Probability Culminating Assignment

Game Fair

Analysing the Four Dice Roll and the Six Dice Roll Games

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> 2023/1/28

Part A. Analysis of the Four Dice Roll game

Four Dice Roll

Instructions

Starting at 1000 points. Each roll of the 4 dice costs 10 points.
You win;

- 10 points if no sixes occur,
- 20 points if 2 sixes occur,
- 50 points if 3 sixes occur, or
- 200 points if 4 sixes occur.

Probabilities & Probability Distribution graph

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

Based on the binomial probability formula above, we can calculate the the probability of zero 6's, one 6, two 6's, three 6's, and four 6's when rollnig 4 dice.
p=1/6
q=5/6
n=4
x=0,1,2,3,4

$$P(x)|_{x=0} = \binom{4}{0} \left(\tfrac{1}{6}\right)^0 \left(\tfrac{5}{6}\right)^{4-0} = 1 \times 1 \times \tfrac{5^4}{6^4} = \tfrac{625}{1296}$$

$$P(x)|_{x=1} = \binom{4}{1} \left(\tfrac{1}{6}\right)^1 \left(\tfrac{5}{6}\right)^{4-1} = 4 \times \tfrac{1}{6} \times \tfrac{5^3}{6^3} = \tfrac{500}{1296}$$

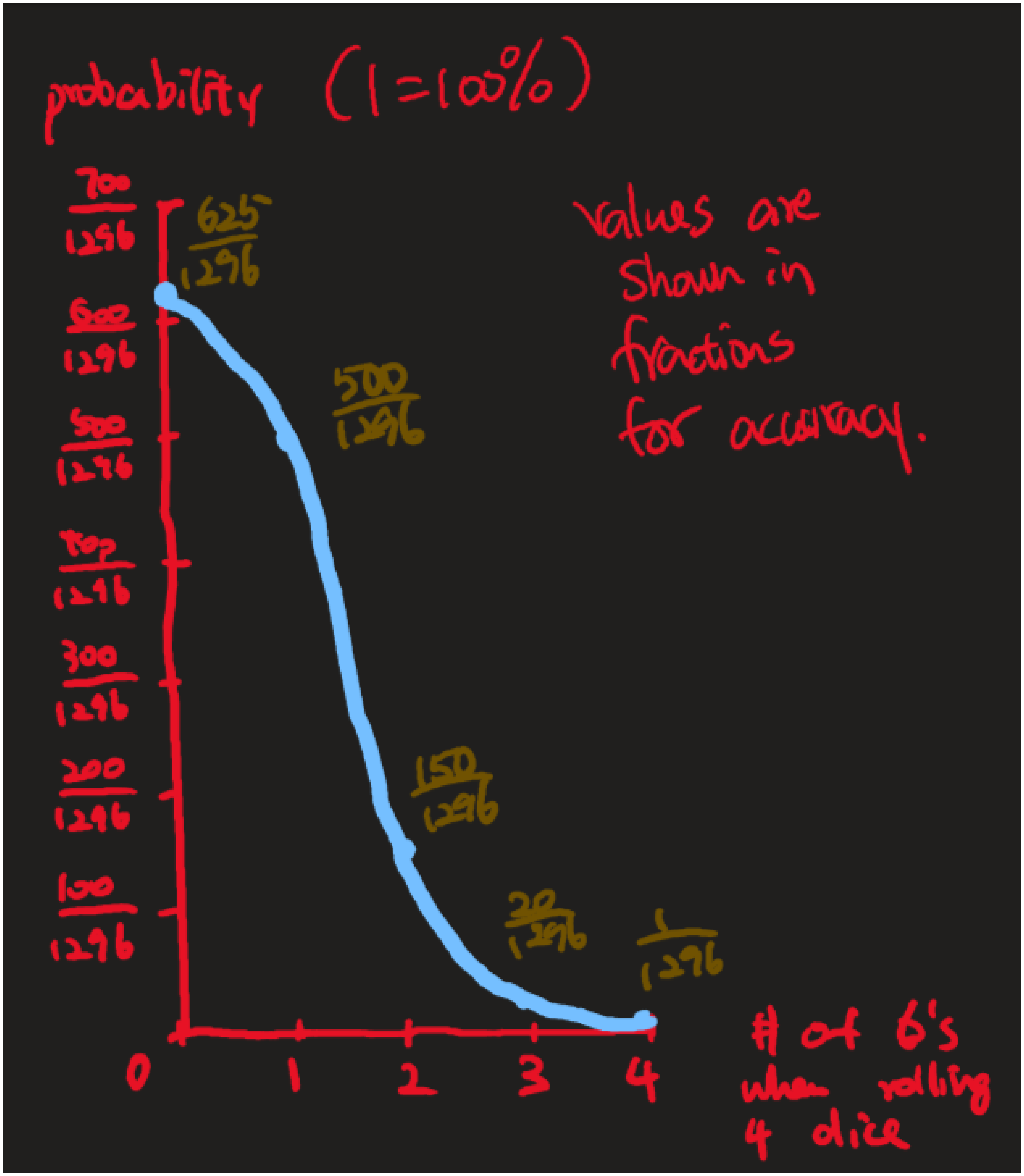
$$P(x)|_{x=2} = \binom{4}{2} \left(\tfrac{1}{6}\right)^2 \left(\tfrac{5}{6}\right)^{4-2} = 6 \times \tfrac{1}{6^2} \times \tfrac{5^2}{6^2} = \tfrac{150}{1296}$$

$$P(x)|_{x=3} = \binom{4}{3} \left(\tfrac{1}{6}\right)^3 \left(\tfrac{5}{6}\right)^{4-3} = 4 \times \tfrac{1}{6^3} \times \tfrac{5}{6} = \tfrac{20}{1296}$$

$$P(x)|_{x=4} = \binom{4}{4} \left(\tfrac{1}{6}\right)^4 \left(\tfrac{5}{6}\right)^{4-4} = 1 \times \tfrac{1}{6^4} \times 1 = \tfrac{1}{1296}$$

$$\tfrac{625}{1296} + \tfrac{500}{1296} + \tfrac{150}{1296} + \tfrac{20}{1296} + \tfrac{1}{1296} = 1$$

The sum of all probabilities is 1, or 100%, which means the values calculated are correct. For accuracy, all probabilities are shown as fractions, but also can be converted to decimal numbers less or equal to 1, or percentage.



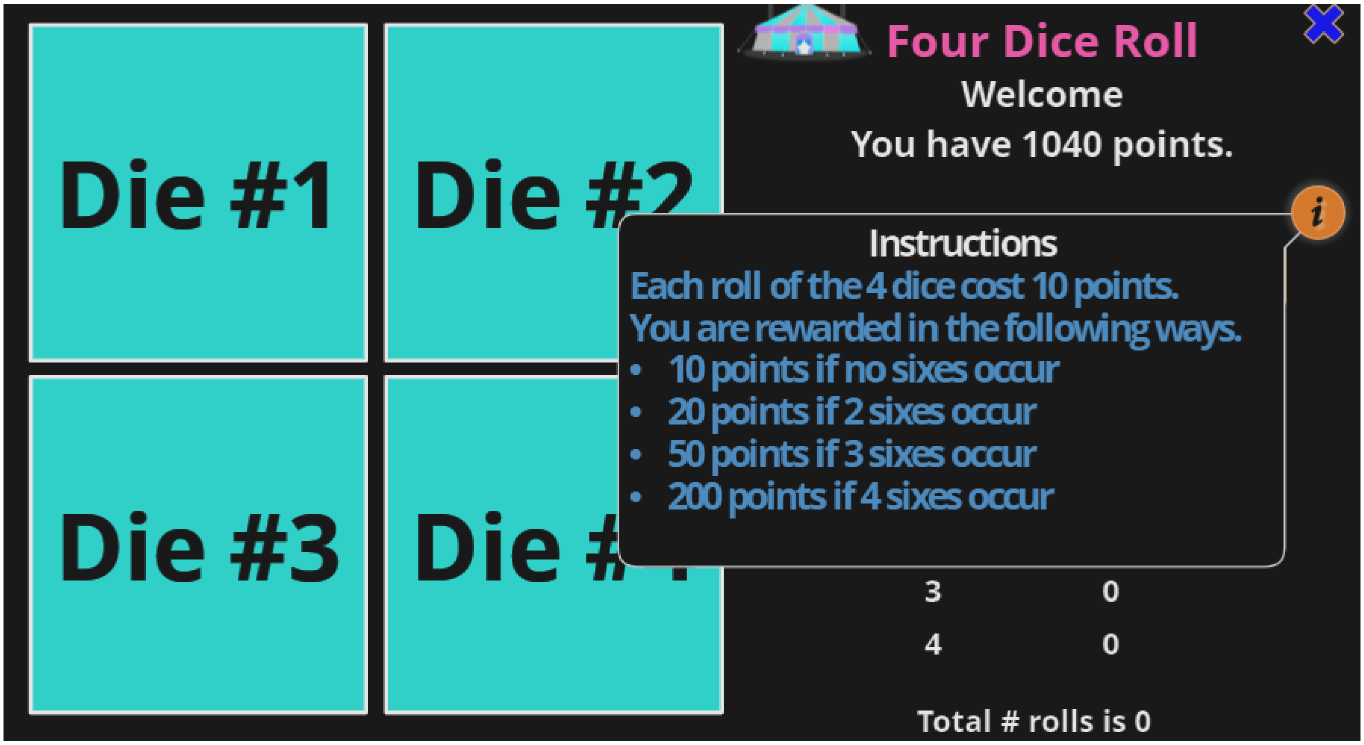
Here is the probability distribution graph. As shown, 0 of 6's when rolling 4 dice is most likely to occur.

Expected Winnings or Losses Regardless of Cost

E(x) = outcome 1 × P(outcome 1) + outcome 2 × P(outcome 2) + outcome 3 × P(outcome 3) + . . .

Using this formula, we can calculate the expected winnings or losses regardless of the cost of every time rolling the dice.

Calculations of P(x)s are already shown previously, so no repeating calculations are shown below for the values of P(x)s.



x (Number of 6's)	y (value of prize in points)	$P(x) = \binom{n}{x} p^x q^{n-x}$	$y \times P(x)$
0	10	$\frac{625}{1296}$	$\frac{3125}{649}$
1	0	$\frac{500}{1296}$	0
2	20	$\frac{150}{1296}$	$\frac{125}{54}$
3	50	$\frac{20}{1296}$	$\frac{125}{162}$
4	200	$\frac{1}{1296}$	$\frac{25}{162}$

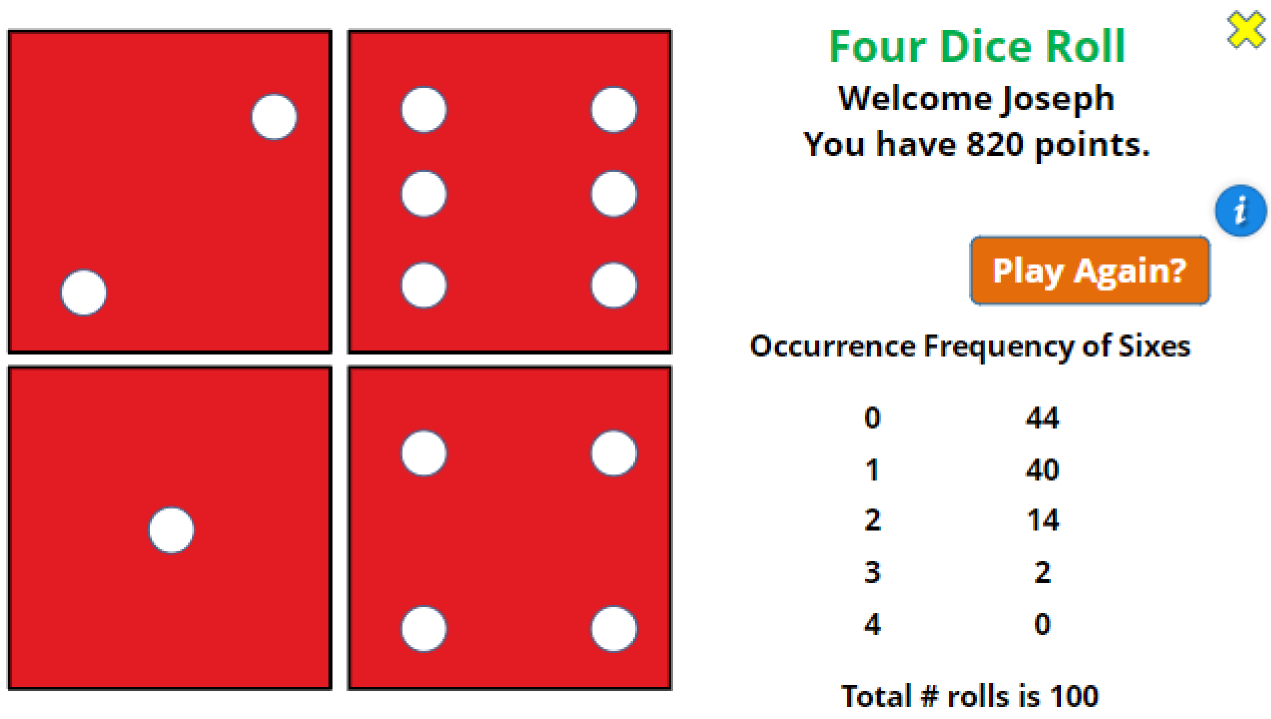
$$E(x) = \sum (y \times P(x)) = \frac{3125}{649} + 0 + \frac{125}{54} + \frac{125}{162} + \frac{25}{162} \approx 8.055840895$$

Expected Winnings or Losses Regarding of Cost

As calculated previously, the E(x) regardless of the cost is approximately 8.055840895 points, or $. But every run costs 10 points to roll, thus, the expted winnings or losses is E(x) = 8.055840895 - 10 = -1.944159105. Since this value is negative and approximately -1.944 points, thus theoretically this game is most likely to loss in a long time.

100 Trials

Here is the result after 100 trials,



Begins with 1000 points, now ended up with 820 points.

Theoretically after 100 rounds, 1000+(100*(-1.944159105)) = 805.5840895 points should be left, and as tried, 820 points left, which the 2 numbers are very close.
In another point of view, the losses per trial of these 100 trials is (820-1000)/100 = -1.8 points per trial, which is close to -1.944 points per trial.

The reason of why the actual losses is differ from the theoretical losses is that nothing happends ideally, because of the lack of attempts, the numbers such as probabilities and losses may be slighly off the theoretical value, but as the number of attempts approching infinity, the values should also be approching the same, such as the probabilities of all frequencies, the average losses per run for the player. However, in reality, all probabilities are not ideal, so this in why the actual losses is less in the case above compared to the theoretical losses.

Part D. Design your own Game of Chance

Name of the Game

Can You Get the Lowest Number? The Six Dice Roller

Description

Roll 6 dice (regular 6-sided dice) at each run (trial). Starting with 800 points. Each run cost 8 points.

1. no point gained when there is no die with the number 1
2. 5 points gained when there is 1 die that have the number 1

3. 15 points gained when there are 2 dice that have the number 1
4. 30 points gained when there are 3 dice that have the number 1
5. 50 points gained when there are 4 dice that have the number 1
6. 70 points gained when there are 5 dice that have the number 1
7. 6666 points gained when there are 6 dice that have the number 1

List of Rules

1. Game starts with 800 points, can still contrinue to play if the player's point is negative or 0.
2. For all trials, 6 six-sided identical dice are thrown on a flat surface, the number appeared at the top is the number of this die in this trial.
3. All sides of the dice from 1 to 6 should all have equal chance to be appeared at the top.
4. Each run cost 8 points.
5. The points gained are shown below in the payout table according to the number of "1" in these 6 dice in the same run.

Payout table

How many "1" in 6 dice	payout in points
0	0
1	5
2	15
3	30
4	50
5	70
6	6666

Probabilities & Probability Distribution graph

This game is a binomial situation, we use the formula to calculate the probabilities,

$$P(x) = \binom{n}{x} p^x q^{n-x}$$

To find the probability of each cases. Which n is the number of runs 6; x is the number of "1" dice 0,1,2,3,4,5,6; p is the probability 1/6, and q is 1-p = 5/6.
p=1/6
q=5/6
n=6
x=0,1,2,3,4,5,6

$$P(x)|_{x=0} = \binom{6}{0}\left(\tfrac{1}{6}\right)^0\left(\tfrac{5}{6}\right)^{6-0} = 1 \times 1 \times \tfrac{5^6}{6^6} = \tfrac{15625}{46656}$$

$$P(x)|_{x=1} = \binom{6}{1}\left(\tfrac{1}{6}\right)^1\left(\tfrac{5}{6}\right)^{6-1} = 6 \times \tfrac{1}{6} \times \tfrac{5^5}{6^5} = \tfrac{18750}{46656}$$

$$P(x)|_{x=2} = \binom{6}{2}\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^{6-2} = 15 \times \tfrac{1}{6^2} \times \tfrac{5^4}{6^4} = \tfrac{9375}{46656}$$

$$P(x)|_{x=3} = \binom{6}{3}\left(\tfrac{1}{6}\right)^3\left(\tfrac{5}{6}\right)^{6-3} = 20 \times \tfrac{1}{6^3} \times \tfrac{5^3}{6^3} = \tfrac{2500}{46656}$$

$$P(x)|_{x=4} = \binom{6}{4}\left(\tfrac{1}{6}\right)^4\left(\tfrac{5}{6}\right)^{6-4} = 15 \times \tfrac{1}{6^4} \times \tfrac{5^2}{6^2} = \tfrac{375}{46656}$$

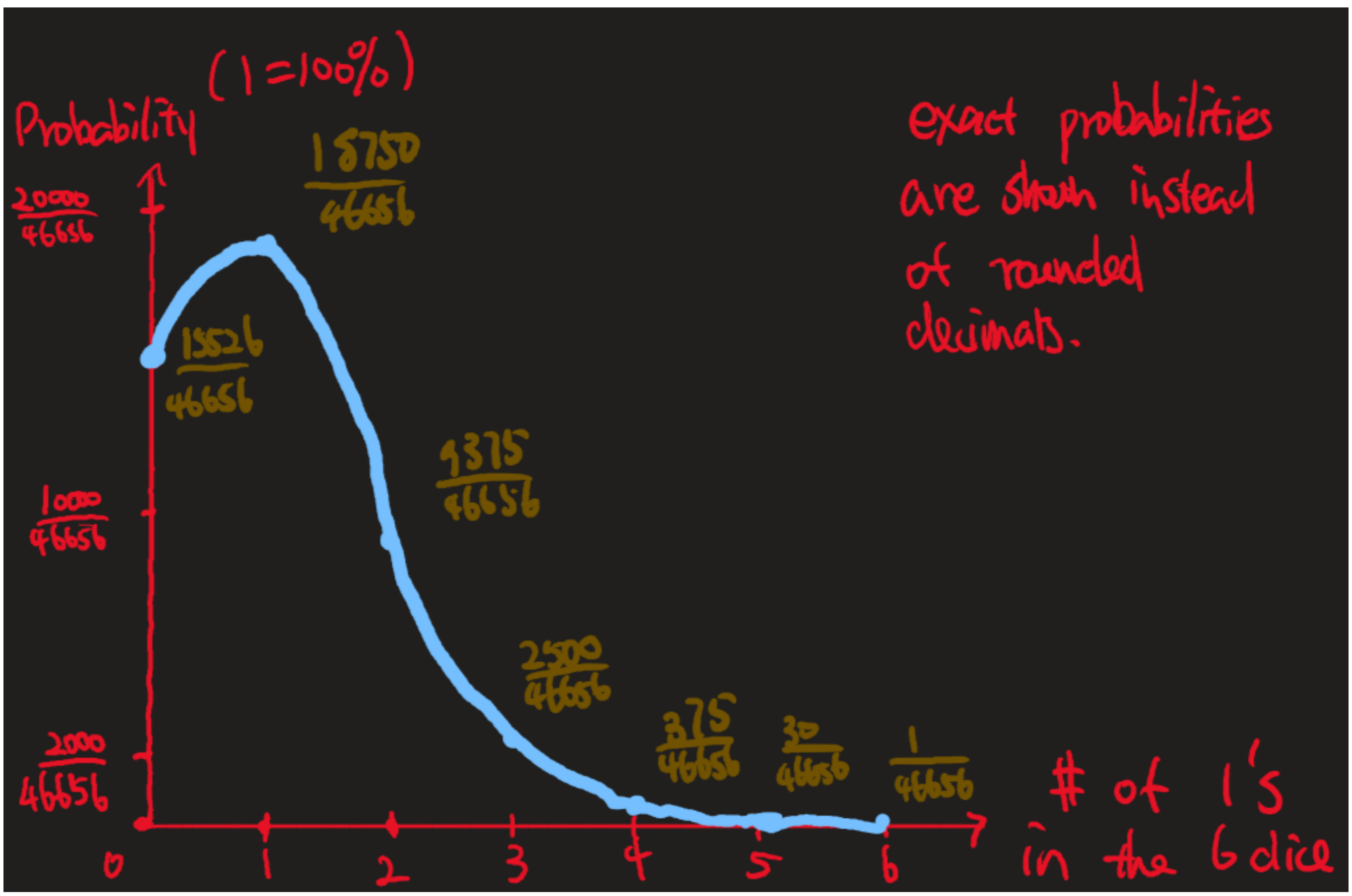
$$P(x)|_{x=5} = \binom{6}{5}\left(\tfrac{1}{6}\right)^5\left(\tfrac{5}{6}\right)^{6-5} = 6 \times \tfrac{1}{6^5} \times \tfrac{5}{6} = \tfrac{30}{46656}$$

$$P(x)|_{x=6} = \binom{6}{6}\left(\tfrac{1}{6}\right)^6\left(\tfrac{5}{6}\right)^{6-6} = 1 \times \tfrac{1}{6^6} \times 1 = \tfrac{1}{46656}$$

$$\tfrac{15625}{46656} + \tfrac{18750}{46656} + \tfrac{9375}{46656} + \tfrac{2500}{46656} + \tfrac{375}{46656} + \tfrac{30}{46656} + \tfrac{1}{46656} = 1$$

The probabilities added up to 1.

Here is the probability dristribution graph:



As shown, 1 1's occurs the most, then no 1's, and so on... Thus, reducing the value of prize of 0 and 1 number of 1's would be rational as well as the increase in prize of the other numbers.

Comparing to the Part A's distribution graph, we can see that the graph seems like shifted to the right because of the increase in number of dice.

Expected Winnings or Losses

x (Number of 1's)	y (value of prize in points)	$P(x) = \binom{n}{x} p^x q^{n-x}$	$y \times P(x)$
0	0	$\frac{15625}{46656}$	0
1	5	$\frac{18750}{46656}$	$\frac{93750}{46656}$
2	15	$\frac{9375}{46656}$	$\frac{140625}{46656}$
3	30	$\frac{2500}{46656}$	$\frac{75000}{46656}$
4	50	$\frac{375}{46656}$	$\frac{18750}{46656}$
5	70	$\frac{30}{46656}$	$\frac{2100}{46656}$
6	6666	$\frac{1}{46656}$	$\frac{6666}{46656}$

$$
\begin{aligned}
&E(x) = \sum (y \times P(x)) \\
&= 0 + \frac{93750}{46656} + \frac{140625}{46656} + \frac{75000}{46656} + \frac{18750}{46656} + \frac{2100}{46656} + \frac{6666}{46656} \\
&\approx 7.220743313
\end{aligned}
$$

Again, the calculations of P(x)s are shown previously, then no repeating calculations are shown.

The Expected winnings or losses regardless of cost is approximately 7.22 points. In order to make a profit of about 10%, the cost of every run needed to be 7.22/90%=8.02 points, approximately 8 points per run. Check reversely, the expected gain for the house when the cost of all runs is 8 points is 8-7.22, divided by 8 points which is the cost of run, 0.0975, or 9.75% is the profit of the house, which is approximately about 10%.

100 Games

The expected winnings or losses for the player is approximately 7.22-8=-0.78 point per run, thus, for 100 games or trials, $100 \times (-0.78) = -78$ points. So, 78 points are expected to be lost for the player in 100 games.

Similarily, the expected winnings or losses for the house is approximately 8-7.22 = 0.78 points. Thus, for 100 games or trials, $100 \times (0.78) = 78$ points is expected to gain for the house in 100 games, which in magnitute, same as the player lost.

20 Games

> Trial 0 -6 3 3 1 5 5 number of 1's: 1

```
Trial 1 -4 6 6 1 1 6 number of 1's: 2
Trial 2 -5 4 5 2 3 1 number of 1's: 1
Trial 3 -5 1 5 2 6 2 number of 1's: 1
Trial 4 -4 1 3 5 1 1 number of 1's: 3
Trial 5 -2 6 5 2 4 4 number of 1's: 0
Trial 6 -4 3 6 3 2 2 number of 1's: 0
Trial 7 -2 4 3 6 1 5 number of 1's: 1
Trial 8 -4 6 2 4 5 1 number of 1's: 1
Trial 9 -3 2 3 2 1 1 number of 1's: 2
Trial 10 -3 2 4 4 1 1 number of 1's: 2
Trial 11 -4 4 4 3 2 4 number of 1's: 0
Trial 12 -1 3 5 6 4 1 number of 1's: 2
Trial 13 -3 4 3 6 6 4 number of 1's: 0
Trial 14 -5 2 4 1 5 5 number of 1's: 1
Trial 15 -3 2 4 6 2 6 number of 1's: 0
Trial 16 -1 2 2 4 5 1 number of 1's: 2
Trial 17 -1 3 4 4 3 3 number of 1's: 1
Trial 18 -6 3 2 5 6 5 number of 1's: 0
Trial 19 -6 2 3 6 1 3 number of 1's: 1
```

Number of 1's	Number of trials	Prize	Points gained
0	6	0	0
1	8	5	40
2	5	15	75
3	1	30	30
4	0	50	0
5	0	70	0
6	0	6666	0

Thus, the total points gained is 40+75+30=145
The total points lost is 20*8=160
The net lost of the point of the player after these 20 games is 145-160=-15 points
For each run, (145-160)/20=-0.75 point is averagely lost for the player
-0.75 and -0.78 are close. As predicted, the player losses after a long run, more accurate practical results are shown below.

1000000 Games

```python
from random import randint

points = [0, 5, 15, 30, 50, 70, 6666]
x = [0,0,0,0,0,0,0]
for i in range(1000000):
    number_of_one = 0
    for j in range(6):
        num = randint(1, 6)
        if num == 1:
            number_of_one += 1
    x[number_of_one] += 1

print("Probabilities of all outcomes:")
for number in x:
    print(number/1000000)

sum = -8 * 1000000
for i in range(0,7):
    sum += x[i] * points[i]
print("The change of the point of the player after 1000000 runs is")
print(sum)

print("The change of point of the player per run is ")
print(sum/1000000)
```

Using the python code above, we can run the simulated game 1000000 times.

Here are the outputs:

```
Probabilities of all outcomes:
0.334361
0.402612
0.200805
0.053533
0.008
0.000668
2.1e-05
The change of the point of the player after 1000000 runs is
-782129
The change of point of the player per run is
-0.782129
```

The probabilities of all outcomes are very close to the calculated probabilities fractions. The change of points of the player per run in 1000000 runs is -0.782129, also very close to the theoretical change -0.78 point per run. Same reason as part A mentioned, ideally as the number of attempts approches infinity, all probabilities, change of total point, and average change of point per run should be close or approching the theoretical values. Because of the uncertainty or the limited amount of results, the values are very liikely to be different, as we can compare the 20 runs and the 1000000 runs results, more fair results create less uncertainly.

Thus, the probabilities, expected values calculated theoretically do match up with the values got pratically.