

Exercise 1

Recall that we have already explained in lecture that all rational function of a single variable can be integrated in finite terms. Starting from there, we introduce integrals of the form, known as the **binomial integrals**:

$$J_{p,q} = \int (a + bz)^p z^q dz, a, b \in \mathbb{R}, p, q \in \mathbb{Q}. \quad (1)$$

This exercise aims at studying the rationalization of the binomial integral, as well as some of its applications.

Question 1

Assume that $p \in \mathbb{Z}$, rationalise the integrand. ?

Since $q \in \mathbb{Q}$, this implies there exists $m \in \mathbb{Z}, n \in \mathbb{N}$ such that $q = \frac{m}{n}$. Then, $J_{p,q}$ can be written as

$$J_{p,q} = \int (a + bz)^p z^{\frac{m}{n}} dz.$$

Let $z = x^n$, then $dz = n \cdot x^{n-1} dx$. Replace z with x we have

$$J_{p,q} = \int (a + b \cdot x^n)^p \cdot (x^n)^{\frac{m}{n}} \cdot n \cdot x^{n-1} dx.$$

Simplify it and we get

$$J_{p,q} = n \int (a + b \cdot x^n)^p \cdot x^{m+n-1} dx.$$

Now, by binomial theorem we can see the integrand is rationalised, as needed. 😊

Question 2

Assume that $q \in \mathbb{Z}$, rationalise the integrand.

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Since $p \in \mathbb{Q}$, this implies there exists $m \in \mathbb{Z}, n \in \mathbb{N}$ such that $p = \frac{m}{n}$. Then, $J_{p,q}$ can be written as

$$J_{p,q} = \int (a + bz)^{\frac{m}{n}} z^q dz.$$

If $b = 0$ then the integrand is trivially rational, thus we consider the case when $b \neq 0$. Let $x^n = a + bz$, then $z = \frac{x^n - a}{b}$ and $n \cdot x^{n-1} dx = b dz$. Replace z with x we have

$$J_{p,q} = \int (x^n)^{\frac{m}{n}} \cdot \left(\frac{x^n - a}{b} \right)^q \cdot \frac{n \cdot x^{n-1}}{b} dx.$$

Simplify it and we get

$$J_{p,q} = \frac{n}{b^{q+1}} \int x^{m+n-1} \cdot (x^n - a)^q dx.$$

Now, by binomial theorem we can see the integrand is rationalised, as needed.

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Question 3

Assume that $p + q \in \mathbb{Z}$, rationalise the integrand.

?

Hint. Write the integrand as

$$\int \left(\frac{a + bz}{z} \right)^p z^{p+q} dz.$$



If $p \in \mathbb{Z}$ then this case is rationalised by Question ?? . Thus we focus on the case when $p \in \mathbb{Q} \setminus \mathbb{Z}$, that is, $p = \frac{m}{n}$ for some $m \in \mathbb{Z}, n \in \mathbb{N}$. First, we change the equation to

$$J_{p,q} = \int \left(\frac{a + bz}{z} \right)^p z^{p+q} dz.$$

Now, let $x^n = \frac{a+bz}{z} = \frac{a}{z} + b$, then $z = \frac{a}{x^n - b}$ and $n \cdot x^{n-1} dx = -\frac{a}{z^2} dz$. Now replace z with x we have

$$\begin{aligned} J_{p,q} &= \int (x^n)^{\frac{m}{n}} \cdot \left(\frac{a}{x^n - b} \right)^{p+q} \cdot \left(\frac{n \cdot x^{n-1} \cdot z^2}{-a} \right) dx \\ &= \int x^m \cdot \frac{a^{p+q}}{(x^n - b)^{p+q}} \cdot \left(\frac{n \cdot x^{n-1} \cdot \frac{a^2}{(x^n - b)^2}}{-a} \right) dx \\ &= n \int x^m \cdot \frac{a^{p+q}}{(x^n - b)^{p+q}} \cdot \left(-\frac{ax^{n-1}}{(x^n - b)^2} \right) dx \\ &= -n \cdot a^{p+q+1} \int x^m \cdot \frac{1}{(x^n - b)^{p+q}} \cdot \left(x^{n-1} \cdot \frac{1}{(x^n - b)^2} \right) dx \\ &= -n \cdot a^{p+q+1} \int x^{m+n-1} \cdot \frac{1}{(x^n - b)^{p+q+2}} dx \\ &= -n \cdot a^{p+q+1} \int x^{m+n-1} \cdot (x^n - b)^{-2-p-q} dx \end{aligned}$$

Now, we can see that the integrand is rationalised, as needed.



Remark 1. So far we have shown an interesting conclusion: if either p , or q , or $p + q$ is an integer, then the function can be integrated in finite terms.



Question 4

Prove that

- If $p \neq -1$, then

$$J_{p,q} = -\frac{(a+bz)^{p+1}z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)}J_{p+1,q}$$

- If $q \neq -1$, then

$$J_{p,q} = \frac{(a+bz)^{p+1}z^{q+1}}{a(q+1)} - b\frac{p+q+2}{a(q+1)}J_{p,q+1}$$

?

Proof.

1) When $p \neq -1$, using integration by parts we have

$$\begin{aligned}
 J_{p,q} &= \int (a + bz)^p \cdot z^q \, dz \\
 &= \int \left(\frac{a + bz}{z} \right)^p \cdot z^{p+q} \, dz \\
 &= \int \left(\frac{a}{z} + b \right)^p \cdot z^{p+q} \, dz \\
 &= \int z^{p+q} \cdot \left(-\frac{z^2}{a} \right) d \left(\frac{\left(\frac{a}{z} + b \right)^{p+1}}{p+1} \right) \\
 &= -\frac{1}{a(p+1)} \int z^{p+q+2} \cdot d \left(\frac{a}{z} + b \right)^{p+1} \\
 &= -\frac{\frac{(a+bz)^{p+1}}{z^{p+1}} \cdot z^{p+q+2}}{a(p+1)} + \frac{1}{a(p+1)} \int \left(\frac{a}{z} + b \right)^{p+1} d(z^{p+q+2}) \\
 &= -\frac{(a+bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)} \int \frac{(a+bz)^{p+1}}{z^{p+1}} \cdot z^{p+q+1} \, dz \\
 &= -\frac{(a+bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)} \int (a+bz)^{p+1} \cdot z^q \, dz \\
 &= -\frac{(a+bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)} J_{p+1,q} \tag{2}
 \end{aligned}$$

2) When $q \neq -1$, we first consider the integral form of $J_{p+1,q}$ in (??), then substitute our result into (??), and the ending result is the desired formula.

To this end, consider $J_{p+1,q} = \int (a + bz)^{p+1} \cdot z^q \, dz$, using integration by parts we have

$$\begin{aligned}
 J_{p+1,q} &= \int (a + bz)^{p+1} \cdot z^q \, dz \\
 &= \int (a + bz)^{p+1} d \left(\frac{z^{q+1}}{q+1} \right) \\
 &= \frac{1}{q+1} (a + bz)^{p+1} \cdot z^{q+1} - \frac{1}{q+1} \int z^{q+1} d(a + bz)^{p+1} \\
 &= \frac{1}{q+1} (a + bz)^{p+1} \cdot z^{q+1} - \frac{b(p+1)}{q+1} \int (a + bz)^p \cdot z^{q+1} \, dz \\
 &= \frac{1}{q+1} (a + bz)^{p+1} \cdot z^{q+1} - \frac{b(p+1)}{q+1} J_{p,q+1} \tag{3}
 \end{aligned}$$

Now, we substitute (??) back to (??), then the equation becomes

$$\begin{aligned}
 J_{p,q} &= -\frac{(a + bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)} J_{p+1,q} \\
 &= -\frac{(a + bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)} \left(\frac{1}{q+1} (a + bz)^{p+1} \cdot z^{q+1} - \frac{b(p+1)}{q+1} J_{p,q+1} \right) \\
 &= -\frac{(a + bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{1}{a} \left[1 + \frac{q+1}{p+1} \right] \cdot \left(\frac{1}{q+1} (a + bz)^{p+1} \cdot z^{q+1} - \frac{b(p+1)}{q+1} J_{p,q+1} \right)
 \end{aligned}$$

Question 5

Based on the previous question, prove that if $p + q \neq -1$, then

$$J_{p,q} = \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{ap}{p+q+1} J_{p-1,q}$$

$$J_{p,q} = \frac{(a+bz)^{p+1} z^q}{b(p+q+1)} - \frac{aq}{b(p+q+1)} J_{p,q-1}$$

?

Proof. We first show the first equation. Consider (??), we will use integration by parts to show the desired equation. That is,

$$\begin{aligned} J_{p,q} &= \int (a+bz)^p z^q dz \\ &= \int \left(\frac{a+bz}{z} \right)^p z^{p+q} dz \\ &= \int \left(\frac{a+bz}{z} \right)^p d \left(\frac{z^{p+q+1}}{p+q+1} \right) \\ &= \frac{(a+bz)^p}{z^p} \cdot \frac{z^{p+q+1}}{p+q+1} - \frac{1}{p+q+1} \int z^{p+q+1} d \left(\frac{a+bz}{z} \right)^p \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} - \frac{1}{p+q+1} \int z^{p+q+1} d \left(\frac{a}{z} + b \right)^p \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} - \frac{1}{p+q+1} \int z^{p+q+1} p \left(\frac{a}{z} + b \right)^{p-1} \left(-\frac{a}{z^2} \right) dz \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{ap}{p+q+1} \int z^{p+q+1} \left(\frac{a+bz}{z} \right)^{p-1} z^{-2} dz \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{ap}{p+q+1} \int (a+bz)^{p-1} z^{p+q+1-p+1-2} dz \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{ap}{p+q+1} \int (a+bz)^{p-1} z^q dz \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{ap}{p+q+1} J_{p-1,q} \end{aligned} \tag{4}$$

With (??) we can show the second equation. Similar to Question ??, we first consider the integral form of $J_{p-1,q}$ in (??), then substitute our result into (??), and the ending result is the desired formula.

$$\begin{aligned} J_{p-1,q} &= \int (a+bz)^{p-1} z^q dz \\ &= \frac{1}{pb} \int z^q d(a+bz)^p \\ &= \frac{1}{pb} (a+bz)^p z^q - \frac{q}{pb} \int (a+bz)^p z^{q-1} dz \\ &= \frac{1}{pb} (a+bz)^p z^q - \frac{q}{pb} J_{p,q-1} \end{aligned} \tag{5}$$

Now, we substitute (??) back to (??), then the equation becomes



Now with all the information obtained above, let's study an example to get some feelings. Define for $m \in \mathbb{Z}$ the integral

$$H_m = \int \frac{x^m}{\sqrt{1-x^2}} dx, \quad x > 0.$$

In the sequel, we will first transform H_m into a binomial integral. Since the binomial integral can be computed by iteration, it in turn provides a recursive relation for $H_m, m \in \mathbb{Z}$. To conclude, we aim at completely (at least formally) solve the problem of integrate H_m in finite terms for any $m \in \mathbb{Z}$.

**Question 6**

By introducing the substitution $z = x^2$, show that

$$H_m = J_{-\frac{1}{2}, \frac{m-1}{2}}.$$

?

Proof. Let $z = x^2$, then $dz = 2x dx$. Replace x with z we have

$$H_m = \int \frac{z^{\frac{m}{2}}}{\sqrt{1-z}} \cdot \frac{dz}{2\sqrt{z}} = \frac{1}{2} \int \frac{z^{\frac{m-1}{2}}}{\sqrt{1-z}} dz = \int \frac{z^{\frac{m-1}{2}}}{\sqrt{4-4z}} dz = \int (4-4z)^{-\frac{1}{2}} \cdot z^{\frac{m-1}{2}} dz.$$

Let $a = 4, b = -4, p = -\frac{1}{2}, q = \frac{m-1}{2}$, then we have

$$H_m = \int (4-4z)^{-\frac{1}{2}} \cdot z^{\frac{m-1}{2}} dz = J_{-\frac{1}{2}, \frac{m-1}{2}}.$$

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Question 7

Prove that $\forall m > 1$,

$$H_m = -\frac{1}{m}x^{m-1}\sqrt{1-x^2} + \frac{m-1}{m}H_{m-2}$$

?

Hint. Perhaps use consequence in question 5.



Proof. Assume $m > 1$. Let $a = 4, b = -4, p = -\frac{1}{2}, q = \frac{m-1}{2}$ (note that here $p + q \neq -1$), $z = x^2$, then we have

$$\begin{aligned} H_m = J_{-\frac{1}{2}, \frac{m-1}{2}} &= \frac{(a+bz)^{p+1}z^q}{b(p+q+1)} - \frac{aq}{b(p+q+1)} J_{-\frac{1}{2}, \frac{(m-2)-1}{2}} \\ &= \frac{(4-4z)^{\frac{1}{2}}z^{\frac{m-1}{2}}}{-4(\frac{m-1}{2} - \frac{1}{2} + 1)} - \frac{4(\frac{m-1}{2})}{-4(\frac{m-1}{2} - \frac{1}{2} + 1)} \int (4-4z)^{-\frac{1}{2}} z^{\frac{(m-2)-1}{2}} dz \\ &= \frac{2(1-x^2)^{\frac{1}{2}}x^{m-1}}{-2(m-1-1+2)} - \frac{2(m-1)}{-2(m-1-1+2)} \int \frac{1}{2}(1-x^2)^{-\frac{1}{2}}x^{(m-2)-1}2x dx \\ &= -\frac{(1-x^2)^{\frac{1}{2}}x^{m-1}}{m} - \frac{m-1}{-m} \int (1-x^2)^{-\frac{1}{2}}x^{m-2} dx \\ &= -\frac{1}{m}x^{m-1}\sqrt{1-x^2} + \frac{m-1}{m} \int \frac{x^{m-2}}{\sqrt{1-x^2}} dx \\ &= -\frac{1}{m}x^{m-1}\sqrt{1-x^2} + \frac{m-1}{m}H_{m-2} \end{aligned}$$

since we have chosen an arbitrary $m > 1$, thus we have shown the equation holds for all $m > 1$, as required.

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**Question 8**Compute H_0, H_1 explicitly.

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$$\begin{aligned} H_0 &= \int \frac{1}{\sqrt{1-x^2}} dx \\ &= \arcsin x + C \end{aligned} \tag{6}$$

Note that the equality follows by formula, which can be verified by taking the derivative of (??). ☺

$$\begin{aligned} H_1 &= \int \frac{x}{\sqrt{1-x^2}} dx \\ &= \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} d(x^2) \\ &= -(1-x^2)^{\frac{1}{2}} + C \end{aligned}$$

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Question 9

Based on the previous sub-question(s), conclude that $\forall m \in \mathbb{Z}_{\geq 0}$, H_m can be integrated in finite terms. ?

Proof. We show by strong induction. From Question ??, we have shown H_0, H_1 can be integrated in finite terms. Let H_0, H_1 be the base cases. Assume H_n holds when $n \in \{0, 1, \dots, m\}$ for some $m \in \mathbb{Z}_{\geq 0}$. Then, we want to show H_{m+1} can also be integrated in finite terms. To this end, consider 2 cases.

When $m + 1 \in \{0, 1\}$, this is immediately covered by our base cases.

When $m + 1 \geq 2$, by Question ??, we have

$$H_{m+1} = -\frac{1}{m+1}x^m\sqrt{1-x^2} + \frac{m}{m+1}H_{m-1},$$

since $m > m - 1 \geq 0$, this means by our inductive hypothesis H_{m-1} can be integrated in finite terms, this gives H_{m+1} can also be integrated in finite terms by our equation.

Therefore, by strong induction, we have shown H_m can be integrated in finite terms for all $m \in \mathbb{Z}_{\geq 0}$, as required.

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Question 10

Prove that $\forall m < -1$,

$$H_m = \frac{x^{m+1}\sqrt{1-x^2}}{m+1} + \frac{2+m}{1+m}H_{m+2}$$

?

Hint. Perhaps use consequence in question 4.



Proof. Choose any $m < -1$, fix $a = 4, b = -4, p = -\frac{1}{2}, q = \frac{m-1}{2}$, let $z = x^2$ where $dz = 2x dx$, then, by formula from Question ?? and Question ?? we have

$$\begin{aligned} H_m = J_{-\frac{1}{2}, \frac{m-1}{2}} &= \frac{(a+bz)^{\frac{1}{2}} z^{\frac{m+1}{2}}}{a(q+1)} - b \frac{p+q+2}{a(q+1)} J_{-\frac{1}{2}, \frac{m+1}{2}} \\ &= \frac{(4-4z)^{\frac{1}{2}} z^{\frac{m+1}{2}}}{4(\frac{m-1}{2}+1)} - (-4) \frac{-\frac{1}{2} + \frac{m-1}{2} + 2}{4(\frac{m-1}{2}+1)} \int (4-4z)^{-\frac{1}{2}} z^{\frac{m+1}{2}} dz \\ &= \frac{2(1-x^2)^{\frac{1}{2}} x^{m+1}}{2(m+1)} + 2 \frac{-1+m-1+4}{2m-2+4} \int \frac{1}{2} (1-x^2)^{-\frac{1}{2}} x^{m+1} 2x dx \\ &= \frac{x^{m+1}\sqrt{1-x^2}}{m+1} + \frac{2+m}{1+m} \int \frac{x^{m+2}}{\sqrt{1-x^2}} dx \\ &= \frac{x^{m+1}\sqrt{1-x^2}}{m+1} + \frac{2+m}{1+m} H_{m+2} \end{aligned}$$

which is precisely the equation required, as needed.

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Question 11

Compute H_{-1}, H_{-2} explicitly.

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Claim 1

$$\int \frac{1}{\sin x} dx = \ln |\csc x - \cot x| + C$$

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Proof of Claim ??.

$$\begin{aligned} \int \frac{1}{\sin x} dx &= \int \frac{-\cos x + 1}{\sin x(-\cos x + 1)} dx \\ &= \int \frac{-\frac{\cos x}{\sin^2 x} + \frac{1}{\sin^2 x}}{\frac{1}{\sin x} - \frac{\cos x}{\sin x}} dx \\ &= \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} dx \\ &= \int \frac{1}{\csc x - \cot x} d(\csc x - \cot x) \\ &= \ln |\csc x - \cot x| + C \end{aligned}$$

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Claim 2

$$\int \frac{1}{\sqrt{x^2 - 1}} dx = \ln |x + \sqrt{x^2 - 1}| + C$$

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Proof of Claim ??. Let $x = \sec u$, then $dx = \sec u \tan u du$, thus we have

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - 1}} dx &= \int \frac{1}{\sqrt{\sec^2 u - 1}} \sec u \tan u du \\ &= \int \sec u du \\ &= \int \frac{1}{\cos u} du \\ &= - \int \frac{1}{\sin\left(\frac{\pi}{2} - u\right)} d\left(\frac{\pi}{2} - u\right) \\ &= - \ln \left| \csc\left(\frac{\pi}{2} - u\right) - \cot\left(\frac{\pi}{2} - u\right) \right| + C \\ &= \ln \left| \frac{1}{x - \sqrt{x^2 - 1}} \right| + C \quad \text{By Pythagorean Theorem} \\ &= \ln \left| \frac{1}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} \right| + C \\ &= \ln |x + \sqrt{x^2 - 1}| + C \end{aligned} \tag{7}$$

If we assume $x \in (-\infty, -1) \cup (1, \infty)$, then $u = \operatorname{arcsec} x \in (0, \pi) \setminus \{\frac{\pi}{2}\}$. Hence,

$$\begin{aligned}
 H_{-1} &= \int \frac{x^{-1}}{\sqrt{1-x^2}} dx \\
 &= \int \frac{1}{x\sqrt{1-x^2}} dx \\
 &= \int \frac{1}{x^2\sqrt{\frac{1}{x^2}-1}} dx \\
 &= -\int \frac{1}{\sqrt{\left(\frac{1}{x}\right)^2-1}} d\left(\frac{1}{x}\right) \\
 &= -\ln \left| \frac{1}{x} + \sqrt{\left(\frac{1}{x}\right)^2-1} \right| + C
 \end{aligned}$$



Using the substitution $x = \sin u$, and the pothagorean theorem, we have

$$\begin{aligned}
 H_{-2} &= \int \frac{x^{-2}}{\sqrt{1-x^2}} dx \\
 &= \int \frac{1}{\sin^2 u \cos u} \cos u du \\
 &= \int \csc^2 u du \\
 &= -\cot u + C \\
 &= -\frac{\sqrt{1-x^2}}{x} + C
 \end{aligned}$$



Question 12

Based on the previous sub-question(s), conclude that $\forall m \in \mathbb{Z}_{<0}$, H_m can be integrated in finite terms. ?

Proof. We perform a variation of strong induction on $\mathbb{Z}_{<0}$. That is, first let H_{-1}, H_{-2} be the base cases which are proven in Question ?? that they can be integrated in finite terms. Now assume the statement holds for all H_n where $n \in \{m, m+1, \dots, -2, -1\}$ for some $m \in \mathbb{Z}_{<0}$. Then, we want to show H_{m-1} can also be integrated in finite terms. To this end, consider 2 cases.

When $m-1 \in \{-2, -1\}$, this is immediately covered by our base cases.

When $m-1 \leq -3$, by Question ??, we have

$$H_{m-1} = \frac{x^m \sqrt{1-x^2}}{m} + \frac{1+m}{m} H_{m+1},$$

since $m < m+1 \leq -1$, this means by our inductive hypothesis H_{m+1} can be integrated in finite terms, this gives H_{m-1} can also be integrated in finite terms by our equation.

Therefore, by strong induction, we have shown H_m can be integrated in finite terms for all $m \in \mathbb{Z}_{<0}$, as required.

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Conclusion: We have shown that for any $m \in \mathbb{Z}$, H_m can be integrated in finite terms.