CSC240 Winter 2024 Homework Assignment 2

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1. Let $\mathcal{S}$ denote the set consisting of all 16 binary connectives. For each binary connective $\star \in \mathcal{S}$, define the following propositional formulas:

$A^\star =$ "$((P \text{ OR } Q) \text{ XOR T}) \text{ IMPLIES } (P \star Q)$"
$B^\star =$ "$(P \text{ OR } Q) \text{ IMPLIES } ((P \star Q) \text{ XOR T})$"
$C^\star =$ "$((P \text{ OR } Q) \text{ XOR } P) \text{ IMPLIES } ((P \text{ OR } Q) \text{ XOR } (P \star Q))$"
$D^\star =$ "$((P \text{ OR } Q) \text{ XOR } Q) \text{ IMPLIES } ((P \text{ OR } Q) \text{ XOR } (P \star Q))$"

(a) For how many binary connectives $\star \in \mathcal{S}$ is the formula $A^\star$ a tautology?
Justify your answer without using a truth table.

Claim. There are 8 possible binary connectives $\star \in \mathcal{S}$ such that the formula $A^\star$ is a tautology.

Justification. For implication, the formula is F only when the hypothesis is T and the conclusion is F. So, to let the formula be tautology, when $((P \text{ OR } Q) \text{ XOR } T)$ is T, $(P \star Q)$ must also assert T. Thus, $P \star Q$ must be T when $(P \text{ OR } Q)$ is F. Becasue $(P \text{ OR } Q)$ is F only when both P and Q are F. Therefore, as we can see there are $2 \times 2 \times 2 = 8$ possible binary connectives $\star \in \mathcal{S}$ such that the formula $A^\star$ is a tautology, as needed (here as long as the truth of $\star$ is T when P is F and Q is F, the formula is still a tautology).

(b) For how many binary connectives $\star \in \mathcal{S}$ is the formula $B^\star$ a tautology?
Justify your answer without using a truth table.

Claim. There are 2 possible binary connectives $\star \in S$ such that the formula $B^\star$ is a tautology.

Justification. For implication, the formula is F only when the hypothesis is T and the conclusion is F. So, to let the formula be tautology, when $(P \text{ OR } Q)$ is T, $((P \star Q) \text{ XOR } T)$ must also assert T, that is, $P \star Q$ must be F when $(P \text{ OR } Q)$ is T. Combining the fact that 3 of the 4 cases of $(P \text{ OR } Q)$ are T, we can see there are only 2 possible binary connectives $\star \in S$ such that the formula $B^\star$ is a tautology, as needed.

(c) For how many binary connectives $\star \in \mathcal{S}$ is the formula $C^\star$ a tautology?
Justify your answer without using a truth table.

Claim. There are 8 possible binary connectives $\star \in S$ such that the formula $C^\star$ is a tautology.

Jusitfication. For implication, the formula is F only when the hypothesis is T and the conclusion is F. So, to let the formula be tautology, when $[(P \text{ OR } Q) \text{ XOR } P]$ is T, $((P \text{ OR } Q) \text{ XOR } (P \star Q))$ must also assert T. So, since $[(P \text{ OR } Q) \text{ XOR } P]$ is T only when P is F and Q is T, and to let $((P \text{ OR } Q) \text{ XOR } (P \star Q))$ be T, $P \star Q$ must be F becasue in this case $(P \text{ OR } Q)$ is T. Hence, because the value of $(P \star Q)$ does not matter in 3 of the 4 cases of P and Q, this gives $2 \times 2 \times 2 = 8$ possible binary connectives $\star \in S$ such that the formula $C^\star$ is a tautology, as needed.

(d) For how many binary connectives $\star \in \mathcal{S}$ is at least one of $A^\star$, $B^\star$, $C^\star$, or $D^\star$ a tautology? Justify your answer without using a truth table.

Claim. There are 14 possible binary connectives $\star \in S$ such that at least one of $A^\star$, $B^\star$, $C^\star$, or $D^\star$ is a tautology.

Justification. For the sake of convinience, we will first count the connectives that are impossible to let any of $A^\star$, $B^\star$, $C^\star$, or $D^\star$ be a tautology, then subtract the number from 16 and get our number. First, consider $C^\star$ and $D^\star$, from part (c) we can see $\star$ is impossible to be a connective that makes $C^\star$ a tautology when it asserts T when P is F and Q is T, similarly we can see $\star$ is impossible to be a connective that makes $D^\star$ a tautology when it asserts T when P is T and Q is F. So, there are $2 \times 2 = 4$ connectives that are impossible to make at least one of $C^\star$ or $D^\star$ a tautology. Combining with part (a), we can see the connectives must also assert F when P is F and Q is F so that $A^\star$ is not a tautology, this eliminated the possibility to only 2 connectives (to be impossible). Now, since these 2 connectives are also impossible to make $B^\star$ a tautology, we conclude these 2 connectives are the only ones that cannot make at least one of $A^\star$, $B^\star$, $C^\star$, or $D^\star$ a tautology. Hence, there are $2 \times 2 \times 2 \times 2 - 2 = 14$ possible binary connectives $\star \in S$ such that at least one of $A^\star$, $B^\star$, $C^\star$, or $D^\star$ is a tautology, as needed.

2. Let U denote a set and let $P : U \times U \to \{\text{T}, \text{F}\}$ denote a binary predicate. Consider the following predicate logic formulas:

$$A_1 = \text{“}\forall u \in U. \forall v \in U.([\forall w \in U.(P(w,u) \text{ IFF } P(w,v)))] \text{ IMPLIES}(u = v))\text{”}$$
$$A_2 = \text{“}\exists u \in U. \forall v \in U.(\text{NOT}(P(v,u)))\text{”}$$
$$A_3 = \text{“}\forall u \in U. \exists v \in U. \forall w \in U.(P(w,v) \text{ IFF } [\exists x \in U.(P(w,x) \text{ AND } P(x,u))])\text{”}$$
$$A_4 = \text{“}\forall u \in U. \forall v \in U. \exists w \in U. \forall x \in U.[P(x,w) \text{ IFF } ((x = u) \text{ OR } (x = v))]\text{”}$$

(a) Consider the interpretation where $U = \mathbb{N}$ and $P(u,v)$ is T if and only if $u < v$.
Which of A_1, A_2, A_3, and A_4 are true under this interpretation? Justify your answer.

Claim. A_1 is true.

Jusitfication. For A_1, consider 3 cases.

If $u < v$, then we can always choose $w = u$ such that $P(w,u)$ is F and $P(w,v)$ is T. So, the hypothesis is F which makes the entire predicate logic formula T.

If $v < u$, similar to the preivous case, then we can always choose $w = v$ such that $P(w,v)$ is F and $P(w,u)$ is T. So the hypothesis is F which makes the entire predicate logic formula T.

If $u = v$, then becasue the conlcusion of implication is T, this makes the entire predicate logic formula T too.

Hence, A_1 is true, as needed.

Claim. A_2 is true.
Jusitfication. For A_2, we can simply fix $u = 0$, then $P(v, u) = P(v, 0)$, since $P(v, 0)$ if and only if $v < 0$, thus $P(v, 0)$ is F for all $v \in \mathbb{N}$. So, the negation of $P(v, u)$ is T for all $v \in \mathbb{N}$. Hence, A_2 is true, as needed.

Claim. A_3 is true.
Jusitfication. We first break the IFF into forward and backward implications, after that, we rename the ambiguous variables, then, we change the formula into prenax normal form. From this form, we justify our claim.
First, A_3 is equivalent to

$$\forall u \in U.\exists v \in U.\forall w \in U.[[P(w, v) \text{ IMPLIES } (\exists x \in U.(P(w, x) \text{ AND } P(x, u)))] \text{ AND } \\ [(\exists x \in U.(P(w, x) \text{ AND } P(x, u)) \text{ IMPLIES } P(w, v))]].$$

Now, to avoid ambiguity, we change the second x to y, that is, A_3 is now equivalent to

$$\forall u \in U.\exists v \in U.\forall w \in U.[[P(w, v) \text{ IMPLIES } (\exists x \in U.(P(w, x) \text{ AND } P(x, u)))] \text{ AND } \\ [(\exists y \in U.(P(w, y) \text{ AND } P(y, u)) \text{ IMPLIES } P(w, v))]].$$

Then, we change the formula into prenax form,

$$\forall u \in U.\exists v \in U.\forall w \in U.\exists x \in U.\forall y \in U.[[P(w, v) \text{ IMPLIES } ((P(w, x) \text{ AND } P(x, u)))] \\ \text{AND } [((P(w, y) \text{ AND } P(y, u)) \text{ IMPLIES } P(w, v))]].$$

Consider 2 cases. When $u = 0$, we can simply fix $v = u = 0$, then $P(w, v) = P(w, 0)$ and $P(y, u) = P(y, 0)$ are both F for all $w, y \in \mathbb{N}$, thus the two implications are vacuously true.
When $u > 0$, that is, $u \geq 1$, we choose $v = x = u - 1$, then $w < v$ IMPLIES $w < x$ and $w < v$ IMPLIES $x < v + 1 = u$, so we can see both implications are always true.
Hence, A_3 is true, as needed.

Claim. A_4 is false.
Jusitfication. We show A_4 is false by showing its negation is T. That is, we want to show that

$$\exists u \in U.\exists v \in U.\forall w \in U.\exists x \in U.[[P(x, w) \text{ AND NOT}((x = u) \text{ OR } (x = v))] \text{ OR } \\ [((x = u) \text{ OR } (x = v)) \text{ AND NOT}(P(x, w))]]$$

holds T.
Consider $u = 100$ and $v = 100$, for all $w \in \mathbb{N}$, if $w = 0$, we choose $x = u$, then $P(x, w)$ is F and $((x = u)$ OR $(x = v))$ is T, showing the formula above holds T; if $w \neq 0$, then choose $x = 0$, then $P(x, w)$ is T and $((x = u)$ OR $(x = v))$ is F, showing the formula above holds T. Hence, A_4 is false, as needed.

(b) Consider the interpretation where $U = \mathbb{N}$ and for $u, v \in \mathbb{N}$, predicate $P(u, v)$ is true if and only if the u^{th} least significant digit in the binary expansion of v is 1.
Which of A_1, A_2, A_3, and A_4 are true under this interpretation? Justify your answer.
Claim. A_1 is true.
Jusitfication. The hypothesis of A_1 means, u has the same w^{th} least significant digit number as v for all $w \in \mathbb{N}$. So, this means $u = v$ which is precisely our conclusion, otherwise one of the digits of u and v must be 0, which contradicts the hypothesis. Hence, A_1 is true, as needed.

Claim. A_2 is true.
Jusitfication. Let $u = 0$, then $P(v, u)$ is F for all $v \in \mathbb{N}$, so the negation of $P(v, u)$ is T for all $v \in \mathbb{N}$. Hence, A_2 is true, as needed.

Claim. A_3 is true.
Jusitfication. From Part (a), we changed A_3 equivalently to

$$\forall u \in U.\exists v \in U.\forall w \in U.\exists x \in U.\forall y \in U.[[P(w, v) \text{ IMPLIES } ((P(w, x) \text{ AND } P(x, u)))] \\ \text{AND } [((P(w, y) \text{ AND } P(y, u)) \text{ IMPLIES } P(w, v))]].$$

Now, becasue there is no such natural number that the w^{th} least significant digit is 1 for all $w \in \mathbb{N}$, thus the first implication is vacuously true becasue of the for all quantifier of w. Moreover, if we fix $v = u$, then $P(y, u)$ IMPLIES $P(y, v)$ is always true. Hence, A_3 is true, as needed.

Claim. A_4 is true.
Jusitfication. Let w be the number where its u^{th} least significant digit is 1 and its v^{th} least significant digit is 1, and all other least significant digits are 0. Then, we can see $P(x, w)$ is T if and only if $x = u$ or $x = v$, which is precisely our if and only if formula, so A_4 is true, as needed.

(c) Is A_4 logically implied by the formula A_1 AND A_2 AND A_3? Justify your answer.
Claim. A_4 is not logically implied by the formula A_1 AND A_2 AND A_3.
Justification. Since in the interpretation of part (A), A_4 is false, but A_1 AND A_2 AND A_3 is true, so A_4 is not logically implied by the formula A_1 AND A_2 AND A_3, as needed.

(d) Is A_2 logically implied by the formula "A_1 AND A_3 AND A_4"? Justify your answer.
Claim. A_2 is not logically implied by the formula "A_1 AND A_3 AND A_4.
Justification. Consider the interpretation where $U = \mathbb{N} - \{0\}$ and for $u, v \in \mathbb{N} - \{0\}$, predicate $P(u, v)$ is true if and only if the $(u-1)^{\text{th}}$ least significant digit in the binary expansion of v is 1.
Then, similar to Part (B),
For A_1, the hypothesis of A_1 is, u, v have the same number for all least significant digits, this directly gives us $u = v$ which is out conclusion, otherwise contradicts the hypothesis. Thus A_1 holds true.
For A_2, becasue there is no such $u \in U$ where all its least significant digits are 0, this implies A_2 is false.

For A_3, first it is equivalent to

$$\forall u \in U.\exists v \in U.\forall w \in U.\exists x \in U.\forall y \in U.[[P(w,v) \text{ IMPLIES } ((P(w,x) \text{ AND } P(x,u)))] \\ \text{AND } [((P(w,y) \text{ AND } P(y,u)) \text{ IMPLIES } P(w,v))]].$$

from part (A) and (B).

Now, since there is no such v that has 1 for all least significant digits (we consider the leading 0's as least significant digits too), the first implication is vacuously true. Moreover, if we fix $v = u$, then $P(y,u)$ IMPLIES $P(y,v)$ is always true. Hence, A_3 is true.

For A_4, let w be the number where its $(u-1)^{\text{th}}$ least significant digit is 1 and its $(v-1)^{\text{th}}$ least significant digit is 1, and all other least significant digits are 0. Then, we can see $P(x,w)$ is T if and only if $x = u$ or $x = v$, which is precisely our if and only if formula, so A_4 is true.

Therefore, in this interpretation A_1, A_3, A_4 are all T, however A_2 is F, showing A_2 is not logically implied by the formula A_1 AND A_3 AND A_4, completing our justification.