CSC240 Winter 2024 Homework Quiz 6

Solve the recurrence

$$T(n) = \begin{cases} 5 & \text{if } n = 1 \\ 8T(\lceil n/3 \rceil) + n^2 & \text{if } n > 1 \end{cases}$$

to within a constant factor. Briefly justify your answer.

Claim. $T(n) \in \Theta(n^2)$.

Proof. Let $a = 8, b = 3, f(n) = n^2$.

Since $\log_b a = \log_3 8 \approx 1.89$, we have $f(n) = n^2 > n^{\log_b a + 0.01}$ for all $n \geq 1$. Let $c = 1$ and $\varepsilon = 0.01$, we can see the first sub-condition of the 3rd case of the Master Theorem is satisfied.

Since $8f(n/3) = 8(n/3)^2 = 8n^2/9 = \frac{8}{9}n^2$, we have $af(n/b) = \frac{8}{9}f(n) \leq \frac{99}{100}f(n)$ for all $n \geq 1$. Let $c = \frac{99}{100}$ we can see the second sub-condition of the 3rd case of the Master Theorem is satisfied.

Hence, $T(n)$ is the third case of the Master Theorem (The MIT Textbook version), hence we conclude $T(n) \in \Theta(f(n)) = \Theta(n^2)$, as needed.

Since we have shown that $T(n) \in \Theta(n^2)$, this implies the value of $T(n)$ is within a constant factor of n^2 by definition of $\Theta(n^2)$. $\square$