Discrete Structures. CSCI-150. Spring 2015.

Homework 10.   Due Thr. Jul 9, 2015.

Problem 1

Prove the inclusion-exclusion formula (it's an extension of the subtraction rule)

$$\begin{aligned} |A \cup B \cup C| = &|A| + |B| + |C| \\ &- |A \cap B| - |A \cap C| - |B \cap C| \\ &+ |A \cap B \cap C|. \end{aligned}$$

To do the proof, let's denote $X = A \cup B$, then

$$|(A \cup B) \cup C| = |X \cup C|,$$

and we can apply the usual subtraction rule (you will have to apply it twice).

Problem 2 (Graded)

Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ that is

(a) one-to-one, but not onto,

(b) onto, but not one-to-one,

(c) neither one-to-one, nor onto.

(d) onto and one-to-one (bijection), which is not the identity function $f(x) = x$.

(e) Function $f : \mathbb{Z} \to \mathbb{Z} \times \{0, 1, 3\}$ that is onto and one-to-one (bijection).

When constructing the functions, try to define them by formulas. (Feel free to use such operations as *absolute value*, *floor*, *ceiling*, *remainder*, in addition to normal arithmetical operations).

For each function, explain (in the best way you can) why they satisfy the required conditions.
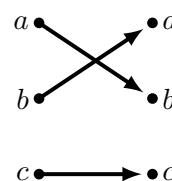
Problem 3 (Graded)

Draw the diagrams (as we did in class) for <u>all bijections</u> $f : A \to A$ when the set A is

(a) $A = \{a\}$

(b) $A = \{a, b\}$

(c) $A = \{a, b, c\}$

(d) For this question, either repeat the task for $A = \{a, b, c, d\}$, or derive a formula for the total number of bijections from A to A, when $|A| = n$. (Explain your answer).

Example of the diagram for a function

$$f : \{a, b, c\} \to \{a, b, c\}$$

Problem 4

(a) Please count how many functions

$$f \ : \ D \to \{0, 1\}$$

can be defined if the domain D is a finite set with the cardinality $|D| = n$.

(b) Can you find a bijection between the set of all such functions and the powerset $\mathcal{P}(D)$?

Problem 5 (Graded). Submodular functions and diminishing returns.

Introduction. Set functions.

Given a set A, consider a function from the powerset $\mathcal{P}(A)$ to reals, $\mathbb{R}$:

$$f : \mathcal{P}(A) \to \mathbb{R}.$$

This function assigns a numerical value to any <u>subset of A</u>. Why is it useful? Let's say that A is a set of items you want to buy. Then the function f may estimate, for example, <u>the happiness you gain</u> from having a subset of those items.

For example, let
$$A = \{ \text{ phone, charger, tea, cups, Rosen's book } \}$$

The function f may be simply equal to the total price of the items in S:

$$f_1(S) = \sum_{item \in S} \text{Price}(item)$$

However in general, f does not have to be so simple. Some objects may be only valuable together with certain other objects but not by themselves. For example:

$$f_2(\{\text{phone}\}) = 50$$
$$f_2(\{\text{charger}\}) = 0$$
$$f_2(\{\text{phone, charger}\}) = 100$$
$$f_2(\{\text{tea, cups}\}) = 80$$
$$f_2(\{\text{phone, cups}\}) = 25$$
$$f_2(\{\text{charger, tea}\}) = 15$$
$$f_2(\{\text{phone, charger, tea, cups}\}) = 150$$
$$\dots$$

Submodularity.

Def. A function $f : \mathcal{P}(A) \to \mathbb{R}$ is called <u>submodular</u> if for any two subsets $S, T \subseteq A$:

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

S   T   is better than   S∪T   S∩T
+   +
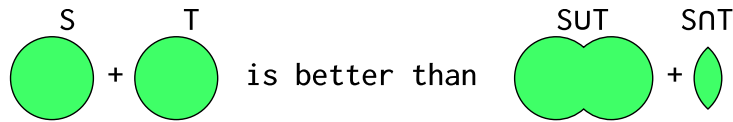
Submodular functions have a natural diminishing returns property which makes them suitable for many applications in Computer Science, Game theory, and Economics.

The diminishing returns property, in the context of submodular function, means that the difference in the value of the function that a single element makes when added to an input set decreases as the size of the input set increases. (Clear?)

Let's consider an example, if you don't know how to play piano, a single lesson may make a significant improvement to your skill. However, a professional pianist will not learn anything new from the same lesson.

Another example, Bill Gates' happiness does not increase much if he earns extra 100 dollars. However, a person with no money will become much happier in the same situation.

The returns diminish if you already have (invested) a lot. Let's prove that this property holds for submodular functions.

We are going to prove that if $X \subseteq Y$ and $a \notin Y$ then for a submodular function f the following inequality holds:
$$f(X \cup \{a\}) - f(X) \geq f(Y \cup \{a\}) - f(Y).$$

In English, if the set X is "smaller" than the set Y (i.e. $X \subseteq Y$), then the addition of a new element $\{a\}$ to X makes a bigger difference than the addition of the same element to the bigger set Y. (Here, X is somebody with no money, and Y is Bill Gates).

To prove this property, use the definition of submodular function (see the previous page). Assume that $S = X \cup \{a\}$ and $T = Y$.