Review

- Definition of $f \in \mathcal{R}[a,b]$
- Criteria for $f \in \mathcal{R}[a,b]$:
  1) 2) 3) 4) 5)
- Proposition of $f \in \mathcal{R}[a,b]$
  - Linearity: $\forall f, g \in \mathcal{R}[a,b]$. $\forall \alpha, \beta \in \mathbb{R}$. $\alpha f + \beta g \in \mathcal{R}[a,b]$ and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$

Remark 1. Riemann integral: $\mathcal{R}[a,b] \xrightarrow{\int_a^b} \mathbb{R}$. $f \mapsto \int_a^b f(x)\,\mathrm{d}x$ can be seen as a linear operator.

Remark 2. If $g \in C[a,b]$, $f \in \mathcal{R}[a,b]$, then $\mathcal{R}[a,b] \xrightarrow{\int_a^b g\,\mathrm{d}x} \mathbb{R}$. $f \mapsto \int_a^b f(x)g(x)\,\mathrm{d}x$ is a linear operator.

Example 1. $L : \mathcal{R}[a,b] \to \mathbb{R}$ is well-defined and linear.

- Composition $g \circ f$.

	$f \in C[a,b]$	$f \in \mathcal{R}[a,b]$
$g \in C[a,b]$	Yes	Yes
$g \in \mathcal{R}[a,b]$	No	No

- Monotonicity

Proposition 1

For any $f, g \in \mathcal{R}[a,b]$, $f \le g$ on $[a,b]$ implies $\int_a^b f \le \int_a^b g$.

Proof. Take a sequence of[a] marked partitions $(\Gamma_n, \eta) \in \Omega^*[a,b]$ s.t. $\|\Gamma_n\| \to 0$. Now,

$$\sum_{k=0}^{m} f(\eta_{n_k})\Delta x_k \le \sum_{k=0}^{m} g(\eta_{n_k})\Delta x_k$$

$$\downarrow n \to \infty \qquad \downarrow n \to \infty$$

$$\xrightarrow[n\to\infty]{} \int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x \xleftarrow[n\to\infty]{}$$

Quod Erat Dem■

Corollary 1

$f \in \mathcal{R}[a,b]$, $f \geq 0$, then $\int_a^b f \geq 0$.
(we compare the average as integral)

Remark 3. If $f \in \mathcal{R}[a,b]$, $g \leq f$, g is bounded, we cannot ensure g is integrable.

Example 2. $f = 2$, $g = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ on $[0,1]$.

Additivity

Observation: If $f : \mathcal{R}[a,b]$, $\forall \overline{a} \leq \overline{b}$ s.t. $[\overline{a}, \overline{b}] \subset [a,b]$, then $f \in \mathcal{R}[a,b]$.

Proposition 2

$f \in \mathcal{R}[a,b]$, $\forall c_1, c_2, c_3 \in [a,b]$, $\int_{c_1}^{c_3} f = \int_{c_1}^{c_2} f + \int_{c_2}^{c_3} f$.

Hint. We did not assume $c_1 \leq c_2 \leq c_3$ due to the convention $\int_b^a = -\int_a^b$.