

# Permutations and Combinations. The Pigeonhole Principle.

# A question

Permutations

Combinations

The Pigeonhole  
Principle

We want to open a door, but it's locked.



It seems that to open the lock, we must press **10 buttons** in the right order.

**Each button must be pressed exactly once.**

How many combinations do we have to try?

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We want to open a door, but it's locked.



It seems that to open the lock, we must press **10 buttons** in the right order.

**Each button must be pressed exactly once.**

How many combinations do we have to try?

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800.$$

If it takes one second to try one combination, we will need 42 days to try each.

# Another question

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There are 7 persons in a team, and 7 distinct individual tasks, in how many ways the tasks can be assigned to the team members?

(Everyone gets exactly one task).

# Another question

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There are 7 persons in a team, and 7 distinct individual tasks, in how many ways the tasks can be assigned to the team members?

(Everyone gets exactly one task).

$$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

# Factorial function

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$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800$$

This function is called *factorial* and denoted by  $n!$ :

$$1! = 1$$

$$2! = 2 \cdot 1$$

$$3! = 3 \cdot 2 \cdot 1$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

...

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 1$$

and by convention,

$$0! = 1$$

# Permutations

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**Def.** A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

The number of permutations of  $n$  objects is

$$P(n) = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n!$$

So, for example, given 6 pictures of cats, there are  $6! = 720$  ways to arrange them in a row.

# Permutations

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There are 15 different magazines on a table.



While waiting, you skimmed through each of them. In how many different orders it was possible to do?



# Permutations

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There are 15 different magazines on a table.



While waiting, you skimmed through each of them. In how many different orders it was possible to do?

$$15! = 1307674368000.$$

# Permutations

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There are 15 different magazines on a table.



*You read three of them.* In how many ways it was possible to do?

# Permutations

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There are 15 different magazines on a table.



*You read three of them.* In how many ways it was possible to do?

$$15 \cdot 14 \cdot 13 = 2730.$$

# Permutations

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An arrangement of 3 objects out of  $n$  is called a 3-permutation.

**Def.** An ordered arrangement of  $r$  elements from a set of  $n$  is called an  *$r$ -permutation*.

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

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Let's prove the last formula.

The number of  $r$ -permutations of the set of  $n$  elements:

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Multiply and divide by  $(n-r) \cdot (n-r-1) \cdot \dots \cdot 2 \cdot 1$ ,

$$P(n, r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \dots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$$

# Permutations

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How many ways are there to select a first-prize winner and a second-prize winner from 100 different people who have entered a contest?

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How many ways are there to select a first-prize winner and a second-prize winner from 100 different people who have entered a contest?

$$P(100, 2) = 100 \cdot 99 = 9900.$$

Alternatively,

$$P(100, 2) = \frac{100!}{(100-2)!} = \frac{1 \cdot 2 \cdot \dots \cdot 100}{1 \cdot 2 \cdot \dots \cdot 98} = 99 \cdot 100 = 9900.$$



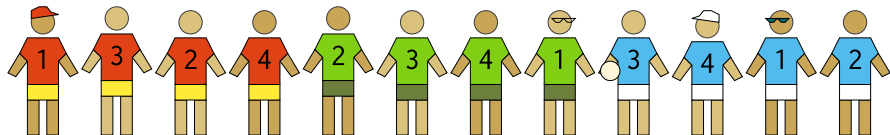
# Permutations

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Three sport teams of four want to take a group photo.



In how many ways can they stand in a row so that all members of the same team are standing together?

# Another problem

Permutations

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There are 4 paintings in a collection. In how many ways can we hang 3 of them on a wall?



# Another problem

Permutations

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There are 4 paintings in a collection. In how many ways can we hang 3 of them on a wall?



We can compute the number of 3-permutations of 4 objects, which is  $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$ .

<i>abc</i>	<i>acb</i>	<i>bac</i>	<i>bca</i>	<i>cab</i>	<i>cba</i>	$\{a, b, c\}$
<i>abd</i>	<i>adb</i>	<i>bad</i>	<i>bda</i>	<i>dab</i>	<i>dba</i>	$\{a, b, d\}$
<i>acd</i>	<i>acb</i>	<i>cad</i>	<i>cda</i>	<i>dac</i>	<i>dca</i>	$\{a, c, d\}$
<i>bcd</i>	<i>bcb</i>	<i>cbd</i>	<i>cdb</i>	<i>dbc</i>	<i>dcb</i>	$\{b, c, d\}$

Observe that there are 4 ways to pick a set of 3 paintings, and each of them can be arranged in  $3! = 6$  many ways, and  $4 \cdot 3! = 24$  too.

# Another problem



Permutations

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$$P(4, 3) = 24 = 4 \cdot 3!$$

When we arrange  $r$  objects out of  $n$ :

$$P(n, r) = \frac{n!}{(n-r)!}$$

But also,

$$P(n, r) = X \cdot r!$$

Where  $X$  is the number of ways to *choose  $r$  objects out of  $n$  without assigning any order* to them. This is exactly what we did when we were selecting 3 paintings.

# Combinations

Permutations

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$$P(n, r) = \frac{n!}{(n-r)!}$$

$$P(n, r) = X \cdot r!$$

$X$  is the number of ways to choose  $r$  objects out of  $n$  without assigning any order to them.

Knowing this  $X$  can be useful. Can we find a formula for it?

# Combinations

Permutations

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Knowing this  $X$  can be useful. Can we find a formula for it?

$$X = \frac{n!}{(n-r)! r!}$$

It is the number of so-called  *$r$ -combinations*.

# Combinations

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Knowing this  $X$  can be useful. Can we find a formula for it?

$$X = \frac{n!}{(n-r)! r!}$$

It is the number of so-called  *$r$ -combinations*.

**Def.** An  *$r$ -combinations* is an unordered selection of  $r$  objects from a set of  $n$  objects.

We write

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)! r!}$$

# Combinations

Permutations

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The number of  $r$ -combinations:

$$\binom{n}{r} = \frac{n!}{(n-r)! r!} \quad (\text{Note that } \binom{n}{r} \text{ reads as “}n \text{ choose } r\text{”}).$$

An  $r$ -combination is a selection of  $r$  objects without specifying the order in which the objects are chosen.

## Exercise:

There are three paintings, you'd like to buy some of them.

In how many ways one painting can be chosen? Two? Three?



# Combinations

Permutations

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In how many ways a team of 6 can be selected from a group of 12 students?

# Combinations

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In how many ways a team of 6 can be selected from a group of 12 students?

$$\binom{12}{6} =$$

# Combinations

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In how many ways a team of 6 can be selected from a group of 12 students?

$$\begin{aligned}\binom{12}{6} &= \frac{12!}{(12-6)! 6!} = \frac{12!}{6! \cdot 6!} = \frac{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{6!} \\ &= \frac{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{7 \cdot 11 \cdot 12}{1} = 924.\end{aligned}$$

# Example with cards

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Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

# Example with cards

Permutations

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The Pigeonhole  
Principle

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)! 5!} = \frac{52!}{47! 5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

# Example with cards

Permutations

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The Pigeonhole  
Principle

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)! 5!} = \frac{52!}{47! 5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

More examples:

$$\binom{52}{2} = \frac{51 \cdot 52}{2} = 1326. \quad \binom{52}{1} = 52.$$

# Combinations

Permutations

Combinations

The Pigeonhole  
Principle

Let's try to prove that for  $1 \leq k \leq n$ :

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

# Summary

Permutations

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Principle

Given a set with  $n$  elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1$$

The number of *r-permutations* of the set:

$$P(n, r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *r-combinations* (“ $n$  choose  $r$ ”):

$$\binom{n}{r} = \frac{P(n, r)}{P(r)} = \frac{n!}{(n-r)! r!}$$



# Solve

Permutations

Combinations

The Pigeonhole  
Principle

## *Problem 1.*

How many permutations of the letters ABCDEFG contain

- (a) the string BCD?
- (b) the string CFGA?
- (c) the strings BA and GF?
- (d) the strings BAC and CED?

# Solve

Permutations

Combinations

The Pigeonhole  
Principle

## *Problem 2.*

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

## *Problem 3.*

How many bit strings contain exactly 8 zeros and 10 ones?

## *Problem 4.*

How many bit strings contain exactly 8 zeros and 10 ones if every zero must be immediately followed by a one?

# Solve: “Knights of the Round Table”

Permutations

Combinations

The Pigeonhole  
Principle

**Def.** A *circular permutation* of  $n$  people is their seating around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

**Def.** Similarly, a *circular  $r$ -permutation* of  $n$  people is a seating of  $r$  of these  $n$  people around a circular table, where, again, the seatings that can be obtained by rotation are considered to be the same.

## *Problem 5.*

In how many ways can King Arthur seat  $n$  different knights at his round table?

## *Problem 6.*

Count the number of circular  $r$ -permutations of  $n$  people.

# “Knights of the Round Table”

Permutations

Combinations

The Pigeonhole  
Principle

**Problem 5.** In how many ways can King Arthur seat  $n$  different knights at his round table?

**Answer:** Because  $n$  normal permutations result in a single circular permutation,  $\frac{n!}{n} = (n-1)!$

**Problem 6.**

Count the number of circular  $r$ -permutations of  $n$  people.

**Answer:**  $r$  normal  $r$ -permutations result in one circular  $r$ -permutation, so we get  $\frac{n!}{(n-r)! r}$ . Transform this formula:

$$\frac{n!}{(n-r)! r} = \frac{n!}{(n-r)! r!} \cdot \frac{r!}{r} = \binom{n}{r} \cdot \frac{r!}{r}$$

So, equivalently, we first choose  $r$  knights out of  $n$ , and then count their circular permutations.

# A typical situation

Permutations

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The Pigeonhole  
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A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?



# A typical situation

Permutations

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Principle

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?



Four is enough.

# The Pigeonhole Principle

Permutations

Combinations

The Pigeonhole  
Principle



If you have 10 pigeons in 9 boxes, at least one box contains two birds.

*The pigeonhole principle.* If  $k$  is a positive integer and  $k + 1$  or more objects are placed into  $k$  boxes, then there is at least one box containing two or more of the objects.

# The Pigeonhole Principle

Permutations

Combinations

The Pigeonhole  
Principle

There are 7 colleges in a city.

What is the minimum number of students to be invited for a survey to make sure that at least two are from the same college?



# The Pigeonhole Principle

Permutations

Combinations

The Pigeonhole  
Principle

There are 7 colleges in a city.

What is the minimum number of students to be invited for a survey to make sure that at least two are from the same college?

$$7 + 1 = 8 \text{ students.}$$

# Generalized Pigeonhole Principle

Permutations

Combinations

The Pigeonhole  
Principle

The ceiling function:

$\lceil x \rceil$  = the smallest integer not less than  $x$

So, for example,

$$\lceil 2.0 \rceil = 2$$

$$\lceil 0.5 \rceil = 1$$

$$\lceil -3.5 \rceil = -3$$

*The Generalized Pigeonhole Principle.*

If  $N$  objects are placed into  $k$  boxes, then there is at least one box containing at least  $\lceil N/k \rceil$  objects.

# Generalized Pigeonhole Principle

Permutations

Combinations

The Pigeonhole  
Principle

*The Generalized Pigeonhole Principle.*

If  $N$  objects are placed into  $k$  boxes, then there is at least one box containing at least  $\lceil N/k \rceil$  objects.

**Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

# Generalized Pigeonhole Principle

Permutations

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## *The Generalized Pigeonhole Principle.*


If  $N$  objects are placed into  $k$  boxes, then there is at least one box containing at least  $\lceil N/k \rceil$  objects.

**Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

“Birds” are cards. “Boxes” are suits,  $k = 4$ .

How many cards,  $N$ , should we take to guarantee that at least three of them fall in the same “box” (suit):

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{N}{4} \right\rceil \geq 3 > 2.$$

The smallest possible  $N = 2 \cdot 4 + 1 = 9$ . 

# Solve

Permutations

Combinations

The Pigeonhole  
Principle

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.