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Recall the recursively defined set NNF of propositional formulas in negation normal form as follows:

Base Case: For any propositional variable P, $P \in NNF$, $(P) \in NNF$.

Constructor Case: If $f, f' \in NNF$, then $(ff') \in NNF$ and $(ff') \in NNF$.

Consider the recursively defined set S as follows:

Base Case: For any propositional variable P, $P \in S$.

Constructor Case: If $f, f' \in S$, then $(ff') \in S$, $(ff') \in S$, and $(f) \in S$.
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Lemma 1

If $g' \in NNF$, then there exist $g \in NNF$ such that $(g')g$ and g', g have the same number of variables. ♠

Proof of Lemma 1 by strong induction.
For any propositional formula f, let $N_v(f)$="# of occurrences of propositional variables in f;" let $N_c(f)$="# of connectors $(, ,)$ in f."
For any $n \in \mathbb{Z}^+$, let $M(n) =$ "$\forall f \in NNF.O(n, f)''$ where

$$O(n, f) = \text{``}[n = N_v(f)[\exists g \in NNF.(N_v(f) = N_v(g))((f)g)]].''$$

We now prove $\forall n \in \mathbb{Z}^+.M(n)$ by strong induction on n.

Base Case: $n = 1$;

Let $f \in NNF$ be arbitrary, then either $1 = N_v(f)$ or $1 \neq N_v(f)$;

Case 1: $1 \neq N_v(f)$;

Since $1 \neq N_v(f)$, the implication of $O(1, f)$ is vacuously true.

For Case 1 we have shown $O(1, f)$.

Case 2: $1 = N_v(f)$;

There can be only 2 possible cases for f: $f = P$ and $f = (P)$ for some propositional variable P.

Case 2.1: $f = P$;

Let $g = (P) \in NNF$;

Since $N_v(f) = N_v(g)$, and $(f) = (P)(P) = g$, we have $O(1, f)$.

For Case 2.1 we have shown $O(1, f)$.

Case 2.2: $f = (P)$;

Let $g = P \in NNF$;

Since $N_v(f) = N_v(g)$, and $(f) = ((P))P = g$, we have $O(1, f)$.

For Case 2.2 we have shown $O(1, f)$.

For all cases of Case 2 we have shown $O(1, f)$, thus $O(1, f)$.

For Case 2 we have shown $O(1, f)$.

For all cases of $n = 1$ we have shown $O(1, f)$, thus $O(1, f)$.

Since f is arbitrary, we have shown $\forall f \in NNF.O(1, f)$, thus $M(1)$.

Let $n \in \mathbb{Z}^+ - \{1\}$ be arbitrary;

Assume $\forall i \in \mathbb{Z}^+.[i < nM(i)]$;

Let $f \in NNF$ be arbitrary, then either $n = N_v(f)$ or $n \neq N_v(f)$;

Case 1: $n \neq N_v(f)$;

Since $n \neq N_v(f)$, the implication of $O(n, f)$ is vacuously true, thus $O(n, f)$.

For Case 1 we have shown $O(n, f)$.

Case 2: $n = N_v(f)$;

Since $N_v(f) = n > 1$, this implies f is not a propositional variable (not constructed from the base case), thus consider 2 cases (of the constructor cases): $f = (f'f'')$ and $f = (f'f'')$ for some $f', f'' \in NNF$.

Case 2.1: $f = (f'f'')$;

Since $1 \leq N_v(f') < n$ and $1 \leq N_v(f'') < n$, apply inductive hypothesis we have there exist $g', g'' \in NNF$ such that $N_v(f') = N_v(g')$ and $N_v(f'') = N_v(g'')$ and $(f')g'$ and $(f'')g''$;
By Demorgan's Law, we have $(f) = (f'f'')((f')(f''))$;
By substition of equivalent formula, we have $(f)(g'g'')$ where $g'g'' \in NNF$ by our constructor cases;
Let $g = (g'g'') \in NNF$, then $N_v(g) = N_v(g'g'') = N_v(g') + N_v(g'') = N_v(f') + N_v(f'') = N_v(f)$;
Thus, for Case 2.1 we have $O(n, f)$.
Case 2.2: $f = (f'f'')$;
Since $1 \leq N_v(f') < n$ and $1 \leq N_v(f'') < n$, apply inductive hypothesis we have there exist $g', g'' \in NNF$ such that $N_v(f') = N_v(g')$ and $N_v(f'') = N_v(g'')$ and $(f')g'$ and $(f'')g''$;
By Demorgan's Law, we have $(f) = (f'f'')((f')(f''))$;
By substition of equivalent formula, we have $(f)(g'g'')$ where $g'g'' \in NNF$ by our constructor cases;
Let $g = (g'g'') \in NNF$, then $N_v(g) = N_v(g'g'') = N_v(g') + N_v(g'') = N_v(f') + N_v(f'') = N_v(f)$;
Thus, for Case 2.2 we have $O(n, f)$.
For all cases of Case 2 we have shown $O(n, f)$, thus $O(n, f)$.
For Case 2 we have shown $O(n, f)$.
For all cases of $N_v(f)$ we have shown $O(n, f)$, thus $O(n, f)$.
Since f is arbitrary, we have shown $\forall f \in NNF.O(n, f)$, thus $M(n)$.
By strong induction, we have shown $\forall n \in \mathbb{Z}^+.M(n)$.

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Proof of Question 1 by strong induction.
Consider $f \in S$, $n \in \mathbb{N}$, define

$$Q(n, f) = \text{“}\exists g \in NNF.[N_c(f) = n((fg)(N_v(f) = N_v(g)))].\text{''}$$

Consider $n \in \mathbb{N}$, define $P(n) = $ “$\forall f \in S.Q(n, f)$.”
To prove $\forall n \in \mathbb{N}.P(n)$, consider the base case when $n = 0$.
Let $f \in S$ be arbitrary;
Consider 2 cases of f: $N_c(f) = 0$ and $N_c(f) \neq 0$.
Case 1: Assume $N_c(f) = 0$;
Since there is no connector in f, this implies f is a propositional variable, which directly gives $f \in NNF$, thus $Q(0, f)$.
For Case 1 we have shown $Q(0, f)$.
Case 2: Assume $N_c(f) \neq 0$;
Since $N_c(f) \neq 0$, the implication of $Q(0, f)$ is vacuously true. Moreover, because NNF is non-empty, by picking any $g \in NNF$ we can conclude $Q(0, f)$.
For Case 2 we have shown $Q(0, f)$.
Thus, since all cases are true, we have shown $Q(0, f)$.
Since f is arbitrary, we have shown $\forall f \in S.Q(0, f)$, thus $P(0)$.
Now, let $n \in \mathbb{Z}^+$ be arbitrary;
Assume $\forall i \in \mathbb{N}.[i < nP(i)]$;
Let $f \in S$ be arbitrary;
Consider 2 cases of f: $N_c(f) = n$ and $N_c(f) \neq n$.
Case 1: Assume $N_c(f) = n$;
Consider 2 cases of f: f is a propositional variable and f is not a propositional variable (i.e. a formula with connectors).
Case 1.1: Assume f is a propositional variable;
Since f is a propositional variable, $f \in NNF$, thus $Q(n, f)$.
For Case 1.1 we have shown $Q(n, f)$.
Case 1.2: Assume f is not a propositional variable;
Since $f \in S$ and is not constructed from the base case, thus consider 2 cases:

$[\exists f' \in S.f = (f')]$, and $[\exists f' \in S.\exists f'' \in S.f = (f' \star f'')]$ where $\star \in \{,\}$.
Case 1.2.1: Assume $\exists f' \in S.f = (f')$;
This implies $N_c(f') = n - 1 \geq 0$, thus by specialization of our inductive hypothesis, $Q(n-1, f')$;
By instantiation, let $g' \in NNF$ be such that $N_c(f') = n - 1((f'g')(N_v(f') = N_v(g')))$;
Since $g' \in NNF$, by Lemma 1 and Demorgan's Law, we have $\exists g \in NNF.((g')g)(N_v(g) = N_v(g'))$, let $g \in NNF$ be such formula;
By use of conjunction $(f')g$; and by substitution of equivalent formula $f = (f') = (g')$, we have fg;
By use of conjunction, we have $N_v(f) = N_v(f') = N_v(g') = N_v(g)$;
For Case 1.2.1 we have shown $Q(n, f)$.
Case 1.2.2: Assume $\exists f' \in S.\exists f'' \in S.f = (f' \star f'')$;
Since $0 \geq N_c(f') < n$ and $0 \geq N_c(f'') < n$, by specialization of our inductive hypothesis, $Q(n-1, f')$ and $Q(n-1, f'')$;
By instantiation, let $g' \in NNF$ be such that $N_c(f') = n - 1((f'g')(N_v(f') = N_v(g')))$;
By instantiation, let $g'' \in NNF$ be such that $N_c(f'') = n-1((f''g'')(N_v(f'') = N_v(g'')))$;
Since $g' \in NNF$ and $g'' \in NNF$, by definition of NNF, $(g' \star g'') \in NNF$;
Now we have $f = (f' \star f''), (f' \star f'')(g' \star g'')$, and $(g' \star g'') \in NNF$;
Also, $N_v(f) = N_v(f') + N_v(f'') = N_v(g') + N_v(g'') = N_v(g)$;
For Case 1.2.2 we have shown $Q(n, f)$.
For all cases of Case 1.2 we have shown $Q(n, f)$, thus $Q(n, f)$.
For Case 1.2 we have shown $Q(n, f)$.
For all cases of Case 1 we have shown $Q(n, f)$, thus $Q(n, f)$.
For Case 1 we have shown $Q(n, f)$.
Case 2: Assume $N_c(f) \neq n$;
Since $N_c(f) \neq n$, the implication of $Q(n, f)$ is vacuously true, moreover, since NNF is non-empty, by picking any $g \in NNF$ we can conclude $Q(n, f)$.
For Case 2 we have shown $Q(n, f)$.
For all cases of f we have shown $Q(n, f)$, thus $Q(n, f)$.
Since f is arbitrary, we have shown $\forall f \in S.Q(n, f)$, thus $P(n)$.
By strong induction, we have shown $\forall n \in \mathbb{N}.P(n)$.

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Solution to Question 2 - Part A.

Fix $n \in \mathbb{Z}^+$. Consider the recursively defined set PP of promising positions as follows:

Let X, Y be sets of truth assignments to a set of n propositional variables $\{P_i \mid 1 \leq i \leq n\}$, and let $t \in^+$.

Base Cases:

1. If $X = \emptyset$ or $Y = \emptyset$, then $(X, Y, s) \in PP$ for all $s \in \mathbb{Z}^+$.
2. If there exists $1 \leq i \leq n$ such that $\tau(P_i) \neq \tau'(P_i)$ for all $\tau \in X$ and $\tau' \in Y$, then $(X, Y, s) \in PP$ for all $s \in \mathbb{Z}^+$.

Constructor Cases:

1. If $(X, Y, t) \in PP$, then $(X, Y, t+1) \in PP$.
2. If $(X, Y, t) \in PP$ and $(X, Y', t') \in PP$, then $(X, Y \cup Y', t + t') \in PP$.
3. If $(X, Y, t) \in PP$ and $(X', Y, t') \in PP$, then $(X \cup X', Y, t + t') \in PP$.

Base Case 1 makes sense since if one of X, Y is empty then Alice wins vacuously due to the quantifiers.

Base Case 2 directly comes from the definition of Alice's winning condition (these cases do overlap thus ambiguous).

Constructor Case 1 is valid since we can simply seperate $(X, Y, t+1)$ into (X, Y, t) and $(\varnothing, \varnothing, 1)$, then by assumption and base cases both are promising positions.

Constructor Cases 2, 3 directly come from the definition of the game.

If X, Y are disjoint, X, Y' are disjoints, so is $X, Y \cup Y'$; same for $X \cup X', Y$.

Note that we can also define the Base Case 1 to be only $(X, Y, 1)$ instead of all $s \in \mathbb{Z}^+$ s.t. (X, Y, s), becasue of our Constructor Case 1, this is valid. However, for the sake of convinience for the proofs relating to PP, we define it as such. ☺

Proof of Question 2 - Part B by structural induction.

Consider $f \in NNF$, define $Q(f) =$ "for any set X of truth assignments that make f true, for any set Y of truth assignments that make f false, and for any integer $t \geq N_v(f)$, (X, Y, t) is a promising position."

Now we prove $\forall f \in NNF.Q(f)$ by structural induction on f.

Let $f \in NNF$ be arbitrary;

Base Case:

Let X be an arbitrary set of truth assignments that make f true;

Let Y be an arbitrary set of truth assignments that make f false;

Consider 2 cases:

Case 1: $f = P_1$ for some propositional variable P_1;

If $X = \varnothing$ or $Y = \varnothing$, then (X, Y, t) is covered by base case 1 of the recursive definition of PP;

If $X \neq \varnothing$ and $Y \neq \varnothing$, it can only be the case that $X = \{\tau\}$ and $Y = \{\tau'\}$ for some truth assignments τ, τ' such that $\tau(P_1) = T$ and $\tau'(P_1) = F$;

So, by letting $i = 1$, we have constructed such index i that $\tau(P_i) \neq \tau'(P_i)$ for all $\tau \in X$ and $\tau' \in Y$, thus $(X, Y, 1) \in PP$;

Since $N_v(f) = 1$, by constructor case 1 of PP, we have $(X, Y, t) \in PP$ for all $t \geq N_v(f)$;

For Case 1 we have shown $(X, Y, t) \in PP$ for all $t \geq N_v(f)$.

Case 2: $f = (P_1)$ for some propositional variable P_1;

If $X = \varnothing$ or $Y = \varnothing$, then (X, Y, t) is covered by base case 1 of the recursive definition of PP;

If $X \neq \varnothing$ and $Y \neq \varnothing$, it can only be the case that $X = \{\tau\}$ and $Y = \{\tau'\}$ for some truth assignments τ, τ' such that $\tau(P_1) = F$ and $\tau'(P_1) = T$;

So, by letting $i = 1$, we have constructed such index i that $\tau(P_i) \neq \tau'(P_i)$ for all $\tau \in X$ and $\tau' \in Y$, thus $(X, Y, 1) \in PP$;

Since $N_v(f) = 1$, by constructor case 1 of PP, we have $(X, Y, t) \in PP$ for all $t \geq N_v(f)$;

For Case 2 we have shown $(X, Y, t) \in PP$ for all $t \geq N_v(f)$.

Since Y is arbitrary, we have shown for all such Y, $(X, Y, t) \in PP$ for all $t \geq N_v(f)$.

Since X is arbitrary, we have shown for all such X, for all such Y, $(X, Y, t) \in PP$ for all $t \geq N_v(f)$. Thus, $Q(f)$.

Constructor Cases:

Assume $Q(f')$ and $Q(f'')$ for $f', f'' \in NNF$ such that $f = (f' \star f'')$, where $\star \in \{,\}$.

Let X be an arbitrary set of truth assignments that make f true;

Let Y be an arbitrary set of truth assignments that make f false;

Consider 2 cases of $\star$: $(\star =)$ and $(\star =)$.

Case 1: Assume $\star =$;

By assumption, for all set X' of truth assignments that make f' true, for all set Y' of truth assignments that make f' false, and for all integer $t' \geq N_v(f')$, (X', Y', t') is a promising position;

By assumption, for all set X'' of truth assignments that make f'' true, for all set Y'' of truth assignments that make f'' false, and for all integer $t'' \geq N_v(f'')$, (X'', Y'', t'') is a promising position;

For any $\tau \in X$, to make f true, τ must make f' true and f'' true, thus $\tau \in X'$ and $\tau \in X''$. This implies $X \subseteq X' \cap X''$;

For any $\tau' \in Y$, to make f false, τ' must make f' false or f'' false, thus $\tau' \in Y'$ or $\tau' \in Y''$. This implies $Y \subseteq Y' \cup Y''$;

By specialization, and $X \subseteq X' \cap X''$ and $Y \subseteq Y' \cup Y''$, we fix $X' = X'' = X$, and choose Y', Y'' such that $Y = Y' \cup Y''$;

Then, by constructor case 2 of PP, we have $(X, Y', t') = (X', Y', t') \in PP$ and $(X, Y'', t'') = (X'', Y'', t'') \in PP$, which implies $(X, Y' \cup Y'', t' + t'') = (X, Y, t' + t'') \in PP$. Moreover, $t' + t'' \geq N_v(f') + N_v(f'') = N_v(f)$;

By constructor case 1 of PP, we have $(X, Y, t) \in PP$ for all $t \geq N_v(f)$;

For Case 1 we have shown $(X, Y, t) \in PP$ for all $t \geq N_v(f)$.

Case 2: Assume $\star =$;

By assumption, for all set X' of truth assignments that make f' true, for all set Y' of truth assignments that make f' false, and for all integer $t' \geq N_v(f')$, (X', Y', t') is a promising position;

By assumption, for all set X'' of truth assignments that make f'' true, for all set Y'' of truth assignments that make f'' false, and for all integer $t'' \geq N_v(f'')$, (X'', Y'', t'') is a promising position;

For any $\tau \in X$, to make f true, τ must make f' true or f'' true, thus $\tau \in X'$ or $\tau \in X''$. This implies $X \subseteq X' \cup X''$;

For any $\tau' \in Y$, to make f false, τ' must make f' false and f'' false, thus $\tau' \in Y'$ and $\tau' \in Y''$. This implies $Y \subseteq Y' \cap Y''$;

By specialization, and $X \subseteq X' \cup X''$ and $Y \subseteq Y' \cap Y''$, we fix $Y' = Y'' = Y$, and choose X', X'' such that $X = X' \cup X''$;

Then, by constructor case 3 of PP, we have $(X', Y, t') = (X', Y', t') \in PP$ and $(X'', Y, t'') = (X'', Y'', t'') \in PP$, which implies $(X' \cup X'', Y, t' + t'') = (X, Y, t' + t'') \in PP$. Moreover, $t' + t'' \geq N_v(f') + N_v(f'') = N_v(f)$;

By constructor case 1 of PP, we have $(X, Y, t) \in PP$ for all $t \geq N_v(f)$;

For Case 2 we have shown $(X, Y, t) \in PP$ for all $t \geq N_v(f)$.

For all cases of $\star$ we have shown $(X, Y, t) \in PP$ for all $t \geq N_v(f)$.

Since Y is arbitrary, we have shown for all such Y, $(X, Y, t) \in PP$ for all $t \geq N_v(f)$.

Since X is arbitrary, we have shown for all such X, for all such Y, $(X, Y, t) \in PP$ for all $t \geq N_v(f)$.

Thus, $Q(f)$.

By structural induction, we have shown $\forall f \in NNF. Q(f)$.

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