1 Review

Riemann Integral. $f \in R[a, b]$

- $[a, b]$ is a closed and bounded
- f is bounded on $[a, b]$

2 Improper Integral

- Unbounded Integral
- Unbounded funciton

3 Unbounded inteval

- $f : [a, +\infty)$
- $\forall b \in (a, +\infty), f \in R[a, b]$

If $\lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$ exists, then we can define

$$\int_a^{+\infty} f(x)\,\mathrm{d}x := \lim_{b\to+\infty} \int_a^b f(x)\,\mathrm{d}x.$$

Similarly we can define

$$\int_{-\infty}^b f(x)\,\mathrm{d}x.$$

Example 1. $\int_0^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x$
Then, $\forall b > 0. \frac{1}{1+x^2} \in R[a, b]$
Moreover,

$$\begin{aligned} \lim_{b\to+\infty} \int_0^b \frac{1}{1+x^2}\,\mathrm{d}x &= \lim_{b\to+\infty} [\arctan x]_0^b \\ &= \lim_{b\to+\infty} \arctan b \\ &= \frac{\pi}{2} \end{aligned}$$

Hence $\lim_0^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}$.

Remark 1.

- By FTC. Rewriting $\lim_{b\to+\infty} \int_a^b f(x)\,\mathrm{d}x$ exists is equivalent of rewriting $\lim_{b\to+\infty} F(b)$ exists, where $F(b) = \int_a^b f(x)\,\mathrm{d}x$.
- By Sequence / Series, $\lim_{b\to+\infty} \int_a^b f(x)\,\mathrm{d}x$ exists is equivalent of rewriting that for all $a_n, (a_0 = a)$ monotonically increasing to positive infinity,

$$\sum_{n=0}^{\infty} \int_{a_n}^{a_{n+1}} f(x)\,\mathrm{d}x < \infty.$$

However, if $f(x) \geq 0$, then $\exists$ is sufficient.

Example 2. $\int_1^{+\infty} \frac{1}{x^\lambda}\,\mathrm{d}x \quad \lambda > 0$
By FTC, $\forall b > 1$,

$$\int_1^b \frac{1}{x^\lambda}\,\mathrm{d}x = \begin{cases} \frac{1}{1-\lambda}\cdot x^{1-\lambda}\Big|_1^b & \lambda \neq 1 \\ \ln x\Big|_1^b & \lambda = 1 \end{cases}$$

Hence,

$$\int_a^{+\infty} \frac{\mathrm{d}x}{x^\lambda} = 1 \quad (\lambda > 1),$$

and is undefined when $\lambda \leq 1$

Remark 2. This is a continuous analogue of p-harmonic series

Example 3. $\int_0^{+\infty} \cos x\,\mathrm{d}x$
By FTC, $\forall b > 0$,

$$\int_a^b \cos x\,\mathrm{d}x = \sin b \quad \text{periodic},$$

Since $\lim_{b\to+\infty} \sin b$ does **<u>NOT</u>** exists, hence $\lim_0^{+\infty} \cos x\,\mathrm{d}x$ is indefined.

3.1 Criteria

- Cauchy Sequence Properties
  $\int_a^{+\infty} f(x)\,\mathrm{d}x$
  $\forall \epsilon > 0. \exists A > 0$ s.t.

$$\forall b_1, b_2 > A, \left|\int_a^{b_1} f(x)\,\mathrm{d}x - \int_a^{b_1} f(x)\,\mathrm{d}x\right| < \epsilon$$
$$\equiv |F(b_1) - F(b_2)| < \epsilon.$$

- Abel and Dirichlet
  $\int_a^{+\infty} f(x)g(x)\,\mathrm{d}x$
  <u>Combo Abel</u>
  1. $\int_a^b f(x)\,\mathrm{d}x < \infty$
  2. g is monotonic and bounded

  <u>Combo Dirichlet</u>
  1. $\left|\int_a^b f(x)\,\mathrm{d}x\right| < M$
  2. g monotonically decreasing to 0.

Example 4. $\int_1^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$
$\int_1^{+\infty} f(x)g(x)\,\mathrm{d}x$, where $f(x)=\sin x$ and $g(x)=\frac{1}{x}$.

1. $\forall b10$, $\left|\int_1^b \sin x\,\mathrm{d}x\right| = |1-\cos b| \le 2$
2. $\frac{1}{x}$ monotonically decreasing to 0 when $x \to +\infty$.

So Dirichlet works.

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Remark 3. (Warning)
If $f \in R[a,b]$, then $|f| \in R[a,b]$. Moreover $\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$.
On the other hand, it is not true that

$$\int_1^{+\infty} |f(x)|\,\mathrm{d}x < \infty \text{ IMPLIES } \int_1^{+\infty} f(x)\,\mathrm{d}x < +\infty,$$

(Counter)-example:

$$\int_1^{+\infty} \frac{|\sin x|}{x}\,\mathrm{d}x \quad \text{is undefined}$$

Actually,

$$|\sin x| \ge \sin^2 x = \frac{1-\cos 2x}{2}$$

$\int_1^{+\infty} \frac{1-\cos 2x}{2}/x\,\mathrm{d}x = \int_1^{+\infty} \frac{1-\cos 2x}{2x}\,\mathrm{d}x$

- $\int_1^{+\infty} \frac{\cos 2x}{2x}\,\mathrm{d}x < \infty$ (By Dirichlet)
- $\int_1^{+\infty} \frac{1}{2x}\,\mathrm{d}x = \infty$

These two imply $\int_1^{+\infty} \frac{1-\cos 2x}{2x}\,\mathrm{d}x$ is undefined.

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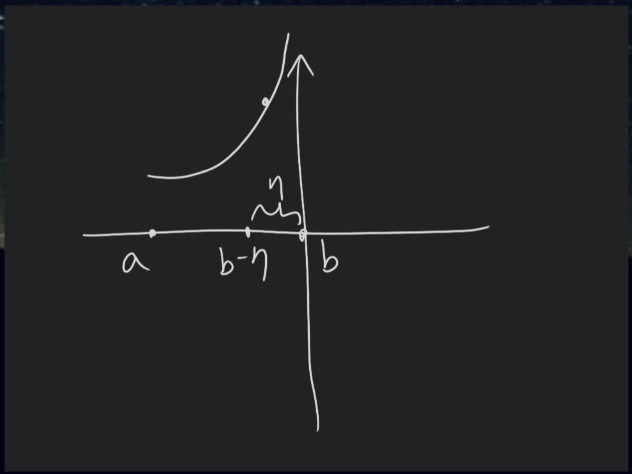
4 Unbounded Functions

$f:[a,b) \to \mathbb{R}$ is unbounded in $[b-\eta, b)$

$f \in R[a,b-\eta], \forall \eta \in (0, b-a)$



If $\lim_{\eta\to 0^+}\int_a^{b-\eta} f(x)\,\mathrm{d}x$ exists, then

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{\eta\to 0^+}\int_a^{b-\eta} f(x)\,\mathrm{d}x$$

Similarly

$$\int_a^b f(x)\,\mathrm{d}x \quad \text{if } f(x) \text{ is unbounded on } (0, a+\eta)$$

Example 5.
$\int_0^1 \frac{1}{x^\lambda}\,\mathrm{d}x \quad \lambda > 0$

$$\int_\eta^1 \frac{1}{x^\lambda}\,\mathrm{d}x = \begin{cases} \left.\frac{x^{1-\lambda}}{1-\lambda}\right|_\eta^1 & \lambda \neq 1 \\ \left.\ln x\right|_\eta^1 & \lambda = 1 \end{cases}$$

Hence,

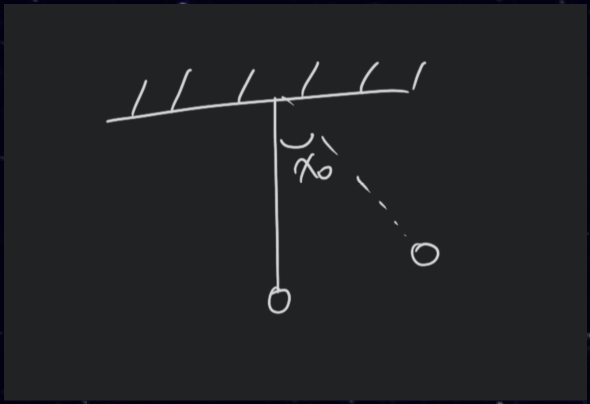
$$\int_0^1 \frac{1}{x^\lambda}\,\mathrm{d}x = \begin{cases} \frac{1}{1-\lambda} & \text{if } 0 < \lambda < 1 \\ \text{undefined otherwise} \end{cases}$$

Remark 4. You can often use Tyalor Expansion near the singularity.

Example 6. $\int_0^1 \frac{\mathrm{d}x}{\sqrt[3]{x(e^x - e^{-x})}}$
Let

$$\begin{aligned} g(x) &= x\left(e^x - e^{-x}\right) \\ &= x\left(1 + x + \frac{x^2}{2} - \left(1 - x + \frac{x^2}{2}\right)\right) \\ &= x(2x + o(x^2)) \approx 2x^2 \end{aligned}$$

Example 7. Consider the pendulum



$$\int_0^{x_0} \frac{\mathrm{d}x}{\sqrt{\cos x - \cos x_0}}$$

$\cos x - \cos x_0 \approx -(x - x_0)\sin x_0$
$\sin x_0 \neq 0 \quad$ when $0 < x_0 < \pi$
Hence

$$\int_0^{x_0} \frac{\mathrm{d}x}{\sqrt{\cos x - \cos x_0}} \leq C \int_0^{x_0} \frac{\mathrm{d}x}{\sqrt{x_0 - x}} < \infty$$

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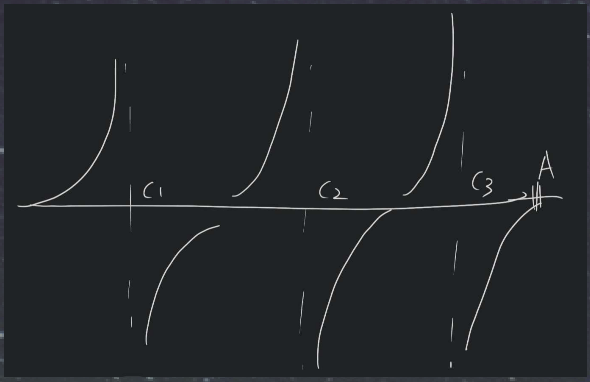
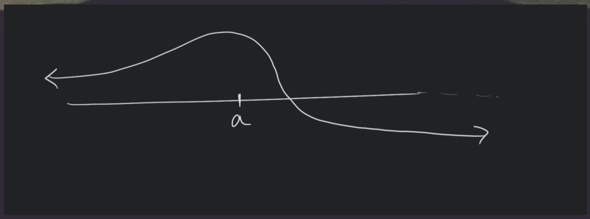
Remark 5. Improper integral for both unbounded interval and unbounded function:



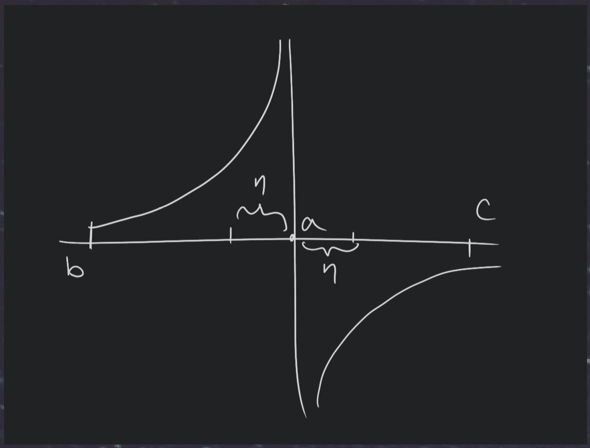
$\int 0^{+\infty} f(x)\,\mathrm{d}x$
If we choose an A that is large enough, and pass A to positive infinity.
However,



$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$$

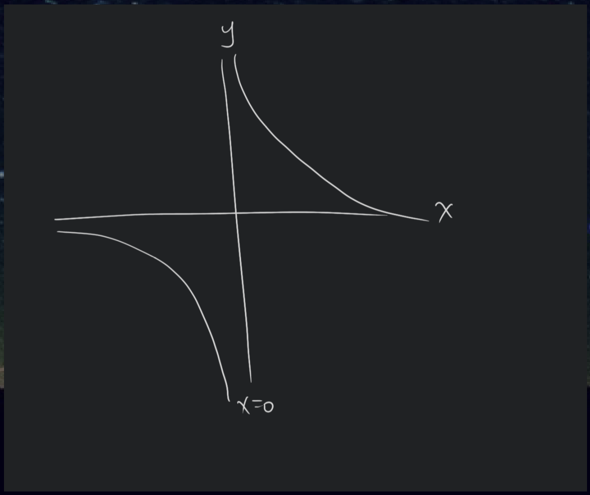
$$\lim_{\substack{b\to+\infty\\ \tilde{b}\to-\infty}} \int_{\tilde{b}}^{b} f(x)\,\mathrm{d}x$$

Warning. b and $\tilde{b}$ are independent of each other.



$$\int_b^c f(x)\,\mathrm{d}x := \lim_{\eta\to 0} \int_b^{a-\eta} f(x)\,\mathrm{d}x + \lim_{\tilde{\eta}\to 0} \int_{a+\tilde{\eta}}^{c} f(x)\,\mathrm{d}x$$

Here η and $\tilde{\eta}$ are independent of each other.



Example 8.

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x}\,\mathrm{d}x &:= \lim_{\eta\to 0}\int_{-1}^{\eta}\frac{1}{x}\,\mathrm{d}x + \lim_{\tilde{\eta}\to 0}\int_{\tilde{\eta}}^{1}\frac{1}{x}\,\mathrm{d}x \\ &= \lim_{\eta\to 0}\ln(-x)\Big|_{-1}^{-\eta} + \lim_{\tilde{\eta}\to 0}\ln x\Big|_{\tilde{\eta}}^{1} \\ &= \lim_{\eta\to 0}\lim_{\tilde{\eta}\to 0}\ln\left(\frac{\eta}{\tilde{\eta}}\right) \\ &= \infty\end{aligned}$$

But, if we force that $\eta = \tilde{\eta}$, then the limit exists, and is equal to 0. The above value is called the Cauchy Principle Value of improper integral.
We should write this as

$$\text{v.p.}\quad \int_{-1}^{1}\frac{1}{x}\,\mathrm{d}x = 0$$

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Example 9.

$$\int_{0}^{+\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

First, it is an improper integral, and we can see the speed it goes to infinity is much slower than any polynomial reverse:

$$\lim_{x\to+\infty}\frac{e^{-x^2}}{\frac{1}{x^2}} = 0$$

since $\int_1^{+\infty}\frac{1}{x^2} < \infty$ IMPLIES $\int_1^{+\infty} e^{-x^2}\,\mathrm{d}x < +\infty$ IMPLIES $\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x < \infty$.
Second, FTC does not work, it is not integralble in finite terms, gg.
Lets do some preparations:

$$J_n := \int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x$$

Then

$$
\begin{aligned}
J_n &= \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \,\mathrm{d}(-\cos x) \\
&= \sin^{n-1} x \cos x \Big|_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \,\mathrm{d}x \\
&= (n-1) \int_0^{\frac{\pi}{2}} \cos x \sin^{n-2} x \cdot \cos x \,\mathrm{d}x \\
&= (n-1) \cdot \left(\int_0^{\frac{\pi}{2}} \sin^{n-2} x \,\mathrm{d}x - J_{n-2} \right) \\
&= (n-1) J_{n-2} - (n-1) J_n \\
J_n &= \frac{n-1}{n} J_{n-2} \\
J_0 &= \frac{\pi}{2} \\
J_1 &= 1
\end{aligned}
$$

By induction,

$$
J_n = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2} & n \text{ is even} \\ \frac{(n-1)!!}{n!!} & n \text{ is odd} \end{cases}
$$

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5 Wallis

Claim 1

$$
\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = 0
$$

♇

Fake proof.

$$
\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \lim_{n\to\infty} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} 0 \,\mathrm{d}x = 0
$$

Quod Erat Dem■

Real proof.
Consider

$$
J_{2n+1} < J_{2n} < J_{2n-1}
$$

i.e.

$$\frac{(2n)!!}{(2n+1)!!} < \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} < \frac{(2n-2)!!}{(2n-1)!!}$$

$$\lim_{n \to \infty} \left(\frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!} \right)^2 = \frac{(2n)!!}{(2n+1)!!} \cdot \frac{(2n-2)!!}{(2n-1)!!}$$

Which implies by Wallis formula, $\frac{\pi}{2} = \lim ...$

Quod Erat Dem■

To be finished :)