

Permutations and Combinations. The Pigeonhole Principle.

A question

Permutations

Combinations

The Pigeonhole
Principle

We want to open a door, but it's locked.



It seems that to open the lock, we must press **10 buttons** in the right order.

Each button must be pressed exactly once.

How many combinations do we have to try?

A question

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We want to open a door, but it's locked.



It seems that to open the lock, we must press **10 buttons** in the right order.

Each button must be pressed exactly once.

How many combinations do we have to try?

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800.$$

If it takes one second to try one combination, we will need 42 days to try each.

Another question

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There are 7 persons in a team, and 7 distinct individual tasks, in how many ways the tasks can be assigned to the team members?

(Everyone gets exactly one task).

Another question

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There are 7 persons in a team, and 7 distinct individual tasks, in how many ways the tasks can be assigned to the team members?

(Everyone gets exactly one task).

$$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

Factorial function

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$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800$$

This function is called *factorial* and denoted by $n!$:

$$1! = 1$$

$$2! = 2 \cdot 1$$

$$3! = 3 \cdot 2 \cdot 1$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

...

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 1$$

and by convention,

$$0! = 1$$

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Def. A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

The number of permutations of n objects is

$$P(n) = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n!$$

So, for example, given 6 pictures of cats, there are $6! = 720$ ways to arrange them in a row.

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There are 15 different magazines on a table.



While waiting, you skimmed through each of them. In how many different orders it was possible to do?

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There are 15 different magazines on a table.



While waiting, you skimmed through each of them. In how many different orders it was possible to do?

$$15! = 1307674368000.$$

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There are 15 different magazines on a table.



You read three of them. In how many ways it was possible to do?

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There are 15 different magazines on a table.



You read three of them. In how many ways it was possible to do?

$$15 \cdot 14 \cdot 13 = 2730.$$

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An arrangement of 3 objects out of n is called a 3-permutation.

Def. An ordered arrangement of r elements from a set of n is called an *r -permutation*.

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

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An arrangement of 3 objects out of n is called a 3-permutation.

Def. An ordered arrangement of r elements from a set of n is called an *r -permutation*.

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}} = \frac{n!}{(n-r)!}$$

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Let's prove the last formula.

The number of r -permutations of the set of n elements:

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Multiply and divide by $(n-r) \cdot (n-r-1) \cdot \dots \cdot 2 \cdot 1$,

$$P(n, r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \dots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$$

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How many ways are there to select a first-prize winner and a second-prize winner from 100 different people who have entered a contest?

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How many ways are there to select a first-prize winner and a second-prize winner from 100 different people who have entered a contest?

$$P(100, 2) = 100 \cdot 99 = 9900.$$

Alternatively,

$$P(100, 2) = \frac{100!}{(100-2)!} = \frac{1 \cdot 2 \cdot \dots \cdot 100}{1 \cdot 2 \cdot \dots \cdot 98} = 99 \cdot 100 = 9900.$$

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Three sport teams of four want to take a group photo.



In how many ways can they stand in a row so that all members of the same team are standing together?

Another problem

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There are 4 paintings in a collection. In how many ways can we hang 3 of them on a wall?



Another problem

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There are 4 paintings in a collection. In how many ways can we hang 3 of them on a wall?



We can compute the number of 3-permutations of 4 objects, which is $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$.

abc	acb	bac	bca	cab	cba	$\{a, b, c\}$
abd	adb	bad	bda	dab	dba	$\{a, b, d\}$
acd	acb	cad	cda	dac	dca	$\{a, c, d\}$
bcd	$bc b$	cbd	cdb	dbc	dcb	$\{b, c, d\}$

Observe that there are 4 ways to pick a set of 3 paintings, and each of them can be arranged in $3! = 6$ many ways, and $4 \cdot 3! = 24$ too.

Another problem



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$$P(4, 3) = 24 = 4 \cdot 3!$$

When we arrange r objects out of n :

$$P(n, r) = \frac{n!}{(n-r)!}$$

But also,

$$P(n, r) = X \cdot r!$$

Where X is the number of ways to *choose r objects out of n without assigning any order* to them. This is exactly what we did when we were selecting 3 paintings.

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$$P(n, r) = \frac{n!}{(n-r)!}$$

$$P(n, r) = X \cdot r!$$

X is the number of ways to choose r objects out of n without assigning any order to them.

Knowing this X can be useful. Can we find a formula for it?

Combinations

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$$P(n, r) = \frac{n!}{(n-r)!}$$

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X is the number of ways to choose r objects out of n without assigning any order to them.

Knowing this X can be useful. Can we find a formula for it?

$$X = \frac{n!}{(n-r)! r!}$$

It is the number of so-called *r -combinations*.

Combinations

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$$P(n, r) = \frac{n!}{(n-r)!}$$

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X is the number of ways to choose r objects out of n without assigning any order to them.

Knowing this X can be useful. Can we find a formula for it?

$$X = \frac{n!}{(n-r)! r!}$$

It is the number of so-called *r -combinations*.

Def. An *r -combinations* is an unordered selection of r objects from a set of n objects.

We write

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)! r!}$$

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The number of r -combinations:

$$\binom{n}{r} = \frac{n!}{(n-r)! r!} \quad (\text{Note that } \binom{n}{r} \text{ reads as “}n \text{ choose } r\text{”}).$$

An r -combination is a selection of r objects without specifying the order in which the objects are chosen.

Exercise:

There are three paintings, you'd like to buy some of them.

In how many ways one painting can be chosen? Two? Three?

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In how many ways a team of 6 can be selected from a group of 12 students?

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In how many ways a team of 6 can be selected from a group of 12 students?

$$\binom{12}{6} =$$

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In how many ways a team of 6 can be selected from a group of 12 students?

$$\begin{aligned}\binom{12}{6} &= \frac{12!}{(12-6)! 6!} = \frac{12!}{6! \cdot 6!} = \frac{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{6!} \\ &= \frac{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{7 \cdot 11 \cdot 12}{1} = 924.\end{aligned}$$

Example with cards

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Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

Example with cards

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Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)! 5!} = \frac{52!}{47! 5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

Example with cards

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Principle

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)! 5!} = \frac{52!}{47! 5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

More examples:

$$\binom{52}{2} = \frac{51 \cdot 52}{2} = 1326. \quad \binom{52}{1} = 52.$$

Combinations

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Let's try to prove that for $1 \leq k \leq n$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Summary

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Given a set with n elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1$$

The number of *r-permutations* of the set:

$$P(n, r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *r-combinations* (“ n choose r ”):

$$\binom{n}{r} = \frac{P(n, r)}{P(r)} = \frac{n!}{(n-r)! r!}$$

Solve

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Problem 1.

How many permutations of the letters ABCDEFG contain

- (a) the string BCD?
- (b) the string CFGA?
- (c) the strings BA and GF?
- (d) the strings BAC and CED?

Solve

Permutations

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Problem 2.

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Problem 3.

How many bit strings contain exactly 8 zeros and 10 ones?

Problem 4.

How many bit strings contain exactly 8 zeros and 10 ones if every zero must be immediately followed by a one?

Solve: “Knights of the Round Table”

Permutations

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Def. A *circular permutation* of n people is their seating around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

Def. Similarly, a *circular r -permutation* of n people is a seating of r of these n people around a circular table, where, again, the seatings that can be obtained by rotation are considered to be the same.

Problem 5.

In how many ways can King Arthur seat n different knights at his round table?

Problem 6.

Count the number of circular r -permutations of n people.

“Knights of the Round Table”

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Problem 5. In how many ways can King Arthur seat n different knights at his round table?

Answer: Because n normal permutations result in a single circular permutation, $\frac{n!}{n} = (n-1)!$

Problem 6.

Count the number of circular r -permutations of n people.

Answer: r normal r -permutations result in one circular r -permutation, so we get $\frac{n!}{(n-r)! r}$. Transform this formula:

$$\frac{n!}{(n-r)! r} = \frac{n!}{(n-r)! r!} \cdot \frac{r!}{r} = \binom{n}{r} \cdot \frac{r!}{r}$$

So, equivalently, we first choose r knights out of n , and then count their circular permutations.

A typical situation

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A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?



A typical situation

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Principle

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?



Four is enough.

The Pigeonhole Principle

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If you have 10 pigeons in 9 boxes, at least one box contains two birds.

The pigeonhole principle. If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

The Pigeonhole Principle

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Principle

There are 7 colleges in a city.

What is the minimum number of students to be invited for a survey to make sure that at least two are from the same college?

The Pigeonhole Principle

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Principle

There are 7 colleges in a city.

What is the minimum number of students to be invited for a survey to make sure that at least two are from the same college?

$$7 + 1 = 8 \text{ students.}$$

Generalized Pigeonhole Principle

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The ceiling function:

$\lceil x \rceil$ = the smallest integer not less than x

So, for example,

$$\lceil 2.0 \rceil = 2$$

$$\lceil 0.5 \rceil = 1$$

$$\lceil -3.5 \rceil = -3$$

The Generalized Pigeonhole Principle.

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Generalized Pigeonhole Principle

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The Generalized Pigeonhole Principle.

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Generalized Pigeonhole Principle

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The Generalized Pigeonhole Principle.


If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

“Birds” are cards. “Boxes” are suits, $k = 4$.

How many cards, N , should we take to guarantee that at least three of them fall in the same “box” (suit):

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{N}{4} \right\rceil \geq 3 > 2.$$

The smallest possible $N = 2 \cdot 4 + 1 = 9$. 

Solve

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Principle

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.