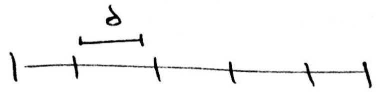
Problem 1

1) Plasma sheet model : uniform ion background → density n_0
evenly spaced electron sheets → spacing between two sheets - δ



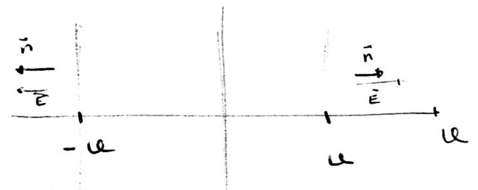
Charge of the electron sheet $= -en_0\delta$

→ Calculate the field created by the ions

· Use poisson's equation and gauss' law

$$\oiint \vec{E}\cdot\vec{n}\,ds = 4\pi Q_{int}$$

· Considering a surface going from $-x$ to x, the charge inside that surface is $2x\,en_0A$



$$\Rightarrow \oiint \vec{E}\cdot\vec{n}\,ds = 2EA$$

· $2E\cancel{A} = 4\pi\, 2x\, en_0 \cancel{A}$

$\Rightarrow$ $E = 4\pi e n_0$

→ Calculate the electric field due to the electron sheet

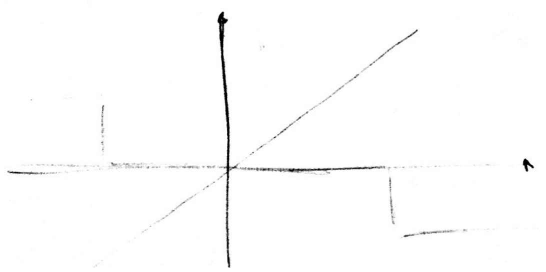
$Q_{int} = -en_0\delta$

$\Rightarrow$ $\oiint \vec{E}\cdot\vec{n}\,ds = 4\pi Q_{int}$

$\Rightarrow$ $2E = -4\pi e n_0\delta$

$\Rightarrow$ $E = -\frac{1}{2}4\pi e n_0\delta$

→ Add the two fields



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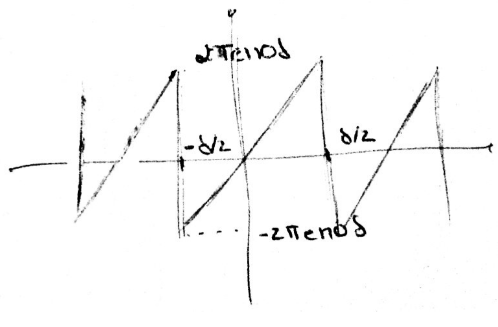
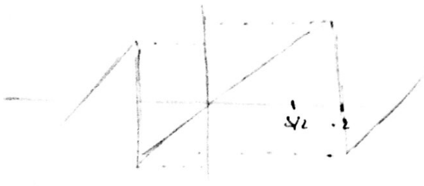


→ Displace an electron sheet by ξ such that its position is given by $x = x_0 + \xi$

→ The electric field is displaced

$$E(x^-) = 4\pi n_0 e \left(\frac{\delta}{2} + \xi\right)$$
$$E(x^+) = 4\pi n_0 e \left(\frac{\delta}{2} + \xi\right) - 4\pi e n_0 \delta$$



→ The electric field at x is given by the average:

$$E = \frac{4\pi n_0 e \left(\frac{\delta}{2} + \xi\right) + 4\pi n_0 e \left(\frac{\delta}{2} + \xi\right) - 4\pi e n_0 \delta}{2}$$

$$\boxed{E = 4\pi n_0 e \xi}$$

2) $$\frac{E}{4\pi n_0 e} = \xi \qquad\qquad \xi = A \sin(k x_0)$$
$$\Rightarrow x_0 = \frac{1}{k} \arcsin\left(\frac{\xi}{A}\right)$$

$$\Leftrightarrow \frac{E}{4\pi n_0 e} = x - x_0$$

$$\Leftrightarrow \frac{E}{4\pi n_0 e} + \frac{1}{k} \arcsin\left(\frac{\xi}{A}\right) = x$$

$$\xi = \frac{E}{4\pi n_0 e} \Rightarrow \boxed{\frac{E}{4\pi n_0 e} + \frac{1}{k} \arcsin\left(\frac{E}{4\pi n_0 e A}\right) = x}$$

3) $$A \ll 1/k \Rightarrow 1 \ll \frac{1}{kA}$$

$$E' + \frac{1}{k} \arcsin(E'/A) = x, \quad E' = \frac{E}{4\pi n_0 e}$$

Taylor expansion of arcsin

$$\Rightarrow E' + \frac{1}{k}\left(\frac{E'}{A} + \frac{E'^3}{6A^3} + \frac{3E'^5}{40A^5} + \dots\right) = x$$

$$\Rightarrow E'\left(1 + \frac{1}{kA} + \frac{E'^2}{6kA^3} + \frac{3E'^4}{40kA^5} + \dots\right) = x$$

approximation in the question

$$\Rightarrow E'\left(\frac{1}{kA} + \frac{E'^2}{6kA^3} + \frac{3E'^4}{40kA^5} + \dots\right) \approx x$$

$$\frac{1}{k}\left(\frac{E'}{A}+\frac{E'^3}{6A^3}+\cdots\right)\simeq u$$

$$\frac{1}{k}\arcsin\left(E'/A\right)\simeq u$$

$$\Rightarrow \quad E' \simeq A \sin(ku)$$

$$\Rightarrow \quad E = 4\pi e n_0 A \sin(ku)$$

$$A \ll 1/k \rightarrow A \ll \lambda$$

For wave amplitudes much smaller than the plasma wave wave-length the E field varies sinusoidally with u.

$$E^* = 4\pi e n_0 A \sin\left(k\left(u-\frac{E}{4\pi n e}\right)\right)$$

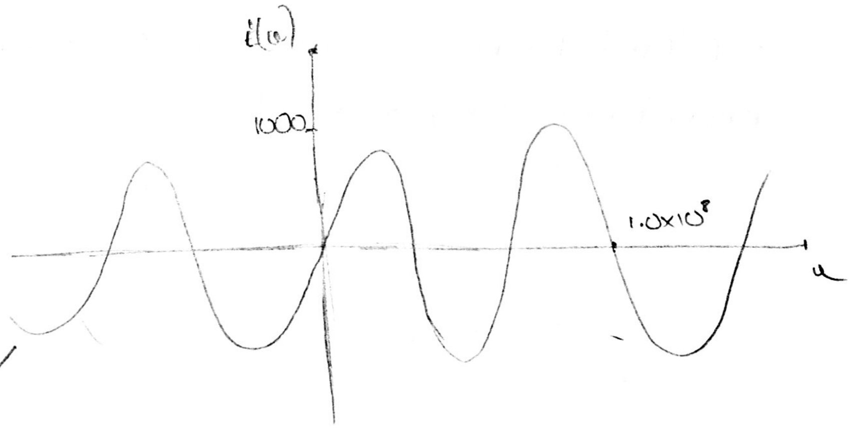
$$\Rightarrow \quad u \gg \frac{E}{4\pi n_0 e} \quad (\Rightarrow$$

4) The plots were made in Mathematica and can be found on H1-01.pdf

for $A \ll 1/k$

$A = 1000$

$k = 0.0000001$



o'simples pomecan sen(ku)

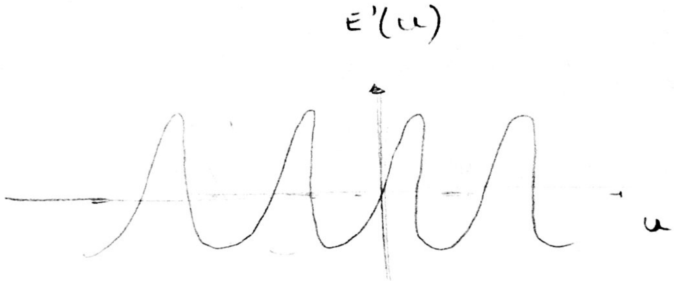
$E'(u)$ seems to be $= A\sin(ku)$ like we expected in the previous question

Small amplitude $\rightarrow$ linear wave

$A \lesssim 1/K$

$A = 1000$

$K = 0.0002$

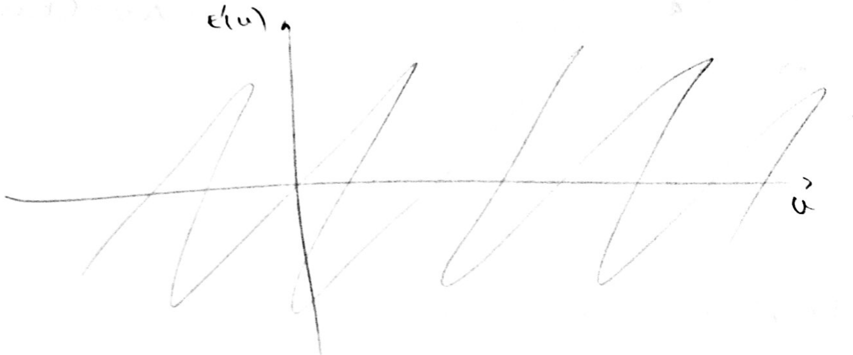


- larger amplitude → nonlinear wave
- The maximum and minimum move (relative to the 1st example) due to non linear effects

$A > 1/K$

$A = 1000$

$K = 001$



- $E'(u)$ becomes multivalued → wavebreaking limit, so the model is no longer valid

5) For a function to be multivalued $\frac{dE}{du} \to \infty$

$$\Rightarrow \quad \frac{dE'}{du} \to \infty \qquad E' = \frac{E}{4\pi e n c}$$

$$\Rightarrow \quad \frac{dE'}{du} + \frac{1}{K} \frac{1}{\sqrt{1-(E'/A)^2}} \frac{1}{A} \frac{dE'}{du} = 1$$

$$\Rightarrow \quad \frac{dE'}{du}\left(1 + \frac{1}{KA}\,\frac{1}{\sqrt{1-\left(\frac{E'}{A}\right)^2}}\right) = 1$$

$$\Rightarrow \quad 1 + \frac{1}{KA}\,\frac{1}{\sqrt{1-\left(\frac{E'}{A}\right)^2}} = 0$$

$$\Rightarrow \quad \frac{1}{K^2A^2}\,\frac{1}{1-E'^2/A^2} = 1$$

$$\Rightarrow \quad A^2K^2 - E'^2K^2 = 1 \qquad\qquad E' = \cancel{KA}\ \mathcal{E}$$

$$\Rightarrow \qquad\qquad\qquad\qquad\qquad\qquad = A\sin(Ku_0)$$

$$A^2K^2 - A^2\sin^2(Ku_0)K^2 = 1$$

$$\Rightarrow \quad A^2 = \frac{1}{K^2(1-\sin^2(Ku_0))} = \frac{1}{K^2\cos^2(Ku_0)}$$

$$\Rightarrow \quad A = \pm\frac{1}{K\cos(Ku_0)}$$

only option that gives $\left(1 + \frac{1}{KA}\frac{1}{\sqrt{\cdots}}\right) = 0$

Condition for wavebraking $\quad \frac{d\mathcal{E}}{du_0} = -1$

$$\Rightarrow \quad AK\cos(Ku_0) = -1 \quad \Rightarrow \quad A = \frac{1}{K\cos(Ku_0)}$$

$\Rightarrow$ $\mathcal{E}$ being multivalued $\Leftrightarrow$ wavebraking

Problem 2

1)

→ Calculate the electric field

$$\oiint \vec{E}.\vec{n}\, dS = 4\pi Q_{int}$$

Using a surface that is a cylinder of radius r and assuming that the plasma density is uniform the Q_{int} is given by:

$$Q_{int} = \underbrace{e n_0 \pi r^2 L}_{\text{charge of the ions inside the cylinder}} - \underbrace{e n_0 \pi r_0^2}_{\text{charge from the electrons}}$$

$$\Rightarrow \quad 2\cancel{\pi}\, r \cancel{L}\, E = 4\cancel{\pi}\, e n_0 \cancel{L} (\pi r^2 - \pi r_0^2)$$

$$\Rightarrow \quad E = 2\pi e n_0 \frac{r^2 - r_0^2}{r}$$

Since $r = r_0 + \Delta r \Rightarrow \frac{\delta^2 r}{\delta t^2} = \frac{\delta^2 \Delta r}{\delta t^2}$

$$m_e \frac{\delta^2 r}{\delta t^2} = -eE$$

$$\Rightarrow \quad m_e \frac{\delta^2 \Delta r}{\delta t^2} = -e \cdot 2\pi e n_0 \frac{r^2 - r_0^2}{r}$$

$$\Rightarrow \quad m_e r_0 \frac{\delta^2 \rho}{\delta t^2} = -2\pi e^2 n_0 \frac{r_0^2}{r} \frac{(r_0 + \Delta r)^2 - r_0^2}{r_0 + \Delta r}$$

$$\Rightarrow \quad \frac{\delta^2 \rho}{\delta t} = -\frac{1}{2} \frac{4\pi e^2 n_0}{m_e} \frac{(1+\rho)^2 - 1}{1+\rho}$$

$$\Rightarrow \quad \frac{\delta^2 \rho}{\delta t} = -\frac{1}{2} \omega_p^2 \frac{(1+\rho)^2 - 1}{1+\rho}$$

2) This equation of motion is for anharmonic oscillators which means that the period of the oscillations depend on their amplitude.

This can be verified by solving the equation numerically, see file H1-Q2.pdf.

In all the numerical solutions the initial velocity $\left(\frac{\delta\rho}{\delta t}\right)$ is considered to be zero, so that the initial position is at the maximum.

In the first solution $\rho(0) = 1$ => $T = 5.8779$
In the second solution $\rho(0) = 0.1$ => $T = 6.2782$
In the third solution $\rho(0) = 2$ => $T = 5.16332$

Since the sheets will have different initial positions r_0 and assuming that the Δr are such that $\rho(0)$ are different, the oscillations of the various sheets will be different
So even if the movements are in phase ~~...~~ at the initial time they will will not be after some time.
If the distance between consecutive ~~...~~ electrons is smaller than the amplitude of the oscillation, at the point in time when ~~...~~ one of the electrons is going out and the next is going in there is going to be sheet crossing. So it is inevitable

In longitudinal oscillations sheet crossing ~~...~~ corresponds to electrostatic plasma wavebreaking, $(n \to \infty)$, E field is a vertical line, $v_{fluid} = v_\phi$

Problem 3

1) Assuming longitudinal waves $\nabla. \rightarrow \frac{\delta}{\delta x}$

equations for electrons

$$\frac{\delta n_e}{\delta t} + \frac{\delta}{\delta x}(n_e v_e) = 0 \quad \rightarrow \text{continuity equation}$$

$$m_e n_e \left(\frac{\delta v_e}{\delta t} + v_e \frac{\delta v_e}{\delta x} \right) = -e n_e E - \frac{\delta P_e}{\delta x} \quad \rightarrow \text{momentum equation}$$

equations for the ions

$$\frac{\delta n_{ij}}{\delta t} + \frac{\delta}{\delta x}(n_{ij} v_{ij}) = 0$$

$$m_{ij} n_{ij} \left(\frac{\delta v_{ij}}{\delta t} + v_{ij} \frac{\delta v_{ij}}{\delta x} \right) = e Z_j n_{ij} E - \frac{\delta P_{ij}}{\delta x}$$

$$\frac{\delta E}{\delta x} = 4\pi e \left(\sum_j^n n_{ij} - n_e \right) \quad - \text{Poisson's equation}$$

- m_e m_{ij} → masses of electrons an ions
- Z_j → is the j ion species' charge normalized to the elementary charge
- n_e n_{ij} → electron an ions density
- v_e, v_{ij} - " " " velocity
- E - electric field
- $\frac{\delta P_e}{\delta x} = \gamma_e T_e \frac{\delta n_e}{\delta x}$,
- $\frac{\delta P_{ij}}{\delta x} = \gamma_{ij} T_{ij} \frac{\delta n_{ij}}{\delta x}$

Linearize:

$n_e = n_{oe} + n_{1e}$, $n_{1e} \ll n_{oe}$

$n_{ij} = n_{oij} + n_{1ij}$, $n_{1ij} < n_{oij}$

$E = E_1$

$v_e = v_{1e}$

$v_{ij} = v_{1ij}$

$$e n_{oe} - \sum_j^n n_{oj} Z_j e = 0 \quad \rightarrow \text{quasineutrality}$$

$$\Rightarrow \begin{cases} \dfrac{\partial n_{e1}}{\partial t} + n_{0e} \dfrac{\partial v_{1e}}{\partial x} = 0 \\ 0 = -e n_{0e} E_1 - \gamma_e T_e \dfrac{\partial n_{1e}}{\partial x} \end{cases}$$

$m_e \to 0$ (electron inertia can be neglected)

$$\begin{cases} \dfrac{\partial n_{1ij}}{\partial t} + n_{0ij} \dfrac{\partial v_{1ij}}{\partial x} = 0 \\ m_{ij} n_{0ij} \dfrac{\partial v_{1ij}}{\partial t} = e Z_j n_{0ij} E_1 - \gamma_{ij} T_{ij} \dfrac{\partial n_{1ij}}{\partial x} \end{cases}$$

$$\frac{\partial E_1}{\partial x} = 4\pi \left(\sum_j^n \cancel{Z_j n_{0j} - n_{0e}} + \sum_j^n Z_j n_{1ij} - n_{1e} \right)$$

$$= 4\pi \left(\sum_j^n Z_j n_{1ij} - n_{1e} \right)$$

↳ neglecting second order terms

Fourier transformations $\dfrac{\partial}{\partial t} \to -i\omega$ $\quad \dfrac{\partial}{\partial x} \to iK$

assuming perturbations $\propto e^{-i(\omega t - Kx)}$

- $0 = -e n_{0e} E_1 - \gamma_e T_e iK n_{1e} \Rightarrow n_{1e} = -\dfrac{e n_{0e} E_1}{\gamma_e T_e iK}$

- $-i\omega n_{1ij} + n_{0ij} iK v_{1ij} = 0 \Rightarrow v_{1ij} = \dfrac{\omega n_{1ij}}{K n_{0ij}}$

- $-m_{ij} n_{0ij} i\omega v_{1ij} = e Z_j n_{0ij} E_1 - iK \gamma_{ij} T_{ij} n_{1ij}$

$$\Rightarrow -m_{ij} n_{0ij} i\omega \frac{\omega n_{1ij}}{K n_{0ij}} = e Z_j n_{0ij} E_1 - iK \gamma_{ij} T_{ij} n_{1ij}$$

$$\Rightarrow n_{1ij} \left(-m_{ij}\, i \frac{\omega^2}{K} + iK \gamma_{ij} T_{ij} \right) = e Z_j n_{0ij} E_1$$

$$\Rightarrow n_{1ij} = \frac{e Z_j n_{0ij} E_1}{-m_{ij}\, i \frac{\omega^2}{K} + iK \gamma_{ij} T_{ij}}$$

- $iK E_1 = 4\pi e \left(\sum_j^n Z_j \dfrac{e Z_j n_{0ij} E_1}{-m_{ij}\, i \frac{\omega^2}{K} + iK \gamma_{ij} T_{ij}} + \dfrac{e n_{0e} E_1}{\gamma_e T_e iK} \right)$

$$\Rightarrow 1 = 4\pi e \left(\sum_j^n Z_j^2 \frac{e n_{0ij}}{m_{ij} \omega^2 - K^2 \gamma_{ij} T_{ij}} - \frac{e n_{0e}}{\gamma_e T_e K^2} \right)$$

$$\omega_p^2 = \frac{4\pi e n_{0e}}{m_e} \qquad \omega_{ij}^2 = \frac{4\pi e n_{0ij}}{m_{ij}}$$

$$\Rightarrow \boxed{1 = \sum_j^n \frac{Z_j^2 \omega_{ij}^2}{\omega^2 - \frac{k^2 \gamma_{ij} T_{ij}}{m_{ij}}} - \frac{\omega_p^2}{\frac{\gamma_e T_e k^2}{m_e}}}$$

2) $T_e \gg T_{ij}$

$$1 + \frac{\omega_p^2}{\gamma_e T_e k^2/m_e} = \sum_j^n \frac{Z_j^2 \omega_{ij}^2}{\omega^2 - \frac{k^2 \gamma_{ij} T_{ij}}{m_{ij}}}$$

$$\Leftrightarrow \omega^2 \left(1 + \frac{\omega_p^2}{\gamma_e T_e k^2/m_e}\right) = \sum_j^n \frac{Z_j^2 \omega_{ij}^2}{1 + \frac{k^2 \gamma_{ij} T_{ij}}{m_{ij}\omega^2}}$$

$$\Leftrightarrow \omega^2 = \sum_j^n \frac{Z_j^2 \omega_{ij}^2}{\left(1 + \frac{k^2 \gamma_{ij} T_{ij}}{m_{ij}\omega^2}\right)\left(1 + \frac{\omega_p^2}{\gamma_e T_e k^2/m_e}\right)}$$

$$\Leftrightarrow \omega^2 = \sum_j^n \frac{Z_j^2 \omega_{ij}^2}{1 + \frac{k^2 \gamma_{ij} T_{ij}}{m_{ij}\omega^2} + \frac{\omega_p^2}{\gamma_e T_e k^2/m_e} + \frac{\omega_p^2 \gamma_{ij} T_{ij} m_e}{m_{ij} T_e \gamma_e \omega^2}}$$

$m_e \ll m_{ij} \Rightarrow \frac{m_e}{m_{ij}} \ll 1$

$T_e \gg T_{ij} \Rightarrow \frac{T_e}{T_{ij}} \gg 1 \Rightarrow \frac{T_{ij}}{T_e} \ll 1$

$$\Leftrightarrow \omega^2 = \sum_j^n \frac{Z_j^2 \omega_{ij}^2}{1 + \frac{k^2 \gamma_{ij} T_{ij}}{m_{ij}\omega^2} + \frac{\omega_p^2}{\gamma_e T_e k^2/m_e}}$$

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