CSC 240 Winter 2024 Homework Assignment 1

due January 18, 2024

My name and student number: Joseph Siu, 1010085701
The list of people with whom I discussed this homework assignment:

Sanchit Manchanda

Serif Wu

Hrithik Parag Shah

Abhi Prajapati

Sepehr Jafari

1. Let U be the set of all people and let $\mathcal{P}(U)$ be the set of all subsets of U.
   Let $In : U \times \mathcal{P}(U) \to \{\text{T,F}\}$ be the predicate such that $In(x, G) = \text{T}$ when person x is in the set of people G.
   Let $size : \mathcal{P}(U) \to \mathbb{N}$ be the function such that $size(G)$ is the number of people in the set G.
   Let $K : U \times U \to \{\text{T,F}\}$ be the predicate such that $K(a, b) = \text{T}$ when person a and person b know each other.
   You may assume that K is symmetric, i.e. $\forall a \in U.\forall b \in U.[K(a, b) \text{ IFF } K(b, a)]$.

   (a) Consider the following predicate about a set of people $G \in \mathcal{P}(U)$:

   $$\exists H \in \mathcal{P}(U).\forall x \in H.\forall y \in H. \\ [(size(H) = 40) \text{ AND } (\text{NOT}(K(x, y)) \text{ OR } (x = y)) \text{ AND } In(x, G)].$$

   Express this predicate using at most 15 English words. The letter G counts as one word. Justify your answer.

   Solution

   The predicate is equivalent to "The set G contains 40 people who don't know each other."

   First we can see the predicate (beside the quantification parts) is mainly formed by 2 AND connections. The first part is "A set of people H has 40 people", the second part is "no one in H knows each other", and the third part is "all people in H are also in G" which means H is a subset of G. Combining these 3 parts with the 3 quantifications, we obtained our sentence.

(b) Consider the following sentence in predicate logic.

$$\begin{aligned}&\exists n \in \mathbb{N}.\forall G \in \mathcal{P}(U).\exists H \in \mathcal{P}(U).\forall x \in H.\forall y \in H.\\ &\quad [((size(H) = 40) \text{ OR } (size(G) < n)) \text{ AND}\\ &\quad (\text{NOT}(K(x, y)) \text{ OR } (x = y) \text{ OR } (size(G) < n)) \text{ AND}\\ &\quad (In(x, G) \text{ OR } (size(G) < n))].\end{aligned}$$

Express this sentence using at most 20 English words. Justify your answer.

Solution

The predicate is equivalent to "Any group of people either has fewer than a fixed number of people, or contains 40 strangers to each other."
First, the predicate is logically equivalent to

$$\begin{aligned}&\exists n \in \mathbb{N}.\forall G \in \mathcal{P}(U).\exists H \in \mathcal{P}(U).\forall x \in H.\forall y \in H.\\ &\quad (size(G) < n) \text{ OR } [(size(H) = 40) \text{ AND}\\ &\quad (\text{NOT}(K(x, y)) \text{ OR } (x = y)) \text{ AND}\\ &\quad (In(x, G))].\end{aligned}$$

Now we move $\exists H \in \mathcal{P}(U).\forall x \in H.\forall y \in H.$ within the second part (right side) of the OR connector, this is allowed becasue these quantifications have no affect on the first (left side) part,

$$\begin{aligned}&\exists n \in \mathbb{N}.\forall G \in \mathcal{P}(U).(size(G) < n) \text{ OR}\\ &\quad [\exists H \in \mathcal{P}(U).\forall x \in H.\forall y \in H.(size(H) = 40) \text{ AND}\\ &\quad (\text{NOT}(K(x, y)) \text{ OR } (x = y)) \text{ AND } (In(x, G))].\end{aligned}$$

We can now see the second (right side) part is precisely our Part A of this question. So, the first two quantifications give "there is a natural number n, for any sets of people G", then the first part of OR gives "the size of the set G is less than the natural number n", and the second part of OR gives "there is a set of 40 people who don't know each other and are all in the set of people G". Combining these 3 parts, we obtained our sentence, completing the justification.

2. A graph consists of a set of vertices V and a set of edges between pairs of vertices. The set of edges can be represented by a symmetric, irreflexive predicate $e : V \times V \to \{T, F\}$, where $e(a, b) = T$ denotes that there is an edge between vertex a and b. Symmetric means $\forall a \in V.\forall b \in V.(e(a, b) \text{ IFF } e(b, a))$ and irreflexive means $\forall a \in V.\text{NOT}(e(a, a)))$.

We say that two vertices a and b are *adjacent* if there is an edge between them.
The *degree of a vertex* is the number of vertices that are adjacent to it.
The *maximum degree of a graph* is the maximum of the degrees of its vertices.

A *colouring* of the graph is a function from V to a set of colours. We will use the natural numbers as the set of colours. Then $\mathbb{N}^V$ denotes the set of all colourings of V.

A *proper colouring* is a *colouring* such that no two adjacent vertices have the same colour.

Using predicate logic, express the following statement about a graph with vertex set V and whose set of edges is represented by the predicate e:
"If the graph has maximum degree two, then it can be properly coloured using at most three colours."
You cannot use any constants. The only predicate (besides e) you can use is binary equality, $=$.

To get full marks, you must use IMPLIES at most once and NOT at most four times.

Use parentheses and brackets when necessary to avoid ambiguity.
Justify your answer by briefly explaining what its various parts mean.

Solution

The statement is logically equivalent to

$$
\begin{aligned}
&[[\forall a \in V. \forall b \in V. \forall c \in V. \forall d \in V.\\
&\quad \text{NOT}(e(a,b) \text{ AND } e(a,c) \text{ AND } e(a,d)) \text{ OR } ((b=c) \text{ OR } (b=d) \text{ OR } (c=d))]\\
&\qquad \text{AND } [\exists g \in V. \exists h \in V. \exists i \in V. e(g,h) \text{ AND } e(g,i) \text{ AND } (\text{NOT}(h=i))]]\\
&\qquad\qquad \text{IMPLIES } [\exists f \in \mathbb{N}^V. \forall w \in V. \forall x \in V. \forall y \in V. \forall z \in V.\\
&\qquad\qquad [(f(w)=f(x)) \text{ OR } (f(w)=f(y)) \text{ OR } (f(w)=f(z))]\\
&\qquad\qquad\qquad \text{AND } [\text{NOT}(e(w,x) \text{ AND } (f(w)=f(x)))]].
\end{aligned}
$$

The part before IMPLIES has two parts connected by AND . First,

$$\forall a \in V. \forall b \in V. \forall c \in V. \forall d \in V.$$

$$\text{NOT}(e(a,b) \text{ AND } e(a,c) \text{ AND } e(a,d)) \text{ OR } ((b=c) \text{ OR } (b=d) \text{ OR } (c=d))$$

is logically equivalent to

$$\forall a \in V. \forall b \in V. \forall c \in V. \forall d \in V.$$

$$(e(a,b) \text{ AND } e(a,c) \text{ AND } e(a,d)) \text{ IMPLIES } ((b=c) \text{ OR } (b=d) \text{ OR } (c=d))$$

, this means if a vertex is adjacent to 3 other vertices (may repeat the same vertex), then at least two of these 3 vertices are the same vertex. What this says is that the maximum degree is at most 2. Second,

$$\exists g \in V.\exists h \in V.\exists i \in V.e(g,h) \text{ AND } e(g,i) \text{ AND } (\text{NOT}(h=i))$$

means there is a vertex adjacent to 2 distinct vertices, this gives the maximum degree of the graph is at least 2. Combining these 2 we have the maximum degree of the graph is precisely 2.

The part after IMPLIES also contains two main parts connected by AND . First,

$$\exists f \in \mathbb{N}^V.\forall w \in V.\forall x \in V.\forall y \in V.\forall z \in V.(f(w)=f(x)) \text{ OR } (f(w)=f(y)) \text{ OR } (f(w)=f(z))$$

means there exists a colouring function which, for any 4 vertices, at least 2 of them must have the same colour, this gives a colouring function of the graph with at most 3 colours in the range of the function. Second,

$$\text{NOT}(e(w,x) \text{ AND } (f(w)=f(x)))$$

gives it cannot be true that two vertices are adjacent and have the same colour, by definition this shows the function f is a proper colouring function. Combining with the first part, the graph needs to have a proper colouring function which only has at most 3 colours in its range.

Therefore, as combining everything before and after IMPLIES , we conclude our predicate is precisely the sentence required.