The Pigeonhole Principle. Permutations and Combinations

A typical situation

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

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A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

Four is enough.

The Pigeonhole Principle

If you have 10 pigeons in 9 boxes, at least one box contains two birds.

The pigeonhole principle. If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

The Pigeonhole Principle

What is the minimum number of students required in a class to be sure that at least two will receive the same grade, if there are five possible grades, A, B, C, D, and F?

The Pigeonhole Principle

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$$5 + 1 = 6 \text{ students.}$$

Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle. If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Ceiling function:

$$\lceil x \rceil = \text{the smallest integer not less that that } x$$

So, for example,

$$\lceil 2.0 \rceil = 2$$

$$\lceil 0.5 \rceil = 1$$

$$\lceil -3.5 \rceil = -3$$

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Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

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Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

"Birds" are cards. "Boxes" are suits, $k = 4$.
How many cards, N, should we take to guarantee that at least three of them fall in the same "box" (suit):

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{N}{4} \right\rceil \geq 3 > 2.$$

The smallest possible $N = 2 \cdot 4 + 1 = 9$. $\clubsuit\clubsuit\diamondsuit\diamondsuit\heartsuit\heartsuit\spadesuit\spadesuit$

Solve

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Permutations

Count the number of ways to arrange the elements of this set:

$$\{a,\ b,\ c,\ d,\ e,\ f\}$$

- There are *6* ways to select the first element,
- *5* ways to select the second element,
- *4* ways to select the third ...
- ... continue the process
- In the end, the only remaining element takes the last position.

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{ways!}$$

Factorial

How large this number is?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This function is called *factorial* and denoted by $n!$:

$$\begin{aligned}
1! &= 1 \\
2! &= 2 \cdot 1 \\
3! &= 3 \cdot 2 \cdot 1 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\
5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
n! &= n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1 \\
&\quad \text{and by convention,} \\
0! &= 1
\end{aligned}$$

Permutations

Def. A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

A finite set of 6 elements has

$$P(6) = 6! = 720 \text{ permutations.}$$

A finite set A with cardinality $|A| = n$ has

$$P(n) = n! \text{ permutations.}$$

Permutations

What if we want to arrange only r elements?

Def. An ordered arrangement of r elements of a set is called an *r-permutation*.

Can we get the formula the the number of r-permutations?

Permutations

Count the number of ways to arrange 4 elements of the set:

$$\{a,\ b,\ c,\ d,\ e,\ f\}$$

- There are *6* ways to select the first element,
- *5* ways to select the second element,
- *4* ways to select the third ...
- *3* ways to select the fourth ...

$$6 \cdot 5 \cdot 4 \cdot 3 \quad \text{ways!}$$

The number of r-permutations of the set of n elements:

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Permutations

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Multiply and divide by $(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1$,

$$P(n,r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$$

Permutations

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$$P(n,r) = \frac{n!}{(n-r)!}$$

The formula makes sense only for $0 \leq r \leq n$, otherwise the notion of r-permutation does not make sense.

Permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970200.$$

Alternatively,

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{1 \cdot 2 \cdot \ldots \cdot 100}{1 \cdot 2 \cdot \ldots \cdot 97} = 98 \cdot 99 \cdot 100 = 970200.$$

What if the order does not matter?

You are given all r-permutations of a set.

Now, let's say that you don't really care about the ordering in each r-permutation.

Concrete example. You are given a group of 4 students, $\{a, b, c, d\}$. How many groups of 3 students can be formed?

The total number of 3-permutations: $P(4,3) = 4 \cdot 3 \cdot 2 = 24$.

All possible groups:
$abc, acb, bac, bca, cab, cba$ the same subset $\{a, b, c\}$

$abd, adb, bad, bda, dab, dba$ the same subset $\{a, b, d\}$

$acd, acb, cad, cda, dac, dca$ the same subset $\{a, c, d\}$

$bcd, bcb, cbd, cdb, dbc, dcb$ the same subset $\{b, c, d\}$

What if the order does not matter?

Concrete example. You are given a group of 4 students, $\{a,\ b,\ c,\ d\}$. How many groups of 3 students can be formed?

The total number of 3-permutations: $P(4,3) = 4 \cdot 3 \cdot 2 = 24$.

Let x be the number of unordered selections of 3 students, such as, for example: $\{a,\ b,\ c\}$, $\{a,\ b,\ d\}$, $\{a,\ c,\ d\}$, $\{b,\ c,\ d\}$.

Each such selection can be realized in $P(3) = 3! = 6$ permutations, e.g. $\{a,\ b,\ c\}$ has the following 6 permutations: abc, acb, bac, bca, cab, cba.

$$P(4,3) = x \cdot P(3).$$

Thus, there are only $x = \dfrac{P(4,3)}{P(3)} = \dfrac{24}{6} = 4$ unordered selections of 3 students.

Combinations

Def. An *r-combination* of elements of a set is an unordered selection of r elements from the set.

The number of r-combinations is

$$\binom{n}{r} = \frac{P(n,r)}{P(r)}$$

This notations reads as "n choose r".

Combinations

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$$\binom{n}{r} = \frac{P(n,r)}{P(r)}$$

Let's express it in terms of n, r, and their factorials:

$$P(n,r) = \frac{n!}{(n-r)!} \text{ and } P(r) = r!, \text{ therefore}$$

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

More examples:

$$\binom{52}{2} = \frac{51 \cdot 52}{2} = 1326. \qquad \binom{52}{1} = 52.$$

Summary

Given a set with n elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

The number of *r-permutations* of the set:

$$P(n,r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *r-combinations* ("n choose r"):

$$\binom{n}{r} = \frac{P(n,r)}{P(r)} = \frac{n!}{(n-r)!\, r!}$$

Solve

Problem 1.

How many permutations of the letters ABCDEFG contain

(a) the string BCD?

(b) the string CFGA?

(c) the strings BA and GF?

(d) the strings BAC and CED?

Solve

Problem 2.

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Problem 3.

How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

Solve: “Knights of the Round Table”

Def. A *circular permutation* of n people is their seating around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

Def. Similarly, a *circular r-permutation* of n people is a seating of r of these n people around a circular table, where, again, the seatings that can be obtained by rotation are considered to be the same.

Problem 4.

In how many ways can King Arthur seat n different knights at his round table?

Problem 5.

Count the number of circular r-permutations of n people.

Let's try to be more systematic

How does $\binom{n}{r}$ change with r?

$$\binom{0}{0} = \frac{0!}{0!\,0!} = 1.$$

$$\binom{1}{0} = \frac{1!}{1!\,0!} = 1, \quad \binom{1}{1} = \frac{1!}{0!\,1!} = 1.$$

$$\binom{2}{0} = \frac{2!}{2!\,0!} = 1, \quad \binom{2}{1} = \frac{2!}{1!\,1!} = 2, \quad \binom{2}{2} = \frac{2!}{0!\,2!} = 1.$$

$$\binom{3}{0} = \frac{3!}{3!\,0!} = 1, \quad \binom{3}{1} = \frac{3!}{2!\,1!} = 3, \quad \binom{3}{2} = \frac{3!}{1!\,2!} = 3, \quad \binom{3}{3} = \frac{3!}{0!\,3!} = 1.$$

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x+y)^n$:

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

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The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x+y)^n$:

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

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$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The general formula is

$$(x+y)^n = \binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}y + \binom{n}{2}\cdot x^{n-2}y^2 + \ldots + \binom{n}{n}\cdot y^n.$$

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$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The general formula is

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}y + \binom{n}{2}\cdot x^{n-2}y^2 + \ldots + \binom{n}{n}\cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} =$$

$$\underbrace{x\cdot x\cdot\ldots\cdot x}_{=x^n} +$$

$$\underbrace{(x\cdot\ldots\cdot x)}_{=x^{n-1}}\cdot y + \ldots + y\cdot(x\cdot\ldots\cdot x) +$$

$$\underbrace{(x\cdot\ldots\cdot x)}_{=x^{n-2}}\cdot(y\cdot y) + \ldots + (y\cdot y)\cdot\underbrace{(x\cdot\ldots\cdot x)}_{=x^{n-2}} +$$

$$\ldots +$$

$$\underbrace{y\cdot y\cdot\ldots\cdot y}_{=y^n}$$

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}y + \binom{n}{2}\cdot x^{n-2}y^2 + \ldots + \binom{n}{n}\cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} =$$

$$1\cdot x^n +$$

$$n\cdot x^{n-1}y +$$

$$\binom{n}{2}\cdot x^{n-2}y^2 +$$

$$\ldots +$$

$$1\cdot y^n$$

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} =$$

$$\binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

Shorter notation for the same thing:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

The Binomial Theorem

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \binom{n}{2} \cdot x^{n-2} y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This result is called *The Binomial Theorem*, and this is why the co-efficients, $\binom{n}{k}$, are also called the *binomial coefficients*.

The Binomial Theorem

Let's prove that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

The Binomial Theorem

Let's prove that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} \cdot 1 = \sum_{k=0}^{n} \binom{n}{k}.$$

Therefore,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

The Binomial Theorem

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

Results like this are not so obvious.

Recall that

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

So,

$$\frac{n!}{n!\,0!} + \frac{n!}{(n-1)!\,1!} + \frac{n!}{(n-2)!\,2!} + \ldots + \frac{n!}{0!\,n!} = 2^n.$$

In this form, the result seems to be much harder to prove.

Pascal's Triangle Again

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Every number in Pascal's triangle is equal to the sum of the two numbers that are immediately above.