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MAT157: Analysis I
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Homework 9

EXERCISE 1

Compute the following limits.

Question 1.

Claim.

$$\lim_{x\to 1} \frac{x + x^2 + x^3 + \ldots + x^n - n}{x - 1} = \frac{n(n+1)}{2}$$

Proof. Replace x with $1 + h$ where $h \to 0$ as $x \to 1$. Then by binomial thoerem we have

$$\begin{aligned}\lim_{x\to 1} \frac{x + x^2 + x^3 + \ldots + x^n - n}{x - 1} &= \lim_{h\to 0} \frac{(1 + h) + (1 + h)^2 + \ldots + (1 + h)^n - n}{h} \\ &= 1 + 2 + \ldots + n \\ &= \frac{n(n+1)}{2}\end{aligned}$$

The constant terms of the binomial terms are cancelled with the $-n$, then we are able to factor out h from all numerator terms, cancel it with the denomonator's h we have many terms with h left and $1+2+\ldots+n$, however as h approaches 0, all terms with h will approach 0 and we are left with $1 + 2 + \ldots + n$, thus giving us $\frac{n(n+1)}{2}$ as needed.

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Question 2.

Claim.

$$\lim_{x\to 1} \left(\frac{m}{1 - x^m} - \frac{n}{1 - x^n} \right) = \frac{m - n}{2}$$

Proof. Using the identity $1 - x^a = (1 - x)(1 + x + \cdots + x^{a-1})$ we have:

$$\begin{aligned}\lim_{x\to 1} \left(\frac{m}{1 - x^m} - \frac{n}{1 - x^n} \right) &= \lim_{x\to 1} \left(\frac{m(1 - x^n) - n(1 - x^m)}{(1 - x^m)(1 - x^n)} \right) \\ &= \lim_{x\to 1} \left(\frac{m(1 - x)(1 + x + \cdots + x^{n-1}) - n(1 - x)(1 + x + \cdots + x^{m-1})}{(1 - x)(1 + x + \cdots + x^{m-1})(1 - x)(1 + x + \cdots + x^{n-1})} \right) \\ &= \lim_{x\to 1} \left(\frac{m(1 - x)(1 + x + \cdots + x^{n-1}) - n(1 - x)(1 + x + \cdots + x^{m-1})}{(1 - x)m(1 - x)n} \right) \text{ since } x \to 1 \\ &= \frac{1}{mn} \lim_{x\to 1} \left(\frac{m(1 + x + \cdots + x^{n-1}) - n(1 + x + \cdots + x^{m-1})}{(1 - x)} \right) \\ &= \frac{1}{mn} \lim_{x\to 1} \left(\frac{m(x + \cdots + x^{n-1}) + m + mn - mn - n - n(x + \cdots + x^{m-1})}{(1 - x)} \right) \\ &= \frac{1}{mn} \lim_{x\to 1} \left(\frac{m(x + \cdots + x^{n-1} - (n - 1)) - n(x + \cdots + x^{m-1} - (m - 1))}{(1 - x)} \right) \\ &= -\frac{1}{mn} \lim_{x\to 1} \left(m\frac{x + \cdots + x^{n-1} - (n - 1)}{x - 1} - n\frac{x + \cdots + x^{m-1} - (m - 1)}{x - 1} \right) \\ &= -\frac{1}{mn} \left(\frac{m(n - 1)n}{2} - \frac{n(m - 1)m}{2} \right) \text{ by E1Q1} \\ &= \frac{m - n}{2}\end{aligned}$$

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Question 3.

Claim.

$$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}=\frac{4}{3}$$

Proof.

$$\begin{aligned}
\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} &= \lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}\cdot\frac{\sqrt{1+2x}+3}{\sqrt{1+2x}+3}\\
&= \lim_{x\to 4}\frac{1+2x-9}{(\sqrt{x}-2)(\sqrt{1+2x}+3)}\\
&= \lim_{x\to 4}\frac{2(x-4)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)}\\
&= \lim_{x\to 4}\frac{2(x-4)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}\\
&= \lim_{x\to 4}\frac{2(x-4)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)}\\
&= \lim_{x\to 4}\frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3}\\
&= \frac{2(\sqrt{4}+2)}{\sqrt{1+2\cdot 4}+3}\\
&= \frac{4}{3}
\end{aligned}$$

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Question 4.

Claim.

$$\lim_{x\to 0}\frac{x^2}{\sqrt{1+5x}-(1+x)}=0$$

Proof.

$$\begin{aligned}
\lim_{x\to 0}\frac{x^2}{\sqrt{1+5x}-(1+x)} &= \lim_{x\to 0}\frac{x^2}{\sqrt{1+5x}-(1+x)}\cdot\frac{\sqrt{1+5x}+(1+x)}{\sqrt{1+5x}+(1+x)}\\
&= \lim_{x\to 0}\frac{x^2(\sqrt{1+5x}+(1+x))}{(1+5x)-(1+x)^2}\\
&= \lim_{x\to 0}\frac{x^2(\sqrt{1+5x}+(1+x))}{1+5x-1-2x-x^2}\\
&= \lim_{x\to 0}\frac{x^2(\sqrt{1+5x}+(1+x))}{-x^2+3x}\\
&= \lim_{x\to 0}\frac{x(\sqrt{1+5x}+(1+x))}{-x+3}\\
&= 0
\end{aligned}$$

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Exercise 2

Definition. Let $f(x)$ be defined on $(a,+\infty)$ for some $a \in \mathbb{R}$. We say that $f(x)$ tends to $k \in \mathbb{R}$ when x tends to $+\infty$, and denote it by

$$\lim_{x\to+\infty} f(x) = k,$$

if $\forall \varepsilon > 0, \exists A \in \mathbb{R},$ s.t. $\forall x > A, |f(x) - k| < \varepsilon.$

In this exercise, we study some limits of average of functions, that are of similar spirits to those of the sequences we encountered before. To this end, assume that

1) $f(x)$ is defined on some interval $(a,+\infty)$, where $a \in \mathbb{R}$ and
2) $\forall b > a, f(x)$ is bounded on (a,b) (be careful: this does not mean that $f(x)$ is bounded on $a,+\infty)$).

Question 1. Prove that if $\lim_{x\to+\infty}(f(x+1) - f(x)) = k$, then

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k.$$

Proof. QUOD ERAT DEM■

Question 2. Prove that if $\forall x \in (a,+\infty), f(x) \geq C > 0$ for some (fixed) positive C, and $\lim_{x\to+\infty} \frac{f(x+1)}{f(x)} = k$, then

$$\lim_{x\to+\infty} (f(x))^{\frac{1}{x}} = k.$$

Exercise 3

Consider the following Riemann function $f : \mathbb{R} \to \mathbb{R}$ defined

$$f(x) = \begin{cases} \frac{1}{n}, & x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\}, \gcd(m, n) = 1, n \in \mathbb{N}; \\ 0, & x \in \mathbb{Q}^c \cup \{0\}. \end{cases}$$

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Question 1. Prove that $f(x)$ is not continuous at any $x \in \mathbb{Q} \setminus \{0\}$.

Proof. We want to show that $\forall a \in \mathbb{Q} \setminus \{0\}, \exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}, |x - a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon$. Fix $a = \frac{m}{n} \in \mathbb{Q} \setminus \{0\}$ where $\gcd(m, n) = 1$, choose $\varepsilon = \frac{f(a)}{2} = \frac{1}{2n}$, then $\forall \delta > 0$, choose $x = \frac{m}{n} + \frac{\delta}{2}$ if δ is irrational otherwise $x = \frac{m}{n} + \frac{\delta}{\sqrt{2}}$ (to ensure x is irrational). Then if δ is irrational, $|x - a| = \frac{\delta}{2} < \delta$ and $|f(x) - f(a)| = |0 - \frac{1}{n}| = \frac{1}{n} \geq \frac{1}{2n} = \varepsilon$; if δ is rational, $|x - a| = \frac{\delta}{\sqrt{2}} < \delta$ and $|f(x) - f(a)| = |0 - \frac{1}{n}| = \frac{1}{n} \geq \frac{1}{2n} = \varepsilon$. Thus by definition $f(x)$ is not continuous at all $a \in \mathbb{Q} \setminus \{0\}$. QUOD ERAT DEM■

Question 2. Prove that $f(x)$ has left and right limit at any $x \in \mathbb{Q}$, i.e.,

$$\forall x \in \mathbb{R}, \lim_{y \to x^+} f(y), \lim_{y \to x^-} f(y) \text{ exist.}$$

Proof. It suffices to show the limit exists, thus by HW8 we know this implies the existence of left and right limits. I claim that the limit of any $a \in \mathbb{Q}$ is 0. Fix $a \in \mathbb{Q}$, fix $\varepsilon > 0$, then by Archimedean Property choose some $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

Choose $\delta = \min(\{|\frac{p}{q} - a| : p \in \mathbb{Z} \cap [-(N-1), (N-1)], q \in \mathbb{N}, q \leq N\})$ (finite and non-empty thus is well-defined), then consider $x \in \mathbb{R}$ s.t. $0 < |x - a| < \delta$: if x is irrational then we immediately get $|f(x) - 0| = 0 < \varepsilon$; if x is rational then x cannot be any of $\pm\frac{1}{1}; \pm\frac{1}{2}; \pm\frac{1}{3}, \pm\frac{2}{3}; \ldots; \pm\frac{1}{N}, \ldots, \pm\frac{N-1}{N}$ by our choose of δ, so, $|f(x) - 0| \leq \frac{1}{N} < \varepsilon$.

Hence, as $\lim_{y \to x} f(y) = 0$ for all $x \in \mathbb{Q}$, this implies the existence of left and right limits, completing our proof.

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Question 3. Prove that $f(x)$ is continuous for all $x \in \mathbb{Q}^c \cup \{0\}$. Moreover, conclude the type of discountinuity for $x \in \mathbb{Q} \setminus \{0\}$.

Proof. (Similar to E3Q2). We want to show that $\forall a \in \mathbb{Q}^c \cup \{0\}, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. Fix $a \in \mathbb{Q}^c \cup \{0\}$, fix $\varepsilon > 0$, then by Archimedean Property choose some $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

Choose $\delta = \min(\{|\frac{p}{q} - a| : p \in \mathbb{Z} \cap [-(N-1), (N-1)], q \in \mathbb{N}, q \leq N\})$ (finite and non-empty thus is well-defined), then consider $x \in \mathbb{R}$ s.t. $0 < |x - a| < \delta$: if x is irrational then we immediately get $|f(x) - f(a)| = |0 - 0| = 0 < \varepsilon$; if x is rational then x cannot be any of $\pm\frac{1}{1}; \pm\frac{1}{2}; \pm\frac{1}{3}, \pm\frac{2}{3}; \ldots; \pm\frac{1}{N}, \ldots, \pm\frac{N-1}{N}$ by our choose of δ, so, $|f(x) - f(a)| = |f(x) - 0| = f(x) \leq \frac{1}{N} < \varepsilon$.

Thus, by definition $f(x)$ is continuous for all $x \in \mathbb{Q}^c \cup \{0\}$. Moreover, since $f(x)$ is not continuous at any $x \in \mathbb{Q} \setminus \{0\}$, and we have shown that $f(x)$ has left and right limits at any $x \in \mathbb{Q}$ (moreover they are equal), thus $f(x)$ is removable discontinuous at all $x \in \mathbb{Q} \setminus \{0\}$, showing it is the first type of discontinuity. QUOD ERAT DEM■

Question 4. Name two functions $f_1(x), f_2(x) : \mathbb{R} \to \mathbb{R}$, satisfying the following conditions, respectively:

- $f_1(x)$ is not continuous at uncountably many points, moreover all discontinuous points are of the second type of discontinuity.
- $f_2(x)$ is not continuous at countably many points, moreover all discontinuous points are of the first type of discontinuity.

Hint: Name two great German mathematicians.

Claim. ...