1 Review

R = set of regular expressions over Σ

Base Cases:

$\varnothing, \lambda \in R$

$\Sigma \subseteq R$

Constructor Cases:

If $r, r' \in R$ then $(r + r'), (r \cdot r'),$ and $r^* \in R$

$L(\varnothing) = \varnothing,\ L(\lambda) = \{\lambda\}$

$L(a) = \{a\}$ for all $a \in \Sigma$

$L((r + r')) = L(r) \cup L(r')$

$L((r \cdot r')) = L(r) \cdot L(r')$

$L(r^*) = (L(r))^*$

A language is regular if $L = L(r)$ for some $r \in R$

Theorem 1

For every regular language L there is an NFA M such that $L = L(M)$.
The converse is also true.

Theorem 2

For every NFA M there is a DFA M' such that $L(M) = L(M')$.

2 Left / Right Quotients

Left quotient of L_1 and L_2

$L_2 \setminus L_1 = \{y \mid \exists x \in L_2.(xy \in L_1)\}$

Right quotient of L_1 and L_2

$L_1 \,/\, L_2 = \{x \mid \exists y \in L_2.(xy \in L_1)\}$

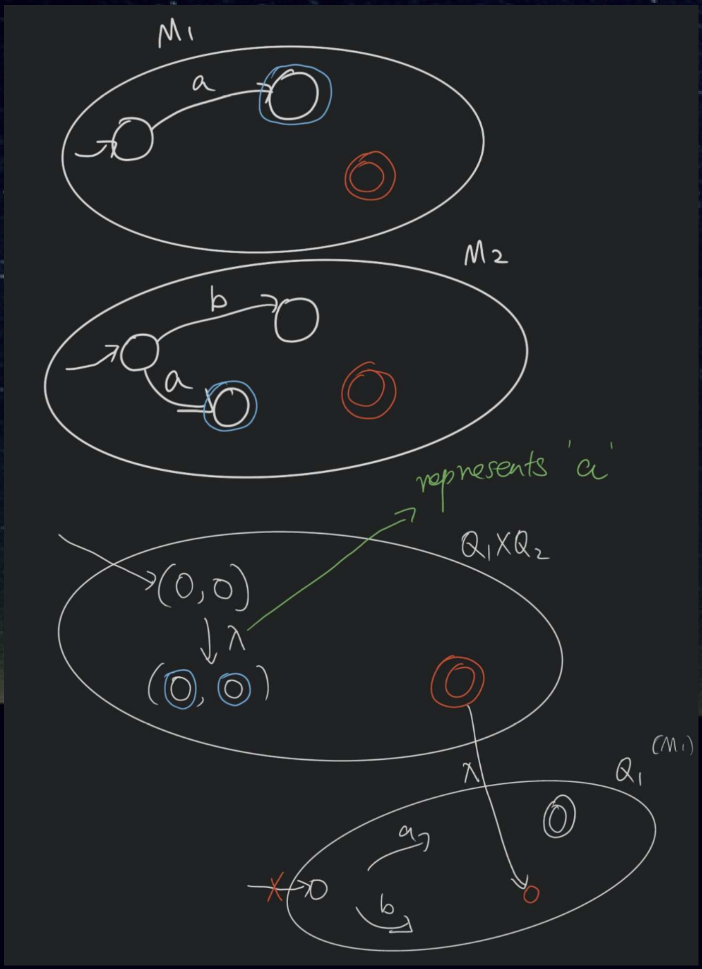
Example 1.
$L_1 = \{o^i 1^j \mid i, j \in \mathbb{Z}^+\}$
$L_2 = \{o^i 1 \mid i \in \mathbb{N}\}$
$L_2 \setminus L_1 = \{1^j \mid j \in \mathbb{N}\}$

Claim 1

If L_1 and L_2 are regular, so are $L_2 \setminus L_1$ and $L_1 \setminus L_2$.

Proof. Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be DFAs such that $L_1 = L(M_1)$ and $L_2 = L(M_2)$ Quod Erat Dem■

Intuitively,



Formally, let $M = (Q, \Sigma, \Delta, (q_1, q_2), F_1)$ where $Q = Q_1 \times Q_2 \cup Q_1$;

For all $q \in Q_1, p \in Q_2$

$\delta((q,p),\lambda) = \{(\delta_1(q,a),\delta_2(p,a)) \mid a \in \Sigma\} \cup \{q \mid p \in F_2\}$ ($\{q\}$ or $\varnothing$)

$\delta((q,p),a) = \varnothing$ for all $a \in \Sigma$, for all $p \in Q_2, q \in Q_1$

For all $q \in Q_1$, $a \in \Sigma$

$\delta(q,a) = \{\delta(q,a)\}$

$\delta(q,\lambda) = \varnothing$

Now we need to prove $L(M) = L_2 \setminus L_1$

For the forward subset,

Let $y \in L_2 \setminus L_1$, then $\exists x \in L_2$ such that $xy \in L_1$ by definition of left quotient

In M_2 there is a path from q_2 to a state $p \in F_2$ labelled by x

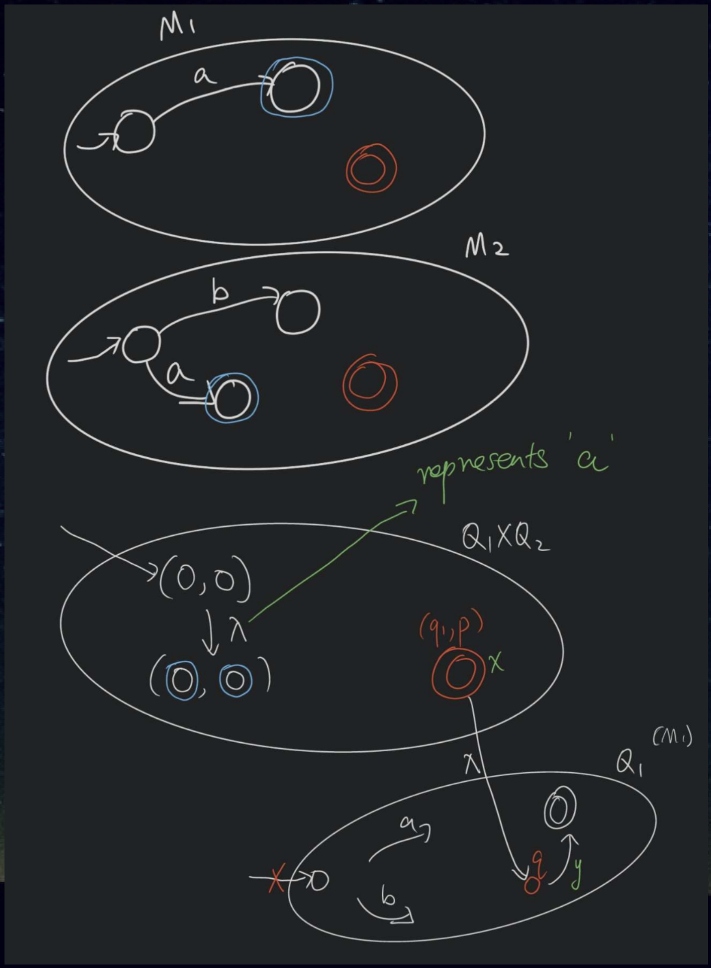
In M_1 there is a path from q_1 to $\delta_1^*(q_1,x) = q$

In $Q_1 \times Q_2$, there is a paht from (q_1,q_2) to (q,p) labelled by λ by construction.

Since $p \in F_2$ there is a λ-transition from (q,p) to q by construction.

Since $xy \in L_1$, $\delta_1^*(q,y) = \delta_1^*(\delta_1^*(q_1,x),y) = \delta_1^*(q_1,xy) \in F_1$

So $y \in L(M)$



Conversely, for the backward subset,

Suppose $y \in L(M)$

Then there is a path from (q_1,q_2) to a state $q' \in F_1$ that is labelled by y

The only way to get from $Q_1 \times Q_2$ to Q_1 is by a λ-transition from a state $(q,p) \in Q_1 \times Q_2$ to $q \in Q_1$ where $p \in F_2$

All the edges between states in $Q_1 \times Q_2$ are labelled by λ

So by definition of δ, there exists a string $x \in \Sigma^*$ such that $q = \delta_1^*(q,x)$ and $p = \delta_2^*(q_2,x)$, since $p \in F_2$, $x \in L(M_1) = L_1$

Edges between states in Q_1 are labelled by letters, $\delta_1^*(q, y) = \delta^*(q, y) = q' \in F_1$

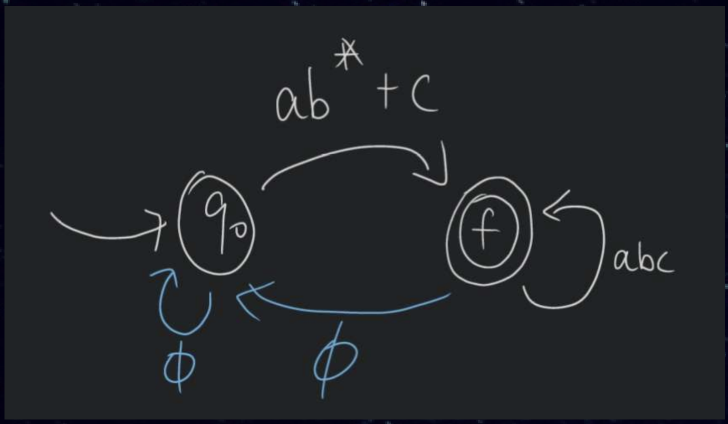
$\delta_1^*(q, xy) = \delta_1^*(\delta_1^*(q_1, x), y) = \delta_1^*(q, y) = q' \in F_1$, so $xy \in L_1$, hence $y \in L_2 \setminus L_1$

3 Generalized Transition Graph GTG

A generalized transition graph is a 5-tuple $G = (Q \text{ (finite set of states)}, \Sigma \text{ (finite alphabet)}, \delta, q_0 \subseteq Q, F \subseteq Q)$

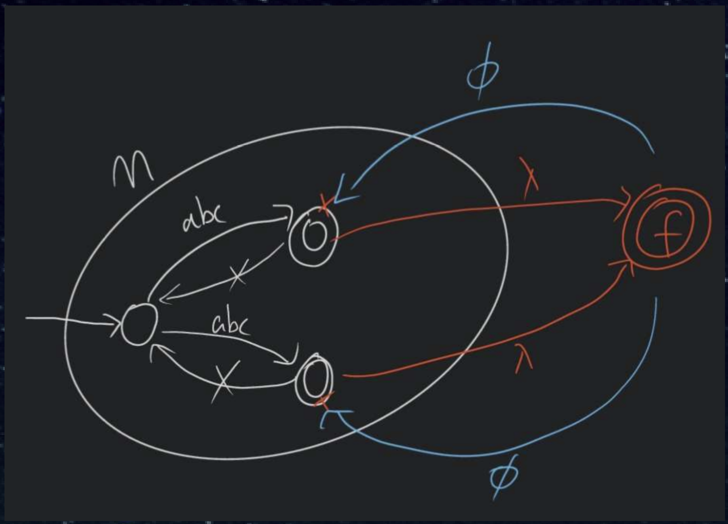
$S : Q \times Q \to R$ where R is the set of regular expressions over Σ.

$L(G) = \{x \in \Sigma^* \mid {}_{\text{described by the concatenation of the labels on the edges of the path}}^{\text{there is a path from } q_0 \text{ to } f \text{ and } x \text{ is in the language}}\}$



Lemma 1

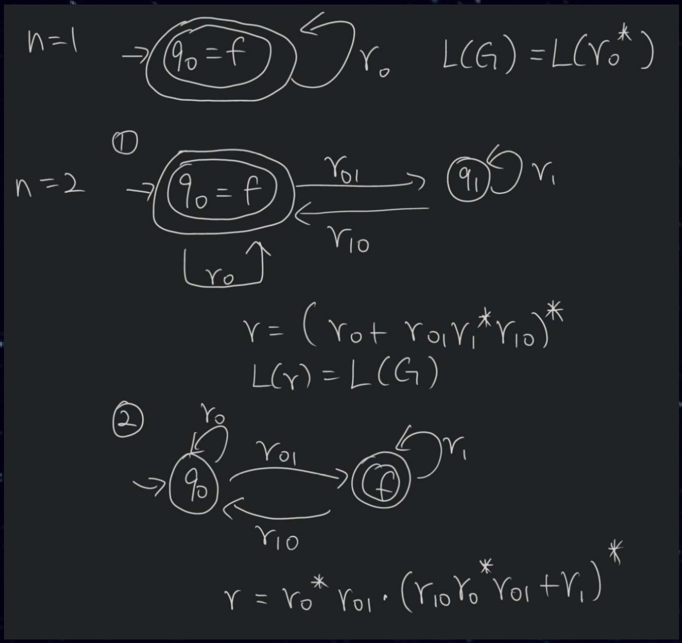
For any NFA $M = (Q, \Sigma, \delta, q_0, F)$, there is a GTG $G = (Q \cup \{f\}, \Sigma, \delta', q_0, f)$ such that $L(M) = L(G)$



Theorem 3

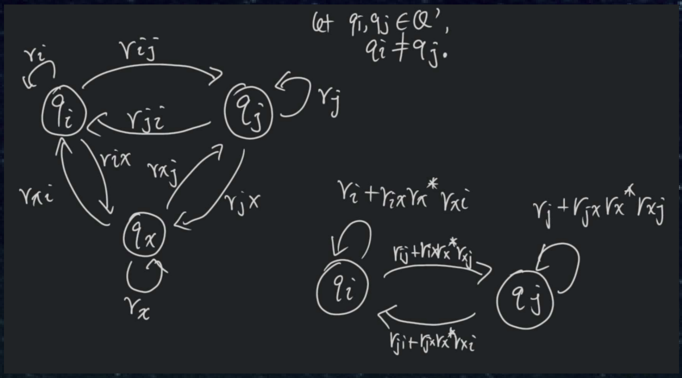
For every GTG G, there is a regular expression r such that $L(G) = L(r)$ (Converse of Theorem 1)

Proof of Theorem 3 by Induction on number of states in G. Define the predicate $P(n)$ ="For every GTG G with n states, there is a regular expression r such that $L(G) = L(r)$"
Base Cases:



Inductive Case:

Let $n \geq 3$ and assume $P(n-1)$
$G = (Q, \Sigma, \delta, q_0, f)$
Let $q_x \in Q - \{q_0, f\}$
$G' = (Q', \Sigma, \delta', q_0, f)$
where $Q' = Q - \{q_x\}$
δ' is defined as follows:
G' has $n-1$ states.
Let $q_i, q_j \in Q'$,



Claim 2

For all $q, q' \in Q$, and all $x \in \Sigma^*$,
there exists a path from q to q' labelled by r in G such that $x \in L(r)$
if and only if
there is a path from q to q' labelled by r' in G' such that $x \in L(r')$

In particular of $q : q_0$ and $q' = f$, then $L(G) = L(G')$
By $P(n-2)$ there is regular expression r' such that $L(G') = L(r')$
By induction, $P(n)$ is true.

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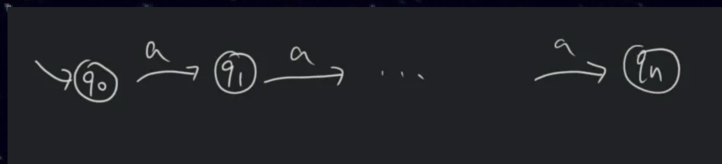
4 Are all languages Regular?

$S = \{a^i b^i \mid i \in \mathbb{Z}^+\}$

Suppose S is regularm then there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $S = L(M)$

Let $n = |Q|$ be the number of states in M, let $q_i = \delta^*(q_0, a^i)$



By pigeonhole principle, there exists $0 \leq i < j \leq n$ such that $q_i = q_j$

$$\begin{aligned}\delta^*(q_0, a^i b^j) &= \delta^*(\delta^*(q_0, a^i), b^j) \\ &= \delta^*(\delta^*(q_0, a^j), b^j) \\ &= \delta^*(q_0, a^j b^j) \in F\end{aligned}$$

Since $a^j b^j \in S \in L(M)$

This is a contradiction since $a^i b^j \notin S$.

Lemma 2 – Pumping Lemma

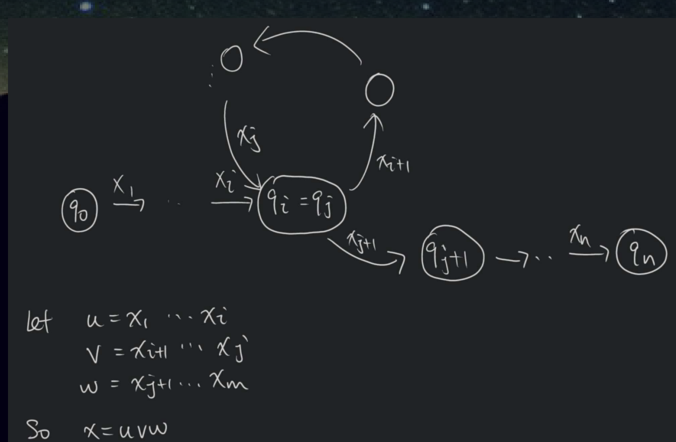
For every regular language $s \subseteq \Sigma^*$, $\exists n \in \mathbb{Z}^+ . \forall x \in S$.

$$[(|x| \geq n) \text{ IMPLIES } \exists u \in \Sigma^* . \exists v \in \Sigma^* . \exists w \in \Sigma^* . [v \neq \lambda \text{ AND } (|uv| \leq n) \text{ AND } x = uvw \text{ AND } \forall k \in \mathbb{N} . uv^k w \in S]]$$

Proof of Pumping Lemma.
Since S is regular it is accepted by a DFA $M = (Q, \Sigma, \delta, q_0, F)$, i.e. $S = L(M)$.
Let $n = |Q|$
Let $x \in S$ such that $|x| = m \geq n$.
Consider the states

$$\begin{aligned}&q_0 \\ &q_1 = \delta(q_0, x_1) \\ &q_i = \delta(q_{i-1}, x_i) \text{ for } 1 \leq i \leq n\end{aligned}$$

By the pigeonhole principle, there exists $0 \leq i < j \leq n$ such that $q_i = q_j$



Since $i < j$, $v \neq \lambda$
Since $j \leq n$, $|uv| \leq n$
$\delta^*(q_0, uv^k) = q_j$ for all $k \in \geq 0$
So

$$\begin{aligned}
\delta^*(q_0, uv^k w) &= \delta^*(\delta^*(q_0, uv^k), w) \\
&= \delta^*(q_j, w) \\
&= \delta^*(\delta^*(q_0, uv), w) \\
&= \delta^*(q_0, uvw) \in F \text{ since } uvw = x \in S
\end{aligned}$$

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Theorem 4

$T = \{a^m b^n \mid m \neq n\}$ is not regular.

Proof by Pumpping Lemma.

Suppose T is regular, then by pumping lemma there exists $n \in \mathbb{N}$ such that $\forall x \in T[(|x| \geq n)$ IMPLIES $\exists u \in \{a,b\}^*.\exists v \in \{a,b\}^*.\exists w \in \{a,b\}^*.[v \neq \lambda$ AND $|uv| \leq n$ AND $(x = uvw)$ AND $uv^k w \in T$ for all $k \in \mathbb{N}]]$
Consider $x = a^{n!} b^{(n+1)!}$ where $n \geq 1$
$x \in T$ since $n! \neq (n+1)!$ for $n \geq 1$.
There exists $u, v, w \in \{a,b\}^*$ such that $v \neq \lambda$, $|uv| \leq n$, $x = uvw$ and $\forall k \in \mathbb{N}$, $(uv^k w) \in T$.
Note that $u = a^i$ and $v = a^j$ for some $i \geq 0$, $j \geq 1$ such that $i + j \leq n$.
Let $k = 1 + n(n!)/j \in \mathbb{Z}^+$, $1 \leq j \leq n$
We claim that $uv^k w \notin T$ since $a^m b^{(n+1)!}$ where $m = n! + (k-1)j$.
$i + j - k + n! - i - j = n! + n(n!) = (n+1)n! = (n+1)!$ since $k - 1 = n(n^i)/j$
So $uv^k w \notin T$, this is a contradiction.

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