Trigonometric Functions and Their Inverses

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Contents

1 Introduction

This article is divided into 4 parts:

- Use graphical representation to observe the concept of inverse functions with examples.
- The graphs, the domains, and the ranges of the graphs $f(x) = sin(x), g(x) = cos(x), k(x) = tan(x), m(x) = sec(x)$, and their inverses
- Derive the formula for $\frac{d}{dx}(sin^{-}(x))$ using Pythagorean identity.
- Find $\frac{dy}{dx}$ of $sin^{-}(xy) = cos^{-}(x - y)$

For all 4 parts, full explanation including graphical explanation will be shown.

Derivation will only be shown once, and will use directly after derived.

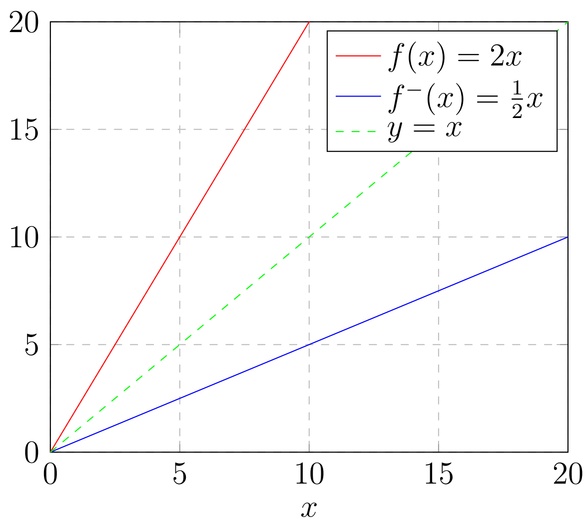
The definitions and explanations won't go beyond the lessons.

The document is made with LaTeX, here is the link to the document.

2 Inverse Functions

2.1 Definition of Inverse Functions

in mathematics, the inverse function of a function f, or the inverse of f, is a function that undoes the operation of f. The inverse of f is usually denoted as f^{-}, for trigonometric functions including sin(x), cos(x), tan(x), csc(x), sec(x), and cot(x), an “arc” before the name of the function can also represent the inverse of this function. For instance, arcsin(x) is the inverse function of sin(x), and arccot(x) is the inverse function of cot(x).



$f(x)$ and $f^{-}(x)$ are shown above. $f(x)$ is defined as $f(x) = 2x$.

x	$f(x)$
0	0
1	2
2	4
3	6
4	8

x	$f^{-}(x)$
0	0
2	1
4	2
6	3
8	4

By looking at the graph, the graph of the inverse of a function is a reflection of its original graph along the line $y = x$. In other words, all coordinates' x and $f(x)$ values are switched.

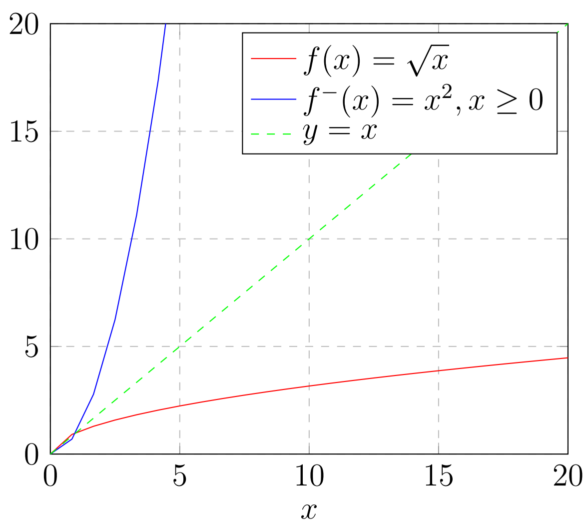
To verify this, we can solve algebraically using the concept that the inverse of the function is formed when all coordinates' independent and the dependent variables on the original function are switched.

$$f(x) = 2x \tag{1}$$
$$x = 2f^{-}(x) \tag{2}$$
$$f^{-}(x) = \frac{1}{2}x \tag{3}$$

all dependent and independent variables are switched when the graph is reflected along the line $y = x$, both approaches result in the same inverse function $f^{-}(x) = \frac{1}{2}x$.

2.2 Inverse of Irrational Functions

As the previous subsection explained generally, for all functions, by graphing, we can find the inverse by reflecting the original graph along the line $y = x$, or in other words, by re-plot all points as their coordinate values are switched.



Same pattern appeared. The only 2 points shared by both graphs are $(0, 0)$ and $(1, 1)$, both lie on the line y=x. which are the invariant points.

$$f(x) = \sqrt{x} \tag{4}$$
$$x = \sqrt{f^{-}(x)} \tag{5}$$

$$\because \sqrt{f^{-}(x)} \in \mathbb{R}$$
$$\therefore f^{-}(x) \geq 0$$
$$\therefore f^{-}(x) = x^2, x \geq 0$$

Which shows the same graph.

2.3 Inverse of Polynomial Functions

Even though polynomial functions are generally defined as $f(x) = a_n x^n + a_{n-1}x^{n-1} \ldots + a_1 x + a_0$. This general form can split into 2 different types of polynomial functions: odd-degree and even-degree polynomials.

To find the inverse of these 2 types of polynomials, simply reflect the graph along the line $y = x$ is the solution, but both types have different domain or range restrictions.

2.3.1 Odd-Degree Polynomials

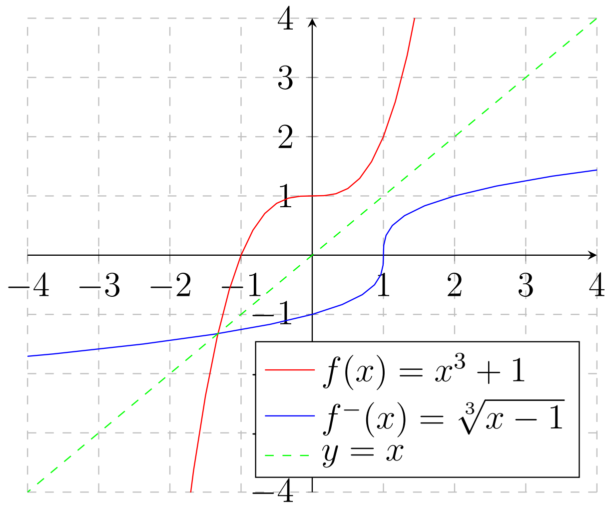
Odd-degree polynomial functions, which n is a positive odd number, have the same domains and ranges:

$$D : \{x \in \mathbb{R}\}$$
$$R : \{f(x) \in \mathbb{R}\}$$

this indicates that the inverse of odd-degree polynomial functions

$$D : \{x \in \mathbb{R}\}$$
$$R : \{f^{-}(x) \in \mathbb{R}\}$$

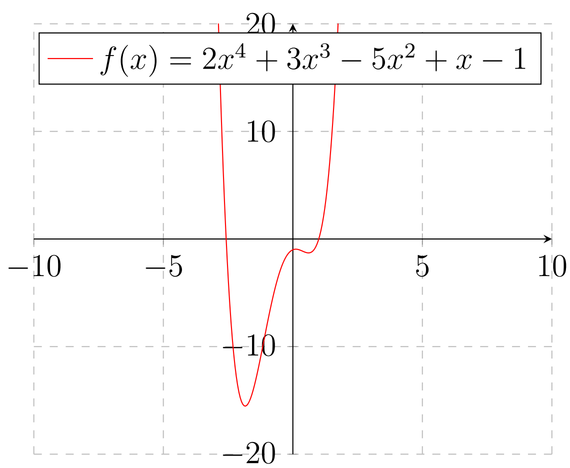
have the same domain and ranges as its original function, as shown:



2.3.2 Even-Degree Polynomials

Even-degree polynomial functions, which n is a positive even number, have only the same domains but not ranges D: $\{x \in \mathbb{R}\}$. The ranges of the functions depend on both the a_n values and the maximum or minumum values of the functions.

Here is an example of an even-degree polynomial function:



Since the value a_n is greater than 0, and the lowest value of $f(x)$ is approximately -15.532, thus we can say the range of this function is approximately R: $\{f(x) \in \mathbb{R} : f(x) \geq -15.532\}$.

2.3.3 Sample Solution of the Inverse of Polynomial Functions

To make the life easier, we define $f(x)$ as $f(x) = 3x^2 + 2x + 1$, to find the inverse of $f(x)$ which is $f^{-}(x)$, we switch x to $f(x)$ and change $f(x)$ to $f^{-}(x)$, then move to where x existed.

$$f(x) = 3x^2 + 2x + 1 \tag{6}$$

$$x = 3\left(f^{-}(x)\right)^2 + 2\left(f^{-}(x)\right) + 1 \tag{7}$$

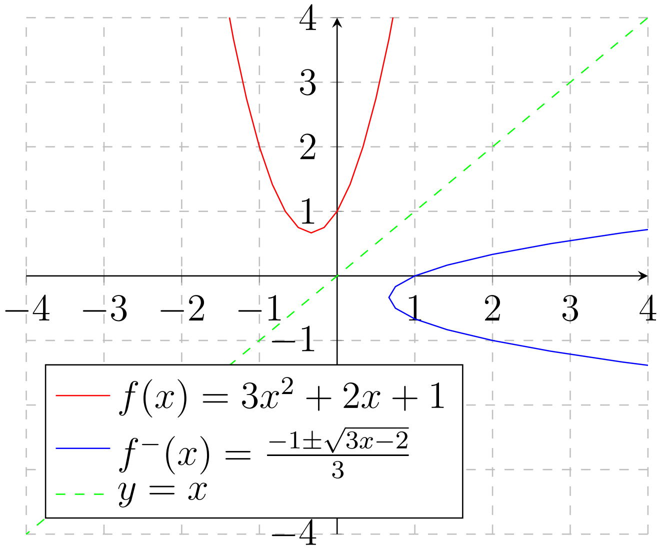
$$0 = 3\left(f^{-}(x)\right)^2 + 2\left(f^{-}(x)\right) + 1 - x \tag{8}$$

$$f^{-}(x) = \frac{-2 \pm \sqrt{4 - 4(3)(1-x)}}{2(3)} \tag{9}$$

$$f^{-}(x) = \frac{-2 \pm \sqrt{4(1 - 3 + 3x)}}{2(3)} \tag{10}$$

$$f^{-}(x) = \frac{-2 \pm 2\sqrt{3x-2}}{2(3)} \tag{11}$$

$$\therefore f^{-}(x) = \frac{-1 \pm \sqrt{3x-2}}{3}$$



The graph verifies the equation of the inverse function.

2.4 Inverse of Rational Functions

Rational functions are functions that are fractions, and both the denominators and the numerators are polynomials. Generally a rational function is defined as

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0} \tag{12}$$

which $m, n \in \mathbb{N}_0$, b_i not all equal to 0.

$f(x) = \frac{x^2+x-2}{2x^2-2x-3}$ is an example of rational functions. To find the inverse of $f(x)$, $f^-(x)$, we follow the same steps as the previous section,

$$f(x) = \frac{x^2 + x - 2}{2x^2 - 2x - 3} \tag{13}$$

$$x = \frac{(f^-(x))^2 + (f^-(x)) - 2}{2(f^-(x))^2 - 2(f^-(x)) - 3} \tag{14}$$

$$2(f^-(x))^2 x - 2(f^-(x))x - 3x = (f^-(x))^2 + (f^-(x)) - 2 \tag{15}$$

$$0 = f^-(x)^2(2x-1) + f^-(x)(-2x-1) - 3x + 2 \tag{16}$$

$$f^-(x) = \frac{-(-2x-1) \pm \sqrt{(-2x-1)^2 - 4(2x-1)(2-3x)}}{2(2x-1)} \tag{17}$$

$$f^-(x) = \frac{2x+1 \pm \sqrt{(4x^2+4x+1) - 4(-6x^2+7x-2)}}{2(2x-1)} \tag{18}$$

$$f^-(x) = \frac{2x+1 \pm \sqrt{4x^2+4x+1+24x^2-28x+8}}{2(2x-1)} \tag{19}$$

$$\therefore f^-(x) = \frac{2x+1 \pm \sqrt{28x^2-24x+9}}{2(2x-1)} \tag{20}$$

Same as even-degree polynomial functions, the inverse of rational function can be a relation rather than a function since different x values can have the same $f(x)$ value.

Because of the denominator $2x^2 - 2x - 3$ of $f(x)$ cannot be 0, then there are 2 vertical asymptotes at

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)} \tag{21}$$

$$x = \frac{2 \pm \sqrt{4 + 24}}{2(2)} \tag{22}$$

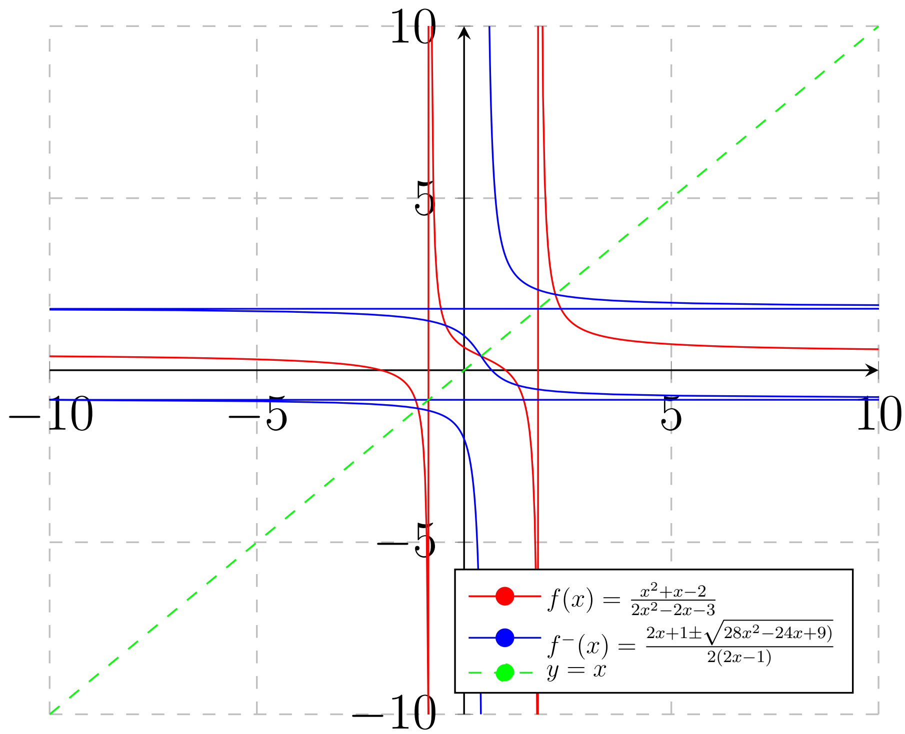
$$x = \frac{2 \pm \sqrt{28}}{2(2)} \tag{23}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{7}}{2} \tag{24}$$

After the x values and $f(x)$ value are switched, the 2 vertical asymptotes of the original equation becomes 2 horizontal asymptotes which are $f^{-}(x) = \frac{1}{2} \pm \frac{\sqrt{7}}{2}$.

To determine the horizontal asymptotes of $f(x)$, we first divide m by n, which in this case is $\frac{1}{1} = 1$. When $\frac{m}{n} < 1$, there is 1 horizontal asymptote at 0; when $\frac{m}{n} = 1$, there is 1 horizontal asymptote at $\frac{a_m}{b_n}$; when $\frac{m}{n} > 1$, there is no horizontal asymptote, but a oblique, or a slant asymptote occurs which can be calculated by dividing the numerator by the denominator, and the quotient is the equation of the asymptote. In this scenario, $\frac{m}{n} = 1$, which the horizontal asymptote is $f(x) = \frac{a_m}{b_n} = \frac{1}{2}$. Which also indicates that the vertical asymptote of $f^{-}(x)$ is $f^{-}(x) = \frac{1}{2}$.

Below is the graph of both $f(x)$ and its inverse $f^{-}(x)$, ignore the line on the asymptotes (due to the graph must be continuous, unable to show the infinity discontinuity found previously),

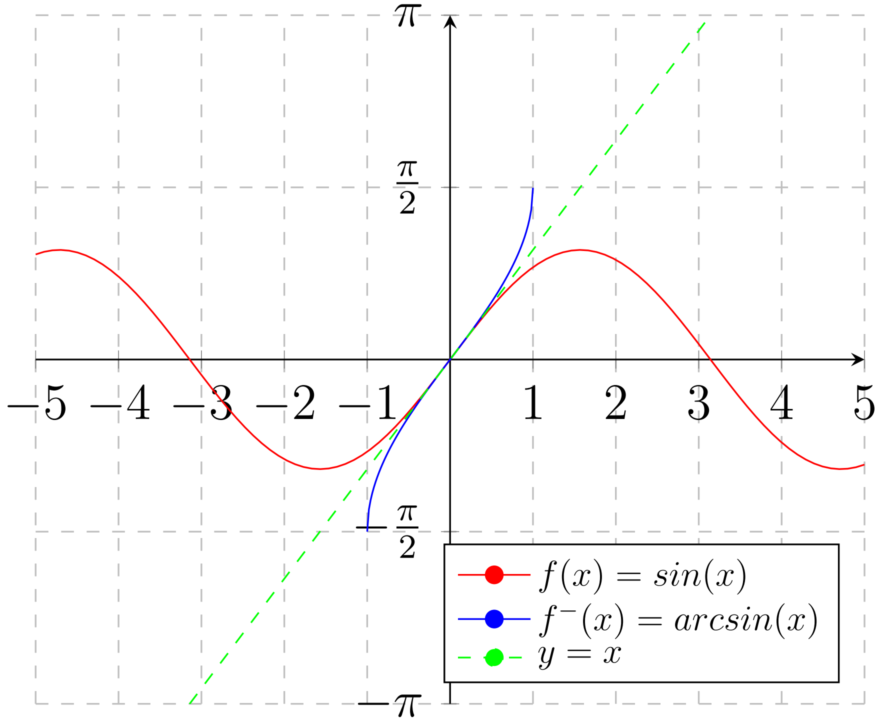


this graph proved the answers of the equation of the asymptotes, and reinforced that for all functions, its inverse is always the original graph but reflected along the line $y = x$.

3 Trigonometric Functions and Their Inverses

Trigonometric Functions are more complicated than the 3 types of functions mentioned in the previous section, simply exchange $f(x)$ and x values and change $f(x)$ to $f^{-}(x)$ does not work with trigonometric functions. This section focuses on the graphs, domains and ranges of functions $f(x) = sin(x)$, $g(x) = cos(x)$, $k(x) = tan(x)$,$m(x) = sec(x)$, and their inverses.

3.1 Sine and Its Inverse Arcsine



As shown, if we simply reflect $f(x)$ along the line $y = x$, then $f^{-}(x)$ turns into a relation not a function, the range of $arcsine(x)$ is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ in order to make it a function.

Thus, the domain and range of $sin(x)$ are

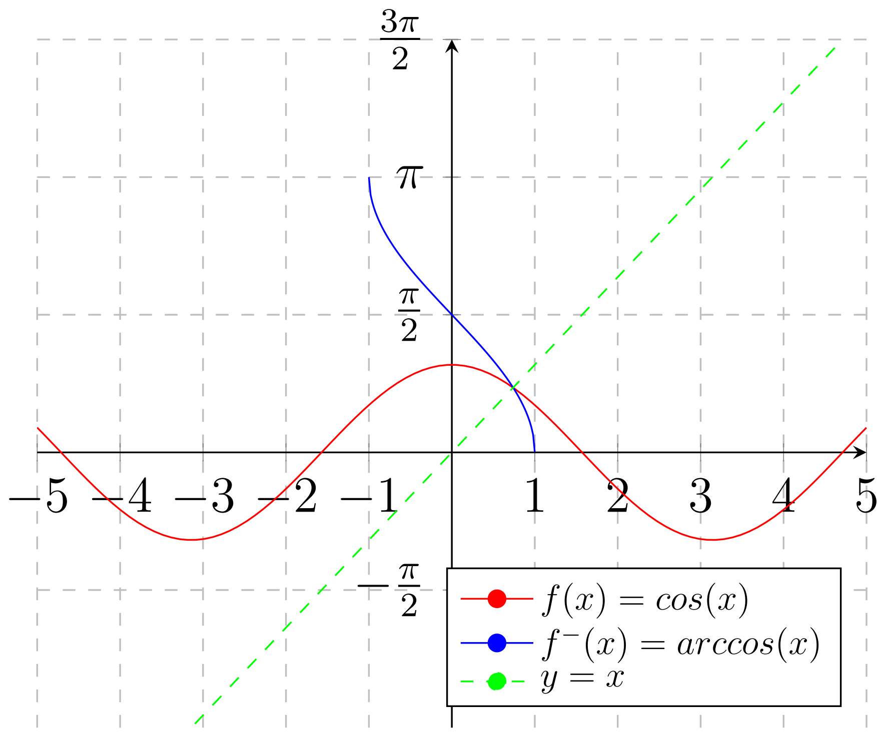
$$D : \{x \in \mathbb{R}\}$$
$$R : \{sin(x) \in \mathbb{R} : sin(x) \in [-1, 1]\}$$

the domain and range of $arcsin(x)$ are

$$D : \{x \in \mathbb{R} : x \in [-1, 1]\}$$
$$R : \{arcsin(x) \in \mathbb{R} : arcsin(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$$

The domain of $arcsin(x)$ is the range of $sin(x)$, and the range of $arcsin(x)$ is half of the period of $sin(x)$.

3.2 Cosine and Its Inverse Arccosine



Same as $sin(x)$ and $arcsin(x)$, the range of $arccos(x)$ is restricted to $[0, \pi]$ to make it a function not a relation.

Thus, the domain and range of $cos(x)$ are

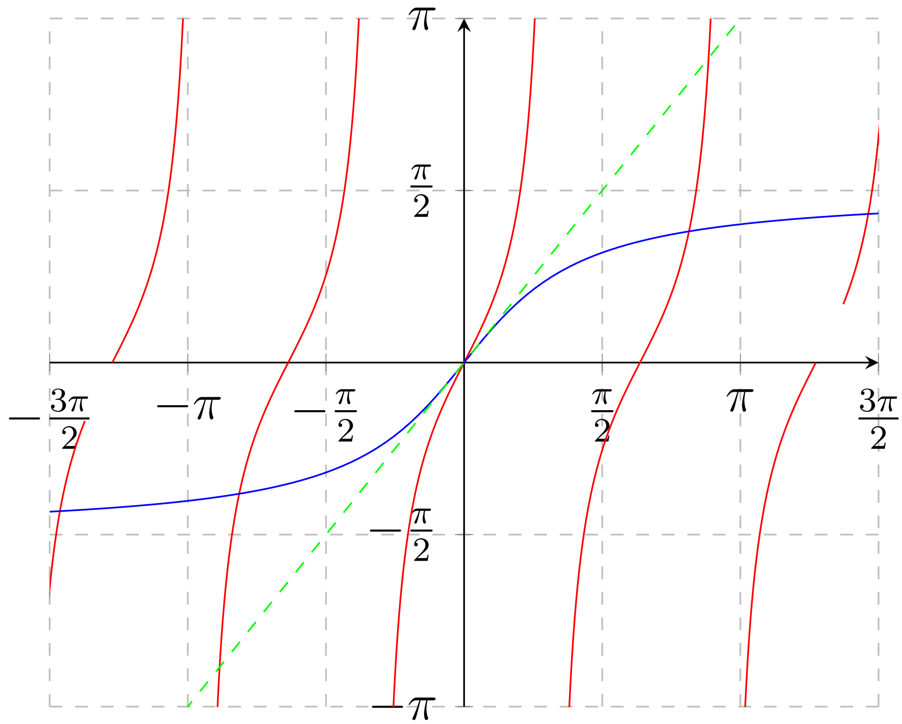
$$D : \{x \in \mathbb{R}\}$$
$$R : \{cos(x) \in \mathbb{R} : cos(x) \in [-1, 1]\}$$

the domain and range of $arccos(x)$ are

$$D : \{x \in \mathbb{R} : x \in [-1, 1]\}$$
$$R : \{arccos(x) \in \mathbb{R} : arccos(x) \in [0, \pi]\}$$

The domain of $arccos(x)$ is the range of $cos(x)$, and the range of $arccos(x)$ is half of the period of $cos(x)$.

3.3 Tangent and Its Inverse Arctangent



Red graph represents $tan(x)$, blue graph represents $arctan(x)$, and green dashed line represents $y = x$. The range of $arctan(x)$ is limited to the interval $[-\frac{pi}{2}, \frac{pi}{2}]$ to avoid relation.

Then, the Domain and Range of $tan(x)$ are:

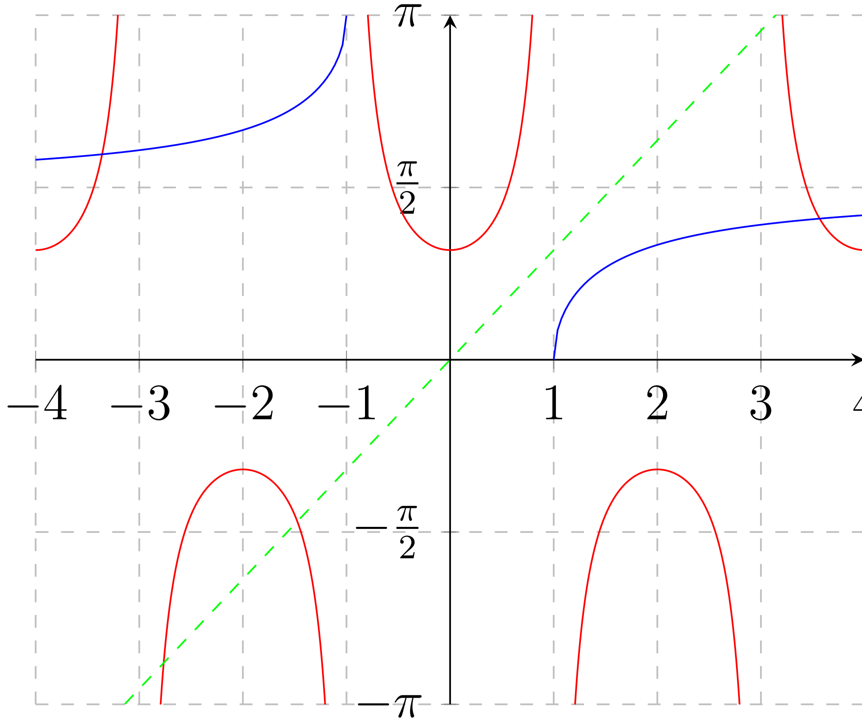
$$D: \left\{x \in \mathbb{R} \ : x \notin \left(\frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}\right\}$$
$$R: \{tan(x) \in \mathbb{R}\}$$

the Domain and Range of $arctan(x)$ are:

$$D: \{x \in \mathbb{R}\}$$
$$R: \left\{arctan(x) \in \mathbb{R}: \ arctan(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$$

The domain of $arctan(x)$ is the range of $tan(x)$.

3.4 Secant and Its Inverse Arcsecant



Red graph represents $sec(x)$, blue graph represents $arcsec(x)$, and green dashed line represents $y = x$. The range of $arcsec(x)$ is limited to the interval $[0, \pi]$ to avoid relation.

Then, the Domain and Range of $sec(x)$ are:

$$D: \left\{x \in \mathbb{R} \ : x \notin \left(\frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}\right\}$$
$$R: \{sec(x) \in \mathbb{R} : sec(x) \in (-\infty, -1] \cup [1, \infty)\}$$

the Domain and Range of $arcsec(x)$ are:

$$D: \{x \in \mathbb{R} : x \in (-\infty, -1] \cup [1, \infty)\}$$
$$R: \{arcsec(x) \in \mathbb{R} : \ arcsec(x) \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\}$$

The domain of $arcsec(x)$ is the range of $sec(x)$.

4 Derivative of Inverse Sine Function

In this section, we will be deriving the formula for $\frac{d}{dx}(sin^{-}(x))$ using the inverse functions theorem, the unit circle, and the Pythagorean identity.

$\dfrac{d}{dx}\left(sin^{-}(x)\right)$ is equivalent to $\dfrac{dy}{dx}$ of the equation $y\ = sin^{-}(x)$, x is the independent variable and y is t he dependent variable,

$$y\ = sin^{-}(x) \tag{25}$$
$$sin(y) = sin\left(sin^{-}(x)\right) \tag{26}$$

using the inverse functions theorem $f\left(f^{-}(x)\right) = x$,

$$sin(y) = x \tag{27}$$

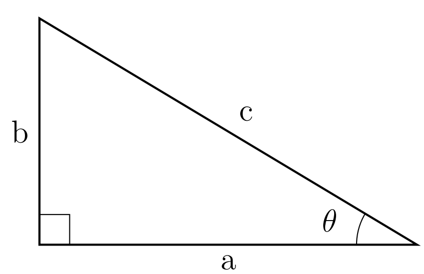
differentiate both sides using the identity $sin'(x) = cos(x) \times (x)'$.

$$(sin(y))^{'} = (x)^{'} \tag{28}$$
$$cos(y) \times \frac{dy}{dx} = 1 \tag{29}$$
$$\frac{dy}{dx} = \frac{1}{cos(y)} \tag{30}$$

we must change $cos(y)$ in terms of x instead of y since y is the dependent variable and x is the independent variable.

4.1 the Pythagorean Theorem

The Pythagorean Theorem states that for all right angle triangles, the sum of the square of 2 right angle sides is equal to the square of the hypotenuse,

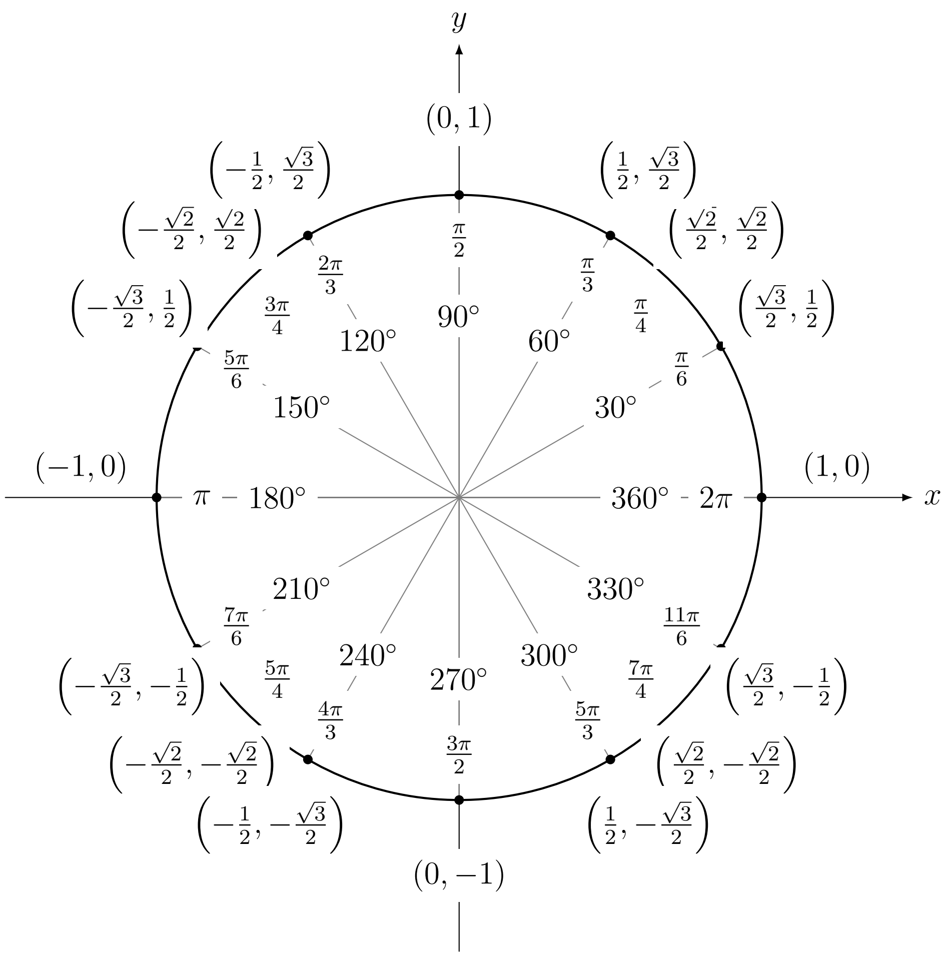


in this case, $a^2 + b^2\ = c^2$. Let θ be any angle in the interval $(0, \frac{\pi}{2})$, then $sin(\theta) = \frac{b}{c}$, $cos(\theta) = \frac{a}{c}$, and $tan(\theta) = \frac{b}{a}$.

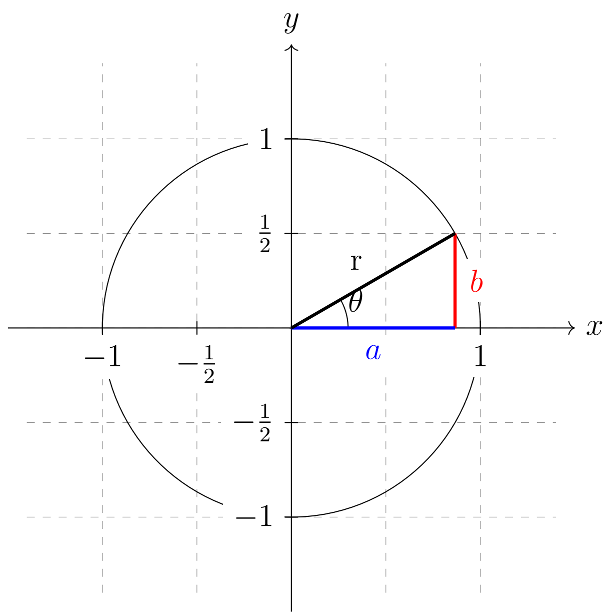
4.2 the Unit Circle and the Pythagorean Identity

Now we can apply the concept of the Pythagorean Theorem into the unit circle.

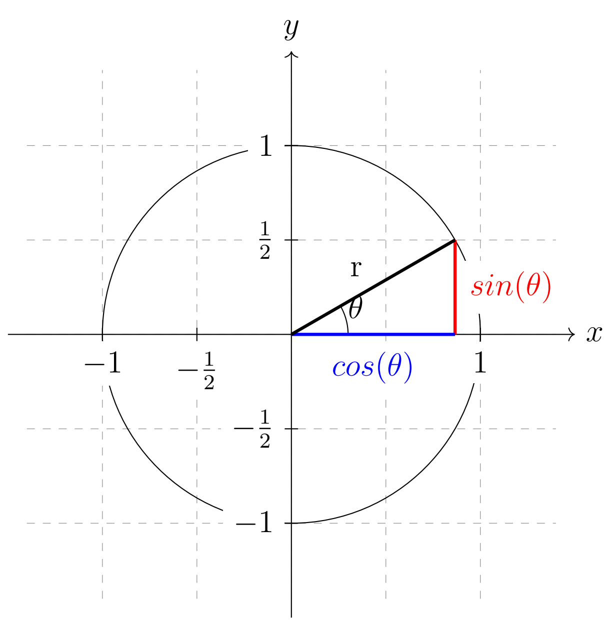
Unit circle, a circle that has a radius of 1,



Choose any related acute angle in the unit circle, let θ be the angle,



since $r = 1$, thus $sin(\theta) = \frac{b}{r} = b$, $cos(\theta) = \frac{a}{r} = a$. Which means we can rewrite the adjacent side length of the triangle formed inside the unit circle with respect to θ as $cos(\theta)$, and the opposite side length as $sin(\theta)$,



Now applying the Pythagorean Theorem,

$$a^2 + b^2 = r^2 \tag{31}$$
$$cos^2(\theta) + sin^2(\theta) = 1 \tag{32}$$

the Pythagorean Identity is proved which is $cos^2(\theta) + sin^2(\theta) = 1$.

4.3 Change $\frac{dy}{dx}$ to in Terms of x

Since $\frac{dy}{dx} = \frac{1}{cos(y)}$, we need to change $cos(y)$ to in terms of x.

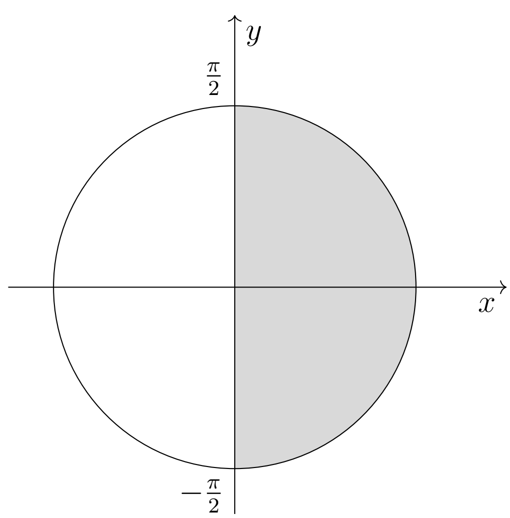
Using the Pythagorean Identity derived previously,

$$cos^2(\theta) + sin^2(\theta) = 1 \tag{33}$$
$$cos^2(\theta) = 1 - sin^2(\theta) \tag{34}$$
$$cos(\theta) = \pm\sqrt{1 - sin^2(\theta)} \tag{35}$$

thus, the general form of $cos(\theta)$ is $\pm\sqrt{1 - sin^2(\theta)}$.

When changing θ to y, there are some restrictions. As shown in Section 3.1, the range of $arcsin(x)$, or $sin^{-}(x)$ is $R : \{arcsin(x) \in \mathbb{R} : arcsin(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$. At (25) we found that y is the arcsine of x, which means the range of y is $R : \{y \in \mathbb{R} : y \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$.

By looking at the unit circle,



as proved in Section 4.2, for any point P on the unit circle, the coordinate of P is $cos(x), sin(x)$, thus within y's range, the x value must ≥ 0, or in other words, $cos(x)$ always ≥ 0.

Also, as derived at (27), we can substitute x to $sin(y)$.

Therefore, we can convert $cos(y)$ to $\sqrt{1-x^2}$.

Substituting $\sqrt{1-x^2}$ into $cos(y)$, we get:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \tag{36}$$

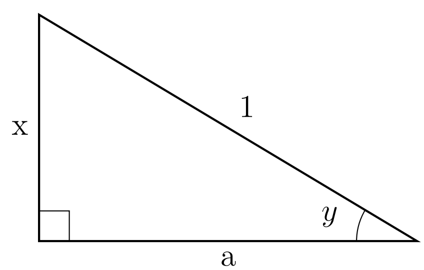
converting back,

$$\frac{d}{dx}(sin^{-}(x)) = \frac{1}{\sqrt{1-x^2}} \tag{37}$$

4.4 Another Approach

Using the concepts of the Pythagorean Theorem, and the Unit Circle, we can find $\frac{d}{dx}(sin^{-}(x))$ in another way.

Since $sin(y) = x$ derived at (27), and the radius of the unit circle is always 1, then,

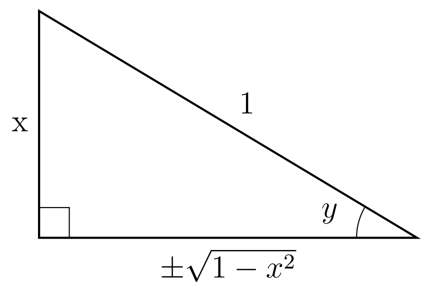


we get this right angle triangle that has a hypotenuse of 1, an angle labelled y, and a opposite side labelled x since $sin(y) = x = \frac{x}{1} = \frac{Opposite}{Hypotenuse}$. Using the Pythagorean Theorem, which explained at Section 4.1,

$$a^2 + x^2 = 1^2 \tag{38}$$

$$a^2 = 1 - x^2 \tag{39}$$

$$\therefore a = \pm\sqrt{1-x^2} \tag{40}$$



now with a we can find $cos(y)$,

$$cos(y) = \frac{Adjacent}{Hypotenuse} \tag{41}$$

$$cos(y) = \pm\frac{\sqrt{1-x^2}}{1} \tag{42}$$

$$\therefore cos(y) = \pm\sqrt{1-x^2} \tag{43}$$

because of the restriction of the values of y, the cosine of y must be positive, thus,

$$cos(y) = \sqrt{1-x^2} \tag{44}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \tag{45}$$

$$\therefore \frac{d}{dx}(sin^{-}(x)) = \frac{1}{\sqrt{1-x^2}} \tag{46}$$

this approach ended up the same answer to the problem.

5 Implicit Differentiation of Inverses of Trigonometric Functions

The final part of this article is to find $\frac{dy}{dx}$ of $sin^{-}(xy) = cos^{-}(x-y)$. To find $\frac{dy}{dx}$, we need to differentiate both side and isolate $\frac{dy}{dx}$, the derivative of $sin^{-}(x)$ is found at (37), to find the derivative of $cos^{-}(x)$, we follow the similar steps.

5.1 Derivative of Inverse Cosine Function

$\dfrac{d}{dx}\left(cos^{-}(x)\right)$ is equivalent to $\dfrac{dy}{dx}$ of the equation $y \; = cos^{-}(x)$, x is the independent variable and y is t he dependent variable,

$$y \; = cos^{-}(x) \tag{47}$$
$$cos(y) = cos\left(cos^{-}(x)\right) \tag{48}$$

using the inverse functions theorem $f\left(f^{-}(x)\right) = x$,

$$cos(y) = x \tag{49}$$

differentiate both sides using the identity $cos'(x) = -sin(x) \times (x)'$.

$$(cos(y))' = (x)' \tag{50}$$
$$-sin(y) \times \frac{dy}{dx} = 1 \tag{51}$$
$$\frac{dy}{dx} = -\frac{1}{sin(y)} \tag{52}$$

we must change $sin(y)$ in terms of x instead of y since y is the dependent variable and x is the independent variable.

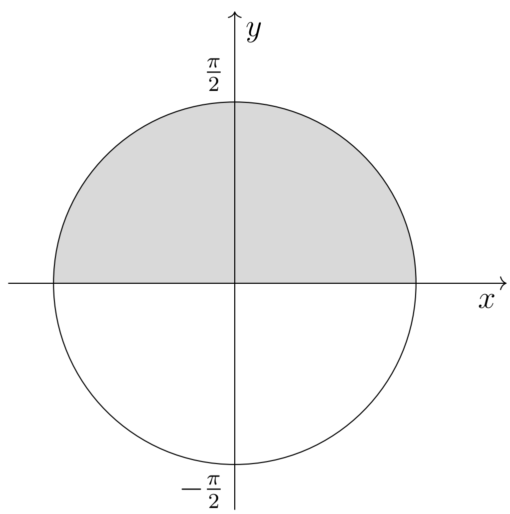
Using the Pythagorean Identity derived previously,

$$cos^2(\theta) + sin^2(\theta) = 1 \tag{53}$$
$$sin^2(\theta) = 1 - cos^2(\theta) \tag{54}$$
$$sin(\theta) = \pm\sqrt{1 - cos^2(\theta)} \tag{55}$$

thus, the general form of $sin(\theta)$ is $\pm\sqrt{1 - cos^2(\theta)}$.

When changing θ to y, there are some restrictions. As shown in Section 3.2, the range of $arccos(x)$, or $cos^{-}(x)$ is $R : \{arccos(x) \in \mathbb{R} : arccos(x) \in [0, \pi]\}$. At (47) we found that y is the arccosine of x, which means the range of y is $R : \{y \in \mathbb{R} : y \in [0, \pi]\}$.

By looking at the unit circle,



as proved in Section 4.2, for any point P on the unit circle, the coordinate of P is $cos(x), sin(x)$, thus within y's range, the y value must ≥ 0, or in other words, $sin(x)$ always ≥ 0.

Also, as derived at (49), we can substitute x to $cos(y)$.

Therefore, we can convert $sin(y)$ to $\sqrt{1 - x^2}$.

Substituting $\sqrt{1 - x^2}$ into $sin(y)$, we get:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \tag{56}$$

converting back,

$$\frac{d}{dx}(cos^{-}(x)) = -\frac{1}{\sqrt{1 - x^2}} \tag{57}$$

5.2 Solving the Implicit Differentiation

Now with the 2 derivatives derived at (37) and at (57), we differentiate $sin^{-}(xy) = cos^{-}(x-y)$ and isolate $\frac{dy}{dx}$. Also, when differentiating a function, we must multiply the derivative of the argument, which needs to apply the product rule to separate $\frac{dy}{dx}$ from $\frac{d(xy)}{dx}$.

$$sin^{-}(xy) = cos^{-}(x-y) \tag{58}$$

$$\frac{d}{dx}\left(sin^{-}(xy)\right) = \frac{1}{\sqrt{1-(xy)^2}} \times \frac{d}{dx}(xy) \tag{59}$$

$$\frac{d}{dx}\left(sin^{-}(xy)\right) = \frac{1}{\sqrt{1-(xy)^2}} \times \left(\left(\frac{dx}{dx}\right)y + \left(\frac{dy}{dx}\right)x\right) \tag{60}$$

$$\frac{d}{dx}\left(sin^{-}(xy)\right) = \frac{y + \left(\frac{dy}{dx}\right)x}{\sqrt{1-(xy)^2}} \tag{61}$$

$$\frac{d}{dx}\left(cos^{-}(x-y)\right) = -\frac{1}{\sqrt{1-(x-y)^2}} \times \frac{d}{dx}(x-y) \tag{62}$$

$$\frac{d}{dx}\left(cos^{-}(x-y)\right) = -\frac{1}{\sqrt{1-(x-y)^2}} \times \left(\frac{dx}{dx} - \frac{dy}{dx}\right) \tag{63}$$

$$\frac{d}{dx}\left(cos^{-}(x-y)\right) = -\frac{\left(1-\frac{dy}{dx}\right)}{\sqrt{1-(x-y)^2}} \tag{64}$$

$$\frac{y + \left(\frac{dy}{dx}\right)x}{\sqrt{1-(xy)^2}} = -\frac{\left(1-\frac{dy}{dx}\right)}{\sqrt{1-(x-y)^2}} \tag{65}$$

$$\left(y + \left(\frac{dy}{dx}\right)x\right)\sqrt{1-(x-y)^2} = -\left(1-\frac{dy}{dx}\right)\sqrt{1-(xy)^2} \tag{66}$$

$$y\sqrt{1-(x-y)^2} + \sqrt{1-(xy)^2} = \frac{dy}{dx}\left(\sqrt{1-(xy)^2} - x\sqrt{1-(x-y)^2}\right) \tag{67}$$

$$\therefore \frac{dy}{dx} = \frac{y\sqrt{1-(x-y)^2} + \sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2} - x\sqrt{1-(x-y)^2}} \tag{68}$$