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MAT157: Analysis I
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Homework 8

In the study of sequences and their limits, we have introduced the Cauchy's completeness theorem. It has an analogue for the limit of a function, which we will study through the following exercise

Exercise 1

Definition. Let $f(x)$ be a function whose domain includes a punctured open interval of x_0. The ocillation $V_\delta(x_0)$ of $f(x)$ in a punctured open interval $I_\delta(x_0) = (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ is defined as

$$V_{\delta,f}(x_0) = \sup_{x_1,x_2 \in I_\delta(x_0)} |f(x_1) - f(x_2)|$$

Question 1. Show that $\forall 0 < \delta_1 < \delta_2, 0 \leq V_{\delta_1}(x_0) \leq V_{\delta_2}(x_0)$.

Proof. Assume $\delta_1 < \delta_2$, by definition of the punctured open interval, we can see $I_{\delta_1}(x_0) \subseteq I_{\delta_2}(x_0)$. As lecture / tutorial has proven, $V_{\delta_1}(x_0) = \sup_{x_1,x_2 \in I_{\delta_1}(x_0)} |f(x_1) - f(x_2)| \leq \sup_{x_1,x_2 \in I_{\delta_2}(x_0)} |f(x_1) - f(x_2)| = V_{\delta_2}(x_0)$.

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Question 2. Show that $\lim_{x \to x_0} f(x)$ exists if and only if $\forall \varepsilon > 0, \exists \delta > 0,$ s.t. $V_{\delta,f}(x_0) < \varepsilon$.

Proof. We first show the forward direction. Assume the limit $\lim_{x \to x_0} f(x)$ exists, denote the limit by $\lim_{x \to x_0} f(x) = L$. then by definition, fix $\frac{\varepsilon}{2} > 0$, there exist some $\delta > 0$ s.t.

$$x \in I_\delta(x_0) \implies |f(x) - L| < \frac{\varepsilon}{2}.$$

Consider $x_1, x_2 \in I_\delta(x_0)$,

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq |f(x_1) - L| + |f(x_2) - L| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, we have shown that $\forall x_1, x_2 \in I_\delta(x_0), |f(x_1) - f(x_2)| < \varepsilon$ where $\varepsilon > 0$, showing ε is an upper bound, by definition of least upper bound this gives $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $V_{\delta,f}(x_0) = \sup_{x_1,x_2 \in I_\delta(x_0)} |f(x_1) - f(x_2)| < \varepsilon$, completing our forward direction.

Now for the backward direction. We first construct a sequence (y_n) defined as:

- Fix $\varepsilon_1 = 2^{-1} > 0$, we know there exists some $\delta_1 > 0$ s.t. $V_{\delta_1,f}(x_0) < \varepsilon_1$, choose $y_1 \in I_{\delta_1}(x_0)$
- Fix $\varepsilon_2 = 2^{-2} > 0$, we know there exists some $\delta_2 > 0$ s.t. $V_{\delta_2,f}(x_0) < \varepsilon_2$, choose $y_2 \in I_{\delta_2}(x_0)$
- ...
- Fix $\varepsilon_n = 2^{-n} > 0$, we know there exists some $\delta_n > 0$ s.t. $V_{\delta_n,f}(x_0) < \varepsilon_n$, choose $y_n \in I_{\delta_n}(x_0)$

So, $\forall x_1, x_2 \in I_{\delta_n}(x_0), |f(x_1) - f(x_2)| \leq V_{\delta,f}(x_0) < \varepsilon_n$

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Using this tool, we reproduce the consequence of the examples discussed in lecture.

Question 3. Give a criterion of $\lim_{x \to x_0} f(x)$ does not exist, in terms of the oscillation.

Claim. If $\exists \varepsilon > 0, \forall \delta > 0, V_{\delta,f}(x_0) \geq \varepsilon$, then $\lim_{x \to x_0} f(x)$ does not exist.

Question 4. Consider

$$g(x) = \sin(\frac{1}{x}), x \neq 0$$

does not have a limit at $x_0 = 0$, by study the oscillation $V_{\delta,f}(0)$.

Exercise 2

Definition. (Left limit) Let $f(x)$ be a function whose domian includes the open interval $(x_0 - c, x_0)$ for some $c > 0$. We say that $l \in \mathbb{R}$ is the left limit of $f(x)$ when x tends to x_0, if

$$\forall\varepsilon > 0, \exists\delta > 0, \text{ s.t. } \forall x \in (x_0 - \delta, x_0), |f(x) - l| < \varepsilon.$$

We denote it by $\lim_{x\to x_0^-} f(x) = l$.

Definition. (Right limit) Let $f(x)$ be a function whose domian includes the open interval $(x_0, x_0 + c)$ for some $c > 0$. We say that $r \in \mathbb{R}$ is the right limit of $f(x)$ when x tends to x_0, if

$$\forall\varepsilon > 0, \exists\delta > 0, \text{ s.t. } \forall x \in (x_0, x_0 + \delta), |f(x) - r| < \varepsilon.$$

We denote it by $\lim_{x\to x_0^+} f(x) = r$.

Question 1. Let $f(x)$ be defined in a punctured open interval of x_0. Prove that

$$\lim_{x\to x_0} f(x) = A \iff \lim_{x\to x_0^-} f(x) = A = \lim_{x\to x_0^+} f(x).$$

Proof. We first show the forward implication. Assume $\lim_{x\to x_0} f(x) = A$, by definiton this means $\forall\varepsilon > 0, \exists\delta > 0, \forall x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}, |f(x) - A| < \varepsilon$. That is, $\forall\varepsilon > 0, \exists\delta > 0, \forall x_1 \in (x_0 - \delta, x_0), \forall x_2 \in (x_0, x_0 + \delta), |f(x_1) - A| < \varepsilon, |f(x_2) - A| < \varepsilon$, by definition of left and right inverses this shows the existence of left and right inverses, moreover they are equal (both limits are A). Namely, showing $\lim_{x\to x_0^-} f(x) = A = \lim_{x\to x_0^+} f(x)$ as needed.

Now we show the backward implication. Assume $\lim_{x\to x_0^-} f(x) = A = \lim_{x\to x_0^+} f(x)$, that is, $\forall\varepsilon > 0, \exists\delta_1, \delta_2 > 0, \forall x_1 \in (x_0 - \delta_1, x_0), \forall x_2 \in (x_0, x_0 + \delta_2), |f(x_1) - A| < \varepsilon, |f(x_2) - A| < \varepsilon$. However, if we set $\delta := \min(\delta_1, \delta_2)$, we have $\forall\varepsilon > 0, \exists\delta > 0, \forall x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}, |f(x) - A| < \varepsilon$, which by definition gives $\lim_{x\to x_0} f(x) = A$ as required.

Thus as two implicatoins are shown, we can conclude that the equivalency holds, which completes our proof. QUOD ERAT DEM■

Question 2. Give a criterion of $\lim_{x\to x_0} f(x)$ does not exist, in terms of the left limit and right limit.

Claim. $\lim_{x\to x_0} f(x)$ does not exist if and only if $\lim_{x\to x_0^-} f(x)$ does not exist or $\lim_{x\to x_0^+} f(x)$ does not exist or $\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x)$

Question 3. Consider the function $f : \mathbb{R}^* \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Prove that $f(x)$ does not have a liimt at $x_0 = x$, by studying $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ respectively.

Proof. $\lim_{x\to 0^-} f(x) = 0$ as $\forall\varepsilon > 0$, choose $\delta = 1$, then $\forall x \in (-1, 0), |f(x) - 0| = 0 < \varepsilon$, showing the left limit of $f(x)$ is 0 when x is approaching 0 (from negative). Also, $\lim_{x\to 0^+} f(x) = 1$ as $\forall\varepsilon > 0$, choose $\delta = 1$, then $\forall x \in (0, 1), |f(x) - 1| = |1 - 1| = 0 < \varepsilon$, showing the right limit of $f(x)$ is 1 when x is approaching 0 (from positive). Thus, from E2Q2 we can see that $\lim_{x\to 0} f(x)$ does not exist as the left limit is not equal to the right limit when x is approaching 0. QUOD ERAT DEM■

Question 4. Consider the function $f : \mathbb{R}^* \to \mathbb{R}$, given by

$$f(x) = \frac{\sin x}{x}$$

Prove that the limit exists, by studying $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ respectively.

Proof. See pages below. (Also notice that here I used the unit circle definition of sin(x) with the assumption that if an area is contained in another shape's area, then the area cannot exceed that shape's)

I also claimed that:

$\lim_{x \to 0} \cos(x) = 1$

Here is the proof:

By using the identity $1 - \cos(x) = 2\sin^2(\frac{x}{2})$ and inequality $\sin(x) \leq x$ for all $x \in \mathbb{R}$, we can see that, $\forall \varepsilon > 0$, choose $\delta = \sqrt{2\varepsilon} > 0$ then $0 < |x| < \delta$ implies $\frac{1}{2}x^2 < \varepsilon$. So,

$$\begin{aligned} |\cos(x) - 1| &= |1 - \cos(x)| \\ &= |2\sin^2(\frac{x}{2})| \\ &\leq 2|\frac{x}{2}|^2 \\ &= \frac{1}{2}x^2 \\ &< \varepsilon \end{aligned}$$

Showing the limit is indeed 1.

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