# Satisfiability.
# Rules of Inference.

# Satisfiability



**Def.** *A proposition is* satisfiable *if some setting of the variables makes the proposition true.*

For example, $p \wedge \neg q$ is satisfiable because the expression is true when $p$ is true and $q$ is false.

# Satisfiability



Determining whether or not a complicated proposition is satisfiable is not so easy.

How about this one?

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q) \wedge (\neg p \vee \neg r) \wedge (\neg r \vee \neg q)$$

The general problem of deciding whether a proposition is satisfiable is called *SAT*. One approach to SAT is to construct a truth table and check whether or not a "$T$" ever appears.

But this approach is not very efficient; a proposition with $n$ variables has a truth table with $2^n$ lines. For a proposition with just 30 variables, that's already over a billion!

*Is there an efficient solution to SAT?*

# Satisfiability



*Is there an efficient solution to SAT?*

No one knows.

An efficient solution to SAT would immediately imply efficient solutions to many, many other important problems involving packing, scheduling, routing, and circuit verification. Decrypting coded messages would also become an easy task (for most codes).

# Tautology and contradiction



**Def.** *A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a* tautology.

Example: $p \vee \neg p$.

| $p$ | $\neg p$ | $p \vee \neg p$ |
|---|---|---|
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |

Example: $p \wedge q \to p$.

| $p$ | $q$ | $p \wedge q$ | $p \wedge q \to p$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ |

**Def.** *A compound proposition that is always false is called a* contradiction.

# New problem



Given a list of true propositions $\left\{ \begin{array}{c} p \rightarrow r \\ \neg p \rightarrow q \\ q \rightarrow s \end{array} \right.$

Need to prove the last one ☞ $\neg r \rightarrow s$

$$\frac{\begin{array}{c} p \rightarrow r \\ \neg p \rightarrow q \\ q \rightarrow s \end{array}}{\neg r \rightarrow s}$$

# Truth tables again?



Truth tables are great[1], but, fortunately for us, more interesting techniques exist.

To prove that a compound proposition is true, we can build an argument, a sequence of true propositions that leads to the proposition we need.

There are inference rules that help us deduce new true propositions.

[1] ha-ha, exponential ($2^n$) table size is not great at all

# Building a formal argument



Given

$$p \to r$$
$$\neg p \to q$$
$$q \to s$$

Derived new true propositions

$$\cdots$$
$$\cdots$$
$$\cdots$$
$$\cdots$$

Need to prove ☞

$$\neg r \to s$$

# Example



Let's first take just one inference rule.

And solve a small problem, using this rule.

# Inference Rules #1.



$$\frac{\begin{array}{c} x \\ x \rightarrow y \end{array}}{y}$$

Meaning:

$$\frac{\begin{array}{c} \text{It is snowing today.} \\ \text{If it snows today, then we will go skiing.} \end{array}}{\text{We will go skiing.}}$$

# Building an argument



Let's prove

$$\begin{array}{c} p \\ p \rightarrow r \\ r \rightarrow s \\ \hline s \end{array}$$

It means that we have to prove that $s$ is true, given that propositions $p$, $(p \rightarrow r)$, and $(r \rightarrow s)$ are true.

# Building an argument



Let's prove

$$\begin{array}{c} p \\ p \rightarrow r \\ r \rightarrow s \\ \hline s \end{array}$$

Our argument:

| | | |
|---|---|---|
| (1) | $p$ | Given. |
| (2) | $p \rightarrow r$ | Given. |
| (3) | $r \rightarrow s$ | Given. |

$\ldots$

The rule

$$\begin{array}{c} x \\ x \rightarrow y \\ \hline y \end{array}$$

# Building an argument



Let's prove

$$\frac{\begin{array}{c} p \\ p \rightarrow r \\ r \rightarrow s \end{array}}{s}$$

The rule

$$\frac{\begin{array}{c} x \\ x \rightarrow y \end{array}}{y}$$

Our argument:

| | | |
|---|---|---|
| (1) | $p$ | Given. |
| (2) | $p \rightarrow r$ | Given. |
| (3) | $r \rightarrow s$ | Given. |
| (4) | $r$ | from (1) and (2). |

...

# Building an argument



Let's prove

$$\begin{array}{c} p \\ p \rightarrow r \\ r \rightarrow s \\ \hline s \end{array}$$

The rule

$$\begin{array}{c} x \\ x \rightarrow y \\ \hline y \end{array}$$

Our argument:

| | | |
|---|---|---|
| (1) | $p$ | Given. |
| (2) | $p \rightarrow r$ | Given. |
| (3) | $r \rightarrow s$ | Given. |
| (4) | $r$ | from (1) and (2). |
| (5) | $s$ | from (3) and (4). |

# How to prove an inference rule?



$$\frac{\begin{array}{c} x \\ x \rightarrow y \end{array}}{y}$$

Inference rules must be always true. In other words, to prove it, we have to show that the conjunction of the premises always implies the conclusion:

$$(x \wedge (x \rightarrow y)) \rightarrow y \quad \text{is a tautology (always true).}$$

| $x$ | $y$ | $x \rightarrow y$ | $x \wedge (x \rightarrow y)$ | $(x \wedge (x \rightarrow y)) \rightarrow y$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |

# Rule 1. Or-Introduction (∨-I)



$$\frac{p}{p \lor q} \qquad \text{“}\lor\text{-I”}$$

Example:

$$\frac{\text{It is sunny.}}{\text{It is sunny or math is hard.}}$$

# Rule 2. And-Introduction (∧-I)



$$\frac{\begin{array}{c} p \\ q \end{array}}{p \wedge q} \quad \text{“}\wedge\text{-I”}$$

Example:

$$\frac{\begin{array}{c} \text{People like cats.} \\ \text{People like dogs.} \end{array}}{\text{People like dogs and cats.}}$$

# Rule 3. And-Elimination ($\wedge$-E)



$$\frac{p \wedge q}{p} \quad \text{“}\wedge\text{-E”}$$

Example:

$$\frac{\text{Alice sent a message to Bob, but Bob did not receive anything.}}{\text{Bob did not receive anything.}}$$

# Rule 4. “Modus Ponens”



$$\frac{\begin{array}{c} p \\ p \to q \end{array}}{q} \qquad \text{“MP”}$$

Example:

$$\frac{\begin{array}{c} \text{It is snowing today.} \\ \text{If it snows today, then we will go skiing.} \end{array}}{\text{We will go skiing.}}$$

# Rule 5. "Modus Tollens"



$$\frac{\begin{array}{c}\neg q \\ p \rightarrow q\end{array}}{\neg p} \qquad \text{"MT"}$$

Example:

$$\frac{\begin{array}{c}\text{I don't need an umbrella.} \\ \text{When it rains, I need an umbrella.}\end{array}}{\text{It is not raining.}}$$

# Rule 6. "Hypothetical Syllogism"



$$\frac{\begin{array}{c} p \rightarrow q \\ q \rightarrow r \end{array}}{p \rightarrow r} \qquad \text{“HS”}$$

Example:

$$\frac{\begin{array}{c} \text{If you want to get an A, get ready for the exam.} \\ \text{To get ready for the exam, do your homeworks.} \end{array}}{\text{If you want to get an A, do your homeworks.}}$$

# Rule 7. “Disjunctive syllogism”



$$\frac{\begin{array}{c} p \lor q \\ \neg q \end{array}}{p} \quad \text{“DS”}$$

Example:

$$\frac{\begin{array}{c} \text{There is too many people in the office, or the AC is broken.} \\ \text{There is not too many people.} \end{array}}{\text{The AC is broken.}}$$

# Rule 8. “Resolution”



$$\frac{\begin{array}{c} p \lor q \\ \neg p \lor r \end{array}}{q \lor r} \quad \text{“Res”}$$

Example:

$$\frac{\begin{array}{c}\text{The list is empty, or the variable is a number.}\\ \text{The list is not empty, or the variable is an array.}\end{array}}{\text{The variable is a number, or it is an array.}}$$

$$\frac{p}{p \vee q} \quad \text{"}\vee\text{-I"}$$

$$\frac{\neg q \quad p \to q}{\neg p} \quad \text{"MT"}$$

$$\frac{p \quad q}{p \wedge q} \quad \text{"}\wedge\text{-I"}$$

$$\frac{p \to q \quad q \to r}{p \to r} \quad \text{"HS"}$$

$$\frac{p \wedge q}{p} \quad \text{"}\wedge\text{-E"}$$

$$\frac{p \vee q \quad \neg q}{p} \quad \text{"DS"}$$

$$\frac{p \quad p \to q}{q} \quad \text{"MP"}$$

$$\frac{p \vee q \quad \neg p \vee r}{q \vee r} \quad \text{"Res"}$$

# Example 1



Prove

$$\frac{p \wedge q}{p \vee q}$$

| | | |
|---|---|---|
| (1) | $p \wedge q$ | Given. |
| (2) | $p$ | 1, $\wedge$-E. |
| (3) | $p \vee q$ | 2, $\vee$-I. |

# Example 2



Prove

$$\frac{\begin{array}{c} p \wedge q \\ p \longrightarrow r \end{array}}{r}$$

| | | |
|---|---|---|
| (1) | $p \wedge q$ | Given. |
| (2) | $p \longrightarrow r$ | Given. |
| (3) | $p$ | 1, $\wedge$-E. |
| (4) | $r$ | 2, 3, MP. |

# Example 3



Prove

$$\begin{array}{c} p \to r \\ \neg p \to q \\ q \to s \\ \hline \neg r \to s \end{array}$$

| | | |
|---|---|---|
| (1) | $p \to r$ | Given. |
| (2) | $\neg p \to q$ | Given. |
| (3) | $q \to s$ | Given. |
| (4) | $\neg p \vee r$ | Equivalent to (1). |
| (5) | $\neg p \to s$ | 2, 3, HS. |
| (6) | $p \vee s$ | Equivalent to (5). |
| (7) | $r \vee s$ | 4, 5, Res |
| (8) | $\neg r \to s$ | Equivalent to (7). |

# Proof by contradiction



To proof by contradiction, we assume $p$, and produce the argument in such a way that at some point we obtain a contradiction[1] (for example, $p \wedge \neg p$).

If we inferred a contradiction, our assumed premise $p$ was false, therefore, its negation $\neg p$ is true.

$$\frac{\text{assuming } p\text{, we infer a contradiction}}{\neg p}$$

[1] by definition, a compound proposition that is always false

# Example 4



Prove

$$\frac{\begin{array}{c} p \to q \\ \neg q \end{array}}{\neg p}$$

| | | | |
|---|---|---|---|
| (1) | $p \to q$ | Given. | |
| (2) | $\neg q$ | Given. | |
| (3) | $p$ | Assume. | \| |
| (4) | $q$ | 1, 3, MP. | \| |
| (5) | $\neg q \wedge q$ | 2, 4, $\wedge$-I. | \| |
| (6) | $\neg p$ | 3–5, by contradiction | |

# Example 5



Prove

$$\frac{p \rightarrow q}{\neg(p \wedge \neg q)}$$

| | | | |
|---|---|---|---|
| (1) | $p \rightarrow q$ | Given. | |
| (2) | $p \wedge \neg q$ | Assume. | \| |
| (3) | $\neg q$ | 2, $\wedge$-E. | \| |
| (4) | $p$ | 2, $\wedge$-E. | \| |
| (5) | $q$ | 1, 3, MP. | \| |
| (6) | $\neg q \wedge q$ | 3, 5, $\wedge$-I. | \| |
| (7) | $\neg(p \wedge \neg q)$ | 2–6, by contradiction | |

# Deduction Theorem ($\rightarrow$-I)



Another interesting rule, is Deduction theorem. This rule says that by assuming $p$ and then deriving $q$, we prove implication $p \rightarrow q$.

$$\frac{\text{assuming } p\text{, we infer } q}{p \rightarrow q} \quad \text{“}{\rightarrow}\text{-I”}$$

Deduction theorem can be called Implication-Introduction. In this sense, Modus Ponens can be called Implication-Elimination.

# Example 6



Prove Hypothetical Syllogism

$$\frac{\begin{array}{c} p \to q \\ q \to r \end{array}}{p \to r}$$

| | | | |
|---|---|---|---|
| (1) | $p \to q$ | Given. | |
| (2) | $q \to r$ | Given. | |
| | | | |
| (3) | $p$ | Assume. | \| |
| (4) | $q$ | 1, 3, MP. | \| |
| (5) | $r$ | 2, 4, MP. | \| |
| (6) | $p \to r$ | 3–5, $\to$-I. | |

# Example 7



Prove

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \\ \hline u \end{array}$$

# Example 7



Prove

$$\begin{array}{c} p \to q \\ q \to (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \\ \hline u \end{array}$$

| | | |
|---|---|---|
| (1) | $p \to q$ | Given. |
| (2) | $q \to (r \wedge s)$ | Given. |
| (3) | $\neg r \vee (\neg t \vee u)$ | Given. |
| (4) | $p \wedge t$ | Given. |
| (5) | $p$ | 4, $\wedge$-E. |
| (6) | $t$ | 4, $\wedge$-E. |
| (7) | $q$ | 1, 5, M.P. |
| (8) | $r \wedge s$ | 3, 7, M.P. |
| (9) | $r$ | 8, $\wedge$-E |
| (10) | $\neg(\neg r)$ | Equivalent to (9) |
| (11) | $\neg t \vee u$ | 3, 10, D.S. |
| (12) | $u$ | 6, 11, D.S. |

# Example 8



Prove

$$\begin{array}{c} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \\ \hline p \end{array}$$

# Example 8



Prove

$$\begin{array}{c} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \\ \hline p \end{array}$$

| | | |
|---|---|---|
| (1) | $(\neg p \vee \neg q) \rightarrow (r \wedge s)$ | Given. |
| (2) | $r \rightarrow t$ | Given. |
| (3) | $\neg t$ | Given. |
| | | |
| (4) | $\neg p$ | Assume |
| (5) | $\neg p \vee \neg q$ | 1, 4, $\vee$-I |
| (6) | $r \wedge s$ | 1, 5, M.P. |
| (7) | $r$ | 6, $\wedge$-E. |
| (8) | $t$ | 2, 7, M.P. |
| (9) | $\neg t \wedge t$ | 3, 8, $\wedge$-I. |
| (10) | $p$ | 4–9, by contradiction |