

Relations. Functions.
Bijection and counting.

Subsets of the Cartesian product

Relations

Functions

Bijection

Inclusion-Exclusion

Infinity

Given two sets

$$A = \{1, 2, 3\}$$

$$B = \{1, 2, 3, 4\}$$

Their Cartesian product

$$\begin{aligned} A \times B = \{ & (1, 1), (2, 1), (3, 1), \\ & (1, 2), (2, 2), (3, 2), \\ & (1, 3), (2, 3), (3, 3), \\ & (1, 4), (2, 4), (3, 4) \} \end{aligned}$$

Consider three subsets of $A \times B$:

$$R_{(id)} = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_{(less)} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

$$R_{(inc)} = \{(1, 2), (2, 3), (3, 4)\}$$

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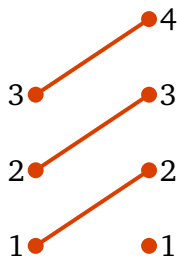
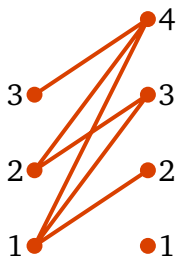
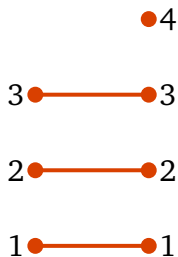
Infinity

Def. A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B .

$$R_{(id)} = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_{(less)} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

$$R_{(inc)} = \{(1, 2), (2, 3), (3, 4)\}$$



Relations

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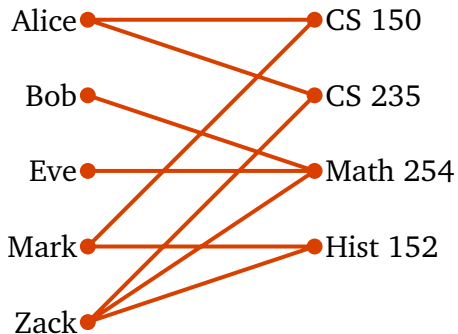
Infinity

Example of a relation:

S = set of students

C = set of classes

$R = \{(s, c) \mid \text{student } s \text{ takes class } c\}$



Functions

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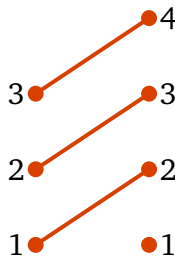
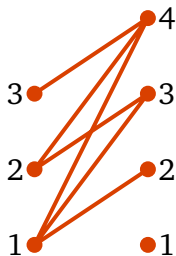
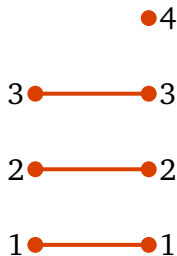
Infinity

Def. A relation $R \subseteq A \times B$ is a *function* (a functional relation) if for every $a \in A$, there is at most one $b \in B$ so that $(a, b) \in R$.

$$R_{(id)} = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_{(less)} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

$$R_{(inc)} = \{(1, 2), (2, 3), (3, 4)\}$$



Functions

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Functional relation $R \subseteq A \times B$ defines a unique way to map each element from the set A to an element from the set B .

There is a well-known and convenient notation for functions:

$$f(a) = b \quad \text{where } a \in A \text{ and } b \in B$$

It maps elements from A to B :

$$f : A \rightarrow B$$

$$A \xrightarrow{f} B$$

Functions

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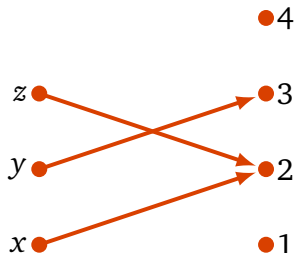
Def. For the function $f : A \rightarrow B$, set A is called *domain*, and set B is called *codomain*.

Def. $f(a)$ is the *image* of $a \in A$.

Def. The *image* of f , denoted by $f(A)$, is the set of the images $f(a)$ for all $a \in A$

$$f(A) = \{x \mid \exists a \in A (f(a) = x)\}.$$

The image of a function is also called *range*.



$$f : A \rightarrow B$$

$$\text{domain}(f) = A = \{x, y, z\}$$

$$\text{codomain}(f) = B = \{1, 2, 3, 4\}$$

$$\text{image}(f) = f(A) = \{2, 3\}$$

Onto

Relations

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Bijection

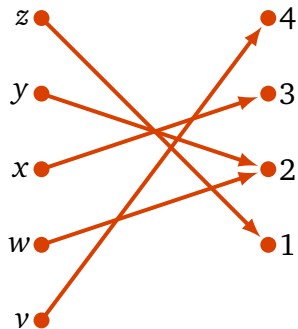
Inclusion-Exclusion

Infinity

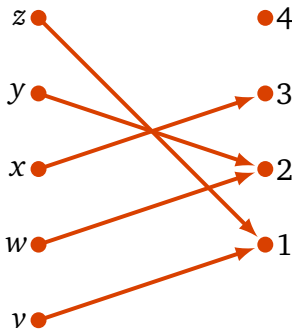
Def. A function $f : A \rightarrow B$ is called *onto* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

In other words, the image $f(A)$ is the whole codomain B .

$f : A \rightarrow B$ is onto



$g : A \rightarrow B$ is not onto



One-to-one

Relations

Functions

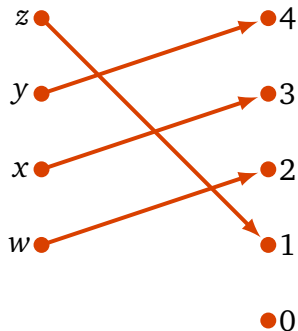
Bijection

Inclusion-Exclusion

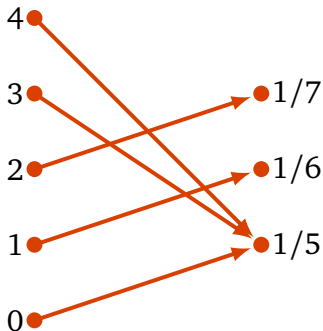
Infinity

Def. A function $f : A \rightarrow B$ is said to be *one-to-one* if and only if $f(x) = f(y)$ implies that $x = y$ for all $x, y \in A$.

$f : A \rightarrow B$ is one-to-one



$g : B \rightarrow C$ is not one-to-one



Bijection

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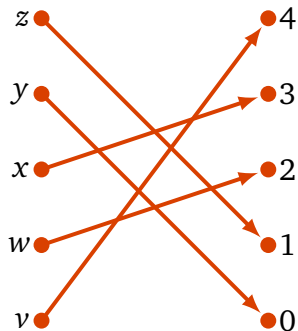
Bijection

Inclusion-Exclusion

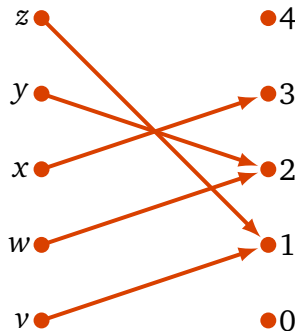
Infinity

Def. The function f is a *bijection* (also called one-to-one correspondence) if and only if it is both one-to-one and onto.

$f : A \rightarrow B$ is a bijection



$g : A \rightarrow B$ is not a bijection



Bijection

Relations

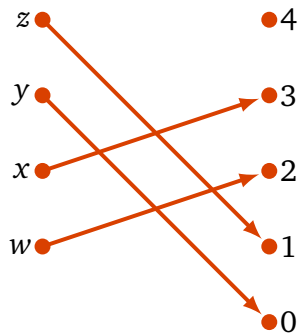
Functions

Bijection

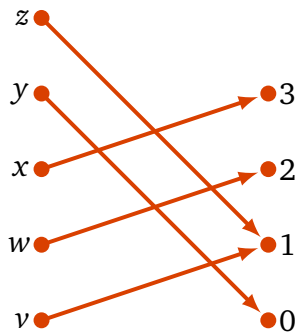
Inclusion-Exclusion

Infinity

$f : A \rightarrow B$ is one-to-one, but
not onto



$g : C \rightarrow D$ is onto, but not
one-to-one



So, both functions are not bijections.

Bijection. Observation

Relations

Functions

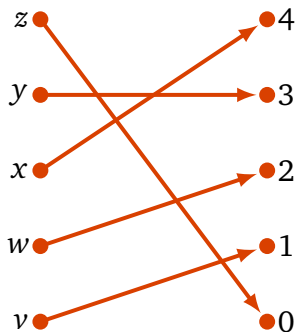
Bijection

Inclusion-Exclusion

Infinity

Bijection Rule. Given two sets A and B , if there exists a bijection

$$f : A \rightarrow B, \quad \text{then } |A| = |B|.$$



We can count the size of the set A , instead of the size of B !

Bijection Rule.

Relations

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Bijection

Inclusion-Exclusion

Infinity

Consider two similar problems:

(a) How many bit strings contain exactly three 1s and two 0s?

11010

(b) How many strings can be composed of three 'A's and five 'b's so that an 'A' is always followed by a 'b'?

AbAbbAbb

We show that this two problems are equivalent by constructing a bijection.

Bijection Rule.

Relations

Functions

Bijection

Inclusion-Exclusion

Infinity

Let X be the set of bit strings

$$X = \{11010, \dots\}$$

and Y be the set of 'A' and 'b' strings

$$Y = \{AbAbbAbb, \dots\}$$

We can construct a bijection $f : X \rightarrow Y$:

1 gets replaced by Ab

0 gets replaced by b

Bijection Rule.

Relations

Functions

Bijection

Inclusion-Exclusion

Infinity

$$f : 11100 \mapsto Ab\ Ab\ Ab\ b\ b$$

$$11010 \mapsto Ab\ Ab\ b\ Ab\ b$$

$$11001 \mapsto Ab\ Ab\ b\ b\ Ab$$

$$10110 \mapsto Ab\ b\ Ab\ Ab\ b$$

$$10101 \mapsto Ab\ b\ Ab\ b\ Ab$$

$$10011 \mapsto Ab\ b\ b\ Ab\ Ab$$

$$01110 \mapsto b\ Ab\ Ab\ Ab\ b$$

$$01101 \mapsto b\ Ab\ Ab\ b\ Ab$$

$$01011 \mapsto b\ Ab\ b\ Ab\ Ab$$

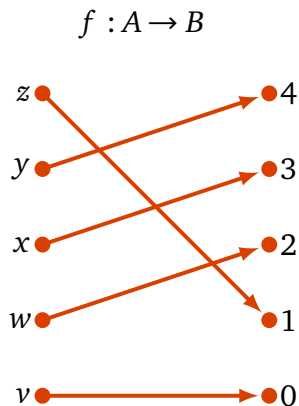
$$00111 \mapsto b\ b\ Ab\ Ab\ Ab$$

Function f is one-to-one and onto, so it is a bijection. Therefore, the cardinalities of two sets are equal: $|X| = |Y| = \binom{5}{3} = 10$.

Bijection. Observation

Observation For every bijection $f : A \rightarrow B$, exists an *inverse* function

$$f^{-1} : B \rightarrow A$$



Relations

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Bijection

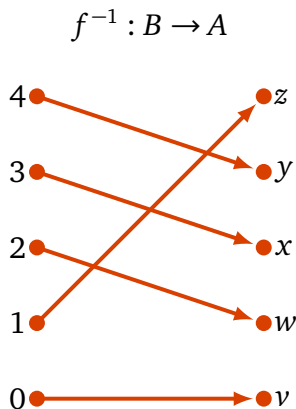
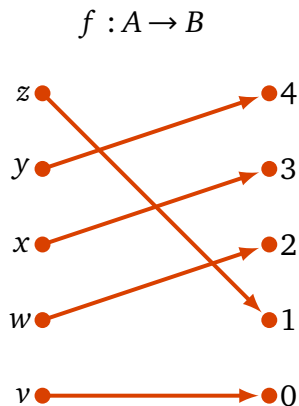
Inclusion-Exclusion

Infinity

Bijection. Observation

Observation For every bijection $f : A \rightarrow B$, exists an *inverse* function

$$f^{-1} : B \rightarrow A$$



The inverse function is a bijection too.

Relations

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Bijection

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Bijection. Observation

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Bijection

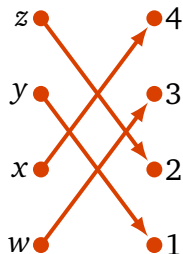
Inclusion-Exclusion

Infinity

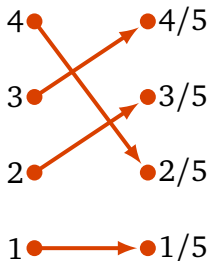
Given two bijections $f : A \rightarrow B$ and $g : B \rightarrow C$.
Consider their composition

$$h(x) = g(f(x))$$

$f : A \rightarrow B$



$g : B \rightarrow C$



Bijection. Observation

Relations

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Bijection

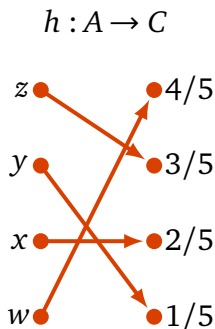
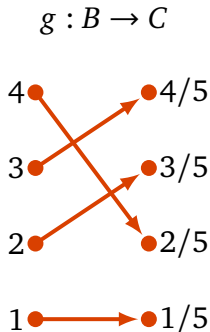
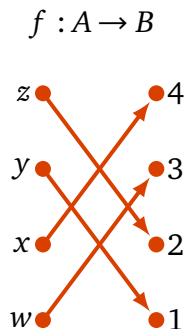
Inclusion-Exclusion

Infinity

Given two bijections $f : A \rightarrow B$ and $g : B \rightarrow C$.

Consider their composition

$$h(x) = g(f(x))$$



$h : A \rightarrow C$ is a bijection, and therefore $|A| = |C|$.

Bijection. Catalan numbers

Relations

Functions

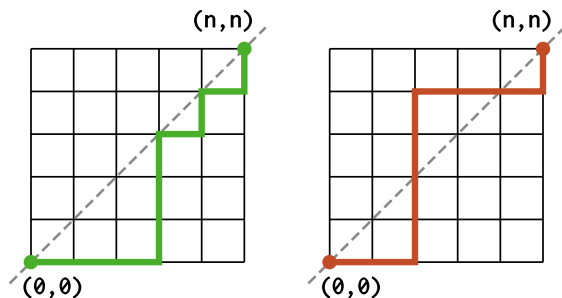
Bijection

Inclusion-Exclusion

Infinity

Let's make a bijection with something more complex.

Catalan numbers provide many nice examples.



Bijection. Catalan numbers

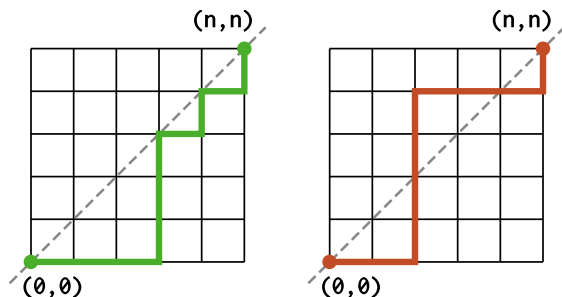
Relations

Functions

Bijection

Inclusion-Exclusion

Infinity



Let's find a bijection between valid paths on the grid of size n and the *strings of n pairs of correctly matching parentheses*:

$((()))()()$ Ok

$((())))^{\text{red lightning bolt}})((()$ Error

Bijection. Catalan numbers

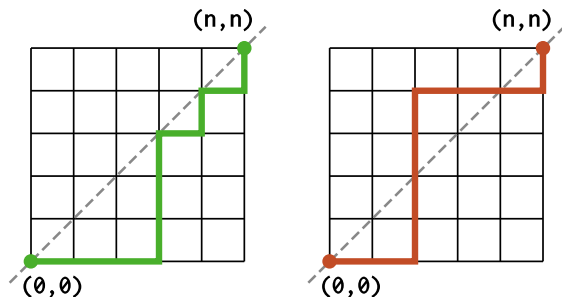
Relations

Functions

Bijection

Inclusion-Exclusion

Infinity



To map a path to a string from $\{ '(', ') ' \}^n$, we go along the path:

- (a) horizontal step $\mapsto ($
- (b) vertical step $\mapsto)$.

By construction, the resulting sequence produces only valid strings of n pairs of parentheses (we never close more than we open). And this is a bijection, because no two different paths map to the same string, and eventually all strings are mapped.

Bijection. Catalan numbers

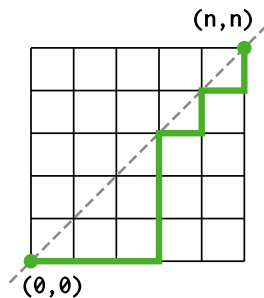
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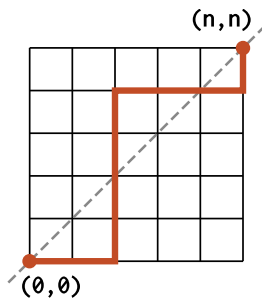
Bijection

Inclusion-Exclusion

Infinity



$((()))()$



$((()))((())$

Bijection. Counting subsets

Relations

Functions

Bijection

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Infinity

Please, count the number of subsets of a set

$$A = \{a, b, c, d, e\}$$

by reducing the problem to counting bit strings.

Bijection. Counting subsets

Relations

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Inclusion-Exclusion

Infinity

Please, count the number of subsets of a set

$$A = \{a, b, c, d, e\}$$

by reducing the problem to counting bit strings.

Let's find a bijection f between the power set

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \dots\}$$

and the set of bit strings of length 5:

$$\{0, 1\}^5 = \{00000, 00001, 00010, 00011, \dots\}$$

Bijection. Counting subsets

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$$f : \mathcal{P}(\{a, b, c, d, e\}) \rightarrow \{0, 1\}^5$$

0s and 1s encode the membership of the five elements of $\{a, b, c, d, e\}$

$$f : \emptyset \mapsto 00000$$

$$\{a\} \mapsto 10000$$

$$\{b\} \mapsto 01000$$

$$\{a, b\} \mapsto 11000$$

$$\{c\} \mapsto 00100$$

$$\{a, c\} \mapsto 10100$$

$$\{b, c\} \mapsto 01100$$

$$\{a, b, c\} \mapsto 11100$$

...skipping um.. 23 subsets

$$\{a, b, c, d, e\} \mapsto 11111$$

The cardinality

$$|\{0, 1\}^5| = 2^5 = 32$$

Therefore, by the bijection rule,

$$|\mathcal{P}(A)| = 2^5$$

Inclusion-Exclusion principle

Relations

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Inclusion-Exclusion

Infinity

We remember the subtraction rule for the union of two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Can it be generalized for a union of n sets

$$|A_1 \cup \dots \cup A_n| = |A_1| + \dots + |A_n| - \langle \text{something} \rangle?$$

Inclusion-Exclusion principle

Relations

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Can it be generalized for a union of n sets

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Of course, it can!

Inclusion-Exclusion principle

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Infinity

Union of three sets

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| = & |A_1| + |A_2| + |A_3| \\ & - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| \\ & + |A_1 \cap A_2 \cap A_3|\end{aligned}$$

$$|\{1, 2, 3\} \cup \{2, 3, 4\} \cup \{3, 4, 1\}| = 3 + 3 + 3 - 2 - 2 - 2 + 1 = 4$$

Inclusion-Exclusion principle

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Union of n sets

$|A_1 \cup \dots \cup A_n| =$ the sum of the sizes of the individual sets
minus the sizes of all two-way intersections
plus the sizes of all three-way intersections
minus the sizes of all four-way intersections
plus the sizes of all five-way intersections
etc.

Infinity?

We know that the set of natural numbers, \mathbb{N} , is infinite, so, definitely, there are sets with infinitely many elements.

How is it possible to construct such sets?

Let's define an operation

$$A^+ = A \cup \{A\}$$

We start with \emptyset and apply this operation:

$$\emptyset = \emptyset$$

$$\emptyset^+ = \{\emptyset\}$$

$$(\emptyset^+)^+ = \{\emptyset, \{\emptyset\}\}$$

$$((\emptyset^+)^+)^+ = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

...

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Infinity

Infinity?

Relations

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Infinity

We know that the set of natural numbers, \mathbb{N} , is infinite, so, definitely, there are sets with infinitely many elements.

How is it possible to construct such sets?

Let's define an operation

$$A^+ = A \cup \{A\}$$

We start with \emptyset and apply this operation:

$$'0' = \emptyset = \emptyset$$

$$'1' = \emptyset^+ = \{\emptyset\} = \{'0'\}$$

$$'2' = (\emptyset^+)^+ = \{\emptyset, \{\emptyset\}\} = \{'0', '1'\}$$

$$'3' = ((\emptyset^+)^+)^+ = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{'0', '1', '2'\}$$

...

This is von Neumann's construction of natural numbers.

Infinity

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Suppose that you have infinitely many one dollar bills (numbered 1, 3, 5, ...) and you come upon the Devil, who is willing to pay two dollars for each of your one-dollar bills.



The Devil is very particular, however, about the order in which the bills are exchanged. The contract stipulates that in each sub-transaction he buys from you your lowest-numbered bill and pays you with higher-numbered bills.

First sub-transaction takes $1/2$ hour, then $1/4$ hour, $1/8$, and so on, so that after one hour the entire exchange will be complete.