

Satisfiability.  
Rules of Inference.

# Satisfiability

Satisfiability

Rules of Inference

Rules of replacement

Proof by  
contradiction

**Def.** A proposition is **satisfiable** if some setting of the variables makes the proposition true.

For example,  $p \wedge \neg q$  is satisfiable because the expression is true when  $p$  is true and  $q$  is false.

# Satisfiability

## Satisfiability

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Determining whether or not a complicated proposition is satisfiable is not so easy.

How about this one?

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee \neg r) \wedge (\neg r \vee \neg q)$$

The general problem of deciding whether a proposition is satisfiable is called **SAT**. One approach to SAT is to construct a truth table and check whether or not a “**T**” ever appears.

But this approach is not very efficient; a proposition with  $n$  variables has a truth table with  $2^n$  lines. For a proposition with just 30 variables, that's already over a billion!

*Is there an efficient solution to SAT?*

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*Is there an efficient solution to SAT?*

No one knows.

An efficient solution to SAT would immediately imply efficient solutions to many, many other important problems involving packing, scheduling, routing, and circuit verification. Decrypting coded messages would also become an easy task (for most codes).

# Tautology and contradiction

**Def.** A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**.

Example:  $p \vee \neg p$ .

$p$	$\neg p$	$p \vee \neg p$
$T$	$F$	$T$
$F$	$T$	$T$

Example:  $p \wedge q \rightarrow p$ .

$p$	$q$	$p \wedge q$	$p \wedge q \rightarrow p$
$T$	$T$	$T$	$T$
$F$	$T$	$F$	$T$
$T$	$F$	$F$	$T$
$F$	$F$	$F$	$T$

**Def.** A compound proposition that is always false is called a **contradiction**.

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# Can we infer new true statements from a list of given statements?

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contradiction

Given that:

- 1) It is raining now.
- 2) If it's raining, it's cloudy.
- 3) When it's cloudy, it's not sunny.

Prove that it is not sunny.

$$\begin{array}{l} r \\ r \rightarrow c \\ c \rightarrow \neg s \end{array}$$

$$\frac{}{\neg s}$$

# Use truth tables again?

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Truth tables are great<sup>1</sup>, but, fortunately for us, more interesting techniques exist.

To prove that a compound proposition is true, we can build an argument, a sequence of true propositions that leads to the proposition we need.

There are inference rules that help us deduce new true propositions.

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<sup>1</sup>ha-ha, exponential ( $2^n$ ) table size is not great at all

# Building a formal argument

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Proof by  
contradiction

Given

(1)  $r$   
(2)  $r \rightarrow c$   
(3)  $c \rightarrow \neg s$


...

Deriving new true propositions

...

...

...

Need to prove 

$\neg s$



# Example

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Let's first take just one inference rule.

And solve a small problem, using this rule.

# Inference Rules #1.

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$$\frac{x \quad x \rightarrow y}{y}$$

Meaning:

$$\frac{\begin{array}{l} \text{It is snowing today.} \\ \text{If it snows today, then we will go skiing.} \end{array}}{\text{We will go skiing.}}$$

# Building an argument

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Proof by  
contradiction

Let's get back to that problem:

- 1) It is raining now.
- 2) If it's raining, it's cloudy.
- 3) When it's cloudy, it's not sunny.

Prove that it is not sunny.

$$\begin{array}{c} r \\ r \rightarrow c \\ c \rightarrow \neg s \\ \hline \neg s \end{array}$$

# Building an argument

Satisfiability

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Proof by  
contradiction

Let's prove

$$\frac{\begin{array}{c} r \\ r \rightarrow c \\ c \rightarrow \neg s \end{array}}{\neg s}$$

The rule

$$\frac{\begin{array}{c} x \\ x \rightarrow y \end{array}}{y}$$

Our argument:

- (1)  $r$       Given.
- (2)  $r \rightarrow c$       Given.
- (3)  $c \rightarrow \neg s$       Given.
- ...

# Building an argument

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Let's prove

$$\frac{\begin{array}{c} r \\ r \rightarrow c \\ c \rightarrow \neg s \end{array}}{\neg s}$$

The rule

$$\frac{\begin{array}{c} x \\ x \rightarrow y \end{array}}{y}$$

Our argument:

- (1)  $r$             Given.
- (2)  $r \rightarrow c$     Given.
- (3)  $c \rightarrow \neg s$    Given.
  
- (4)  $c$             from 1 and 2.

# Building an argument

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Let's prove

$$\frac{\begin{array}{c} r \\ r \rightarrow c \\ c \rightarrow \neg s \end{array}}{\neg s}$$

The rule

$$\frac{\begin{array}{c} x \\ x \rightarrow y \end{array}}{y}$$

Our argument:

- (1)  $r$             Given.
- (2)  $r \rightarrow c$     Given.
- (3)  $c \rightarrow \neg s$    Given.
  
- (4)  $c$             from 1 and 2.
- (5)  $\neg s$         from 3 and 4.

# How to prove an inference rule?

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contradiction

$$\frac{x \quad x \rightarrow y}{y}$$

Inference rules must be always true. In other words, to prove it, we have to show that the conjunction of the premises always implies the conclusion:

$(x \wedge (x \rightarrow y)) \rightarrow y$  is a tautology (always true).

$x$	$y$	$x \rightarrow y$	$x \wedge (x \rightarrow y)$	$(x \wedge (x \rightarrow y)) \rightarrow y$
$T$	$T$	$T$	$T$	$T$
$F$	$T$	$T$	$F$	$T$
$T$	$F$	$F$	$F$	$T$
$F$	$F$	$T$	$F$	$T$

# Rule 1. Or-Introduction ( $\vee$ -I)

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Proof by  
contradiction

$$\frac{p}{p \vee q} \quad \text{“}\vee\text{-I”}$$

Example:

$$\frac{\text{It is sunny.}}{\text{It is sunny or math is hard.}}$$



## Rule 2. And-Introduction ( $\wedge$ -I)

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Proof by  
contradiction

$$\frac{\begin{array}{c} p \\ q \end{array}}{p \wedge q} \quad \text{“}\wedge\text{-I”}$$

Example:

$$\frac{\begin{array}{c} \text{People like cats.} \\ \text{People like dogs.} \end{array}}{\text{People like dogs and cats.}}$$

# Rule 3. And-Elimination ( $\wedge$ -E)

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Proof by  
contradiction

$$\frac{p \wedge q}{p} \quad \text{“}\wedge\text{-E”}$$

Example:

Alice sent a message to Bob, but Bob did not receive anything.

---

Bob did not receive anything.

# Rule 4. “Modus Ponens”

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Proof by  
contradiction

$$\frac{p \quad p \rightarrow q}{q} \quad \text{“MP”}$$

Example:

$$\frac{\begin{array}{l} \text{It is snowing today.} \\ \text{If it snows today, then we will go skiing.} \end{array}}{\text{We will go skiing.}}$$

# Rule 5. “Modus Tollens”

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Proof by  
contradiction

$$\frac{\neg q \quad p \rightarrow q}{\neg p} \quad \text{“MT”}$$

Example:

$$\frac{\begin{array}{l} \text{I don't need an umbrella.} \\ \text{When it rains, I need an umbrella.} \end{array}}{\text{It is not raining.}}$$

# Rule 6. “Hypothetical Syllogism”

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Proof by  
contradiction

$$\frac{p \rightarrow q \quad q \rightarrow r}{p \rightarrow r} \quad \text{“HS”}$$

Example:

If you want to get an A, get ready for the exam.  
To get ready for the exam, do your homeworks.

---

If you want to get an A, do your homeworks.

# Rule 7. “Disjunctive syllogism”

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Proof by  
contradiction

$$\frac{p \vee q \quad \neg q}{p} \quad \text{“DS”}$$

Example:

There is too many people in the office, or the AC is broken.

There is not too many people.

---

The AC is broken.

# Rule 8. “Resolution”

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Proof by  
contradiction

$$\frac{p \vee q \quad \neg p \vee r}{q \vee r} \quad \text{“Res”}$$

Example:

The list is empty, or the variable is a number.
The list is not empty, or the variable is an array.
<hr/>
The variable is a number, or it is an array.

$$\frac{p}{p \vee q} \quad \text{"}\vee\text{-I"}$$

$$\frac{\neg q \quad p \rightarrow q}{\neg p} \quad \text{"MT"}$$

$$\frac{p \quad q}{p \wedge q} \quad \text{"}\wedge\text{-I"}$$

$$\frac{p \rightarrow q \quad q \rightarrow r}{p \rightarrow r} \quad \text{"HS"}$$

$$\frac{p \wedge q}{p} \quad \text{"}\wedge\text{-E"}$$

$$\frac{p \vee q \quad \neg q}{p} \quad \text{"DS"}$$

$$\frac{p \quad p \rightarrow q}{q} \quad \text{"MP"}$$

$$\frac{p \vee q \quad \neg p \vee r}{q \vee r} \quad \text{"Res"}$$

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Proof by  
contradiction



# Example 1

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Rules of replacement

Proof by  
contradiction

Prove

$$\frac{(p \rightarrow r) \wedge p}{r \wedge p}$$

- |     |                              |                    |
|-----|------------------------------|--------------------|
| (1) | $(p \rightarrow r) \wedge p$ | Given.             |
| (2) | $p \rightarrow r$            | $\wedge$ -E, 1.    |
| (3) | $p$                          | $\wedge$ -E, 1.    |
| (4) | $r$                          | MP, 2, 3.          |
| (5) | $r \wedge p$                 | $\wedge$ -I, 3, 4. |

## Example 2

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Rules of replacement

Proof by  
contradiction

Prove

$$\frac{\begin{array}{c} (s \wedge p) \rightarrow r \\ r \rightarrow t \\ \neg t \end{array}}{\neg(s \wedge p)}$$

- |     |                              |           |
|-----|------------------------------|-----------|
| (1) | $(s \wedge p) \rightarrow r$ | Given.    |
| (2) | $r \rightarrow t$            | Given.    |
| (3) | $\neg t$                     | Given.    |
| (4) | $(s \wedge p) \rightarrow t$ | HS, 1, 2. |
| (5) | $\neg(s \wedge p)$           | MT, 4, 3. |

# Example 3

Satisfiability

Rules of Inference

Rules of replacement

Proof by  
contradiction

Prove

$$\frac{s \rightarrow \neg(p \wedge q) \quad q \wedge s}{\neg p}$$

- |     |                                  |                  |
|-----|----------------------------------|------------------|
| (1) | $s \rightarrow \neg(p \wedge q)$ | Given.           |
| (2) | $q \wedge s$                     | Given.           |
| (3) | $s$                              | $\wedge$ -E, 2.  |
| (4) | $\neg(p \wedge q)$               | MP, 1, 3.        |
| (5) | $\neg p \vee \neg q$             | Equivalent to 4  |
| (6) | $q$                              | $\wedge$ -E, 2.  |
| (7) | $\neg(\neg q)$                   | Equivalent to 6. |
| (8) | $\neg p$                         | DS, 5, 7.        |

# Using the equivalence formulas

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The equivalence formulas provide rules of replacement.

For example, if  $p$  is true then  $\neg(\neg p)$  is true:

$$\frac{p}{\neg(\neg p)}$$

or a biconditional formula can be replaced by the conjunction of two implications

$$\frac{p \leftrightarrow q}{(p \rightarrow q) \wedge (q \rightarrow p)}$$

*Each* equivalence formula gives you two rules.

# Example 4

Prove

$$\begin{array}{l} p \rightarrow r \\ \neg p \rightarrow q \\ q \rightarrow s \\ \hline \neg r \rightarrow s \end{array}$$

- (1)  $p \rightarrow r$      Given.
- (2)  $\neg p \rightarrow q$      Given.
- (3)  $q \rightarrow s$      Given.
  
- (4)  $\neg p \vee r$      Equivalent to (1).
- (5)  $\neg p \rightarrow s$      2, 3, HS.
- (6)  $p \vee s$      Equivalent to (5).
- (7)  $r \vee s$      4, 5, Res
- (8)  $\neg r \rightarrow s$      Equivalent to (7).

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Proof by  
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# Example 5

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Rules of replacement

Proof by  
contradiction

Prove

$$\frac{\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \end{array}}{u}$$

# Example 5

Satisfiability

Rules of Inference

Rules of replacement

Proof by  
contradiction

Prove

- |     |                               |        |
|-----|-------------------------------|--------|
| (1) | $p \rightarrow q$             | Given. |
| (2) | $q \rightarrow (r \wedge s)$  | Given. |
| (3) | $\neg r \vee (\neg t \vee u)$ | Given. |
| (4) | $p \wedge t$                  | Given. |

- |      |                 |                   |
|------|-----------------|-------------------|
| (5)  | $p$             | 4, $\wedge$ -E.   |
| (6)  | $t$             | 4, $\wedge$ -E.   |
| (7)  | $q$             | 1, 5, M.P         |
| (8)  | $r \wedge s$    | 3, 7, M.P         |
| (9)  | $r$             | 8, $\wedge$ -E    |
| (10) | $\neg(\neg r)$  | Equivalent to (9) |
| (11) | $\neg t \vee u$ | 3, 10, D.S.       |
| (12) | $u$             | 6, 11, D.S.       |

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ \hline p \wedge t \\ \hline u \end{array}$$

# Proof by contradiction

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Proof by  
contradiction

To proof by contradiction, we assume  $p$ , and produce the argument in such a way that at some point we obtain a contradiction<sup>1</sup> (for example,  $p \wedge \neg p$ ).

If we inferred a contradiction, our assumed premise  $p$  was false, therefore, its negation  $\neg p$  is true.

assuming  $p$ , we infer a contradiction

---

 $\neg p$ 

---

<sup>1</sup>by definition, a compound proposition that is always false



# Example 6

Satisfiability

Rules of Inference

Rules of replacement

Proof by  
contradiction

Prove

$$\frac{p \rightarrow q \quad \neg q}{\neg p}$$

(1)  $p \rightarrow q$     Given.

(2)  $\neg q$     Given.

(3)  $p$     Assume.    |

(4)  $q$     1, 3, MP    |

(5)  $\neg q \wedge q$     2, 4,  $\wedge$ -I.    |

(6)  $\neg p$     3–5, by contradiction

# Example 7

Satisfiability

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Proof by  
contradiction

Prove

$$\frac{p \rightarrow q}{\neg(p \wedge \neg q)}$$

- |     |                         |                       |  |
|-----|-------------------------|-----------------------|--|
| (1) | $p \rightarrow q$       | Given.                |  |
| (2) | $p \wedge \neg q$       | Assume.               |  |
| (3) | $\neg q$                | 2, $\wedge$ -E.       |  |
| (4) | $p$                     | 2, $\wedge$ -E.       |  |
| (5) | $q$                     | 1, 3, MP              |  |
| (6) | $\neg q \wedge q$       | 3, 5, $\wedge$ -I.    |  |
| (7) | $\neg(p \wedge \neg q)$ | 2-6, by contradiction |  |

# Deduction Theorem ( $\rightarrow$ -I)

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Proof by  
contradiction

Another interesting rule, is Deduction theorem. This rule says that by assuming  $p$  and then deriving  $q$ , we prove implication  $p \rightarrow q$ .

$$\frac{\text{assuming } p, \text{ we infer } q}{p \rightarrow q} \quad \text{"}\rightarrow\text{-I"}$$

Deduction theorem can be called Implication-Introduction. In this sense, Modus Ponens can be called Implication-Elimination.

# Example 8

Satisfiability

Rules of Inference

Rules of replacement

Proof by  
contradiction

Prove Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

- |     |                   |                        |  |
|-----|-------------------|------------------------|--|
| (1) | $p \rightarrow q$ | Given.                 |  |
| (2) | $q \rightarrow r$ | Given.                 |  |
| (3) | $p$               | Assume.                |  |
| (4) | $q$               | 1, 3, MP.              |  |
| (5) | $r$               | 2, 4, MP.              |  |
| (6) | $p \rightarrow r$ | 3-5, $\rightarrow$ -I. |  |

# Example 8

Satisfiability

Rules of Inference

Rules of replacement

Proof by  
contradiction

Prove

$$(\neg p \vee \neg q) \rightarrow (r \wedge s)$$

$$r \rightarrow t$$

$$\neg t$$

---

$$p$$

# Example 8

Prove

$$\frac{\begin{array}{c} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \end{array}}{p}$$

- |      |   |                       |  |
|------|---|-----------------------|--|
| (1)  | $(\neg p \vee \neg q) \rightarrow (r \wedge s)$ | Given.                |  |
| (2)  | $r \rightarrow t$                               | Given.                |  |
| (3)  | $\neg t$  | Given.                |  |
| (4)  | $\neg p$  | Assume                |  |
| (5)  | $\neg p \vee \neg q$                            | 1, 4, $\vee$ -I       |  |
| (6)  | $r \wedge s$                                    | 1, 5, M.P.            |  |
| (7)  | $r$   | 6, $\wedge$ -E.       |  |
| (8)  | $t$   | 2, 7, M.P.            |  |
| (9)  | $\neg t \wedge t$                               | 3, 8, $\wedge$ -I.    |  |
| (10) | $p$   | 4–9, by contradiction |  |

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# Common logical errors

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Proof by  
contradiction

$$\frac{p \rightarrow q \quad q}{p}$$

The fallacy of affirming the conclusion.

$((p \rightarrow q) \wedge q) \rightarrow p$  is not a tautology.

# Common logical errors

Satisfiability

Rules of Inference

Rules of replacement

Proof by  
contradiction

$$\frac{p \rightarrow q \quad \neg p}{\neg q}$$

The fallacy of denying the hypothesis.

$((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$  is not a tautology.