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Question 1

Give a well-structured informal proof by induction that, for each positive integer n and each sequence $r = \{r_i\}_{i=1}^n$ of n positive real numbers,

$$\prod_{i=1}^{n} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n} r_i}{1+\sum_{i=1}^{n} r_i}.$$

Lemma 1

For any $x \in \mathbb{R}^+$ we have

$$-1 < \frac{1-x}{1+x} < 1 \text{ and } 0 \leq \left|\frac{1-x}{1+x}\right| < 1.$$

Proof of Lemma **??**.

Let $x \in \mathbb{R}^+$ be arbitrary;

Then we have $x > 0$ and $2x > 0$, which gives $-1 < 1 < 1 + 2x$;

Subtract x, then divide by $x + 1$ (which is positive) we get $-1 < \frac{1-x}{1+x} < 1$;

Consider 2 cases of $-1 < \frac{1-x}{1+x} < 1$: $-1 < \frac{1-x}{1+x} < 0$ and $0 \leq \frac{1-x}{1+x} < 1$;

Case 1: assume $-1 < \frac{1-x}{1+x} < 0$;

By definition of absolute value we have $\left|\frac{1-x}{1+x}\right| = -\frac{1-x}{1+x}$;

Multiply our inequality by -1 and we have $1 > -\frac{1-x}{1+x} > 0$;

Substitute the absolute value and we have $0 < \left|\frac{1-x}{1+x}\right| < 1$ which gives $0 \leq \left|\frac{1-x}{1+x}\right| < 1$;

For Case 1 we have shown that $0 \leq \left|\frac{1-x}{1+x}\right| < 1$;

Case 2: assume $0 \leq \frac{1-x}{1+x} < 1$;

By definition of absolute value we have $\left|\frac{1-x}{1+x}\right| = \frac{1-x}{1+x}$;

Substitute the absolute value and we have $0 \leq \left|\frac{1-x}{1+x}\right| < 1$;

For Case 2 we have shown that $0 \leq \left|\frac{1-x}{1+x}\right| < 1$;

Since we have shown that $0 \leq \left|\frac{1-x}{1+x}\right| < 1$ for all cases, we conclude $0 \leq \left|\frac{1-x}{1+x}\right| < 1$;

Since our $x \in \mathbb{R}^+$ is arbitrary, hence we conclude that for any $x \in \mathbb{R}^+$ we have $-1 < \frac{1-x}{1+x} < 1$ and $0 \leq \left|\frac{1-x}{1+x}\right| < 1$.

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Lemma 2

For any $n \in \mathbb{Z}^+$. For any sequence $r = \{r_i\}_{i=1}^n$ of n positive real numbers. If for some sequence s that is an rearranged version of r (e.g. rearrange r_i in r to non-increasing order) we have $\prod_{i=1}^{n} \frac{1-s_i}{1+s_i} \geq \frac{1-\sum_{i=1}^{n} s_i}{1+\sum_{i=1}^{n} s_i}$, then $\prod_{i=1}^{n} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n} r_i}{1+\sum_{i=1}^{n} r_i}$ holds for r.

Proof of Lemma **??**.

Let $n \in \mathbb{Z}^+$ be arbitrary.

Let $r = \{r_i\}_{i=1}^n$ be an arbitrary sequence of n positive real numbers.

Assume $\prod_{i=1}^{n} \frac{1-s_i}{1+s_i} \geq \frac{1-\sum_{i=1}^{n} s_i}{1+\sum_{i=1}^{n} s_i}$ holds for some sequence s that is an rearranged version of r;

Because s is an rearranged version of the finite sequence r, and since addition and multiplication are commutative for real numbers, we can rearrange the terms in the inequality to get $\prod_{i=1}^{n} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n} r_i}{1+\sum_{i=1}^{n} r_i}$.

Because r is arbitrary, the implication holds for all such sequence r.

Because n is arbitrary, the implication holds for all $n \in \mathbb{Z}^+$, for all such sequence r.

We conclude our lemma holds, as needed.

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Lemma 3

$$\forall x \in \mathbb{R}^+ . \forall y \in \mathbb{R}^+ . \left[\frac{1-x+y}{1+x+y} \geq \frac{1-x}{1+x} \right].$$

Proof of Lemma **??**.

Let $x \in \mathbb{R}^+$ be arbitrary;

Let $y \in \mathbb{R}^+$ be arbitrary;

Since $x > 0, y > 0$, by arithmetic we have

$$\begin{gathered} yx \geq -yx \\ 1 - x + y + x - x^2 + yx \geq 1 + x + y - x - x^2 - yx \\ (1+x)(1-x+y) \geq (1-x)(1+x+y) \\ \frac{1-x+y}{1+x+y} \geq \frac{1-x}{1+x} \end{gathered}$$

So, $\forall y \in \mathbb{R}^+ . \left[\frac{1-x+y}{1+x+y} \geq \frac{1-x}{1+x} \right]$.

Hence, $\forall x \in \mathbb{R}^+ . \forall y \in \mathbb{R}^+ . \left[\frac{1-x+y}{1+x+y} \geq \frac{1-x}{1+x} \right]$.

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Lemma 4

Let $n \in \mathbb{Z}^+$. Define the predicate $P(n) =$"For any sequence $r = \{r_i\}_{i=1}^n$ of n positive real numbers such that $r_1 > 1$, $\left|\prod_{i=1}^{n} \frac{1-r_i}{1+r_i}\right| \le \left|\frac{1-r_1}{1+r_1}\right| \le \left|\frac{1-\sum_{i=1}^n r_i}{1+\sum_{i=1}^n r_i}\right|$." Then $P(n)$ holds for all $n \in \mathbb{Z}^+$.

Proof of Lemma **??** *by induction.*

Consider the base case when $n = 1$;

Let $r = \{r_i\}_{i=1}^1$ be an arbitrary sequence of 1 positive real numbers such that $r_1 > 1$;

$\left|\prod_{i=1}^{1} \frac{1-r_i}{1+r_i}\right| = \left|\frac{1-r_1}{1+r_1}\right| = \left|\frac{1-\sum_{i=1}^1 r_i}{1+\sum_{i=1}^1 r_i}\right|$;

Since r is arbitrary, $P(1)$ holds.

Now let $n \in \mathbb{Z}^+$ be arbitrary;

Assume $P(n)$;

Let $r = \{r_i\}_{i=1}^{n+1} = \{r_i\}_{i=1}^{n} \circ \{r_i\}_{i=n+1}^{n+1}$ (concatenation in course note) be an arbitrary sequence of $n+1$ positive real numbers such that $r_1 > 1$;

Since $\{r_i\}_{i=1}^n$ is covered by $P(n)$, we have $\left|\prod_{i=1}^{n} \frac{1-r_i}{1+r_i}\right| \le \left|\frac{1-r_1}{1+r_1}\right|$;

Since $r_{n+1} \in \mathbb{R}^+$, by Lemma **??** we have $\left|\frac{1-r_{n+1}}{1+r_{n+1}}\right| < 1$;

Combine the above 2 inequalities we get

$\left|\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i}\right| = \left|\frac{1-r_{n+1}}{1+r_{n+1}}\right| \left|\prod_{i=1}^{n} \frac{1-r_i}{1+r_i}\right| \le \left|\frac{1-r_1}{1+r_1}\right| \left|\frac{1-r_{n+1}}{1+r_{n+1}}\right| \le \left|\frac{1-r_1}{1+r_1}\right|$;

Now, since the sum of positive numbers is positive, $n+1 \ge 2$, $r_1 > 1$, and by arithmetic we have the following:

$$\begin{aligned}
-\textstyle\sum_{i=2}^{n+1} r_i &< \textstyle\sum_{i=2}^{n+1} r_i \\
(r_1 - 1)\textstyle\sum_{i=2}^{n+1} r_i &< (r_1 + 1)\textstyle\sum_{i=2}^{n+1} r_i \qquad (\text{add } r_1 \textstyle\sum_{i=2}^{n+1} r_i) \\
\left|(r_1-1)(r_1+1) + (r_1-1)\textstyle\sum_{i=2}^{n+1} r_i\right| &< \left|(r_1-1)(r_1+1) + (r_1+1)\textstyle\sum_{i=2}^{n+1} r_i\right| \\
\left|(1-r_1)(1+r_1) + (1-r_1)\textstyle\sum_{i=2}^{n+1} r_i\right| &< \left|(1-r_1)(1+r_1) - (1+r_1)\textstyle\sum_{i=2}^{n+1} r_i\right| \\
\left|(1-r_1)(1+r_1+\textstyle\sum_{i=2}^{n+1} r_i)\right| &< \left|(1+r_1)(1-r_1-\textstyle\sum_{i=2}^{n+1} r_i)\right| \\
\left|\frac{1-r_1}{1+r_1}\right| &< \left|\frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}\right|
\end{aligned}$$

Combining 2 inequalities: $\left|\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i}\right| \le \left|\frac{1-r_1}{1+r_1}\right| \le \left|\frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}\right|$;

Since r is arbitrary such sequence, $P(n+1)$ holds.

Hence, by induction, $P(n)$ holds for all $n \in \mathbb{Z}^+$.

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*Proof of Question **??** by induction.*

Define the predicate $Q(n) =$ "For each sequence $r = \{r_i\}_{i=1}^n$ of n positive real numbers, $\prod_{i=1}^n \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^n r_i}{1+\sum_{i=1}^n r_i}$."

Base Case: $n = 1$;

Let $r = \{r_i\}_{i=1}^n$ be an arbitrary sequence of n positive real numbers;

$\prod_{i=1}^1 \frac{1-r_i}{1+r_i} = \frac{1-r_1}{1+r_1} = \frac{1-\sum_{i=1}^1 r_i}{1+\sum_{i=1}^1 r_i}$;

Since r is arbitrary, $Q(1)$ holds.

Let $n \in \mathbb{Z}^+$ be arbitrary;

Assume $Q(n)$ holds;

Let $r = \{r_i\}_{i=1}^{n+1}$ be arbitrary sequence of $n+1$ positive real numbers;

Let $s = \{r_i\}_{i=1}^{n+1}$ be an rearranged sequence of r such that for all $i \in [1,n] \cap \mathbb{N}$, $s_i \geq s_{i+1}$ (i.e. non-increasing sequence);

Consider 2 cases: $s_1 \leq 1$ and $s_1 > 1$;

Case 1: assume $s_1 \leq 1$;

Since $Q(n)$, we have $\prod_{i=1}^n \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^n r_i}{1+\sum_{i=1}^n r_i}$;

Since s is non-increasing, this implies $r_{n+1} \leq s_1 \leq 1$;

So, we get $r_{n+1} \leq 1$ which implies $\frac{1-r_{n+1}}{1+r_{n+1}} \geq 0$;

Apply inductive hypothesis we get

$$\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i} = \frac{1-r_{n+1}}{1+r_{n+1}} \prod_{i=1}^{n} \frac{1-r_i}{1+r_i} \geq \frac{1-r_{n+1}}{1+r_{n+1}} \frac{1-\sum_{i=1}^n r_i}{1+\sum_{i=1}^n r_i} = \frac{1-\sum_{i=1}^{n+1} r_i + r_{n+1}\sum_{i=1}^n r_i}{1+\sum_{i=1}^{n+1} r_i + r_{n+1}\sum_{i=1}^n r_i}$$

By Lemma **??** we have $\frac{1-\sum_{i=1}^{n+1} r_i + r_{n+1}\sum_{i=1}^n r_i}{1+\sum_{i=1}^{n+1} r_i + r_{n+1}\sum_{i=1}^n r_i} \geq \frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}$;

Hence, we get $\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}$;

We have shown that when $s_1 \leq 1$, $\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}$ holds.

Case 2: assume $s_1 > 1$;

By Lemma **??**, for all $m \in \mathbb{Z}^+$. $\left|\prod_{i=1}^{m+1} \frac{1-s_i}{1+s_i}\right| \leq \left|\frac{1-\sum_{i=1}^{m+1} s_i}{1+\sum_{i=1}^{m+1} s_i}\right|$;

$s_1 > 1$ implies $\sum_{i=1}^{m+1} s_i > 1$, thus $\frac{1-\sum_{i=1}^{m+1} s_i}{1+\sum_{i=1}^{m+1} s_i} < 0$, to remove the absolute value sign we multiply both sides by -1, now $\prod_{i=1}^{m+1} \frac{1-s_i}{1+s_i} \geq -\left|\prod_{i=1}^{m+1} \frac{1-s_i}{1+s_i}\right| \geq -\left|\frac{1-\sum_{i=1}^{m+1} s_i}{1+\sum_{i=1}^{m+1} s_i}\right| = \frac{1-\sum_{i=1}^{m+1} s_i}{1+\sum_{i=1}^{m+1} s_i}$;

By Lemma **??**, this implies $\forall m \in \mathbb{Z}^+. \prod_{i=1}^{m+1} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{m+1} r_i}{1+\sum_{i=1}^{m+1} r_i}$;

By specialization, we get $\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}$;

We have shown that when $s_1 > 1$, $\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}$ holds.

We conclude $\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}$ holds.

Since r is arbitrary, $Q(n+1)$ holds.

Hence, by induction, $Q(n)$ holds for all $n \in \mathbb{Z}^+$.

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Question 2

Proof of Question ?? by induction.
Define the predicate $P(n)$ ="There exists an n-bit gradually changing sequence."
Base Case $n = 1$: Consider $0, 1$, since the length of this sequence is $2^1 = 2$, all strings are unique, and satisfies the definitions (first and last strings differ in 1 position, and consecutive strings differ in 1 position), this sequence is a 1-bit gradually changing sequence, thus $P(1)$ holds.

Let $n \in \mathbb{Z}^+$ be arbitrary;
Assume $P(n)$;
By instantiation of $P(n)$, let $s = \{s_i\}_{i=1}^{2^n}$ be an n-bit gradually changing sequence;
Let $s^R = \{s_i^R\}_{i=1}^{2^n}$ be the reversal of s: since s is gradually changing sequence, the uniqueness and length follows; the differ of consecutive strings follows as all strings still have the same strings next to them; the differ of first and last strings follows as the first and last strings are now swapped. Thus we conclude s^R is also a n-bit gradually changing sequence;
Consider $s' = \{0s_i\}_{i=1}^{2^n} \circ \{1s_i^R\}_{i=1}^{2^n}$:

1. since s and s^R are gradually changing sequence, we have consecutive strings in $\{0s_i\}_{i=1}^{2^n}$ and $\{1s_i^R\}_{i=1}^{2^n}$ differ in only 1 position respectively;
2. by definition of s^R the last string $0s_n$ in $\{0s_i\}_{i=1}^{2^n}$ and the first string $1s_n$ in $\{1s_i^R\}_{i=1}^{2^n}$ differ only in the first position;
3. by definition of s^R the first string $0s_1$ in $\{0s_i\}_{i=1}^{2^n}$ and the last string $1s_1$ in $\{1s_i^R\}_{i=1}^{2^n}$ also differ only in the first position;
4. s' is a sequence of $2^n + 2^n = 2^{n+1}$ strings by our concatenation;
5. No string in $\{0s_i\}_{i=1}^{2^n}$ is in $\{1s_i^R\}_{i=1}^{2^n}$ and vice versa becasue of their first bit $0 \neq 1$. Moreover, all strings in $\{0s_i\}_{i=1}^{2^n}$ and $\{1s_i^R\}_{i=1}^{2^n}$ are unique respectively becasue s and s^R are gradually changing sequence. Combining these two facts, we have all strings in $\{0s_i\}_{i=1}^{2^n} \circ \{1s_i^R\}_{i=1}^{2^n}$ are unique;
6. All strings in $\{0s_i\}_{i=1}^{2^n}$ or $\{1s_i^R\}_{i=1}^{2^n}$ are $n+1$ bits long becasue of our concatenation $0s_i$ and $1s_i^R$ for all $i \in [1, 2^n] \cap \mathbb{N}$;

Hence, since all definitions are satisfied, we conclude s' is a $n+1$-bit gradually changing sequence, which by construction $P(n+1)$ holds.
Hence, by induction, $P(n)$ holds for all $n \in \mathbb{Z}^+$.

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