Exercise 1

Recall that we have already explained in lecture that all rational function of a single variable can be integrated in finite terms. Starting from there, we introduce integrals of the form, known as the **binomial integrals**:

$$J_{p,q} = \int (a+bz)^p z^q \, \mathrm{d}z, a, b \in \mathbb{R}, p, q \in \mathbb{Q}. \tag{1}$$

This exercise aims at studying the rationalization of the binomial integral, as well as some of its applications.

Question 1

Assume that $p \in \mathbb{Z}$, rationalise the integrand. ?

Since $q \in \mathbb{Q}$, this implies there exists $m \in \mathbb{Z}, n \in \mathbb{N}$ such that $q = \frac{m}{n}$. Then, $J_{p,q}$ can be written as

$$J_{p,q} = \int (a+bz)^p z^{\frac{m}{n}} \, \mathrm{d}z.$$

Let $z = x^n$, then $\mathrm{d}z = n \cdot x^{n-1} \, \mathrm{d}x$. Replace z with x we have

$$J_{p,q} = \int (a + b \cdot x^n)^p \cdot (x^n)^{\frac{m}{n}} \cdot n \cdot x^{n-1} \, \mathrm{d}x.$$

Simplify it and we get

$$J_{p,q} = n \int (a + b \cdot x^n)^p \cdot x^{m+n-1} \, \mathrm{d}x.$$

Now, by binomial theorem we can see the integrand is rationalised, as needed. ☺

Question 2

Assume that $q \in \mathbb{Z}$, rationalise the integrand. ?

Since $p \in \mathbb{Q}$, this implies there exists $m \in \mathbb{Z}, n \in \mathbb{N}$ such that $p = \frac{m}{n}$. Then, $J_{p,q}$ can be written as

$$J_{p,q} = \int (a+bz)^{\frac{m}{n}} z^q \, \mathrm{d}z.$$

If $b = 0$ then the integrant is trivially rational, thus we consider the case when $b \neq 0$. Let $x^n = a + bz$, then $z = \frac{x^n - a}{b}$ and $n \cdot x^{n-1} \, \mathrm{d}x = b \, \mathrm{d}z$. Replace z with x we have

$$J_{p,q} = \int (x^n)^{\frac{m}{n}} \cdot \left(\frac{x^n - a}{b} \right)^q \cdot \frac{n \cdot x^{n-1}}{b} \, \mathrm{d}x.$$

Simplify it and we get

$$J_{p,q} = \frac{n}{b^{q+1}} \int x^{m+n-1} \cdot (x^n - a)^q \, \mathrm{d}x.$$

Now, by binomial theorem we can see the integrand is rationalised, as needed. ☺

Question 3

Assume that $p+q\in\mathbb{Z}$, rationalise the integrand. ?

Hint. Write the integrand as

$$\int\left(\frac{a+bz}{z}\right)^p z^{p+q}\,\mathrm{d}z.$$

♡

If $p\in\mathbb{Z}$ then this case is rationalised by Question 1. Thus we focus on the case when $p\in\mathbb{Q}\setminus\mathbb{Z}$, that is, $p=\frac{m}{n}$ for some $m\in\mathbb{Z},n\in\mathbb{N}$. First, we change the equation to

$$J_{p,q}=\int\left(\frac{a+bz}{z}\right)^p\cdot z^{p+q}\,\mathrm{d}z.$$

Now, let $x^n=\frac{a+bz}{z}=\frac{a}{z}+b$, then $z=\frac{a}{x^n-b}$ and $n\cdot x^{n-1}\,\mathrm{d}x=-\frac{a}{z^2}\,\mathrm{d}z$. Now replace z with x we have

$$\begin{aligned}
J_{p,q}&=\int(x^n)^{\frac{m}{n}}\cdot\left(\frac{a}{x^n-b}\right)^{p+q}\cdot\left(\frac{n\cdot x^{n-1}\cdot z^2}{-a}\right)\mathrm{d}x\\
&=\int x^m\cdot\frac{a^{p+q}}{(x^n-b)^{p+q}}\cdot\left(\frac{n\cdot x^{n-1}\cdot\frac{a^2}{(x^n-b)^2}}{-a}\right)\mathrm{d}x\\
&=n\int x^m\cdot\frac{a^{p+q}}{(x^n-b)^{p+q}}\cdot\left(-\frac{ax^{n-1}}{(x^n-b)^2}\right)\mathrm{d}x\\
&=-n\cdot a^{p+q+1}\int x^m\cdot\frac{1}{(x^n-b)^{p+q}}\cdot\left(x^{n-1}\cdot\frac{1}{(x^n-b)^2}\right)\mathrm{d}x\\
&=-n\cdot a^{p+q+1}\int x^{m+n-1}\cdot\frac{1}{(x^n-b)^{p+q+2}}\,\mathrm{d}x\\
&=-n\cdot a^{p+q+1}\int x^{m+n-1}\cdot(x^n-b)^{-2-p-q}\,\mathrm{d}x
\end{aligned}$$

Now, we can see that the integrand is rationalised, as needed. ☺

Remark 1. So far we have shown an interesting conclusion: if either p, or q, or $p+q$ is an integer, then the function can be integrated in finite terms.

△

Question 4

Prove that

- If $p \neq -1$, then
$$J_{p,q} = -\frac{(a+bz)^{p+1}z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)}J_{p+1,q}$$
- If $q \neq -1$, then
$$J_{p,q} = \frac{(a+bz)^{p+1}z^{q+1}}{a(q+1)} - b\frac{p+q+2}{a(q+1)}J_{p,q+1}$$

?

Proof.

1) When $p \neq -1$, using integration by parts we have

$$\begin{aligned}
J_{p,q} &= \int (a+bz)^p \cdot z^q \,\mathrm{d}z \\
&= \int \left(\frac{a+bz}{z}\right)^p \cdot z^{p+q}\,\mathrm{d}z \\
&= \int \left(\frac{a}{z}+b\right)^p \cdot z^{p+q}\,\mathrm{d}z \\
&= \int z^{p+q}\cdot\left(-\frac{z^2}{a}\right)\mathrm{d}\left(\frac{\left(\frac{a}{z}+b\right)^{p+1}}{p+1}\right) \\
&= -\frac{1}{a(p+1)}\int z^{p+q+2}\cdot \mathrm{d}\left(\frac{a}{z}+b\right)^{p+1} \\
&= -\frac{\frac{(a+bz)^{p+1}}{z^{p+1}}\cdot z^{p+q+2}}{a(p+1)} + \frac{1}{a(p+1)}\int\left(\frac{a}{z}+b\right)^{p+1}\mathrm{d}\left(z^{p+q+2}\right) \\
&= -\frac{(a+bz)^{p+1}\cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)}\int\frac{(a+bz)^{p+1}}{z^{p+1}}\cdot z^{p+q+1}\,\mathrm{d}z \\
&= -\frac{(a+bz)^{p+1}\cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)}\int (a+bz)^{p+1}\cdot z^{q}\,\mathrm{d}z \\
&= -\frac{(a+bz)^{p+1}\cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)}J_{p+1,q} \qquad (2)
\end{aligned}$$

2) When $q \neq -1$, we first consider the integral form of $J_{p+1,q}$ in (1), then substitute our result into (2), and the ending result is the desired formula.
To this end, consider $J_{p+1,q} = \int (a+bz)^{p+1} \cdot z^q \, \mathrm{d}z$, using integration by parts we have

$$\begin{aligned}
J_{p+1,q} &= \int (a+bz)^{p+1} \cdot z^q \, \mathrm{d}z \\
&= \int (a+bz)^{p+1} \, \mathrm{d}\left(\frac{z^{q+1}}{q+1}\right) \\
&= \frac{1}{q+1}(a+bz)^{p+1} \cdot z^{q+1} - \frac{1}{q+1} \int z^{q+1} \, \mathrm{d}(a+bz)^{p+1} \\
&= \frac{1}{q+1}(a+bz)^{p+1} \cdot z^{q+1} - \frac{b(p+1)}{q+1} \int (a+bz)^{p} \cdot z^{q+1} \, \mathrm{d}z \\
&= \frac{1}{q+1}(a+bz)^{p+1} \cdot z^{q+1} - \frac{b(p+1)}{q+1} J_{p,q+1} \qquad (3)
\end{aligned}$$

Now, we substitute (3) back to (2), then the equation becomes

$$\begin{aligned}
J_{p,q} &= -\frac{(a+bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)} J_{p+1,q} \\
&= -\frac{(a+bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{p+q+2}{a(p+1)} \left(\frac{1}{q+1}(a+bz)^{p+1} \cdot z^{q+1} - \frac{b(p+1)}{q+1} J_{p,q+1}\right) \\
&= -\frac{(a+bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{1}{a}\left[1 + \frac{q+1}{p+1}\right] \cdot \left(\frac{1}{q+1}(a+bz)^{p+1} \cdot z^{q+1} - \frac{b(p+1)}{q+1} J_{p,q+1}\right) \\
&= -\frac{(a+bz)^{p+1} \cdot z^{q+1}}{a(p+1)} + \frac{1}{a}\left[\frac{z^{q+1}}{q+1} \cdot (a+bz)^{p+1} - \frac{b(p+1)}{q+1} J_{p,q+1}\right] \\
&\qquad\qquad + \frac{q+1}{a(p+1)}\left[\frac{z^{q+1}}{q+1}(a+bz)^{p+1} - \frac{b(p+1)}{q+1} J_{p,q+1}\right] \\
&= -\frac{(a+bz)^{p+1} z^{q+1}}{a(p+1)} + \frac{1}{a(q+1)}(a+bz)^{p+1} z^{q+1} - \frac{b(p+1)}{a(q+1)} J_{p,q+1} \\
&\qquad\qquad + \frac{(a+bz)^{p+1} z^{q+1}}{a(p+1)} - \frac{b}{a} J_{p,q+1} \\
&= \frac{(a+bz)^{p+1} z^{q+1}}{a(q+1)} - \frac{b(p+1+q+1)}{a(q+1)} J_{p,q+1} \\
&= \frac{(a+bz)^{p+1} z^{q+1}}{a(q+1)} - b\frac{p+q+2}{a(q+1)} J_{p,q+1}
\end{aligned}$$

which is precisely the equation required, as needed.

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Question 5

Based on the previous question, prove that if $p+q\neq -1$, then

$$J_{p,q}=\frac{(a+bz)^p z^{q+1}}{p+q+1}+\frac{ap}{p+q+1}J_{p-1,q}$$
$$J_{p,q}=\frac{(a+bz)^{p+1} z^{q}}{b(p+q+1)}-\frac{aq}{b(p+q+1)}J_{p,q-1}$$

?

Proof. We first show the first equation. Consider (1), we will use integration by parts to show the desired equation. That is,

$$\begin{aligned}
J_{p,q}&=\int (a+bz)^p z^q \,\mathrm{d}z\\
&=\int \left(\frac{a+bz}{z}\right)^p z^{p+q}\,\mathrm{d}z\\
&=\int \left(\frac{a+bz}{z}\right)^p \mathrm{d}\left(\frac{z^{p+q+1}}{p+q+1}\right)\\
&=\frac{(a+bz)^p}{z^p}\cdot\frac{z^{p+q+1}}{p+q+1}-\frac{1}{p+q+1}\int z^{p+q+1}\,\mathrm{d}\left(\frac{a+bz}{z}\right)^p\\
&=\frac{(a+bz)^p z^{q+1}}{p+q+1}-\frac{1}{p+q+1}\int z^{p+q+1}\,\mathrm{d}\left(\frac{a}{z}+b\right)^p\\
&=\frac{(a+bz)^p z^{q+1}}{p+q+1}-\frac{1}{p+q+1}\int z^{p+q+1}p\left(\frac{a}{z}+b\right)^{p-1}\left(-\frac{a}{z^2}\right)\mathrm{d}z\\
&=\frac{(a+bz)^p z^{q+1}}{p+q+1}+\frac{ap}{p+q+1}\int z^{p+q+1}\left(\frac{a+bz}{z}\right)^{p-1}z^{-2}\,\mathrm{d}z\\
&=\frac{(a+bz)^p z^{q+1}}{p+q+1}+\frac{ap}{p+q+1}\int (a+bz)^{p-1}z^{p+q+1-p+1-2}\,\mathrm{d}z\\
&=\frac{(a+bz)^p z^{q+1}}{p+q+1}+\frac{ap}{p+q+1}\int (a+bz)^{p-1}z^{q}\,\mathrm{d}z\\
&=\frac{(a+bz)^p z^{q+1}}{p+q+1}+\frac{ap}{p+q+1}J_{p-1,q} \qquad (4)
\end{aligned}$$

With (4) we can show the second equation. Similar to Question 4, we first consider the integral form of $J_{p-1,q}$ in (1), then substitute our result into (4), and the ending result is the desired formula.

$$\begin{aligned} J_{p-1,q} &= \int (a+bz)^{p-1} z^q \,\mathrm{d}z \\ &= \frac{1}{pb}\int z^q \,\mathrm{d}(a+bz)^p \\ &= \frac{1}{pb}(a+bz)^p z^q - \frac{q}{pb}\int (a+bz)^p z^{q-1}\,\mathrm{d}z \\ &= \frac{1}{pb}(a+bz)^p z^q - \frac{q}{pb} J_{p,q-1} \end{aligned} \tag{5}$$

Now, we substitute (5) back to (4), then the equation becomes

$$\begin{aligned} J_{p,q} &= \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{ap}{p+q+1} J_{p-1,q} \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{ap}{p+q+1}\left(\frac{1}{pb}(a+bz)^p z^q - \frac{q}{pb} J_{p,q-1}\right) \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{a}{p+q+1}\left(\frac{1}{b}(a+bz)^p z^q - \frac{q}{b} J_{p,q-1}\right) \\ &= \frac{(a+bz)^p z^{q+1}}{p+q+1} + \frac{a}{p+q+1}\frac{(a+bz)^p z^q}{b} - \frac{aq}{b(p+q+1)} J_{p,q-1} \\ &= \frac{bz(a+bz)^p z^q}{b(p+q+1)} + \frac{a(a+bz)^p z^q}{b(p+q+1)} - \frac{aq}{b(p+q+1)} J_{p,q-1} \\ &= \frac{(a+bz)^{p+1} z^q}{b(p+q+1)} - \frac{aq}{b(p+q+1)} J_{p,q-1} \end{aligned}$$

which is precisely the equation required, as needed.

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Now with all the information obtained above, let's study an example to get some feelings. Define for $m \in \mathbb{Z}$ the integral

$$H_m = \int \frac{x^m}{\sqrt{1-x^2}}\,\mathrm{d}x, \quad x > 0.$$

In the sequel, we will first transform H_m into a binomial integral. Since the binoimal integral can be computed by iteration, it in turn provides a recursive relation for $H_m, m \in \mathbb{Z}$. To conclude, we aim at completely (at least formally) solve the prolem of integrate H_m in finite terms for any $m \in \mathbb{Z}$.

Question 6

By introducing the substitution $z = x^2$, show that

$$H_m = J_{-\frac{1}{2},\frac{m-1}{2}}.$$

?

Proof. Let $z = x^2$, then $\mathrm{d}z = 2x\,\mathrm{d}x$. Replace x with z we have

$$H_m = \int \frac{z^{\frac{m}{2}}}{\sqrt{1-z}} \cdot \frac{\mathrm{d}z}{2\sqrt{z}} = \frac{1}{2}\int \frac{z^{\frac{m-1}{2}}}{\sqrt{1-z}}\,\mathrm{d}z = \int \frac{z^{\frac{m-1}{2}}}{\sqrt{4-4z}}\,\mathrm{d}z = \int (4-4z)^{-\frac{1}{2}} \cdot z^{\frac{m-1}{2}}\,\mathrm{d}z.$$

Let $a = 4, b = -4, p = -\frac{1}{2}, q = \frac{m-1}{2}$, then we have

$$H_m = \int (4-4z)^{-\frac{1}{2}} \cdot z^{\frac{m-1}{2}}\,\mathrm{d}z = J_{-\frac{1}{2},\frac{m-1}{2}}.$$

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Question 7

Prove that $\forall m > 1$,

$$H_m = -\frac{1}{m}x^{m-1}\sqrt{1-x^2} + \frac{m-1}{m}H_{m-2}$$

?

Hint. Perhaps use consequence in question 5.

♡

Proof. Assume $m > 1$. Let $a = 4, b = -4, p = -\frac{1}{2}, q = \frac{m-1}{2}$(note that here $p+q \neq -1$), $z = x^2$, then we have

$$\begin{aligned}
H_m = J_{-\frac{1}{2},\frac{m-1}{2}} &= \frac{(a+bz)^{p+1}z^q}{b(p+q+1)} - \frac{aq}{b(p+q+1)}J_{-\frac{1}{2},\frac{(m-2)-1}{2}} \\
&= \frac{(4-4z)^{\frac{1}{2}}z^{\frac{m-1}{2}}}{-4(\frac{m-1}{2}-\frac{1}{2}+1)} - \frac{4\left(\frac{m-1}{2}\right)}{-4(\frac{m-1}{2}-\frac{1}{2}+1)}\int (4-4z)^{-\frac{1}{2}}z^{\frac{(m-2)-1}{2}}\,\mathrm{d}z \\
&= \frac{2(1-x^2)^{\frac{1}{2}}x^{m-1}}{-2(m-1-1+2)} - \frac{2(m-1)}{-2(m-1-1+2)}\int \frac{1}{2}(1-x^2)^{-\frac{1}{2}}x^{(m-2)-1}2x\,\mathrm{d}x \\
&= -\frac{(1-x^2)^{\frac{1}{2}}x^{m-1}}{m} - \frac{m-1}{-m}\int (1-x^2)^{-\frac{1}{2}}x^{m-2}\,\mathrm{d}x \\
&= -\frac{1}{m}x^{m-1}\sqrt{1-x^2} + \frac{m-1}{m}\int \frac{x^{m-2}}{\sqrt{1-x^2}}\,\mathrm{d}x \\
&= -\frac{1}{m}x^{m-1}\sqrt{1-x^2} + \frac{m-1}{m}H_{m-2}
\end{aligned}$$

since we have chosen an arbitrary $m > 1$, thus we have shown the equaiton holds for all $m > 1$, as required.

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Question 8

Compute H_0, H_1 explicitely. ?

$$\begin{aligned} H_0 &= \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= \arcsin x + C \end{aligned} \tag{6}$$

Note that the equality follows by formula, which can be verified by taking the derivative of (6). ☺

$$\begin{aligned} H_1 &= \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= \frac{1}{2}\int \left(1-x^2\right)^{-\frac{1}{2}}\mathrm{d}\left(x^2\right) \\ &= -\left(1-x^2\right)^{\frac{1}{2}} + C \end{aligned}$$

☺

Question 9

Based on the previous sub-question(s), conclude that $\forall m \in \mathbb{Z}_{\geq 0}$, H_m can be integrated in finite terms. ?

Proof. We show by strong induction. From Question 8, we have shown H_0, H_1 can be integrated in finite terms. Let H_0, H_1 be the base cases. Assume H_n holds when $n \in \{0, 1, \cdots, m\}$ for some $m \in \mathbb{Z}_{\geq 0}$. Then, we want to show H_{m+1} can also be integrated in finite terms. To this end, consider 2 cases.

When $m + 1 \in \{0, 1\}$, this is immediately covered by our base cases.

When $m + 1 \geq 2$, by Question 7, we have

$$H_{m+1} = -\frac{1}{m+1} x^m \sqrt{1 - x^2} + \frac{m}{m+1} H_{m-1},$$

since $m > m - 1 \geq 0$, this means by our inductive hypothesis H_{m-1} can be integrated in finite terms, this gives H_{m+1} can also be integrated in finite terms by our equation.

Therefore, by strong induction, we have shown H_m can be integrated in finite terms for all $m \in \mathbb{Z}_{\geq 0}$, as required. QUOD ERAT DEM■

Question 10

Prove that $\forall m < -1$,

$$H_m = \frac{x^{m+1}\sqrt{1-x^2}}{m+1} + \frac{2+m}{1+m}H_{m+2}$$

?

Hint. Perhaps use consequence in question 4.

♡

Proof. Choose any $m < -1$, fix $a = 4, b = -4, p = -\frac{1}{2}, q = \frac{m-1}{2}$, let $z = x^2$ where $\mathrm{d}z = 2x\,\mathrm{d}x$, then, by formula from Question 4 and Question 6 we have

$$\begin{aligned}
H_m = J_{-\frac{1}{2},\frac{m-1}{2}} &= \frac{(a+bz)^{\frac{1}{2}}z^{\frac{m+1}{2}}}{a(q+1)} - b\frac{p+q+2}{a(q+1)}J_{-\frac{1}{2},\frac{m+1}{2}} \\
&= \frac{(4-4z)^{\frac{1}{2}}z^{\frac{m+1}{2}}}{4(\frac{m-1}{2}+1)} - (-4)\frac{-\frac{1}{2}+\frac{m-1}{2}+2}{4(\frac{m-1}{2}+1)}\int (4-4z)^{-\frac{1}{2}}z^{\frac{m+1}{2}}\,\mathrm{d}z \\
&= \frac{2(1-x^2)^{\frac{1}{2}}x^{m+1}}{2(m+1)} + 2\frac{-1+m-1+4}{2m-2+4}\int \frac{1}{2}(1-x^2)^{-\frac{1}{2}}x^{m+1}2x\,\mathrm{d}x \\
&= \frac{x^{m+1}\sqrt{1-x^2}}{m+1} + \frac{2+m}{1+m}\int \frac{x^{m+2}}{\sqrt{1-x^2}}\,\mathrm{d}x \\
&= \frac{x^{m+1}\sqrt{1-x^2}}{m+1} + \frac{2+m}{1+m}H_{m+2}
\end{aligned}$$

which is precisely the equation required, as needed. QUOD ERAT DEM■

Question 11

Compute H_{-1}, H_{-2} explicitely. ?

Claim 1

$$\int \frac{1}{\sin x}\,\mathrm{d}x = \ln|\csc x - \cot x| + C$$

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Proof of Claim 1.

$$\begin{aligned}
\int \frac{1}{\sin x}\,\mathrm{d}x &= \int \frac{-\cos x + 1}{\sin x(-\cos x + 1)}\,\mathrm{d}x \\
&= \int \frac{-\frac{\cos x}{\sin^2 x} + \frac{1}{\sin^2 x}}{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}\,\mathrm{d}x \\
&= \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x}\,\mathrm{d}x \\
&= \int \frac{1}{\csc x - \cot x}\,\mathrm{d}\left(\csc x - \cot x\right) \\
&= \ln|\csc x - \cot x| + C
\end{aligned}$$

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Claim 2

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2-1}\right| + C$$

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Proof of Claim 2. Let $x = \sec u$, then $\mathrm{d}x = \sec u \tan u \,\mathrm{d}u$, thus we have

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{\sec^2 u - 1}} \sec u \tan u \,\mathrm{d}u \\
&= \int \sec u \,\mathrm{d}u \qquad (7) \\
&= \int \frac{1}{\cos u}\,\mathrm{d}u \\
&= -\int \frac{1}{\sin\left(\frac{\pi}{2}-u\right)}\,\mathrm{d}\left(\frac{\pi}{2}-u\right) \\
&= -\ln\left|\csc\left(\frac{\pi}{2}-u\right) - \cot\left(\frac{\pi}{2}-u\right)\right| + C \\
&= \ln\left|\frac{1}{x-\sqrt{x^2-1}}\right| + C \quad \text{By Pythagorean Theorem} \\
&= \ln\left|\frac{1}{x-\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right| + C \\
&= \ln\left|x+\sqrt{x^2-1}\right| + C
\end{aligned}$$

If we assume $x \in (-\infty,-1)\cup(1,\infty)$, then $u = \operatorname{arcsec} x \in (0,\pi)\setminus\{\frac{\pi}{2}\}$. Hence, consider 2 cases.

When $u \in (0,\pi/2)$, we have $\tan u > 0$, thus our step (7) is valid. When $u \in (\pi/2,\pi)$, despite $\tan u < 0$, in this case $x \in (-\infty,-1)$, becasue the funciton $\frac{1}{\sqrt{x^2-1}}$ is even, the integral of this case is same as the first case, hence we are allowed to change our $\frac{\tan u}{|\tan u|}$ into 1.

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$$\begin{aligned}
H_{-1} &= \int \frac{x^{-1}}{\sqrt{1-x^2}}\,\mathrm{d}x \\
&= \int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x \\
&= \int \frac{1}{x^2\sqrt{\frac{1}{x^2}-1}}\,\mathrm{d}x \\
&= -\int \frac{1}{\sqrt{\left(\frac{1}{x}\right)^2-1}}\,\mathrm{d}\left(\frac{1}{x}\right) \\
&= -\ln\left|\frac{1}{x}+\sqrt{\left(\frac{1}{x}\right)^2-1}\right| + C
\end{aligned}$$

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Using the substitution $x = \sin u$, and the pothagorean theorem, we have

$$\begin{aligned}
H_{-2} &= \int \frac{x^{-2}}{\sqrt{1-x^2}}\,\mathrm{d}x \\
&= \int \frac{1}{\sin^2 u \cos u} \cos u \,\mathrm{d}u \\
&= \int \csc^2 u \,\mathrm{d}u \\
&= -\cot u + C \\
&= -\frac{\sqrt{1-x^2}}{x} + C
\end{aligned}$$

☺

Question 12

Based on the previous sub-question(s), conclude that $\forall m \in \mathbb{Z}_{<0}$, H_m can be integrated in finite terms. ?

Proof. We perform a variation of strong induction on $\mathbb{Z}_{<0}$. That is, first let H_{-1}, H_{-2} be the base cases which are proven in Question 11 that they can be integrated in finite terms. Now assume the statement holds for all H_n where $n \in \{m, m+1, \cdots, -2, -1\}$ for some $m \in \mathbb{Z}_{<0}$. Then, we want to show H_{m-1} can also be integrated in finite terms. To this end, consider 2 cases.
When $m-1 \in \{-2, -1\}$, this is immediately covered by our base cases.
When $m-1 \leq -3$, by Question 10, we have

$$H_{m-1} = \frac{x^m\sqrt{1-x^2}}{m} + \frac{1+m}{m}H_{m+1},$$

since $m < m+1 \leq -1$, this means by our inductive hypothesis H_{m+1} can be integrated in finite terms, this gives H_{m-1} can also be integrated in finite terms by our equation. Therefore, by strong induction, we have shown H_m can be integrated in finite terms for all $m \in \mathbb{Z}_{<0}$, as required. QUOD ERAT DEM■

Conclusion: We have shown that for any $m \in \mathbb{Z}$, H_m can be integrated in finite terms.