MAT240: Algebra I

Definition Sheet

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Info: Most of the numbers are correct. This sheet also does not include the examples.

Chapter 1: Fields and Polynomials

Definition 1 – Field

A field $\mathbb{F}$ is a set with two special elements "$0 \in \mathbb{F}$" and "$1 \in \mathbb{F}$" and two operations "$+$" and "$\cdot$" which satisfy the following axioms.

1. (Commutativity) For all $x, y \in \mathbb{F}$ we have: $x + y = y + x$ and $x \cdot y = y \cdot x$.
2. (Associativity) For all $x, y, z \in \mathbb{F}$ we have: $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
3. (Distributivity) For all $x, y, z \in \mathbb{F}$ we have: $x \cdot (y + z) = x \cdot y + x \cdot z$.
4. (Identities) For all $x \in \mathbb{F}$ we have: $x + 0 = x$ and $x \cdot 1 = x$.
5. (Inverses) For all $x \in \mathbb{F}$ there exists $y \in \mathbb{F}$ such that $x + y = 0$. For all $x \in \mathbb{F} \setminus \{0\}$ there exists $z \in \mathbb{F}$ such that $x \cdot z = 1$.

$\mathbb{Q}$ is a field where $\mathbb{Z}$ is not a field becasue of the absence of some multiplicative inverse.

Corollary 1 – Field

Let $\mathbb{F}$ be a field and $a, b, c \in \mathbb{F}$.

1. If $a + c = b + c$, then $a = b$.
2. If $c \neq 0$ and $c \cdot a = c \cdot b$, then $a = b$.
3. The field elements $0, 1$ are unique.
4. The elements y and z from Axoim 5 are unique. (From now on, we will denote the additive inverse of x by $-x$, and the multiplicative inverse of x by x^{-1}.)
5. $a \cdot 0 = 0$.
6. $(-a) \cdot (b) = -(a \cdot b) = (a) \cdot (-b)$.
7. $-(-a) = a$. If $a \neq 0$, then $\left(a^{-1}\right)^{-1} = a$.
8. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Theorem 1 – Equivalence

1. $\sim$ is an equivalence relation.
2. $a \sim b$ if and only if a and b have the same remainder when divided by n.
3. There are exactly n equivalence classes for this relation: $[0], [1], \dots, [n-1]$ - one for each possible remainder for division by n.

Definition 2 – $\mathbb{Z}_n$

Let $\mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$ be the est of equivalence classes for this equivalence relation. We define $+, \cdot$ on $\mathbb{Z}_n$ as follows:

$$[a] + [b] = [a + b],$$

$$[a] \cdot [b] = [a \cdot b].$$

Theorem 2 – Quadratic Formula

Let $a, b, c \in \mathbb{R}$. The quadratic equation $ax^2 + bx + c = 0$ (where $a \neq 0$) has:

1. Solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if $b^2 - 4ac \geq 0$.
2. No solutions if $b^2 - 4ac < 0$.

Definition 3 – Complex

Let $i = \sqrt{-1}$. I.e. i is a number with the property that $i^2 = -1$.
Let $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$. We call $\mathbb{C}$ the set of complex numbers and we define addition and multiplication $+, \cdot$ in the obvious ways:

$$(x + yi) + (a + bi) = (x + a) + (y + b)i$$

$$(x + yi) \cdot (a + bi) = ax + bi + ayi + byi^2 = (ax - by) + (ay + bx)i$$

$\mathbb{C}$ is a field.

Given a complex number $z = x + yi$, we define its **conjugate** by:

$$\overline{z} = x - yi.$$

We define the **length (or modulus)** of a complex number by:

$$|z| = \sqrt{x^2 + y^2}.$$

Note that we in the xy-plane, we obtain $\overline{z}$, the conjugate of z, by reflecting z in the x-axis, and the length of a complex number is just the usual distance from z to the origin in the xy-plane.

Theorem 3 – Complex

For any $z, w \in \mathbb{C}$ we have:

1. $\overline{z + w} = \overline{z} + \overline{w}$.
2. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.
3. $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$ (provided $w \neq 0$).
4. $\overline{\overline{z}} = z$.
5. $z\overline{z} = |z|^2$.
6. $z^{-1} = \frac{\overline{z}}{|z|^2}$ (provided $z \neq 0$).
7. $|zw| = |z||w|$.
8. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ (provided $w \neq 0$).
9. $|z + w| \leq |z| + |w|$ ("Triangle nequality for Complex Numbers").

Definition 4 – Polar Form

For $z = x + yi$, we define its **polar form** as $z = re^{i\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ and θ is the angle between z and the positive x axis (measured counterclockwise, in radians). The angle θ is called the **argument** of z, and r is called the **length (or modulus)** of z.

Theorem 4 – Polar Form

Let $z = re^{i\theta}$, $w = Re^{i\phi}$.

$$zw = rRe^{i(\theta+\phi)}$$

$$z^n = r^n e^{in\theta}$$

Definition 5 – Polynomial

A polynomial p with coefficient from $\mathbb{F}$ is an expression

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

where $c_i \in \mathbb{F}$. We call the field elements $c_0, \ldots, c_n$ the "coefficients" of p.
The largest exponent n so that $c_n \neq 0$ is called the **degree** of p, and we typically write $\deg p = n$. Constant polynomials are degree 0.
The set of all polynomials over $\mathbb{F}$ is denoted by $P(\mathbb{F})$.
The set of all polynomials of degree **less than or equal** to n is denoted by $P_n(\mathbb{F})$.

Theorem 5 – Polynomial

Let $\mathbb{F}$ be a field, and $f, g \in P(\mathbb{F})$ be non-zero polynomials. Then there exist unique polynomials $q, r \in P(\mathbb{F})$ so that:

1. $f(x) = q(x)g(x) + r(x)$.
2. $\deg r < \deg g$ if $\deg g \neq 0$.
3. $r = 0$ if $\deg g = 0$.

Definition 6 – Polynomial Cont.

Let $\mathbb{F}$ be a field and $f, g \in P(\mathbb{F})$. We say that g divides f if $f(x) = q(x)g(x)$ for some polynomial $q \in P(\mathbb{F})$.
We say that a non-constant polynomial $p \in P(\mathbb{F})$ is "irreducible" if we **cannot** express p as a product of polynomials of smaller degree.
I.e. p is irreducible if we **cannot** write $p(x) = g(x)q(x)$ for any polynomials $g, q \in P(\mathbb{F})$ with the property that both $\deg g, \deg q < \deg p$.

$f(x) = x^2 - 2$ is irreducible over $\mathbb{Q}$ but not over $\mathbb{R}$.

Theorem 6 – Polynomial Cont.

Let $\mathbb{F}$ be a field, $p \in P(\mathbb{F})$ and $\deg p \geq 1$. Then $a \in \mathbb{F}$ is a root of p if and only if $x - a$ divides p.

Theorem 7 – Fundamental Theorem of Algebra

Every non-constant polynomial has a root over $\mathbb{C}$.
In fact, every non-constant polynomial factors completely into a product of linear terms over $\mathbb{C}$.

Chapter 2: Linear Systems

Definition 9 – Linear

Let $\mathbb{F}$ be a field and $b, c_1, \ldots, c_n \in \mathbb{F}$. An equation in the variables $x_1, \ldots, x_n$ is called **linear** if it can be expressed as $c_1x_1 + c_2x_2 + \cdots + c_nx_n = b$.

Definition 10 – System of Equations

Let $\mathbb{F}$ be a field, and $a_{ij} \in \mathbb{F}$ (where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$). A **system of linear equations** in variables $x_1, x_2, \ldots, x_n$ is a finite collection of linear equations in $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

A system of m equations with n unkowns is called an $m \times n$ **system**.

Definition 11 – Solutions

A **solution** to a linear equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = b$ is a choice of field elements $s_1, s_2, \ldots, s_n \in \mathbb{F}$, so that when we substitute them for $x_1, x_2, \ldots, x_n$ respectively, the resulting equation is true.
That is, we have $c_1s_1 + c_2s_2 + \cdots + c_ns_n = b$ (i.e. the left- and right-hand sides are equal.)
A **solution to a system** is a choice of field elements $s_1, s_2, \ldots, s_n$ which solves *every* equation of the system.

Definition 12 – Consistent

If a system of equations has at least one solution, we say it is **consistent**.
If a system of equations has no solutions, we say it is **inconsistent**.

Definition 13 – Matrix

An $m \times n$ **matrix** over $\mathbb{F}$ is a rectangular array of field elements consisting of m rows and n columns.
We denote the j^{th} entry in row i of matrix A, by a_{ij}, and call it the ij^{th} **entry** of A.

Definition 14 – Augmented Matrix

Consider a system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We defie the **augmented matrix** *corresponding to* the system of equations above to be:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right),$$

Definition 15 – RREF

We say a matrix A is in **reduced row echelon form** if *all* of the following conditions are met:

1. All zero rows are at the bottom of the matrix A.
2. The first non-zero entry in each non-zero row is a 1. (Such entries are called "leading 1's".)
3. The leading 1's move to the right, as we go down the rows of A.
4. All entries above and below a leading 1 are 0.

We will use the abbreviation "RREF" for "row-reduced echelon form", for the rest of the text.

All matrices have a unique RREF.

Theorem 11 – Gaussian Elimination

To "row reduce" a matrix perform the following steps:

1. If the matrix consists entire of 0's, stop. It's already row-reduced.
2. Find the first column with a non-zero entry and move the corresponding row to the top. (We will call the first non-zero entry a.)
3. Divide the row by the number a to obtain a leading one.
4. Subtract multiples of ths row from the rows above and below, in order to make each entry above and below the leading 1 equal to 0.
5. Repeat 1-4 on the matrix consisting of the remaining rows.

Definition 16 – Variables

Suppose that R is a matrix in RREF. We say that x_i is a **leading variable** if column i contains a leading one. If a variable is not "leading" we call it a **non-leading variable**.

Remark 1. To solve a system:

1. Row reduce the augmented coefficient matrix.
2. If there is a row of the form $\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right)$ then there are no solutions.
3. Otherwise, assign the non-leading variables (if any) parameters, and use the equatoins coming from the rows of the RREF to solve for each variable interms of the parameters.

Definition 17 – Homogeneous

A system of equations is called **homogeneous** if it is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

In other words, it is homogeneous if the constant term (or right hand side) of ***each*** equation in the system is 0.

- $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is ***always*** a solution to any homogeneous equation.
- We call this solution the ***trivial*** solution.
- Any other solution is called a ***non-trivial*** solution.

Chapter 3: Vector Spaces

Definition 18 – Vector Operators

Given two vectors **v,w** we define **their sum** **v+w** using “tip to tail” addition (or the “parallelogram law of addition”). In the diagram in the margin, the vector **v+w** is diagonal in the parallelogram spanned by v, w that shares its tail with v and w.
We can also define their **difference** **v-w** geometrically using the same parallelogram: v- w is the diagonal going from the tip of w to the tip of v.
Finally, given a vector v and real number $a \in \mathbb{R}$, we can **scale v by** a as follows:

- $0\mathbf{v} = \mathbf{0}$.
- If $a > 0$, then $a\mathbf{v}$ is a vector pointing in the same direction as v with length scaled by a
- If $\alpha < 0$, then α is a vector pointing in the opposite direction as v with length scaled by $|\alpha|$.

If $v = (x, y, z)$ and $w = (p, q, r)$, then $v + w = (x + p, y + q, z + r)$, $av = (ax, ay, az)$.

Definition 20 – $\mathbb{F}^n$

Let $\mathbb{F}$ be a field. Consider the set $\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{F}\}$. We can define two operations on $\mathbb{F}^n$ which we call ”vector addition” which is a map $\mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$, and ”scaling” which is a map $\mathbb{F} \times \mathbb{F}^n \to \mathbb{F}^n$ as follows.
For $v = (x_1, x_2, \ldots, x_n), w = (y_1, y_2, \ldots, y_n) \in \mathbb{F}^n$, and $c \in \mathbb{F}$ we define:

$$\begin{aligned}
v + w &= (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) && \text{(vector addition)} \\
&= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\
cv &= c(x_1, x_2, \ldots, x_n) && \text{(scaling)} \\
&= (cx_1, cx_2, \ldots, cx_n)
\end{aligned}$$

Theorem 13 – $\mathbb{F}^n$

Let $\mathbb{F}$ be a field. Set $0 = (0, 0, \ldots, 0)$. For any $v, w, u \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$ we have:

1. $v + w = w + v$.
2. $v + (w + u) = (v + w) + u$.
3. $a(v + w) = av + aw$.
4. $(a + b)v = av + bv$.
5. $(ab)v = a(bv)$.
6. $1v = v$.
7. $0 + v = v$.
8. For every $v \in V$ there exists $w \in V$ so that $v + w = 0$.

Definition 21 – Vector Space

Let $\mathbb{F}$ be a field. A vector space V over $\mathbb{F}$ is a non-empty set, containing a special element 0, with two operatoins $V \times V \to V$ (called vector addition) and $\mathbb{F} \times V \to V$ (called scaling) so that for all $v, w, u \in V$ and $a, b \in \mathbb{F}$:

1. $v + w = w + v$.
2. $v + (w + u) = (v + w) + u$.
3. $a(v + w) = av + aw$.
4. $(a + b)v = av + bv$.
5. $(ab)v = a(bv)$.
6. $1v = v$.
7. $0 + v = v$.
8. For every $v \in V$ there exists $w \in V$ so that $v + w = 0$.

$P(\mathbb{F}), P_n(\mathbb{F})$ and $\mathbb{F}^n$ are vector spaces.

Definition 22 – Matrix Cont.

Let $\mathbb{F}$ be a field. An $m \times n$ **matrix** M **with entries in** $\mathbb{F}$ is a rectangular array of elements of $\mathbb{F}$ consisting of m rows and n columns.
We denote the entry in the i row and j column of a matrix m by m_{ij}.
The set of all $m \times n$ matrices with coefficients in $\mathbb{F}$ is denoted by $\mathcal{M}_{m\times n}(\mathbb{F})$.
For example, a 2×3 matrix looks like $\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{F})$, wile a 3×2 matrix N looks like $\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{pmatrix} \in \mathcal{M}_{3\times 2}(\mathbb{F})$.

$\mathcal{M}_{m\times n}(\mathbb{F})$, with pointwise addition and scaling is a vector space over $\mathbb{F}$;

Corollary 2 – Vector Space

Let $\mathbb{F}$ be a field, and V a vector space over $\mathbb{F}$. THen for any $v, w, u \in V$ and $a \in \mathbb{F}$ we have:

1. If $v + w = v + u$, then $w = u$.
2. If $a \neq 0$ and $av = aw$, then $v = w$.
3. The element $0 \in V$ is unique.
4. Additive inverses in V are unique. (This means that for each $v \in V$ there is only one element $w \in V$ which satisfies the condition of Axiom 8.)
5. $(-a)v = -(av)$. In particupar $(-1)v = -v$.
6. $0v = 0$.
7. $a0 = 0$.

Definition 23 – Subspace

Let $\mathbb{F}$ be a field and V a vector space over $\mathbb{F}$. We say that a subset $W \subseteq V$ is **a subspace of** V if W is also a vector space over $|F$ using the same operations defined in V.

$P_n(\mathbb{F})$ is a subspace of $P(\mathbb{F})$.

$P_n(\mathbb{F})$ is a subspace of $P_m(\mathbb{F})$ if $n < m \in \mathbb{N}$.

Theorem 19 – Subspace

Let V be a vector space over a field $\mathbb{F}$. A **non-empty** subset $W \subseteq V$ is a subspace of V if and only if

1. For all $v, w \in W$ we have $v + w \in W$.
2. For all $v \in W$ and $c \in \mathbb{F}$ we have $cv \in W$.

Definition 24 – Trivial / Non-Trivial Subspace

Let V be a vector space over a field $\mathbb{F}$. The subspaces $\{0\}$ and V are called the **trivial subspaces** of V. Any other subspace W of V is called a **non-trivial subspace of** V.
In particular, we say that a subspace W is a non-trivial subspace of V if $W \neq \{0\}$ and $W \neq V$.

Chapter 4: Bases and Dimension

Definition 25 – Linear Combinatoin of Vectors

Let V be a vector space over $\mathbb{F}$, and $v_1, v_2, \ldots, v_k \in V$. A vector of the form $a_1v_1 + a_2v_2 + \cdots + a_kv_k \in V$ is called a linear combination of the vectors $v_1, v_2, \ldots, v_k$.

Definition 26 – Span

Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. We define the **span of** S, denoted $\operatorname{span} S$, as follows:

1. If $S = \varnothing$ is empty, then $\operatorname{span} S = \{0\}$.
2. Otherwise, $\operatorname{span} S = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k | a_i \in \mathbb{F} v_i \in S\}$ is the set of all possible linear combinations of vectors from S.

Theorem 20 – Span as Subspace

Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$ be **any** subset of vectors. Then the subset $\operatorname{span} S \subseteq V$ is a subspace of V.

Definition 27 – S spans V

Let V be a vector space over $\mathbb{F}$. We say that a subset $S \subseteq V$ is **a spanning set for** V (or "S **spans** V") if $V = \operatorname{span} S$.

Definition 28 – Linearly Independnet

Let V be a vector space over $\mathbb{F}$. We say that a set S is **linearly independent** if for any vectors $v_1, v_2, \ldots, v_k \in S$:

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \implies c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

Otherwise, we say that S is **linearly dependent**.

Definition 29 – Basis

Let V be a vector space over $\mathbb{F}$. A subset $\beta \subseteq V$ is called a **basis** if:

1. β spans V
2. β is linearly independent.

Vector space over $\mathbb{F}$ also has basis.

Finite spanning set for V also contains basis for V.

Let $\mathbb{F}$ be a field.

1. The set $\{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{F}^n$.
2. The set $\{E_{ij} | 1 \leq i \leq m, 1 \leq n\}$ is a basis for $\mathcal{M}_{m\times n}(\mathbb{F})$.
3. The set $\{1, x, x^2, x^3, \ldots\}$ is a basis for $P(\mathbb{F})$.
4. The set $\{1, x, x^2, x^3, \ldots, x^n\}$ is a basis for $P_n(\mathbb{F})$.

Theorem 23 – Unique Expression from Basis

Let V be a vector space over $\mathbb{F}$ and β a basis of V. Then any $\mathbf{v} \in V$ has a unique expression

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$$

where $\mathbf{v}_i \in \beta$ and $a_i \in \mathbb{F}$.

Theorem 24 – The Replacement Theorem

Suppose that $\beta = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for V and $I = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ an independent subset of V. Then for any $i \in \{1, ..., k\}$, we can obtain a new basis by replacing i elements of β with $\{\mathbf{w}_1, \ldots, \mathbf{w}_i\}$. So after relabelling the elements* $\mathbf{v}_j \in \beta$ we have that the set $\beta_i = \{\mathbf{w}_1, \ldots, \mathbf{w}_i, \mathbf{v}_{i+1}, \ldots \mathbf{v}_n\}$ is a basis for V.

Corollary 3

Suppose that V is a vector space over $\mathbb{F}$ with a finite basis. Suppose that β is any basis of V and I is any independent set. Then $|I| \leq |B|$.

Theorem 25 – Size of Bases

Let V be a vector space over a field $\mathbb{F}$. If V has a finite basis, then all bases of V have the same size.

Definition 30 – Dimension

Let V be a vector space over $\mathbb{F}$ with a finite basis. We define the **dimension of** V to be the size of a basis for V.
In this case, we say that V **is finite dimensional**. Otherwise, we say that V is infinite dimenisonal.

1. $\dim \mathbb{F}^n = n$.
2. $\dim \mathscr{M}_{m\times n}(\mathbb{F}) = mn$.
3. $\dim P_n(\mathbb{F}) = n + 1$.
4. $P(\mathbb{F})$ is infinite dimensional.

Theorem 26

Let V be a finite dimensional vector space over a field $\mathbb{F}$. Suppose taht S is a spanning set and I is a linearly independent set. Then:

1. There exists a subset $\beta \subseteq S$ which is a basis for V; that is every spanning set contains a basis.
2. There exists a basis β of B so that $I \subseteq \beta$; that is every linearly independent set can be extended to a basis.

Corollary 4

Let V be a finite dimensional vector space over $\mathbb{F}$. S any spanning set for V, I any independent set in V, and β any basis. Then

$$|I| \leq |\beta| \leq |S|.$$

Chapter 5: Linear Transformations

Definition 31 – Linear Transformation

Let V and W be vector spaces over $\mathbb{F}$. A map $T : V \to W$ is called a **linear transformation** if:

$$T(v + w) = T(v) + T(w) \qquad \text{for all } v, w \in V$$
$$T(cv) = cT(v) \qquad \text{for all } v \in V \text{ and } c \in \mathbb{F}$$

$T(x, y, z) = (2x - 4y + z, 3x - y + 2x)$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$T(p) = \frac{\mathrm{d}}{\mathrm{d}x}p$ is a linear transformation.

Theorem 27 – Properties of Linearity

Let V, W be vector spaces over $\mathbb{F}$.

1. If $T : V \to W$ is linear, then $T(0_v) = 0_w$.
2. The map $O : V \to W$ given by $O(v) = 0_w$ for all $v \in V$ is linear. This map is called the "zero map."
3. The map $I_V : V \to V$ given by $I_V(v) = v$ for all $v \in V$ is linear. This map is called the "identity map."

Theorem 28

Let V e a finite dimensional vector space over $\mathbb{F}$ and $\beta = \{v_1, \ldots, v_n\}$ a basis of V. A linear map $T : V \to W$ is uniquely determined by the values $T(v_1), T(v_2), \ldots, T(v_n) \in W$.

Corollary 5 – Extending by Linearity

Let V, W be vector spaces over $\mathbb{F}$, and $\beta = \{v_1, \ldots, v_n\}$ a basis for V. Given a list of (not necessarily distinct) vectors $w_1, \ldots, w_n \in W$ there is exactly one linear map $T : V \to W$ so that $W(v_i) = w_i$.

This map is defined for all $v \in V$ as follows. Writing $v = \sum_{i=1}^{n} a_i v_i$, we then set $T(v) = \sum_{i=1}^{n} a_i w_i$.

This process is called "extending by linearity".

Theorem 29 – Composition of Linear Maps

Let V, W, X be vector spaces over $\mathbb{F}$. If $T : V \to W$ and $S : W \to X$ are linear maps, then the composition $S \circ T : V \to X$ is linear.

Theorem 30 – Null Space / Image

Let V, W be vector spaces over $\mathbb{F}$ and $T : V \to W$ a linear transformation. The sets:

$$N(T) = \{v \in V \mid T(v) = 0\} \subseteq V$$

$$\mathrm{im}(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\} \subseteq W$$

are subspaces of V, W respectively.
The subspace $N(T)$ is called the **null space of** T.
The subspace $\mathrm{im}(T)$ is called the **image of** T.

Definition 32 – rank

Let V, W be vector spaces over $\mathbb{F}$ and $T : V \to W$ linear. We define the **rank** of T by $\operatorname{rank} T = \dim \mathrm{im}(T)$.

Theorem 31 – The Dimension Theorem

Let V, W be finite dimenisonal vector spaces over $\mathbb{F}$. If $T : V \to W$ linear, then

$$\dim V = \dim N(T) + \dim \operatorname{im}(T)$$

.

Definition 33 – $\mathcal{L}(V, W)$

Let V, W be vector spaces over a field $\mathbb{F}$. We denote by $\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is linear}\}$ the set of all linear transformations from V to W.
We define addition and scaling of transformations by:

$$(T + S)(v) = T(v) + S(v) \qquad \text{(we can add } T(v), S(v) \in W\text{)}$$
$$(cT)(v) = cT(v) \qquad \text{(we can scale the vector } T(v) \in W \text{ by } c \in \mathbb{F}\text{)}$$

Theorem 32

Let V, W be vector spaces over a field $\mathbb{F}$. If $S, T \in \mathcal{L}(V, W)$, and $c \in \mathbb{F}$ then $T + S, cT \in \mathcal{L}(V, W)$.

Theorem 33

Let V, W be vector spaces over a field $\mathbb{F}$.
1. The set $\mathcal{L}(V, W)$ is a vector space over $\mathbb{F}$.
2. If $\dim V = n, \dim W = m$, then $\dim \mathcal{L}(V, W) = nm$.

Definition 34 – Invertibility

Let A, B be sets and $f : A \to B$ be a function.
1. We say that f is injective if for all $x, y \in A, f(x) = f(y) \implies x = y$.
2. We say that f is surjective if $\operatorname{im} f = f(A) = B$.
3. We say that f is bijective if it is both injective and surjective.
4. We say that f is **invertible**, if there exists a function $g : W \to V$ so that $g \circ f = I_A : A \to A$ and $f \circ g = I_B : B \to B$.
   In this case we call the map g the **inverse** of f and denote it by f^{-1}.

Theorem 34

Let V, W be vector spaces over $\mathbb{F}$. If $T : V \to W$ is linear and bijective, then the inverse $T^{-1} : W \to V$ is also linear.

Definition 35 – Isomorphism

Let V, W be vector spaces over the field $\mathbb{F}$. We say that a linear map $T : V \to W$ is an **isomorphism** if it is bijective.
We say that V is **isomorphic to** W, and write $V \simeq W$, if there exists an isomorphism $T : V \to W$.

Theorem 35

Let V, W be vector spaces over $\mathbb{F}$ and $T : V \to W$ be linear. Then T is injective if and only if $N(T) = \{0_v\}$.

Theorem 36

Let V, W be finite dimensional vector spaces over $\mathbb{F}$. Then $V \simeq W$ if and only if $\dim V = \dim W$.

Theorem 37

Let V, W be finite dimensional vector spaces over the field $\mathbb{F}$.
If $T : V \to W$ is linear and $\dim V = \dim W$, then the following are equivalent:

1. T is injective.
2. T is surjective.
3. T is an isomorphism.

Theorem 38

The maps $R_\theta, \text{proj}_L, \mathbb{R}_L : \mathbb{R}^2 \to \mathbb{R}^2$ are linear.

Theorem 39

The maps $R_{\theta,L}, \text{proj}_L, R_P : \mathbb{R}^3 \to \mathbb{R}^3$ are linear.

Chapter 6: Coordinates

Theorem 40

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$. Define $T_A : \mathbb{F}^n \to \mathbb{F}^m$ by $T_A(x) = Ax$.

1. The map T_A is linear.
2. The map $F : \mathscr{M}_{m\times n}(\mathbb{F}) \to \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ given by $F(A) = T_A$ is an isomorphism of vector spaces. That is, every linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ is given by matrix multiplication for some matrix A.

Definition 36

Let V be a finite dimensional vector space over $\mathbb{F}$. An **ordered basis** for V is a basis β, together with a **fixed** order for listing its elements: $\beta = \{v_1, \ldots, v_n\}$.

Theorem 41

Let V be a vctor space over $\mathbb{F}$ of dimension n, and $\beta = \{v_1, \ldots, v_n\}$ a basis of V. The map $\phi_\beta : V \to \mathbb{F}^n$ defined above is an isomorphism of vector spaces.
We will denote by $[v]_\beta$ the element $\phi_\beta(v)$. I.e. $\phi_\beta(v) = [v]_\beta$.
We will call a choice of basis on V, together with the isomorphism $\phi_beta : V \to \mathbb{F}^n$ a "**coordinate system**" on V.

Definition 37

Let V, W be finite dimensional vector spaces over $\mathbb{F}$. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W. For $T \in \mathcal{L}(V, W)$ we define the matrix $[T]_\beta^\gamma \in \mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$ as follows:

1. The columns of $[T]_\beta^\gamma$ are given by: $[T(v_1)]_\gamma, \ldots, [T(v_n)]_\gamma$.Alternatively: we can express $T(v_j) \in W$ using the basis γ to obtain an expression $T(v_j) = \sum_{i=1}^{m} A_{ij} w_i$. We then define $\left([T]_\beta^\gamma\right)_{ij} = A_{ij}$.

When $T : V \to V$ we denote by $[T]_\beta = [T]_\beta^\beta$.

Theorem 42

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, $\beta = \{v_1, \ldots, v_n\}$ a basis of V, and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W.
The map $\phi_\beta^\gamma : \mathcal{L}(V, W) \to \mathscr{M}_{m\times n}(\mathbb{F})$ given by $\phi_\beta^\gamma(T) = [T]_\beta^\gamma$ is an isomorphism of vector spaces.

Theorem 43

Let V, W, X be finite dimensional vector spaces over $\mathbb{F}$. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V, $\gamma = \{w_1, \ldots, w_m\}$ a basis of W, and $\delta = \{x_1, \ldots, x_p\}$ a basis of X.
For all $T \in \mathscr{L}(V, W)$ and $S \in \mathscr{L}(W, X)$ we have that

$$[S \circ T]_\beta^\delta = [S]_\gamma^\delta [T]_\beta^\gamma.$$

In other words, when using compatible coordinate systems, the composition of linear transformations corresponds to matrix multiplication.

Theorem 44

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, $T : V \to W$ be a linear map, $\beta = \{v_1, \ldots, v_n\}$ a basis of V, and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible.
Moreover, we have

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$$

if T is invertible.

Theorem 45

Let V, W be finite dimensional vector spaces over a field $\mathbb{F}$, $\beta = \{v_1, \ldots, v_n\}$ a basis of V, and $\gamma = \{w_1, \ldots, w_m\}$ a basis of W. Then we have

$$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma$$

for all $v \in V$.

Theorem 46

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, β, β' bases for V, and γ, γ' bases for W. Let $T : V \to W$ be a linear map.

1. $[v]_{\beta'} = [I_V]_\beta^{\beta'} [v]_\beta$.
2. $[T]_{\beta'}^\gamma = [T]_\beta^\gamma [I_V]_{\beta'}^\beta$.
3. $[T]_\beta^{\gamma'} [T]_\beta^\gamma$.
4. $[T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'} [T]_\beta^\gamma [I_V]_{\beta'}^\beta$.

Corollary 6 – Change of Bases

Let V be finite dimensional vector spaces over $\mathbb{F}$, and β, β' bases for V. If $T : V \to V$ is linear then we have that

$$[T]_{\beta'} = [I_V]_\beta^{\beta'} [T]_\beta [I_V]_{\beta'}^\beta.$$

Denoting $[I_V]_{\beta'}^\beta = Q, A = [T]_\beta$, and $B = [T]_{\beta'}$, and recalling that $[I_V]_{\beta'}^\beta = \left([I_V]_\beta^{\beta'}\right)^{-1}$ we have that

$$B = Q^{-1} A Q.$$

Definition 38 – Change of Bases Matrix

Let V be a finite dimensional vector spaces over $\mathbb{F}$, and β, β' bases for V.
The matrix $Q = [I_V]_\beta^{\beta'}$ is called the **change of coordinate matrix from** β to β' coordinates.

Theorem 47

Let V be finite dimensional vector spaces over $\mathbb{F}$, and β, β' bases for V.
If $Q = [I_V]_\beta^{\beta'}$ is the change of coordinate matrix from β to β' coordinates, then Q^{-1} is the change of coordinate matrix from β' to β coordinates.

Chapter 7: Rank, Invertibility and Systems of Equations

Definition 39 – rank

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$ and $T_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ the asscotiated transformation. We define the **rank of** A denoted $\operatorname{rank} A$ by $\operatorname{rank} A = \operatorname{rank} T_A$.

Lemma 1

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_n\}$ bases of V, W respectively, and $T : V \to W$ a linear map. We have $\operatorname{im}(T) = \operatorname{span}[T(v_1), \dots, T(v_n)]$.
As a consequence we have that if $A \in \mathscr{M}_{m\times n}(\mathbb{F})$, then $\operatorname{im}(T_A)$ is spanned by the columns of A.

Theorem 48

Let V, W, X be finite dimensional vector spaces over $\mathbb{F}$ and $T : V \to W$, $S : W \to X$ linear transformations.

1. $\operatorname{im}(S \circ T) \subseteq \operatorname{im} S$.
2. $\operatorname{im}(S \circ T) \subseteq S(\operatorname{im} T)$.
3. If S is an isomorphism, then $\operatorname{im} S \circ T \simeq \operatorname{im} T$.
4. If T is an isomorphism, then $\operatorname{im} S \circ T = \operatorname{im} S$.

Corollary 7

Let V, W, X be finite dimensional vector spaces over $\mathbb{F}$ and $T : V \to W$, $S : W \to X$ linear transformations.

1. $\operatorname{rank} T \leq \min[\dim V, \dim W]$.
2. $\operatorname{rank}(S \circ T) \leq \operatorname{rank} S$.
3. $\operatorname{rank}(S \circ T) \leq \operatorname{rank} T$.
4. If S is an isomorphism, then $\operatorname{rank} S \circ T = \operatorname{rank} T$.
5. If T is an isomorphism, then $\operatorname{rank} S \circ T = \operatorname{rank} S$.

Theorem 49 – Rank of change of bases

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ bases of V, W respectively, and $T : V \to W$ a linear map. Then we have $\operatorname{rank} T = \operatorname{rank}[T]_\beta^\gamma$.

Corollary 8

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F}), B \in \mathscr{M}_{n\times k}(\mathbb{F})$.

1. $\operatorname{rank} A \leq \min\{m, n\}$.
2. $\operatorname{rank} AB \leq \operatorname{rank} A$.
3. $\operatorname{rank} AB \leq \operatorname{rank} B$.
4. If A is invertible, then $\operatorname{rank} AB = \operatorname{rank} B$.
5. If B is invertible, then $\operatorname{rank} AB = \operatorname{rank} A$.

Corollary 9 – Invertibility and Rank

A square matrix $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ is invertible if and only if $\operatorname{rank} A = n$.

Definition 40 – Elementary Matrix

Let $E \in \mathscr{M}_{n\times n}(\mathbb{F})$ be a matrix obtained from 1_n by performing a row/column operation. We call E an **elementary matrix**. We say that E is of Type I, II, or, III if it is obtianed using a Type I, II, or III operation.

Theorem 50

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$ and $E \in \mathscr{M}_{n\times n}(\mathbb{F})$, $E' \in \mathscr{M}_{m\times m}(\mathbb{F})$ be elementary matrices. Set $B = AE$ and $C = E'A$.

1. If E is obtained from 1_n by performing a column operation, then B is obtained from A by performing **the same** column operation.
2. If E is obtained from 1_n by performing a row operation, then B is obtained from A by performing **the same** row operation.
3. E is invertible, and its inverse is an elementary matrix of the same type. (The same is true of E').

Theorem 51

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$ and $E \in \mathscr{M}_{n\times n}(\mathbb{F})$, $E' \in \mathscr{M}_{m\times m}(\mathbb{F})$ be elementary matrices. Set $B = AE$ and $C = E'A$.

1. $\operatorname{rank} A = \operatorname{rank} B$.
2. $\operatorname{rank} A = \operatorname{rank} C$.
3. $Ax = y$ if and only if $Cx = Cy$. In other words, row operations do not change the solutions in a system.
4. $\operatorname{im}(T_A) = \operatorname{im}(T_B)$. In other words, column operations do not change the image.

Definition 41 – RREF

We say a matrix A is in **row-reduced echelon form** if *all* of the following conditions are met:

1. All zero rows are at the bottom of the matrix A.
2. The first non-zero entry of a non-zero row is 1. (Such entries are called "leding 1's".)
3. The leading 1's move to the right, as we go down the rows of A.
4. All entries above and below a leading 1 are 0.

We use the abbreviation "RREF" for "row-reduced echelon form".

Theorem 52 – Gaussian Elimination

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$.

1. If the matrix consists entirely of 0's, stop. It's already row-reduced.
2. Find the first column with a non-zero entry and move the corresopnding row to the top. (We will call the first non-zero entry a.)
3. Divide the row by the number a to obtain a leading one.
4. Subtract multiples of this row from the rows above and below, in order to make each entry above and below the leading 1 equal to 0.
5. Repeat 1-4 on the matrix consisting of the remaining rows.

At the end of the process, we obtain a matrix $R \in \mathscr{M}_{m\times n}(\mathbb{F})$ which is in RREF.
As a consequence, every matrix has a RREF.

Lemma 2

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$ and let $R \in \mathscr{M}_{m\times n}(\mathbb{F})$ be its RREF.

1. If k is the number of zero rows in R, then the columns of R consisting the leading ones are the first $m-k$ elements $\{e_1, \dots, e_{m-k}\}$ of the standard basis $\{e_1, e_2, \dots, e_m\}$ for $\mathbb{F}^m$.
2. $\operatorname{rank} A = \operatorname{rank} R = \#$ of leading 1's.
3. $\operatorname{rank} A = \operatorname{rank} R = \#$ of non-zero rows in R.

Corollary 10

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$. Then there exists an invertible matrix $b \in \mathscr{M}_{m\times m}(\mathbb{F})$ so that $BA = R$ is in RREF.

Corollary 11

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. Then A is invertible if and only if the RREF of A is 1_n.

Theorem 53

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$ be a matrix and $r = \operatorname{rank} A$.
If $r = 0$, then $A = O_{mn}$ is the zero matrix.
If $r > 0$, then there exists a sequence of row operations an column operations, which transform A into a matrix of the form $\left(\begin{array}{c|c} \mathrm{I}_r & 0 \\ \hline 0 & 0 \end{array}\right)$.

Theorem 54

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$, R denote its $RREF$, and $T_A : \mathbb{F}^n \to \mathbb{F}^m$ te associated linear transformation. Let $a_1, \ldots, a_n$ denote the columns of A.

1. The set of $n - r$ "basic solutions" for to $Ax = 0$ is a basis for $N(T_A)$.
2. If the leading 1's in R appear in columns $k_1, \ldots, k_r$, then the columns $a_{k_1}, a_{k_2}, \ldots, a_{k_r}$ of A provide a basis for $\operatorname{im}(T_A)$.
3. If we apply the Gaussian Algorithm replacing "row operations" with "column operations" at every step, then the non-zero columns of the resulting matrix provide a basis for $\operatorname{im}(T_A)$.

Theorem 55 – Invertibility

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. Then A is invertible if and only if A is a product of elementary matrices.

Theorem 56 – Computing the Inverse

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. Consider the augmented matrix $(A|I_n)$. Row reduce the augmented matrix to obtain $(R \mid B)$, where R is the RREF of A.
Then A is invertible if and only if this procedure results in the augmented matrix $(I_n \mid B)$. In this case, $B = A^{-1}$.

Chapter 8: Determinants

Definition 42

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{F})$. We define the **determinant** of A by $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \in \mathbb{F}$.

Theorem 57

The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{F})$ is invertible if and only if $\det A = ad - bc \neq 0$.

In this case A is invertible we have $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Definition 43

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. For each $i, j \in \{1, \ldots, n\}$ we define $\tilde{A}_{ij}$ to be the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j of A.
We define $\det A \in \mathbb{F}$ as follows:

1. For $n = 1$, $A = (a) \in \mathbb{F}$, and we set $\det(a) = a$.
2. For $n = 2$, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
3. For $n \geq 2$, we set $\det A = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det\left(\tilde{A}_{1j}\right)$

Theorem 58

Consider the determinant as a function on the rows of a matrix.
If $\mathbf{r}_1, \ldots, \mathbf{r}_n, \mathbf{a}, \mathbf{b} \in \mathbb{F}^n$ and $c \in \mathbb{F}$, then we have:

$$\det \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ c\mathbf{a}+\mathbf{b} \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = c \det \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{a} \\ \vdots \\ \mathbf{r}_n \end{pmatrix} + \det \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{b} \\ \vdots \\ \mathbf{r}_n \end{pmatrix}.$$

In other words, for each $i \in \{1, \ldots, n\}$, the determinant is a linear function of row i, **with all the other rows fixed.**

Theorem 59

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ with $n \geq 2$.

1. If A has a row of 0's, then $\det A = 0$.
2. For any $i \in \{1, \ldots, n\}$, $\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det\left(\tilde{A}_{ij}\right)$. (This is called the "row i expansion of $\det A$".)
3. If A has two identical rows, then $\det A = 0$.
4. If B is obtained from A by swapping two rows, then $\det B = -\det A$.
5. If B is obtained from A by scaling row i by $a \in \mathbb{F}$, then $\det B = a \det A$.
6. If B is obtained from A by adding a multiple of one row to another, then $\det B = \det A$. (Here $n \geq 2$ so that the statement makes sense.)
7. If A is upper (or lower) triangular, then $\det A$ is the product of the diagonal entries of A.

Theorem 60

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. If $\operatorname{rank} A < n$, then $\det A = 0$.

Theorem 61

Let $E \in \times(\mathbb{F})$ be an elementary matrix.

1. If E is of Type I, then $\det E = -1$.
2. If E is of Type II, and scales by $a \in F$, then $\det E = a$.
3. If E is of Type III, then $\det E = 1$.

Theorem 62

Let $A, B \in \mathscr{M}_{n\times n}(\mathbb{F})$. Then $\det(AB) = \det(A)\det(B)$.

Theorem 63

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. Then A is invertible if and only if $\det A \neq 0$.

Theorem 64

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. Then $\det(A^t) = \det A$.

Corollary 12

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$, then $\det(A) = \sum_{i=1}^{n}(-1)^{i+1}A_{i1}\det\left(\tilde{A}_{i1}\right)$. (This is called the "column j expansion of $\det A$").

Theorem 65

Let $A, B \in \mathscr{M}_{n\times n}(\mathbb{F})$. If $A \sim B$, then $\det A = \det B$.

Definition 44

Let V be finite dimensional vector space over $\mathbb{F}$, and $T : V \to V$ a linear map. We define

$$\det T = \det[T]_\beta,$$

where β is any basis of V.

Theorem 66 – Geometric meaning of Determinants

Let $: \mathbb{R}^2 \to \mathbb{R}^2$ be linear. Then $|\det T| =$Area(P), where P is the parallelogram spanned by $T(e_1), T(e_2)$.

Theorem 67

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be linear. Then $|\det T| =$Vol(P), were P is the parallelepiped spanned by $T(e_1), T(e_2), T(e_3)$.

Theorem 68

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. Then $|\det T|$=Vol(P), where P is the n-parallelepiped spanned by $T(e_1), \dots, T(e_n)$.

Chapter 9: Diagonalization

Definition 45

Recall, we say that a square matrix $D \in \mathscr{M}_{n\times n}(\mathbb{F})$ is diagonal if $D_{ij} = 0$ for $i \neq j$.

We will denote the diagonal matrix $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ by $D = \operatorname{diag}(d_{11}, \ldots, d_{nn})$.

For example $\operatorname{diag}(1,2,3)$ denotes the diagonal matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Definition 46

Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ a linear map.
We say that $\mathbf{v} \in V$ is **an eigenvector for** T if $\mathbf{v} \neq \mathbf{0}$ and $T(\mathbf{v}) = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{F}$.
We call the scalar $\lambda \in \mathbb{F}$ an **eigenvalue** of T.
Let $E_\lambda = \{\mathbf{v} \in V \mid T(\mathbf{v} = \lambda\mathbf{v})\}$. We call E_λ the **eigenspace for eigenvalue** λ.

Theorem 69

Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. For any $\lambda \in \mathbb{F}$, the set E_λ is a subspace.

Corollary 13

Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. For any $\lambda \in \mathbb{F}$ we have $E_\lambda = N(\lambda\mathbf{I}_V - T)$.

Definition 47

Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map.
Set $C_T(x) = \det(x\mathbf{I}_V - T)$. We call C_T the **characteristic polynomial** of T.
Similarly, for $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ we define $C_A(x) = \det(x\mathbf{I}_V - A)$, and call C_A the characteristic polynomial of A.

Theorem 70

Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. Let $n = \dim V$.

1. C_T is a polynomial of degree n and the coefficient of x^n is 1.
2. $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if $C_T(\lambda) = 0$. That is, the roots of C_T are precisely the eigenvalues of T.
3. For all eigenvalues $\lambda \in \mathbb{F}$, the corresponding eigenvectors can be found by solving the homogeneous system $(\lambda\mathbf{I} - T)(\mathbf{x}) = \mathbf{0}_V$.

Definition 48

Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. We say that T is **diagonalizable** if there exists a basis β of B so that $[T]_\beta$ is a diagonal matrix.
We say that $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ is **diagonalizable** if there eixsts an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$.

Theorem 71 – Alternate Characterizations of Diagonalizability

1. Let V be a finite dimensional vector space over F and $T{:}V \to V$ a linear map. T is diagonalizable if and only if there exists a basis of V consisting of eigenvectors for T.
2.$A \in \mathscr{M}_{\mathbf{n}\times\mathbf{n}}(\mathrm{F})$ is diagonalizable if and only if A is similar to a diagonal matrix.
3.$A \in \mathscr{M}_{\mathbf{n}\times\mathbf{n}}(\mathcal{F})$ is diagonalizable if and only if $T_A{:}\mathbb{F}^n \to \mathbb{F}^n$ is a diagonalizable linear map.

Theorem 72 – Test for Diagonalization

Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map.
T is diagonalizable if and only if all the conditions here are met:
1. The characteristic polynomial $C_T(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i}$ splits over $\mathbb{F}$.*
2. For each eigenvalue λ_i we have $m_i = \dim E_i$.

If these conditions are met, then we obtain a diagonalizing basis β as follows:
1. For each eigenvalue λ_i find a basis β_i of $E_i = N(\lambda_i \mathrm{I}_V - T)$ by solving the corresponding system.
2. Set $\beta = \beta_1 \cup \cdots \cup \beta_k$.
3. Then $[T]_\beta = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$

Corollary 14 – The "Matrix" Test for Diagonalization

Let $A \in \mathscr{M}_{\mathbf{n}\times\mathbf{n}}(\mathbb{F})$. Then A is diagonalizable if and only if all the following conditions are met:
1. The characteristic polynomial $C_A(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i}$ splits over $\mathbb{F}$.
2. For each eigenvalue λ_i we have $m_i = \dim E_i$.

If these conditions are met, then we obtain P, D as follows:
1. For each eigenvalue λ_i find a basis β_i of $E_i = N(\lambda_i \mathrm{I} - A)$ by solving the corresponding system.
2. Moreover, the matrix P has columns $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$, where $\mathbf{v}_i \in \beta$ are eigenvectors (expressed as columns in "standard coordinates"). The matrix $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.*

Definition 49

Suppose that $T : V \to V$ is linear and $\lambda \in \mathbb{F}$ is an eigenvalue. Then we know that λ is a root of C_T, so we have $C_T(x) = (x - \lambda)^m p(x)$ for some $m \in \mathbb{N}$ and $p \in \mathrm{P}(\mathbb{F})$ so that $p(\lambda) \neq 0$.
We call m the multiplicity of λ.

Lemma 3

Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map.
If λ is an eigenvalue of T, then we have $1 \leq \dim E_\lambda \leq m_\lambda$.

Lemma 4

Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map.
If λ_1, λ_2 are distinct eigenvalues for T, and $S_1 \subseteq E_1$, $S_2 \subseteq E_2$ linearly independent subsets of eigenvectors, then $S_1 \cup S_2$ is linearly independent.

Theorem 73 – Invariant Subspaces and Diagonalization

Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ be a linear map. The following two conditions are equivalent:

1. T is diagonalizable.
2. There exists one-dimensional invariant suspaces $W_1, \ldots, W_n$ so that $V = W_1 \oplus \cdots \oplus W_n$.

Chapter 10: Matrix Algebra (Appendix A)

Definition 50 – Matrix Multiplication

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$ and $B \in \mathscr{M}_{n\times k}(\mathbb{F})$. We define their product $AB \in \mathscr{M}_{m\times k}(\mathbb{F})$ as follows: for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k\}$ the ij-entry of the product AB is given by

$$(AB)_{ij} = \sum_{l=1}^{n} A_{il} B_{lj}.$$

$AB \neq BA.$

Definition 51 – Special Matrices

For each $n, m \in \mathbb{N}$ we define the following matrices:

1. $O_{m,n} \in \mathscr{M}_{m\times n}(\mathbb{F})$ - the matrix consisting of all 0's. In other words $(O_{m,n})_{i,j} = 0$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.
2. $I_n \in \mathscr{M}_{n\times n}(\mathbb{F})$ - the matrix with 1's on the diagonals, and 0 in all other entries. In other words

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$O_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem 74

Let $\mathbb{F}$ be a field, $A, A_1, A_2 \in \mathscr{M}_{\mathbf{m}\times\mathbf{n}}(\mathbb{F}), B, B_1, B_2 \in \mathcal{M}_{n\times k}(\mathbb{F}), C \in \mathcal{M}_{k\times p}(\mathbb{F})$ and $c \in \mathbb{F}$.

1. $A(BC) = (AB)C$
2. $(A_1 + A_2)B = A_1B + A_2B$
3. $A(B_1 + B_2) = AB_1 + AB_2$
4. $I_mA = A = AI_n$
5. $\mathrm{O}_{rm}A = \mathrm{O}_{rn}$ for any $r \in \mathbb{N}$.
6. $A(cB) = c(AB) = (c\mathrm{I}_m)AB = AB(c\mathrm{I}_k) = A(c\mathrm{I}_n)B.$

Definition 52 – Invertibility

Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. We say that A is **invertible** if there exists a matrix $B \in \mathscr{M}_{n\times n}(\mathbb{F})$ so that $AB = I_n = BA$.

Theorem 75

Let $A, B \in \mathscr{M}_{n\times n}(\mathbb{F})$.

1. If A is invertible, then the inverse of A is unique.
2. If A is invertible, then A^{-1} is also invertible.
3. If A and B are invertible, then AB is invertible.
4. I_n is invertible.
5. If $AB = I_n$, then A is invertible and $B = A^{-1}$.

Definition 53 – A^t

Let $A \in \mathscr{M}_{m\times n}(\mathbb{F})$. We define the matrix $A^t \in \mathscr{M}_{n\times m}(\mathbb{F})$ by:

$$(A^t)_{ij} = A_{ji}.$$

In other words, to obtain A^t we "swap the rows and columns of A."

Definition 54 – Symmetric

We say that $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ is **symmetric** if $A^t = A$. We denot ethe set of all symmetric matrices by $\mathbf{Sym}_n(\mathbb{F})$. We say that $A \in \mathscr{M}_{n\times n}(\mathbb{F})$ is **skew-symmetric** if $A^t = -A$. We denote the set of all skew-symmetric matrices by $\mathbf{Sk}_n(\mathbb{F})$.

Theorem 76

Let $A, B \in \mathscr{M}_{\mathbf{m}\times\mathbf{n}}(\mathbb{F}), C \in \mathscr{M}_{n\times k}(\mathbb{F})$ and $c \in \mathbb{F}$.
1. $(A+B)^t = A^t + B^t$
2. $(cA)^t = cA^t$
3. $(A^t)^t = A$
4. $(AC)^t = C^t A^t$.
5. In the case that $m = n$, we also have that if $A \in \mathscr{M}_{\mathbf{n}\times\mathbf{n}}(\mathbb{F})$ is invertible, then $A^t \in \mathscr{M}_{\mathbf{n}\times\mathbf{n}}(\mathbb{F})$ is invertible and $(A^t)^{-1} = \left(A^{-1}\right)^t$.

Definition 55 – Diagonal and Triangular

We say that A is **diagonal** if $A_{ij} = 0$ for all $i \neq j$.
Let $A \in \mathscr{M}_{n\times n}(\mathbb{F})$. We say that A is **upper triangular** if $A_{ij} = 0$ for all $i > j$. This means that all entries below the diagonal of A must be 0.
Similarily, we say that A is **lower triangular** if $A_{ij} = 0$ for all $i < j$. This means that all entries above the diagonal of A must be 0.
We say that A is **strictly upper-triangular** if $A_{ij} = 0$ for all $i \geq j$.