Math 336 Homework 6

a lipson

May 21, 2025

Problem 1. Poisson summation formula.

(a) Fix τ with $\operatorname{Im}\tau > 0$. Apply Poisson summation formula to

$$f(z) = (\tau + z)^{-k}$$

for $2 \leq k \in \mathbb{N}$ to obtain

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}e^{2\pi i m\tau}.$$

(b) Set $k = 2$ in the above identity and use $\operatorname{Im}\tau > 0$ to show

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) Can we conclude the above identity holds when τ is any complex number and not just an integer?

Proof of (a). Using residues, we will show that when $\xi < 0$, we have $\hat{f}(\xi) = 0$, and for $\xi > 0$, we have

$$\hat{f}(\xi) = \frac{(-2\pi i)^k}{(k-1)!}\xi^{k-1}e^{2\pi i\xi\tau}.$$

Since $\operatorname{Im}\tau > 0$ and f has a pole of order k at $z = -\tau$, then this pole is not on the real axis, and f is holomorphic in some strip of width $a < \tau$.

Since f satisfies sufficient decay conditions for $k \geq 2$, and is holomorphic within a strip, then we can apply the Fourier transform. So, $\forall x \in \mathbb{R}$,

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\, dx$$

Let

$$g(z) = f(z)e^{-2\pi i z\xi} = \frac{e^{2\pi i z\xi}}{(\tau+z)^k}.$$

Since $e^{-2\pi i z\xi}$ is holomorphic, then g also has only a pole of order k at $z = -\tau$.

Since $\operatorname{Im}\tau > 0$, then $\operatorname{Im}(-\tau) < 0$, so the pole of g pole is the lower half-plane.

We will consider the three cases where $\xi < 0$, $\xi = 0$, and $\xi > 0$.

For $\xi < 0$, the exponential term decays in the upper half-plane, so a semicircular contour there will enclose no poles, contributing no residues to the integral $\oint g dz$, thus $\hat{f}(\xi) = 0$ for $\xi < 0$.

For $\xi = 0$,

$$\int_{\mathbb{R}} \frac{dx}{(\tau + x)^k}$$

, since the integrand decays at sufficient magnitude of z, then can use the upper semicircular contour, again with no residues, or the lower semicircular contour, where the residue will vanish. So, the integral vanishes when $\xi = 0$.

For $\xi > 0$, the exponential term will vanish in the lower half-plane, so we will construct a semicircular contour there, picking up the residue from the pole at $z = -\tau$. Note that the positively oriented contour traverses the real axis in the opposite direction, so we pick up a negative sign. So, we have

$$-\hat{f}(\xi) = \int_{\mathbb{R}} g(x)\, dx = 2\pi i \mathrm{res}_{-\tau} g(z).$$

For the residue, we have

$$\begin{aligned} \mathrm{res}_{-\tau} g(z) &= \lim_{z \to -\tau} \frac{1}{(k-1)!} \left(\frac{d}{dx}\right)^{k-1} e^{-2\pi i z \xi} \\ &= \lim_{z \to -\tau} \frac{(-2\pi i \xi)^{k-1}}{(k-1)!} e^{-2\pi i z \xi} \\ &= \frac{(-2\pi i)^{k-1}}{(k-1)!} \xi^{k-1} e^{2\pi i \tau \xi}. \end{aligned}$$

Therefore, $\forall \xi > 0$,

$$\hat{f}(\xi) = \frac{(-2\pi i)^k}{(k-1)!} \xi^{k-1} e^{2\pi i \tau \xi},$$

and $\hat{f}(\xi)$ vanishes otherwise.

Now, with the Poisson summation formula, we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

Since $\forall \xi \leq 0,\ \hat{f}(\xi) = 0$, then the above becomes

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \in \mathbb{Z}^+} n^{k-1} e^{2\pi i n \tau},$$

as desired. $\square$

Proof of (b). For $k = 2$, we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^2} = \frac{(-2\pi i)^2}{(2-1)!} \sum_{n \in \mathbb{Z}^+} n^{2-1} e^{2\pi i n \tau} = -4\pi^2 \sum_{n \in \mathbb{Z}^+} n e^{2\pi i n \tau}.$$

Note that

$$\sum_{n\in\mathbb{Z}_{\geq 0}} z^n = \frac{1}{1-z} \implies \sum_{n\in\mathbb{Z}^+} nz^n = \frac{z}{(1-z)^2}.$$

Let $w = e^{\pi i\tau}$,

$$\sin \pi t = \frac{e^{\pi i\tau} - e^{-\pi i\tau}}{2i} = \frac{w - w^{-1}}{2i}.$$

Let $w^2 = z = e^{2\pi i\tau}$,

$$\sin^2 \pi t = -\frac{1}{4}\left(w - w^{-1}\right)^2 = -\frac{1}{4}\left(w^2 - 2 + w^{-2}\right) = -\frac{z^2 - 2z + 1}{4z} = -\frac{(z-1)^2}{4z}.$$

Therefore we have $\sum_{n\in\mathbb{Z}^+} nz^n = -\frac{1}{4\sin^2 \pi\tau}$, which implies

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^2} = -4\pi^2 \sum_{n\in\mathbb{Z}^+} ne^{2\pi in\tau} = \frac{\pi^2}{\sin^2 \pi\tau},$$

which is what we wanted to show. □

Proof of (c). Both functions in the identity of part (b) are meromorphic functions of τ which agree on the open half-plane and have identical poles at integer values.

We will show that the principle part of both functions matches at the poles.

The principle part of the series near integer τ poles is 1.

We will take the Taylor expansion of $\frac{\pi^2}{\sin^2 \pi\tau}$ near for τ near integers m,

$$\sin \pi\tau = \sin(\pi m + \pi(\tau - m)) \approx \pi(\tau - m)$$

where the approximation holds by the Fundamental Theorem of Engineering.

Therefore we have $\sin^2 \pi\tau \to \pi^2(\tau-m)^2$ near integers m, so $\frac{\pi^2}{\sin^2 \pi\tau} \to \frac{1}{(\tau-m)^2}$ there as well, which has principle part 1 as well.

Since the principle parts agree near poles, then we have matching analytic continuations both functions on all $\tau \in \mathbb{C}$.

Thus, the identity holds for all complex τ. □

Problem 2. Suppose $\hat{f}$ has compact support in $[-M, M]$ and $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$.

(a) Show

$$a_n = \frac{(2\pi i)^n}{n!} \int_{-M}^{M} \hat{f}(\xi)\xi^n \, d\xi.$$

(b) Show

$$\limsup_{n\to\infty} \left(n! \, |a_n|\right)^{\frac{1}{n}} \leq 2\pi M.$$

(c) In the converse direction, show that if f is analytic with the limit supremum condition, then f is entire and

$$\forall \varepsilon > 0, \exists A_\varepsilon > 0 : |f(z)| \leq A_\varepsilon e^{2\pi(M+\varepsilon)|z|}.$$

Proof of (a). Since f has compact support, then f and $\hat{f}$ have moderate decay.

Since f is entire, then $f \in \mathcal{F}_a$, so Fourier inversion holds.

Therefore, by the compact support of $\hat{f}$

$$f(z) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i z\xi} \, d\xi = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i z\xi} \, d\xi.$$

By the Cauchy integral formula for series coefficients,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} \, dz.$$

Since f is entire, then the integrand has a pole of order $n+1$ at $z = 0$.

Using residues,

$$\begin{aligned} a_n &= \lim_{z\to 0} \frac{1}{n!} \left(\frac{d}{dz}\right)^n f(z) \\ &= \lim_{z\to 0} \frac{1}{n!} \left(\frac{d}{dz}\right)^n \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i z\xi} \, d\xi \\ &= \int_{-M}^{M} \frac{1}{n!} \hat{f}(\xi) \lim_{z\to 0} \left(\frac{d}{dz}\right)^n e^{2\pi i z\xi} \, d\xi \\ &= \int_{-M}^{M} \lim_{z\to 0} \frac{1}{n!} \hat{f}(\xi) (2\pi i\xi)^n e^{2\pi i z\xi} \, d\xi \\ &= \frac{(2\pi i)^n}{n!} \int_{-M}^{M} \hat{f}(\xi) \xi^n \, d\xi, \end{aligned}$$

where the interchange of limit processes is justified by the finite integral and continuity of the integrand in both ξ and z.

Note that we can also arrive at this conclusion much faster by using the series expansion of $e^{2\pi i z\xi}$ inside the Fourier inversion integral for f. □

Proof of (b). With $|\xi| \le M$ by the bounds of the integral and $\hat{f}(\xi)$ bounded by some constant C from compact support, we have

$$\begin{aligned} n!|a_n| &= (2\pi)^n \left| \int_{-M}^{M} \hat{f}(\xi) \xi^n \, d\xi \right| \\ &\le (2\pi)^n \int_{-M}^{M} \left|\hat{f}(\xi)\right| |\xi|^n \, d\xi \\ &\le (2\pi)^n M^n \int_{-M}^{M} \left|\hat{f}(\xi)\right| d\xi \\ &\le C(2\pi M)^n. \end{aligned}$$

So,

$$\limsup_{n\to\infty} \left(n! \ |a_n|\right)^{1/n} \le \limsup_{n\to\infty} C^{1/n} 2\pi M = 2\pi M.$$

□

Proof of (c). For all $\varepsilon > 0$, there exists an N_ε such that for all $n > N_\varepsilon$, we have

$$(n!\ |a_n|)^{1/n} < 2\pi(M+\varepsilon) \implies |a_n| < \frac{(2\pi(M+\varepsilon))^n}{n!}.$$

We can split the series of f at N_ε,

$$|f(z)| \leq \sum_{n=0}^{N_\varepsilon} |a_n|\ |z|^n + \sum_{n=N_\varepsilon+1}^{\infty} |a_n|\ |z|^n.$$

The first sum is bounded by $C_1\ |z|^{N_\varepsilon}$.

For second sum with $n > N_\varepsilon$, we have the bound

$$\sum_{n=N_\varepsilon+1}^{\infty} \frac{(2\pi(M+\varepsilon))^n}{n!},$$

which is part of the series expansion for the exponential function, hence we also have the bound $C_2 e^{2\pi(M+\varepsilon)|z|}$.

Combining the bounds,

$$|f(z)| < C_1 |z|^{N_\varepsilon} + C_2 e^{2\pi(M+\varepsilon)|z|}.$$

Since exponential functions grow faster than polynomials, then, for some A_ε, we have

$$|f(z)| \leq A_\varepsilon e^{2\pi(M+\varepsilon)|z|},$$

which is what we wanted to show. □

Problem 3. We will show results similar to Phragmén-Lindelöf.

(a) Suppose F is holomorphic is the right half-plane and extends continuously to the imaginary axis boundary. Given the boundary condition $\forall y \in \mathbb{R},\ |F(iy)| \leq 1$ and the growth condition $|F(z)| \leq C \exp(c|z|^\gamma)$ for $c, C > 0$ and $\gamma > 1$, prove $|F(z)| \leq 1$ for all z in the right half-plane.

(b) Let S be the sector with vertex at the origin, forming an angle of $\frac{\pi}{\beta}$. Suppose F is holomorphic in S and continuous on the boundary, $|F(z)| \leq 1$ on ∂S, and $|F(z)| \leq C \exp(c|z|^\alpha)$ for all z in S, with $c, C > 0$ and $0 < \alpha < \beta$. Prove $\forall z \in S,\ |F(z)| \leq 1$.

Proof of (b). Note that we will first prove part (b) as a more general case of part (a).

We have that $|\arg z| < \frac{\pi}{2\beta}$.

Assume that $\beta > 1$, i.e. the sector remains in the right half-plane.[1]

We will take the principal log branch cut on $\mathbb{R}^-$.

Consider the function $\exp(-\varepsilon z^\beta)$ where $z = \mathrm{Re}^{i\theta}$,

$$|\exp(-\varepsilon z^\beta)| = \exp(-\varepsilon r^\beta \cos \beta\theta).$$

[1]We need $\cos(\arg z) > 0$ for decay.

But, $|\beta\theta| < \frac{\pi}{2\beta}$, so $|\theta| < \frac{\pi}{2}$ and $\cos\theta > 1$.

Let $F_\varepsilon(z) = F(z)\exp\left(-\varepsilon z^\beta\right)$. So,

$$|F_\varepsilon(z)| \leq |F(z)|\,\left|\exp\left(-\varepsilon z^\beta\right)\right| \leq C\exp\left(c|z|^\alpha - \varepsilon z^\beta\right).$$

Since $\alpha < \beta$, then $|F_\varepsilon| \to 0$ as $|z| \to \infty$.

Therefore

$$\sup_{\partial S \cap C_R} |F_\varepsilon| \to 0 \text{ as } R \to \infty.$$

Consider the compact region $\overline{S} \cap D_R$, the intersection between the closure of S and the closed disk of radius R.

On ∂S, since $|F| \leq 1$, then $|F_\varepsilon| \leq \exp\left(-\varepsilon z^\beta\right) \leq 1$.

On the outer arc, with $\alpha < \beta$ and for sufficiently large R, $|F_\varepsilon| \leq C\exp\left(cR^\alpha - \varepsilon R^\beta\right) < 1$.

By MMP, as $R \to \infty$,

$$\sup_{\overline{S} \cap D_R} |F_\varepsilon| \leq \sup_{\partial\left(\overline{S} \cap D_R\right)} |F_\varepsilon| \leq 1.$$

Since F_ε is continuous in ε, then as $\varepsilon \to 0$,

$$\sup_{\overline{S} \cap D_R} |F_\varepsilon| \to \sup_{\overline{S} \cap D_R} |F| \leq 1,$$

which is what we wanted to show. $\square$

Proof of (a). Let S be the right half-plane sector. On S, we have $\arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so we will use the principal log branch.

For $\varepsilon > 0$, let $F_\varepsilon(z) = F(z)\exp(-\varepsilon z)$.

On S, the real part of z is be positive. So, $|F_\varepsilon| \leq C\exp(c|z|^\gamma - \varepsilon z)$ will vanish as $|z| \to \infty$ because $\gamma < 1$, and F_ε is bounded.

On ∂S, $|F_\varepsilon| = |F|\,\left|e^{-\varepsilon z}\right| \leq 1$.

On the outer arc $|F_\varepsilon| \leq 1$ from the decay demonstrated above.

So, as $R \to \infty$, we have $|F_\varepsilon| \leq 1$ on the boundary $\partial\left(\overline{S} \cap D_R\right)$, which bounds F_ε on the interior by MMP.

Then, as $\varepsilon \to 0$, we have that $|F| \leq 1$ on S, which is what we wanted to show. $\square$

Problem 4. A function and its Fourier transform cannot both be too small at infinity, this is illustrated by the following theorem by Hardy. If the function f on $\mathbb{R}$ satisfies

$$f(x) = O\left(e^{-\pi x^2}\right) \wedge \hat{f}(\xi) = O\left(e^{-\pi\xi^2}\right),$$

then $f = ce^{-\pi x^2}$. As a result, $f(x) = O\left(e^{-\pi A x^2}\right)$ and $\hat{f}(\xi) = O\left(e^{-\pi B\xi^2}\right)$. When $AB > 1$ and $A, B > 0$, then f is identically zero.

(a) Show that for f even, $\hat{f}$ extends to an even entire function. Let $g(z) = \hat{f}\left(z^{1/2}\right)$, which satisfies

$$|g(x)| \leq ce^{-\pi x} \wedge |g(z)| \leq c \exp(\pi R \sin^2 \theta/2) \leq ce^{\pi\, |z|},$$

when $x, \theta \in \mathbb{R}$, $R \geq 0$, and $z = Re^{i\theta}$.

(b) Apply the Phragmén-Lindelöf principle to

$$F(z) = g(z)e^{\gamma z} \text{ where } \gamma = i\pi \frac{e^{-\pi/(2\beta)}}{\sin \pi/(2\beta)}$$

and the sector $0 \leq \theta \leq \pi/\beta < \pi$.

Let $\beta \to 1$ to deduce $e^{\pi z} g(z)$ is bounded in the closed upper half-plane, and the same in the lower half-plane. By Liouville, $e^{\pi z} g(z)$ is constant, as desired.

(c) If f is odd, then $\hat{f}(0) = 0$. Apply the above argument to $\frac{\hat{f}(z)}{z}$ to deduce $f = \hat{f} = 0$. Write f as an appropriate sum of an even and odd function.

Note that there is likely a typo in the suggested solution, the original erroneous statement was taking $\beta \to \pi$.

Proof of (a). We will consider

$$\hat{f}(-\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x \xi}\, dx.$$

Note that flipping the differential and direction of integration cancel out opposing signs.

Since f is even, then with the map $x \mapsto -x$, we have

$$\hat{f}(-\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx = \hat{f}(\xi),$$

which implies that $\hat{f}$ is even.

Since $f(x) = O\left(e^{-\pi x^2}\right)$, then the integral

$$\hat{f}(z) = \int_{\mathbb{R}} f(x) e^{2\pi i x z}\, dx$$

will converge for all $z \in \mathbb{C}$, hence $\hat{f}$ is entire.

Now, $\forall x \in \mathbb{R}$, we have $\hat{f}(\xi) = O\left(e^{-\pi \xi^2}\right)$.

Let $g(x) = \hat{f}\left(x^{1/2}\right)$. So, $|g(x)| = \hat{f}\left(x^{1/2}\right) \leq ce^{-\pi\left(x^{1/2}\right)^2} = ce^{-\pi x}$.

With $z = Re^{i\theta}$, $\cos \theta = 1 - 2\sin^2 \theta/2$, and $\sin^2 \theta/2 < 1$ for all θ,

$$\begin{aligned}
|g(z)| &\leq |c\exp\bigl(-\pi R e^{i\theta}\bigr)| \\
&= c\exp(-\pi R\cos\theta) \\
&= c\exp\bigl(-\pi R(1-\sin^2\theta/2)\bigr) \\
&= c\exp\bigl(\pi R(2\sin^2\theta/2-1)\bigr) \\
&\leq c\exp\bigl(\pi R\sin^2\theta/2\bigr) \\
&\leq ce^{\pi R} \\
&= ce^{\pi\,|z|}.
\end{aligned}$$

□

Proof of (b). We will examine $|F|$ on the boundaries of the sector ∂S.

When $\theta = 0$ on $\mathbb{R}^+$,

$$|F(x)| = |g(x)e^{\gamma x}| \leq ce^{-\pi x}e^{\gamma x} = ce^{((\gamma-\pi)}.$$

We have

$$\gamma = i\pi\frac{e^{-\pi/(2\beta)}}{\sin\pi/(2\beta)} = \frac{\pi}{\sin\pi/(2\beta)}\left(i\cos-\frac{\pi}{2\beta}-\sin-\frac{\pi}{2\beta}\right),$$

by the oddness of sine, Re $\gamma = \pi$ for all β.

So $|F(x)| \leq c$.

For $\theta = \pi/\beta$, $z = |z|\ e^{i\pi/\beta}$.

$$\begin{aligned}
|F(z)| = |g(z)e^{\gamma z}| &\leq ce^{\pi\,|z|}e^{\gamma z} \\
&= c\exp\bigl(\pi\ |z| + \gamma\ |z|\ e^{i\pi/\beta}\bigr) \\
&= c\exp\bigl(\bigl(\pi+\gamma e^{i\pi/\beta}\bigr)|z|\bigr).
\end{aligned}$$

We have

$$e^{i\pi/\beta}\gamma = i\pi\frac{e^{\pi/(2\beta)}}{\sin\pi/(2\beta)} = \frac{\pi}{\sin\pi/(2\beta)}\left(i\cos\frac{\pi}{2\beta}-\sin\frac{\pi}{2\beta}\right).$$

Now, taking Re $\left(e^{i\pi/\beta}\gamma\right) = -\pi$, the exponential argument of the bound above will vanish, so $|F|$ is bounded by c.

So $|F| \leq c$ on ∂S.

Since S, we have

$$|F(z)| \leq ce^{\pi\,|z|}|e^{\gamma z}| \leq ce^{2\pi\,|z|}$$

because Re $\gamma = \pi$, which gives us a sufficient global bound to apply Phragmén-Lindelöf.

We will apply the PL principle to a normalized version of $\frac{F(z)}{c}$, which does not change the holomorphicity or growth conditions of F.

Therefore, we have that $F(z) \leq c$ inside S.

Now, as $\beta \to 1$, the sector S becomes the upper half-plane,[2] and $\lim_{\beta\to 1}\gamma = i\pi\frac{e^{i\pi/2}}{\sin\pi/2} = \pi$.

[2]We take the limit to retain the principle log branch cut used in the relation of g to $\hat{f}$.

So $F(z) \to e^{\pi z} g(z)$ is bounded by c in the upper half-plane.

For the lower half-plane, we can use the same bound on the positive real axis, and consider the sector with opening angle $-\pi/\beta$.

Since F is bounded in the lower half-plane as well, and F is entire given that $\hat{f}$ was entire, then F must be constant by Liouville.

Since F is constant, then

$$g(z) = Ce^{-\pi z} = \hat{f}\left(z^{1/2}\right) \Longrightarrow \hat{f}(z) = Ce^{-\pi z^2}.$$

Since the Gaussian function is its own Fourier transform, then we have $f(x) = Ce^{-\pi x^2}$, as desired. □

Proof of (c). If f is odd, then $\hat{f}(0) = \int_{\mathbb{R}} f(x)\, dx = 0$.

Consider $\frac{\hat{f}(z)}{z}$, which has a removable singularity at $z = 0$ by L'Hôpital with $\hat{f}(0) = 0$.

Let $\tilde{g}(z) = \frac{\hat{f}\left(z^{1/2}\right)}{z^{1/2}}$.

As in part (b), we wish to show that $e^{\pi z} \tilde{g}(z)$ is constant.

Now,

$$\tilde{g} = Be^{-\pi z} \Longrightarrow \hat{f}(z) = Bze^{-\pi z^2}.$$

However, to satisfy our decay condition on $\hat{f}$, we must have $B = 0$, so $f = \hat{f} = 0$.

Any function can be written as a sum of even and odd parts,

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f_{e(x)} + f_{o(x)}.$$

By the linearity of the Fourier transform, we can consider $\hat{f}_e$ and $\hat{f}_o$ independently. □