1 Eigenvalues and eigenvectors

Eigenvalues are the roots of the characteristic polynomial $p_T(z) = \det(zI - T)$.
Algebric multiplicity of λ is the multiplicity of λ as a root of $p_T(z)$.
Geometric multiplicity of λ is the dimension of the eigenspace $E_\lambda(T) = \ker(T - \lambda I)$.

1.1 Diagonalizability

T is diagonalizable if and only if the characteristic polynomial splits, and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.
That is, there exists an eigenbasis of T, or equivalently, V is a direct sum of eigenspaces of T.

1.2 Compute eigenvectors, eigenvalues, and multiplicities

For $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $p_A(z) = \det(zI - A) = \det \begin{pmatrix} z-1 & -1 & -1 \\ 0 & z-1 & 0 \\ 0 & -1 & z-2 \end{pmatrix} = (z-1)^2(z-2)$.
Eigenvalues are $\lambda = 1, 2$.
Algebric multiplicity of $\lambda = 1$ is 2, and of $\lambda = 2$ is 1.
Geometric multiplicity of $\lambda = 1$ is: $\dim(\ker(A - I))$, since $\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$ has rank 1, the geometric dimension is 2.
To find the eigenvectors of $\lambda = 1$, solve the homogenous system, $\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $(1, 0, 0), (0, 1, -1)$ are the eigenvectors.
Geometric multiplicity of $\lambda = 2$ is 1 since it is at least 1 and at most the algebric multiplicity 1.
To find the eigenvectors of $\lambda = 2$, solve the homogenous system, $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $(1, 0, 1)$ is the eigenvector.

♍

2 Normal forms for linear transformations

2.1 Invariant subspaces, cyclic subspaces

T-invariant if $T(W) \subseteq W$.
T-cyclic if $W = \text{span}\{v, T(v), T^2(v), ...\}$.
For finite dimension W, $W = \text{span}\{v, T(v), T^2(v), ..., T^{k-1}(v)\}$ for some k.

2.2 Companion matrix

If V admits a T-cyclic vector v, then with basis $\beta = \{v, T(v), T^2(v), ...\}$, $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$, with $p_T(z) = a_0 + ... + z^n$.

2.3 Compute Jordan form

T admites a unique Jordan canonical form (up to permutation) if and only if the characteristic polynomial splits.

Jordan block $J(\lambda, k) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \in M_{k \times k}(\mathbb{F})$
N nilpotent if $N^k = 0$ for some k.
Find JCF:

1. Compute eigenvalues of T (roots of $p_T(z)$).
2. Compute geometric multiplicities of eigenvalues (number of Jordan blocks of type λ).
3. Compute algebric multiplicities of eigenvalues (sum of sizes of Jordan blocks of type λ).
4. Largest Jordan block of type λ with size k is the smallest k such that $\text{rank}(T - \lambda I)^k = \text{rank}(T - \lambda I)^{k+1}$.

For $A = \begin{pmatrix} 2 & 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$, $p_A(z) = (z-2)^5$.

Then since $A - 2I = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ has rank 3 and nullity 2, the geometric multiplicity is 2, so there are 2 Jordan blocks of type 2 (total size of 5).

Since $(A - 2I)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $(A - 2I)^3 = 0$, the largest Jordan block of type 2 is of size 3.

Then the remaining one has size 2.

Giving $J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

♍

2.4 Find Jordan basis

For $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $p_A(z) = (z-1)^4$, $A - I = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, nullity is 2, so there are 2 Jordan blocks of type 1 (with total size 4).
Since $(A-I)^2 = 0$, the sizes are 2 and 2.

To find the Jordan basis, first pick $v = (0,1,0,0)$ not in the kernel of $A - I$, then v with $(A-I)(v) = (1,0,0,0)$ generate the first Jordan block.
Then pick $w = (0,0,0,1)$ such that w and $(A-I)(w) = (1,-1,1,0)$ are linearly independent with $v, (A-I)v$, and w is not in the kernel of $A - I$.

Let $C = (v, (A-I)v, w, (A-I)w)$, then $C^{-1}AC = J$ where J is the Jordan form. ♍

3 Minimal polynomial

3.1 Cayley-Hamilton theorem

The characteristic polynomial $p_T(z)$ satisfies $p_T(T) = 0$.

3.2 Minimal polynomial

Let k be the smallest such that $I, T, T^2, ..., T^k$ are linearly dependent. Then exists unique coefficients such that $a_0 I + a_1 T + ... + a_k T^k = 0$.
$q_T(z) = a_0 + a_1 z + ... + a_k z^k$ the minimal polynomial of T.
$p_T(z)$ may not be the minimal polynomial.
The minimal polynomial divides the characteristic polynomial.
Eigenvalues of T are roots of the minimal polynomial.
T has T-cyclic vector v if and only if $q_T(z) = p_T(z)$.

3.3 Read minimal polynomial from JCF

If $p_T(z)$ splits, then $p_T(z)$ has powers the algebric multiplicities; $q_T(z)$ has powers the size of the largest Jordan blocks.

T diagonalizable if and only if $q_T(z)$ splits with no powers.
So, if there is a unique Jordan block of each type, then $p_T(z) = q_T(z)$.

3.4 Get JCF from minimal polynomial

Compute each $(A - \lambda I)^k$ and find the largest size of the Jordan block of type λ, then if all k matches the algebraic multiplicity, then $p_T(z) = q_T(z)$.

4 Inner product spaces

Real inner product is a symmetric positive definite bilinear map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.

Complex inner product is a positive definite map to $\mathbb{C}$ s.t.: linear in the first argument, $\langle v, w \rangle = \overline{\langle w, v \rangle}$ (so conjugate linear in the second argument).
Inner product space has norm $||v|| = \sqrt{\langle v, v \rangle}$.

4.1 Projection

v, w orthogonal if $\langle v, w \rangle = 0$.
From $0 = \langle v - aw, w \rangle = \langle v, w \rangle - a\,||w||^2$, we have $a = \frac{\langle v, w \rangle}{||w||^2}$.
So, define projection $\text{proj}_w(v) = \frac{\langle v, w \rangle}{||w||^2} w$.

4.2 Triangle inequality and Cauchy-Schwarz inequality

Cauchy-Schwarz: $|\langle v, w \rangle| \le ||v||\,||w||$, equality holds if and only if v and w are linearly dependent (derived from splitting v to $\text{proj}_w(v)$ and $v - \text{proj}_w(v)$ and apply Pythagorean).
Triangle Inequality: $||v+w||^2 = ||v||^2 + 2\,\text{Re}\,\langle v, w \rangle + ||w||^2 \le ||v||^2 + 2|\langle v, w \rangle| + ||w||^2 \le (||v|| + ||w||)^2$.
So, $||v+w|| \le ||v|| + ||w||$.
Equality holds if and only if $w = 0$ or $v = \lambda w$ for $\lambda \ge 0$.

4.3 Orthogonal and orthonormal bases

Orthogonal basis is a basis $\beta = \{v_1, ..., v_n\}$ such that $\langle v_i, v_j \rangle = 0$ for $i \ne j$.
Orthonormal basis can be obtained from orthogonal basis by replacing v with $\frac{v}{||v||}$.

For orthonormal basis $v_1, ..., v_n$, $v = a_1 v_1 + ... + a_n v_n = \sum_{j=1}^n \langle v, v_j \rangle v_j$
$\langle v, w \rangle = \sum_{j=1}^n a_j \overline{b_j}$

4.4 Gram-Schmidt algorithm

Orthogonal basis:

1. Let $y_1, ..., y_n$ be any basis of V.
2. $u_1 = y_1$.
3. $u_{k+1} = y_{k+1} - \sum_{j=1}^k \text{proj}_{u_j}(y_{k+1})$.

We may normalize throughout or after the algorithm to get orthonormal basis.

For $= P_2(\mathbb{R})$, $\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$.
Fix basis x, x^2.
Let $u_1 = \frac{x}{||x||} = \frac{x}{\sqrt{5}}$.
And $u_2' = x^2 - \text{proj}_{u_1}(x^2) = x^2 - \frac{\langle x^2, u_1 \rangle}{||u_1||^2} u_1 = x^2 - \langle x^2, u_1 \rangle u_1 = x^2 - \frac{9}{\sqrt{5}} \frac{x}{\sqrt{5}} = x^2 - \frac{9}{5}x$, which gives $u_2 = \frac{u_2'}{||u_2'||} = \frac{\sqrt{5}}{2} x^2 - \frac{2\sqrt{5}}{10} x$ ♍

4.5 Orthogonal complements

Let $W^\perp = \{v \in V : \langle v, w \rangle = 0, \forall w \in W\}$ be the orthogonal subspace of W.

$W \cap W^\perp = \{0\}$, $W \subseteq (W^\perp)^\perp$, with equality holds if W is finite dimensional.

4.6 Orthogonal projections

Every direct sum $V = V_1 \oplus V_2$ matches with a projection P where $\ker(P) = V_1$ and $\mathrm{ran}(P) = V_2$.
Call P_W which matches $V = W \oplus W^\perp$ the orthogonal projection onto W.

4.7 Minimizing distance

$P_W v$ is the unique point in W closest to v.
$||P_W v - v|| \leq ||x - v||$ for all $x \in W$, with equality if and only if $P_W v = x$.
For orthonormal basis $v_1, ..., v_m$ of W,

$$P_W(v) = \sum_{j=1}^{m} \langle v, v_j \rangle v_j.$$

5 Adjoint operators

5.1 Riesz representation theorem

For finite dimension V, $V \to V^*, y \mapsto \ell_y$ is a bijection. I.e. every linear functional $\ell \in V^*$ is of the form $\ell(v) = \langle v, y \rangle$ for some $y \in V$.
Specifically, $y = \sum_{j=1}^n \overline{\ell(v_j)} v_j$ where $\beta = \{v_1, ..., v_n\}$ is an orthonormal basis of V.

5.2 Adjoint operator

For finite dimension V, for any $T \in \mathcal{L}(V, W)$, exists unique T^* (adjoint operator to T) such that $\langle Tv, w \rangle = \langle v, T^* w \rangle$ for all $v \in V, w \in W$.
$A^* = \overline{A}^T$
$V = \ker(T) \oplus \mathrm{ran}(T^*), W = \ker(T^*) \oplus \mathrm{ran}(T)$

5.3 Self-Adjoint operator

Call P self-adjoint if $P = P^*$.
P corresponds to $V = W \oplus U$ and P^* corresponds to $V = U^\perp \oplus W^\perp$.
T is self-adjoint if and only if P is an orthogonal projection.

5.4 Spectral theorem for self-adjoint operators

Since $A = A^* = \overline{A}^T$, for $\mathbb{F} = \mathbb{R}$, A is symmetric, and for $\mathbb{F} = \mathbb{C}$, diagonal entries of A are real.
All eigenvalues of A are real (implies at least one eigenvector).
Eigenvectors with different eigenvalues are orthogonal.
Self-adjoint A is diagonalizable with real eigenvalues (by finding eigenvalues on induction).

5.5 Quadratic forms and their diagonalization

Define quadratic form $q(v) = \langle Tv, v \rangle = \lambda_1 y_1^2 + ... + \lambda_n y_n^2$ for self-adjoint T.
Every quadratic form (real symmetric matrix and thus self-adjoint) can be put into diagonal form (diagonalized) by an orthogonal change of coordinates (eigenvectors).

5.6 Skew-adjoint operators

Call S skew-adjoint if $S* = -S$.
If S is complex skew-adjoint, then $T = \frac{1}{i} S$ is self-adjoint.
If S is real skew-adjoint, then diagonal entries are all 0.

5.7 Unitary operators

Call T unitary if $T^*T = TT^* = I$.
Unitary if and only if T is invertible and $T^{-1} = T^*$.

$\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in V$.
$||Tv|| = ||v||$ for all $v \in V$.

T unitary if and only if its columns are an orthonormal basis of $\mathbb{F}^n$.
eigenvalues of T satisfy $|\lambda| = 1$.

6 Normal operators

Call T normal if $T^*T = TT^*$ commutes.
Self-adjoint, skew-adjoint, and unitary are normal.
Normal if and only if $||Tv|| = ||T^*v||$ for all $v \in V$.
If T normal, then eigenvector v of T is also an eigenvector of T^*, with $\lambda' = \overline{\lambda}$.

6.1 Spectral theorem for normal operators

T normal if and only if exists orthonormal basis $\beta = \{v_1, ..., v_n\}$ of V such that T is diagonal in β.

6.2 Spectral resolution (decomposition)

T normal if and only if $T = \sum_{j=1}^n \lambda_j P_j$, where λ_j are eigenvalues of T and P_j are orthogonal projections onto the eigenspaces of T.

6.3 Functional calculus

Let spectrum be the set of eigenvalues of T.
If T normal, then $T = \sum_{j=1}^n \lambda_j P_j$, which impies $T^* = \sum_{j=1}^n \overline{\lambda_j} P_j$.

If T normal, then

- self-adjoint if and only if $\lambda_j \in \mathbb{R}$.
- unitary if and only if $\lambda_j \in \mathbb{C}$ with $|\lambda_j| = 1$.

Using $P_\lambda P_\mu = 0$ for $\lambda \neq \mu$, $q(T) = \sum_{j=1}^n q(\lambda_j) P_j$ for $q(z)$ polynomial.
This holds more generally for any complex f s.t.. $f(z) = \sum_{j=1}^n f(\lambda_j) P_j$.

6.4 Positive operators

T is positive if it is self-adjoint and eigenvalues are non-negative.
For complex T, positive if and only if $\langle Tv, v \rangle \geq 0$ for all $v \in V$.

For 2×2 positive matrix, find square root:

1. Find eigenvalues λ_1, λ_2.
2. Find normalized eigenvectors v_1, v_2.
3. Let U be unitary change of coordinate matrix with columns v_1, v_2, then

$$U^* A U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with

$$\sqrt{A} = U \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} U^*$$

7 Decomposition theorems

7.1 Polar decomposition of operators

For invertible T, exists unique unitary U and positive R such that $T = UR$.

7.2 Singular value decomposition