# MAT159: Analysis II

# Term Test 2 Problems


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Info: Good luck!

# 1 Riemann Sums

**Question 1**

$f(x), \varphi(x) \in \mathcal{C}^{(1)}[a,b]$. Prove that

$$\lim_{\max|\Delta x_i|\to 0} \sum_{i=1}^{n} f(\xi_i)\varphi(\theta_i)\Delta x_i = \int_a^b f(x)\cdot\varphi(x)\,\mathrm{d}x,$$

where $x_{i-1} \le \xi_i \le x_i$, $x_{i-1} \le \theta_i \le x_i$ $(i = 1, \cdots n)$, and $\Delta x_i = x_i - x_{i-1}(x_0 = a, x_n = b)$.

*Proof.* Fix partition $\Gamma$. Let $M$ be the upper bound of $|f|$ on $[a,b]$, let $\omega_i$ be the oscillation of $\varphi$ on $[x_{i-1}, x_n]$ of $\Gamma$, then we have

$$\begin{aligned} 0 \le \left|\sum_{i=1}^{n} f(\xi_i)\varphi(\theta_i)\Delta x_i - \sum_{i=1}^{n} f(\xi_i)\varphi(\xi_i)\Delta x_i\right| &\le \sum_{i=1}^{n} |f(\xi_i)|\cdot|\varphi(\theta_i)-\varphi(\xi_i)|\Delta x_i \\ &\le M\sum_{i=1}^{n}\omega_i\Delta x_i \to 0. \end{aligned}$$

Here as $||\Gamma|| \to 0$, since $\varphi$ is integrable, by squeeze theorem we have the differences of the sums go to 0, which proves the statement.

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**Question 2**

Let $f(x) \in \mathcal{C}^{(1)}[a,b]$,

$$\Delta_n = \int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + k\frac{b-a}{n}\right),$$

find $\lim_{n\to\infty} n\Delta_n$.

*Proof.* We partition $[a,b]$ equally, and mark $x_k = a + k\frac{b-a}{n}$, $k = 0, 1, \cdots, n$. Then we have $\Delta x_k = \frac{b-a}{n}(k = 1, \cdots, n)$. Now, split $n\Delta_n$:

$$\begin{aligned} n\Delta n &= n\left(\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{n}\sum_{k=1}^{n} f(x_k)\right) \\ &= n\left(\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}[f(x)-f(x_k)]\,\mathrm{d}x\right) \end{aligned}$$

By MVT, we have $f(x) - f(x_k) = f'(\xi_k)(x - x_k)$ for some $\xi_k \in [x_{k-1}, x_k]$, thus we have

$$= n\left(\sum_{k=1}^{n}\int_{k-1}^{k} f'(\xi_k)(x-x_k)\,\mathrm{d}x\right)$$

Let $m_k$ and $M_k$ be the minimum and maximum of $f'(x)$ on $[x_{k-1}, x_k]$, then we have

$$n\sum_{k=1}^{n} m_k \int_{k-1}^{k} (x - x_k)\,\mathrm{d}x \le n\Delta_n = n\left(\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} f'(\xi_k)(x - x_k)\,\mathrm{d}x\right)$$
$$\le n\sum_{k=1}^{n} M_k \int_{x_{k-1}}^{x_k} (x - x_k)\,\mathrm{d}x.$$

So, for $k = 1, \cdots, n$ we have[a]

$$\int_{x_{k-1}}^{x_k} (x_k - x)\,\mathrm{d}x = \frac{1}{2}(\Delta x_k)^2 = \frac{b-a}{2n}\Delta x_k,$$

which gives the inequality

$$-\frac{b-a}{2}\sum_{k=1}^{n} m_k\Delta x_k \le n\Delta_n \le -\frac{b-a}{2}\sum_{k=1}^{n} M_k\Delta x_k.$$

Now take limit to infinity and we have

$$-\frac{b-a}{2}\int_a^b f'(x)\,\mathrm{d}x \le \lim_{n\to\infty} n\Delta_n \le -\frac{b-a}{2}\int_a^b f'(x)\,\mathrm{d}x$$

Finally, we get

$$\lim_{n\to\infty} n\Delta_n = -\frac{b-a}{2}\int_a^b f'(x)\,\mathrm{d}x = -\frac{b-a}{2}[f(b) - f(a)].$$

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[a] Here we have $x_k - x$ instead of $x - x_k$ so we take out -1 from the integral to get the new inequality

**Question 3**

Let $f(x) \in \mathfrak{R}[a, b]$, and

$$f_n(x) = \sup f(x) \;\; (x_{i-1} \le x < x_i,\; i = 1, \cdots, n),$$

where $x + i = a + \frac{i}{n}(b-a)(i = 0, 1, \cdots, n; n = 1, 2, \cdots)$. Prove:

$$\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

*Proof.* Let $\omega_i$ be the oscillation of $f$ on $[x_{i-1}, x_i]$, then we have

$$\begin{aligned}\left|\int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x\right| &\leq \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} |f_n(x) - f(x)|\,\mathrm{d}x \\ &\leq \sum_{i=1}^{n} \omega_i \Delta x_i \to 0\end{aligned}$$

So, since $f$ is integrable on $[a, b]$ we know the summation of oscillations go to 0, therefore the statement is proved.

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**Question 4**

Prove that if $f(x) \in \mathfrak{R}[a, b]$, then there exists a continuous function sequence $\varphi_n(x)$ $(n = 1, 2, \cdots)$ such that when $a \leq c \leq b$ we have

$$\int_a^c f(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_a^c \varphi_n(x)\,\mathrm{d}x.$$

*Proof.* Partition $[a, b]$ to n intervals, let $x_i = a + \frac{i}{n}(b - a)$, $i = 0, 1, \cdots, n$. Define

$$\varphi_n(x) = f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}}[f(x_i) - f(x_{i-1})], x_{i-1} \leq x \leq x_i.$$

Here $f(x_{i-1})$ is the left point of $\varphi_n(x)$, $\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}$ is the slope of the line, and $x - x_{i-1}$ is the distance from $x_{i-1}$. (that is, we are constructing a line passing through the left and right end points of each $i$ intervals)
Then, we have

$$\begin{aligned}\left|\int_a^c \varphi_n(x)\,\mathrm{d}x - \int_a^c f(x)\,\mathrm{d}x\right| &\leq \int_a^c |\varphi_n(x) - f(x)|\,\mathrm{d}x \\ &\leq \int_a^b |\varphi_n(x) - f(x)|\,\mathrm{d}x \\ &\leq \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} |\varphi_n(x) - f(x)|\,\mathrm{d}x \\ &\leq \sum_{i=1}^{n} \omega_i \Delta x_i \to 0,\end{aligned}$$

where $\omega_i$ is the oscillation of $f$ on $[x_{i-1}, x_i]$.
Therefore as $f$ is integrable on $[a, b]$, by letting $n \to \infty$ we can see the limit is 0, which proves the statement.

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**Question 5**

Let $f(x) \in \mathfrak{R}[A, B]$, prove the function has integral continuity, that is,

$$\lim_{h \to 0} \int_a^b |f(x+h) - f(x)| \, \mathrm{d}x = 0,$$

where $[a, b] \subset (A, B)$.

*Proof.* Too long (lazy) :( Quod Erat Dem■

**Question 6**

Compute

$$\int_0^\pi \ln(1 - 2\alpha \cos x + \alpha^2) \, \mathrm{d}x,$$

where consider 2 cases:

1. $|\alpha| < 1$,
2. $|\alpha| > 1$.

*Proof.* To hard (trivial). Quod Erat Dem■

## 2 Riemann Integrity

**Question 7**

Prove that if $f(x) \in \mathfrak{R}[a, b]$ and is bounded, then $|f(x)| \in \mathfrak{R}[a, b]$. Moreover,

$$\left| \int_a^b f(x) \, \mathrm{d}x \right| \leq \int_a^b |f(x)| \, \mathrm{d}x.$$

*Proof.* Since all continuous points of $f$ are also continuous points of $|f|$, thus by Lebesgue Theorem $|f|$ is integrable. Moreover,

$$\left| \sum_{i=1}^n f(\xi_i) \Delta x_i \right| \leq \sum_{i=1}^n |f(\xi_i)| \Delta x_i$$

Take the limit and we have

$$\left| \int_a^b f(x) \, \mathrm{d}x \right| \leq \int_a^b |f(x)| \, \mathrm{d}x.$$

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**Question 8**

Let $f(x) \in \mathfrak{R}[a,b]$, prove that

$$\int_a^b f^2(x)\,\mathrm{d}x = 0$$

holds if and only if $f \equiv 0$ almost everywhere. That is, at all continuous points $x \in [a,b]$ we have $f(x) = 0$.

*Proof.* For the forward direction, to obtain a contradiction assume there exists a continuous point $x_0$ such that $f(x_0) \neq 0$. Since $x_0$ is continuous, this implies there exists a delta such that for all $x \in (x_0 - \delta, x_0 + \delta)$ we have $f(x) \neq 0$. Thus, for all partition if we refine such interval into the partition, then since all terms are non-zero, and there exists a positive term, the sum of squares will be strictly positive, which contradicts the integral being 0 (And we assumed that $x_0$ produces a fixed value $f(x_0)$).
For the backward impication, for all partitions, we choose the marked points to be the continuous points of $f$. Then, since the discontinuous points form a null set, we can always make the sum of the intervals containing all such points to be less than epsilon, which gives a summation $0 + \epsilon$ for all $\epsilon > 0$, which shows the limit is 0.

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**Question 9 – Homework Question**

1. Prove that if $\varphi(x) \in \mathcal{C}[a,b]$ and $f(x) \in \mathfrak{R}[a,b]$ such that when $a \le x \le b$ we have $A \le f(x) \le B$. Prove that $\varphi(f(x)) \in \mathfrak{R}[a,b]$.
2. If $\varphi(x), f(x) \in \mathfrak{R}[a,b]$, is $f(\varphi(x))$ also integrable?

*Proof.* For the first one, since all continuous points of $f$ are also continuous points of $\varphi$, and so $\varphi$ is also continuous almost everywhere, by Lebesgue Theorem $\varphi(f(x))$ is integrable.
For the second one, let $\varphi$ be the indicator function of 0, and let $f$ to be the Thomae-Riemann function, then the composition is not integrable.

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## 3 Newton-Liebniz Formula

**Question 10**

Find $\displaystyle\int_{\sinh 1}^{\sinh 2} \frac{\mathrm{d}x}{\sqrt{1+x^2}}$.

*Proof.* The explicite form of the primitive is easy to find:

$$\int_{\sinh 1}^{\sinh 2} \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \operatorname{arcsinh}(x)\Big|_{\sinh 1}^{\sinh 2} = 2 - 1 = 1.$$

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**Question 11**

Compute

$$I(\alpha) = \int_0^{\pi} \frac{\sin^2 x}{1 + 2\alpha\cos x + \alpha^2}\,\mathrm{d}x.$$

*Proof.* When $\alpha = 0, \pm 1$, we can directly compute the integral to be $\frac{\pi}{2}$.
Using the trig substitution $t = \tan\frac{x}{2}$, we can transform the integral to rational... Then get a long formula, here it is:

$$= -\frac{1}{2\alpha}\sin x + \left(\frac{1}{2} + \frac{1}{2\alpha^2}\right)\frac{x}{2} - \frac{|1-\alpha^2|}{2\alpha^2}\arctan\left(\frac{|1-\alpha|}{|1+\alpha|}\tan\frac{x}{2}\right) + C.$$

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**Question 12**

Find this limit using definite integral:

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\right].$$

*Proof.* Let $x_k = a + k\frac{b-a}{n}$, then $\Delta x_k = \frac{b-a}{n}$. Then when $f$ is integrable we have

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\right] &= \frac{1}{b-a}\lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\right] \\ &= \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.\end{aligned}$$

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# 4 Other

**Question 13**

Find

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_a^b \sin t^2 \,\mathrm{d}t$
2. $\dfrac{\mathrm{d}}{\mathrm{d}a}\displaystyle\int_a^b \sin x^2 \,\mathrm{d}x$
3. $\dfrac{\mathrm{d}}{\mathrm{d}b}\displaystyle\int_a^b \sin x^2 \,\mathrm{d}x$

*Proof.*

1. Since the integral is a constant with respect to $x$, it is equal to 0.
2. By FTC, we have it to be $-\frac{\mathrm{d}}{\mathrm{d}a}\int_b^a \sin x^2 \,\mathrm{d}x = -\sin a^2$, we plugged $a$ into the function.
3. Similarly, we have it to be $\sin b^2$.

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**Question 14**

Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{x^2}^{x^3} \frac{\mathrm{d}t}{\sqrt{1+t^4}}$.

*Proof.* By FTC, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{x^3} \frac{\mathrm{d}t}{\sqrt{1+t^4}} &= \left(\frac{1}{\sqrt{1+t^4}}\right)\Bigg|_{x^3} \cdot \frac{\mathrm{d}x^3}{\mathrm{d}x} - \left(\frac{1}{\sqrt{1+t^4}}\right)\Bigg|_{x^2} \cdot \frac{\mathrm{d}x^2}{\mathrm{d}x} \\ &= \frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}}.\end{aligned}$$

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**Question 15**

Let $f(x) \in \mathcal{C}^{(1)}[0,+\infty)$, $\lim\limits_{x\to+\infty} f(x) = A$, find

$$\lim_{n\to+\infty}\int_0^1 f(nx)\,\mathrm{d}x.$$

*Proof.* Let $t = nx$, then $x = \frac{1}{n}t$, $\mathrm{d}x = \frac{1}{n}\,\mathrm{d}t$, and when $x$ is from 0 to 1, we have $t$ is from 0 to $n$, then we have

$$\lim_{n\to\infty}\int_0^1 f(nx)\,\mathrm{d}x=\lim_{n\to\infty}\frac{\int_0^n f(t)\,\mathrm{d}t}{n},$$

which we may apply L'Hopital's rule and get

$$\lim_{n\to\infty}\frac{\int_0^n f(t)\,\mathrm{d}t}{n}=\lim_{n\to+\infty}f(n)=A.$$

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**Question 16**

Let $f(x)\in\mathcal{C}^{(1)}[0,+\infty)$, $\lim\limits_{x\to+\infty}f(x)=A$, find

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t.$$

*Proof.* Same as the previous question.

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**Question 17**

Compute $\displaystyle\int_{-1}^{1}\frac{x\,\mathrm{d}x}{\sqrt{5-4x}}$.

*Proof.* Let $\sqrt{5-4x}=t$, then $x=\frac{1}{4}(5-t^2)$, $\mathrm{d}x=-\frac{1}{2}t\,\mathrm{d}t$. Then we have

$$\begin{aligned}\int\frac{x\,\mathrm{d}x}{\sqrt{5-4x}}&=\int\frac{5-t^2}{4t}\cdot\left(-\frac{t}{2}\right)\mathrm{d}t=-\frac{1}{8}\int(5-t^2)\,\mathrm{d}t\\&=-\frac{5}{8}(5-4x)^{\frac{1}{2}}+\frac{1}{24}(5-4x)^{\frac{3}{2}}+C.\end{aligned}$$

By Newton-Liebniz formula we have

$$\begin{aligned}\int_{-1}^{1}\frac{x\,\mathrm{d}x}{\sqrt{5-4x}}&=\left[-\frac{5}{8}(5-4x)^{\frac{1}{2}}+\frac{1}{24}(5-4x)^{\frac{3}{2}}\right]\Bigg|_{-1}^{1}\\&=\frac{1}{6}.\end{aligned}$$

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**Question 18**

Prove $\lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = 0.$

*Proof.* Fixed $\epsilon > 0$, take $\delta = \frac{\epsilon}{2}$, then split the integral and we can see

$$0 < \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}-\delta} \sin^n x \,\mathrm{d}x + \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x$$

Since $\sin^n x$ is monotonely increasing from 0 to $\frac{\pi}{2}$, take the upper bound which is the right most value, times the length of the whole interval, and we get

$$\leq \frac{\pi}{2}\sin^n\left(\frac{\pi}{2} - \delta\right) + \delta$$

above for the second integral, the length is delta, and is bounded by 1.

$$\leq \frac{\pi}{2}\cos^n x + \frac{\epsilon}{2}$$

So, since there exists $N$ such that for all $n > N$ we have $0 < \cos^n x < \frac{\epsilon}{2}$, we can see the limit is 0 by definition.

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**Question 19**

Let $\varphi(x), \psi(x), \varphi^2(x), \psi^2(x) \in \mathfrak{R}[a, b]$, prove that

$$\left\{\int_a^b \varphi(x)\psi(x)\,\mathrm{d}x\right\}^2 \leq \int_a^b \varphi^2(x)\,\mathrm{d}x \int_a^b \psi^2(x)\,\mathrm{d}x.$$

*Proof.* Apply Cauchy inequality to the Riemann sums and we directly get the results.

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**Question 20**

Fix $p > 0$, show

$$\lim_{n\to\infty}\int_n^{n+p} \frac{\sin x}{x}\,\mathrm{d}x = 0.$$

*Proof.* If we treat $p$ to be a constant, then

$$0 \leq \left|\int_n^{n+p} \frac{\sin x}{x}\,\mathrm{d}x\right| \leq \int_n^{n+p} \left|\frac{\sin x}{x}\right| \mathrm{d}x \leq \frac{p}{n} \to 0.$$

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