Discrete Structures. CSCI-150. Fall 2013.

Review

Problem 1

First, prove that $k(k+1)$ is even for any $k \in \mathbb{Z}$.
Then, for positive $n \in \mathbb{Z}$, prove that if n is odd then $8 \mid (n^2 - 1)$.

Problem 2

Modular exponentiation. Prove that $3^{23} \equiv 5 \pmod{11}$.

(RSA encryption/decryption is done similarly)

Problem 3

Given two numbers,

$$a_0 = 135, \quad a_1 = 129,$$

Find $a_k = \gcd(a_0, a_1)$ and Bezout's coefficients x_k and y_k, i.e. the numbers such that the following equation is satisfied:

$$a_k = \gcd(a_0, a_1) = x_k a_0 + y_k a_1$$

If it's possible, find the multiplicative inverse of a_1 modulo a_0.

(Note that the multiplicative iniverse exists if and only if a_1 and a_0 are relative primes)

Problem 4

Given three sets

$$A = \{1, 2, 3\}, \quad B = \{1, 2, 3, 4\}, \quad C = \{1, 2, 3, 4, 5\},$$

construct the following functions (or prove that they don't exist):

1. one-to-one function from A to B
2. one-to-one function from C to B
3. onto function from C to B
4. bijection from A to B

Problem 5

Draw the Hasse diagram for divisibility on the set $\{2, 4, 5, 6, 10, 12\}$.

Find the maximal elements. Find the minimal elements.

Construct a topological sort of this poset.

Problem 6
Find two incomparable elements in these posets.

(a) $(\mathcal{P}(0, 1, 2), \subseteq)$, where $\mathcal{P}(X)$ denotes the powerset of X.

(b) $(\{1, 2, 4, 6, 8\}, |)$

Problem 7
Use the Schröder-Bernstein theorem to show that there is a bijection between two intervals $[0, 1] \subseteq \mathbb{R}$ and $[1, \infty) \subseteq \mathbb{R}$, thus they have the same cardinality.

What about the sets $(0, 5)$ and $(10, 20)$? Is there a bijection between them? (Don't need the Schröder-Bernstein theorem here). Similarly, consider $[0, \infty)$ and $[1, \infty)$.

Problem 8
Prove by diagonalization that the interval $[0, 1] \subseteq \mathbb{R}$ is uncountable.

Problem 9
Given a multigraph G, does it have an Eulerian cycle? Does it have an Eulerian path? (A drawing of the graph will be supplied).

Problem 10
We define the complement G^c of a graph G as the graph on the same vertex set in which two vertices are joined by an edge if and only if they are not joined by an edge in G. Prove that it cannot happen that both G and G^c are disconnected.

Problem 11
How many leaves does a full 3-ary tree with 100 vertices have?

Problem 12
Use Huffman coding to encode these symbols with given frequencies:
A: 0.05, B: 0.07, C: 0.08, D: 0.10, E: 0.15, F : 0.25, G: 0.30.

What is the average number of bits required to encode a symbol?

Problem 13
By rolling a six-sided die 6 times, a strictly increasing sequence of numbers was obtained, what is the probability of such an event?

Problem 14
Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

(a) the probability of at least two successes

(b) the probability of at least two failure