My name and student number: Joseph Siu, 1010085701. Sanchit Manchanda, Ali Zaki Rashid.

For any language $S \subseteq \Sigma^*$, define $C(S) = \{x \in \Sigma^* \mid \exists w \in \Sigma^*.\exists y \in \Sigma^*.(xwxy \in S)\}$.

For example, if $S = \{ababc, aabaab\}$, then $C(S) = \{\lambda, a, aa, ab, aab\}$.

Question 1. Describe the language $S = L(((01)^* + 1^*)^*) = \{z \in \{0,1\}^* \mid \dots\}$ by replacing the ... with at most 10 words. (z counts as one word.) Briefly justify your answer.

Here ... is equivalent to "all occurences of 0 in z are followed by 1."

Justification. We will (briefly) show the equality by double subset. For the forward subset, let $s \in L(((01)^* + 1^*)^*)$ be arbitrary, then by definition of *, s is concatenated by strings in $(01)^* + 1^*$, let string $w \in (01)^* + 1^*$ be an arbitrary string in such concatenation. Then w is either a string of 1's or a string of 01's, for both cases we can see that all occurences of 0 in w are followed by 1. Thus $s \in \{z \in \{0,1\}^* \mid \text{ all occurences of 0 in } z \text{ are followed by } 1\}$.

For the backward subset, let $s \in \{z \in \{0,1\}^* \mid \text{ all occurences of 0 in } z \text{ are followed by } 1\}$ be arbitrary, then the string s must be a concatenation of strings in $(01)^* + 1^*$ if we group all 0's followed by 1's together. Hence by definition of *, $s \in L(((01)^* + 1^*)^*)$.

Therefore, $S = L(((01)^* + 1^*)^*) = \{z \in \{0,1\}^* \mid \text{ all occurences of 0 in } z \text{ are followed by } 1\}$.

Question 2. Describe the language $T = L\left(\overline{\overline{\phi} \cdot 00 \cdot \overline{\phi}}\right) = \{x \in \{0,1\}^* \mid \dots\}$ by replacing the ... with at most 10 words. (x counts as one word.) Briefly justify your answer.

Here ... is equivalent to "there are no consecutive 0's in string x."

Justification. We will (briefly) show the equality by double subset. To show $\overline{L\left(\overline{\phi} \cdot 00 \cdot \overline{\phi}\right)} = L\left(\overline{\overline{\phi} \cdot 00 \cdot \overline{\phi}}\right) = \{x \in \{0,1\}^* \mid \text{ there are no consecutive 0's in string } x\}$, by definition of complement it is equivalent to show $L\left(\overline{\phi} \cdot 00 \cdot \overline{\phi}\right) = \{x \in \{0,1\}^* \mid \text{ there are consecutive 0's in string } x\}$.

For the forward subset, let $s \in L\left(\overline{\phi} \cdot 00 \cdot \overline{\phi}\right)$ be arbitrary, then by definition of carcatenation, since $\overline{\phi} = \Sigma^*$, string $s = \alpha \cdot 00 \cdot \beta$ for some $\alpha \in \Sigma^*$, for some $\beta \in \Sigma^*$, so s contains consecutive 0's. Thus $s \in \{x \in \{0,1\}^* \mid \text{ there are consecutive 0's in string } x\}$.

For the backward subset, let $s \in \{x \in \{0,1\}^* \mid \text{ there are consecutive 0's in string } x\}$ be arbitrary, then this means there exists $\alpha \in \Sigma^*$ and $\beta \in \Sigma^*$ such that $s = \alpha \cdot 00 \cdot \beta$, so $s \in L\left(\overline{\phi} \cdot 00 \cdot \overline{\phi}\right)$.

Therefore, as $L\left(\overline{\phi} \cdot 00 \cdot \overline{\phi}\right) = \{x \in \{0,1\}^* \mid \text{ there are consecutive 0's in string } x\}$, we conclude that $T = L\left(\overline{\overline{\phi} \cdot 00 \cdot \overline{\phi}}\right) = \{x \in \{0,1\}^* \mid \text{ there are no consecutive 0's in string } x\}$.

Question 3. Explain why $C(S) = T$.

Both $C(S)$ and T are precisely the sets that contain all strings that do not have consecutive 0's. We will show they are equal by strong induction on the length of the string x.

Proof of Question 3 by induction.
Define the predicate $P(n) =$"For all strings $x \in \{0,1\}^*$ of length n, $x \in C(S)$ if and only if $x \in T$."
Let $n \in \mathbb{N}$ be arbitrary;
Assume $\forall i \in \mathbb{N}.((i < n) \text{ IMPLIES } P(i))$.
Case 1. $n = 0$.
Then only string $x = \lambda$ is length 0, by our condition, $\lambda\lambda\lambda\lambda \in S$ so $\lambda = x \in C(S)$ and $x \in T$ since x has no consecutive 0's.
In Case 1 $P(n)$ holds.
Case 2. $n = 1$.
Only strings $x = 0$ and $x = 1$ are length 1:
Subcase (1): $x = 0$: Then by condition, since $0101 \in S$, $0 \in C(S)$ and $0 \in T$ since 0 has no

consecutive 0's. Thus $P(1)$ holds. For Case 2.1 $P(n)$ holds.

Subcase (2): $x = 1$: Then by condition, since $1111 \in S$, $1 \in C(S)$ and $1 \in T$ since 1 has no consecutive 0's. Thus $P(1)$ holds. For Case 2.2 $P(n)$ holds.

In Case 2 $P(n)$ holds.

Case 3. $n \geq 2$.

Let string $x \in \{0,1\}^*$ of length n be arbitrary. Since $n \geq 2$, we know $x = y \cdot \alpha$ for some $y \in \{0,1\}^*$ of length $n-1$ and $\alpha \in \{0,1\}$.

Subcase (1): x has consecutive 0's.

Then by condition this implies $x \notin T$.

To obtain a contradiction, assume $x \in C(S)$. This implies there exists $w' \in \Sigma^*$ and $y' \in \Sigma^*$ such that $xw'xy' \in S$. However by definition of S, $xw'xy' \in S$ implies all occurences of 0 in $xw'x'y'$ are followed by 1. This contradicts the fact that x has consecutive 0's.

Thus $x \notin C(S)$.

For this subcase $x \in C(S)$ if and only if $x \in T$.

Subcase (2): x has no consecutive 0's.

Then by deifnition $x \in T$. Since y has no consecutive 0's, by definition $y \in T$.

Moreover, by inductive hypothesis, specialization and modus ponens, $y \in C(S)$. By definition this means there exists $w' \in \Sigma^*$ and $y' \in \Sigma^*$ such that $yw'yy' \in S$.

Combining the facts that $y\alpha, \alpha 1, 1w', w'y, 1y'$ have no consecutive 0's by definition of S, we have $y\alpha 1w'y\alpha 1y' \in S$ since the entire string cannot have consecutive 0's, so $x = y\alpha \in C(S)$ as $1w' \in \Sigma^*$ and $1y' \in \Sigma^*$.

For this subcase $x \in C(S)$ if and only if $x \in T$.

Since for all subcases of Case 3 $x \in C(S)$ if and only if $x \in T$, for Case 3 $x \in C(S)$ if and only if $x \in T$.

Since x is arbitrary, $P(n)$ holds for Case 3.

Since all cases $P(n)$ hold, we conclude $P(n)$ holds.

By strong induction on n, we conclude that for all $n \in \mathbb{N}$, $P(n)$ holds. Thus for all $x \in \{0,1\}^*$, $x \in C(S)$ if and only if $x \in T$. By definition of set equality we conclude $C(S) \subseteq T$ and $T \subseteq C(S)$, so $C(S) = T$.

QUOD ERAT DEM■

Question 4. Give any deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$, construct a finite automaton $M' = (Q', \Sigma, \delta', q_0', F')$ such that $L(M') = C(L(M))$.

Let $Q' = Q, q_0' = q_0, \delta' = \delta, F' = \{q \in Q' \mid \forall x \in \Sigma^*. \exists w \in \Sigma^*. \exists y \in \Sigma^*. [\delta'^*(q_0', x) = q \text{ IMPLIES } \delta'^*(q, wxy) \in F]\}$.

Then $M' = (Q', \Sigma, \delta', q_0', F')$ is the desired finite automaton.

Question 5. briefly describe how M' works.

The only differences between M and M' are the accepted states. By our construction of F', we can see that string x is accepted by M' implies there exists strings w and y such that $xwxy$ is accepted by M. And by our condition of C, $xwxy$ is accepted by M implies $x \in C(L(M))$. Thus $L(M') \subseteq C(L(M))$.

Question 6. Prove that $L(M') = C(L(M))$.

Proof. First by construction

$$\begin{aligned} F' &= \{q \in Q' \mid \forall x \in \Sigma^*. \exists w \in \Sigma^*. \exists y \in \Sigma^*. [\delta'^*(q_0', x) = q \text{ IMPLIES } \delta'^*(q, wxy) \in F]\} \\ &= \{q \in Q \mid \forall x \in \Sigma^*. \exists w \in \Sigma^*. \exists y \in \Sigma^*. [\delta^*(q_0, x) = q \text{ IMPLIES } \delta^*(q, wxy) \in F]\} \end{aligned}$$

So, let $x \in L(M')$ be arbitrary, then

$$\begin{aligned}
x \in L(M') &\text{ IMPLIES } \delta'^*(q_0', x) \in F' \\
&\text{ IMPLIES } \delta^*(q_0, x) \in F' \\
&\text{ IMPLIES } \exists q \in F'.\exists w \in \Sigma^*.\exists y \in \Sigma^*. [\delta^*(q_0, x) = q \text{ IMPLIES } \delta^*(q, wxy) \in F] \\
&\text{ IMPLIES } \exists q \in F'.\exists w \in \Sigma^*.\exists y \in \Sigma^*. [\delta^*(q_0, x) = q \text{ IMPLIES } \delta^*(\delta^*(q_0, x), wxy) \in F] \\
&\text{ IMPLIES } \exists q \in F'.\exists w \in \Sigma^*.\exists y \in \Sigma^*. [\delta^*(q_0, x) = q \text{ IMPLIES } \delta^*(q_0, xwxy) \in F] \\
&\text{ IMPLIES } \exists w \in \Sigma^*.\exists y \in \Sigma^*. xwxy \in L(M) \\
&\text{ IMPLIES } x \in C(L(M))
\end{aligned}$$

We have shown that $x \in L(M')$ implies $x \in C(L(M))$.
Since x is arbitrary, we conclude $L(M') \subseteq C(L(M))$.
To show the other direction, we need to construct another automaton : (

Quod Erat Dem■