Relations. Partial orders.
Infinity. Cardinality.
Pairing function. Diagonalization.

Relations

Remember that a relation is a subset of the Cartesian Product of two sets.

For example,

$$R = \{(a, b) \in A \times B \mid \text{some property holds}\}$$

$$R \subseteq A \times B$$

For convenience, we adopt the following infix notation:

$$\text{when} \quad (a, b) \in R, \quad \text{we write} \quad aRb$$

Relations. Infix notation

It is originated from the relations like $=$, $\leq$, $\geq$, $<$, and $>$.

$$(1,2) \in R_{(<)} \quad \text{we usually write} \quad 1 < 2$$

$$(3,3) \in R_{(=)} \quad \text{we usually write} \quad 3 = 3$$

Divisibility is a relation on $\mathbb{N}$ too. And we use infix notation:

$$(15,60) \in R_{(divides)} \quad \text{we write} \quad 15 \mid 60$$

Relations on the same set

What if the sets A and B are the same?

$$R \subseteq A \times A$$

For example, $=$, $\leq$, $\geq$, $<$, $>$ are relations on $\mathbb{N}$. That is, these relations are subsets of $\mathbb{N} \times \mathbb{N}$.

Def. A relation on the set A is

- *reflexive* if $\forall x \in A : xRx$.
- *symmetric* if $\forall x, y \in A : xRy \to yRx$.
- *antisymmetric* if $\forall x, y \in A : (xRy \land yRx) \to x = y$.
- *transitive* if $\forall x, y, z \in A : (xRy \land yRz) \to xRz$.

Relations on the same set

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	reflexive?	symmetric?	antisymmetric?	transitive?
$x \equiv y \pmod 5$	Yes	Yes	No	Yes
$x \mid y$	Yes	No	Yes	Yes
$x \leq y$	Yes	No	Yes	Yes

Partial orders

Def. A relation is a *partial order* if it is reflexive, antisymmetric, and transitive.

An example, the "divides" relation on the natural numbers is a partial order:

- It is reflexive because $x \mid x$.
- It is antisymmetric because $x \mid y$ and $y \mid x$ implies $x = y$.
- It is transitive because $x \mid y$ and $y \mid z$ implies $x \mid z$.

The $\leq$ relation on the natural numbers is also a partial order. However, the $<$ relation is not a partial order, because it is not reflexive; no number is less than itelf.

Partial orders

Often a partial order relation is denoted with the symbol

$$\preceq$$

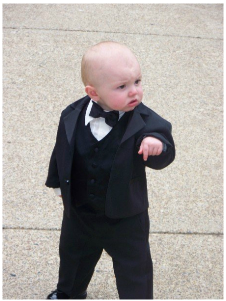
instead of a letter, like R.

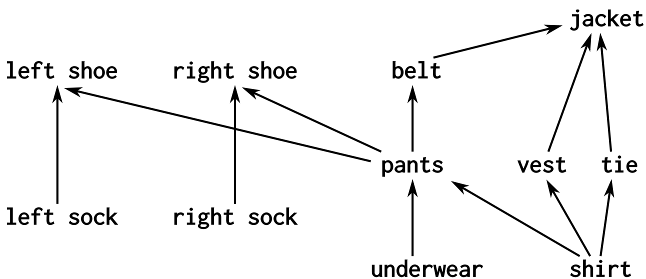
This makes sense since the symbol calls to mind $\leq$, which is one of the most common partial orders.

$x \preceq y$ it reads as "x precedes y".

Partially ordered sets

Def. If $\preceq$ is a partial order on the set A, then
the pair $(A, \preceq)$ is called a *partially-ordered set* or *poset*.

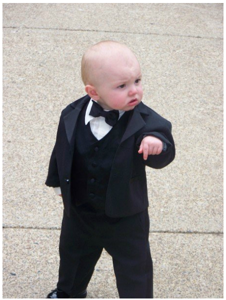


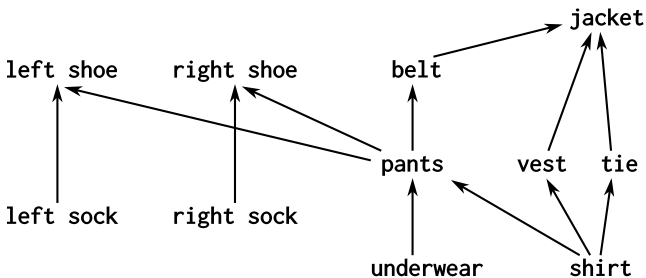


Def. The elements x and y of a poset $(A, \preceq)$ are called *comparable*
if either $x \preceq y$ or $x \preceq y$.

When x and y are elements of A such that neither $x \preceq y$ nor $y \preceq x$,
x and y are called *incomparable*.

Hasse diagram





This graph is called the *Hasse diagram* for the poset $(A, \preceq)$.

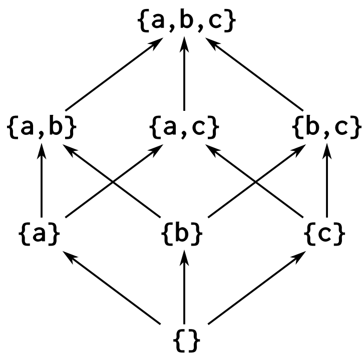
For a and b from A, we draw an edge from a to b if $a \preceq b$.

Self-loops and edges implied by transitivity are omitted.

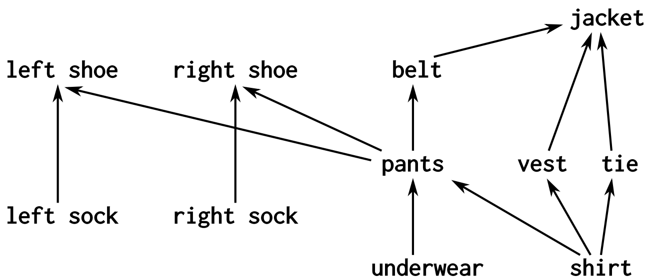
Hasse diagram

Consider a poset $(\mathscr{P}(A), \subseteq)$ for $A = \{a, b, c\}$.

Its Hasse diagram:



Mminimal and maximal elements



In a poset $(A, \preceq)$, an element $x \in A$ is *minimal* if there is no other element $y \in A$ such that $y \preceq x$.

Similarly, an element $x \in A$ is *maximal* if there is no other element $y \in A$ such that $x \preceq y$.

There are four minimal elements.

Partially ordered sets

Theorem. A poset $(A, \preceq)$ has no directed cycles other than self-loops, that is, there is no sequence of $n \geq 2$ distinct elements $a_i \in A$ such that

$$a_1 \preceq a_2 \preceq a_3 \preceq a_4 \preceq \ldots \preceq a_{n-1} \preceq a_n \preceq a_1$$

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Proof. Suppose that for some $n \geq 2$ such sequence $a_1 \ldots a_n$ exists.

Recall that the partial order is a transitive, antisymmetric, and refelxive relation.

Partially ordered sets

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Proof. Suppose that for some $n \geq 2$ such sequence $a_1 \ldots a_n$ exists.

Recall that the partial order is a transitive, antisymmetric, and refelxive relation.

Since it's transitive: $a_1 \preceq a_2$ and $a_2 \preceq a_3$, therefore $a_1 \preceq a_3$.

Similarly, we prove that $a_1 \preceq a_4$, $a_1 \preceq a_5$, $\ldots$, $a_1 \preceq a_n$.

Thus $a_1 \preceq a_n$ and $a_n \preceq a_1$.

But $\preceq$ is antisymmetric, and therefore $a_1 = a_n$. This contradicts the supposition that $a_1, \ldots a_n$ are $n \geq 2$ distinct elements! Thus there is no such directed cycle.

Total order

Def. A *total order* is a partial order in which every pair of elements is comparable.

$(A, \preceq)$ is a total order if for every $x, y \in A$, either $x \preceq y$ or $y \preceq x$.

The $\leq$ relation on natural numbers is a total order. However, the "divides" relation on the same set $\mathbb{N}$ is not.

Question: Given a parially ordered set $(A, \preceq)$, can we make a total order $\preceq_T$ that is "compatible" with the given partial order $\preceq$? (Compatible in the sense that the total order never violates the given partial order)

Topological sort

Def. A *topological sort* of a poset $(A, \preceq)$ is a total order $\preceq_T$ s.t.

$$x \preceq y \quad \text{implies} \quad x \preceq_T y.$$

Theorem. Every finite poset has a topological sort.

Lemma. Every finite poset has a minimal element.

Infinite sets

Consider three sets:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Even_N = \{0, 2, 4, 6, 8, \ldots\}$$
$$Odd_N = \{1, 3, 5, 7, 9, \ldots\}$$
$$\mathbb{Z}^- = \{-1, -2, -3, -4, \ldots\}$$

Can we compare their cardinalities?

Infinite sets

Consider three sets:

$$\begin{aligned}\mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\}\\ Even_N &= \{0, 2, 4, 6, 8, \ldots\}\\ Odd_N &= \{1, 3, 5, 7, 9, \ldots\}\\ \mathbb{Z}^- &= \{-1, -2, -3, -4, \ldots\}\end{aligned}$$

Can we compare their cardinalities?

We need a definition for the cardinality of an infinite set.

Cardinality of an infinite set

Def. The sets A and B have the same cardinality if and only if there is a bijection from A to B.

When A and B have the same cardinality, we write $|A| = |B|$.

$$\begin{aligned} \mathbb{N} &= \{0, 1, 2, 3, 4, \ldots\} \\ Even_N &= \{0, 2, 4, 6, 8, \ldots\} \\ Odd_N &= \{1, 3, 5, 7, 9, \ldots\} \\ \mathbb{Z}^- &= \{-1, -2, -3, -4, \ldots\} \end{aligned}$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Even_N = \{0, 2, 4, 6, 8, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to Even_N$$

$$0\bullet \quad 1\bullet \quad 2\bullet \quad 3\bullet \quad \cdots$$

$$\bullet 0 \quad \bullet 2 \quad \bullet 4 \quad \bullet 6 \quad \cdots$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Even_N = \{0, 2, 4, 6, 8, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to Even_N$$

$$\begin{tikzcd}
0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & \cdots \\
0 & 2 & 4 & 6 & \cdots
\end{tikzcd}$$

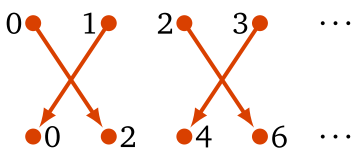
$$f(x) = 2x$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Even_N = \{0, 2, 4, 6, 8, \ldots\}$$

Alternatively

$$f : \mathbb{N} \to Even_N$$



$$\ldots$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Odd_N = \{1, 3, 5, 7, 9, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to Odd_N$$

$$0\bullet \quad 1\bullet \quad 2\bullet \quad 3\bullet \quad \cdots$$

$$\bullet 1 \quad \bullet 3 \quad \bullet 5 \quad \bullet 7 \quad \cdots$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
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Find a bijection

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$$\begin{tikzcd}
0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & \cdots \\
1 & 3 & 5 & 7 & \cdots
\end{tikzcd}$$

$$f(x) = 2x + 1$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$\mathbb{Z}^- = \{-1, -2, -3, -4, -5, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to \mathbb{Z}^-$$

$$0\bullet \quad 1\bullet \quad 2\bullet \quad 3\bullet \quad \cdots$$

$$\bullet{-1} \quad \bullet{-2} \quad \bullet{-3} \quad \bullet{-4} \cdots$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
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Find a bijection

$$f : \mathbb{N} \to \mathbb{Z}^-$$

$$\begin{tikzcd}
0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & \cdots \\
-1 & -2 & -3 & -4 & \cdots
\end{tikzcd}$$

$$f(x) = -x - 1$$

Cardinality of an infinite set

Therefore, all these sets have the same cardinality

$$|\mathbb{N}| = |Even_N| = |Odd_N| = |\mathbb{Z}^-|$$

Countable sets

Therefore, all these sets have the same cardinality

$$|\mathbb{N}| = |Even_N| = |Odd_N| = |\mathbb{Z}^-|$$

Def. A set S is called *countable* if $|S| = |\mathbb{N}|$ or if S is a finite set.

Countable sets

Since $\mathbb{N}$ is an infinite set, the cardinality $|\mathbb{N}|$ is greater than any natural number. We need a way to denote the cardinality of this set.

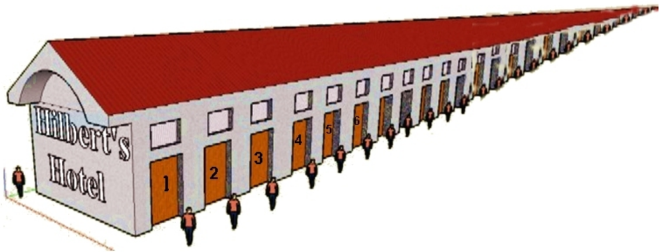
The following symbol is used

$$|\mathbb{N}| = \aleph_0$$

It reads as "aleph naught", "aleph null", "aleph zero".

All infinite countable sets have the same cardinality $\aleph_0$.

Hilbert’s Hotel

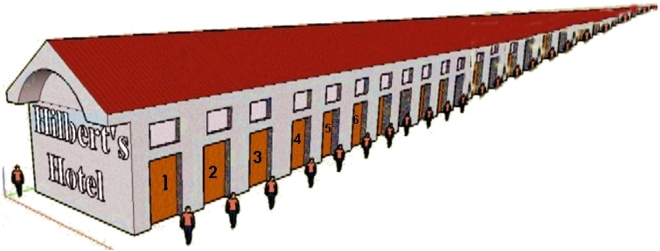


Imagine a hotel with a countably infinite number of rooms.

Each room is occupied by a guest.

Question: Can it accomodate one more guest?

Hilbert’s Hotel

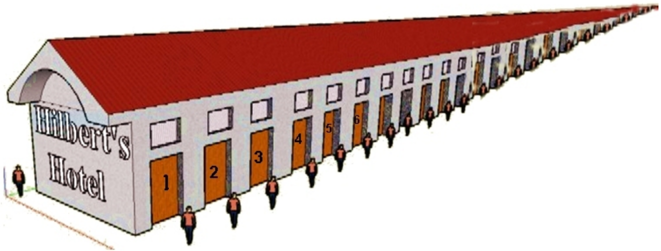


There is a bijection between $\{x\} \cup \mathbb{N}$ (guests) and $\mathbb{N}$ (rooms)

$$x\bullet \quad 0\bullet \quad 1\bullet \quad 2\bullet \quad 3\bullet \quad \cdots$$

$$\bullet 0 \quad \bullet 1 \quad \bullet 2 \quad \bullet 3 \quad \cdots$$

Hilbert's Hotel



There is a bijection between $\{x\} \cup \mathbb{N}$ (guests) and $\mathbb{N}$ (rooms)

$$
\begin{tikzcd}
x \arrow[dr] & 0 \arrow[dr] & 1 \arrow[dr] & 2 \arrow[dr] & 3 \arrow[dr] & \cdots \\
& 0 & 1 & 2 & 3 & \cdots
\end{tikzcd}
$$

More complex cases

We want to prove that $B = \mathbb{N} \times \{T, F\}$ is countable.

Can we find a bijection between $\mathbb{N}$ and $B = \mathbb{N} \times \{T, F\}$?

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

$$B = \{(0, T), (1, T), (2, T), \ldots (0, F), (1, F), (2, F), \ldots\}$$

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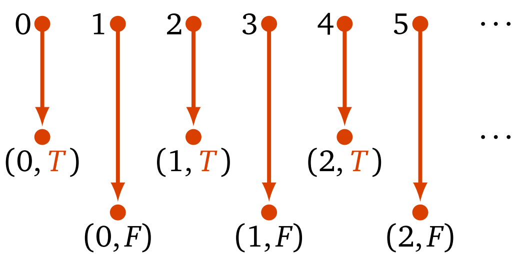


$$(0, T), (0, F), (1, T), (1, F), (2, T), (2, F), \ldots$$

More complex cases

Similarly, there is a bijection between $\mathbb{N}$ and $\mathbb{Z}$

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

$$\mathbb{Z} = \{\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots\}$$

We just rearrange the order of integers:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

In general, if there is a way *to list* the elements of a given set in linear order, then it is *countable* (i.e. there is a bijection between this set and $\mathbb{N}$).

More complex cases

Find a bijection $h : A \to B$, where

$$A = \mathbb{N} \times \{T, F\}$$

$$B = \mathbb{Z}$$

More complex cases

Find a bijection $h : A \to B$, where

$$A = \mathbb{N} \times \{T, F\}$$

$$B = \mathbb{Z}$$

A and B are countable, and we know how to construct the following two bijections

$$f : \mathbb{N} \to A$$
$$g : \mathbb{N} \to B$$

Since f is a bijection, there exist an inverse function $f^{-1} : A \to \mathbb{N}$, which is a bijection too, and we can find it, so

$$h(x) = g(f^{-1}(x))$$

$A \times \mathbb{N}$

We have shown that $\mathbb{Z}$ is countable, $\mathbb{N} \times \{T, F\}$ is countable.

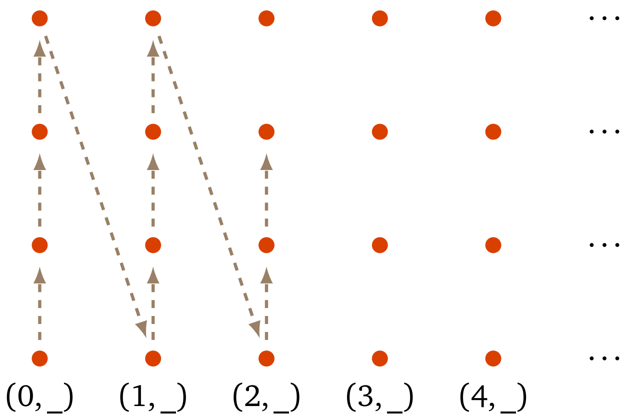
Similarly, it's not hard to show that for any *finite* set A, its Cartesian products

$$A \times \mathbb{N} \text{ and } \mathbb{N} \times A \text{ are countable.}$$

$\mathbb{N} \times A$ and $A \times \mathbb{N}$ when A is finite

Similarly, it's not hard to show that for any *finite* set A, its Cartesian products

$$A \times \mathbb{N} \text{ and } \mathbb{N} \times A \text{ are countable.}$$



Is the set $\mathbb{N} \times \mathbb{N}$ countable?

Can we find a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$? If yes, then the set of ordered pairs of natural numbers, $\mathbb{N} \times \mathbb{N}$, is a countable set.

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & \cdots \\
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & \cdots
\end{array}
$$

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(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & \cdots \\
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & \cdots
\end{array}
$$

Pairing function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$P(x,y) = \frac{1}{2}(x+y)(x+y+1) + y$$

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & \cdots \\
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & \cdots
\end{array}$$

The set of rational numbers, $\mathbb{Q}$

We can define the set of rational numbers as the set of all quotients p/q such that $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$:

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z} \wedge q \in \mathbb{Z}^+ \right\}$$

We can prove that $\mathbb{Q}$ is countable. The argument is similar to the proof for $\mathbb{N} \times \mathbb{N}$.

Power set

Is the power set $\mathscr{P}(\mathbb{N})$ countable?

Power set

Theorem. The power set $\mathscr{P}(\mathbb{N})$ is not countable.

Power set

Proof. (by contradiction)

Assume that $\mathscr{P}(\mathbb{N})$ is countable, so all subsets of $\mathbb{N}$ can be listed:

$$A_0, A_1, A_2, \ldots$$

We know that subsets can be encoded by sitrings of 1s and 0s.

Subset	0	1	2	3	4	5	...
A_0	0	0	0	1	0	0	...
A_1	1	1	1	0	0	1	...
A_2	1	1	1	1	1	1	...
A_3	0	0	0	0	0	1	...
A_4	1	0	0	0	0	1	...
A_5	1	1	0	0	1	1	...

Now, we want to construct a counter-example subset $C \subseteq \mathbb{N}$ that is different from each A_i.

Power set

Subset	0	1	2	3	4	5	...
A_0	*0*	0	0	1	0	0	...
A_1	1	*1*	1	0	0	1	...
A_2	1	1	*1*	1	1	1	...
A_3	0	0	0	*0*	0	1	...
A_4	1	0	0	0	*0*	1	...
A_5	1	1	0	0	1	*1*	...
...							
C	1	0	0	1	1	0	...

We construct a counter-example set C that is different from each subset A_i. How can we do it?

For all $i = 0, 1, 2, 3 \dots$: Whenever $i \in A_i$, we choose $i \notin C$, and vice versa, when $i \notin A_i$, we choose $i \in C$. Thus, by construction, C is different from each A_i. Effectively, the set C inverts the diagonal.

Power set

Since $C \neq A_i$ for all i, and C is obviously a subset of $\mathbb{N}$ by construction, the list of subsets A_i does not contain all subsets of $\mathbb{N}$ (it does not contain C, for example), therefore, our assumption was incorrect: the subsets of $\mathbb{N}$ are not countable.

That is, *the power set $\mathscr{P}(\mathbb{N})$ is uncountable*. $\square$

This proof strategy is called diagonalization.

Similarly, we can show that the *unit interval* $0 \leq x \leq 1$ of real numbers is uncountable. (Also, see Rosen's book for the proof). And because you can make a bijection between this interval, $[0, 1]$, and $\mathbb{R}$, the set of all real number is uncountable.

More results about cardinality

Theorem. If A and B are countable sets, then their union $A \cup B$ is also countable.

Proof. Wihtout loss of generality, we can assume that A and B are disjoint. (If they are not, we continue the proof with A and $B \setminus A$)

If at least one of the sets is finite, we first list this set, then the other set.

Otherwise, if both are infinite countable sets, we list both sets by alternating elements:

$$a_0, b_0, a_1, b_1, a_2, b_2, \ldots$$

where $a_i \in A$ and $b_i \in B$. $\square$

Cardinality, one-to-one and onto

Mapping rules

If there is a *one-to-one* function $f : A \rightarrow B$ then

$$|A| \leq |B|.$$

If there is an *onto* function $g : A \rightarrow B$ then

$$|A| \geq |B|.$$

If there is a *bijection* $h : A \rightarrow B$ then

$$|A| = |B|.$$

Schröder-Bernstein Theorem

Theorem (Schröder-Bernstein). Given two sets A and B, if there exist one-to-one functions $f : A \to B$ and $g : B \to A$, then there is a bijection between A and B.

In other words, to prove existence of a bijection, it's enough to prove existence of two one-to-one functions:

Once you have found a one-to-one function $f : A \to B$, instead of proving that f is onto, you can prove that there exists another one-to-one function that maps B to A.