I, Joseph Siu, affirm that this assignment represents entirely my own efforts. I confirm that:

- I have not copied any portion of this work.
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- This is the final version of my assignment and not a draft.
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Theorem 1

Recall that we have proved the following theorem in class:
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then the following statements are equivalent:

1. $f \in \mathfrak{R}[a, b]$.
2. (Darboux Criterion) $\overline{I}(f) = \underline{I}(f)$.
3. $\forall \varepsilon > 0, \forall \delta > 0, \exists \Gamma \in \Omega_{[a,b]}$ s.t.
$$\sum_{\substack{0\leq k\leq n-1\\ V_f(I_k)>\varepsilon}} |I_k| < \delta.$$
4. (Du Bois Raymond Criterion) $\forall \varepsilon > 0, \forall \delta > 0, \exists n \in \mathbb{N}$ and $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, such that
$$\left(D_f(\varepsilon, [a, b]) \subseteq \bigcup_{1\leq k\leq n} (a_k, b_k)\right) \wedge \left(\sum_{k=1}^{n} (b_k - a_k) < \delta\right).$$
5. (Lebesgue Criterion) f is continuous almost everywhere on $[a, b]$.

Question 1

Prove that $5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 5$. In each part, you should not use criterion other than the two involved in the statement. *?*

Lemma 1

Fix $\varepsilon > 0$. If $x \in D_f(\varepsilon, [a, b])$, then $V_f([c, d]) > \varepsilon$ for any $[c, d] \subseteq [a, b]$ containing x. ♠

Proof. If $x \in D_f(\varepsilon, [a, b])$, by definition we have $V_f(x) > \varepsilon$, that is, $\eta := \lim_{\delta\to 0} V_f(I_\delta(x)) > \varepsilon$. Fix an open neighborhood $I_\varsigma(x) \subseteq [c, d]$ of x. Since for all $0 < \delta' < \delta$ we have $I_\delta(x) \supseteq I_{\delta'}(x)$, which implies $V_f(I_\delta(x)) \geq V_f(I_{\delta'}(x))$. Thus, since all sequences of decreasing δ are monotonely decreasing, we have that $\eta < V_f(I_\delta(x))$ for all $\delta > 0$. Thus, $V_f(I_\varsigma(x)) \geq \eta > \varepsilon$, this implies our proof. *quod erat dem*■

Lemma 2

If $\forall x \in [a,b], V_f(x) \leq \frac{\varepsilon}{3}$, then for all positive epsilon there exists a finite partition Γ of $[a,b]$ such that $V_f(I_i) \leq \varepsilon$ for all partitioned intervals I_i of Γ.

♠

Proof. $V_f(x) \leq \frac{\varepsilon}{3}$ implies there exists an open neighborhood $I_\delta(x)$ of x such that $V_f(I_\delta(x)) \leq \frac{\varepsilon}{3}$. Then, we can see $\{I_\delta(x)\}_{x\in[a,b]}$ forms an open cover of $[a,b]$. Moreover, since $[a,b]$ is closed and bounded thus compact, by Borel-Lebesgue / Heine-Borel there exists a finite subcover $\{J_i\}_{1\leq i\leq n}$ of $\{I_\delta(x)\}_{x\in[a,b]}$. Then, construct a partition Γ based on the endpoints of the intervals in the finite subcover $\{J_i\}_{1\leq i\leq n}$.

Now, for any partitioned interval $[\alpha,\beta]$ of Γ, we have $(\alpha,\beta) \subseteq J_i$ for some $1 \leq i \leq n$. Since $V_f(I_\delta(\alpha)) \leq \frac{\varepsilon}{3}$, $V_f(J_i) \leq \frac{\varepsilon}{3}$, and $V_f(I_\delta(\beta)) \leq \frac{\varepsilon}{3}$, we have $V_f([\alpha,\beta]) \leq \sup\limits_{x,y\in I_\delta(\alpha)\cup J_i\cup I_\delta(\beta)} |f(x)-f(y)| \leq \dfrac{\varepsilon}{3}+\dfrac{\varepsilon}{3}+\dfrac{\varepsilon}{3} = \varepsilon$.

Since we have consturcted such finite partition Γ for arbitrary $\varepsilon > 0$, this completes our proof. *quod erat dem* ■

Proof. $(5 \Rightarrow 4)$

Assume f is continuous almost everywhere on $[a,b]$. By definition of almost everywhere, this implies the set of discontinuous points of f forms a null set D.

By definition of continuity, we have

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I_\delta(x), |f(x)-f(x_0)| < \varepsilon.$$

Fix $\varepsilon = \frac{\xi}{4}$, then $V_f(I_\delta(x)) \leq \frac{\xi}{2}$.

For continuous points x in $[a,b]$, we construct the open neighborhood $I_\delta(x)$ as above, let I_x denote such open neighborhood of x. For discontinuous points x' in $[a,b]$, since $x' \in D$ and D forms a null set, by definition this implies

$$\forall \delta > 0, \exists \{I_i\}_{i\in\mathbb{N}}, \left(D \subseteq \bigcup_{i\in\mathbb{N}} I_i\right) \wedge \left(\sum_{i=1}^{\infty} |I_i| < \delta\right).$$

Fix $\delta = \frac{\xi}{4}$, then we have $\exists \{I_i\}_{i\in\mathbb{N}}, \left(D \subseteq \bigcup_{i\in\mathbb{N}} I_i\right) \wedge \left(\sum_{i=1}^{\infty} |I_i| < \frac{\xi}{4}\right)$, fix such $\{I_i\}_{i\in\mathbb{N}}$. Now, let $I_{x'}$ be the open interval that covers x'.

So, since $[a,b]$ is closed and bounded that compact, and $\{I_x\}_{x\in[a,b]\setminus D} \cup \{I_{x'}\}_{x'\in D}$ forms a cover of $[a,b]$, hence by Borel-Lebesgue / Heine-Borel Theorem this implies a finite subcover $\{J_i\}_{1\leq i\leq N}$ of $\{I_x\}_{x\in[a,b]\setminus D} \cup \{I_{x'}\}_{x'\in D}$ where N is the number of intervals in the finite subcover.

Now, let Γ be the partition based on the endpoints of the intervals in the finite subcover $\{J_i\}_{1\leq i\leq N}$, consider 2 of the cases of the partitioned intervals $[\alpha,\beta]$:

Case 1. $(\alpha,\beta) \subseteq I_x$ for some $x \in [a,b] \setminus D$.

In this case we have $V_f([\alpha,\beta]) \leq V_f(I_x) \leq \frac{\xi}{2} < \xi$. That is, $D_f(\xi,[a,b]) \cap [\alpha,\beta] = \varnothing$. We may ignore these intervals for proving Du Bois Raymond Criterion (4).

Case 2. $(\alpha,\beta) \subseteq I_{x'}$ for some $x' \in D$.

In this case we have

$$([\alpha,\beta] \subseteq I_{x'}) \wedge \left(|I_{x'}| < \frac{\xi}{4}\right) \implies (\beta-\alpha) < \frac{\xi}{4}.$$

Let M denote the number of all such $[\alpha,\beta]$, since the cover $\{J_i\}_{1\leq i\leq N}$ is finite, this implies the partition of Γ is finite, thus M also needs to be finite.

Let $[\alpha_i,\beta_i]$ denote the i^{th} such interval where $1 \leq i \leq M$, this is allowed because of the order of the partition of Γ.

Since there are only finitely many such α and β by our partition Γ, let N be the set containing all partitions of Γ, i.e. , containing a, b, and all α, β. Then, since N is a collection of finitely many points, we can see N is also a null set. Hence, by definition of null set, we have

$$\forall \delta > 0, \exists \{I_i\}_{i\in\mathbb{N}}, \left(N \subseteq \bigcup_{i\in\mathbb{N}} I_i\right) \wedge \left(\sum_{i=1}^{\infty} |I_i| < \delta\right).$$

Fix $\delta = \frac{\xi}{4}$, let $\{K_i\}_{i\in\mathbb{N}}$ be the open cover that covers N and $\sum_{i=1}^{\infty} |K_i| < \frac{\xi}{4}$. Moreover, since N is finite and closed thus bounded. By Borel-Lebesgue / Heine-Borel Theorem, we can find a finite subcover $\{K_i'\}_{1\le i\le P}$ of $\{K_i\}_{i\in\mathbb{N}}$ where P is the number of intervals in the finite subcover.

Now, combining the above 2 cases we have shown that

$$\left(D_f(\xi, [a,b]) \subseteq \left(\bigcup_{1\le i\le M} (\alpha_i, \beta_i)\right) \bigcup \left(\bigcup_{1\le i\le P} K_i'\right)\right) \wedge \left(\sum_{i=1}^{M} (\beta_i - \alpha_i) + \sum_{i=1}^{P} |K_i'| < \frac{\xi}{4} + \frac{\xi}{4} < \xi\right).$$

Hence, since our ξ is arbitrary, for arbitrary $\varepsilon, \delta > 0$ by letting $\xi := \min\{\varepsilon, \delta\}, n := M + P$, leting $(a_1, b_1), \ldots, (a_n, b_n)$ be the intervals that covers $D_f(\xi, [a, b])$, we have shown that

$$\left(D_f(\varepsilon, [a,b]) \subseteq D_f(\xi, [a,b]) \subseteq \bigcup_{1\le k\le n} (a_k, b_k)\right) \wedge \left(\sum_{k=1}^{n} (b_k - a_k) < \xi \le \delta\right).$$

Therefore, $\forall \varepsilon > 0, \forall \delta > 0, \exists n \in \mathbb{N}$ and $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, such that

$$\left(D_f(\varepsilon, [a,b]) \subseteq \bigcup_{1\le k\le n} (a_k, b_k)\right) \wedge \left(\sum_{k=1}^{n} (b_k - a_k) < \delta\right),$$

this completes our proof.

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Proof. $(4 \Rightarrow 3)$

Fix $\frac{\varepsilon}{3} > 0$, $\delta > 0$. By Criterion 4 there exist a natural number n and a finite open cover $\{J_k\}_{1\le k\le n}$ such that

$$\left(D_f(\frac{\varepsilon}{3}, [a,b]) \subseteq \bigcup_{1\le k\le n} J_k\right) \wedge \left(\sum_{k=1}^{n} |J_k| < \delta\right).$$

Fix such $n \in \mathbb{N}$, then there exists $n_1 \in \mathbb{N}$ and a finite open cover $\{J_k'\}_{1\le k\le n_1}$ such that

$$\left(D_f(\frac{\varepsilon}{3}, [a,b]) \subseteq \bigcup_{1\le k\le n_1} J_k'\right) \wedge \left(\sum_{k=1}^{n_1} |J_k'| < \frac{\delta}{n}\right).$$

Let Γ be the partition based on the endpoints of $\{J_k'\}_{1\le k\le n}$. Split the partitioned intervals $[\alpha, \beta]$ into 2 parts:

Case 1. $[\alpha, \beta] \cap D_f(\frac{\varepsilon}{3}, [a, b]) = \varnothing$.

By Lemma 2, this implies there exists a finite partition of $[\alpha, \beta]$ such that for all partitioned interval $[\gamma, \zeta]$ we have $V_f([\gamma, \zeta]) \le \varepsilon$. Refine our Γ to include these partitions, let Γ^* denote the refined partition.

Let m_1 denote the total number of all such intervals' partitioned intervals. Let $[\alpha_k, \beta_k]$ denote the endpoints of each interval where $0 \le k \le m_1 - 1$. After we refine our original Γ to include the finite partitions for all the "$[\alpha, \beta]$", we can see $\sum_{\substack{0\le k\le m_1-1 \\ V_f([\alpha_k,\beta_k])>\varepsilon}} |\beta_k - \alpha_k| = 0$ since any partitioned interval $[\alpha_k, \beta_k]$ has the property that $V_f([\alpha_k, \beta_k]) \le \varepsilon$.

Case 2. $[\alpha', \beta'] \cap D_f(\frac{\varepsilon}{3}, [a, b]) \neq \varnothing$, that is, $(\alpha', \beta') \subseteq J_i \cup J_j \cup J_k$ for some $1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n$ by our construction of partitions.

Since there can only be n such intervals. By assumption the total length of all such intervals is less than δ. Namely, let m_2 denote the number of all such intervals of Γ (clearly $m_2 \leq n$), then $\sum\limits_{1 \leq k \leq m_2} |\beta'_k - \alpha'_k| \leq m_2 \cdot \sum\limits_{k=1}^{n_1} |J'_k| < n \cdot \dfrac{\delta}{n} = \delta$.

Since these 2 cases cover the entire interval, we conclude

$$\sum_{\substack{0 \leq k \leq n-1 \\ V_f(I_k) > \varepsilon}} |I_k| \leq 0 + \sum_{\substack{0 \leq k \leq m_2 - 1 \\ V_f(\alpha'_k, \beta'_k]) > \varepsilon}} |\beta'_k - \alpha'_k| < \delta.$$

Since ε and δ are arbitrary, and we have constructed such Γ, this completes our proof. *quod erat dem* ■

Proof. $(3 \Rightarrow 2)$

For all $\varepsilon > 0$ and $\delta > 0$, by Criterion 3, there exists a partition Γ such that

$$\sum_{\substack{0 \leq k \leq n-1 \\ V_f(I_k) > \delta}} |I_k| < \frac{\varepsilon}{8|M| + 1},$$

where f is bounded by M (integrable implies boundedness).

Fix $\delta = \frac{\varepsilon}{4(b-a)+1}$.

Then, by the supremum and infimum definition we have

$$\overline{I}(f) \leq \sum_{k=0}^{n-1} M_{x_k, x_{k+1}} \Delta x_k$$

$$\sum_{k=0}^{n-1} m_{x_k, x_{k+1}} \Delta x_k \leq \underline{I}(f)$$

$$\begin{aligned} \overline{I}(f) - \underline{I}(f) &\leq \sum_{k=0}^{n-1} M_{x_k, x_{k+1}} \Delta x_k - \sum_{k=0}^{n-1} m_{x_k, x_{k+1}} \Delta x_k \\ &\leq \sum_{k=0}^{n-1} (M_{x_k, x_{k+1}} - m_{x_k, x_{k+1}}) \Delta x_k \\ &\leq \sum_{k=0}^{n-1} (V_f([x_k, x_{k+1}])) \Delta x_k, \end{aligned}$$

now we seperate the intervals into $V_f(I_k) > \delta$ and $V_f(I_k) \leq \delta$ where $I_k = [x_k, x_{k+1}]$, then:

$$\leq \sum_{\substack{0 \leq k \leq n-1 \\ V_f(I_k) > \delta}} (V_f(I_k)) \Delta x_k + \sum_{\substack{0 \leq k \leq n-1 \\ V_f(I_k) \leq \delta}} (V_f(I_k)) \Delta x_k,$$

since f is bounded by M, by our assumption then we have:

$$
\begin{aligned}
&\leq 2M \sum_{\substack{0\leq k\leq n-1 \\ V_f(I_k)>\delta}} \Delta x_k + \sum_{\substack{0\leq k\leq n-1 \\ V_f(I_k)\leq\delta}} (V_f(I_k))\Delta x_k \\
&\leq \frac{\varepsilon}{4} + \sum_{\substack{0\leq k\leq n-1 \\ V_f(I_k)\leq\delta}} (V_f(I_k))|I_k|, \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4(b-a)+1} \sum_{\substack{0\leq k\leq n-1 \\ V_f(I_k)\leq\delta}} |I_k| \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&\leq \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we conclude $\overline{I}(f) - \underline{I}(f) = 0$, therefore $\overline{I}(f) = \underline{I}(f)$ as needed.

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Proof. $(2 \Rightarrow 1)$

By Proposition 2.3 we have for any two partitions $\Gamma_1, \Gamma_2 \in \Omega[a,b]$, $\underline{S}(f,\Gamma_1) \leq \overline{S}(f,\Gamma_2)$. So, by definition for any marked partition $(\Gamma,\eta) \in \Omega^*[a,b]$ we have

$$\underline{S}(f,\Gamma_1) \leq \sum_{i=0}^{n-1} f(\eta_i)\Delta x_i \leq \overline{S}(f,\Gamma_2).$$

By letting $||\Gamma|| \to 0$, we have

$$\underline{I}(f) \leq \sum_{i=0}^{n-1} f(\eta_i)\Delta x_i \leq \overline{I}(f).$$

Since $\overline{I}(f) = \underline{I}(f)$, we can claim that $f \in \Re[a,b]$, moreover $\int_a^b f(x)\,\mathrm{d}x = \overline{I}(f) = \underline{I}(f)$:

When $||\Gamma|| < \delta$, this is equivalent to $||\Gamma|| \to 0$, and by squeeze theorem we have that $0 - \frac{\varepsilon}{2} \leq \sum_{i=0}^{n-1} f(\eta_i)\Delta x_i - \int_a^b f(x)\,\mathrm{d}x \leq 0 + \frac{\varepsilon}{2}$, which gives $\left|\sum_{i=0}^{n-1} f(\eta_i)\Delta x_i - \int_a^b f(x)\,\mathrm{d}x\right| \leq \frac{\varepsilon}{2} < \varepsilon$, since (Γ,η) is arbitrary, this gives the definition of Riemann integrability, as needed.

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Proof. $(1 \Rightarrow 5)$

We will prove the contrapositive. Assume the discontinuous points of f do not form a null set N, namely

$$\exists \varepsilon_1 > 0, \forall \{J_i\}_{i\in\mathbb{N}} \text{ we have } \left(N \not\subseteq \bigcup_{i\in\mathbb{N}} J_i\right) \vee \left(\sum_{i=1}^{\infty} |J_i| \geq \varepsilon_1\right).$$

Fix such $\varepsilon_1 > 0$. Let $\delta > 0$ be arbitrary, let (Γ,η) be arbitrary such that $||\Gamma|| \leq \delta$, let M denotes the number of partitioned intervals.

Let $\{I_k\}_{k\in M}$ be the partitioned intervals of Γ. Let $a = \min\{V_f(I_k)\}_{k\in M} \geq 0$, here $a \neq 0$ because of the non-empty discontinuous set N. then, consider $\varepsilon := a(\varepsilon_1) > 0$.

By specialization, and ignore a, b if they are discontinuous, we may consturct an open cover $\{J_i\}_{i\in\mathbb{N}}$ of the discontinuous set $N \setminus \{a,b\}$ such that $\sum_{i=1}^{\infty} |J_i| \geq \varepsilon_1$ and does not cover a, b, since N do not form a null set, so is $N \setminus \{a,b\}$.

Since $\sum_{i=0}^{M-1}|I_i| \geq \sum_{i=1}^{\infty}|J_i|$ (becasue of the forall quantifier, we may assume the union of the cover $\{J_i\}_{i\in\mathbb{N}}$ is contained within $[a,b]$), we can see that

$$\begin{aligned}\sum_{i=0}^{M-1} V_f(\varepsilon, I_i)|I_i| &\geq a\sum_{i=0}^{M-1}|I_i| \\ &\geq a\sum_{i=1}^{\infty}|J_i| \\ &\geq a\varepsilon_1 \geq \varepsilon\end{aligned}$$

So, by the negation of the Riemann integrability in terms of Aggregated Oscillation, we have shown that

$$\exists\varepsilon>0, \forall\delta>0, \exists(\Gamma,\eta)\in\Omega^*_{[a,b]}, ||\Gamma||\leq\delta\wedge\sum_{i=0}^{n-1}V_f(I_i)|I_i|\geq\varepsilon,$$

hence the contrapositive is true, which implies $(1 \Rightarrow 5)$ as needed.

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Question 2

Using Lebesgue Criterion to study the Riemann integrability for the five examples in assignment 4. ?

1. Since $D(x)$ is continuous nowhere (as proven in MAT157 Homework), then the sum of any open cover covering $[0,1]$ must be greater than $\varepsilon = \frac{1}{2}$. By Lebesgue Criterion this implies that $D(x)$ is not Riemann integrable.
2. Since $T(x)$ is only discontinuous when $x \in \mathbb{Q}$, and $\mathbb{Q}$ is a countable set, thus is also a null set. Hence, by definition of null set and continuous almost everywhere, we conclude $T(x)$ is Riemann integrable by Lebesgue Criterion.
3. Similarly, we can also see that $H(x)$ is only discontinuous when $x = \frac{1}{n}$ for some $n \in \mathbb{N}$ or $x = 0$ (by definition of floor function). Since $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ is a countable set, and $\{0\}$ is finite, the union of these sets is countable thus a null set. Hence, $H(x)$ is contnuous almost everywhere and is Riemann integrable by Lebesgue Criterion.
4. $G(x)$ is also discontinuous whenever $x = \frac{1}{n}$ for some $n \in \mathbb{N}$ or $x = 0$ (by observering the values that $\sin\left(\frac{\pi}{x}\right)$ changes its sign), same as $H(x)$, we may conclude that $G(x)$ is Riemann integrable by Lebesgue Criterion.
5. $\ln(\frac{1}{x}) = -\ln(x)$ is also continuous everywhere except at $x = 0$. So, by MAT157 since $\sin x$ is continuous everywhere, we know $\sin\left(\ln\left(\frac{1}{x}\right)\right)$ is continuous everywhere on $(0, 1]$, thus it can be discontinuous at most at $x = 0$ which is a null set. Hence, $\sin\left(\ln\left(\frac{1}{x}\right)\right)$ is Riemann integrable by Lebesgue Criterion.

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Question 3

Let $f, g : [a, b] \to [a, b]$. Fill in the blanks below regarding the integrability of $g \circ f$ and justify your answers, by giving either a proof or a counter-example.

	$f \in \mathcal{C}[a,b]$	$f \in \mathfrak{R}[a,b]$
$g \in \mathcal{C}[a,b]$	Yes	Yes
$g \in \mathfrak{R}[a,b]$	No	No

Table 1: Integrability of $g \circ f$ under different assumptions.

?

Proof.

1. $f \in \mathcal{C}[a,b]$:
   (a). $g \in \mathcal{C}[a,b]$: Since the composition of continuous functions is continuous, so $g \circ f$ is continuous everywhere. Thus, by Lebesgue Criterion, $g \circ f$ is Riemann integrable.
   (b). $g \in \mathfrak{R}[a,b]$: See below.
2. $f \in \mathfrak{R}[a,b]$:
   (a). $g \in \mathcal{C}[a,b]$: Let $x \in [a,b]$ be such that $V_f(x) = 0$, then $g \circ f$ is also continuous at x. Since such x are almost everywhere, we conclude $g \circ f$ is continuous almost everywhere and thus Riemann integrable.
   (b). $g \in \mathfrak{R}[a,b]$: Consider the example:

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q}, q > 0, \gcd(p,q) = 1 \\ 0, & \text{otherwise} \end{cases}, g(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}.$$

   Then, we can see that both $f, g \in \mathfrak{R}[a,b]$, but $(g \circ f)(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$, which is not integrable (the Dirichlet function, as shown in the previous homework).

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For $1.b.$, we show $g \circ f$ is false by constructing a counter-example:

Let $[a, b] = [0, 1]$. We want to show that for some $g \in \mathfrak{R}[0, 1]$, $f \in \mathcal{C}[0, 1]$, $g \circ f$ is not Riemann integrable. To this end, we first define

$$g : [0, 1] \to [0, 1], g(y) = \begin{cases} 1, & y \neq 0 \\ 0, & y = 0 \end{cases}.$$

Then, we want to construct a function f that is both continuous on $[0, 1]$ and has uncountably disconnected many points $x \in [0, 1]$ such that $f(x) = 0$, or $f(x) = 1$ seperately. (so that $g \circ f$ is discontinuous uncountably many points thus does not satisfy Lebesgue Criterion for Riemann Integrability). So, for simplicity we will construct the case when $f(x) = 0$ based on the fat cantor set (the Smith–Volterra–Cantor set) FC.

Consider the recursively defined set FC as follows:

$FC_0 = [0, 1]$.

1. We take out $(\frac{3}{8}, \frac{5}{8})$, i.e. $\frac{1}{4}$ from the middle of FC_0: $FC_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$ (the length of $(\frac{3}{8}, \frac{5}{8})$ is same as $[\frac{3}{8}, \frac{5}{8}]$ due to $\{\frac{3}{8}, \frac{5}{8}\}$ is a null set / measure zero).

2. For each interval in FC_1, we take out the middle $\frac{1}{16}$ of each interval: $FC_2 = [0, \frac{5}{32}] \cup [\frac{7}{32}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{25}{32}] \cup [\frac{27}{32}, 1]$.

$\vdots$

n. For each interval in FC_{n-1}, we take out the middle $\frac{1}{4^n}$ of each interval (totally 2^{n-1} such intervals), the remaining set is FC_n.

In this way if we let $FC = \bigcap_{n=0}^{\infty} FC_n$, then FC is the fat cantor set.

We can verify the following properties of FC:

Lemma 1

The 'length' of FC on [0,1] is $\frac{1}{2}$.

Proof. We consider the length of the intervals removed at each step of the construction. At the n-th step, the length of 2^{n-1} intervals removed is $\frac{2^{n-1}}{4^n} = \frac{1}{2^{n+1}}$, thus the total length of the intervals removed is $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$, which implies the length of FC is $\frac{1}{2}$.

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Lemma 2

FC is totally disconnected and is closed.

Proof. Let $x, y \in FC$ be arbitrary such that $x \neq y$, w.l.o.g. we let $x < y$. Moreover all points in FC are endpoints of the intervals in the construction thus so are x, y.

To obtain a contradiction, assume x and y are connected, that is, $[x, y] \subseteq FC$. However, by our construction of $[x, y] \subseteq FC$, such $[x, y]$ always has to take out a middle interval from $[x, y]$ by some positive length interval to get a new set FC' such that $FC' \subsetneq FC$, which contradicts the fact that FC is the intersection of all FC_n. Thus, FC is totally disconnected.

Moreover, since FC is constructed by taking out open intervals from $[0, 1]$, this implies FC is closed.

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Now, we construct $f(x)$ as follows:

$$f(x) = \begin{cases} 0, & x \in FC \\ -(x - x_1)(x - x_2), & x \notin FC \text{ where } x \in (x_1, x_2) \subseteq [0, 1] \setminus FC \text{ s.t. } x_1, x_2 \in FC \end{cases}.$$

Claim 1

f is defined for all $x \in [0,1]$.

Proof. It suffices to show that whenever $x \notin FC$, there always exists an open interval (x_1, x_2) such that $x_1, x_2 \in FC$ and $(x_1, x_2) \subseteq [0,1] \setminus FC$.

By our construction of FC, if $x \notin FC$, this implies there exists an open interval (x_1, x_2) such that this entire open interval is 'removed' from FC, thus $(x_1, x_2) \subseteq [0,1] \setminus FC$, showing all the middle points are also removed from FC.

Moreover, since by our construction, we can see the boundary / end points of FC are all in FC (we are always keeping the endpoints from the previous generation), thus we have $x_1, x_2 \in FC$.

Since both conditions must be satisfied when $x \notin FC$, we conclude that $f(x)$ is defined for all $x \in [0,1]$.

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Claim 2

f is continuous on $[0,1]$.

Proof. We consider the cases when $x \in FC$ and $x \notin FC$ separately.

1. When $x \in FC$, since by our Lemma 2 FC is totally disconnected and is closed, this implies there exists $x_1, x_3 \in FC$ such that $x_1 < x < x_3$, and $(x_1, x) \subseteq [0,1] \setminus FC$, $(x, x_3) \subseteq [0,1] \setminus FC$, $x_1, x, x_3 \in FC$. Then, we can see the left limit of $f(x)$ is $\lim_{x' \to x^-} f(x') = -(x' - x_1)(x' - x) = 0$ and the right limit of $f(x)$ is $\lim_{x' \to x^+} f(x') = -(x' - x)(x' - x_3) = 0$, thus since both the limit of $f(x)$ is 0 and $f(x) = 0$, we conclude $f(x)$ is continuous at x.

2. When $x \notin FC$, we have $f(x) = -(x - x_1)(x - x_2)$ for some $x_1, x_2 \in FC$ such that $x \in (x_1, x_2) \subseteq [0,1] \setminus FC$. Then, since (x_1, x_2) is open, we can always find an open neighborhood of x such that $f(x') = -(x' - x_1)(x' - x_2)$ for all $x' \in I_\delta(x)$, since polynomial is continuous everywhere by MAT157, we conclude $f(x)$ is continuous at x locally.

Since $x \in [0,1]$ is arbitrary, we conclude that f is continuous on $[0,1]$.

quod erat dem■

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Claim 3

$f([0,1]) \subseteq [0,1]$ (so that f is a function $f : [0,1] \to [0,1]$).

Proof. If $x \in FC$, then $f(x) = 0 \in [0,1]$.

If $x \notin FC$, then there exist $x_1, x_2 \in FC$ such that $x \in (x_1, x_2) \subseteq [0,1] \setminus FC$. Now, we can see

$$f(x) = -(x - x_1)(x - x_2),$$

since this parabola achieves its maximum at the midpoint of the interval, so we have:

$$\begin{aligned}
&\leq -\left(\frac{x_1 + x_2}{2} - x_1\right)\left(\frac{x_1 + x_2}{2} - x_2\right) \\
&= -\left(\frac{x_2 - x_1}{2}\right)\left(\frac{x_1 - x_2}{2}\right) \\
&= \frac{1}{4}(x_2 - x_1)^2 \\
&\leq 1.
\end{aligned}$$

Also $x_1 < x, x < x_2$ imply $f(x) = -(x - x_1)(x - x_2) \geq 0$, thus $f(x) \in [0,1]$.

Since $x \in [0,1]$ is arbitrary, we conclude $f([0,1]) \subseteq [0,1]$. *quod erat dem*■

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Claim 4

$f(x) = 0$ if and only if $x \in FC$.

Proof. The backward direction holds by our definition of f.

For the forward direction, we will prove the contrapositive. Assume $x \notin FC$, then by our construction of f we have $f(x) = -(x - x_1)(x - x_2)$ for some $x_1, x_2 \in FC$ such that $x \in (x_1, x_2) \subseteq [0,1] \setminus FC$. Then, $x \neq x_1, x \neq x_2$ imply $f(x) = -(x - x_1)(x - x_2) \neq 0$, thus $f(x) \neq 0$ which shows the contrapositive of the forward direction holds.

Hence we conclude $f(x) = 0$ if and only if $x \in FC$. *quod erat dem*■

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Now, consider $g \circ f$, by Claim 4 we have

$$(g \circ f)(x) = \begin{cases} 0, & x \in FC \\ 1, & x \notin FC \end{cases}.$$

Since by Lemma 1 FC has 'length' $\frac{1}{2}$, to show it does not satisfy Lebesgue Criterion it suffices to show the discontinuous points of $g \circ f$ do not form a null set. Namely,

$$\exists \varepsilon > 0, \forall \{(a_i, b_i)\}_{i \in \mathbb{N}} \text{ we have } \left(FC \not\subseteq \bigcup_{i \in \mathbb{N}} (a_i, b_i)\right) \vee \left(\sum_{i=1}^{\infty} (b_i - a_i) \geq \varepsilon\right),$$

which is equivalent to

$$\exists \varepsilon > 0, \forall \{(a_i, b_i)\}_{i \in \mathbb{N}} \text{ we have } \left(FC \subseteq \bigcup_{i \in \mathbb{N}} (a_i, b_i)\right) \implies \left(\sum_{i=1}^{\infty} (b_i - a_i) \geq \varepsilon\right).$$

Since FC is totally disconnected, we can see the set FC contains the discontinuous points of $g \circ f$ (all points in FC are also discontinuous points of $g \circ f$), thus it is enough to show FC does not form a null set.

So, fix $\varepsilon = \frac{1}{8} > 0$. Let $\{(a_i, b_i)\}_{i \in \mathbb{N}}$ be an arbitrary open cover of FC. Since FC has a total length of $\frac{1}{2}$, and we know the total length of the cover is at least the length of FC, i.e. $\frac{1}{2}$, thus we have $\sum_{i=1}^{\infty}(b_i - a_i) \geq \frac{1}{2} \geq \frac{1}{8} = \varepsilon$.

Since our open cover is arbitrary, and we have constructed such $\varepsilon > 0$, we conclude that $g \circ f$ does not satisfy Lebesgue Criterion, and thus is not Riemann integrable. Moreover, since f is a continuous function from $[0,1]$ to $[0,1]$ as shown in Claim 2 and Claim 3, and $g : [0,1] \to [0,1]$ is Riemann integrable, we thus found a counter-example to show that $g \circ f$ is not Riemann integrable when $f \in \mathcal{C}[a,b]$ and $g \in \mathfrak{R}[a,b]$. ☺