CSC240 Winter 2024 Homework Assignment 3

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The list of people with whom I discussed this homework assignment:

1. Let $\mathcal{F}$ be the set of all functions from D to D, where D is a nonempty set.
Consider the following two predicates with domain $\mathcal{F} \times \mathcal{F}$:

$$\begin{aligned} P(f,g) &= \exists y \in D.\forall x \in D.[f(g(x)) \neq y] \text{ and} \\ Q(f,g) &= \exists v \in D.[\forall u \in D.(f(u) \neq v) \text{ OR } \forall u \in D.(g(u) \neq v)]. \end{aligned}$$

Formally prove that $\forall f \in \mathcal{F}.\forall g \in \mathcal{F}.(P(f,g) \text{ IMPLIES } Q(f,g))$.
Remember to number all lines, indent properly, and justify all your steps, including references to the appropriate line numbers, as described in Proof Outlines. Only do one step of the proof per line.

Proof.

1 Let $f \in \mathcal{F}$ be arbitrary;
2 Let $g \in \mathcal{F}$ be arbitrary;
3 Assume $P(f,g)$;
4 $P(f,g)$ IFF $[\exists a \in D.\forall b \in D.(f(g(b)) \neq a)]$;
5 $\exists a \in D.\forall b \in D.(f(g(b)) \neq a)$; modus ponens
6 Let $y \in D$ such that $\forall b \in D.(f(g(b)) \neq y)$; instantiation
7 $(\forall d \in D.f(d) \neq y)$ OR NOT$(\forall d \in D.f(d) \neq y)$; tautology
8 Case 1. $\forall d \in D.f(d) \neq y$;
9 $\forall d \in D.(f(d) \neq y)$ OR $\forall d \in D.(g(d) \neq y)$; proof of disjunction
10 $\exists y \in D.[\forall d \in D.(f(d) \neq y)$ OR $\forall d \in D.(g(d) \neq y)]$
11 Case 2. NOT$(\forall d \in D.f(d) \neq y)$;
12 NOT$(\forall d \in D.f(d) \neq y)$ IFF $(\exists d \in D.f(d) = y)$; tautology
13 $\exists d \in D.f(d) = y$; modus ponens
14 Let $x \in D$ such that $f(x) = y$; instantiation
15 To reach contradiction, assume $\exists e \in D.x = g(e)$;
16 $f(x) = f(x)$; tautology
17 $f(x) = f(g(e))$; substitution
18 $f(g(e)) \neq y$;
19 $f(x) \neq y$; substitution
20 $\forall p \in D.x \neq g(p)$; proof by contradiction
21 $\forall p \in D.f(p) \neq x$ OR $\forall p \in D.g(p) \neq x$; proof of disjunction
22 $\exists x \in D.[\forall p \in D.f(p) \neq x$ OR $\forall p \in D.g(p) \neq x]$;
23 $\exists v \in D.[\forall p \in D.f(p) \neq x$ OR $\forall p \in D.g(p) \neq x]$; proof by cases
24 $Q(f,g)$;
25 $P(f,g)$ IMPLIES $Q(f,g)$;
26 $\forall g \in \mathcal{F}.[P(f,g)$ IMPLIES $Q(f,g)]$;
27 $\forall f \in \mathcal{F}.\forall g \in \mathcal{F}.[P(f,g)$ IMPLIES $Q(f,g)]$;

□

2. Recall that, if p is a polynomial of degree $m \geq 1$, then there exist coefficients a_i for $0 \leq i \leq m$ such that $a_m \neq 0$ and, for all numbers n,

$$p(n) = \sum_{i=0}^{m} a_i n^i.$$

Give a well-structured informal proof that, for any polynomial p of degree at least 1 whose coefficients are natural numbers, there is a natural number n such that $p(n)$ is not prime.

Proof.

Assume p is a polynomial of degree $m \geq 1$ whose coefficients are natural numbers.

Case 1: $a_0 = 0$ or $a_0 = 1$;

Let $n = 0 \in \mathbb{N}$;

$$p(n) = \sum_{i=0}^{m} a_i n^i = a_0 \cdot 0^0 + \sum_{i=1}^{m} a_i 0^i = a_0$$

Since 0 is not prime and 1 is not prime, these imply $p(n)$ is also not prime.

For Case 1, we have shown that there exists a natural number n such that $p(n)$ is not prime.

Case 2: $a_0 \geq 2$;

Let $n = a_0 \in \mathbb{N}$;

$$p(n) = a_0 \cdot a_0^0 + \sum_{i=1}^{m} a_i a_0^i = a_0 + a_0 \sum_{i=1}^{m} a_i a_0^{i-1} = a_0 \left(1 + \sum_{i=1}^{m} a_i a_0^{i-1}\right)$$

Since $a_0 \in \mathbb{N}$, $1 + \sum_{i=1}^{m} a_i a_0^{i-1} \in \mathbb{N}$, $a_0 \neq 1$, and $\left(1 + \sum_{i=1}^{m} a_i a_0^{i-1}\right) \geq (1 + a_m a_0^{m-1}) \geq (1 + a_m) \geq 2 \neq 1$, these imply $p(n)$ is not prime.

For Case 2, we have shown that there exists a natural number n such that $p(n)$ is not prime.

For all cases, we have shown that there is such natural number n which makes $p(n)$ not a prime.

Therefore, we conclude if p is a polynomial of degree $m \geq 1$ whose coefficients are natural numbers, then there is a natural number n suc hthat $p(n)$ is not prime. □