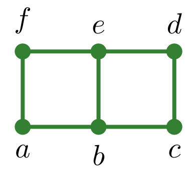
Discrete Structures. CSCI-150. Spring 2015.

Homework 11. **Due Fri. May 15, 2015.**

Graphs and trees

Problem 1



What are the adjacency matrix and the adjacency list of a graph?
Find the adjacency matrix of the graph shown in the figure.
Find the adjacency list of the graph.

Problem 2 (Graded)

Given a graph with n vertices, prove that if the degree of each vertex is at least $(n-1)/2$ then the graph is connected.

Hint: First, you may consider a small graph, for example a graph with 5 vertices, can you make it disconnected?

Problem 3 (Graded)

A simple graph is called n-regular if every vertex of the graph has degree n.

Show that if a bipartite graph $G = (V, E)$ with a bipartition of the vertex set (V_1, V_2) is n-regular for some positive integer n then $|V_1| = |V_2|$.

Problem 4

Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 faces, how many vertices does this graph have?

Problem 5

How many edges does a full binary tree with 10000 internal vertices have?

Problem 6 (Graded)

Suppose 1000 people enter a chess tournament. Use a rooted tree model of the tournament to determine how many games must be played to determine a champion, if a player is eliminated after one loss and games are played until only one entrant has not lost. (Assume there are no ties.)

Problem 7 (Graded)

Use Huffman coding to encode these symbols with given frequencies:
A: 0.05, B: 0.07, C: 0.08, D: 0.10, E: 0.15, F: 0.25, G: 0.30.

Show all intermediate steps. What is the average number of bits required to encode a symbol?

Probability

A bit of theory first

The complement of an event A, is the event $\overline{A} = \Omega \setminus A$, thus the following properties hold: $A \cap \overline{A} = \varnothing$ and $A \cup \overline{A} = \Omega$.

$$P(\overline{A}) \equiv P(\Omega \setminus A) = 1 - P(A)$$

The formula can be very useful, because sometimes it is much easier to compute $P(A)$ rather than $P(\overline{A})$, or the other way around.

Example of A and $\overline{A}$: Five cards are drawn from a standard deck. What is the probability that there is at least one ace among them?

A: there is at least one ace. $\qquad$ $\overline{A}$: there are no aces.

(This is not needed for the problems 1 and 2, by the way).

Problem 1 (Graded)

Given six cards:

$$A\spadesuit, J\spadesuit, 2\spadesuit, A\heartsuit, 2\heartsuit, 2\diamondsuit,$$

you pick one card at random.

Consider two events:

$$\begin{aligned} A &: \quad \text{the chosen card is an ace} \\ S &: \quad \text{the chosen card is a spade} \end{aligned}$$

(a) What is the sample space Ω?

(b) Compute the probabilities $P(A)$ and $P(S)$.

(c) Are the events A and S independent?

(d) Can you find any (other?) pair of independent events for the given set of cards?

Problem 2

Three cards are drawn from a standard 52-card deck.

Each combination of three cards was equally likely, find the probability that the following hand is obtained: $\{K\spadesuit, Q\heartsuit, J\diamondsuit\}$ (this is a set, the order does not matter).

Problem 3 (Graded)

A project was implemented by three developers: Alice, Bob, and Carol. They used four languages: C, C++, Python, and JavaScript. The table summarizes what fraction of the code was written by each person in each language.

	C	C++	Python	JavaScript
Alice	5/24	1/8	1/6	0
Bob	1/24	1/8	1/12	0
Carol	0	0	1/12	1/6

You pick a piece of code at random.

(a) Who is most likely to be the author of that piece of code?

(b) Who is most likely to be the author given that it was written in JS?

(c) Who is most likely to be the author given that it was written in C or C++?

(d) What is the probability that it was written by Bob? Does the probability change if we know that the code is in Python? Are the events *Python* and *Bob* independent or not?

(e) Are the events *Alice* and *C* independent?

(f) The same question for *Carol* and *JS*.

Problem 4

A fair six-sided die is rolled twice. What is the probability that the outcome of the second roll is the same as the outcome of the first roll?

Problem 5 (Graded)

Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

(a) the probability of no successes

(b) the probability of at least one success

(c) the probability of at most one success

(d) the probability of at least two successes

Problem 6

By rolling a six-sided die 6 times, a strictly increasing sequence of numbers was obtained, what is the probability of such an event?