Joseph Siu
MAT157: Analysis I
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Homework 10

Exercise 1

In this exercise, we aim at proving that each open set in $\mathbb{R}$ is the countable union of disjoint open intervals. To this end, assume that $U \subseteq \mathbb{R}$ is a non-empty open set.

Question 1. Prove that in $\mathbb{R}$ each connected open set is an open interval.

Question 2. Let $x \in U$. Introduce the relation on $U \times U$ as the following:

$$x \sim y := (x = y) \vee ((x > y) \wedge ((y, x) \subseteq U)) \vee ((y > x) \wedge ((x, y) \subseteq U)).$$

Prove that the relation is an equivalence relation.

Question 3. Prove that $\forall x \in U$, $[x]$ (its equivalence class) is connected.

Question 4. Based on all the previous sub-questions, conclude that each open set in $\mathbb{R}$ is the countable union of disjoint open intervals.

Exercise 2

Let $I_0 = [0,1]$ be the closed interval. Consider the following construction:

- Take off the open interval of $1/3$ length of I_0 in the middle, and let I_1 be the remaining set, which consists of 2 closed intervals.
- Take off the open interval of $1/3$ length in the middle of each component of I_1, and let I_2 be the remaining set, which consists of 4 closed intervals.
- Take off the open interval of $1/3$ length in the middle of each component of I_2, and let I_3 be the remaining set, which consists of 8 closed intervals.
- ...
- Take off the open interval of $1/3$ length in the middle of each component of I_n, and let I_{n+1} be the remaining set, which consists of 2^{n+1} closed intervals.

Now let

$$I^* = \bigcap_{i \in \mathbb{N}} I_n$$

Question 1. Prove that I^* is non-empty.

Proof. To show I^* is non-empty, it suffices to show that two end points of $[0,1]$ are contained in all $I_n, n \in \mathbb{N} \cup \{0\}$ and thus are in I^*.

First $0, 1 \in [0,1]$ showing $0, 1 \in I_0$. Assume $0, 1 \in I_n$ for some $n \in \mathbb{N} \cup \{0\}$, we want to show $0, 1 \in I_n \implies 0, 1 \in I_{n+1}$. That is, first 0 in I_n means 0 is in one of the closed intervals of I_n, and since I_{n+1} is the remaining set after taking off the open interval of $1/3$ length in the middle of each component of I_n, then 0 is still in one of the closed intervals of I_{n+1}, thus $0 \in I_{n+1}$. Similarly, 1 in I_n means 1 is in one of the closed intervals of I_n, and since I_{n+1} is the remaining set after taking off the open interval of $1/3$ length in the middle of each component of I_n, then 1 is still in one of the closed intervals of I_{n+1}, thus $1 \in I_{n+1}$. Thus by induction $0, 1 \in I_n, \forall n \in \mathbb{N} \cup \{0\}$, thus $0, 1 \in I^*$. From these 2 cases, we can also generalize that all end points of $I_n, n \in \mathbb{N} \cup \{0\}$ are contained in I^*, thus I^* is non-empty. QUOD ERAT DEM■

Question 2. Prove that I^* has length equal to 0. (Without introducing the measure, it is not clear what does length mean. However, in this question you can interpret the length in a naive way.)

Proof. Denote the length of the interval I using $\operatorname{len}(I)$. First since I^* is the intersection of all I_n, we have $\operatorname{len}(I^*) \leq \operatorname{len}(I_n)$ for all $n \in \mathbb{N}$. We show that $\operatorname{len}(I_n) = (\frac{2}{3})^n$ by induction on n, then this implies

$$\lim_{n\to\infty} \operatorname{len}(I^*) \leq \lim_{n\to\infty} \operatorname{len}(I_n) = \lim_{n\to\infty} (\frac{2}{3})^n = 0,$$

and since "length" cannot be negative, by squeeze theorem we have $\operatorname{len}(I^*) = 0$.

To this end, we show that $\operatorname{len}(I_n) = (\frac{2}{3})^n$ by induction on n.

Base case: $n = 0$, then $I_0 = [0,1]$ which has length $\operatorname{len}(I_0) = 1 = (\frac{2}{3})^0 = 1$, the equality holds.

Inductive hypothesis: $\operatorname{len}(I_n) = (\frac{2}{3})^n$ for some $n \in \mathbb{N} \cup \{0\}$.

Inductive step: Suppose $\operatorname{len}(I_n) = (\frac{2}{3})^n$, then because I_{n+1} is the remaining set after taking off the open interval of $1/3$ length in the middle of each component of I_n, and I_n consists of 2^n closed intervals, then I_{n+1} consists of 2^{n+1} closed intervals, and so, $\operatorname{len}(I_{n+1}) = \frac{2}{3}\operatorname{len}(I_n) = \frac{2}{3}(\frac{2}{3})^n = (\frac{2}{3})^{n+1}$.

By induction we have shown that $\operatorname{len}(I_n) = (\frac{2}{3})^n$ for all $n \in \mathbb{N}$, and thus $\operatorname{len}(I^*) \leq \operatorname{len}(I_n) = (\frac{2}{3})^n$ for all $n \in \mathbb{N}$.

As a consequence, we can see the length of I^* is 0. QUOD ERAT DEM■

Question 3. Prove that I^* is uncountable.

Proof. Assuming the definition of base 2 and base 3 decimal expressions (base 10 is shown by tutorial, similarily we may see that these expressions are well covering all real numbers uniquely). To show I^* is uncountable, we show the existence of a surjective function f from I^* to $[0, 1]$. To this end, we first represent every $x \in [0, 1]$ in base 3 notation (ternary). And so, since we take the middle 1/3 off to construct I_{n+1} from I_n, the remaining elements need to be in the form 0.000xxx... or 0.22202xxx... where $x \in \{0, 2\}$, in other words $x \neq 1$ where x is any digit of any element in I^* otherwise contradicting the construction of I^*. Thus, we construct f as a composition of g and h where g is a function that maps every element in I^* to a sequence of 0s and 2s, and h is a function that maps every sequence of 0s and 2s to a real number in $[0, 1]$ in base 2 notation, namely we define h as the function that changes any occurence of 2s to 1s and remain the occurence of 0. To show f is surjective, take any $y \in [0, 1]$, then there exists a sequence of 0s and 1s that represents y in base 2 notation, then we can change every occurence of 1s to 2s and remain the occurence of 0s, then we have a sequence of 0s and 2s, which is in the form of 0.000xxx... or 0.22202xxx... where $x \in \{0, 2\}$, in other words $x \neq 1$, thus it is in I^* (otherwise it must be excluded by I_n for all $n \geq N$ for some $N \in \mathbb{N}$, and however it cannot be excluded as end point cannot be excluded), and so, pick this sequence to be a which $f(a)$ gives the y as needed. Thus, f is surjective, and so, I^* is uncountable since the interval $[0, 1]$ is uncountable.

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Exercise 3

Definition. (Darboux function). A function $f : D \subseteq \mathbb{R}$ is called a Darboux function if
$\forall a, b \in D$ s.t. $f(b) > f(a), \forall y \in (f(a), f(b)), \exists c$ between a and b s.t. $f(c) = y$

For example, by intermediate value theorem, any continuous function defined on a closed interval is a Darboux function. However, Darboux function can be more general.

Question 1. Consider the function defined on closed interval $[-1, 1]$.

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \in [-1,1] \setminus \{0\} \\ 0 & x = 0 \end{cases}.$$

Prove that $f(x)$ is not continuous at $x = 0$, but it is still a Darboux function.

Proof. We first prove $f(x)$ is not continuous at $x = 0$ by showing $\lim_{x\to 0} \sin(1/x)$ does not exist. Consider $\lim_{x\to 0^+} \sin(1/x)$, let $h = \frac{1}{x}$, then $h \to \infty$ as $x \to 0^+$. Then we have $\lim_{x\to 0^+} \sin(1/x) = \lim_{h\to\infty} \sin(h)$ which clearly does not exist. Thus, by HW8 $\lim_{x\to 0} \sin(1/x)$ does not exist showing it is not continuous at $x = 0$.

Now we show $f(x)$ is continuous at all $x \in [-1,1]\setminus\{0\}$. Let $x \in [-1,1]\setminus\{0\}$, then $f(x) = \sin(1/x)$ which is continuous at x since $\sin(x)$ is continuous everywhere. Thus, $f(x)$ is continuous at all $x \in [-1,1] \setminus \{0\}$.

Since by IVT, every continuous function is a Darboux function, and $f(x)$ is continuous at all $x \in [-1,1] \setminus \{0\}$, then consider cases:

If $a, b \in [-1, 0)$, then because $f(x)$ is continuous on $[-1, 0)$, then by IVT, $\forall y \in (f(a), f(b)), \exists c$ between a and b s.t. $f(c) = y$. Thus, $f(x)$ is a Darboux function on $[-1, 0)$.

If $a, b \in (0, 1]$, then because $f(x)$ is continuous on $(0, 1]$, then by IVT, $\forall y \in (f(a), f(b)), \exists c$ between a and b s.t. $f(c) = y$. Thus, $f(x)$ is a Darboux function on $(0, 1]$.

If $a \in [-1, 0), b \in (0, 1]$, choose some $c_1 \in (a, 0), c_2 \in (0, b)$ such that $f(c_1) = \sin(1/c_1) = 1, f(c_2) = \sin(1/c_2) = -1$, this is allowed as $\sin(x)$ is a continouus function on all $x \in \mathbb{R}$ and it is periodically oscillating between -1 nad 1 inclusively. Thus, by the first 2 cases f is continouus on (a, c_1) and (c_2, b), also $f(c_1) \geq f(b)$ and $f(c_2) \leq f(a)$, giving $\forall y \in (f(a), f(b)), \exists c \in (a, b), f(c) = y$.

If $b \in [-1, 0), a \in (0, 1]$, similar to the previous case, choose some $c_1 \in (b, 0), c_2 \in (0, a)$ such that $f(c_1) = \sin(1/c_1) = -1, f(c_2) = \sin(1/c_2) = 1$, this is allowed as $\sin(x)$ is a continuous function on all $x \in \mathbb{R}$ and it is periodically oscillating between -1 and 1 inclusively. Thus, by the first 2 cases f is continuous on (b, c_1) and (c_2, a), also $f(c_1) \leq f(a)$ and $f(c_2) \geq f(b)$, giving $\forall y \in (f(a), f(b)), \exists c \in (a, b), f(c) = y$.

If $a = 0, b \in (0, 1] \cup [-1, 0)$. If $b \in (0, 1]$ we choose $c \in (a, b)$ such that $f(c) = \sin(1/c) = -1$, then by Case 2 we are done. If $b \in [-1, 0)$ we choose $c \in (b, a)$ such that $f(c) = \sin(1/c) = -1$, then by Case 1 we are done.

If $b = 0, a \in (0, 1] \cup [-1, 0)$. If $a \in (0, 1]$ we choose $c \in (b, a)$ such that $f(c) = \sin(1/c) = 1$, then by Case 2 we are done. If $a \in [-1, 0)$ we choose $c \in (a, b)$ such that $f(c) = \sin(1/c) = 1$, then by Case 1 we are done.

Therefore for all cases we have shown that the function $f(x)$ is indeed a Darboux funciton.

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Question 2. Consider the function defined on closed interval $[-1, 1]$.

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \in [-1,1] \setminus \{0\} \\ 0 & x = 0 \end{cases}.$$

Is the function continuous in $(-1, 1)$? Is the function differentiable in $(-1, 1)$? Explain your answer.

Proof. The function is continous in $(-1, 1)$. We show it by cases: when $x = 0$, we have the inequality $-|x| \leq f(x) \leq |x|$ since $\forall x \in \mathbb{R}, \sin(x) \in [-1, 1]$ and $-|x| \leq 0 \leq |x|, x = 0$. By squeeze theorem we have $0 = \lim_{x\to 0} -|x| \leq \lim_{x\to 0} f(x) \leq \lim_{x\to 0} |x| = 0$, showing the $\lim_{x\to 0} f(x) = f(0)$, thus continuous at $x = 0$. When

$x \in (-1,1) \setminus \{0\}$, g(x)=x is continouus and h(x)=sin(1/x) is continuous (by E3Q1) give $f(x) = x\sin(1/x)$ is continuous too (product of continuous functions is also continuous).

We have shown that the function is continuous in $(-1,1)$, however, f is not differentiable at $x = 0$:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h) - 0}{h} \\ &= \lim_{h\to 0} \frac{h\sin(1/h)}{h} \\ &= \lim_{h\to 0} \sin(1/h) \\ &= \text{DNE by E3Q1} \end{aligned}$$

But f is actually differentiable at $x \in (-1,1) \setminus \{0\}$. When $x \in (-1,1) \setminus \{0\}$, define $g(x) = x, h(x) = \sin(x), k(x) = 1/x$ where $f(x) = g(x)h(k(x))$, by Product Rule and Chain Rule $f'(x) = (g(x)h(k(x)))' = g'(x)h(k(x)) + g(x)h'(k(x))k'(x)$, that is, because all g, h, k are differentiable on $(-1,1) \setminus \{0\}$ ($\sin(1/x)$ is differentiable when $x = \neq 0$ since $sin(y)$ is differentiable everywhere where $y \in \mathbb{R}$; $k(x) = 1/x$ is differentiable when $x \neq 0$ since $a \in (-1,1) \setminus \{0\}, \lim_{h\to 0} \frac{k(a+h)-k(a)}{h} = \lim_{h\to 0} \frac{a-a-h}{ah(a+h)} = -\frac{1}{a^2}$), showing f is also differentiable (derivative exists implies differentiability).

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Question 3. Consider the function defined on closed interval $[-1,1]$.

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \in [-1,1] \setminus \{0\} \\ 0 & x = 0 \end{cases}.$$

Prove that the function is differentiable in $(-1,1)$. Is the derivative of the function $f'(x)$ continuous in $(-1,1)$? Is $f'(x)$ a Darboux function in $(-1,1)$? Explain your answer.

Proof. We first prove the function is differentiable. If $x = 0$,

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2\sin(1/h)}{h} \\ &= \lim_{h\to 0} h\sin(1/h) \\ &= 0 \text{ by E3Q2} \end{aligned}$$

If $x \in (-1,1) \setminus \{0\}$, by E3Q2, since $g(x) = x$ and $h(x) = x\sin(1/x)$ are differentiable thus by product rule their product $f(x) = g(x)h(x)$ must also be differentiable.

Now, to show f' is not continuous on its domain, we just need to show that $\lim_{x\to 0} f'(x) = \text{DNE}$.

When $x \in (-1,1) \setminus \{0\}$, by product rule and chain rule we have $f'(x) = (x^2\sin(1/x))' = 2x\sin(1/x) + x^2\cos(1/x)(-1/x^2) = 2x\sin(1/x) - \cos(1/x)$.

So,

$$\begin{aligned}
\lim_{x\to 0} f'(x) &= \frac{f'(x)-f'(0)}{x-0} \\
&= \lim_{x\to 0} \frac{f'(x)}{x} \\
&= \lim_{x\to 0} \frac{f'(x)}{x} \\
&= \lim_{x\to 0} \frac{2x\sin(1/x)-\cos(1/x)}{x} \\
&= \lim_{x\to 0} \frac{2x\sin(1/x)}{x} - \frac{\cos(1/x)}{x} \\
&= \lim_{x\to 0} 2\sin(1/x) - \frac{\cos(1/x)}{x} \\
&= \text{DNE by E3Q1}
\end{aligned}$$

Thus, f' is not continuous at $x = 0$ / in the domain $(-1, 1)$. Nonetheless, we can still show that $f'(x)$ is continuous at all $x \in (-1, 1) \setminus \{0\}$:

When $x \in (-1, 1) \setminus \{0\}$, by product rule and chain rule we have $f'(x) = (x^2 \sin(1/x))' = 2x\sin(1/x) + x^2\cos(1/x)(-1/x^2) = 2x\sin(1/x) - \cos(1/x)$, and $f'(x)$ is continuous at $x \in (-1, 1) \setminus \{0\}$ since $2x\sin(1/x)$ and $\cos(1/x)$ are continuous at $x \in (-1, 1) \setminus \{0\}$ (by E3Q2 and the fact that $\cos(x)$ is also continuous at all $x \in \mathbb{R}$).

Also, by similar argument as E3Q1, we can see that $f'(x) = 2x\sin(1/x) - \cos(1/x)$ is also a Darboux function since it is continous everywhere except $x = 0$.

That is, it is sufficient to show $f'(x)$ is an even function, then if a, b are not both in $(-1, 0)$ or $(0, 1)$, w.l.o.g. if $a \in (-1, 0)$ and $b \in (0, 1)$, then $f'(a) = f'(-a)$, giving $-a \in (0, 1)$ and $b \in (0, 1)$ and $y \in (f'(a), f'(b)) \iff y \in (f'(-a), f'(b))$, then because of the continuity of the function $f'(x)$ on the interval $(0, 1)$, apply IVT we are able to conclude it is indeed a Darboux function.

To this end, we show $f'(x)$ is an even function. Let $x \in (-1, 1) \setminus \{0\}$, then $f'(-x) = 2(-x)\sin(1/(-x)) - \cos(1/(-x)) = 2x\sin(1/x) - \cos(1/x) = f'(x)$, thus $f'(x)$ is an even function.

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