Joseph Siu
MAT157: Analysis I
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Homework 11

Exercise 1

Let $f(x)$ be a function defined near 0 and $\lim_{x \to 0} f(x) = 0$.

Question 1. Prove that if $g(x) = o(\mathcal{O}(f(x))),$ then $g(x) = o(f(x))$.

Question 2. Prove that if $g(x) = \mathcal{O}(o((f(x))),$ then $g(x) = o(f(x))$.

Exercise 2

Let the angle $\angle AOB = x$. Find $n \in \mathbb{N}$ so that the following quantity $g(x)$ satisfies that $g(x) = \mathcal{O}(x^n)$ and $x^n = \mathcal{O}(g(x))$.

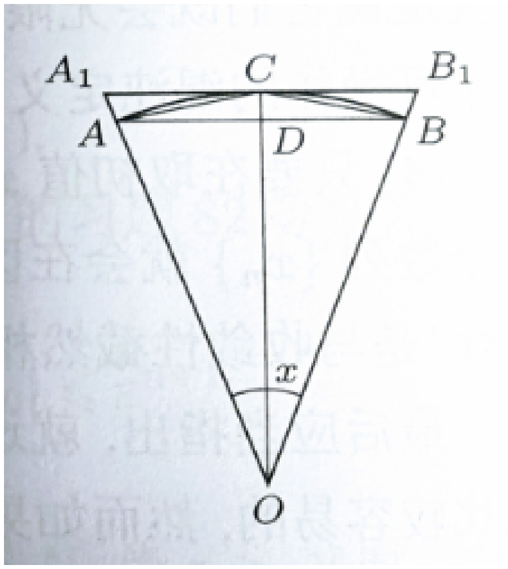

Figure 1. Exercise 2

Question 1. The chord length $|AB|$.

Definition. We denote $f(x) \sim g(x)$ when $x \to 0$ if $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$.

Lemma. $2\sin(\frac{x}{2}) \sim x$ when $x \to 0$.

Proof. As proven, we have $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, thus

$$\begin{aligned}\lim_{x\to 0} \frac{2\cdot\sin(\frac{x}{2})}{x} &= \lim_{x\to 0} 2\cdot\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\cdot\frac{\frac{x}{2}}{x} \\ &= \lim_{x\to 0} 2\cdot\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\cdot\frac{1}{2} \\ &= 2\cdot\frac{1}{2} \\ &= 1\end{aligned}$$

by definition showing $2\sin(\frac{x}{2}) \sim x$ when $x \to 0$. QUOD ERAT DEM■

Lemma. If $f(x), g(x)$ are bounded and continuous functions when $x \in [-1, 1]$, then

$$\limsup_{x\to 0} |f(x)g(x)| \leq \limsup_{x\to 0} |f(x)| \cdot \limsup_{x\to 0} |g(x)|$$

Proof. By Bolzano-Weierstrass Theorem, there must exist a convergent seqeunce $(x_n y_n)$ where $\forall n \in \mathbb{N}, x_n \in |f([-1,1])|, y_n \in |g([-1,1])|$ s.t. $\lim_{n\to\infty}(x_n y_n) = \limsup_{x\to 0} |f(x)g(x)| \in \mathbb{R}$ (by lecture [-1,1] is closed and f, g are continuous mean the supremum $|f(x)g(x)|$ must be achieved).

Thus, $0 \leq \lim_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \limsup_{x\to 0} |f(x)|$ and $0 \leq \lim_{n\to\infty} y_n \leq \limsup_{n\to\infty} y_n \leq \limsup_{x\to 0} |g(x)|$ give

$$\limsup_{x\to 0} |f(x)g(x)| = \lim_{n\to\infty} (x_n y_n) \leq \limsup_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n \leq \limsup_{x\to 0} |f(x)| \cdot \limsup_{x\to 0} |g(x)|,$$

as required. QUOD ERAT DEM■

Lemma. Assume f, g are continuous functions. If $x \to 0, f(x) \sim g(x), f(x), g(x) \neq 0$, then when $x \to 0$, $f(x) = \mathcal{O}(x^n) \iff g(x) = \mathcal{O}(x^n)$ for some fixed $n \in \mathbb{N}$.

Proof. Since $f(x) \sim g(x) \iff g(x) \sim f(x)$, w.l.o.g. we just need to show $f(x) = \mathcal{O}(x^n) \implies g(x) = \mathcal{O}(x^n)$. By definition, we have

$$\limsup_{x\to 0} |\frac{f(x)}{x^n}| \leq M, M \geq 0,$$

and f is bounded since x^n is bounded on $[-1, 1]$, which also implies g is bounded as f is bounded and $f(x) \sim g(x)$ when $x \to 0$. Then, by the previous lemma

$$\begin{aligned}
\limsup_{x\to 0} |\frac{f(x)}{x^n}| &= \limsup_{x\to 0} |\frac{f(x)}{x^n}| \cdot \limsup_{x\to 0} |1| \\
&= \limsup_{x\to 0} |\frac{f(x)}{x^n}| \cdot \limsup_{x\to 0} |\frac{g(x)}{f(x)}| \\
&\geq \limsup_{x\to 0} |\frac{f(x)}{x^n} \cdot \frac{g(x)}{f(x)}| \\
&= \limsup_{x\to 0} |\frac{g(x)}{x^n}|
\end{aligned}$$

showing

$$\limsup_{x\to 0} |\frac{g(x)}{x^n}| \leq \limsup_{x\to 0} |\frac{f(x)}{x^n}| \leq M$$

, which means $g(x) = \mathcal{O}(x^n)$, as required. QUOD ERAT DEM■

Lemma. When $x \to 0$, $Cx^n = \mathcal{O}(x^n)$ for all $C \in \mathbb{R}$, for all $n \in \mathbb{N}$.

Proof. $\limsup_{x\to 0} |\frac{Cx^n}{x^n}| = |C| \leq |C|$ where $|C| \geq 0$, by definition showing $Cx^n = \mathcal{O}(x^n)$ for any $n \in \mathbb{N}$ and any $C \in \mathbb{R}$. QUOD ERAT DEM■

Proof. Assume $x \to 0$. Denote the radius $R = AO = CO = BO > 0$, then by lemmas and formula of triangle we have:

$g(x) = |AB| = 2R\sin(\frac{x}{2}) \sim Rx$, since $R \in \mathbb{R}$, choose $n = 1$ we have $g(x) \sim Rx = \mathcal{O}(x)$ and $x = \mathcal{O}(Rx) \implies x = \mathcal{O}(g(x))$ since

$$\begin{aligned}
\limsup_{x\to 0} \frac{x}{Rx} &= \limsup_{x\to 0} \frac{x}{Rx} \cdot \limsup_{x\to 0} 1 \\
&= \limsup_{x\to 0} \frac{x}{Rx} \cdot \limsup_{x\to 0} \frac{Rx}{g(x)} \\
&\geq \limsup_{x\to 0} \frac{x}{Rx} \cdot \frac{Rx}{g(x)} \\
&= \limsup_{x\to 0} \frac{x}{g(x)}
\end{aligned}$$

which gives $x = \mathcal{O}(g(x))$. QUOD ERAT DEM■

Question 2. The arch height $|CD|$.

Lemma. $1 - \cos(\frac{x}{2}) \sim \frac{x^2}{8}$ when $x \to 0$.

Proof. By l'hopital's rule we have

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(\frac{x}{2})}{\frac{x^2}{8}} &= \lim_{x\to 0} \frac{8 - 8\cos(\frac{x}{2})}{x^2} \\ &= \lim_{x\to 0} \frac{4\sin(\frac{x}{2})}{2x} \\ &= \lim_{x\to 0} \cos(\frac{x}{2}) \\ &= 1\end{aligned}$$

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Proof. Assume $n \to 0$ and D is the mid point of line AB. Then, $g(x) = |CD| = R - R\cos(\frac{x}{2}) \sim \frac{Rx^2}{8}$, choose $n = 2$, then similar to E2Q1 we have $g(x) = \mathcal{O}(x^2)$ and $x^2 = \mathcal{O}(g(x))$.

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Question 3. Area of the sector AOB.

Proof. Denote A as the area of the sector AOB. Then, $g(x) = A = \frac{1}{2}R^2x$, choose $n = 1$, similar to E2Q1 we have $g(x) = \mathcal{O}(x)$ and $x = \mathcal{O}(g(x))$.

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Question 4. Area of the triangle $\triangle ACB$.

Lemma. $\sin(\frac{x}{2})(1 - \cos(\frac{x}{2})) \sim \frac{x^3}{16}$ when $x \to 0$.

Proof. By the previous lemmas we have

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(\frac{x}{2})(1-\cos(\frac{x}{2}))}{\frac{x^3}{16}} &= \lim_{x\to 0} \frac{\sin(\frac{x}{2})}{\frac{x}{2}} \cdot \lim_{x\to 0} \frac{1-\cos(\frac{x}{2})}{\frac{x^2}{8}} \\ &= 1 \cdot 1 \\ &= 1\end{aligned}$$

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Proof. Assume $x \to 0$ and D is the mid point of line AB. Denote A as the function returning the area of a triangle. Then, $A(\triangle ACB) = A(\triangle ACO) + A(\triangle BCO) - A(\triangle ABO)$. That is, $g(x) = A(\triangle ACB) = 2 \cdot \frac{1}{2}R^2\sin(\frac{x}{2}) - \frac{1}{2}R^2\sin(x) = R^2\sin(\frac{x}{2}) - R^2\sin(\frac{x}{2})\cos(\frac{x}{2}) = R^2\sin(\frac{x}{2})(1-\cos(\frac{x}{2})) \sim \frac{R^2x^3}{16}$, choose $n = 3$, then similar to E2Q1 we have $g(x) = \mathcal{O}(x^3)$ and $x^3 = \mathcal{O}(g(x))$.

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Exercise 3

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = e^{x^2 + \frac{\sin(x)}{1+x^2}}$$

Question 1. Compute the approximation of the value $f(1.001)$ by using linear approximation.

$$\begin{aligned}
f'(x) &= (e^{x^2 + \frac{\sin(x)}{1+x^2}})' \\
&= (x^2 + \frac{\sin(x)}{1+x^2})' e^{x^2 + \frac{\sin(x)}{1+x^2}} \\
&= ((x^2)' + (\frac{\sin(x)}{1+x^2})') e^{x^2 + \frac{\sin(x)}{1+x^2}} \\
&= (2x + (\frac{\sin(x)}{1+x^2})') e^{x^2 + \frac{\sin(x)}{1+x^2}} \\
&= (2x + \frac{\cos(x)(1+x^2) - \sin(x)(2x)}{(1+x^2)^2}) e^{x^2 + \frac{\sin(x)}{1+x^2}}
\end{aligned}$$

Using the formula $f(x + \Delta x) - f(x) = f'(x)\Delta x + o(\Delta x)$, where $x = 1, \Delta x = 0.001$, isolate $f(x + \Delta x)$ we have

$$\begin{aligned}
f(x + \Delta x) &= f(x) + f'(x)\Delta x + o(\Delta x) \\
&\approx f(1) + 0.001 \cdot f'(1) \\
&\approx ...
\end{aligned}$$

Question 2. Now suppose that you need to ensure the tolerance of error is less or equal to the scale of 10^{-17}. Normally speaking, how many terms in the Taylor expansion approximation do you need, given that in our scenario $\Delta x = 0.001$?

Proof. Since Δx is the differences of e^x and the Taylor expansion approximation of e^x $\Delta x = e^x$. QUOD ERAT DEM■