Discrete Structures. CSCI-150. Spring 2015.

Homework 4. Due Mon. Jun. 15, 2015.

Problem 1

Ellen draws 5 cards from a standard deck of 52 cards.

(a) In how many ways can her selection result in a hand with no clubs?

(b) A hand with at least one club?

Problem 2

How many bit strings of length 10 contain at least three 1s and at least three 0s?
(Answer: 912).

Problem 3 (Graded)

How many bit strings contain exactly seven 0s and nine 1s such that every 0 is immediately followed by a 1?

Hint: Observe that "0" can be found in a string only as a combination "01". So, in some sense it's similar to the problem with the blocks of letters, but it's not exactly like that problem.

Problem 4 (Graded)

A computer science professor has nine different programming books on a bookshelf. Four of the books deal with the programming language C++, the other five with LISP. In how many ways can the professor arrange these books on the shelf

(a) if there are no restrictions?

(b) if the languages should alternate?

(c) if all the C++ books must be next to each other?

(d) if all the C++ books must be next to each other and all the LISP books must be next to each other?

Note that all the books are distinct. Not for grade, you may consider the case when two of the C++ books are identical copies. Then, what if all the C++ books and all the LISP books are identical copies?

Proving identities and Double counting

Problem 5 (Graded)

In this problem, you have to find two proofs for the following identity:

$$(2n)! = \binom{2n}{n} \cdot (n!)^2$$

(a) The first proof is algebraic. Using the fact that $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$, show that the equation is always true. It involves some factorial manipulation, but almost everything should cancel out.

Remember: when proving the identity (or anything else in general), don't prove it "backwards", it's a logically inconsistent and faulty technique.

You may consider the left-hand side and the right-hand side separately, showing that they are equal to the same formula. *However, don't make it look like a "backwards" proof!*

(b) For the second part, prove the same identity using the technique called "Double counting" or "Combinatorial argument". It's a combinatorial proof technique for showing that two expressions are equal by demonstrating that they are two ways of counting the size of one set.

In this particular case, show that two formulas: $(2n)!$ and $\binom{2n}{n} \cdot (n!)^2$ describe two counting procedures that count the same set.

A hint: We know that the first formula, $(2n)!$, counts the number of ways to order $2n$ objects. Show that the second formula, $\binom{2n}{n} \cdot (n!)^2$, counts the same thing.

Problem 6

Find "double counting" proofs for the following identities:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

$$\binom{n-1+2}{2} = n + \binom{n}{2}$$

$$\binom{n-1+3}{3} = n + \binom{n}{2} \cdot 2 + \binom{n}{3}$$

If you try proving the last two identities, think of selection with repetition.

Pigeonhole principle

Problem 7

Prove that among any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair at the distance not greater than 1/2.

Problem 8 (Graded)

In a room there are 10 people, none of them is younger than 1 or older than 100 years. Prove that among them, one can always find two groups of people (possibly intersecting, but different) the sums of whose ages are the same.

Hint: Use the pigeonhole principle. How many different groups of people can be picked? How large the sum of their ages can be?

Selection with repetition

In the following two problems, you have to use the selection with repetition formula. Please read the book about this topic.

Problem 9 (Graded)

In how many ways can 101 identical computers be distributed among 5 computer stores if

(a) there are no restrictions?

(b) each store gets at least one?

(c) the largest store gets at least 51?

(d) each store gets at least 20?

Problem 10

Find the number of integer solutions to the equation

$$w + x + y + z = 20,$$

where the variables are positive integers.