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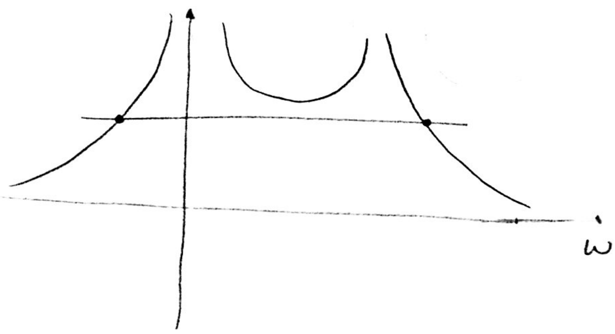
Problem 2 → Two-Stream Instability

a) Notebook : H3 - Q2 - 84364.nb

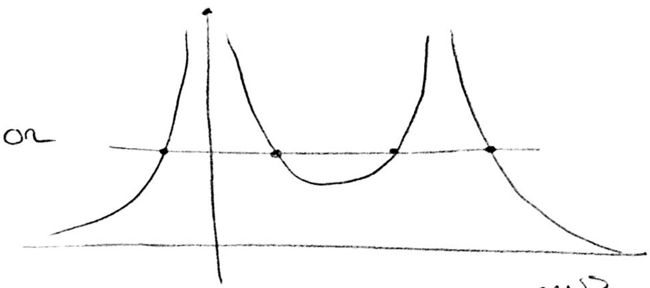
Dispersion relation

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - k v_0)^2} = 0$$

This equation has 4 roots (2 complex conjugate 2 real or 4 real)



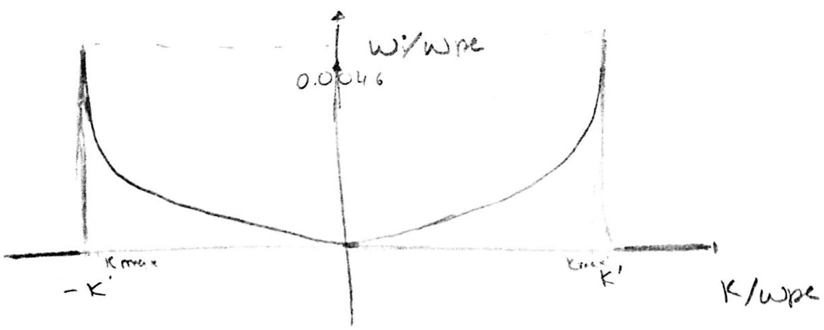
or



Since $\exp(-i\omega t) = \exp(\omega_i t)\exp(i\omega_r t)$ → instability = grows exponentially

the growth rate is given by the imaginary parte of the solution that is positive.

the plot was done in the notebook H3 - Q2.nb

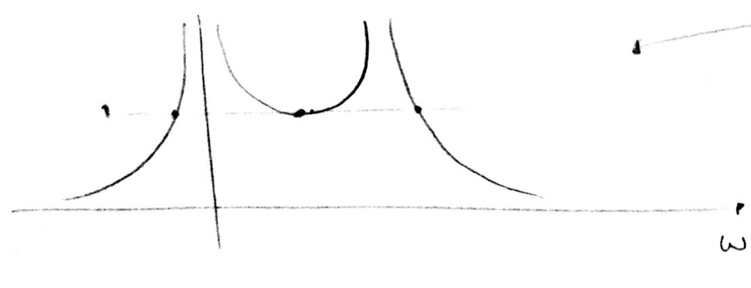


$k_{max} = \pm 1.00005$

↓

from find max of table.

To get the value of k', look for the first value of k that has no imaginary ω => look for k that:



$$\Rightarrow \quad \varepsilon(k,\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - k v_0)^2}$$

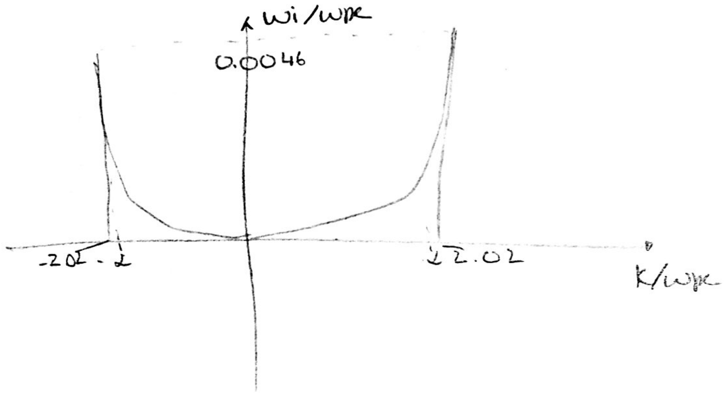
$$\begin{cases} \varepsilon(k,\omega) = 0 \\ \dfrac{\delta \varepsilon(k,\omega)}{\delta \omega} = 0 \end{cases} \qquad k \approx 1.01$$

$$\frac{k v_0}{\omega} = 1.01$$

The values of k that lead to instability $[-1.01, 1.01]$ except $k = 0$.

b)

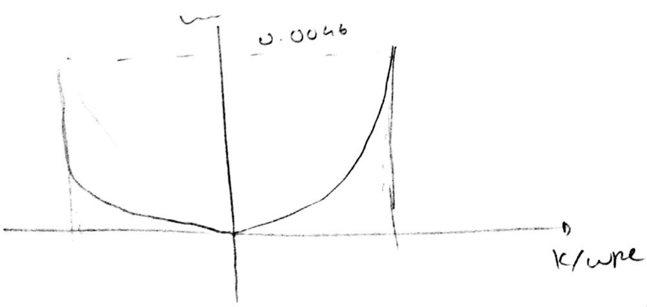
$v = 0.5$



$k \in [-2.02, 2.02]$ → explained in previous question

$K_{max} = \pm 2$

$v = 2$



$k \in [-0.505, 0.505]$

$k_{max} = 0.5$

c) $1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{(kv_0-\omega)^2} = 0$

$\omega_{pi} \to 0$ — from the question

$\Rightarrow \quad kv_0 - \omega = \pm\omega_{pe}$

$\Rightarrow \quad \omega = kv_0 \pm \omega_{pe}$

The doppler shifted frequency in the frame moving with the electrons is given by

$\omega' = \omega - kv_0 = kv_0 \pm \omega_{pe} - kv_0 = \pm\omega_{pe}$

$\Rightarrow$ The electrons see an oscillation at nearly their natural frequency of oscillation $\Rightarrow$ resonance.

Problem 3 → The KdV equation

a) • $x \to x - c_s t$ → frame that travels at the ion accustic sound velocity

• neglet dispersive effects
• Ion accustic non linear waves: longitudinal oscillation
Continuity equation: $m_e \to 0$???

$$\frac{\delta n_s}{\delta t} + \vec{\nabla}.(n_s \vec{v_s}) = 0$$

Force equation:

$$m_s n_s \left[\frac{\delta \vec{v_s}}{\delta t} + \vec{v_s}\, \vec{\nabla}. \vec{v_s} \right] = q n_s (\vec{E} + \vec{v_s} \times \vec{B}) - \vec{\nabla} P_s$$

Poisson's equation:

$$\vec{\nabla}.\vec{E} = 4\pi \sum q n_s$$

electrons

linearization, fourier transform, neglet electron inertia ($m_e \to 0$)

$$-i\omega n_{1e} + iK n_0 v_e = 0 \quad (1)$$

$$0 = -iK T_e n_{1e} - e n_0 E \quad (2)$$

Ions

Keep non linear terms, neglet ion pressure ($T_i \to 0$) → force equation

linearization, fourier transformation → continuity equation

$$-i\omega n_{1i} + iK n_0 v_i = 0 \quad (3)$$

$$m_i \frac{\delta v_i}{\delta t} + m_i v_i \frac{\delta v_i}{\delta x} = eE \quad (4)$$

Poisson:

$$iKE = 4\pi e (n_{1i} - n_{e1}) \quad (5)$$

→ from (2) $$n_{1e} = -\frac{e n_0 E}{iKT_e} = \frac{i e n_0 E}{KT_e}$$

→ (2) → (5) $$+iKE = -\frac{4\pi e^2 n_0 E i}{KT_e} + 4\pi e n_{1i}$$

$$\Rightarrow \quad E\left(iK + \frac{i e^2 n_0 4\pi}{T_e K}\right) = 4\pi e n_{i1}$$

$$\Rightarrow \quad E = \frac{4\pi e n_{i1}}{iK - \frac{\omega_{pi}^2 m_i}{i T_e K}} \qquad (6)$$

$\rightarrow$ (6) $\rightarrow$ (4)

$$m_i \frac{\delta v_i}{\delta t} + m_i v_i \frac{\delta v_i}{\delta x} = \frac{4\pi e^2 n_{i1}}{iK - \frac{\omega_{pi}^2 m_i}{i T_e K}} \qquad (7)$$

$\rightarrow$ (2) $\rightarrow$ $$n_{i1} = \frac{iK n_0 v_i}{i\omega} = \frac{K n_0 v_i}{\omega}$$

$\rightarrow$ (2) $\rightarrow$ (7)

$$m_i \frac{\delta v_i}{\delta t} + m_i v_i \frac{\delta v_i}{\delta x} = \frac{4\pi e^2 K n_0 v_i/\omega}{iK - \frac{\omega_{pi}^2 m_i}{i T_e K}}$$

$$\Rightarrow \quad \frac{\delta v_i}{\delta t} + v_i \frac{\delta v_i}{\delta x} = \boxed{-\frac{\omega_{pi}^2 K v_i}{\omega} \quad \frac{i}{K + \omega_{pi}^2 m_i / T_e K}} \qquad (8)$$

$$c_s^2 = \frac{T_e}{m_i}$$

$$\lambda_{De}^2 = \frac{T_e}{m_e} \frac{1}{\omega_{pe}^2} = \frac{T_e}{m_e m_i} \frac{m_i}{\omega_{pe}^2} = \frac{T_e}{m_i} \frac{1}{\omega_{pi}^2}$$

$$\rightarrow (8) \quad - \frac{\omega_{pi}^2 K v_i}{\omega} \quad \frac{1/\omega_{pi}^2 \quad K \quad i}{K/\omega_{pi}^2 \left(K + \omega_{pi}^2 m_i / T_e K\right)}$$

$$= - \frac{i K^2 v_i}{\omega} \quad \frac{1}{K^2/\omega_{pi}^2 + m_i/T_e}$$

$$= - \frac{i K^2 v_i}{\omega} \quad \frac{T_e/m_i}{K^2 T_e/\omega_{pi}^2 m_i - 1}$$

$$= - \frac{i K^2 c_s^2}{\omega} \left(K^2 \lambda_{De}^2 + 1\right)^{-1} v_i$$

Ion accustic disperhon relation: $\omega = K c_s \left(1 + K^2 \lambda_{De}^2\right)^{-1/2}$

Assuming that the disperhons relations for ion acoustic waves and non linear waves are simillar:

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = - \frac{i k^2 c_s^2}{k c_s (1+k^2\lambda_{De}^2)^{-1/2}} (1+k^2\lambda_{De})^{-1} v_i$$

$$\Rightarrow \quad \frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = - i k c_s (1+k^2\lambda_{De}^2)^{-1/2} v_i$$

For small $k\lambda_{De}^2$ such that $1 \gg k^2\lambda_{De}^2$

$$\Rightarrow \quad \frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = - i k c_s \left(1 - \frac{1}{\alpha} k^2 \lambda_{De}^2\right) v_i$$

$$ik \to \frac{\partial}{\partial x}$$
$$-ik^3 \to \frac{\partial^3}{\partial x^3}$$

$$\Rightarrow \quad \frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = - c_s \frac{\partial v_i}{\partial x} - \frac{c_s}{\alpha} \lambda_{De}^2 \frac{\partial^3 v_i}{\partial x^3}$$

$$\Rightarrow \quad \frac{\partial v_i}{\partial t} + \underbrace{(v_i + c_s)\frac{\partial v_i}{\partial x}}_{\text{non linear term}} + \underbrace{\frac{c_s \lambda_{De}^2}{\alpha} \frac{\partial^3 v_i}{\partial x^3}}_{\text{dispersive term}} = 0$$

since $x' = x - c_s t$, $t' = t$

$$\frac{\partial x'}{\partial t} = \frac{\partial x}{\partial t} - c_s = v_i - c_s = v_i'$$

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -c_s \frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}$$

neglecting the dispersive term $\frac{c_s \lambda_{De}^2}{\alpha} \frac{\partial^3 v_i}{\partial x^3} \to 0$

$$\Rightarrow \quad \frac{\partial v_i}{\partial t'} - \cancel{c_s \frac{\partial v_i}{\partial x'}} + (v_i + \cancel{c_s}) \frac{\partial v_i}{\partial x'} = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial v_i}{\partial t'} + v_i \frac{\partial v_i}{\partial x'} = 0}$$

b)

$$\begin{cases} \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0, \quad t > 0 \\ v(x,0) = v_{in}(x) \end{cases}$$

$$\begin{cases} \frac{dx}{dt} = v(x(t), t), \quad t > 0 \\ x(0) = x_0 \end{cases}$$

$$\frac{d v(x(t),t)}{dt} = \frac{\partial v(x(t),t)}{\partial t} + \frac{dx}{dt} \frac{\partial v(x(t),t)}{\partial x}$$

$$= \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0$$

v ~~can~~ is constant along the characteristic curve $x(t)$

$$\Rightarrow v(x(t), t) = v(x(0), 0) = v_{in}(x_0)$$

since $\frac{dx}{dt} = v(x(t), t)$ (v is contant along x)

$\Rightarrow x(t) = v_{in}(x_0)\, t + \text{const} \Rightarrow x(0) = x_0.$

$\Rightarrow x(t) = v_{in}(x_0)\, t + x_0$

$\Rightarrow x_0$ can be written as $x_0(x, t)$

$x_0(x, t) = x - v_{in}(x_0)\, t$

$\Rightarrow v(x, t) = v(x_0(x,t)) = v_{in}(x - v_0(x_0)\, t)$

e) $$v(u,0) = v_0\left(1 - \left(\frac{u}{\sigma_u}\right)^2\right) = v_{in}(u)$$

$$v(u,t) = v(u_0(u,t)) = v(u - v_{in}(u_0)t)$$

$$u_0 = u - v_0\left(1 - \left(\frac{u_0}{\sigma_u}\right)^2\right)t$$

$$\Rightarrow \quad u_0 = u - v_0 t + v_0 t \frac{u_0^2}{\sigma_u^2}$$

$$\Rightarrow \quad u_0^2 \frac{v_0 t}{\sigma_u^2} - u_0 + (u - v_0 t) = 0$$

$$\Rightarrow \quad u_0 = \frac{1 \pm \sqrt{1^2 - 4(u - v_0 t) \cdot v_0 t / \sigma_u^2}}{2 v_0 t / \sigma_u^2}$$

$$u_0^2 = \frac{\sigma_u^4}{4(v_0 t)^2}\left(1 \pm 2\sqrt{1 + \frac{4(v_0 t - u)v_0 t}{\sigma_u^2}} + 1 + \frac{4(v_0 t - u)v_0 t}{\sigma_u^2}\right)$$

$$= \frac{\sigma_u^4}{2(v_0 t)^2}\left(1 \pm \sqrt{1 + \frac{4(v_0 t - u)v_0 t}{\sigma_u^2}} + \frac{2(v_0 t - u)v_0 t}{\sigma_u^2}\right)$$

$$\Rightarrow \quad v_{in}(u_0(u,t)) = v_0\left(1 - \frac{u_0^2}{\sigma_u^2}\right) =$$

$$= v_0\left(1 - \frac{\sigma_u^2}{2(v_0 t)^2}\left(1 \pm \cdots + \frac{2(v_0 t - u)v_0 t}{\sigma_u^2}\right)\right)$$

$$= v_0\left(1 - \frac{v_0 t - u}{v_0 t} - \frac{\sigma_u^2}{2(v_0 t)^2}(1 \pm \cdots)\right)$$

$$= v_0\left(1 - 1 + \frac{u}{v_0 t} - \cdots\right)$$

$$= \left(\frac{u}{t} - \frac{\sigma_u^2}{2 v_0 t^2}(1 \pm \cdots)\right)$$

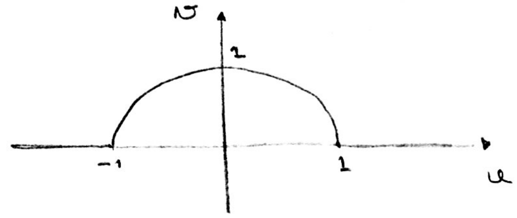
$$= \frac{\sigma_u}{t}\left(\frac{u}{\sigma_u} - \frac{\sigma_u}{2 v_0 t}(1 \pm \cdots)\right)$$

$$= \frac{\sigma u}{t} \quad \frac{\sigma u}{2 v_0 t}\left[\frac{2 v_0 t}{\sigma u}\quad\frac{u}{\sigma u} - \left(1 \pm \sqrt{1 + \frac{4 v_0 t\,(v_0 t - u)}{\sigma u^2}}\right)\right.$$

d) Notebook: H3-Q3-84364.nb

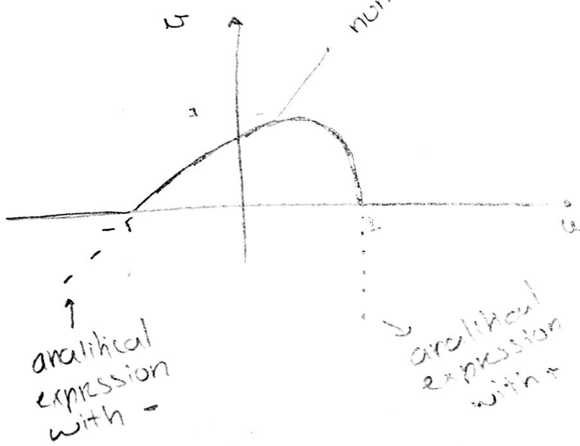
$\boxed{t=0}$ the function (s) does not wont $\to \infty$

Plot of $v(u,0) = v_0\left(1 - \frac{u^2}{\sigma^2}\right)$



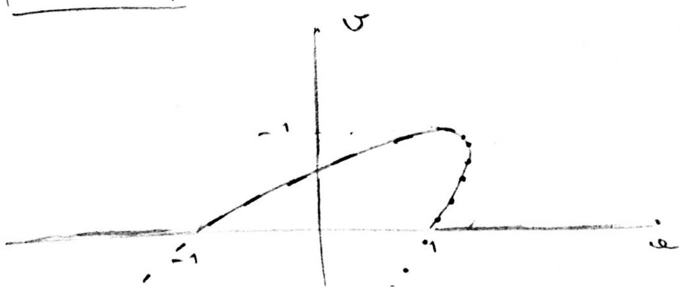
$\boxed{t = 0.5}$

since the velocity of each particle is constant and the position varies with $u = u_0 + v_0(u,t)\,t$ (see previus question) we can plot the velocity distribuition as seen in the notbook.



If we consider the $v > 0$ then the two solutions seem to be the same.

$\boxed{t = 1.5}$ the two solutions are also similar



e) to determine the time at which there is a shock we consider two particles with inicial positions u_1 and $u_2 = u_1 + \Delta u$

The two particles cross when their positions are the same

$$u_1(t) = u_2(t)$$

$$\Rightarrow \quad u_1 + v_{in}(u_1)t = u_2 + v_{in}(u_2)t$$

$$\Rightarrow \quad t = \frac{u_1 - u_2}{v_{in}(u_1) - v_{in}(u_2)}$$

$$= \frac{\Delta u}{v_{in}(u_1) - v_{in}(u_1 + \Delta u)}$$

when $\Delta u \to \infty$ $\quad t = \dfrac{1}{-v_{in}'(u)}$

$$\Rightarrow \quad t = \min\left(-\frac{1}{v_{in}'(u)}\right) = \min\left(+\frac{\sigma_u^2}{2u\, v_0}\right)$$

since $\sigma_u = 1$ $\quad v_0 = 1$

$$t = \min\left(\frac{1}{2u}\right)$$

since $v_0(u)$ is $\neq 0$ from $u \in [-1, 1]$

$$t = \frac{1}{2} = 0.5$$

Numerically the first t at which there are two particle with the same u is 0.5.

so it's confirmed

Notebook: H3 - Q3 - 84364.nb

To get the time at which the function becomes multivalued at a certain position, that the shock gets to position u:

$$\Rightarrow \frac{\delta v}{\delta u} \to \infty \quad \to \text{multivalued}$$

$$\frac{\delta v}{\delta u} = \frac{\sigma^2}{2\omega t^2}\left[\frac{2\omega t}{\sigma^2} \pm \frac{1}{2}\frac{1}{\sqrt{\ldots}} - \frac{4\omega t}{\sigma^2}\right]$$

$$= \frac{1}{t} \pm \frac{1}{t\sqrt{1+\frac{4\omega t(\omega t-u)}{\sigma^2}}}$$

$$= \frac{\sqrt{1+\frac{4\omega t(\omega t-u)}{\sigma^2}} \pm 1}{t\sqrt{}} \to \infty$$

quando $t\sqrt{1+\frac{4\omega t(\omega t-u)}{\sigma^2}} \to 0$

$$\Rightarrow 1+\frac{4\omega t(\omega t-u)}{\sigma^2} = 0$$

$$\Rightarrow \sigma^2 + 4\omega^2 t^2 - 4\omega t u = 0 \quad (2)$$

$$\Rightarrow t = \frac{4\omega u \pm \sqrt{16\omega^2 u^2 - 16\omega^2\sigma^2}}{8\omega^2}$$

$$= \frac{u}{2\omega} \pm \frac{1}{2}\sqrt{\frac{u^2}{\omega^2} - \frac{\sigma^2}{\omega^2}}$$

$$= \frac{1}{2\omega}\left(u \pm \sqrt{u^2-\sigma^2}\right)$$

$$= \frac{u}{2\omega}\left(1 \pm \sqrt{1-\frac{\sigma^2}{u^2}}\right)$$

$\Rightarrow t = 1.866$

$t = 0.134 \to$ não faz sentido, a função só fica multivalued para qualquer u em $t > 0.5$ (primeira parte da pergunta) confirmed by the numerical calculations

$t = 1.87$

$u = 1.99949$

$v = 0.9516$ and $v = 0$