The Pigeonhole Principle. Permutations and Combinations

A typical situation

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

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A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

Four is enough.

The Pigeonhole Principle

If you have 10 pigeons in 9 boxes, at least one box contains two birds.

The pigeonhole principle. If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

The Pigeonhole Principle

What is the minimum number of students required in a class to be sure that at least two will receive the same grade, if there are five possible grades, A, B, C, D, and F?

The Pigeonhole Principle

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$$5 + 1 = 6 \text{ students.}$$

Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle. If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Ceiling function:

$$\lceil x \rceil = \text{the smallest integer not less that that } x$$

So, for example,

$$\lceil 2.0 \rceil = 2$$

$$\lceil 0.5 \rceil = 1$$

$$\lceil -3.5 \rceil = -3$$

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Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

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Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

"Birds" are cards. "Boxes" are suits, $k = 4$.
How many cards, N, should we take to guarantee that at least three of them fall in the same "box" (suit):

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{N}{4} \right\rceil \geq 3 > 2.$$

The smallest possible $N = 2 \cdot 4 + 1 = 9$. $\clubsuit\clubsuit\diamondsuit\diamondsuit\heartsuit\heartsuit\spadesuit\spadesuit$

Permutations

Count the number of ways to arrange the elements of this set:

$$\{a, b, c, d, e, f\}$$

- There are *6* ways to select the first element,
- *5* ways to select the second element,
- *4* ways to select the third ...
- ... continue the process
- In the end, the only remaining element takes the last position.

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{ways!}$$

Factorial

How large this number is?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This function is called *factorial* and denoted by $n!$:

$$\begin{aligned}
1! &= 1 \\
2! &= 2 \cdot 1 \\
3! &= 3 \cdot 2 \cdot 1 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\
5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
n! &= n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1 \\
&\quad \text{and by convention,} \\
0! &= 1
\end{aligned}$$

Permutations

Def. A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

A finite set of 6 elements has

$$P(6) = 6! = 720 \text{ permutations.}$$

A finite set A with cardinality $|A| = n$ has

$$P(n) = n! \text{ permutations.}$$

Permutations

What if we want to arrange only r elements?

Def. An ordered arrangement of r elements of a set is called an *r-permutation*.

Can we get the formula the the number of r-permutations?

Permutations

Count the number of ways to arrange 4 elements of the set:

$$\{a,\ b,\ c,\ d,\ e,\ f\}$$

- There are *6* ways to select the first element,
- *5* ways to select the second element,
- *4* ways to select the third ...
- *3* ways to select the fourth ...

$$6 \cdot 5 \cdot 4 \cdot 3 \quad \text{ways!}$$

The number of r-permutations of the set of n elements:

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Permutations

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Multiply and divide by $(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1$,

$$P(n,r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$$

Permutations

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$$P(n,r) = \frac{n!}{(n-r)!}$$

The formula makes sense only for $0 \leq r \leq n$, otherwise the notion of r-permutation does not make sense.

Permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970200.$$

Alternatively,

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{1 \cdot 2 \cdot \ldots \cdot 100}{1 \cdot 2 \cdot \ldots \cdot 97} = 98 \cdot 99 \cdot 100 = 970200.$$

What if the order does not matter?

You are given all r-permutations of a set.

Now, let's say that you don't really care about the ordering in each r-permutation.

Concrete example. You are given a group of 4 students, $\{a,\ b,\ c,\ d\}$. How many groups of 3 students can be formed?

The total number of 3-permutations: $P(4,3) = 4 \cdot 3 \cdot 2 = 24$.

All possible groups:
$abc, acb, bac, bca, cab, cba$ the same subset $\{a,\ b,\ c\}$

$abd, adb, bad, bda, dab, dba$ the same subset $\{a,\ b,\ d\}$

$acd, acb, cad, cda, dac, dca$ the same subset $\{a,\ c,\ d\}$

$bcd, bcb, cbd, cdb, dbc, dcb$ the same subset $\{b,\ c,\ d\}$

What if the order does not matter?

Concrete example. You are given a group of 4 students, $\{a,\ b,\ c,\ d\}$. How many groups of 3 students can be formed?

The total number of 3-permutations: $P(4,3) = 4 \cdot 3 \cdot 2 = 24$.

Let x be the number of unordered selections of 3 students, such as, for example: $\{a,\ b,\ c\}$, $\{a,\ b,\ d\}$, $\{a,\ c,\ d\}$, $\{b,\ c,\ d\}$.

Each such selection can be realized in $P(3) = 3! = 6$ permutations, e.g. $\{a,\ b,\ c\}$ has the following 6 permutations: abc, acb, bac, bca, cab, cba.

$$P(4,3) = x \cdot P(3).$$

Thus, there are only $x = \dfrac{P(4,3)}{P(3)} = \dfrac{24}{6} = 4$ unordered selections of 3 students.

Combinations

Def. An *r-combination* of elements of a set is an unordered selection of r elements from the set.

The number of r-combinations is

$$\binom{n}{r} = \frac{P(n,r)}{P(r)}$$

This notations reads as “n choose r”.

Combinations

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Let's express it in terms of n, r, and their factorials:

$$P(n,r) = \frac{n!}{(n-r)!} \text{ and } P(r) = r!, \text{ therefore}$$

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{48\cdot 49\cdot 50\cdot 51\cdot 52}{5!} = 2598960.$$

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)!\, 5!} = \frac{52!}{47!\, 5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

More examples:

$$\binom{52}{2} = \frac{51 \cdot 52}{2} = 1326. \qquad \binom{52}{1} = 52.$$

Summary

Given a set with n elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

The number of *r-permutations* of the set:

$$P(n,r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *r-combinations* ("n choose r"):

$$\binom{n}{r} = \frac{P(n,r)}{P(r)} = \frac{n!}{(n-r)!\, r!}$$

Let's be more systematic

How does $\binom{n}{r}$ change with r?

$$\binom{0}{0} = \frac{0!}{0!\,0!} = 1.$$

$$\binom{1}{0} = \frac{1!}{1!\,0!} = 1, \quad \binom{1}{1} = \frac{1!}{0!\,1!} = 1.$$

$$\binom{2}{0} = \frac{2!}{2!\,0!} = 1, \quad \binom{2}{1} = \frac{2!}{1!\,1!} = 2, \quad \binom{2}{2} = \frac{2!}{0!\,2!} = 1.$$

$$\binom{3}{0} = \frac{3!}{3!\,0!} = 1, \quad \binom{3}{1} = \frac{3!}{2!\,1!} = 3, \quad \binom{3}{2} = \frac{3!}{1!\,2!} = 3, \quad \binom{3}{3} = \frac{3!}{0!\,3!} = 1.$$

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x + y)^n$:

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

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The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x+y)^n$:

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

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$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The general formula is

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

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$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The general formula is

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \binom{n}{2} \cdot x^{n-2} y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)}_{n \text{ times}} =$$

$$\underbrace{x \cdot x \cdot \ldots \cdot x}_{=x^n} +$$

$$(\underbrace{x \cdot \ldots \cdot x}_{=x^{n-1}}) \cdot y + \ldots + y \cdot (x \cdot \ldots \cdot x) +$$

$$(\underbrace{x \cdot \ldots \cdot x}_{=x^{n-2}}) \cdot (y \cdot y) + \ldots + (y \cdot y) \cdot (\underbrace{x \cdot \ldots \cdot x}_{=x^{n-2}}) +$$

$$\ldots +$$

$$\underbrace{y \cdot y \cdot \ldots \cdot y}_{=y^n}$$

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}y + \binom{n}{2}\cdot x^{n-2}y^2 + \ldots + \binom{n}{n}\cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n\text{ times}} =$$

$$1\cdot x^n +$$

$$n\cdot x^{n-1}y +$$

$$\binom{n}{2}\cdot x^{n-2}y^2 +$$

$$\ldots +$$

$$1\cdot y^n$$

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \binom{n}{2} \cdot x^{n-2} y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)}_{n \text{ times}} =$$

$$\binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \binom{n}{2} \cdot x^{n-2} y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

Shorter notation for the same thing:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

The Binomial Theorem

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This result is called *The Binomial Theorem*, and this is why the coefficients, $\binom{n}{k}$, are also called the *binomial coefficients*.

The Binomial Theorem

Let's prove that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

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$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} \cdot 1 = \sum_{k=0}^{n} \binom{n}{k}.$$

Therefore,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

The Binomial Theorem

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

Results like this are not so obvious.

Recall that

$$\binom{n}{k} = \frac{n!}{(n-k)!\, k!}$$

So,

$$\frac{n!}{n!\, 0!} + \frac{n!}{(n-1)!\, 1!} + \frac{n!}{(n-2)!\, 2!} + \ldots + \frac{n!}{0!\, n!} = 2^n.$$

In this form, the result seems to be much harder to prove.

Pascal's Triangle Again

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Every number in Pascal's triangle is equal to the sum of the two numbers that are immediately above.