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MAT157: Analysis I
November 8, 2023

Homework 7

Throughout this exercise, we assume that $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence.

Exercise 1

Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence. Define

$$i_k = \inf_{n\geq k} a_n, \quad s_k = \sup_{n\geq k} a_n$$

Question 1. Prove that $\lim_{k\to\infty} i_k$ and $\lim_{k\to\infty} s_k$ exist (as finite real numbers). Moreover, if we denote them by

$$\lim_{k\to\infty} i_k = a_*, \quad \lim_{k\to\infty} s_k = a^*,$$

then

$$a_* \leq a^*$$

Theorem (Monotone Convergence Theorem)**.** Suppose $\{a_n\}_{n\in\mathbb{N}}$ is monotone. Then $\{a_n\}_{n\in\mathbb{N}}$ converges if and only if it is bounded. Moreover,

- If $\{a_n\}_{n\in\mathbb{N}}$ is increasing, then either $\{a_n\}_{n\in\mathbb{N}}$ diverges to ∞ or
$$\lim_{n\to\infty} a_n = \sup(\{a_n : n \in \mathbb{N}\}).$$
- If $\{a_n\}_{n\in\mathbb{N}}$ is decreasing, then either $\{a_n\}_{n\in\mathbb{N}}$ diverges to $-\infty$ or
$$\lim_{n\to\infty} a_n = \inf(\{a_n : n \in \mathbb{N}\}).$$

Proof. In lecture we have shown monotone+converges implies bounded.

Now we show that if a sequence is bounded and monotonic, then it also converges: Assume for the sake of contradiction that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded below by C and above by D, and monotonic decreasing, and does not converge. That is, because it is monotonic decreasing, it is only possible that the sequence diverges to $-\infty$, by definition $\forall M < 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N, a_n < M$, however $\forall n \in \mathbb{N}, a_n \geq C$ contradicting the case when $M = \min(C, -C, -1) < 0$, it must be True that a sequence is bounded and monotonic implies it is also convergent.

Now we show the limit is the supremum or the infimum. Assume $\{a_n\}_{n\in\mathbb{N}}$ is increasing, $\alpha = \sup(\{a_n : n \in \mathbb{N}\})$ and we want to show $\alpha = \lim_{n\to\infty} a_n$. That is, we want to show for any $\varepsilon > 0$, there exists some N such that $\forall n \in \mathbb{N}, n > N, |a_n - \alpha| < \varepsilon$. Fix $\varepsilon > 0$, since $\alpha = \sup(\{a_n : n \in \mathbb{N}\})$, by tutorial (suprema analytically theorem) there exists some $a_N > \alpha - \varepsilon$, and since $\{a_n\}_{n\in\mathbb{N}}$ is monotonically increasing we see that for any $n > N$ we have $a_n \geq a_N > \alpha - \varepsilon$, and $a_n \leq \alpha$ due to α being the supremum. Thus, we have found such N, for any $\varepsilon > 0$, for all $n > N$, we have $\alpha - \varepsilon < a_n \leq \alpha < \alpha + \varepsilon$ and hence $|a_n - \alpha| < \varepsilon$ as required.

Similarly for the other case. Assume $\{a_n\}_{n\in\mathbb{N}}$ is decreasing, $\beta = \inf(\{a_n : n \in \mathbb{N}\})$ and we want to show $\beta = \lim_{n\to\infty} a_n$. That is, we want to show for any $\varepsilon > 0$, there exists some N such that $\forall n \in \mathbb{N}, n > N, |a_n - \beta| < \varepsilon$. Fix $\varepsilon > 0$, since $\beta = \inf(\{a_n : n \in \mathbb{N}\})$, by tutorial (suprema analytically theorem) there exists some $a_N < \beta + \varepsilon$, and since $\{a_n\}_{n\in\mathbb{N}}$ is monotonically decreasing we see that for any $n > N$ we have $a_n \leq a_N < \beta + \varepsilon$, and $a_n \geq \beta$ due to β being the infimum. Thus, we have found such N, for any $\varepsilon > 0$, for all $n > N$, we have $\beta + \varepsilon > a_n \geq \beta > \beta - \varepsilon$ and hence $|a_n - \beta| < \varepsilon$ as required. QUOD ERAT DEM■

Proof. First we want to show both $\{i_k\}_{k\in\mathbb{N}}$ and $\{s_k\}_{k\in\mathbb{N}}$ are convergent. By the Least Upper Bound / Greatest Lower Bound Theorem, since $\{a_n\}_{n\in\mathbb{N}}$ is bounded, this implies the existence of the infimum and supremum, denote them as $\sup\{a_n\}_{n\in\mathbb{N}}, \inf\{a_n\}_{n\in\mathbb{N}}$. That is, $\forall n \in \mathbb{N}, \inf\{a_n\}_{n\in\mathbb{N}} \leq a_n \leq \sup\{a_n\}_{n\in\mathbb{N}}$, this implies $\forall k \in \mathbb{N}, \inf\{a_n\}_{n\in\mathbb{N}} \leq \inf_{n\geq k} a_n \leq \sup_{n\geq k} a_n \leq \sup\{a_n\}_{n\in\mathbb{N}}$, showing both sequences $\{i_k\}_{k\in\mathbb{N}}, \{s_k\}_{k\in\mathbb{N}}$ are bounded.

Now we show that the sequence $\{i_k\}_{k\in\mathbb{N}}$ monotonically increases and $\{s_k\}_{k\in\mathbb{N}}$ monotonically decreases. It is obvious that $\inf_{n\geq k} a_n \leq \inf_{n\geq k+1} a_n$ since $\{a_n\}_{n\geq k} = \{a_k\} \cup \{a_n\}_{n\geq k+1}$, we have $\inf_{n\geq k} a_n = \min(a_k, \inf\{a_n\}_{n\geq k+1})$. Similarly we can see $\sup_{n\geq k} a_n \geq \sup_{n\geq k+1} a_n$ as $\{a_n\}_{n\geq k} = \{a_k\} \cup \{a_n\}_{n\geq k+1}$ and $\sup_{n\geq k} a_n = \max(a_k, \sup\{a_n\}_{n\geq k+1})$.

Thus, by Monotone Convergence Theorem (MCT) two sequences $\{i_k\}_{k\in\mathbb{N}}$ and $\{s_k\}_{k\in\mathbb{N}}$ must converge to some finite real number respectively, as denoted above,

$$\lim_{k\to\infty} i_k = a_*, \quad \lim_{k\to\infty} s_k = a^*.$$

Moreover, we have $a_* = \sup(\{i_k\}_{k\in\mathbb{N}})$ and $a^* = \inf(\{s_k\}_{k\in\mathbb{N}})$, we want to show $a_* \leq a^*$. Assume for contradiction $a_* > a^*$, pick $\varepsilon_1 = a_* - a^* > 0$, consider the open neighborhood $(2a^* - a_*, a_*)$, since the sequence $\{s_k\}_{k\in\mathbb{N}}$ converges to a^*, there must be infinite elements of the sequence within the open neighborhood of a^*: $(2a^* - a_*, a_*)$. Then, there are some $K \in \mathbb{N}$ s.t. $\forall n \geq K, 2a^* - a_* < s_n < a_*$ (since it is monotone).

However, fix $\varepsilon_2 = \varepsilon_1 = a_* - a^* > 0$, the open neighborhood $(a^*, 2a_* - a^*)$ also contains infinite elements of $\{i_k\}_{k\in\mathbb{N}}$, this gives $\exists N \in \mathbb{N}, i_N > a_*$, but $\{i_k\}_{k\in\mathbb{N}}$ monotonically increases, giving $\forall k \geq N, i_k > a_*$, we pick $M := \max(K, N)$, then showing $\forall m \geq M, \sup_{n\geq m} a_n = s_m < a_* < i_m = \inf_{n\geq m} a_n$, this is contradicting $\inf_{n\geq m} a_n < \sup_{n\geq m} a_n$, therefore it must be true that $a_* \leq a^*$ as required. QUOD ERAT DEM■

Remark. We say that a_* is the limit inferior of the sequence, and a^* is the limit superior of the sequence. From now on, we denote them by

$$\liminf_{k\to\infty} a_k := a_*,$$
$$\limsup_{k\to\infty} a_k := a^*,$$

respectively.

Question 2. Compute the $\liminf_{k\to\infty} a_k$ and $\limsup_{k\to\infty} a_k$ for the following sequences:

1) $a_k = (-1)^k$.
2) $b_k = \frac{(-1)^k}{k}$.

Claim.

$$\liminf_{k\to\infty} a_k = -1,$$
$$\limsup_{k\to\infty} a_k = 1,$$
$$\liminf_{k\to\infty} b_k = 0,$$
$$\limsup_{k\to\infty} b_k = 0.$$

Proof. 1) Since it is clear that when k is odd $a_k = -1$, when k is even $a_k = 1$, thus for any $N \in \mathbb{N}$, for all $n > N$, n is either odd (showing $a_n = -1$) or even (showing $a_n = 1$), giving the infimum is always -1, and the supremum is always 1, thus the limit of the constant sequences are the constants themselves, giving $\liminf_{k\to\infty} a_k = -1$ and $\limsup_{k\to\infty} a_k = 1$.

2) We want to show that $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|b_n| < 0+\varepsilon, \forall n \geq N$. That is, when the sequence converges to 0, the limit superior and limit inferior also converge to 0. First the sequence is converging to 0 as $\forall \varepsilon > 0, \exists M \in \mathbb{N}$ s.t. $M > \frac{1}{\varepsilon}, \forall m > M, |b_m - 0| = \frac{|(-1)^m|}{m} = \frac{1}{|m|} = \frac{1}{m} < \frac{1}{M} < \varepsilon$.

Now, we want to show the sequence converges to 0 implies the limit superior and limit inferior also converge to 0. First from E1Q1 we can see the supremum sequence $\{\sup_{n\geq k} b_n\}_{k\in\mathbb{N}}$ and the infimum sequence $\{\inf_{n\geq k} b_n\}_{k\in\mathbb{N}}$ both converges (It must be true that $b_* \leq 0 \leq b^*$ otherwise contradicting (b_k) converges to 0). That is, assume for the sake of contradiction that $b_* < b^*$ (they are not equal), choose $\varepsilon = b^* - b_* > 0$, because we know the sequence converges to 0, choose $\varepsilon_1 = \min(|\frac{b_*}{2}|, |\frac{b^*}{2}|)$ where $\varepsilon_1 < -b_*$ (b_* is at most 0), then $\exists M \in \mathbb{N}$ s.t. $\forall m > M, |b_m| < \varepsilon_1$, however this shows $\forall m > M, b_* < -\varepsilon_1 \leq \inf_{n\geq m} b_n$ (greatest lower bound), thus contradicting the fact that the limit inferior sequence monotonically increases and is bounded above by b_* (from E1Q1), therefore we conclude that $b_* = b^* = 0$ as required.

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Question 3. Prove that for a bounded sequence $\{a_n\}_{n\in\mathbb{N}}$, the sequence is convergent if and only if

$$\liminf_{k\to\infty} a_k = \limsup_{k\to\infty} a_k.$$

Proof. From E1Q2 we have shown that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges implies $\liminf_{k\to\infty} a_k = \limsup_{k\to\infty} a_k$. Now we just need to show $\liminf_{k\to\infty} a_k = \limsup_{k\to\infty} a_k = a$ implies $\{a_n\}_{n\in\mathbb{N}}$ converges.

By definition the equality gives $\forall\varepsilon > 0, \exists N, M \in \mathbb{N}, a-\varepsilon < \inf_{k\geq N} a_k < a+\varepsilon, a-\varepsilon < \sup_{k\geq M} a_k < a+\varepsilon$, thus we pick $K := \max(N, M)$, showing

$$\forall n > K, a-\varepsilon < \inf_{k\geq K} a_k \leq a_n \leq \sup_{k\geq K} a_k < a+\varepsilon$$

Thus it also converges. Therefore, the equivalency holds, completing our proof.

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Question 4. Using the consequence of the previous sub-question, discuss the convergence of $\{a_n\}_{n\in\mathbb{N}}$ where

1) $a_k = (-1)^k$.
2) $b_k = \frac{(-1)^k}{k}$.

Claim.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (-1)^k = \text{ D.N.E. (diverges)},$$

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{(-1)^k}{k} = 0.$$

Proof. From E1Q2 we have shown $a_* = -1$ and $a^* = 1$, by E1Q3 and E1Q2 these mean the sequence does not converge, that is, diverges.

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Proof. From E1Q2 we have already shown that (b_k) converges to 0.

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Exercise 2

This exercise is the continuation of the previous exercise.

Question 1. Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded sequence. For any convergent sub-sequence $\{a_{k_n}\}_{n\in\mathbb{N}}$ of the sequence $\{a_n\}_{n\in\mathbb{N}}$ (whose existence is ensured by the Bolzano-Weierstrass theorem). Assume that

$$\lim_{n\to\infty} a_{k_n} = c.$$

Prove that

$$\liminf_{k\to\infty} a_k \le c \le \limsup_{k\to\infty} a_k.$$

Proof. Assume for the sake of contradiction that $c < a_* \vee c > a^*$, we want to show both inequality lead to contradiction.

Assume $c < a_*$, that is, $\lim_{n\to\infty} a_{k_n} = c < a_* = \sup(\{\inf_{n\ge k} a_n\}_{k\in\mathbb{N}})$ by E1 and MCT. This implies $\exists k \in \mathbb{N}$ s.t. $c < \inf_{n\ge k} a_n$ by suprema analytically, however, consider $\varepsilon = \frac{\inf_{n\ge k} a_n - c}{2} > 0$, then $\exists M \in \mathbb{N}, \forall m \ge M, a_{k_m} \in (c-\varepsilon, c+\varepsilon)$ by definition of c. We choose $N = \max(k, M)$, then showing $a_{k_N} < \inf_{n\ge N} a_n$, thus contradicting the definition of a greatest lower bound, thus, giving $c \ge a_*$ as required.

Assume $c > a^*$, that is, $\lim_{n\to\infty} a_{k_n} = c > a^* = \inf(\{\sup_{n\ge k} a_n\}_{k\in\mathbb{N}})$ by E1 and MCT. This implies $\exists k \in \mathbb{N}$ s.t. $c > \sup_{n\ge k} a_n$ by suprema analytically, however, consider $\varepsilon = \frac{c - \sup_{n\ge k} a_n}{2} > 0$, then $\exists M \in \mathbb{N}, \forall m \ge M, a_{k_m} \in (c-\varepsilon, c+\varepsilon)$ by definition of c. We choose $N = \max(k, M)$, then showing $a_{k_N} > \sup_{n\ge N} a_n$, thus contradicting the definition of a least upper bound, thus, giving $c \le a^*$ as required.

Therefore, we conclude that $a_* \le c \le a^*$ as required.

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Question 2. Using the consequence of the previous sub-question (E2Q1) as well as (E1Q3), give another proof that any sub-sequence of a convergent sequence is also converging to the same limit.

Proof. By E1Q3 when a sequence (a_n) is converging, we have $\limsup_{k\to\infty} a_k = \liminf_{k\to\infty} a_k$, by E2Q1 and Squeeze Theorem we have $\limsup_{k\to\infty} a_k = c = \liminf_{k\to\infty} a_k$ where c is the limit of any sub-sequence, therefore showing all sub-sequences are converging to a single point, including the original sequence (a_n).

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Exercise 3

This exercise is the continuation of the previous exercises.

Question 1. Let b^* be a real number. Prove that $b^* = \limsup_{k\to\infty} a_k$ if and only if the following two conditions are satisfied at the same time:

1) There exists a subsequence $\{a_{k_n}\}_{n\in\mathbb{N}}$ that converges to b^*, and
2) For any $\varepsilon > 0, \exists M \in \mathbb{N}$ s.t. $\forall k > M, a_k < b^* + \varepsilon$.

Proof. The forward direction is trivial as by definition (2) is satisfied, and from E2Q1 we get (1) as a consequence. Consider the backward implication, since a_k converges, and one of the subsequence converges to b^*, by E2Q1 this gives the entire sequence is also converging to b^*.

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Remark. E2Q1 tells us that any convergent sub-sequence of $\{a_n\}_{n\in\mathbb{N}}$ has a limit inferior than $\limsup_{k\to\infty} a_k$. However it is not clear whether there is a sub-sequence of $\{a_n\}_{n\in\mathbb{N}}$ that indeed converges to $\limsup_{k\to\infty} a_k$. This is now confirmed by E3Q1.

Question 2. Now let b_* be a real number. State the necessary and sufficient condition for $b_* = \liminf_{k\to\infty} a_k$, by using an analogue of E3Q1 (no proof is needed.)

Remark. In our discussion in lecture, we have already mentioned how sub-sequences can given useful information about the convergence of the original sequence. However, a sequence has infinitely many sub-sequences, it might not be clear which sub-sequence is the correct candidate to investigate. These exercises indicate that

$$\liminf_{k\to\infty} a_k, \quad \limsup_{k\to\infty} a_k$$

as well as the sub-sequences achieving them (convinced by E3Q1) might be the extremely useful.