Chapter 1: Fields and Polynomials

Definition 1 – Field

A field $\mathbb{F}$ is a set with two special elements "$0 \in \mathbb{F}$" and "$1 \in \mathbb{F}$" and two operations "$+$" and "$\cdot$" which satisfy the following axioms.

1) (Commutativity) For all $x, y \in \mathbb{F}$ we have: $x + y = y + x$ and $x \cdot y = y \cdot x$.
2) (Associativity) For all $x, y, z \in \mathbb{F}$ we have: $(x+y)+z = x+(y+z)$ and $(x\cdot y)\cdot z = x\cdot(y\cdot z)$.
3) (Distributivity) For all $x, y, z \in \mathbb{F}$ we have: $x \cdot (y + z) = x \cdot y + x \cdot z$.
4) (Identities) For all $x \in \mathbb{F}$ we have: $x + 0 = x$ and $x \cdot 1 = x$.
5) (Inverses) For all $x \in \mathbb{F}$ there exists $y \in \mathbb{F}$ such that $x + y = 0$. For all $x \in \mathbb{F} \setminus \{0\}$ there exists $z \in \mathbb{F}$ such that $x \cdot z = 1$.

$\mathbb{Q}$ is a field where $\mathbb{Z}$ is not a field becasue of the absence of some multiplicative inverse.

Corollary 1 – Field

Let $\mathbb{F}$ be a field and $a, b, c \in \mathbb{F}$.

1) If $a + c = b + c$, then $a = b$.
2) If $c \neq 0$ and $c \cdot a = c \cdot b$, then $a = b$.
3) The field elements $0, 1$ are unique.
4) The elements y and z from Axoim 5 are unique. (From now on, we will denote the additive inverse of x by $-x$, and the multiplicative inverse of x by x^{-1}.)
5) $a \cdot 0 = 0$.
6) $(-a) \cdot (b) = -(a \cdot b) = (a) \cdot (-b)$.
7) $-(-a) = a$. If $a \neq 0$, then $(a^{-1})^{-1} = a$.
8) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Theorem 1 – Equivalence

1) $\sim$ is an equivalence relation.
2) $a \sim b$ if and only if a and b have the same remainder when divided by n.
3) There are exactly n equivalence classes for this relation: $[0], [1], \ldots, [n-1]$ - one for each possible remainder for division by n.

Definition 2 – $\mathbb{Z}_n$

Let $\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$ be the est of equivalence classes for this equivalence relation. We define $+, \cdot$ on $\mathbb{Z}_n$ as follows:

$$[a] + [b] = [a + b],$$
$$[a] \cdot [b] = [a \cdot b].$$

Theorem 2 – Quadratic Formula

Let $a, b, c \in \mathbb{R}$. The quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) has:

1) Solutions $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ if $b^2 - 4ac \geq 0$.
2) No solutions if $b^2 - 4ac < 0$.

Definition 3 – Complex

Let $i = \sqrt{-1}$. I.e. i is a number with the property that $i^2 = -1$.
Let $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$. We call $\mathbb{C}$ the set of complex numbers and we define addition and multiplication $+, \cdot$ in the obvious ways:

$$(x + yi) + (a + bi) = (x + a) + (y + b)i$$
$$(x + yi) \cdot (a + bi) = ax + bi + ayi + byi^2 = (ax - by) + (ay + bx)i$$

$\mathbb{C}$ is a field.

Given a complex number $z = x + yi$, we define its **conjugate** by:

$$\overline{z} = x - yi.$$

We define the **length (or modulus)** of a complex number by:

$$|z| = \sqrt{x^2 + y^2}.$$

Note that we in the xy-plane, we obtain $\overline{z}$, the conjugate of z, by reflecting z in the x-axis, and the length of a complex number is just the usual distance from z to the origin in the xy-plane.

Theorem 3 – Complex

For any $z, w \in \mathbb{C}$ we have:

1) $\overline{z + w} = \overline{z} + \overline{w}$.
2) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.
3) $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$ (provided $w \neq 0$).
4) $\overline{\overline{z}} = z$.
5) $z\overline{z} = |z|^2$.
6) $z^{-1} = \frac{\overline{z}}{|z|^2}$ (provided $z \neq 0$).
7) $|zw| = |z||w|$.
8) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ (provided $w \neq 0$).
9) $|z + w| \leq |z| + |w|$ ("Triangle nequality for Complex Numbers").

Definition 4 – Polar Form

For $z = x + yi$, we define its **polar form** as $z = re^{i\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ and θ is the angle between z and the positive x axis (measured counterclockwise, in radians). The angle θ is called the **argument** of z, and r is called the **length (or modulus)** of z.

Theorem 4 – Polar Form

Let $z = re^{i\theta}$, $w = Re^{i\phi}$.

$$zw = rRe^{i(\theta+\phi)}$$
$$z^n = r^n e^{in\theta}$$

Definition 5 – Polynomial

A polynomial p with coefficient from $\mathbb{F}$ is an expression

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

where $c_i \in \mathbb{F}$. We call the field elements $c_0, \ldots, c_n$ the "coefficients" of p.
The largest exponent n so that $c_n \neq 0$ is called the **degree** of p, and we typically write $\deg p = n$.
Constant polynomials are degree 0.
The set of all polynomials over $\mathbb{F}$ is denoted by $P(\mathbb{F})$.
The set of all polynomials of degree **less than or equal** to n is denoted by $P_n(\mathbb{F})$.

Theorem 5 – Polynomial

Let $\mathbb{F}$ be a field, and $f, g \in P(\mathbb{F})$ be non-zero polynomials. Then there exist unique polynomials $q, r \in P(\mathbb{F})$ so that:

1) $f(x) = q(x)g(x) + r(x)$.
2) $\deg r < \deg g$ if $\deg g \neq 0$.
3) $r = 0$ if $\deg g = 0$.

Definition 6 – Polynomial Cont.

Let $\mathbb{F}$ be a field and $f, g \in P(\mathbb{F})$. We say that g divides f if $f(x) = q(x)g(x)$ for some polynomial $q \in P(\mathbb{F})$.
We say that a non-constant polynomial $p \in P(\mathbb{F})$ is "irreducible" if we **cannot** express p as a product of polynomials of smaller degree.
I.e. p is irreducible if we **cannot** write $p(x) = g(x)q(x)$ for any polynomials $g, q \in P(\mathbb{F})$ with the property that both $\deg g, \deg q < \deg p$.

$f(x) = x^2 - 2$ is irreducible over $\mathbb{Q}$ but not over $\mathbb{R}$.

Theorem 6 – Polynomial Cont.

Let $\mathbb{F}$ be a field, $p \in P(\mathbb{F})$ and $\deg p \geq 1$. Then $a \in \mathbb{F}$ is a root of p if and only if $x - a$ divides p.

Theorem 7 – Fundamental Theorem of Algebra

Every non-constant polynomial has a root over $\mathbb{C}$.
In fact, every non-constant polynomial factors completely into a product of linear terms over $\mathbb{C}$.

Chapter 2: Linear Systems

Definition 9 – Linear

Let $\mathbb{F}$ be a field and $b, c_1, \ldots, c_n \in \mathbb{F}$. An equation in the variables $x_1, \ldots, x_n$ is called **linear** if it can be expressed as $c_1x_1 + c_2x_2 + \cdots + c_nx_n = b$.

Definition 10 – System of Equations

Let $\mathbb{F}$ be a field, and $a_{ij} \in \mathbb{F}$ (where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$). A **system of linear equations** in variables $x_1, x_2, \ldots, x_n$ is a finite collection of linear equations in $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

A system of m equations with n unkowns is called an $m \times n$ **system**.

Definition 11 – Solutions

A **solution** to a linear equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = b$ is a choice of field elements $s_1, s_2, \ldots, s_n \in \mathbb{F}$, so that when we substitute them for $x_1, x_2, \ldots, x_n$ respectively, the resulting equation is true.
That is, we have $c_1s_1 + c_2s_2 + \cdots + c_ns_n = b$ (i.e. the left- and right-hand sides are equal.)
A **solution to a system** is a choice of field elements $s_1, s_2, \ldots, s_n$ which solves *every* equation of the system.

Definition 12 – Consistent

If a system of equations has at least one solution, we say it is **consistent**.
If a system of equations has no solutions, we say it is **inconsistent**.

Definition 13 – Matrix

An $m \times n$ **matrix** over $\mathbb{F}$ is a rectangular array of field elements consisting of m rows and n columns.
We denote the j^{th} entry in row i of matrix A, by a_{ij}, and call it the $ij^{\textbf{th}}$ **entry** of A.

Definition 14 – Augmented Matrix

Consider a system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We defie the **augmented matrix** *corresponding to* the system of equations above to be:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right),$$

Definition 15 – RREF

We say a matrix A is in **reduced row echelon form** if *all* of the following conditions are met:

1) All zero rows are at the bottom of the matrix A.
2) The first non-zero entry in each non-zero row is a 1. (Such entries are called "leading 1's".)
3) The leading 1's move to the right, as we go down the rows of A.
4) All entries above and below a leading 1 are 0.

We will use the abbreviation "RREF" for "row-reduced echelon form", for the rest of the text.

All matrices have a unique RREF.

Theorem 11 – Gaussian Elimination

To "row reduce" a matrix perform the following steps:

1) If the matrix consists entire of 0's, stop. It's already row-reduced.
2) Find the first column with a non-zero entry and move the corresponding row to the top. (We will call the first non-zero entry a.)
3) Divide the row by the number a to obtain a leading one.
4) Subtract multiples of ths row from the rows above and below, in order to make each entry above and below the leading 1 equal to 0.
5) Repeat 1-4 on the matrix consisting of the remaining rows.

Definition 16 – Variables

Suppose that R is a matrix in RREF. We say that x_i is a **leading variable** if column i contains a leading one. If a variable is not "leading" we call it a **non-leading variable**.

Remark 1. To solve a system:

1) Row reduce the augmented coefficient matrix.
2) If there is a row of the form $\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right)$ then there are no solutions.
3) Otherwise, assign the non-leading variables (if any) parameters, and use the equatoins coming from the rows of the RREF to solve for each variable interms of the parameters.

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Definition 17 – Homogeneous

A system of equations is called **homogeneous** if it is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

In other words, it is homogeneous if the constant term (or right hand side) of ***each*** equation in the system is 0.

- $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is ***always*** a solution to any homogeneous equation.
- We call this solution the ***trivial*** solution.
- Any other solution is called a ***non-trivial*** solution.

Chapter 3: Vector Spaces

Definition 18 – Vector Operators

Given two vectors **v,w** we define **<u>their sum</u>** **v+w** using "tip to tail" addition (or the "parallelogram law of addition"). In the diagram in the margin, the vector **v+w** is diagonal in the parallelogram spanned by v, w that shares its tail with v and w.
We can also define their **<u>difference</u>** **v-w** geometrically using the same parallelogram: v- w is the diagonal going from the tip of w to the tip of v.
Finally, given a vector v and real number $a \in \mathbb{R}$, we can **<u>scale v by a</u>** as follows:

- $0\mathbf{v} = \mathbf{0}$.
- If $a > 0$, then $a\mathbf{v}$ is a vector pointing in the same direction as v with length scaled by a
- If $\alpha < 0$, then αvis a vector pointing in the opposite direction as v with length scaled by $|\alpha|$.

If $v = (x, y, z)$ and $w = (p, q, r)$, then $v + w = (x + p, y + q, z + r)$, $av = (ax, ay, az)$.

Definition 20 – $\mathbb{F}^n$

Let $\mathbb{F}$ be a field. Consider the set $\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{F}\}$. We can define two operations on $\mathbb{F}^n$ which we call "vector addition" which is a map $\mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$, and "scaling" which is a map $\mathbb{F} \times \mathbb{F}^n \to \mathbb{F}^n$ as follows.
For $v = (x_1, x_2, \ldots, x_n), w = (y_1, y_2, \ldots, y_n) \in \mathbb{F}^n$, and $c \in \mathbb{F}$ we define:

$$\begin{aligned} v + w &= (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) && \text{(vector addition)} \\ &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ cv &= c(x_1, x_2, \ldots, x_n) && \text{(scaling)} \\ &= (cx_1, cx_2, \ldots, cx_n) \end{aligned}$$

Theorem 13 – $\mathbb{F}^n$

Let $\mathbb{F}$ be a field. Set $0 = (0, 0, \ldots, 0)$. For any $v, w, u \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$ we have:

1) $v + w = w + v$.
2) $v + (w + u) = (v + w) + u$.
3) $a(v + w) = av + aw$.
4) $(a + b)v = av + bv$.
5) $(ab)v = a(bv)$.
6) $1v = v$.
7) $0 + v = v$.
8) For every $v \in V$ there exists $w \in V$ so that $v + w = 0$.

Definition 21 – Vector Space

Let $\mathbb{F}$ be a field. A vector space V over $\mathbb{F}$ is a non-empty set, containing a special element 0, with two operatoins $V \times V \to V$ (called vector addition) and $\mathbb{F} \times V \to V$ (called scaling) so that for all $v, w, u \in V$ and $a, b \in \mathbb{F}$:

1) $v + w = w + v$.
2) $v + (w + u) = (v + w) + u$.
3) $a(v + w) = av + aw$.
4) $(a + b)v = av + bv$.
5) $(ab)v = a(bv)$.
6) $1v = v$.
7) $0 + v = v$.
8) For every $v \in V$ there exists $w \in V$ so that $v + w = 0$.

$P(\mathbb{F}), P_n(\mathbb{F})$ and $\mathbb{F}^n$ are vector spaces.

Definition 22 – Matrix Cont.

Let $\mathbb{F}$ be a field. An $m \times n$ **matrix** M **with entries in** $\mathbb{F}$ is a rectangular array of elements of $\mathbb{F}$ consisting of m rows and n columns.
We denote the entry in the i row and j column of a matrix m by m_{ij}.
The set of all $m \times n$ matrices with coefficients in $\mathbb{F}$ is denoted by $\mathcal{M}_{m\times n}(\mathbb{F})$.
For example, a 2×3 matrix looks like $\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{F})$, wile a 3×2 matrix N looks like $\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{pmatrix} \in \mathcal{M}_{3\times 2}(\mathbb{F})$.

$\mathcal{M}_{m\times n}(\mathbb{F})$, with pointwise addition and scaling is a vector space over $\mathbb{F}$;

Corollary 2 – Vector Space

Let $\mathbb{F}$ be a field, and V a vector space over $\mathbb{F}$. THen for any $v, w, u \in V$ and $a \in \mathbb{F}$ we have:

1) If $v + w = v + u$, then $w = u$.
2) If $a \neq 0$ and $av = aw$, then $v = w$.
3) The element $0 \in V$ is unique.
4) Additive inverses in V are unique. (This means that for each $v \in V$ there is only one element $w \in V$ which satisfies the condition of Axiom 8.)
5) $(-a)v = -(av)$. In particupar $(-1)v = -v$.
6) $0v = 0$.
7) $a0 = 0$.

Definition 23 – Subspace

Let $\mathbb{F}$ be a field and V a vector space over $\mathbb{F}$. We say that a subset $W \subseteq V$ is **a subspace of** V if W is also a vector space over $|F$ using the same operations defined in V.

$P_n(\mathbb{F})$ is a subspace of $P(\mathbb{F})$.
$P_n(\mathbb{F})$ is a subspace of $P_m(\mathbb{F})$ if $n < m \in \mathbb{N}$.

Theorem 19 – Subspace

Let V be a vector space over a field $\mathbb{F}$. A **non-empty** subset $W \subseteq V$ is a subspace of V if and only if

1) For all $v, w \in W$ we have $v + w \in W$.
2) For all $v \in W$ and $c \in \mathbb{F}$ we have $cv \in W$.

Definition 24 – Trivial / Non-Trivial Subspace

Let V be a vector space over a field $\mathbb{F}$. The subspaces $\{0\}$ and V are called the **trivial subspaces** of V. Any other subspace W of V is called a **non-trivial subspace of** V.
In particular, we say that a subspace W is a non-trivial subspace of V if $W \neq \{0\}$ and $W \neq V$.

Chapter 4: Bases and Dimension

Definition 25 – Linear Combinatoin of Vectors

Let V be a vector space over $\mathbb{F}$, and $v_1, v_2, \ldots, v_k \in V$. A vector of the form $a_1v_1 + a_2v_2 + \cdots + a_kv_k \in V$ is called a linear combination of the vectors $v_1, v_2, \ldots, v_k$.

Definition 26 – Span

Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. We define the **span of** S, denoted $\operatorname{span} S$, as follows:

1) If $S = \varnothing$ is empty, then $\operatorname{span} S = \{0\}$.
2) Otherwise, $\operatorname{span} S = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k | a_i \in \mathbb{F} v_i \in S\}$ is the set of all possible linear combinations of vectors from S.

Theorem 20 – Span as Subspace

Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$ be **any** subset of vectors. Then the subset $\operatorname{span} S \subseteq V$ is a subspace of V.

Definition 27 – S spans V

Let V be a vector space over $\mathbb{F}$. We say that a subset $S \subseteq V$ is **a spanning set for** V (or "S **spans** V") if $V = \operatorname{span} S$.

Definition 28 – Linearly Independnet

Let V be a vector space over $\mathbb{F}$. We say that a set S is **linearly independent** if for any vectors $v_1, v_2, \ldots, v_k \in S$:

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \implies c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

Otherwise, we say that S is **linearly dependent**.

Definition 29 – Basis

Let V be a vector space over $\mathbb{F}$. A subset $\beta \subseteq V$ is called a **basis** if:

1) β spans V
2) β is linearly independent.

Vector space over $\mathbb{F}$ also has basis.
Finite spanning set for V also contains basis for V.

Let $\mathbb{F}$ be a field.

1) The set $\{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{F}^n$.
2) The set $\{E_{ij} | 1 \leq i \leq m, 1 \leq n\}$ is a basis for $\mathcal{M}_{m\times n}(\mathbb{F})$.
3) The set $\{1, x, x^2, x^3, \ldots\}$ is a basis for $P(\mathbb{F})$.
4) The set $\{1, x, x^2, x^3, \ldots, x^n\}$ is a basis for $P_n(\mathbb{F})$.

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Theorem 23 – Unique Expression from Basis

Let V be a vector space over $\mathbb{F}$ and β a basis of V. Then any $\mathbf{v} \in V$ has a unique expression

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$$

where $\mathbf{v}_i \in \beta$ and $a_i \in \mathbb{F}$.

Theorem 24 – The Replacement Theorem

Suppose that $\beta = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for V and $I = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ an independent subset of V. Then for any $i \in \{1, ..., k\}$, we can obtain a new basis by replacing i elements of β with $\{\mathbf{w}_1, \ldots, \mathbf{w}_i\}$. So after relabelling the elements* $\mathbf{v}_j \in \beta$ we have that the set $\beta_i = \{\mathbf{w}_1, \ldots, \mathbf{w}_i, \mathbf{v}_{i+1}, \ldots \mathbf{v}_n\}$ is a basis for V.

Corollary 3

Suppose that V is a vector space over $\mathbb{F}$ with a finite basis. Suppose that β is any basis of V and I is any independent set. Then $|I| \leq |B|$.

Theorem 25 – Size of Bases

Let V be a vector space over a field $\mathbb{F}$. If V has a finite basis, then all bases of V have the same size.

Definition 30 – Dimension

Let V be a vector space over $\mathbb{F}$ with a finite basis. We define the **dimension of** V to be the size of a basis for V.
In this case, we say that V **is finite dimensional**. Otherwise, we say that V is infinite dimenisonal.

1) $\dim \mathbb{F}^n = n$.
2) $\dim \mathscr{M}_{m\times n}(\mathbb{F}) = mn$.
3) $\dim P_n(\mathbb{F}) = n + 1$.
4) $P(\mathbb{F})$ is infinite dimensional.

Corollary 4

Let V be a finite dimensional vector space over $\mathbb{F}$. S any spanning set for V, I any independent set in V, and β any basis. Then

$$|I| \leq |\beta| \leq |S|.$$

Chapter 5: Linear Transformations

Definition 31 – Linear Transformation

Let V and W be vector spaces over $\mathbb{F}$. A map $T : V \to W$ is called a **linear transformation** if:

$$T(v + w) = T(v) + T(w) \qquad \text{for all } v, w \in V$$
$$T(cv) = cT(v) \qquad \text{for all } v \in V \text{ and } c \in \mathbb{F}$$

$T(x, y, z) = (2x - 4y + z, 3x - y + 2x)$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.
$T(p) = \frac{\mathrm{d}}{\mathrm{d}x}p$ is a linear transformation.

Theorem 26 – Properties of Linearity

Let V, W be vector spaces over $\mathbb{F}$.

1) If $T : V \to W$ is linear, then $T(0_v) = 0_w$.
2) The map $O : V \to W$ given by $O(v) = 0_w$ for all $v \in V$ is linear. This map is called the "zero map."
3) The map $I_V : V \to V$ given by $I_V(v) = v$ for all $v \in V$ is linear. This map is called the "identity map."

Theorem 27

Let V e a finite dimensional vector space over $\mathbb{F}$ and $\beta = \{v_1, \ldots, v_n\}$ a basis of V. A linear map $T : V \to W$ is uniquely determined by the values $T(v_1), T(v_2), \ldots, T(v_n) \in W$.

Corollary 5 – Extending by Linearity

Let V, W be vector spaces over $\mathbb{F}$, and $\beta = \{v_1, \ldots, v_n\}$ a basis for V. Given a list of (not necessarily distinct) vectors $w_1, \ldots, w_n \in W$ there is exactly one linear map $T : V \to W$ so that $W(v_i) = w_i$.

This map is defined for all $v \in V$ as follows. Writing $v = \sum_{i=1}^{n} a_i v_i$, we then set $T(v) = \sum_{i=1}^{n} a_i w_i$.

This process is called "extending by linearity".

Theorem 28 – Composition of Linear Maps

Let V, W, X be vector spaces over $\mathbb{F}$. If $T : V \to W$ and $S : W \to X$ are linear maps, then the composition $S \circ T : V \to X$ is linear.

Theorem 29 – Null Space / Image

Let V, W be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear transformation. The sets:

$$N(T) = \{v \in V \mid T(v) = 0\} \subseteq V$$
$$\operatorname{im}(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\} \subseteq W$$

are subspaces of V, W respectively.
The subspace $N(T)$ is called the **null space of** T.
The subspace $\operatorname{im}(T)$ is called the **image of** T.

Definition 32 – rank

Let V, W be vector spaces over $\mathbb{F}$ and $T : V \to W$ linear. We define the **rank** of T by $\operatorname{rank} T = \dim \operatorname{im}(T)$.

Theorem 30 – The Dimension Theorem

Let V, W be finite dimenisonal vector spaces over $\mathbb{F}$. If $T : V \to W$ linear, then

$$\dim V = \dim N(T) + \dim \operatorname{im}(T)$$

.

Definition 33 – $\mathcal{L}(V, W)$

Let V, W be vector spaces over a field $\mathbb{F}$. We denote by $\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is linear}\}$ the set of all linear transformations from V to W.
We define addition and scaling of transformations by:

$$(T + S)(v) = T(v) + S(v) \qquad \text{(we can add } T(v), S(v) \in W)$$
$$(cT)(v) = cT(v) \qquad \text{(we can scale the vector } T(v) \in W \text{ by } c \in \mathbb{F})$$

Theorem 31

Let V, W be vector spaces over a field $\mathbb{F}$. If $S, T \in \mathcal{L}(V, W)$, and $c \in \mathbb{F}$ then $T + S, cT \in \mathcal{L}(V, W)$.

Theorem 32

Let V, W be vector spaces over a field $\mathbb{F}$.

1) The set $\mathcal{L}(V, W)$ is a vector space over $\mathbb{F}$.
2) If $\dim V = n, \dim W = m$, then $\dim \mathcal{L}(V, W) = nm$.

Definition 34 – Invertibility

Let A, B be sets and $f : A \to B$ be a function.

1) We say that f is injective if for all $x, y \in A, f(x) = f(y) \implies x = y$.
2) We say that f is surjective if $\operatorname{im} f = f(A) = B$.
3) We say that f is bijective if it is both injective and surjective.
4) We say that f is **invertible**, if there exists a function $g : W \to V$ so that $g \circ f = I_A : A \to A$ and $f \circ g = I_B : B \to B$.
   In this case we call the map g the **inverse** of f and denote it by f^{-1}.

Theorem 33

Let V, W be vector spaces over $\mathbb{F}$. If $T : V \to W$ is linear and bijective, then the inverse $T^{-1} : W \to V$ is also linear.

Definition 35 – Isomorphism

Let V, W be vector spaces over the field $\mathbb{F}$. We say that a linear map $T : V \to W$ is an **isomorphism** if it is bijective.
We say that V is **isomorphic to** W, and write $V \simeq W$, if there exists an isomorphism $T : V \to W$.

Theorem 34

Let V, W be vector spaces over $\mathbb{F}$ and $T : V \to W$ be linear. Then T is injective if and only if $N(T) + \{0_v\}$.

Theorem 35

Let V, W be finite dimensional vector spaces over $\mathbb{F}$. Then $V \simeq W$ if and only if $\dim V = \dim W$.

Theorem 36

Let V, W be finite dimensional vector spaces over the field $\mathbb{F}$.
If $T : V \to W$ is linear and $\dim V = \dim W$, then the following are equivalent:

1) T is injective.
2) T is surjective.
3) T is an isomorphism.

Theorem 37

The maps $R_\theta, \operatorname{proj}_L, \mathbb{R}_L : \mathbb{R}^2 \to \mathbb{R}^2$ are linear.

Theorem 38

The maps $R_{\theta,L}, \operatorname{proj}_L, R_P : \mathbb{R}^3 \to \mathbb{R}^3$ are linear.

Chapter 6: Coordinates

Theorem 39

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$. Define $T_A : \mathbb{F}^n \to \mathbb{F}^m$ by $T_A(x) = Ax$.

1) The map T_A is linear.
2) The map $F : \mathscr{M}_{m\times n}(\mathbb{F}) \to \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ given by $F(A) = T_A$ is an isomorphism of vector spaces. That is, every linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ is given by matrix multiplication for some matrix A.

Definition 36

Let V be a finite dimensional vector space over $\mathbb{F}$. An **ordered basis** for V is a basis β, together with a **fixed** order for listing its elements: $\beta = \{v_1, \ldots, v_n\}$.

Theorem 40

Let V be a vctor space over $\mathbb{F}$ of dimension n, and $\beta = \{v_1, \ldots, v_n\}$ a basis of V. The map $\phi_\beta : V \to \mathbb{F}^n$ defined above is an isomorphism of vector spaces.
We will denote by $[v]_\beta$ the element $\phi_\beta(v)$. I.e. $\phi_\beta(v) = [v]_\beta$.
We will call a choice of basis on V, together with the isomorphism $\phi_beta : V \to \mathbb{F}^n$ a "**coordinate system**" on V.

Definition 37

Let V, W be finite dimensional vector spaces over $\mathbb{F}$. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W. For $T \in \mathcal{L}(V, W)$ we define the matrix $[T]_\beta^\gamma \in \mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$ as follows:

1) The columns of $[T]_\beta^\gamma$ are given by: $[T(v_1)]_\gamma, \ldots, [T(v_n)]_\gamma$.Alternatively: we can express $T(v_j) \in W$ using the basis γ to obtain an expression $T(v_j) = \displaystyle\sum_{i=1}^{m} A_{ij} w_i$. We then define $\left([T]_\beta^\gamma\right)_{ij} = A_{ij}$.

When $T : V \to V$ we denote by $[T]_\beta = [T]_beta^\beta$.

Theorem 41

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, $\beta = \{v_1, \ldots, v_n\}$ a basis of V, and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W.
The map $\phi_\beta^\gamma : \mathcal{L}(V, W) \to \mathscr{M}_{m\times n}(\mathbb{F})$ given by $\phi_\beta^\gamma(T) = [T]_\beta^\gamma$ is an isomorphism of vector spaces.

Theorem 42

Let V, W, X be finite dimensional vector spaces over $\mathbb{F}$. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V, $\gamma = \{w_1, \ldots, w_m\}$ a basis of W, and $\delta = \{x_1, \ldots, x_p\}$ a basis of X.
For all $T \in \mathscr{L}(V, W)$ and $S \in \mathscr{L}(W, X)$ we have that

$$[S \circ T]_\beta^\delta = [S]_gamma^\delta [T]_\beta^\gamma.$$

In other words, when using compatible coordinate systems, the composition of linear transformations corresponds to matrix multiplication.

Theorem 43

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, $T : V \to W$ be a linear map, $\beta = \{v_1, \ldots, v_n\}$ a basis of V, and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible.
Moreover, we have

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$$

if T is invertible.

Theorem 44

Let V, W be finite dimensional vector spaces over a field $\mathbb{F}$, $\beta = \{v_1, \ldots, v_n\}$ a basis of V, and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W. Then we have

$$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma$$

for all $v \in V$.

Chapter 7: Matrix Algebra (Appendix A)

Definition 38 – Matrix Multiplication

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$ and $B \in \mathscr{M}_{n\times k}(\mathbb{F})$. We define their product $AB \in \mathscr{M}_{m\times k}(\mathbb{F})$ as follows: for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k\}$ the ij-entry of the product AB is given by

$$(AB)_{ij} = \sum_{l=1}^{n} A_{il}B_{lj}.$$

$AB \neq BA.$

Definition 39 – Special Matrices

For each $n, m \in \mathbb{N}$ we define the following matrices:

1) $O_{m,n} \in \mathscr{M}_{m\times n}(\mathbb{F})$ - the matrix consisting of all 0's. In other words $(O_{m,n})_{i,j} = 0$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
2) $I_n \in \mathscr{M}_{n\times n}(\mathbb{F})$ - the matrix with 1's on the diagonals, and 0 in all other entries. In other words

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$O_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Theorem 45

Let $\mathbb{F}$ be a field, $A, A_1, A_2 \in \mathscr{M}_{\mathbf{m}\times\mathbf{n}}(\mathbb{F}), B, B_1, B_2 \in \mathcal{M}_{n\times k}(\mathbb{F}), C \in \mathcal{M}_{k\times p}(\mathbb{F})$ and $c \in \mathbb{F}$.

1. $A(BC) = (AB)C$
2. $(A_1 + A_2)B = A_1B + A_2B$
3. $A(B_1 + B_2) = AB_1 + AB_2$
4. $I_mA = A = AI_n$
5. $\mathrm{O}_{rm}A = \mathrm{O}_{rn}$ for any $r \in \mathbb{N}$.
6. $A(cB) = c(AB) = (c\mathrm{I}_m)AB = AB(c\mathrm{I}_k) = A(c\mathrm{I}_n)B.$

Definition 40 – Invertibility

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. We say that A is **invertible** if there exists a matrix $B \in \mathscr{M}_{n\times n}(\mathbb{F})$ so that $AB = I_n = BA$.

Theorem 46

Let $A, B \in \mathscr{M}_{n\times n}(\mathbb{F})$.

1) If A is invertible, then the inverse of A is unique.
2) If A is invertible, then A^{-1} is also invertible.
3) If A and B are invertible, then AB is invertible.
4) I_n is invertible.
5) If $AB = I_n$, then A is invertible and $B = A^{-1}$.

Definition 41 – A^t

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$. We define the matrix $A^t \in \mathscr{M}_{n\times m}(\mathbb{F})$ by:

$$(A^t)_{ij} = A_{ji}.$$

In other words, to obtain A^t we "swap the rows and columns of A."

Definition 42 – Symmetric

We say that $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ is **symmetric** if $A^t = A$. We denot ethe set of all symmetric matrices by $\mathbf{Sym}_n(\mathbb{F})$.
We say that $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ is **skew-symmetric** if $A^t = -A$. We denote the set of all skew-symmetric matrices by $\mathbf{Sk}_n(\mathbb{F})$.

Theorem 47

Let $A, B \in \mathscr{M}_{\mathbf{m}\times\mathbf{n}}(\mathbb{F}), C \in \mathscr{M}_{n\times k}(\mathbb{F})$ and $c \in \mathbb{F}$.
1. $(A+B)^t = A^t + B^t$
2. $(cA)^t = cA^t$
3. $(A^t)^t = A$
4. $(AC)^t = C^t A^t$.
5. In the case that $m = n$, we also have that if $A \in \mathscr{M}_{\mathbf{n}\times\mathbf{n}}(\mathbb{F})$ is invertible, then $A^t \in \mathscr{M}_{\mathbf{n}\times\mathbf{n}}(\mathbb{F})$ is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Definition 43 – Diagonal and Triangular

We say that A is **diagonal** if $A_{ij} = 0$ for all $i \neq j$.
Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. We say that A is **upper triangular** if $A_{ij} = 0$ for all $i > j$. This means that all entries below the diagonal of A must be 0.
Similarily, we say that A is **lower triangular** if $A_{ij} = 0$ for all $i < j$. This means that all entries above the diagonal of A must be 0.
We say that A is **strictly upper-triangular** if $A_{ij} = 0$ for all $i \geq j$.