CSC240 Winter 2024 Quiz 3

Let $a, b, c, d \in \mathbb{N}$ be such that $a^2 + b^2 + c^2 = d^2$.

Give a well-structured informal proof that if d is even, then a, b, and c are all even.

You may use the following facts without proof:

1. The sum of two even numbers is even.
2. The sum of two odd numbers is even.
3. The sum of an even number and an odd number is odd.
4. The product of an even number and any natural number is even.

Proof. We will prove the contrapositive of the statement, that is, if a, b, and c are not all even, then d is not even.

First, we consider 3 cases:

Case 1: Only one of a, b, and c is odd. W.l.o.g. we assume a is odd. Then, $a = 2k + 1$ for some $k \in \mathbb{N}$, $b = 2j$, $c = 2i$, for some $i, j \in \mathbb{N}$. We rewrite the equation as $(2k+1)^2 + (2j)^2 + (2i)^2 = d^2$, which can be expanded as $2(2k^2 + 2k + 2j^2 + 2i^2) + 1 = d^2$, here d must be odd so that d^2 is odd, otherwise d^2 will be even which is impossible.

Case 2: Two of a, b, c are odd. W.l.o.g. we assume a, b are odd. Then, $a = 2k + 1$, $b = 2j + 1$, $c = 2i$, for some $i, j, k \in \mathbb{N}$. We rewrite the equation as $(2k + 1)^2 + (2j + 1)^2 + (2i)^2 = d^2$, which can be expanded as $4(k^2 + k + j^2 + j + i^2) + 2 = d^2$, here d must be odd, otherwise if $d = 2m$ for some $m \in \mathbb{N}$, then $d^2 = 4m^2$, however the left hand side is not divisible by 4.

Case 3: a, b, c are all odd. Then, $a = 2k + 1$, $b = 2j + 1$, $c = 2i + 1$, for some $i, j, k \in \mathbb{N}$. We rewrite the equation as $(2k + 1)^2 + (2j + 1)^2 + (2i + 1)^2 = d^2$, which can be expanded as $4(k^2 + k + j^2 + j + i^2 + i + j) + 3 = d^2$, here d must be odd, otherwise if $d = 2m$ for some $m \in \mathbb{N}$, then $d^2 = 4m^2$, however the left hand side is not divisible by 4.

Thus, as all possible cases lead to d being odd, we have proved the contrapositive of the statement, that is, if a, b, and c are not all even, then d is not even, completing our proof.

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