# 1. Induction

**Definition 1 – Induction**

Let $p : \mathbb{N} \to \{T, F\}$ be a predicate, to prove $\forall n \in \mathbb{N}, p(n)$:
L1: $p(0)$
L2: Let $n \in \mathbb{N}$ be arbitrary
L3: Assume $p(n)$
⋮
L4: $p(n+1)$
L5: $p(n)$ IMPLIES $P(n+1)$; direct proof L3, L4
L6: $\forall n \in \mathbb{N}.[p(n)$ IMPLIES $p(n+1)]$; generalization L5
L7: $\forall n \in \mathbb{N} p(n)$; by induction L1, L6

**Theorem 1**

Consider any square chessboard whose side have length a poewr of 2. If any one square is removed, then the resulting shape can be tiled using 3 square L-shape tiles.

For any $n \in \mathbb{N}$, let $p(n)$ be the predicate: for any $2^n \times 2^n$ chessboard with one square removed can be tiled using L-tiles.
Let $C_n$= the set of all $2^n \times 2^n$ chess boards with 1 square removed.
$|C_n| = 2^{2n} = 2^n \times 2^n$
$p(n) = \forall c \in C_n . c$ can be tiled using L-tiles. ☺

*Proof.* Base Case: $P(0)$ is true, becasue a $2^0 \times 2^0$ chessboard with 1 tile remove has no squares. Hence can be tiled with 0 L-tiles.
Let $n \in \mathbb{N}$ be arbitrary.
Suppose $p(n)$ is true. (We want to show $\forall n \in \mathbb{N}.P(n)$ by induction, that is, $p(n+1) = \forall c \in C_{n+1}$. c can be tiled using L-tiles, which is by generalization).
Now, let $c \in C_{n+1}$ be arbitrary.
Divide $c$ into 4 equal $2^n \times 2^n$ chessboards, one has a square removed, so it is in $C_n$. By the induction hypothesis (and specialization) it can be tiled with L-tiles. The other 3 chessboards, each has 1 tile in the middle of c. With that tile removed the induction hypothesis says it can be tiled using L-tiles. The 3 squares we removed from the center can be tiled with 1 L-tile

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**Theorem 2**

All square chessboards whose sides have length $a$ power of 2 and has 1 square removed from the middle can be tiled using L-tiles.

Let $C'_n$= set of all $2^n \times 2^n$ chessboards with 1 square removed from the middle.
$P'_n = \forall c \in C'_n.c$ can be tiled using nay L-tiles ☺

**Remark 1.** We can't use $P'(n)$ to prove $P'(n+1)$. Sometimes using a more general result is easier to do (when doing a proof by induction) because strengthening the induction hypothesis makes the induction step easier.

△

**Theorem 3**

If $M = \{m \in \mathbb{N} \mid m \geq 3\}$, then

$$\forall m \in M.(2m + 1 \leq 2^m).$$

**Remark 2.** We can prove this using 3 ways:

1) $\forall n \in \mathbb{N}.[p(n) \text{ IMPLIES } P(n+1)], p(n) := q(n+3)$
2) $\forall m \in M.[q(m) \text{ IMPLIES } q(m+1)], q(n) := 2n + 1 \leq 2^n$
3) $\forall n \in \mathbb{N}.[r(n)], r(n) := \forall n \in \mathbb{N}.(n \geq 3 \text{ IMPLIES } q(n))$

△

*Proof of 2.* For all $n \in M$, let $q(n) = 2n + 1 \leq 2^n$.

$q(3)$ is true (trivial).

Let $m \in M$ be arbitrary.

Assume $q(m)$

$\vdots$

$q(m+1)$

$\forall m \in M.q(m)$

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*Proof of 3.* $r(0)$ is true vacuously since $n \geq 3$ is false when $n = 0$.

Let $n \in \mathbb{N}$ be arbitrary.

Assume $r(n)$

$n \geq 3$ IMPLIES $q(n)$; by Definition

Assume $n + 1 \geq 3$

$\vdots$

$q(n+1)$

$(n+1) \geq 3$ IMPLIES $q(n+1)$

$r(n+1)$

$\forall n \in \mathbb{N}.r(n)$; by induction.

To do this, we consider 2 cases:

Case 1: $n + 1 = 3 (n = 2)$

$2(n+1) + 1 = 2 \times 3 + 1 = 6 \leq 2^3 = 2^{n+1}$

So $q(n+1)$ is true.

Case 2: $n + 1 > 3 (n \geq 3)$

$q(n)$; by modus ponens

Then $2n + 1 \leq 2^n$; by definition of $q$

$2 \leq 8 = 2^3 \leq 2^n$

$2(n+1) + 1 = 2n + 2 + 1 = (2n+1) + 2 \leq 2^n + 2^n = 2^{n+1}$; arithematic + Substitution

$q(n+1)$ is true.

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Suppose we want to prove $q(n)$ is true for all even natural numbers. $\forall n \in \mathbb{N}.(\text{even})$ IMPLIES $q(n)$.
To this end, let $p(k) = q(2k)$.

$\forall k \in \mathbb{N}.p(k)$
Base case $p(0)$
Let $k \in \mathbb{N}$ be arbitrary
  Assume $p(k)$
  $\vdots$
  $p(k+1)$
It is not sufficient to prove $q(0)$ and $\forall n \in \mathbb{N}.(q(n)$ IMPLIES $q(n+2))$, since this can be false but $p(n)$ cna be true.
However, we can write it as $\forall n \in \mathbb{N}.$(n is even AND $q(n)$) IMPLIES $q(n+2)$

Prove $\forall i \in M, p(i)$ where $M = \{i \in \mathbb{N} | 0 \leq i \leq n\}$
$\vdots$

$p(0)$
Let $i \in M - \{n\}$ be arbitrary
  Assume $p(i)$

  $\vdots$
  $p(i+1)$

$p(i)$ IMPLIES $p(i+1)$
$\forall i \in M.p(i)$ ☺

☺

Arithmetic Mean $(\sum_{i=1}^{n} a_i)/n$
Geometric Mean $(\prod_{i=1}^{n} a_i)^{1/n}$ ☺

**Theorem 4**

(Cauchy 1821) For all positiv eintegers, the geometric mean of $n$ positive real numbers is less than or euqal to their arithemetic mean,
$\forall n \in \mathbb{Z}^+$, let $P(n) = \forall a \in \mathbb{R}.(\sum_{i=1} a_i)/n \leq (\prod_{i=1} a_i)^{1/n}$.

*Proof.* Base case $n = 2$.
Induction steps:
Let $n$ be arbitrary integer $\geq 2$,
Assume $P(n)$
$\vdots$
$P(n-1)$
$P(n)$ IMPLIES $P(n-1)$
Assume $p(i)$
$\vdots$
$P(2i)$
$P(i)$ IMPLIES $P(2i)$
$\forall n \in \mathbb{Z}^+.P(n)$ by induction
$P(m)$
Consider the smallest power of 2 that is at least $m$.
$2^{k-1} < m \leq 2^k$
$P(2)P(4)...P(k)$

With the template above, we now consider the base case $n = 2$.
$\forall a_1 \in \mathbb{R}^+.\forall a_2 \in \mathbb{R}^+.\left(\sqrt{a_1 a_2} \leq (a_1 + a_2)/2\right).$
Let $a_1, a_2 \in \mathbb{R}^+$ be arbitrary
Then $a_1^2 - 2a_1a_2 + a_2^2 = (a_1 - a_2)^2 \geq 0$
so $a_1^2 + a_2^2 \geq 2a_1a_2$
Hence $\left(\frac{a_1+a_2}{2}\right)^2 = \frac{a_1^2+2a_1a_2+a_2^2}{4} \geq \frac{2a_1a_2+2a_1a_2}{4} = a_1a_2$
hence $(a_1a_2)^{1/2} \leq \frac{a_1+a_2}{2}$
Now, assume $P(n)$
$\forall a \in (\mathbb{R}^+)^n, (\sum_{i=1} a_i)/n \leq (\prod_{i=1}^n a_i)^{1/n}$
For $1 \leq i \leq n-1$, let $a_i \in \mathbb{R}^+$ be arbitrary
Let $b_i = a_i$ for $1 \leq i \leq n-1$
and let $b_n = (\sum_{i=1}^{n-1} a_i)/(n-1)$, then $\sum_{i=1}^n a_i = b_n(n-1)$
By specialization of $P(n)$, we have

$$\begin{aligned}
(\prod_{i=1}^n b_i)^{1/n} &\leq (\sum_{i=1}^n b_1)/n \\
&= (b_n + \sum_{i=1}^{n-1} a_i)/n \\
&= (b + (n-1)b_n)/n \\
&= (nb_n)/n \\
&= b_n
\end{aligned}$$

Then, ...

We first show $P(n)$ IMPLIES $P(n-1)$, that is,

$$\begin{aligned}\left(\prod_{i=1}^{n-1} a_i\right)^{1/n-1} = \left(\prod_{i=1}^{n-1} b_i\right)^{1/n-1} &= \left(\frac{\prod_{i=1}^{n} b_i}{b_n}\right)^{1/n-1} \\ &\leq \left(\frac{(b_n)^n}{b_n}\right)^{1/n-1} \\ &= \left({b_n}^{n-1}\right)^{1/n-1} \\ &= b_n \\ &= \left(\sum_{i=1}^{n-1} a_i\right)/(n-1)\end{aligned}$$

Now we show $P(n)$ IMPLIES $P(2n)$.
To this end, assume $P(n)$;
$\forall a \in \left(\mathbb{R}^+\right)^n, \quad \left(\prod_{i=1}^{n} ai\right)^{1/n} \leq \left(\sum_{i=1}^{n} a_i\right)/n;$
for $1 \leq i \leq 2n$ let $a_i \in \mathbb{R}^+$ be arbitrary;
let $b_1 = \left(\sum_{i=1}^{n} a_i\right)/h$ and $b_2 = \left(\sum_{i=n+1}^{2n} a_i\right)/n$
By specialization of $P(n)$
$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \left(\sum_{i=1}^{n} a_i\right)/n$
and $\left(\prod_{i=n+1}^{n} a_i\right)^{1/n} \leq \left(\sum_{i=n+1}^{2n} a_i\right)/n$
$P(2n)$; generalization

By speciclization of $P(n)$

$$
\begin{aligned}
&(b_1 b_2)^n = (b_1 b_2)^{2n/2} \leq ((b_1 + b_2)/2)^{2n} \\
&\prod_{i=1}^{2n} a_i = \left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=n+1}^{2n} a_i\right) \\
&\leq \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^n \left(\frac{\sum_{i=n+1}^{2n} a_i}{n}\right)^n \\
&= (b_1 b_2)^n \leq \left(\frac{b_1 b_2}{2}\right)^{2n} \\
&= \left(\frac{\sum_{i=1}^{n} a_i}{2n} + \frac{\sum_{i=n+1}^{2n} a_i}{2n}\right)^{2n} \\
&= \left(\frac{\sum_{i=1}^{2n} a_i}{2n}\right)^{2n}
\end{aligned}
$$

Hence

$$
\left(\prod_{i=1}^{2n} a_i\right)^{1/2n} \leq \frac{\sum_{i=1}^{2n} a_i}{2n}
$$

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**Definition 2 – Strong / Complete Induction**

To prove $\forall i \in \mathbb{N}.P(i)$.
$p(0)$
Assume $p(0)$, prove $p(1)$
Asusme $p(0), p(1)$, prove $p(2)$
Assume $p(0), p(1), p(2)$, prove $p(3)$
$\vdots$
$\forall i \in \mathbb{N}.p(i)$