Strong Induction.
Catalan Numbers.

A game

The game starts with a stack of n coins. In each move, you divide one stack into two nonempty stacks.

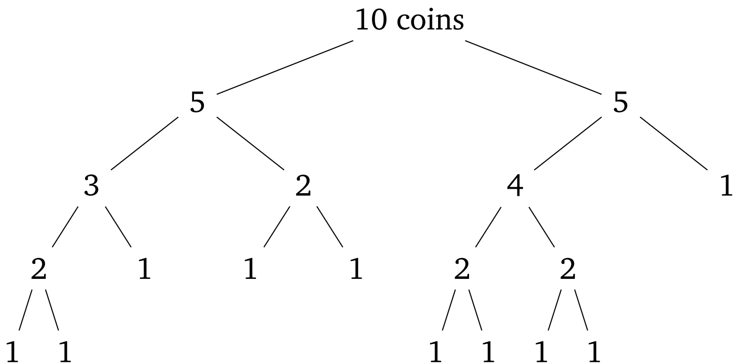
$$\begin{aligned} ||||| &\to ||| + || \\ &\to ||| + | + | \\ &\to || + | + | + | \\ &\to | + | + | + | + | \end{aligned}$$

If the new stacks have height a and b, then you score ab points for the move.

$$||||| \to ||| + || \qquad \text{you get } 3 \cdot 2 = 6 \text{ points}$$

What is the maximum score you can get?

A game



The total score: $25 + 6 + 4 + 2 + 1 + 4 + 1 + 1 + 1 = 45$ points.

Can we find a better strategy?

A game

Theorem. Every way of unstacking n coins gives a score of

$$S(n) = \frac{n(n-1)}{2} \text{ points.}$$

And we want to prove it by induction.

Let $P(n)$ be the proposition that every way of unstacking n coins gives a score of $S(n)$.

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$$S(n) = S(k) + S(n-k) + k(n-k) = \frac{n(n-1)}{2}.$$

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We would like to have something a little “stronger” than ordinary induction.

Strong induction

Principle of Strong Induction. Let $P(n)$ be a predicate. If

- $P(0)$ is true, and
- for all $n \in \mathbb{N}$, $P(0), P(1), \ldots, P(n)$ imply $P(n+1)$,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Strong induction *allows you to assume $P(0), \ldots, P(n)$* in the inductive step, whereas in ordinary induction, you assume $P(n)$ only.

Strong induction

Strong induction is *no more powerful* than ordinary induction.

Consider a predicate

$$Q(n) = \forall k \left(0 \le k \le n \to P(k)\right)$$

$Q(n)$ says that $P(k)$ is true for all $0 \le k \le n$.

Ordinary induction on the predicate $Q(n)$ is equivalent to strong induction on $P(n)$.

Any theorem that can be proved with strong induction can also be proved with ordinary induction. However, an apeal to the strong induction principle can make some proofs a bit simpler.

Unstacking n coins

Theorem. Every way of unstacking n coins gives a score of

$$\frac{n(n-1)}{2} \text{ points.}$$

Proof. By strong induction. Let $P(n)$ be the proposition that every way of unstacking n coins gives a score of $S(n) = n(n-1)/2$.

The base case:
When $n = 1$, no moves is possible, so the score is $S(1) = 0$. The formula works, so $P(1)$ is true.

The inductive step:

$$\begin{aligned} S(n) &= S(k) + S(n-k) + n(n-k) \\ &= \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} + k(n-k) \\ &= \frac{1}{2}(k^2 - k + n^2 - nk - n - nk + k^2 + k + 2kn - 2k^2) = \frac{n(n-1)}{2}. \end{aligned}$$

Another game

A game
Strong Induction
Another game
Catalan Numbers

Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches.

The player who removes the last match wins the game.

Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

Another game

Let $P(n)$ be the proposition that the second player has a winning strategy if each pile contains n matches.

The base case:
$P(1)$ is true, because if each pile contains just 1 match, there is an obvious strategy for the second player.

The inductive step: When $n > 1$.

Another game

Let $P(n)$ be the proposition that the second player has a winning strategy if each pile contains n matches.

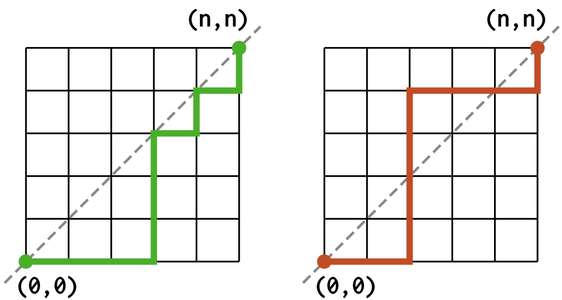
The base case:
$P(1)$ is true, because if each pile contains just 1 match, there is an obvious strategy for the second player.

The inductive step: When $n > 1$.
If the first player removes n matches from the first pile, the second player removes n matches from the other pile and wins.

Otherwise, if the first player removes $k < n$ from the first pile, the second player is doing the same, removing k matches from the other pile, so there are $n-k$ matches remain in both piles. And now, by the inductive hypothesis, for $n-k$ matches the second player has a winning strategy.

Count the number of good paths, C_n

Paths should go entirely below the diagonal line



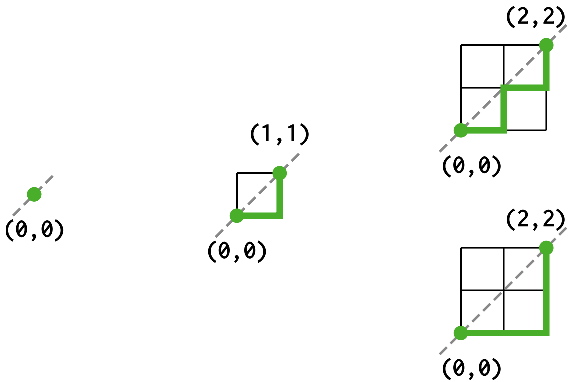
Good paths are below the diagonal

Bad paths cross it

The number of such paths, C_n, is the n^{th} Catalan number.

Count the number of good paths, C_n

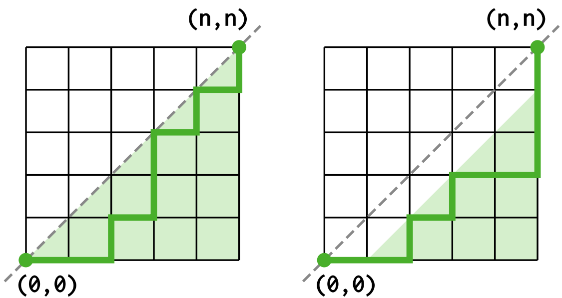
The cases when n is small: 0, 1, 2.



$C_0 = 1$ $\quad$ $C_1 = 1$ $\quad$ $C_2 = 2$

Recurrent formula

C_n is the number of paths that go below the diagonal (or touch the diagonal).



Introduce D_n, the number of paths that don't touch the diagonal in the middle points of the path.

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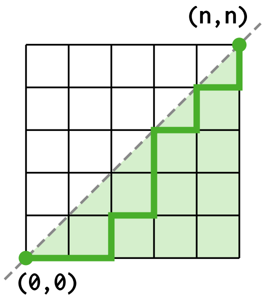
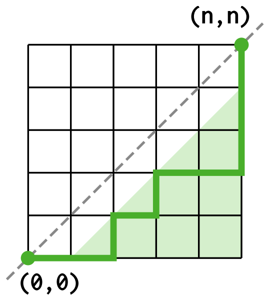


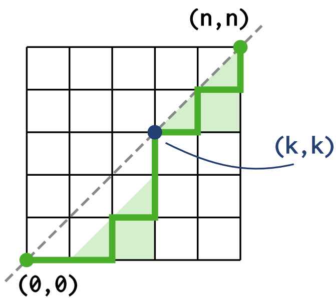


Introduce D_n, the number of paths that don't touch the diagonal in the middle points of the path.

$$D_n = C_{n-1}$$

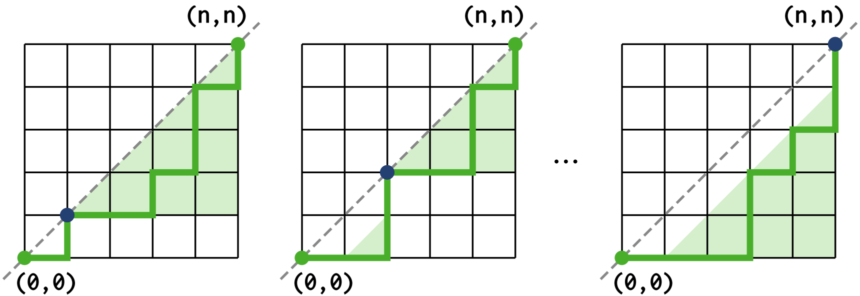
Recurrent formula

(k,k) be the first point of the given path that is on the diagonal and $k \neq 0$.



Given (k,k), the number of paths is $D_k C_{n-k}$

Recurrent formula

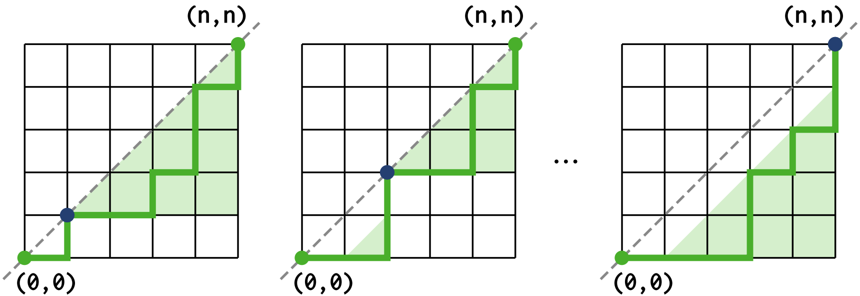


The diagonal point can be anywhere: $(1,1), (2,2), \ldots, (n,n)$

So, to count the total number of paths, we add up these n cases:

$$C_n = D_1 C_{n-1} + D_2 C_{n-2} + \ldots + D_n C_0 = \sum_{k=1}^{n} D_k C_{n-k}$$

Recurrent formula



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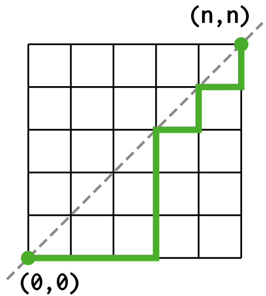
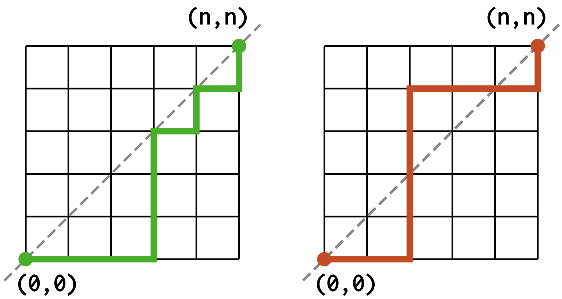
$$\text{since } D_k = C_{k-1}, \text{ we get } C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

Recurrent formula



$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

Closed form formula



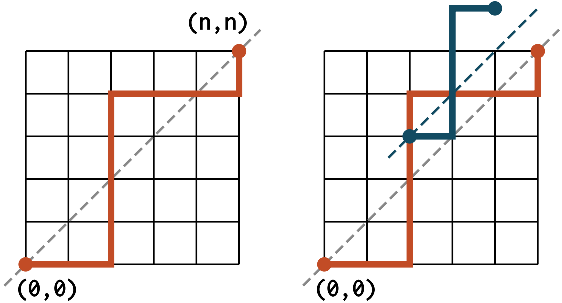
Good paths are below the diagonal

Bad paths cross it

We already know that the number of paths from the bottom-left to the top-right corner is $B_n = \binom{2n}{n}$

Let's try to count the number of paths that cross the diagonal, there is $B_n - C_n$ of them.

Closed form formula

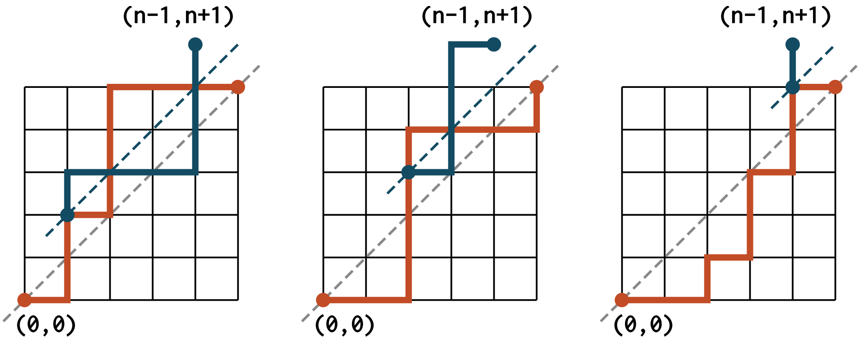


Consider a bad path that crosses the diagonal.

Lets say that the point $P = (k, k+1)$ is the first point above the diagonal. We mirror the remaining part of the path (shown in blue).

We can construct such new path for any invalid path.

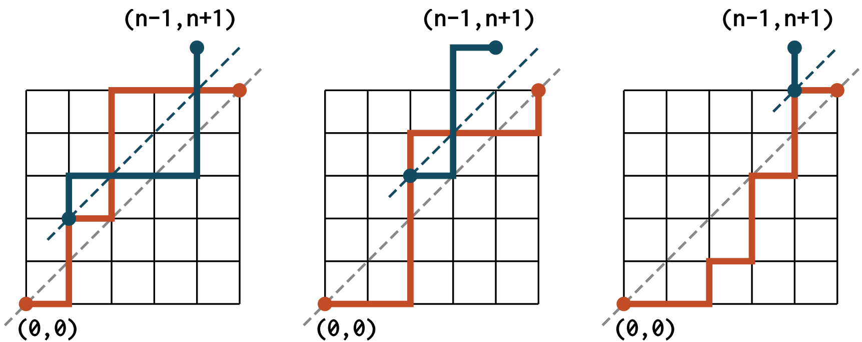
Closed form formula



Since we mirror the path starting at $P = (k, k+1)$, the remaining part of the path consisted of $(n-k, n-k-1)$ horizontal and vertical moves. Once reflected, it contains $(n-k-1, n-k)$ moves.

So, the resulting path ends up at the point
$Z = (k+n-k-1, k+1+n-k) = (n-1, n+1)$.
It does not depend on k.

Closed form formula

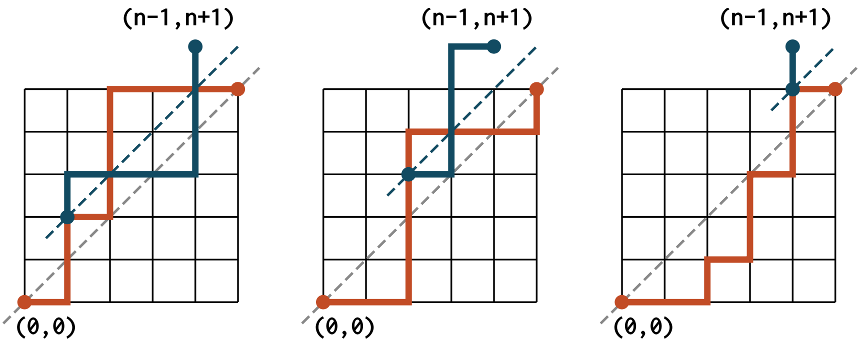


Every invalid paths becomes a path with $(n-1, n+1)$ horizontal and vertical moves.

So there is

$$B_n - C_n = \binom{n-1+n+1}{n+1} = \binom{2n}{n+1} \quad \text{of them.}$$

Closed form formula

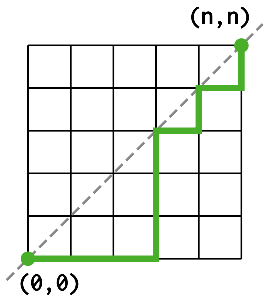


$$C_n = B_n - \binom{2n}{n+1}$$

Therefore,

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{2} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}$$

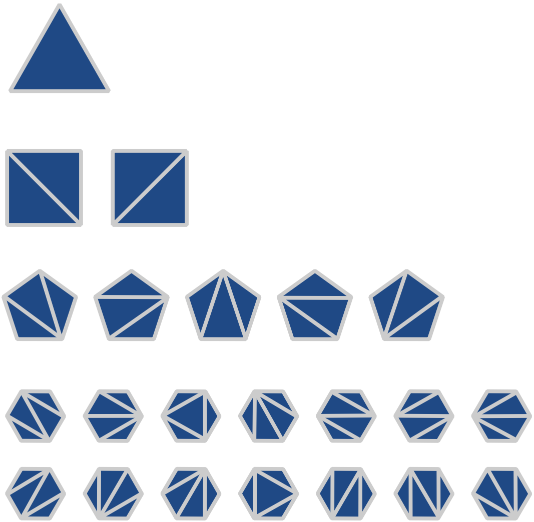
Three formulas for C_n



$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

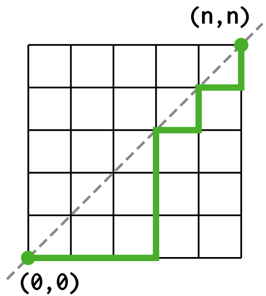
$$C_n = \binom{2n}{n} - \binom{2n}{n+1} \qquad C_n = \frac{1}{n+1}\binom{2n}{n}$$

Catalan numbers are more than that



The number of ways to triangulate convex polygons:
1, 2, 5, 14, . . .

Catalan numbers are more than that

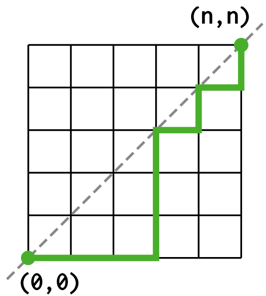


Let's encode the path with bits, $\{0, 1\}$.
If every move to the right is 1, and and every move up is 0:

$$1110001010$$

Well, not particularly interesting ...

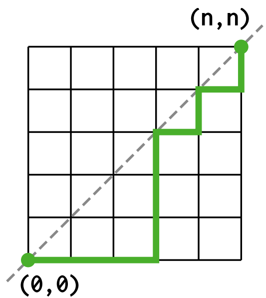
Catalan numbers are more than that



Let's encode the path with parentheses, {*(*,*)*}.
If every move to the right is *(*, and and every move up is *)*:

((( ))) ( ) ( )

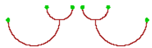
Catalan numbers are more than that



$$((( ))) ( ) ( )$$

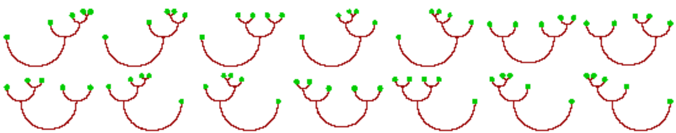
C_n is the number of strings made of n pairs of correctly balanced parentheses.

Catalan numbers are more than that



$C_2 = 2$

$C_3 = 5$



$C_4 = 14$

C_n is the number of full binary trees with $n+1$ leaves:

$$2, 5, 14, \ldots$$

(A rooted binary tree is full if every internal node has two children)

Alternative “proof”

All paths from the bottom-left to the top-right corner:

$$B_n = \binom{2n}{n}$$

$$B_n = \sum_{k=1}^{n} 2D_k B_{n-k} = \sum_{k=1}^{n} 2C_{k-1} B_{n-k}$$

Count the paths that cross the diagonal (not valid paths):

$$B_n - C_n = \sum_{k=0}^{n-1} C_k \cdot 1 \cdot \binom{2(n-k)-1}{(n-k)-1}$$

Fitst $2k$ moves were valid, and you ended up on the diagonal, you could do so in C_k ways. At the move $2k+1$, you cross the diagonal by going up. And after that simply finish the remaining $2n-2k-1$ moves so that $n-k-1$ times you are going up.

Alternative "proof"

Put it all together

$$\text{valid paths:} \qquad C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

$$\text{all paths:} \qquad B_n = \sum_{k=1}^{n} 2C_{k-1} B_{n-k}$$

$$\text{invalid paths:} \qquad B_n - C_n = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

Alternative "proof"

Adjust the indices

$$C_n = \sum_{k=0}^{n-1} C_k C_{(n-k)-1}$$

$$B_n = \sum_{k=0}^{n-1} 2 C_k B_{(n-k)-1}$$

$$B_n - C_n = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

Alternative "proof"

$$C_n = \sum_{k=0}^{n-1} C_k C_{(n-k)-1}$$

$$B_n = \sum_{k=0}^{n-1} 2 C_k B_{(n-k)-1}$$

$$B_n - C_n = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

Substitute B_n and C_n in the last formula

$$\sum_{k=0}^{n-1} 2 C_k B_{(n-k)-1} - \sum_{k=0}^{n-1} C_k C_{(n-k)-1} = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

Alternative "proof"

$$\sum_{k=0}^{n-1} 2C_k B_{(n-k)-1} - \sum_{k=0}^{n-1} C_k C_{(n-k)-1} = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

Rewrite

$$\sum_{k=0}^{n-1} C_k \left(2B_{(n-k)-1} - C_{(n-k)-1} \right) = \sum_{k=0}^{n-1} C_k \left(\binom{2(n-k)-1}{(n-k)-1} \right)$$

Warning! Handwaving here ... I failed at this point.

These sums are equal only if their coefficients are equal:

$$2B_{(n-k)-1} - C_{(n-k)-1} = \binom{2(n-k)-1}{(n-k)-1}$$

Replace $m = (n-k)-1$

$$2B_m - C_m = \binom{2m+1}{m}$$

Alternative “proof”

$$2B_n - C_n = \binom{2n+1}{n}$$

Remember that $B_n = \binom{2n}{n}$:

$$2\binom{2n}{n} - C_n = \binom{2n+1}{n}$$

By Pascal’s Identity, $\binom{2n+1}{n} = \binom{2n}{n} + \binom{2n}{n-1}$, so

$$2\binom{2n}{n} - C_n = \binom{2n}{n} + \binom{2n}{n-1}$$

Finally,

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$