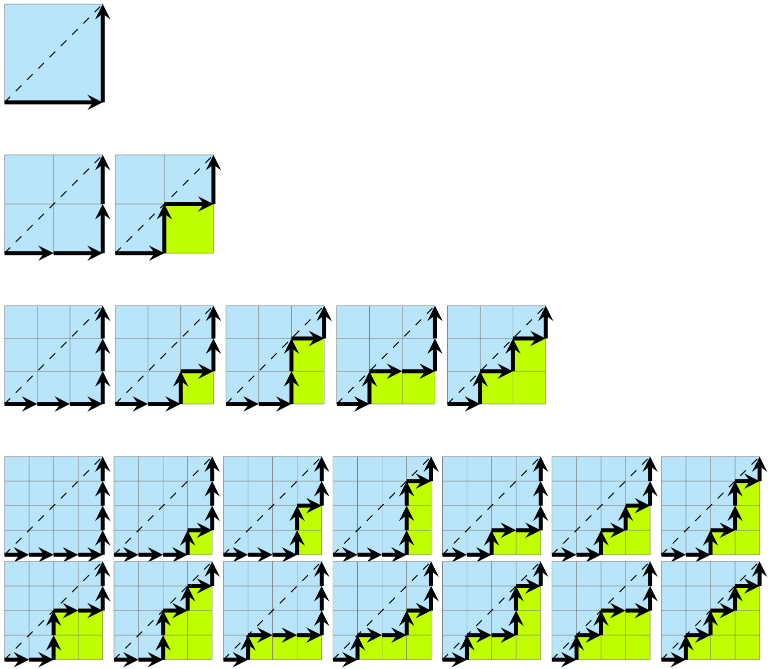
Counting

What can we count?

- In how many ways can we paint 6 rooms, choosing from 15 available colors?
- What if we want all rooms painted with different colors?
- In how many different ways 10 books can be arranged on a shelf?
- What if 2 of those 10 books are identical copies?

1, 2, 5, 14, . . .



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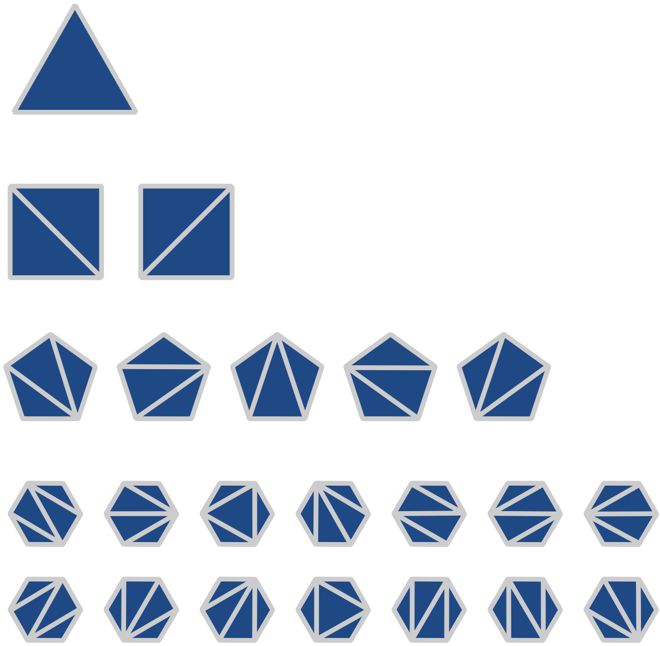

Problem

The chairs of an auditorium are to be labeled with an uppercase letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

$$\text{A-1, A-2, } \ldots \text{ Z-100}$$

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

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$$26 \cdot 100$$

The Rule of Product

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2. 100 ways to assign a number.

$$26 \cdot 100$$

The Rule of Product. Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Chairs again

Consider the same problem about the labels for chairs

1. 26 ways to choose a letter and
2. 100 ways to choose a number.

Write a program that prints all 2600 labels?

Chairs again

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1. 26 ways to choose a letter and
2. 100 ways to choose a number.

Write a program that prints all 2600 labels?

```
for a := A to Z do
    for n := 1 to 100 do
        print_label(a, n)
```

Generalized Product Rule

If a procedure consists of *k sub-tasks*, and the sub-tasks can be performed in $n_1, \ldots, n_k$ ways, then the procedure can be performed

$$\text{in } (n_1 \cdot n_2 \cdot \ldots \cdot n_k) \text{ ways.}$$

Example:
Count the number of different bit strings of length seven.

Generalized Product Rule

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Example:
Count the number of different bit strings of length seven.

The value for each bit can be chosen in two ways (0 or 1). Therefore:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$$

License plates

How many different license plates of this format can be made?

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How many different license plates of this format can be made?

$$26^3 \cdot 10^4 = 175,760,000$$

Another counting problem

A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

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$$40 + 50 = 90 \quad \text{this is called the rule of sum}$$

The Rule of Sum

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

The Rule of Sum

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Write an algorithm for a robot to read all the books in the library:

for $b := 1$ **to** 40 **do**
  $read(\texttt{Sociology}, b)$
for $b := 1$ **to** 50 **do**
  $read(\texttt{Anthropology}, b)$

The Rule of Sum

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

The Rule of Sum. If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Note that it's important that the two groups don't have common elements (We say that they are disjoint sets).

Problem

You have 3 textbooks, 5 novels, 4 magazines, and 2 comic books.

You want to pick only one book to read in the subway. How many options do you have?

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You want to pick only one book to read in the subway. How many options do you have?

$$3 + 5 + 4 + 2 = 14.$$

Problem

NYC whats to change the license plates format, allowing

3 letters + 3 digits; 2 letters + 2 digits; and 1 letter + 1 digit.

AAA 111
AA 11
A 1

How many license plates can be made?

Problem

NYC whats to change the license plates format, allowing up to 3 letters followed by up to 3 digits.

A 1	A 11	A 111
AA 1	AA 11	AA 111
AAA 1	AAA 11	AAA 111

How many license plates can be made?

A new object

Def. A *set* is an unordered collection of objects. The objects are called elements.

If e is an element of the set A, we write $a \in A$.

Otherwise, if it's not in A, we write $a \notin A$.

Example:

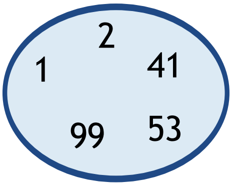
$$A = \{1, 2, 97, 3, 15\}.$$

$$1 \in A.$$

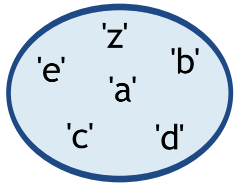
$$4 \notin A.$$

$$\{1, 2, 3, 3, 2, 1, 1, 1, 1\} = \{1, 2, 3\}$$

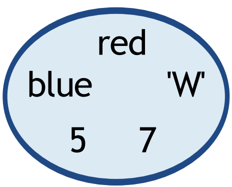
Sets



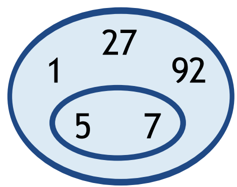
A = {1, 2, 41, 53, 99}



B = {'a', 'z', 'e', 'd', 'c', 'b'}



C = {'W', blue, 5, red, 7}



D = {27, 1, {5,7}, 92}

Some important sets

Natural numbers

$$\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ \ldots\}$$

Integer numbers

$$\mathbb{Z} = \{\ldots,\ -3,\ -2,\ -1,\quad 0,\quad 1,\quad 2,\quad 3,\ \ldots\}$$

Empty set

$$\varnothing = \{\ \}$$

Set Builder Notation

We can describe sets using predicates:

$$A = \{x \mid P(x)\}$$

"Set A is such that $x \in A$ if and only if $P(x)$."

Example. Positive integers:

$$Z^+ = \left\{n \in \mathbb{Z} \mid n > 0\right\} = \{1, 2, 3, \ldots\}$$

More complex predicates are fine too. Odd and even numbers:

$$Even = \left\{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}\,(n = 2k)\right\}$$
$$Odd = \left\{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}\,(n = 2k + 1)\right\}$$

Union, ∪

$A \cup B$ denotes all things that are *members of either A or B*:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Equivalently:

x belongs to $A \cup B$ if and only if $x \in A$ or $x \in B$.

Examples:

$$\{1,\ 2\} \cup \{\text{‘a’},\ \text{‘b’}\} = \{1,\ 2,\ \text{‘a’},\ \text{‘b’}\}$$
$$\{1,\ 2,\ 3\} \cup \{2,\ 3,\ 5\} = \{1,\ 2,\ 3,\ 5\}$$

Intersection, $\cap$

$A \cap B$ denotes all things that are *members of both A and B*:

$$A \cap B = \{x \mid (x \in A) \land (x \in B)\}$$

Equivalently:

x belongs to $A \cap B$ if and only if $x \in A$ and $x \in B$.

Examples:

$$\{1, 2\} \cap \{\text{‘a’}, \text{‘b’}\} = \varnothing$$
$$\{1, 2, 3\} \cap \{2, 3, 5\} = \{2, 3\}$$

Sets A and B are called *disjoint* if their intersection is empty: $A \cap B = \varnothing$.

Number of the elements of a finite set

Def. If set A is finite, and there are exactly n elements in S, then n is the *cardinality* of the set A. We write

$$|A| = n.$$

Examples:

$$\begin{aligned} A &= \{3,\ 4,\ 5,\ 6\} \\ |A| &= 4 \\ B &= \big\{\{3,\ 4\},\ \{5,\ 6\},\ 7\big\} \\ |B| &= 3 \\ |\varnothing| &= 0 \end{aligned}$$

Question

$$A = \{1,\ 2,\ 4,\ 5\}$$

$$B = \{20,\ 21,\ 22,\ 23,\ 24\}$$

$$A \cup B = \{1,\ 2,\ 4,\ 5,\ 20,\ 21,\ 22,\ 23,\ 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| =$$

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If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| = 9, \quad \text{and} \quad |A| + |B| = 4 + 5 = 9.$$

$$|A \cup B| = |A| + |B| = 4 + 5 = 9.$$

Question

You are given k disjoint sets $A_1, \ldots A_k$:

It means that $A_i \cap A_j = \varnothing$ when $i \neq j$.

What is the cardinality if their union $A_1 \cup \ldots \cup A_k$?

$$|A_1 \cup \ldots \cup A_k| =$$

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This is the same *rule of sum*, right?

Question

Why do we insist on the sets being disjoint?

Really, who cares?

Because

$$A = \{1,\ 2,\ 3\}$$

$$B = \{3,\ 4\}$$

$$\text{Their union: } A \cup B = \{1,\ 2,\ 3,\ 4\}$$

$$|A| + |B| = \left|\{1,\ 2\ ,3\}\right| + \left|\{3,\ 4\}\right| = 3 + 2 = 5$$

$$|A \cup B| = \left|\{1,\ 2,\ 3,\ 4\}\right| = 4$$

So, in general, $|A \cup B| \neq |A| + |B|$, and if we try to use the sum rule when the sets are not disjoint, we *overcount*, <u>and this is really bad</u>.

Overcounting

$$A = \{1,\ 2,\ 3\}$$

$$B = \{3,\ 4\}$$

$$A \cup B = \{1,\ 2,\ 3,\ 4\}$$

We were overcounting, because the common elements of A and B were counted twice:

$$|A| + |B| = \big|\{1,\ 2,\ 3\}\big| + \big|\{3,\ 4\}\big| = 5,$$
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Therefore, to get the correct value of $|A \cup B|$, the number of common elements must be subtracted:

$$|A \cup B| = |A| + |B| - \left|\{3\}\right| = 3 + 2 - 1 = 4.$$

The Subtraction Rule

$$A = \{1,\ 2,\ 3\}$$

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The *Subtraction Rule* for two arbitrary sets A and B:

$$|A \cup B| =$$

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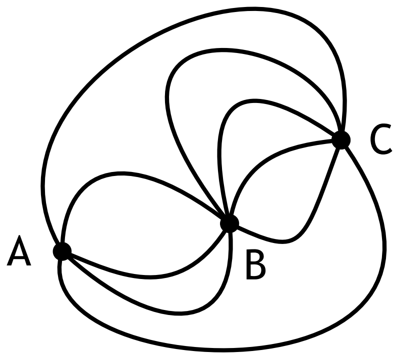
The *Subtraction Rule* for two arbitrary sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting paths

This is a map with three cities, connected by roads.

Count the number of paths from city A to city C, such that each city is visited not more than once.



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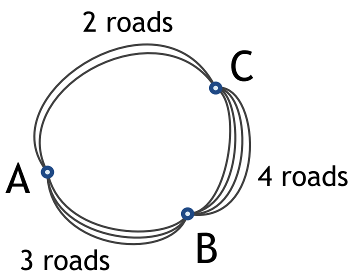

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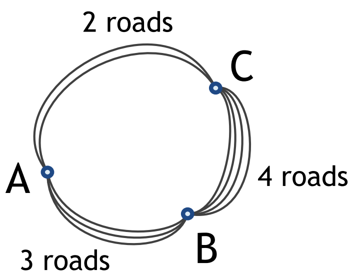


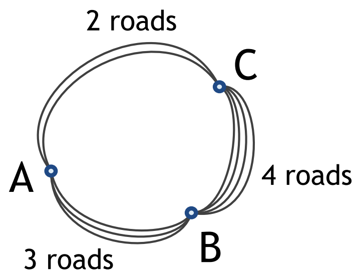
$A \to C$ or $A \to B \to C$:

$$2 + 3 \cdot 4 = 14$$

Counting round trips

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that cities A and C are visited not more than once.

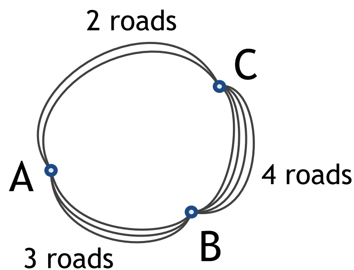


Counting round trips

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that cities A and C are visited not more than once.

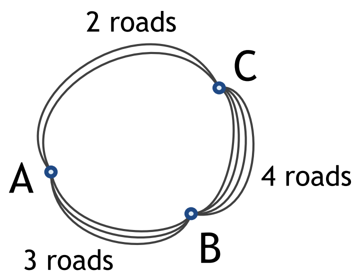
$$\begin{array}{cc} B \to A \to B & 3 \cdot 3 = 9 \\ B \to C \to B & 4 \cdot 4 = 16 \\ B \to A \to C \to B & 3 \cdot 2 \cdot 4 = 24 \\ B \to C \to A \to B & 4 \cdot 2 \cdot 3 = 24 \\ \hline \text{Total} & 73 \end{array}$$



Counting round trips II

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that
1) cities A and C are visited not more than once, and
2) each road is used not more than once during a trip.



Counting round trips II

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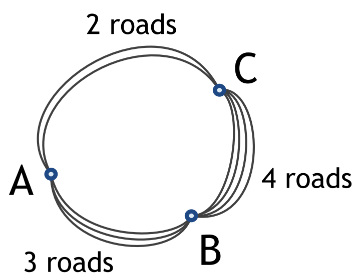
Count the number of round trips starting in city B, such that
1) cities A and C are visited not more than once, and
2) each road is used not more than once during a trip.

$B \to A \to B$	$3 \cdot 2 = 6$
$B \to C \to B$	$4 \cdot 3 = 12$
$B \to A \to C \to B$	$3 \cdot 2 \cdot 4 = 24$
$B \to C \to A \to B$	$4 \cdot 2 \cdot 3 = 24$
Total	66

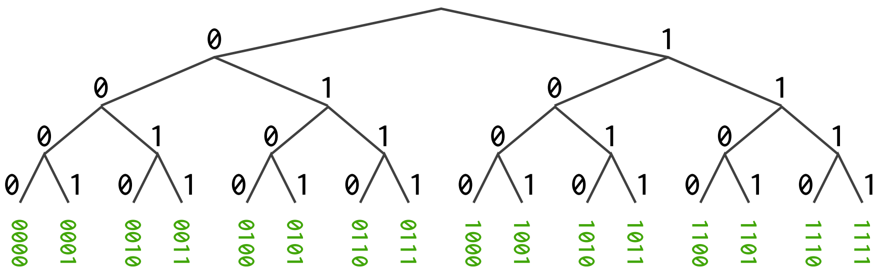


Tree Diagrams

Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.
Each possible outcome is a leaf of the tree (an endpoint that does not branch).

Example: Count all bit strings of length four.



16 strings.

Tree Diagrams

Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.
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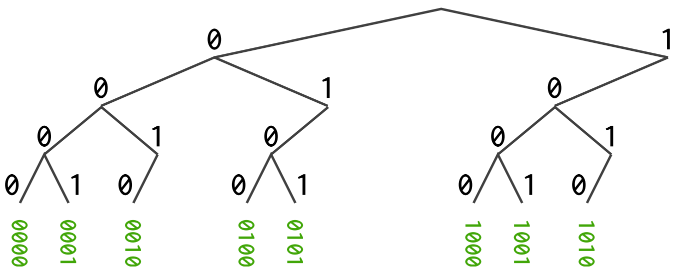
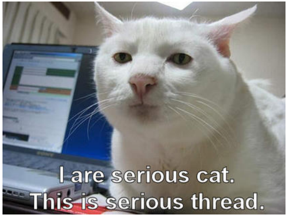
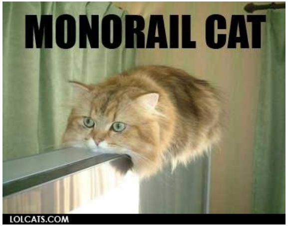
Example 2: Count all bit strings of length four that *do not have two consecutive 1s*.



8 strings.

Ranking cats





Ranking cats

In how many different ways can you rank a set of 6 cats:

$$\{a, b, c, d, e, f\}$$

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In how many different ways can you rank a set of 6 cats:

$$\{a, b, c, d, e, f\}$$

- There are 6 ways to select the first cat,
- 5 ways to select the second cat among the remaining five,
- 4 ways to select the third cat ...
- ... continue the process
- In the end, the only remaining cat takes the last position in the rank.

Ranking cats

In how many different ways can you rank a set of 6 cats:

$$\{a, b, c, d, e, f\}$$

- There are *6* ways to select the first cat,
- *5* ways to select the second cat among the remaining five,
- *4* ways to select the third cat ...
- ... continue the process
- In the end, the only remaining cat takes the last position in the rank.

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{ways!}$$

Factorial

How large this number is?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This function is called *factorial* and denoted by $n!$:

$$\begin{aligned}
1! &= 1 \\
2! &= 2 \cdot 1 \\
3! &= 3 \cdot 2 \cdot 1 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\
5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
n! &= n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1 \\
&\text{and by convention,} \\
0! &= 1
\end{aligned}$$