



MAT240: Algebra I

Definition Sheet

Author: Joseph Siu

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Info: Most of the numbers are correct.
This sheet also does not include the examples.



Chapter 1: Fields and Polynomials

Definition 1 – Field

A field \mathbb{F} is a set with two special elements " $0 \in \mathbb{F}$ " and " $1 \in \mathbb{F}$ " and two operations "+" and "·" which satisfy the following axioms.

1. (Commutativity) For all $x, y \in \mathbb{F}$ we have: $x + y = y + x$ and $x \cdot y = y \cdot x$.
2. (Associativity) For all $x, y, z \in \mathbb{F}$ we have: $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
3. (Distributivity) For all $x, y, z \in \mathbb{F}$ we have: $x \cdot (y + z) = x \cdot y + x \cdot z$.
4. (Identities) For all $x \in \mathbb{F}$ we have: $x + 0 = x$ and $x \cdot 1 = x$.
5. (Inverses) For all $x \in \mathbb{F}$ there exists $y \in \mathbb{F}$ such that $x + y = 0$. For all $x \in \mathbb{F} \setminus \{0\}$ there exists $z \in \mathbb{F}$ such that $x \cdot z = 1$.

\mathbb{Q} is a field where \mathbb{Z} is not a field because of the absence of some multiplicative inverse.



Corollary 1 – Field

Let \mathbb{F} be a field and $a, b, c \in \mathbb{F}$.

1. If $a + c = b + c$, then $a = b$.
2. If $c \neq 0$ and $c \cdot a = c \cdot b$, then $a = b$.
3. The field elements $0, 1$ are unique.
4. The elements y and z from Axiom 5 are unique. (From now on, we will denote the additive inverse of x by $-x$, and the multiplicative inverse of x by x^{-1} .)
5. $a \cdot 0 = 0$.
6. $(-a) \cdot (b) = -(a \cdot b) = (a) \cdot (-b)$.
7. $-(-a) = a$. If $a \neq 0$, then $(a^{-1})^{-1} = a$.
8. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.



Theorem 1 – Equivalence

1. \sim is an equivalence relation.
2. $a \sim b$ if and only if a and b have the same remainder when divided by n .
3. There are exactly n equivalence classes for this relation: $[0], [1], \dots, [n-1]$ - one for each possible remainder for division by n .

Definition 2 – \mathbb{Z}_n

Let $\mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$ be the set of equivalence classes for this equivalence relation. We define $+, \cdot$ on \mathbb{Z}_n as follows:

$$[a] + [b] = [a + b],$$

$$[a] \cdot [b] = [a \cdot b].$$



Theorem 2 – Quadratic Formula

Let $a, b, c \in \mathbb{R}$. The quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) has:

1. Solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if $b^2 - 4ac \geq 0$.
2. No solutions if $b^2 - 4ac < 0$.



Definition 3 – Complex

Let $i = \sqrt{-1}$. I.e. i is a number with the property that $i^2 = -1$.

Let $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$. We call \mathbb{C} the set of complex numbers and we define addition and multiplication $+, \cdot$ in the obvious ways:

$$(x + yi) + (a + bi) = (x + a) + (y + b)i$$

$$(x + yi) \cdot (a + bi) = ax + bi + ayi + byi^2 = (ax - by) + (ay + bx)i$$

\mathbb{C} is a field.



Given a complex number $z = x + yi$, we define its **conjugate** by:

$$\bar{z} = x - yi.$$

We define the **length (or modulus)** of a complex number by:

$$|z| = \sqrt{x^2 + y^2}.$$

Note that we in the xy -plane, we obtain \bar{z} , the conjugate of z , by reflecting z in the x -axis, and the length of a complex number is just the usual distance from z to the origin in the xy -plane.

**Theorem 3 – Complex**

For any $z, w \in \mathbb{C}$ we have:

1. $\bar{z} + \bar{w} = \bar{z} + \bar{w}$.
2. $\bar{z} \cdot \bar{w} = \bar{z} \cdot \bar{w}$.
3. $\frac{\bar{z}}{\bar{w}} = \frac{\bar{z}}{\bar{w}}$ (provided $w \neq 0$).
4. $\frac{w}{\bar{z}} = z$.
5. $z\bar{z} = |z|^2$.
6. $z^{-1} = \frac{\bar{z}}{|z|^2}$ (provided $z \neq 0$).
7. $|zw| = |z||w|$.
8. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ (provided $w \neq 0$).
9. $|z + w| \leq |z| + |w|$ ("Triangle inequality for Complex Numbers").

**Definition 4 – Polar Form**

For $z = x + yi$, we define its **polar form** as $z = re^{i\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ and θ is the angle between z and the positive x axis (measured counterclockwise, in radians). The angle θ is called the **argument** of z , and r is called the **length (or modulus)** of z .

**Theorem 4 – Polar Form**

Let $z = re^{i\theta}$, $w = Re^{i\phi}$.

$$zw = rRe^{i(\theta+\phi)}$$

$$z^n = r^n e^{in\theta}$$



Definition 5 – Polynomial

A polynomial p with coefficient from \mathbb{F} is an expression

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

where $c_i \in \mathbb{F}$. We call the field elements c_0, \dots, c_n the "coefficients" of p .

The largest exponent n so that $c_n \neq 0$ is called the **degree** of p , and we typically write $\deg p = n$. Constant polynomials are degree 0.

The set of all polynomials over \mathbb{F} is denoted by $P(\mathbb{F})$.

The set of all polynomials of degree **less than or equal** to n is denoted by $P_n(\mathbb{F})$.

**Theorem 5 – Polynomial**

Let \mathbb{F} be a field, and $f, g \in P(\mathbb{F})$ be non-zero polynomials. Then there exist unique polynomials $q, r \in P(\mathbb{F})$ so that:

1. $f(x) = q(x)g(x) + r(x)$.
2. $\deg r < \deg g$ if $\deg g \neq 0$.
3. $r = 0$ if $\deg g = 0$.

**Definition 6 – Polynomial Cont.**

Let \mathbb{F} be a field and $f, g \in P(\mathbb{F})$. We say that g divides f if $f(x) = q(x)g(x)$ for some polynomial $q \in P(\mathbb{F})$.

We say that a non-constant polynomial $p \in P(\mathbb{F})$ is "irreducible" if we **cannot** express p as a product of polynomials of smaller degree.

I.e. p is irreducible if we **cannot** write $p(x) = g(x)q(x)$ for any polynomials $g, q \in P(\mathbb{F})$ with the property that both $\deg g, \deg q < \deg p$.

$$f(x) = x^2 - 2 \text{ is irreducible over } \mathbb{Q} \text{ but not over } \mathbb{R}.$$

**Theorem 6 – Polynomial Cont.**

Let \mathbb{F} be a field, $p \in P(\mathbb{F})$ and $\deg p \geq 1$. Then $a \in \mathbb{F}$ is a root of p if and only if $x - a$ divides p .

**Theorem 7 – Fundamental Theorem of Algebra**

Every non-constant polynomial has a root over \mathbb{C} .

In fact, every non-constant polynomial factors completely into a product of linear terms over \mathbb{C} .



Chapter 2: Linear Systems

Definition 9 – Linear

Let \mathbb{F} be a field and $b, c_1, \dots, c_n \in \mathbb{F}$. An equation in the variables x_1, \dots, x_n is called **linear** if it can be expressed as $c_1x_1 + c_2x_2 + \dots + c_nx_n = b$.

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Definition 10 – System of Equations

Let \mathbb{F} be a field, and $a_{ij} \in \mathbb{F}$ (where $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$). A **system of linear equations** in variables x_1, x_2, \dots, x_n is a finite collection of linear equations in x_1, x_2, \dots, x_n :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m. \end{aligned}$$

A system of m equations with n unknowns is called an $m \times n$ **system**.

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Definition 11 – Solutions

A **solution** to a linear equation $c_1x_1 + c_2x_2 + \dots + c_nx_n = b$ is a choice of field elements $s_1, s_2, \dots, s_n \in \mathbb{F}$, so that when we substitute them for x_1, x_2, \dots, x_n respectively, the resulting equation is true. That is, we have $c_1s_1 + c_2s_2 + \dots + c_ns_n = b$ (i.e. the left- and right-hand sides are equal.) A **solution to a system** is a choice of field elements s_1, s_2, \dots, s_n which solves *every* equation of the system.

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Definition 12 – Consistent

If a system of equations has at least one solution, we say it is **consistent**.

If a system of equations has no solutions, we say it is **inconsistent**.

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Definition 13 – Matrix

An $m \times n$ **matrix** over \mathbb{F} is a rectangular array of field elements consisting of m rows and n columns.

We denote the j^{th} entry in row i of matrix A , by a_{ij} , and call it the ij^{th} **entry** of A .

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Definition 14 – Augmented Matrix

Consider a system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

We define the **augmented matrix** corresponding to the system of equations above to be:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right),$$

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Definition 15 – RREF

We say a matrix A is in **reduced row echelon form** if *all* of the following conditions are met:

1. All zero rows are at the bottom of the matrix A .
2. The first non-zero entry in each non-zero row is a 1. (Such entries are called “leading 1’s”.)
3. The leading 1’s move to the right, as we go down the rows of A .
4. All entries above and below a leading 1 are 0.

We will use the abbreviation ”RREF” for ”row-reduced echelon form”, for the rest of the text.

All matrices have a unique RREF.

**Theorem 11 – Gaussian Elimination**

To ”row reduce” a matrix perform the following steps:

1. If the matrix consists entirely of 0’s, stop. It’s already row-reduced.
2. Find the first column with a non-zero entry and move the corresponding row to the top. (We will call the first non-zero entry a .)
3. Divide the row by the number a to obtain a leading one.
4. Subtract multiples of this row from the rows above and below, in order to make each entry above and below the leading 1 equal to 0.
5. Repeat 1-4 on the matrix consisting of the remaining rows.

**Definition 16 – Variables**

Suppose that R is a matrix in RREF. We say that x_i is a **leading variable** if column i contains a leading one. If a variable is not ”leading” we call it a **non-leading variable**.



Remark 1. To solve a system:

1. Row reduce the augmented coefficient matrix.
2. If there is a row of the form $(\ 0 \ 0 \ \cdots \ 0 \ | \ 1)$ then there are no solutions.
3. Otherwise, assign the non-leading variables (if any) parameters, and use the equations coming from the rows of the RREF to solve for each variable in terms of the parameters.



Definition 17 – Homogeneous

A system of equations is called **homogeneous** if it is of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0.$$

In other words, it is homogeneous if the constant term (or right hand side) of *each* equation in the system is 0.

- $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is *always* a solution to any homogeneous equation.
- We call this solution the *trivial* solution.
- Any other solution is called a *non-trivial* solution.



Chapter 3: Vector Spaces

Definition 18 – Vector Operators

Given two vectors \mathbf{v}, \mathbf{w} we define their sum $\mathbf{v} + \mathbf{w}$ using “tip to tail” addition (or the “parallelogram law of addition”). In the diagram in the margin, the vector $\mathbf{v} + \mathbf{w}$ is diagonal in the parallelogram spanned by \mathbf{v}, \mathbf{w} that shares its tail with \mathbf{v} and \mathbf{w} .

We can also define their difference $\mathbf{v} - \mathbf{w}$ geometrically using the same parallelogram: $\mathbf{v} - \mathbf{w}$ is the diagonal going from the tip of \mathbf{w} to the tip of \mathbf{v} .

Finally, given a vector \mathbf{v} and real number $a \in \mathbb{R}$, we can scale \mathbf{v} by a as follows:

- $0\mathbf{v} = \mathbf{0}$.
- If $a > 0$, then $a\mathbf{v}$ is a vector pointing in the same direction as \mathbf{v} with length scaled by a .
- If $a < 0$, then $a\mathbf{v}$ is a vector pointing in the opposite direction as \mathbf{v} with length scaled by $|a|$.

If $\mathbf{v} = (x, y, z)$ and $\mathbf{w} = (p, q, r)$, then $\mathbf{v} + \mathbf{w} = (x + p, y + q, z + r)$, $a\mathbf{v} = (ax, ay, az)$.



Definition 20 – \mathbb{F}^n

Let \mathbb{F} be a field. Consider the set $\mathbb{F}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{F}\}$. We can define two operations on \mathbb{F}^n which we call ”vector addition” which is a map $\mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}^n$, and ”scaling” which is a map $\mathbb{F} \times \mathbb{F}^n \rightarrow \mathbb{F}^n$ as follows.

For $\mathbf{v} = (x_1, x_2, \dots, x_n), \mathbf{w} = (y_1, y_2, \dots, y_n) \in \mathbb{F}^n$, and $c \in \mathbb{F}$ we define:

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) && \text{(vector addition)} \\ &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ c\mathbf{v} &= c(x_1, x_2, \dots, x_n) && \text{(scaling)} \\ &= (cx_1, cx_2, \dots, cx_n) \end{aligned}$$



Theorem 13 – \mathbb{F}^n

Let \mathbb{F} be a field. Set $\mathbf{0} = (0, 0, \dots, 0)$. For any $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$ we have:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
2. $\mathbf{v} + (\mathbf{w} + \mathbf{u}) = (\mathbf{v} + \mathbf{w}) + \mathbf{u}$.
3. $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$.
4. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
5. $(ab)\mathbf{v} = a(b\mathbf{v})$.
6. $1\mathbf{v} = \mathbf{v}$.
7. $\mathbf{0} + \mathbf{v} = \mathbf{v}$.
8. For every $\mathbf{v} \in V$ there exists $\mathbf{w} \in V$ so that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.



Definition 21 – Vector Space

Let \mathbb{F} be a field. A vector space V over \mathbb{F} is a non-empty set, containing a special element 0 , with two operations $V \times V \rightarrow V$ (called vector addition) and $\mathbb{F} \times V \rightarrow V$ (called scaling) so that for all $v, w, u \in V$ and $a, b \in \mathbb{F}$:

1. $v + w = w + v$.
2. $v + (w + u) = (v + w) + u$.
3. $a(v + w) = av + aw$.
4. $(a + b)v = av + bv$.
5. $(ab)v = a(bv)$.
6. $1v = v$.
7. $0 + v = v$.
8. For every $v \in V$ there exists $w \in V$ so that $v + w = 0$.

$P(\mathbb{F})$, $P_n(\mathbb{F})$ and \mathbb{F}^n are vector spaces.

**Definition 22 – Matrix Cont.**

Let \mathbb{F} be a field. An $m \times n$ **matrix** M with entries in \mathbb{F} is a rectangular array of elements of \mathbb{F} consisting of m rows and n columns.

We denote the entry in the i row and j column of a matrix m by m_{ij} .

The set of all $m \times n$ matrices with coefficients in \mathbb{F} is denoted by $\mathcal{M}_{m \times n}(\mathbb{F})$.

For example, a 2×3 matrix looks like $\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \in \mathcal{M}_{2 \times 3}(\mathbb{F})$, while a 3×2 matrix N looks like $\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{pmatrix} \in \mathcal{M}_{3 \times 2}(\mathbb{F})$.

$\mathcal{M}_{m \times n}(\mathbb{F})$, with pointwise addition and scaling is a vector space over \mathbb{F} ;

**Corollary 2 – Vector Space**

Let \mathbb{F} be a field, and V a vector space over \mathbb{F} . Then for any $v, w, u \in V$ and $a \in \mathbb{F}$ we have:

1. If $v + w = v + u$, then $w = u$.
2. If $a \neq 0$ and $av = aw$, then $v = w$.
3. The element $0 \in V$ is unique.
4. Additive inverses in V are unique. (This means that for each $v \in V$ there is only one element $w \in V$ which satisfies the condition of Axiom 8.)
5. $(-a)v = -(av)$. In particular $(-1)v = -v$.
6. $0v = 0$.
7. $a0 = 0$.



Definition 23 – Subspace

Let \mathbb{F} be a field and V a vector space over \mathbb{F} . We say that a subset $W \subseteq V$ is a **subspace of V** if W is also a vector space over \mathbb{F} using the same operations defined in V .

$P_n(\mathbb{F})$ is a subspace of $P(\mathbb{F})$.

$P_n(\mathbb{F})$ is a subspace of $P_m(\mathbb{F})$ if $n < m \in \mathbb{N}$.

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Theorem 19 – Subspace

Let V be a vector space over a field \mathbb{F} . A **non-empty** subset $W \subseteq V$ is a subspace of V if and only if

1. For all $v, w \in W$ we have $v + w \in W$.
2. For all $v \in W$ and $c \in \mathbb{F}$ we have $cv \in W$.

▼

Definition 24 – Trivial / Non-Trivial Subspace

Let V be a vector space over a field \mathbb{F} . The subspaces $\{0\}$ and V are called the **trivial subspaces** of V . Any other subspace W of V is called a **non-trivial subspace of V** .

In particular, we say that a subspace W is a non-trivial subspace of V if $W \neq \{0\}$ and $W \neq V$.

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Chapter 4: Bases and Dimension

Definition 25 – Linear Combinatoin of Vectors

Let V be a vector space over \mathbb{F} , and $v_1, v_2, \dots, v_k \in V$. A vector of the form $a_1v_1 + a_2v_2 + \dots + a_kv_k \in V$ is called a linear combination of the vectors v_1, v_2, \dots, v_k .



Definition 26 – Span

Let V be a vector space over \mathbb{F} and $S \subseteq V$. We define the **span of S** , denoted $\text{span } S$, as follows:

1. If $S = \emptyset$ is empty, then $\text{span } S = \{0\}$.
2. Otherwise, $\text{span } S = \{a_1v_1 + a_2v_2 + \dots + a_kv_k \mid a_i \in \mathbb{F}, v_i \in S\}$ is the set of all possible linear combinations of vectors from S .



Theorem 20 – Span as Subspace

Let V be a vector space over \mathbb{F} and $S \subseteq V$ be **any** subset of vectors. Then the subset $\text{span } S \subseteq V$ is a subspace of V .



Definition 27 – S spans V

Let V be a vector space over \mathbb{F} . We say that a subset $S \subseteq V$ is a **spanning set for V** (or " S spans V ") if $V = \text{span } S$.



Definition 28 – Linearly Independnet

Let V be a vector space over \mathbb{F} . We say that a set S is **linearly independent** if for any vectors $v_1, v_2, \dots, v_k \in S$:

$$c_1v_1 + c_2v_2 + \dots + c_kv_k = 0 \implies c_1 = 0, c_2 = 0, \dots, c_k = 0.$$

Otherwise, we say that S is **linearly dependent**.



Definition 29 – Basis

Let V be a vector space over \mathbb{F} . A subset $\beta \subseteq V$ is called a **basis** if:

1. β spans V
2. β is linearly independent.

Vector space over \mathbb{F} also has basis.

Finite spanning set for V also contains basis for V .



Let \mathbb{F} be a field.

1. The set $\{e_1, e_2, \dots, e_n\}$ is a basis for \mathbb{F}^n .
2. The set $\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $\mathcal{M}_{m \times n}(\mathbb{F})$.
3. The set $\{1, x, x^2, x^3, \dots\}$ is a basis for $P(\mathbb{F})$.
4. The set $\{1, x, x^2, x^3, \dots, x^n\}$ is a basis for $P_n(\mathbb{F})$.



Theorem 23 – Unique Expression from Basis

Let V be a vector space over \mathbb{F} and β a basis of V . Then any $\mathbf{v} \in V$ has a unique expression

$$\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i$$

where $\mathbf{v}_i \in \beta$ and $a_i \in \mathbb{F}$.

**Theorem 24 – The Replacement Theorem**

Suppose that $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V and $I = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ an independent subset of V . Then for any $i \in \{1, \dots, k\}$, we can obtain a new basis by replacing i elements of β with $\{\mathbf{w}_1, \dots, \mathbf{w}_i\}$. So after relabelling the elements* $\mathbf{v}_j \in \beta$ we have that the set $\beta_i = \{\mathbf{w}_1, \dots, \mathbf{w}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$ is a basis for V .

Corollary 3

Suppose that V is a vector space over \mathbb{F} with a finite basis. Suppose that β is any basis of V and I is any independent set. Then $|I| \leq |\beta|$.

**Theorem 25 – Size of Bases**

Let V be a vector space over a field \mathbb{F} . If V has a finite basis, then all bases of V have the same size.

**Definition 30 – Dimension**

Let V be a vector space over \mathbb{F} with a finite basis. We define the **dimension of V** to be the size of a basis for V .

In this case, we say that V is **finite dimensional**. Otherwise, we say that V is infinite dimensional.

1. $\dim \mathbb{F}^n = n$.
2. $\dim \mathcal{M}_{m \times n}(\mathbb{F}) = mn$.
3. $\dim P_n(\mathbb{F}) = n + 1$.
4. $P(\mathbb{F})$ is infinite dimensional.

**Corollary 4**

Let V be a finite dimensional vector space over \mathbb{F} . S any spanning set for V , I any independent set in V , and β any basis. Then

$$|I| \leq |\beta| \leq |S|.$$



Chapter 5: Linear Transformations

Definition 31 – Linear Transformation

Let V and W be vector spaces over \mathbb{F} . A map $T : V \rightarrow W$ is called a **linear transformation** if:

$$\begin{aligned} T(v + w) &= T(v) + T(w) && \text{for all } v, w \in V \\ T(cv) &= cT(v) && \text{for all } v \in V \text{ and } c \in \mathbb{F} \end{aligned}$$

$T(x, y, z) = (2x - 4y + z, 3x - y + 2z)$ is a linear transformation from \mathbb{R}^3 to \mathbb{R}^2 .

$T(p) = \frac{d}{dx}p$ is a linear transformation.



Theorem 26 – Properties of Linearity

Let V, W be vector spaces over \mathbb{F} .

1. If $T : V \rightarrow W$ is linear, then $T(0_v) = 0_w$.
2. The map $O : V \rightarrow W$ given by $O(v) = 0_w$ for all $v \in V$ is linear. This map is called the "zero map."
3. The map $I_V : V \rightarrow V$ given by $I_V(v) = v$ for all $v \in V$ is linear. This map is called the "identity map."



Theorem 27

Let V be a finite dimensional vector space over \mathbb{F} and $\beta = \{v_1, \dots, v_n\}$ a basis of V . A linear map $T : V \rightarrow W$ is uniquely determined by the values $T(v_1), T(v_2), \dots, T(v_n) \in W$.



Corollary 5 – Extending by Linearity

Let V, W be vector spaces over \mathbb{F} , and $\beta = \{v_1, \dots, v_n\}$ a basis for V . Given a list of (not necessarily distinct) vectors $w_1, \dots, w_n \in W$ there is exactly one linear map $T : V \rightarrow W$ so that $T(v_i) = w_i$.

This map is defined for all $v \in V$ as follows. Writing $v = \sum_{i=1}^n a_i v_i$, we then set $T(v) = \sum_{i=1}^n a_i w_i$.

This process is called "extending by linearity".



Theorem 28 – Composition of Linear Maps

Let V, W, X be vector spaces over \mathbb{F} . If $T : V \rightarrow W$ and $S : W \rightarrow X$ are linear maps, then the composition $S \circ T : V \rightarrow X$ is linear.



Theorem 29 – Null Space / Image

Let V, W be vector spaces over \mathbb{F} and $T : V \rightarrow W$ a linear transformation. The sets:

$$N(T) = \{v \in V \mid T(v) = 0\} \subseteq V$$

$$\text{im}(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\} \subseteq W$$

are subspaces of V, W respectively.

The subspace $N(T)$ is called the **null space** of T .

The subspace $\text{im}(T)$ is called the **image** of T .



Definition 32 – rank

Let V, W be vector spaces over \mathbb{F} and $T : V \rightarrow W$ linear. We define the **rank** of T by $\text{rank } T = \dim \text{im}(T)$.



Theorem 30 – The Dimension Theorem

Let V, W be finite dimensional vector spaces over \mathbb{F} . If $T : V \rightarrow W$ linear, then

$$\dim V = \dim N(T) + \dim \text{im}(T)$$

**Definition 33 – $\mathcal{L}(V, W)$**

Let V, W be vector spaces over a field \mathbb{F} . We denote by $\mathcal{L}(V, W) = \{T : V \rightarrow W \mid T \text{ is linear}\}$ the set of all linear transformations from V to W .

We define addition and scaling of transformations by:

$$(T + S)(v) = T(v) + S(v)$$

(we can add $T(v), S(v) \in W$)

$$(cT)(v) = cT(v)$$

(we can scale the vector $T(v) \in W$ by $c \in \mathbb{F}$)

**Theorem 31**

Let V, W be vector spaces over a field \mathbb{F} . If $S, T \in \mathcal{L}(V, W)$, and $c \in \mathbb{F}$ then $T + S, cT \in \mathcal{L}(V, W)$.

**Theorem 32**

Let V, W be vector spaces over a field \mathbb{F} .

1. The set $\mathcal{L}(V, W)$ is a vector space over \mathbb{F} .
2. If $\dim V = n, \dim W = m$, then $\dim \mathcal{L}(V, W) = nm$.

**Definition 34 – Invertibility**

Let A, B be sets and $f : A \rightarrow B$ be a function.

1. We say that f is injective if for all $x, y \in A, f(x) = f(y) \implies x = y$.
2. We say that f is surjective if $\text{im } f = f(A) = B$.
3. We say that f is bijective if it is both injective and surjective.
4. We say that f is **invertible**, if there exists a function $g : B \rightarrow A$ so that $g \circ f = I_A : A \rightarrow A$ and $f \circ g = I_B : B \rightarrow B$.

In this case we call the map g the **inverse** of f and denote it by f^{-1} .

**Theorem 33**

Let V, W be vector spaces over \mathbb{F} . If $T : V \rightarrow W$ is linear and bijective, then the inverse $T^{-1} : W \rightarrow V$ is also linear.

**Definition 35 – Isomorphism**

Let V, W be vector spaces over the field \mathbb{F} . We say that a linear map $T : V \rightarrow W$ is an **isomorphism** if it is bijective.

We say that V is **isomorphic to W** , and write $V \simeq W$, if there exists an isomorphism $T : V \rightarrow W$.

**Theorem 34**

Let V, W be vector spaces over \mathbb{F} and $T : V \rightarrow W$ be linear. Then T is injective if and only if $N(T) = \{0_v\}$.



Theorem 35

Let V, W be finite dimensional vector spaces over \mathbb{F} . Then $V \simeq W$ if and only if $\dim V = \dim W$.

**Theorem 36**

Let V, W be finite dimensional vector spaces over the field \mathbb{F} .

If $T : V \rightarrow W$ is linear and $\dim V = \dim W$, then the following are equivalent:

1. T is injective.
2. T is surjective.
3. T is an isomorphism.

**Theorem 37**

The maps $R_\theta, \text{proj}_L, \mathbb{R}_L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are linear.

**Theorem 38**

The maps $R_{\theta,L}, \text{proj}_L, R_P : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are linear.



Chapter 6: Coordinates

Theorem 39

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$. Define $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ by $T_A(x) = Ax$.

1. The map T_A is linear.
2. The map $F : \mathcal{M}_{m \times n}(\mathbb{F}) \rightarrow \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ given by $F(A) = T_A$ is an isomorphism of vector spaces. That is, every linear map $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$ is given by matrix multiplication for some matrix A .



Definition 36

Let V be a finite dimensional vector space over \mathbb{F} . An **ordered basis** for V is a basis β , together with a **fixed order** for listing its elements: $\beta = \{v_1, \dots, v_n\}$.



Theorem 40

Let V be a vector space over \mathbb{F} of dimension n , and $\beta = \{v_1, \dots, v_n\}$ a basis of V . The map $\phi_\beta : V \rightarrow \mathbb{F}^n$ defined above is an isomorphism of vector spaces.

We will denote by $[v]_\beta$ the element $\phi_\beta(v)$. I.e. $\phi_\beta(v) = [v]_\beta$.

We will call a choice of basis on V , together with the isomorphism $\phi_\beta : V \rightarrow \mathbb{F}^n$ a “**coordinate system**” on V .



Definition 37

Let V, W be finite dimensional vector spaces over \mathbb{F} . Let $\beta = \{v_1, \dots, v_n\}$ be a basis of V and $\gamma = \{w_1, \dots, w_m\}$ a basis of W . For $T \in \mathcal{L}(V, W)$ we define the matrix $[T]_\beta^\gamma \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ as follows:

1. The columns of $[T]_\beta^\gamma$ are given by: $[T(v_1)]_\gamma, \dots, [T(v_n)]_\gamma$. Alternatively: we can express $T(v_j) \in W$ using the basis γ to obtain an expression $T(v_j) = \sum_{i=1}^n A_{ij} w_i$. We then define $([T]_\beta^\gamma)_{ij} = A_{ij}$.

When $T : V \rightarrow W$ we denote by $[T]_\beta = [T]_\beta^\gamma$.



Theorem 41

Let V, W be finite dimensional vector spaces over \mathbb{F} , $\beta = \{v_1, \dots, v_n\}$ a basis of V , and $\gamma = \{w_1, \dots, w_m\}$ a basis of W .

The map $\phi_\beta^\gamma : \mathcal{L}(V, W) \rightarrow \mathcal{M}_{m \times n}(\mathbb{F})$ given by $\phi_\beta^\gamma(T) = [T]_\beta^\gamma$ is an isomorphism of vector spaces.



Theorem 42

Let V, W, X be finite dimensional vector spaces over \mathbb{F} . Let $\beta = \{v_1, \dots, v_n\}$ be a basis of V , $\gamma = \{w_1, \dots, w_m\}$ a basis of W , and $\delta = \{x_1, \dots, x_p\}$ a basis of X .

For all $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, X)$ we have that

$$[S \circ T]_\beta^\delta = [S]_\gamma^\delta [T]_\beta^\gamma.$$

In other words, when using compatible coordinate systems, the composition of linear transformations corresponds to matrix multiplication.



Theorem 43

Let V, W be finite dimensional vector spaces over \mathbb{F} , $T : V \rightarrow W$ be a linear map, $\beta = \{v_1, \dots, v_n\}$ a basis of V , and $\gamma = \{w_1, \dots, w_m\}$ a basis of W . Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible.

Moreover, we have

$$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$$

if T is invertible.



Theorem 44

Let V, W be finite dimensional vector spaces over a field \mathbb{F} , $\beta = \{v_1, \dots, v_n\}$ a basis of V , and $\gamma = \{w_1, \dots, w_m\}$ a basis of W . Then we have

$$[T]_{\beta}^{\gamma}[v]_{\beta} = [T(v)]_{\gamma}$$

for all $v \in V$.

**Theorem 45**

Let V, W be finite dimensional vector spaces over \mathbb{F} , β, β' bases for V , and γ, γ' bases for W . Let $T : V \rightarrow W$ be a linear map.

1. $[v]_{\beta'} = [I_V]_{\beta}^{\beta'} [v]_{\beta}$.
2. $[T]_{\beta'}^{\gamma} = [T]_{\beta}^{\gamma} [I_V]_{\beta'}^{\beta}$.
3. $[T]_{\beta}^{\gamma'} [T]_{\beta}^{\gamma}$.
4. $[T]_{\beta'}^{\gamma'} = [I_W]_{\gamma'}^{\gamma} [T]_{\beta}^{\gamma} [I_V]_{\beta'}^{\beta}$.

**Corollary 6 – Change of Bases**

Let V be finite dimensional vector spaces over \mathbb{F} , and β, β' bases for V . If $T : V \rightarrow V$ is linear then we have that

$$[T]_{\beta'} = [I_V]_{\beta}^{\beta'} [T]_{\beta} [I_V]_{\beta'}^{\beta}.$$

Denoting $[I_V]_{\beta'}^{\beta} = Q$, $A = [T]_{\beta}$, and $B = [T]_{\beta'}$, and recalling that $[I_V]_{\beta'}^{\beta} = ([I_V]_{\beta}^{\beta'})^{-1}$ we have that

$$B = Q^{-1} A Q.$$

**Definition 38 – Change of Bases Matrix**

Let V be a finite dimensional vector spaces over \mathbb{F} , and β, β' bases for V .

The matrix $Q = [I_V]_{\beta}^{\beta'}$ is called the **change of coordinate matrix from β to β' coordinates**.

**Theorem 46**

Let V be finite dimensional vector spaces over \mathbb{F} , and β, β' bases for V .

If $Q = [I_V]_{\beta}^{\beta'}$ is the change of coordinate matrix from β to β' coordinates, then Q^{-1} is the change of coordinate matrix from β' to β coordinates.



Chapter 7: Rank, Invertibility and Systems of Equations

Definition 39 – rank

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$ and $T_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ the associated transformation. We define the **rank of A** denoted $\text{rank } A$ by $\text{rank } A = \text{rank } T_A$.



Lemma 1

Let V, W be finite dimensional vector spaces over \mathbb{F} , $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ bases of V, W respectively, and $T : V \rightarrow W$ a linear map. We have $\text{im}(T) = \text{span}[T(v_1), \dots, T(v_n)]$.



As a consequence we have that if $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, then $\text{im}(T_A)$ is spanned by the columns of A .

Theorem 47

Let V, W, X be finite dimensional vector spaces over \mathbb{F} and $T : V \rightarrow W, S : W \rightarrow X$ linear transformations.

1. $\text{im}(S \circ T) \subseteq \text{im } S$.
2. $\text{im}(S \circ T) \subseteq S(\text{im } T)$.
3. If S is an isomorphism, then $\text{im } S \circ T \simeq \text{im } T$.
4. If T is an isomorphism, then $\text{im } S \circ T = \text{im } S$.



Corollary 7

Let V, W, X be finite dimensional vector spaces over \mathbb{F} and $T : V \rightarrow W, S : W \rightarrow X$ linear transformations.

1. $\text{rank } T \leq \min[\dim V, \dim W]$.
2. $\text{rank}(S \circ T) \leq \text{rank } S$.
3. $\text{rank}(S \circ T) \leq \text{rank } T$.
4. If S is an isomorphism, then $\text{rank } S \circ T = \text{rank } T$.
5. If T is an isomorphism, then $\text{rank } S \circ T = \text{rank } S$.



Theorem 48 – Rank of change of bases

Let V, W be finite dimensional vector spaces over \mathbb{F} , $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ bases of V, W respectively, and $T : V \rightarrow W$ a linear map. Then we have $\text{rank } T = \text{rank}[T]_{\beta}^{\gamma}$.



Corollary 8

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F}), B \in \mathcal{M}_{n \times k}(\mathbb{F})$.

1. $\text{rank } A \leq \min\{m, n\}$.
2. $\text{rank } AB \leq \text{rank } A$.
3. $\text{rank } AB \leq \text{rank } B$.
4. If A is invertible, then $\text{rank } AB = \text{rank } B$.
5. If B is invertible, then $\text{rank } AB = \text{rank } A$.



Corollary 9 – Invertibility and Rank

A square matrix $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ is invertible if and only if $\text{rank } A = n$.



Definition 40 – Elementary Matrix

Let $E \in \mathcal{M}_{n \times n}(\mathbb{F})$ be a matrix obtained from 1_n by performing a row/column operation. We call E an **elementary matrix**. We say that E is of Type I, II, or, III if it is obtained using a Type I, II, or III operation.



Theorem 49

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$ and $E \in \mathcal{M}_{n \times n}(\mathbb{F})$, $E' \in \mathcal{M}_{m \times m}(\mathbb{F})$ be elementary matrices. Set $B = AE$ and $C = E'A$.

1. If E is obtained from 1_n by performing a column operation, then B is obtained from A by performing **the same** column operation.
2. If E is obtained from 1_n by performing a row operation, then B is obtained from A by performing **the same** row operation.
3. E is invertible, and its inverse is an elementary matrix of the same type. (The same is true of E').

**Theorem 50**

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$ and $E \in \mathcal{M}_{n \times n}(\mathbb{F})$, $E' \in \mathcal{M}_{m \times m}(\mathbb{F})$ be elementary matrices. Set $B = AE$ and $C = E'A$.

1. $\text{rank } A = \text{rank } B$.
2. $\text{rank } A = \text{rank } C$.
3. $Ax = y$ if and only if $Cx = Cy$. In other words, row operations do not change the solutions in a system.
4. $\text{im}(T_A) = \text{im}(T_B)$. In other words, column operations do not change the image.

**Definition 41 – RREF**

We say a matrix A is in **row-reduced echelon form** if *all* of the following conditions are met:

1. All zero rows are at the bottom of the matrix A .
2. The first non-zero entry of a non-zero row is 1. (Such entries are called "leading 1's".)
3. The leading 1's move to the right, as we go down the rows of A .
4. All entries above and below a leading 1 are 0.

We use the abbreviation "RREF" for "row-reduced echelon form".

**Theorem 51 – Gaussian Elimination**

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$.

1. If the matrix consists entirely of 0's, stop. It's already row-reduced.
2. Find the first column with a non-zero entry and move the corresponding row to the top. (We will call the first non-zero entry a .)
3. Divide the row by the number a to obtain a leading one.
4. Subtract multiples of this row from the rows above and below, in order to make each entry above and below the leading 1 equal to 0.
5. Repeat 1-4 on the matrix consisting of the remaining rows.

At the end of the process, we obtain a matrix $R \in \mathcal{M}_{m \times n}(\mathbb{F})$ which is in RREF.

As a consequence, every matrix has a RREF.

**Lemma 2**

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$ and let $R \in \mathcal{M}_{m \times n}(\mathbb{F})$ be its RREF.

1. If k is the number of zero rows in R , then the columns of R consisting the leading ones are the first $m - k$ elements $\{e_1, \dots, e_{m-k}\}$ of the standard basis $\{e_1, e_2, \dots, e_m\}$ for \mathbb{F}^m .
2. $\text{rank } A = \text{rank } R = \#$ of leading 1's.
3. $\text{rank } A = \text{rank } R = \#$ of non-zero rows in R .

**Corollary 10**

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$. Then there exists an invertible matrix $b \in \mathcal{M}_{m \times m}(\mathbb{F})$ so that $BA = R$ is in RREF.

**Corollary 11**

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. Then A is invertible if and only if the RREF of A is 1_n .



Theorem 52

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$ be a matrix and $r = \text{rank } A$.

If $r = 0$, then $A = O_{mn}$ is the zero matrix.

If $r > 0$, then there exists a sequence of row operations and column operations, which transform A into a matrix

of the form $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)$.

**Theorem 53**

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, R denote its RREF, and $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ be associated linear transformation. Let a_1, \dots, a_n denote the columns of A .

1. The set of $n - r$ “basic solutions” for $Ax = 0$ is a basis for $N(T_A)$.
2. If the leading 1’s in R appear in columns k_1, \dots, k_r , then the columns $a_{k_1}, a_{k_2}, \dots, a_{k_r}$ of A provide a basis for $\text{im}(T_A)$.
3. If we apply the Gaussian Algorithm replacing “row operations” with “column operations” at every step, then the non-zero columns of the resulting matrix provide a basis for $\text{im}(T_A)$.

**Theorem 54 – Invertibility**

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. Then A is invertible if and only if A is a product of elementary matrices.

**Theorem 55 – Computing the Inverse**

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. Consider the augmented matrix $(A|I_n)$. Row reduce the augmented matrix to obtain $(R|B)$, where R is the RREF of A .

Then A is invertible if and only if this procedure results in the augmented matrix $(I_n | B)$. In this case, $B = A^{-1}$.



Chapter 8: Determinants

Definition 42

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{F})$. We define the **determinant** of A by $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \in \mathbb{F}$.


Theorem 56

The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{F})$ is invertible if and only if $\det A = ad - bc \neq 0$.

In this case A is invertible we have $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.


Definition 43

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. For each $i, j \in \{1, \dots, n\}$ we define \tilde{A}_{ij} to be the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of A .

We define $\det A \in \mathbb{F}$ as follows:

1. For $n = 1$, $A = (a) \in \mathbb{F}$, and we set $\det(a) = a$.
2. For $n = 2$, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
3. For $n \geq 2$, we set $\det A = \sum_{j=1}^n (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j})$


Theorem 57

Consider the determinant as a function on the rows of a matrix.

If $\mathbf{r}_1, \dots, \mathbf{r}_n, \mathbf{a}, \mathbf{b} \in \mathbb{F}^n$ and $c \in \mathbb{F}$, then we have:

$$\det \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ c\mathbf{a} + \mathbf{b} \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = c \det \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{a} \\ \vdots \\ \mathbf{r}_n \end{pmatrix} + \det \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{b} \\ \vdots \\ \mathbf{r}_n \end{pmatrix}.$$

In other words, for each $i \in \{1, \dots, n\}$, the determinant is a linear function of row i , with all the other rows fixed.


Theorem 58

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ with $n \geq 2$.

1. If A has a row of 0's, then $\det A = 0$.
2. For any $i \in \{1, \dots, n\}$, $\det A = \sum_{j=1}^n (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})$. (This is called the “row i expansion of $\det A$ ”.)
3. If A has two identical rows, then $\det A = 0$.
4. If B is obtained from A by swapping two rows, then $\det B = -\det A$.
5. If B is obtained from A by scaling row i by $a \in \mathbb{F}$, then $\det B = a \det A$.
6. If B is obtained from A by adding a multiple of one row to another, then $\det B = \det A$. (Here $n \geq 2$ so that the statement makes sense.)
7. If A is upper (or lower) triangular, then $\det A$ is the product of the diagonal entries of A .


Theorem 59

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. If $\text{rank } A < n$, then $\det A = 0$.



Theorem 60

Let $E \in \times(\mathbb{F})$ be an elementary matrix.

1. If E is of Type I, then $\det E = -1$.
2. If E is of Type II, and scales by $a \in F$, then $\det E = a$.
3. If E is of Type III, then $\det E = 1$.

Theorem 61

Let $A, B \in \mathcal{M}_{n \times n}(\mathbb{F})$. Then $\det(AB) = \det(A)\det(B)$.

Theorem 62

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. Then A is invertible if and only if $\det A \neq 0$.

Theorem 63

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. Then $\det(A^t) = \det A$.

Corollary 12

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$, then $\det(A) = \sum_{i=1}^n (-1)^{i+1} A_{i1} \det(\tilde{A}_{i1})$. (This is called the “column j expansion of $\det A$ ”).

**Theorem 64**

Let $A, B \in \mathcal{M}_{n \times n}(\mathbb{F})$. If $A \sim B$, then $\det A = \det B$.

Definition 44

Let V be finite dimensional vector space over \mathbb{F} , and $T : V \rightarrow V$ a linear map. We define

$$\det T = \det[T]_\beta,$$

where β is any basis of V .

**Theorem 65 – Geometric meaning of Determinants**

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be linear. Then $|\det T| = \text{Area}(P)$, where P is the parallelogram spanned by $T(e_1), T(e_2)$.

**Theorem 66**

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be linear. Then $|\det T| = \text{Vol}(P)$, where P is the parallelepiped spanned by $T(e_1), T(e_2), T(e_3)$.

Theorem 67

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear. Then $|\det T| = \text{Vol}(P)$, where P is the n -parallelepiped spanned by $T(e_1), \dots, T(e_n)$.



Chapter 9: Diagonalization

Definition 45

Recall, we say that a square matrix $D \in \mathcal{M}_{n \times n}(\mathbb{F})$ is diagonal if $D_{ij} = 0$ for $i \neq j$.

We will denote the diagonal matrix $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ by $D = \text{diag}(d_{11}, \dots, d_{nn})$.
 For example $\text{diag}(1, 2, 3)$ denotes the diagonal matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. ¶

Definition 46

Let V be a vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map.

We say that $\mathbf{v} \in V$ is an **eigenvector** for T if $\mathbf{v} \neq \mathbf{0}$ and $T(\mathbf{v}) = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{F}$.

We call the scalar $\lambda \in \mathbb{F}$ an **eigenvalue** of T .

Let $E_\lambda = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \lambda\mathbf{v}\}$. We call E_λ the **eigenspace for eigenvalue λ** . ¶

Theorem 68

Let V be a vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map. For any $\lambda \in \mathbb{F}$, the set E_λ is a subspace. ▼

Corollary 13

Let V be a vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map. For any $\lambda \in \mathbb{F}$ we have $E_\lambda = N(\lambda\mathbf{I}_V - T)$. ¶

Definition 47

Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map.

Set $C_T(x) = \det(x\mathbf{I}_V - T)$. We call C_T the **characteristic polynomial** of T .

Similarly, for $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ we define $C_A(x) = \det(x\mathbf{I}_V - A)$, and call C_A the characteristic polynomial of A . ¶

Theorem 69

Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map. Let $n = \dim V$.

1. C_T is a polynomial of degree n and the coefficient of x^n is 1.
 2. $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if $C_T(\lambda) = 0$. That is, the roots of C_T are precisely the eigenvalues of T .
 3. For all eigenvalues $\lambda \in \mathbb{F}$, the corresponding eigenvectors can be found by solving the homogeneous system $(\lambda\mathbf{I} - T)(\mathbf{x}) = \mathbf{0}_V$.
- ▼

Definition 48

Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map. We say that T is **diagonalizable** if there exists a basis β of B so that $[T]_\beta$ is a diagonal matrix.

We say that $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ is **diagonalizable** if there exists an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$. ¶

Theorem 70 – Alternate Characterizations of Diagonalizability

1. Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map. T is diagonalizable if and only if there exists a basis of V consisting of eigenvectors for T .
 2. $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ is diagonalizable if and only if A is similar to a diagonal matrix.
 3. $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ is diagonalizable if and only if $T_A : \mathbb{F}^n \rightarrow \mathbb{F}^n$ is a diagonalizable linear map.
- ▼

Theorem 71 – Test for Diagonalization

Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map.

T is diagonalizable if and only if all the conditions here are met:

1. The characteristic polynomial $C_T(x) = \prod_{i=1}^k (x - \lambda_i)^{m_i}$ splits over \mathbb{F} .*
2. For each eigenvalue λ_i we have $m_i = \dim E_i$.

If these conditions are met, then we obtain a diagonalizing basis β as follows:

1. For each eigenvalue λ_i find a basis β_i of $E_i = N(\lambda_i I_V - T)$ by solving the corresponding system.
2. Set $\beta = \beta_1 \cup \dots \cup \beta_k$.
3. Then $[T]_\beta = \text{diag}(\lambda_1, \dots, \lambda_n)$

**Corollary 14 – The "Matrix" Test for Diagonalization**

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. Then A is diagonalizable if and only if all the following conditions are met:

1. The characteristic polynomial $C_A(x) = \prod_{i=1}^k (x - \lambda_i)^{m_i}$ splits over \mathbb{F} .
2. For each eigenvalue λ_i we have $m_i = \dim E_i$.

If these conditions are met, then we obtain P, D as follows:

1. For each eigenvalue λ_i find a basis β_i of $E_i = N(\lambda_i I - A)$ by solving the corresponding system.
2. Moreover, the matrix P has columns $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, where $\mathbf{v}_i \in \beta$ are eigenvectors (expressed as columns in "standard coordinates"). The matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$.*

**Definition 49**

Suppose that $T : V \rightarrow V$ is linear and $\lambda \in \mathbb{F}$ is an eigenvalue. Then we know that $|\lambda|$ is a root of C_T , so we have $C_T(x) = (x - \lambda)^m p(x)$ for some $m \in \mathbb{N}$ and $p \in P(\mathbb{F})$ so that $p(\lambda) \neq 0$.

We call m the multiplicity of λ .

**Lemma 3**

Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map.

If λ is an eigenvalue of T , then we have $1 \leq \dim E_\lambda \leq m_\lambda$.

**Lemma 4**

Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map.

If λ_1, λ_2 are distinct eigenvalues for T , and $S_1 \subseteq E_1, S_2 \subseteq E_2$ linearly independent subsets of eigenvectors, then $S_1 \cup S_2$ is linearly independent.

**Theorem 72 – Invariant Subspaces and Diagonalization**

Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ be a linear map. The following two conditions are equivalent:

1. T is diagonalizable.
2. There exists one-dimensional invariant subspaces W_1, \dots, W_n so that $V = W_1 \oplus \dots \oplus W_n$.



Chapter 10: Matrix Algebra (Appendix A)

Definition 50 – Matrix Multiplication

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$ and $B \in \mathcal{M}_{n \times k}(\mathbb{F})$. We define their product $AB \in \mathcal{M}_{m \times k}(\mathbb{F})$ as follows: for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, k\}$ the ij -entry of the product AB is given by

$$(AB)_{ij} = \sum_{l=1}^n A_{il}B_{lj}.$$

$$AB \neq BA.$$



Definition 51 – Special Matrices

For each $n, m \in \mathbb{N}$ we define the following matrices:

1. $O_{m,n} \in \mathcal{M}_{m \times n}(\mathbb{F})$ - the matrix consisting of all 0's. In other words $(O_{m,n})_{i,j} = 0$ for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.
2. $I_n \in \mathcal{M}_{n \times n}(\mathbb{F})$ - the matrix with 1's on the diagonals, and 0 in all other entries. In other words

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$O_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Theorem 73

Let \mathbb{F} be a field, $A, A_1, A_2 \in \mathcal{M}_{m \times n}(\mathbb{F})$, $B, B_1, B_2 \in \mathcal{M}_{n \times k}(\mathbb{F})$, $C \in \mathcal{M}_{k \times p}(\mathbb{F})$ and $c \in \mathbb{F}$.

1. $A(BC) = (AB)C$
2. $(A_1 + A_2)B = A_1B + A_2B$
3. $A(B_1 + B_2) = AB_1 + AB_2$
4. $I_m A = A = A I_n$
5. $O_{rm} A = O_{rn}$ for any $r \in \mathbb{N}$.
6. $A(cB) = c(AB) = (cI_m)AB = AB(cI_k) = A(cI_n)B$.



Definition 52 – Invertibility

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. We say that A is **invertible** if there exists a matrix $B \in \mathcal{M}_{n \times n}(\mathbb{F})$ so that $AB = I_n = BA$.



Theorem 74

Let $A, B \in \mathcal{M}_{n \times n}(\mathbb{F})$.

1. If A is invertible, then the inverse of A is unique.
2. If A is invertible, then A^{-1} is also invertible.
3. If A and B are invertible, then AB is invertible.
4. I_n is invertible.
5. If $AB = I_n$, then A is invertible and $B = A^{-1}$.

**Definition 53 – A^t**

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$. We define the matrix $A^t \in \mathcal{M}_{n \times m}(\mathbb{F})$ by:

$$(A^t)_{ij} = A_{ji}.$$

In other words, to obtain A^t we “swap the rows and columns of A .”

**Definition 54 – Symmetric**

We say that $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ is **symmetric** if $A^t = A$. We denote the set of all symmetric matrices by $\text{Sym}_n(\mathbb{F})$.

We say that $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ is **skew-symmetric** if $A^t = -A$. We denote the set of all skew-symmetric matrices by $\text{Sk}_n(\mathbb{F})$.

**Theorem 75**

Let $A, B \in \mathcal{M}_{m \times n}(\mathbb{F}), C \in \mathcal{M}_{n \times k}(\mathbb{F})$ and $c \in \mathbb{F}$.

1. $(A + B)^t = A^t + B^t$
2. $(cA)^t = cA^t$
3. $(A^t)^t = A$
4. $(AC)^t = C^t A^t$.
5. In the case that $m = n$, we also have that if $A \in \mathcal{M}_{n \times n}(\mathbb{F})$ is invertible, then $A^t \in \mathcal{M}_{n \times n}(\mathbb{F})$ is invertible and $(A^t)^{-1} = (A^{-1})^t$.

**Definition 55 – Diagonal and Triangular**

We say that A is **diagonal** if $A_{ij} = 0$ for all $i \neq j$.

Let $A \in \mathcal{M}_{n \times n}(\mathbb{F})$. We say that A is **upper triangular** if $A_{ij} = 0$ for all $i > j$. This means that all entries below the diagonal of A must be 0.

Similarly, we say that A is **lower triangular** if $A_{ij} = 0$ for all $i < j$. This means that all entries above the diagonal of A must be 0.

We say that A is **strictly upper-triangular** if $A_{ij} = 0$ for all $i \geq j$.

