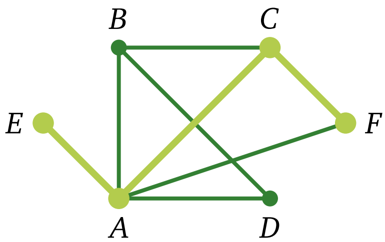
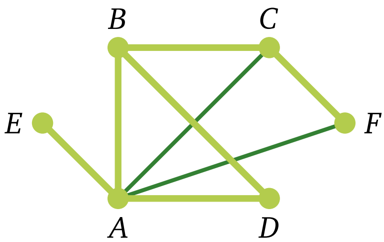
Paths. Connectivity.
Euler and Hamilton Paths.
Planar graphs.

Path





Def. A *path* from s to t is a sequence of edges

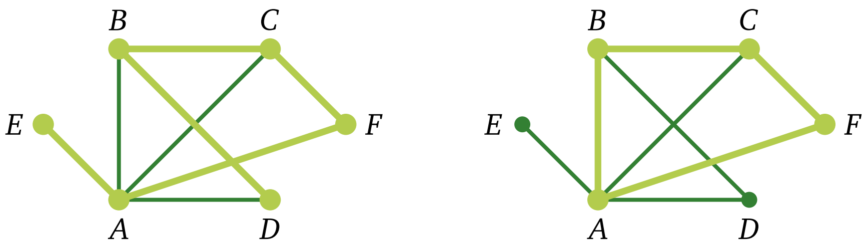
$$\{x_0, x_1\}, \ \{x_1, x_2\}, \ \ldots \ \{x_{n-1}, x_n\},$$

where $x_0 = s$, and $x_n = t$.

Def. The *length* of a path is the number of edges in it.

$$\{E,A\} \ \{A,B\} \ \{B,D\} \ \{D,A\} \ \{A,B\} \ \{B,C\} \ \{C,F\}$$

Simple path. Cycle

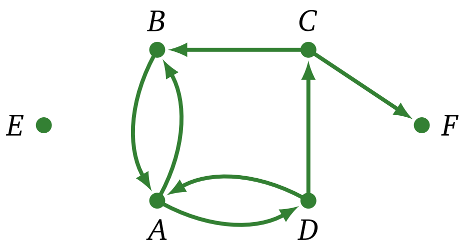


Def. A *simple path* is a path that does not contain the same edge more than once.

Def. A path is called a *cycle* (or *circuit*) if its first and last vertices are the same, and its length is greater than 0.

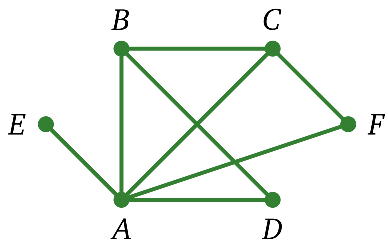
Def. A *simple sycle* is a cycle that does not contain the same edge more than once.

Paths and cycles in directed graphs?



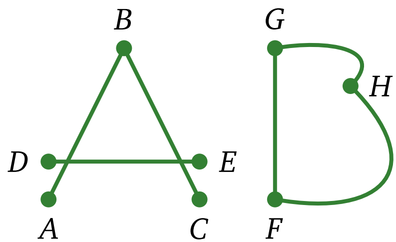
There are similar definitions for paths and cycles in directed graphs.

Connected graph



Def. An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph.

Connected compoments

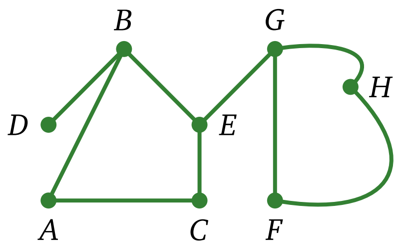


Def. A *connected component* of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G.

(So, a connected component is a maximal connected subgraph)

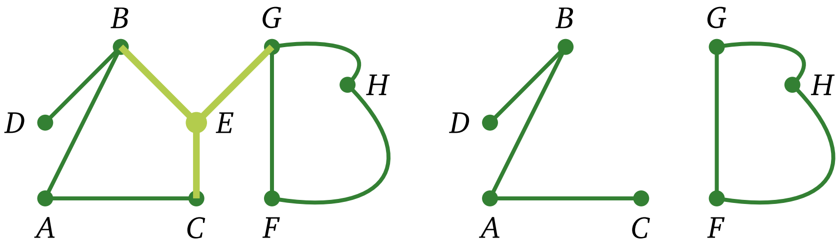
Question: How many connected components is in the graph?

Vertex cut



Def. A *vertex cut* V' is a subset of vertices, such that the graph becomes disconnected, if V' and their incident edges are removed.

Vertex cut

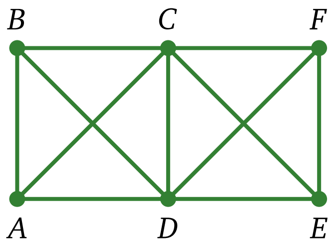


Def. A *vertex cut* V' is a subset of vertices, such that the graph becomes disconnected, if V' and their incident edges are removed.

Example: $V' = \{E\}$.

This is one of three minimum vertex cuts in this graph. Can you find the other two?

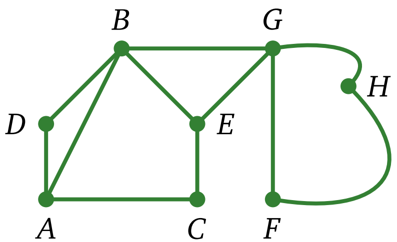
Vertex cut



Def. A *vertex cut* V' is a subset of vertices, such that the graph becomes disconnected, if V' and their incident edges are removed.

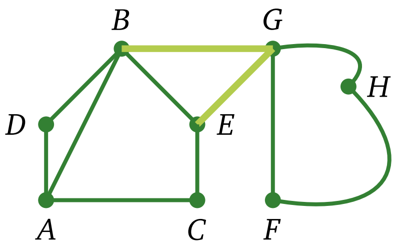
Find a vertex cut.

Edge cut



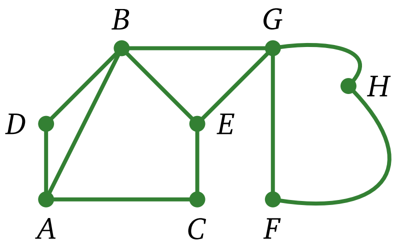
Def. An *edge cut* E' is a subset of edges, such that the graph becomes disconnected, if the edges E' are removed.

Edge cut



Def. An *edge cut* E' is a subset of edges, such that the graph becomes disconnected, if the edges E' are removed.

Distance and diameter



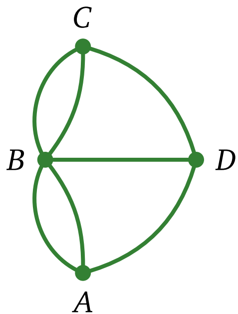
Def. The *distance* between two vertices in a graph is the length of the shortest path between them.

$$\text{distance}(A, H) = 3$$

Def. The *diameter* of a graph is the distance between the two vertices that are farthest apart.

$$\text{diameter} = 3$$

Euler path and cycle



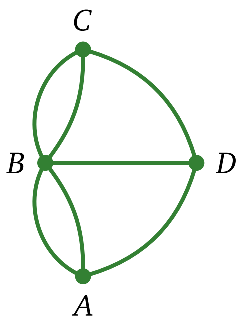
Def. An *Euler cycle* in a graph G is a simple cycle containing every edge of G.

Similarly, an *Euler path* in G is a simple path containing every edge of G.

(In a simple path (or cycle), edges are not repeated)

Euler cycle

Walk across all the bridges once.
And get back to the original location.



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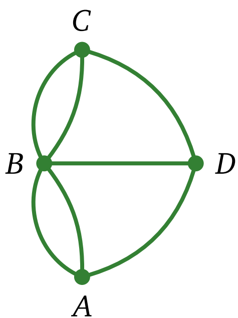
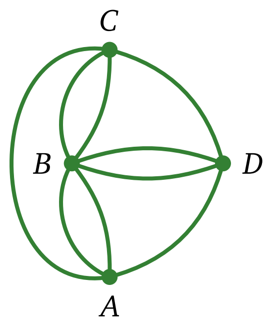
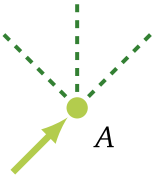




What if we build two new bridges?

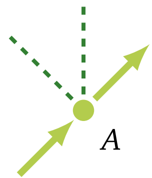
Observation



Let's say that *we cross a bridge to the vertex A*.

What is the condition to continue walking?

Observation

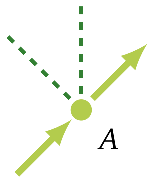


Let's say that *we cross a bridge to the vertex A*.

What is the condition to continue walking?

There should be *at least one more bridge* at the vertex A.

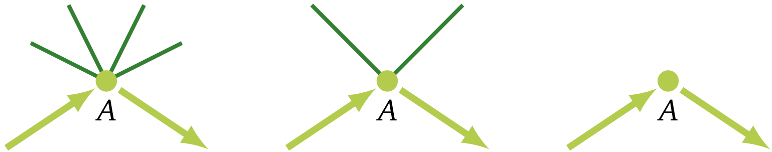
Observation



When we enter a vertex and then leave it, we use *two bridges*.

So, every time we visit a vertex, two bridges are gone.

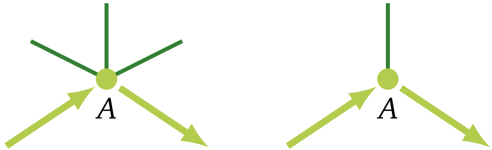
Finding an Euler cycle



If we visit a vertex, we use two bridges.

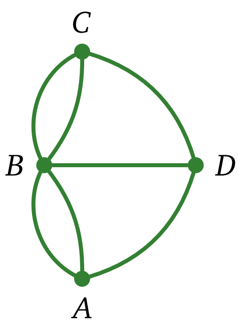
If there is an even number of bridges at the vertex A, then after our visit, there is still an even number of bridges.

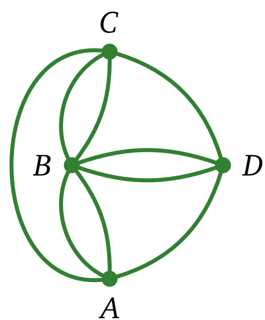
If a vertex has only one bridge, it can be only the final point in the path.



Necessary and sufficient condition for Euler cycles

Theorem. A connected multigraph with at least two vertices has an *Euler cycle* if and only if each of its vertices has *even degree*.





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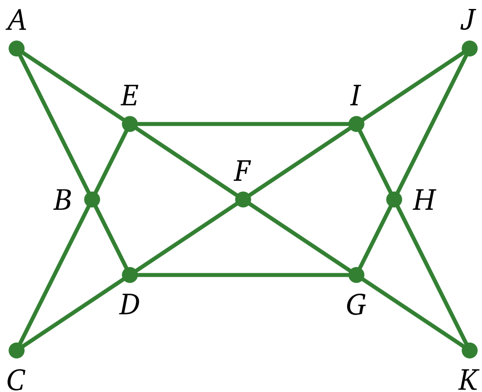
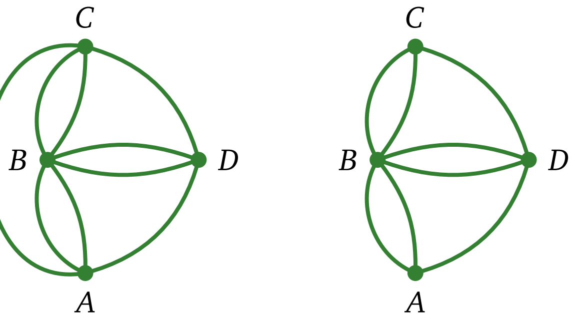


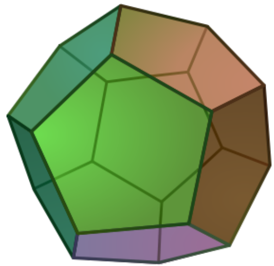
Constructing an Eulerian cycle takes linear time in the number of edges! This is efficient.

Euler path



Theorem. A connected multigraph has an *Euler path* but not an Euler cycle if and only if it has *exactly two vertices of odd degree*.

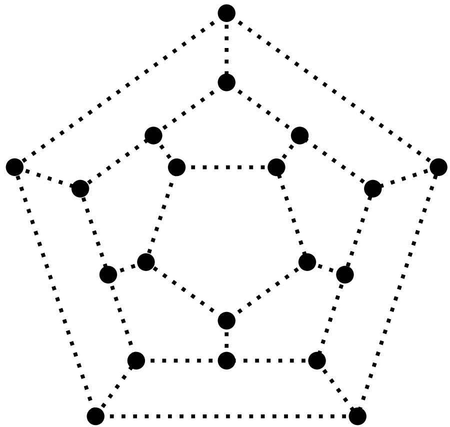
Icosian Puzzle



A puzzle invented in 1857 by *Sir William Rowan Hamilton*:

The task is to travel along the edges of a dodecahedron, visit each of 20 vertices exactly once, and end back at the first vertex.

Icosian Puzzle



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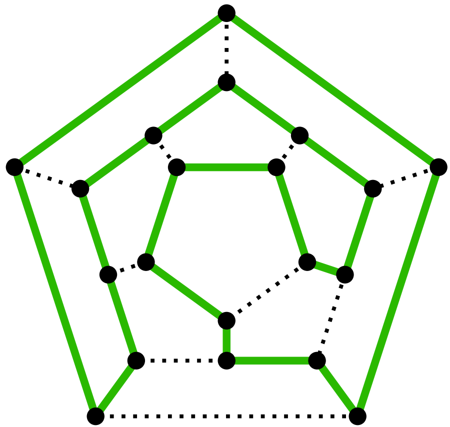

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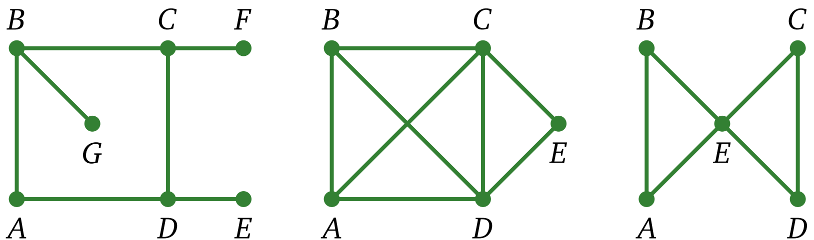
Hamilton path



Def. A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*.

And a simple cycle in a graph G that passes through every vertex exactly once is called a *Hamilton cycle*.

Sufficient conditions for a cycle

Theorem (Dirac's theorem). If G is a simple graph with n vertices with $n \geq 3$ such that

the degree of every vertex in G is at least $n/2$,

then G has a Hamilton cycle.

Theorem (Ore's theorem). If G is a simple graph with n vertices with $n \geq 3$ such that

$$\deg(u) + \deg(v) \geq n$$

for every pair of *nonadjacent* vertices u and v in G, then G has a Hamilton cycle.

Algorithm for finding a cycle?

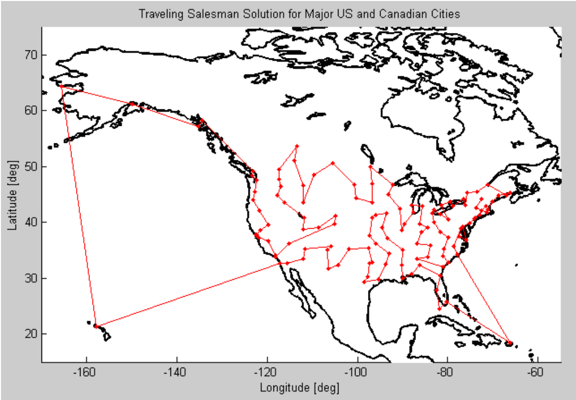
The best algorithms known for finding a Hamilton cycle in a graph or determining that no such cycle exists have *exponential worst-case time* complexity in the number of vertices of the graph.

In fact, this is an NP-complete problem.

More Hamilton cycles

The famous Traveling Salesperson Problem (TSP):

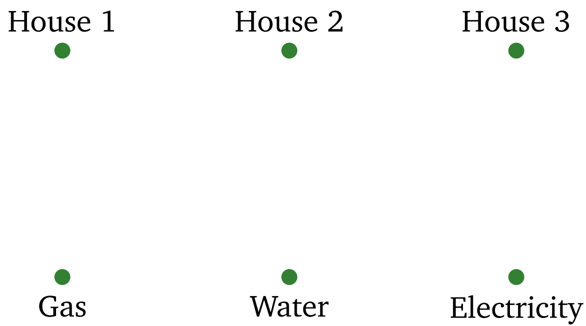
Find the shortest route a traveling salesperson should take to visit a given set of cities.



It reduces to finding a Hamilton cycle on a complete graph such that the total weight of the path is the smallest.

Planar graphs

Question: Is it possible to join these houses and utilities so that none of the connections cross?



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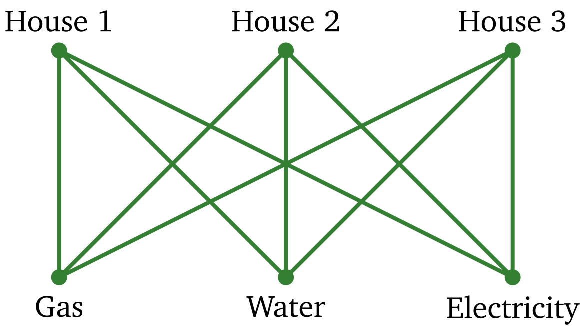
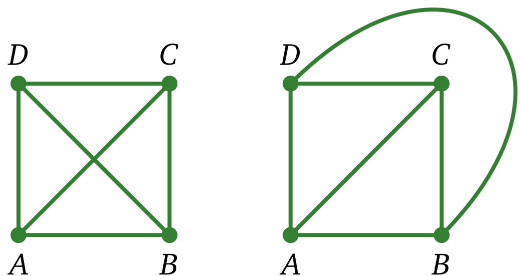


This is a complete bipartite graph, denoted by $K_{3,3}$.

Planar graphs

Def. A graph is called *planar* if it can be drawn in the plane without any edges crossing.

Complete graph K_4 is planar:



Planar graphs

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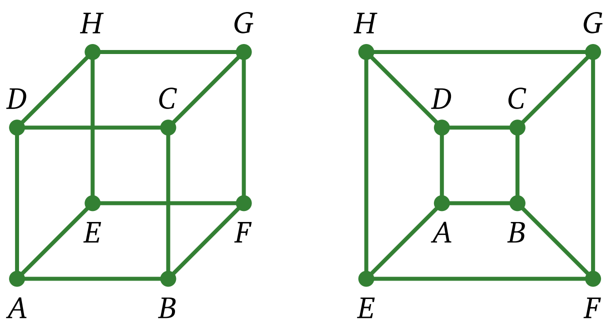
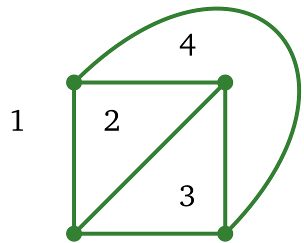
3-dimensional hypercube graph, Q_3, is planar:



Euler formula

A drawing of a planar graph divides the plane into *faces*, regions bounded by edges of the graph.

four faces:



Theorem (Euler formula). Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of faces in a planar representation of G. Then

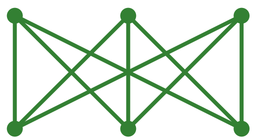
$$v - e + f = 2.$$

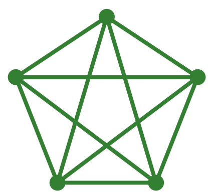
Planar graphs

Theorem (Kuratowski). A graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

What is a *subdivision*? Inserting a new vertex into an existing edge of a graph is called subdividing the edge, and one or more subdivisions of edges create a subdivision of the original graph.

$K_{3,3}$





K_5