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MAT157: Analysis I
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# Homework 12

## Exercise 1

This exercise aims at derive the Taylor formula with the Roche-Schlomilch Remainder. To this end, assume that for some $H > 0$ fixed, the continuous function

$$f(x) : [x_0 - H, x_0 + H] \subseteq \mathbb{R} \to \mathbb{R}$$

satisfy that

- $\forall 1 \leq k \leq n, f^{(k)}(x)$ exists and is continuous $\forall x \in I_H(x_0)$.
- $f^{(n+1)}(x)$ exists $\forall x \in \mathring{I}_H(x_0)$.

As usual denote the remainder $r_n(x)$ for $x \in \mathring{I}_H(x_0)$ by

$$r_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

Now, fix some $x \in (x_0, x_0 + H)$, define for $z \in [x_0, x]$ the auxiliary function

$$\phi(z) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(z)(x - z)^k}{k!}$$

**Question 1.** Show that $\phi(x_0) = r_n(x)$ and $\phi(x) = 0$.

*Proof.* By definition, we have

$$\begin{aligned}
\phi(x_0) &= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!} \\
r_n(x) &= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!} \\
&= \phi(x_0)
\end{aligned}$$

also, as we define $0^k = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$, and $0! = 1$, then, we have

$$\begin{aligned}
\phi(x) &= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)(x - x)^n}{k!} \\
\phi(x) &= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)0^k}{k!} \\
\phi(x) &= f(x) - \frac{f^{(0)}(x)0^0}{0!} \\
\phi(x) &= f(x) - f(x) \\
\phi(x) &= 0
\end{aligned}$$

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**Question 2.** Prove that

$$\phi'(z) = -\frac{f^{(n+1)}(z)}{n!}(x-z)^n$$

.

*Proof.* By definition and derivative rules, we have

$$\begin{aligned}
\phi(z) &= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(z)(x-z)^k}{k!} \\
&= f(x) - \frac{f^{(0)}(z)(x-z)^0}{0!} - \sum_{k=1}^{n} \frac{f^{(k)}(z)(x-z)^k}{k!} \\
&= f(x) - f(z) - \sum_{k=1}^{n} \frac{f^{(k)}(z)(x-z)^k}{k!} \\
\phi'(z) &= -f'(z) - \sum_{k=1}^{n} \frac{d}{dz}\left[\frac{f^{(k)}(z)(x-z)^k}{k!}\right] \\
&= -f'(z) - \sum_{k=1}^{n} \left[\frac{f^{(k+1)}(z)(x-z)^k}{k!} - \frac{f^{(k)}(z)k(x-z)^{k-1}}{k!}\right] \\
&= -f'(z) + \sum_{k=1}^{n} \left[\frac{f^{(k)}(z)(x-z)^{k-1}}{(k-1)!} - \frac{f^{(k+1)}(z)(x-z)^k}{k!}\right] \\
&= -f'(z) + \sum_{k=1}^{n} \frac{f^{(k)}(z)(x-z)^{k-1}}{(k-1)!} - \sum_{k=1}^{n} \frac{f^{(k+1)}(z)(x-z)^k}{k!} \\
&= \sum_{k=2}^{n} \frac{f^{(k)}(z)(x-z)^{k-1}}{(k-1)!} - \sum_{k=1}^{n} \frac{f^{(k+1)}(z)(x-z)^k}{k!} \\
&= \sum_{k=1}^{n-1} \frac{f^{(k+1)}(z)(x-z)^k}{k!} - \sum_{k=1}^{n} \frac{f^{(k+1)}(z)(x-z)^k}{k!} \\
&= -\frac{f^{(n+1)}(z)(x-z)^n}{n!}
\end{aligned}$$

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Now consider a function $\psi : [x_0, x] \to \mathbb{R}$, s.t.

- $\psi(z)$ is continuous in $[x_0, x]$.
- $\psi'(z) \neq 0$ in $(x_0, x)$.

**Question 3.** Verify that the function $\psi : [x_0, x] \to \mathbb{R}$ given by

$$\psi(z) = (x-z)^p$$

satisfies the conditions above.

*Proof.* Fix $p > 0$ and given $\psi(z) = (x-z)^p$. Then, it is clear that $\psi(z)$ is continuous (can be shown using the fact that exp function is continuous). And when $z \in (x_0, x)$, $x \neq z \implies (x-z)^{p-1} \neq 0$, so combining $p > 0$ we have $\psi'(z) = p(x-z)^{p-1} \neq 0$, satisfying the 2 conditions, as required.

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**Question 4.** Prove that $\exists c \in (x_0, x)$ s.t.

$$r_n(x) = \frac{\psi(x) - \psi(x_0)}{\psi'(c)} \frac{f^{(n+1)}(c)}{n!}(x-c)^n.$$

Hint: Apply the Cauchy theorem to the pair $\phi, \psi$.

*Proof.* By Cauchy MVT, we have $\exists c \in (x_0, x)$ s.t.

$$[\phi(x) - \phi(x_0)]\psi'(c) = [\psi(x) - \psi(x_0)]\phi'(c)$$

As proven in E1Q1, we have

$$[0 - r_n(x)]\psi'(c) = [\psi(x) - \psi(x_0)]\phi'(c)$$

Rearrange and isolate $r_n(x)$ we have

$$\begin{aligned} r_n(x) &= \frac{\psi(x_0) - \psi(x)}{\psi'(c)}\phi'(c) \\ &= \frac{\psi(x_0) - \psi(x)}{\psi'(c)} \cdot \left(-\frac{f^{(n+1)}(c)}{n!}(x-c)^n\right) \\ &= \frac{\psi(x) - \psi(x_0)}{\psi'(c)} \frac{f^{(n+1)}(c)}{n!}(x-c)^n \end{aligned}$$

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**Question 5.** Plug in $\psi(z) = (x-z)^p$ in the result above, and derive that

$$r_n(x) = \frac{f^{(n+1)}(c)}{n!p}(x-c)^{n+1-p}(x-x_0)^p.$$

*Proof.* From E1Q4 and by definition we get

$$\begin{aligned} r_n(x) &= \frac{\psi(x) - \psi(x_0)}{\psi'(c)} \frac{f^{(n+1)}(c)}{n!}(x-c)^n \\ \psi'(c) &= -p(x-c)^{p-1} \\ \psi(x) &= (x-x)^p = 0 \\ \psi(x_0) &= (x-x_0)^p \end{aligned}$$

So,

$$\begin{aligned} r_n(x) &= \frac{\psi(x) - \psi(x_0)}{\psi'(c)} \frac{f^{(n+1)}(c)}{n!}(x-c)^n \\ &= \frac{0 - ((x-x_0)^p)}{-p(x-c)^{p-1}} \frac{f^{(n+1)}(c)}{n!}(x-c)^n \\ &= \frac{f^{(n+1)}(c)}{n!p}(x-c)^{n+1-p}(x-x_0)^p \end{aligned}$$

as required.

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**Question 6.** Now let $c = x_0 + \theta(x - x_0)$, with $\theta \in (0,1)$, prove that $\exists 0 < \theta < 1$ s.t.

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x-x_0))}{n!p}(1-\theta)^{n+1-p}(x-x_0)^{n+1}, \quad \text{(Roche-Scholomilch Remainder)}$$

*Proof.* Isolate $\theta = \frac{c-x_0}{x-x_0}$, we have that $1-\theta = \frac{x-x_0}{x-x_0} + \frac{x_0-c}{x-x_0} = \frac{x-c}{x-x_0}$. Then, by E1Q5, we have

$$\begin{aligned} r_n(x) &= \frac{f^{(n+1)}(c)}{n!p}(x-c)^{n+1-p}(x-x_0)^p \\ &= \frac{f^{(n+1)}(c)}{n!p}\left(\frac{x-c}{x-x_0}\right)^{n+1-p}(x-x_0)^{n+1-p}(x-x_0)^p \\ &= \frac{f^{(n+1)}(c)}{n!p}(1-\theta)^{n+1-p}(x-x_0)^{n+1} \\ &= \frac{f^{(n+1)}(x_0+\theta(x-x_0))}{n!p}(1-\theta)^{n+1-p}(x-x_0)^{n+1} \end{aligned}$$

as needed. QUOD ERAT DEM■

**Question 7.** Choose distinct value of $p$, so that the Roche-Scholomilch Remainder becomes

- The Lagrange remainder.
- The Cauchy remainder.

*Proof.* Choose $p = 1$, then we have

$$r_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-x_0)$$

By definition, this is the cauchy remainder.

Choose $p = n+1$, then we have

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

By definition, this is the Lagrange remainder. QUOD ERAT DEM■