Fibonacci Numbers.
Solving Linear Recurrences

Consider a recurrence

An example:

$$f(n) = f(n-1) + f(n-2)$$

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$$f(0) = 0$$

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\begin{aligned}
f(n) &= f(n-1) + f(n-2) \\
f(0) &= 0 \\
f(1) &= 1
\end{aligned}
$$

Consider a recurrence

An example:

$$f(n) = f(n-1) + f(n-2)$$
$$f(0) = 0$$
$$f(1) = 1$$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ Fibonacci numbers

Consider a recurrence

An example:

$$\begin{aligned} f(n) &= f(n-1) + f(n-2) \\ f(0) &= 0 \\ f(1) &= 1 \end{aligned}$$

When solving a linear recurrence like this, first, we are looking for a solution in the form

$$f(n) = x^n$$

We get

$$x^n = x^{n-1} + x^{n-2}$$

Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$
$$f(0) = 0$$
$$f(1) = 1$$

$$x^n = x^{n-1} + x^{n-2} \qquad \text{divide by } x^{n-2}$$
$$x^2 = x + 1$$

So, we have to solve the quadratic equation

$$x^2 - x - 1 = 0$$

It is called the *characteristic equation* of the recurrence.

Quadratic equations

Recall that to solve a quadratic equation

$$ax^2 + bx + c = 0$$

We compute the discriminant

$$\Delta = b^2 - 4ac.$$

If $\Delta \geq 0$ there are two solutions (roots):

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a} \qquad \text{and} \qquad x_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

If $\Delta < 0$, there is no solutions.

Note that if $\Delta = 0$, $x_1 = x_2$.

Consider a recurrence

Solve the characteristic equation

$$x^2 - x - 1 = 0$$

The discriminant:

$$\Delta = (-1)^2 - 4 \cdot 1 \cdot (-1) = 1 + 4 = 5 \geq 0$$

So, the solutions (roots) are

$$x_1 = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad x_2 = \frac{1 - \sqrt{5}}{2}$$

Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$
$$f(0) = 0$$
$$f(1) = 1$$

$$x_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad x_2 = \frac{1-\sqrt{5}}{2}$$

So were were looking for the solution of the first equation of the recurrence in the form $f(n) = x^n$. We found two:

$$f(n) = {x_1}^n = \left(\frac{1+\sqrt{5}}{2}\right)^n \qquad f(n) = {x_2}^n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Good, but this is not the end. We have to satisfy the boundary conditions too.

Consider a recurrence

$$
\begin{aligned}
f(n) &= f(n-1) + f(n-2) \\
f(0) &= 0 \\
f(1) &= 1
\end{aligned}
$$

$$x_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad x_2 = \frac{1-\sqrt{5}}{2}$$

Consider a linear combination of x_1^n and x_2^n with yet unknown coefficients b and c:

$$f(n) = bx_1^n + cx_2^n$$

Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$
$$f(0) = 0$$
$$f(1) = 1$$

$$f(n) = bx_1^n + cx_2^n$$

$$f(n) = b\left(\frac{1+\sqrt{5}}{2}\right)^n + c\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This $f(n)$ satisfies the first equation it the recurrence too. Let's show that.

Consider a recurrence

$$f(n) = bx_1^n + cx_2^n$$

x_1 and x_2 are the roots of the characteristic equation:

$$x_1^n = x_1^{n-1} + x_1^{n-2}$$

$$x_2^n = x_2^{n-1} + x_2^{n-2}$$

Multiply the equations by b and c, respectively, and add them up:

$$\underbrace{bx_1^n + cx_2^n}_{=f(n)} = \underbrace{bx_1^{n-1} + cx_2^{n-1}}_{=f(n-1)} + \underbrace{bx_1^{n-2} + cx_2^{n-2}}_{=f(n-2)}$$

Therefore, the linear combination $f(n) = bx_1^n + cx_2^n$ satisfies the first equation of the recurrence too:

$$f(n) = f(n-1) + f(n-2)$$

Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$
$$f(0) = 0$$
$$f(1) = 1$$

The proposed solution

$$f(n) = bx_1^n + cx_2^n = b\Big(\frac{1+\sqrt{5}}{2}\Big)^n + c\Big(\frac{1-\sqrt{5}}{2}\Big)^n$$

has to satisfy the boundary conditions

$$f(0) = b\Big(\frac{1+\sqrt{5}}{2}\Big)^0 + c\Big(\frac{1-\sqrt{5}}{2}\Big)^0 = 0$$

$$f(1) = b\Big(\frac{1+\sqrt{5}}{2}\Big)^1 + c\Big(\frac{1-\sqrt{5}}{2}\Big)^1 = 1$$

Consider a recurrence

$$f(0) = b\left(\frac{1+\sqrt{5}}{2}\right)^0 + c\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$$

$$f(1) = b\left(\frac{1+\sqrt{5}}{2}\right)^1 + c\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

So, this is a system of two equations and two unknowns b and c

$$\begin{cases} b + c = 0 \\ b\dfrac{1+\sqrt{5}}{2} + c\dfrac{1-\sqrt{5}}{2} = 1 \end{cases}$$

From the first equation, $c = -b$. Therefore,

$$b\frac{1+\sqrt{5}}{2} + (-b)\frac{1-\sqrt{5}}{2} = 1; \qquad b\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

Consider a recurrence

$$b\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

$$b\frac{2\sqrt{5}}{2} = 1$$

$$b = \frac{1}{\sqrt{5}}$$

$$c = -\frac{1}{\sqrt{5}}$$

$$f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

General Linear Recurrence

A general homogeneous linear recurrence

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)$$

with boundary conditions:

$$\begin{aligned} f(0) &= z_1, \\ f(1) &= z_2, \\ &\ldots \\ f(d-1) &= z_d \end{aligned}$$

$f(n)$ is a linear combinations of $f(n-1), \ldots f(n-d)$.

$a_1, \ldots a_d$ and $z_1 \ldots z_d$ are constants (numbers, in fact).

General Linear Recurrence

Step 1. Find the roots, x_i, of the characteristic equation.
Take the recurrence,

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)$$

First, assume $f(n) = x^n$:

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_d x^{n-d}$$

Divide by x^{n-d} to obtain the characteristic equation:

$$x^d = a_1 x^{d-1} + a_2 x^{d-2} + \ldots a_d$$

After solving the equation, we get its roots x_1, x_2, $\ldots x_d$.

General Linear Recurrence

Step 2A. If all roots are distinct, then
The solution of the recurrence is a linear combination of x_i^n:

$$f(n) = b_1 x_1^n + b_2 x_2^n + \ldots + b_d x_d^n$$

We find the unknown coefficients $b_1, \ldots, b_d$ from the boundary conditions.

Step 2B. If not all roots are distinct:
If a root x_i has multiplicity two, then instead of $b_i x_i^n$, it contributes

$$b_i x_i^n + c_i n x_i^n \quad \text{to the sum.}$$

If a root x_i has multiplicity three, it contributes

$$b_i x_i^n + c_i n x_i^n + d_i n^2 x_i^n.$$

b_i, c_i, d_i are constants, we find them from the boundary conditions.