Exercise 1

This exercise aims at computing the following indefinite integral

$$J_n = \int \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x, n \in \mathbb{N}$$

Question 1

Compute J_1.

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$$\begin{aligned} J_1 &= \int \frac{\mathrm{d}x}{x^2+a^2} \\ &= \int \frac{1}{a^2} \cdot \frac{\mathrm{d}x}{(\frac{x}{a})^2+1} \\ &= \frac{1}{a} \int \frac{\mathrm{d}(\frac{x}{a})}{(\frac{x}{a})^2+1} \\ &= \frac{1}{a} \arctan(\frac{x}{a}) + C \end{aligned}$$

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Question 2

Compute J_2.

Hint. Consider the substitution $x = \tan t$.

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$$J_2 = \int \frac{\mathrm{d}x}{(x^2+a^2)^2}$$

$$\text{Substitute } x = \tan t$$

$$\mathrm{d}x = \sec^2 t \cdot \mathrm{d}t$$

$$J_2 = \int \frac{\sec^2 t}{(\tan^2 t + a^2)^2}\,\mathrm{d}t$$

$$\mathrm{d}(\frac{1}{\tan^2 t + a^2}) = -\frac{2\tan t \sec^2 t}{(\tan^2 t + a^2)^2}\,\mathrm{d}t$$

$$J_2 = -\int \frac{1}{2\tan t}\,\mathrm{d}(\frac{1}{\tan^2 t + a^2})$$

$$= -\int \frac{1}{2x}\,\mathrm{d}(\frac{1}{x^2+a^2})$$

By integration by parts, we have

$$\begin{aligned}
J_2 &= -\frac{1}{2x(x^2+a^2)} + \int \frac{1}{x^2+a^2}\,\mathrm{d}(\frac{1}{2x}) \\
&= -\frac{1}{2x(x^2+a^2)} + \frac{1}{2}\int \frac{1}{x^2+a^2}\,\mathrm{d}(\frac{1}{x}) \\
&= -\frac{1}{2x(x^2+a^2)} - \frac{1}{2}\int \frac{1}{x^2+a^2}\frac{1}{x^2}\,\mathrm{d}x
\end{aligned}$$

By partial fractions, we have

$$\begin{aligned}
J_2 &= -\frac{1}{2x(x^2+a^2)} - \frac{1}{2}\int -\frac{1}{a^2(x^2+a^2)} + \frac{1}{a^2x^2}\,\mathrm{d}x \\
&= -\frac{1}{2x(x^2+a^2)} + \frac{1}{2}\int \frac{1}{a^2(x^2+a^2)}\,\mathrm{d}x - \frac{1}{2}\int \frac{1}{a^2x^2}\,\mathrm{d}x \\
&= -\frac{1}{2x(x^2+a^2)} + \frac{1}{2a^2}\int \frac{1}{x^2+a^2}\,\mathrm{d}x - \frac{1}{2a^2}\int \frac{1}{x^2}\,\mathrm{d}x \\
&= -\frac{1}{2x(x^2+a^2)} + \frac{1}{2a^3}\arctan(\frac{x}{a}) + \frac{1}{2a^2x} + C \\
&= \frac{x^2+a^2-a^2}{2xa^2(x^2+a^2)} + \frac{1}{2a^3}\arctan(\frac{x}{a}) + C \\
&= \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\arctan(\frac{x}{a}) + C
\end{aligned}$$

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Question 3

Prove that $\forall n \in \mathbb{N}$,

$$J_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx$$

Hint. Consider integration by parts.

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Proof. Pick an arbitrary $n \in \mathbb{N}$, we have that

$$\mathrm{d}\left(\frac{1}{(x^2+a^2)^n}\right) = -\frac{2nx}{(x^2+a^2)^{n+1}}\,\mathrm{d}x$$

$$\begin{aligned}\frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}\,\mathrm{d}x &= \frac{x}{(x^2+a^2)^n} + 2n\int -\frac{x}{2n}\,\mathrm{d}\left(\frac{1}{(x^2+a^2)^n}\right) \\ &= \frac{x}{(x^2+a^2)^n} - \int x\,\mathrm{d}\left(\frac{1}{(x^2+a^2)^n}\right)\end{aligned}$$

By integration by parts,

$$\begin{aligned}\frac{x}{(x^2+a^2)^n} - \int x\,\mathrm{d}\left(\frac{1}{(x^2+a^2)^n}\right) &= \frac{x}{(x^2+a^2)^n} - \frac{x}{(x^2+a^2)^n} + \int \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x \\ &= \int \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x \\ &= J_n\end{aligned}$$

Thus we have shown for any $n \in \mathbb{N}$ the equality holds, completing our proof. QUOD ERAT DEM■

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Question 4

Based on the previous sub-question, establish the recursive relation:

$$J_{n+1} = \frac{2n-1}{2n}\frac{1}{a^2}J_n + \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n}$$

From there, conclude that J_n is an integral of finite terms (i.e., it is an elementary function)

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$$\begin{aligned}
J_{n+1} &= \int \frac{1}{(x^2+a^2)^{n+1}}\,\mathrm{d}x \\
&= \frac{1}{a^2}\int \frac{(x^2+a^2)-x^2}{(x^2+a^2)^{n+1}}\,\mathrm{d}x \\
&= \frac{1}{a^2}\int \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x - \frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^{n+1}}\,\mathrm{d}x \\
&= \frac{1}{a^2}J_n - \frac{1}{a^2}\int -\frac{x}{2n}\,\mathrm{d}(\frac{1}{(x^2+a^2)^n}) \\
&= \frac{1}{a^2}J_n + \frac{1}{2na^2}\int x\,\mathrm{d}(\frac{1}{(x^2+a^2)^n}) \\
&= \frac{1}{a^2}J_n + \frac{1}{2na^2}\left[\frac{x}{(x^2+a^2)^n} - \int \frac{1}{(x^2+a^2)^n}\,\mathrm{d}x\right] \\
&= \frac{1}{a^2}J_n + \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} - \frac{1}{2na^2}J_n \\
&= J_n[\frac{1}{a^2} - \frac{1}{2na^2}] + \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} \\
&= (1-\frac{1}{2n})\frac{1}{a^2}J_n + \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} \\
&= \frac{2n-1}{2n}\frac{1}{a^2}J_n + \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n}
\end{aligned}$$

as needed. ☺

Question 5

Compute J_3.

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By Q4's formula, combining with our result of J_2, we have that

$$\begin{aligned}
J_3 &= \frac{4-1}{4}\frac{1}{a^2}J_2 + \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} \\
&= \frac{4-1}{4}\frac{1}{a^2}[\frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\arctan(\frac{x}{a}) + C] + \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} \\
&= \frac{3}{4a^2}\frac{x}{2a^2(x^2+a^2)} + \frac{3}{4a^2}\frac{1}{2a^3}\arctan(\frac{x}{a}) + \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + C \\
&= \frac{1}{4a^2}\left[\frac{3}{2a^3}\arctan(\frac{x}{a}) + \frac{3x}{2a^2(x^2+a^2)} + \frac{x}{(x^2+a^2)^2}\right] + C
\end{aligned}$$

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