Math 336 Homework 9

a lipson

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Lemma (*Schwarz-Pick*). $f : \mathbb{D} \to \mathbb{D}$ holomorphic,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \ z \in \mathbb{D}.$$

Problem 1. The Schwarz-Pick lemma is the infinitesimal version of an important observation in complex analysis and geometry. For complex numbers $w \in \mathbb{C}$ and $z \in \mathbb{D}$ we define the *hyperbolic length* of w at z by

$$\|w\|_z = \frac{|w|}{1 - |z|^2},$$

where $|w|$ and $|z|$ denote the usual absolute values. This length is sometimes referred to as the Poincaré metric, and as a Riemann metric it is written as

$$ds^2 = \frac{|dz|^2}{(1 - |z|^2)^2}$$

The idea is to think of w as a vector lying in the tangent space at z. Observe that for a fixed w, its hyperbolic length grows to infinity as z approaches the boundary of the disk. We pass from the infinitesimal hyperbolic length of tangent vectors to the global hyperbolic distance between two points by integration.

(a) Given two complex numbers z_1 and z_2 in the disk, we define the *hyperbolic distance* between them by

$$d(z_1, z_2) = \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_{\gamma(t)} \, dt,$$

where the infimum is taken over all smooth curves $\gamma : [0, 1] \to \mathbb{D}$ joining z_1 and z_2. Use the Schwarz-Pick lemma to prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2), \ \forall z_1, z_2 \in \mathbb{D}.$$

In other words, holomorphic functions are distance-decreasing in the hyperbolic metric.

(b) Prove that automorphisms of the unit disk preserve the hyperbolic distance, namely

$$d(\varphi(z_1), \varphi(z_2)) = d(z_1, z_2), \ \forall z_1, z_2 \in \mathbb{D}.$$

and any automorphism φ. Conversely, if $\varphi : \mathbb{D} \to \mathbb{D}$ preserves the hyperbolic distance, then either φ or $\overline{\varphi}$ is an automorphism of $\mathbb{D}$.

(c) Given two points $z_1, z_2 \in \mathbb{D}$, show that there exists an automorphism φ such that $\varphi(z_1) = 0$ and $\varphi(z_2) = s$ for some s on the segment $[0, 1) \subset \mathbb{R}$.

(d) Prove that the hyperbolic distance between 0 and $s \in [0, 1)$ is

$$d(0,s) = \frac{1}{2}\log\frac{1+s}{1-s}$$

(e) Find a formula for the hyperbolic distance between any two points in the unit disk.

Proof of (a). Let $z_1, z_2 \in \mathbb{D}$. Let $\gamma : [0,1] \to \mathbb{D}$ be a smooth curve joining z_1, z_2.

Since $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and hence continuous, then $\gamma_f = f \circ \gamma$ is smooth curve joining $f(z_1), f(z_2)$.

By Schwarz-Pick, since $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then for all $z \in \mathbb{D}$,

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}.$$

In particular, for all $z \in \gamma \subset \mathbb{D}$,

$$\|\gamma_{f'}\|_{\gamma_f} = \frac{|(f(\gamma))'|}{1-|f(\gamma)^2|} = \frac{|\gamma'||f'(\gamma)|}{1-|f(\gamma)|^2} \le \frac{|\gamma'|}{1-|\gamma|^2} = \|\gamma'\|_\gamma.$$

Thus, $\|\gamma_{f'}(t)\|_{\gamma_f(t)} \le \|\gamma(t)\|_{\gamma(t)}$, which implies $d(f(z_1), f(z_2)) \le d(z_1, z_2)$ for all $z_1, z_2 \in \mathbb{D}$. $\square$

Proof of (b). (i) Let $\varphi \in \mathrm{Aut}(\mathbb{D})$. By Theorem 8.2.2, φ must be a composition of a conformal translation ψ_α and a rotation. We will show that hyperbolic distance is preserved in each of these cases, and hence in their composition as well.

If φ is a rotation, then hyperbolic distance is preserved; we have $|\varphi(z)| = |z|$ and $|\varphi'(z)| = 1$ for all $z \in \mathbb{D}$, which gives that the hyperbolic length of any curve joining z_1 and z_2 is equal to the length of its image under φ.

If φ is the conformal translation ψ_α, then φ preserves the Reimann metric. We have

$$\psi_\alpha'(z) = \frac{|\alpha|^2 - 1}{(1-\overline{\alpha}z)^2}.$$

Let $w = \varphi(z)$ so $dw = \varphi'(z)dz$. We will show

$$\frac{|dw|^2}{(1-|w|^2)^2} = \frac{|dz|^2}{(1-|z|^2)^2}.$$

First,

$$|dw|^2 = |\varphi'(z)|^2 |dw|^2 = \frac{\left|1-|\alpha|^2\right|^2}{|1-\overline{\alpha}z|^4}|dz|^2.$$

Then,

$$1-|w|^2 = 1 - \left|\frac{\alpha - z}{1-\overline{\alpha}z}\right|^2 = \frac{(1-|\alpha|^2)(1-|z|^2)}{|1-\overline{\alpha}z|^2}.$$

Substituting back,

$$\frac{|dw|^2}{(1-|w|^2)^2} = \frac{|1-\overline{\alpha}z|^4}{((1-|\alpha|^2)(1-|z|^2))^2}\frac{\left|1-|\alpha|^2\right|^2}{|1-\overline{\alpha}z|^4}|dz|^2 = \frac{|dz|^2}{(1-|z|^2)^2},$$

as desired.

Since hyperbolic distance is preserved in both cases, then it is preserved for any composition, which are all automorphisms of the unit disk.

(ii) Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a map that preserves hyperbolic distance. $\square$

Proof of (c). Let $z_1, z_2 \in \mathbb{D}$.

We will use the conformal translation to map z_1 to the origin and then a rotation to rotate the image of z_2 under the first map to the positive real axis.

Consider the conformal translation automorphism of the unit disk ψ_{z_1} which maps $z_1 \mapsto 0$. Let $\psi_{z_1}(z_2) = w = Re^{i\theta}$ with $R \in [0, 1)$ be the image of z_2 under this map.

Then, consider the automorphism rotation map r defined by $z \mapsto e^{-i\theta} z$, which maps $Re^{i\theta} \mapsto R$. Now, let $s = R$ lies on the positive real axis inside the unit disk.

Since the map $\varphi = r \circ \varphi_{z_1}$ is a composition of automorphisms, then it is also an automorphism.

Thus, we have an automorphisms φ which maps $z_1 \mapsto 0$ and $z_2 \mapsto s \in [0, 1)$, as desired. $\square$

Proof of (d). Let γ be a path from 0 to s. We will consider the straight line path on the real axis because the geodesic paths include straight lines through the origin. Furthermore, any γ that deviates from the real axis can be reflected across the real axis to get another path of the same hyperbolic length. The "average" of these two paths lies closer to the real axis, suggesting the real axis path is optimal.

Therefore, we have

$$\begin{aligned} d(0, s) &= \inf_{\gamma} \int_0^1 \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} \, dt \\ &= \int_0^s \frac{dt}{1 - t^2}. \\ &= \frac{1}{2} \int_0^s \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt \\ &= \frac{1}{2} \left[\log \frac{1+t}{1-t} \right]_0^s \\ &= \frac{1}{2} \log \frac{1+s}{1-s}, \end{aligned}$$

which is what we wanted to show. $\square$

Proof of (e). Let φ be an automorphism as in part (c) which maps $(z_1, z_2) \mapsto (0, s)$, and by part (b), φ preserves hyperbolic distance.

Since $s = |\varphi(z_2)| = \psi_{z_1}(z_2)$, then, by part (d), we have

$$d(z_1, z_2) = \frac{1}{2} \log \frac{1 + \psi_{z_1}(z_2)}{1 - \psi_{z_1}(z_2)},$$

which provides the hyperbolic distance for any two points z_1, z_2 in the unit disk. $\square$

Problem 2.

Proof.

□

Problem 3. Let $\Omega \subset \mathbb{C}$ be a region and suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of holomorphic functions that is uniformly bounded on compact subsets of Ω.

Let f be a holomorphic function on Ω, and let $S := \left\{z \in \Omega \mid \lim_{n\to\infty} f_n(z) = f(z)\right\}$.

Prove that if S has a limit point in Ω, then $f_n \to f$ uniformly on compact subsets of Ω.

Proof. By Montel, $\{f_n\}$ is a normal family $\mathcal{F}$, so every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function.

Suppose for a contradiction that f_n does not converge uniformly on compact subsets of Ω. So, there exists a $K \subset\subset \Omega$ and $\varepsilon > 0$ such that for all $n \in \mathbb{N}$,

$$\sup_{z\in K} |f_{n(z)} - f(z)| \geq \varepsilon.$$

We will consider a subsequence $\left\{f_{n_k}\right\}_{k\in\mathbb{N}}$. By the normality of $\mathcal{F}$, this subsequence f_{n_k} must converge uniformly on compact subsets of Ω to a holomorphic function $g : \Omega \to \mathbb{C}$.

Now, for all $z \in S$, we have that $\lim_{n\to\infty} f_n(z) = f(z)$. In particular, since f_{n_k} is a subsequence, then $\lim_{k\to\infty} f_{n_k}(z) = f(z)$ for all $z \in S$.

But, we have that $\lim_{k\to\infty} f_{n_k}(z) = g(z)$ for all $z \in \Omega$.

Consider the holomorphic difference function $h = g - f$ which has zeros in S because, for all $z \in S$, $g(z) = \lim_{k\to\infty} f_{n_k}(z) = f(z)$.

Since S has a limit point in Ω, which is open and connected, and h is holomorphic, then by Theorem 2.4.8, we must have that h accumulates zeros and is identically zero, which means that $f = g$ in Ω.

Since $f = g$, then $f_{n_k} \to f$ uniformly on compact subsets of Ω. In particular, we have on K,

$$\lim_{k\to\infty} \sup_{z\in K} \left|f_{n_k}(z) - f(z)\right| = 0,$$

which is a contradiction with the assumption that f_n did not converge uniformly on compact subsets of Ω.

Thus, $f_n \to f$ uniformly on compact subsets of Ω. □