MAT159

Week 10

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Review.

If f_n is a sequence of functions

$$f_n : D \subseteq \mathbb{R} \to \mathbb{R},$$

and f is a function

$$f : D \subseteq \mathbb{R} \to \mathbb{R}.$$

Convergence:

- Pointwise $f_n \to f$
- Uniform $f_n \rightrightarrows f$

In practice, this type of problem comes often from the mission of represent f by series (usually a basis of an infinite dimensional space)

Example.

- Polynomials

$$f \stackrel{?}{=} \sum_{n=0}^{\infty} a_n x^n \quad f_n = \sum_{n=0}^{m} a_n x^n.$$

- Trigonometric

$$f \stackrel{?}{=} \sum_{n=0}^{\infty} a_n \sin(nx) \quad f_n = \sum_{n=0}^{m} a_n \sin(nx).$$

Assume that one have a function series

$$\sum_{n=1}^{\infty} u_n(x) \quad u_n : D \subseteq \mathbb{R} \to \mathbb{R}.$$

Now if we want to study the partial sum,

$$f_n : \sum_{n=1}^{m} u_n \quad \text{series} \to \text{sequence}$$

$$f_m : \underbrace{(f_n - f_{n-1})}_{u_n} + \underbrace{(f_{n+1} - f_{n-2})}_{u_{n-1}} + \cdots + \underbrace{(f_2 - f_1)}_{u_2} + \underbrace{(f_1 - f_0)}_{u_1} + \underbrace{f_0}_{u_0}$$

$$f_m : \sum_{j=1}^{n} u_j \quad \text{sequence} \to \text{series}$$

Theorem 1 – Recall Cauchy Criterion

$\displaystyle\sum_{n=1}^{\infty} u_n(x)$ converges to f if

$$\forall x \in D. \forall \epsilon > 0. \exists N_0 \in \mathbb{N} \text{ s.t. } \forall m > n > N_0, \left| \sum_{n+1}^{m} u_j(x) \right| < \epsilon.$$

Remark 1. If the above choice of N_0 is independent of $x \in D$, then the convergence is uniform.

Theorem 2 – Abel Identity

$$\sum_{i=n+1}^{m} a_i b_i$$

By change of variable:

$$B_k = \sum_{i=1}^{k} b_i, \quad B_0 = 0,$$

The above is equal to

$$\begin{aligned}\sum_{i=n+1}^{m} a_i b_i = \sum_{i=1}^{m} a_i(B_i - B_{i-1}) &= a_1B_1 + a_2(B_2 - B_1) + a_3(B_3 - B_2) + \cdots + a_m(B_m - B_{m-1}) \\ &= (a_1 - a_2)B_1 + (a_2 - a_3)B_2 + (a_3 - a_4)B_3 + \cdots + (a_{m-1} - a_m)B_{m-1} + a_m B_m \\ &= a_m B_m - \sum_{i=1}^{m-1}(a_{i+1} - a_i)B_i.\end{aligned}$$

Remark 2. This is just an integration by parts (discrete version).
"Morally speaking"

$$a_i = f(x_i) \quad b_i = g(x_i) \quad B_i \approx \int_a^{x_i} g(x_i)\,\mathrm{d}x_i = G(x_i)$$

$$\sum a_i b_i \approx \int f(x)g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}G(x) = f(x)G(x)\Big|_a^b - \int_a^b G(x)\,\mathrm{d}f(x) \approx a_m B_m - \sum_{i=1}^{m-1}(a_{i+1} - a_i)B_i.$$

Now, remember what you want to control. Back to analysis of convergence. Given a constant seris $\sum_{i=1}^{m} a_i b_i$. By Cauchy we need to have

$$\left|\sum_{i=n+1}^{m} a_i b_i\right| \xrightarrow{m,n\to\infty} 0$$

According to Abel identity,

$$\begin{aligned}\sum_{i=n+1}^{m} a_i b_i &= a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + \cdots + a_m b_m \\ &= a_{n+1}(B_{n+1} - B_n) + a_{n+2}(B_{n+1} - B_{n+2}) + \cdots + a_m(B_m - B_{m-1}) \\ &= \left|a_m B_m - \sum_{i=n}^{m-1}(a_{i+1} - a_i)B_i - a_n B_n\right|\end{aligned}$$

Observe that $|B_n| \leq L$,

$$\begin{aligned}R_{m,n} &\leq \left|a_m B_m - a_n B_m + a_n B_m - a_n B_n - \sum_{i=n}^{m-1}(a_{i+1} - a_i)B_i\right| \\ &\leq |a_m - a_n|\,L + |a_n|\,|B_m - B_n| + L\sum_{i=n}^{m-1}|a_{i+1} - a_i|\end{aligned}$$

Since a_n is monotonic (increasing or decreasing)

$$R_{m,n} \leq |a_m - a_n|L + |a_n||B_m - B_n| + L|a_m - a_n|$$

How to force $R_{m,n} \to 0$?

Combo 1 $B_n \to 0$ and a_n bounded. We can see that $R_{m,n} \to 0$ since a_n converges. This is called the Abel Test.

Combo 2 B_n bounded and $a_n \to 0$. This is called the Dirichlet Test.

Example 1.

$$\sum_{n=1}^{\infty} \underbrace{(-1)^{n+1}}_{B_n} \cdot a_n \quad a_n \text{ is decreasing to } 0$$

Since $B_n \in \{-1, 1\}$, by dirichlet the series converges.

Remark 3.

$$\sum_{i=1}^{\infty} a_i(x) b_i(x) \to f(x)$$

We can solve the above using Abel / Dirichlet Test.

$$\sum_{i=1}^{\infty} a_i(x) b_i(x) \rightrightarrows f(x)$$

For the above, we can simply change the conditions to uniform, e.g. bounded to bounded uniformly, converge to converge uniformly.

Example 2. Let's consider

$$\sum_{n=1}^{\infty} \frac{a_n}{n^x} \quad a_n \text{ is arbitrary.}$$

If $a_n \equiv 1$ then this is just the p-harmonic series, ez.
If $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges for some x_0. Then $\forall x > x_0, \sum_{n=1}^{\infty} \frac{a_n}{n^x} = \sum_{n=1}^{\infty} \underbrace{\frac{a_n}{n^{x_0}}}_{\text{converge}} \cdot \underbrace{\frac{1}{n^{x-x_0}}}_{\text{monotonic and bounded}} < \infty$ by Abel Test it converges.