = 35.
\end{aligned}
$$

Hence $|9z^2| > |z^4 - 2x^3 + z - 1|$ on $|z| = 2$.

So, $9z^2$ and $p(z)$ have the same number of zeros inside $|z| = 2$ by Rouché.

Since $9z^2$ has one zero of multiplicity two, then $p(z)$ has two zeros inside $|z| = 2$. □

**Problem** 3. Show that

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating over the appropriate branch of the multi-valued function

$$f(z) = \frac{z^{-\frac{1}{2}}}{z^2+1} = \frac{e^{-\frac{1}{2}\log z}}{z^2+1}$$

over

(a) the closed contour $\gamma$ which bounds the region $\{p < |z| < R, \mathrm{Im}\ z > 0\}$, and

(b) the closed contour $\gamma_\varepsilon$ which bounds the region $\{p < |z| < R\} \setminus \{\mathrm{Re}\ z > 0, |\mathrm{Im}\ z| < \varepsilon\}$.

*Proof of (a).* Let $I$ be the given integral. We wish to show that $I = \frac{\pi}{\sqrt{2}}$. Note that $f$ has poles at 0 and $\pm i$.

For $\gamma$, we will consider the logarithm function branch cut in the negative imaginary axis.

This yields the following definition of log:

$$\log z = \log|z| + i \arg z, \text{ where } \arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

Let the pieces of the indented semicircular contour $\gamma$ be:[1]

- $C_R^+$, for the upper curve with radius $R$;
- $C_\rho^+$, for the lower curve with radius $\rho$;
- $L_-$, for the horizontal line on the negative real axis from $\rho$ to $R$; and
- $L_+$, for the horizontal line on the positive real axis from $-R \to -\rho$.

On $L_+$, we find our desired integral, $\int\limits_{L_+} f \to I$ as $R \to \infty$.

On $L_-$, we have that $\log(-x) = \log(xe^{i\pi}) = \log x + i\pi$, which is a valid argument for our branch cut. So,

$$\int\limits_{L_-} f = \int\limits_{L_+} \frac{e^{-\frac{1}{2}(\log x + i\pi)}}{z^2+1}\, dz = e^{-i\frac{\pi}{2}} \int\limits_{L_+} f = -iI.$$

On $C_\rho^+$,

$$|f| = \left| \frac{e^{-\frac{1}{2}(\log|z|\ + i \arg z)}}{z^2+1} \right| = \frac{\rho^{-\frac{1}{2}}}{\rho^2+1}.$$

So,

$$\left| \int\limits_{C_\rho^+} f \right| \leq \pi\rho \frac{\rho^{-\frac{1}{2}}}{\rho^2+1} = \pi \frac{\rho^{\frac{1}{2}}}{\rho^2+1} \to 0, \text{ as } \ \rho \to 0.$$

Similarly, on $C_R^+$,

[1] i recently switched from LaTeX to a different typesetting language called Typst. i am still learning how to format diagrams.

$$\left|\int_{C_R^+} f\right| \le \pi R \frac{R^{-\frac{1}{2}}}{R^2+1} = \pi \frac{R^{\frac{1}{2}}}{R^2+1} \to 0, \text{ as } \quad R \to \infty.$$

In total, since $f$ only have one pole at $i$ inside $\gamma$,

$$\oint_\gamma f = \int_{L_+} f + \int_{L_-} f + \int_{C_R^+} f + \int_{C_\rho^+} f = 2\pi i \operatorname{res}_i f.$$

But, the integrals on the curve vanish and we can relate the straight line integrals to our desired integral $I$,

$$I - iI = 2\pi i \operatorname{res}_i f.$$

Then,

$$\operatorname{res}_i f = \lim_{z\to i} \frac{e^{-\frac{1}{2}\log z}}{z+i} = \frac{e^{-\frac{1}{2}(\log|i| \, + i \arg i)}}{2i} = \frac{e^{-\frac{1}{2}(i\frac{\pi}{2})}}{2i} = \frac{e^{-\frac{\pi}{4}}}{2i} = \frac{1-i}{2\sqrt{2}i}.$$

Thus,

$$I - iI = \pi \frac{1-i}{\sqrt{2}} \implies I = \frac{\pi}{\sqrt{2}},$$

which is what we wanted to show. □

*Proof of (b).* For $\gamma_\varepsilon$, we will consider the logarithm function branch cut in the positive real axis.

Let the pieces of the keyhole contour $\gamma_\varepsilon$ be:
- $C_R$, for the outer circle of radius $R$ with $\varepsilon$ height removed from the part on the positive real axis;
- $C_\varepsilon$, for the inner semicircle circle of radius $\varepsilon$, facing the negative real axis;
- $L_+$, for the horizontal line $\varepsilon$ above the positive real axis from 0 to about $R$; and
- $L_-$, for the horizontal line $\varepsilon$ below the positive real axis from about $R$ to zero.

On $L_+$, since $\arg z \to 0$ as $\varepsilon \to 0$, then $\int\limits_{L_+} f \to I$ as $\varepsilon \to 0$.

On $L_-$, since $\arg z \to 2\pi$ as $\varepsilon \to 0$ by the branch cut, then $\frac{e^{-\frac{1}{2}(\log|z| \, + 2\pi)}}{z^2+1} = e^{-i\pi} f = -f$. But, $L_-$ is traversed in the opposite direction of $L_+$, therefore we have $\int\limits_{L_-} f = \int\limits_{L_+} f = I$ as $\varepsilon \to 0$.

The line integrals on $C_R$ and $C_\varepsilon$ will vanish as in part (a) because we have merely doubled the path length with the same limit $R \to \infty$ but replacing $\varepsilon \to 0$ for $\rho \to 0$.

So, with the appropriate limits,

$$\oint_{\gamma_\varepsilon} f = \int_{L_+} f + \int_{L_-} f + \int_{C_R} f + \int_{C_\varepsilon} f = 2I = 2\pi i \sum_{\text{poles}} \operatorname{res}_{z_i} f.$$

However, now we have both resides at $-i$ and $i$ inside $\gamma_\varepsilon$. So,

$$\operatorname{res}_{-i} f = \lim_{z\to -i} \frac{e^{-\frac{1}{2}\log z}}{z-i} = \frac{e^{-\frac{1}{2}(\log|-i| + i\arg(-i))}}{-2i} = \frac{e^{-\frac{1}{2}(i\frac{3\pi}{2})}}{-2i} = \frac{e^{-\frac{3\pi}{4}}}{-2i} = \frac{-1-i}{-2\sqrt{2}i} = \frac{1+i}{2\sqrt{2}i}.$$

Thus,

$$2I = 2\pi i\left(\frac{1-i}{2\sqrt{2}i} + \frac{1+i}{2\sqrt{2}i}\right) = \frac{2\pi}{\sqrt{2}} \Rightarrow I = \frac{\pi}{\sqrt{2}},$$

as desired. $\square$