Permutations and Combinations. The Pigeonhole Principle.

A question

We want to open a door, but it's locked.

It seems that to open the lock, we must press **10 buttons** in the right order.
Each button must be pressed exactly once.

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$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800.$$

If it takes one second to try one combination, we will need 42 days to try each.

Another question

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$$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

Factorial function

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3628800$$

This function is called *factorial* and denoted by $n!$:

$$
\begin{aligned}
&1! = 1\\
&2! = 2 \cdot 1\\
&3! = 3 \cdot 2 \cdot 1\\
&4! = 4 \cdot 3 \cdot 2 \cdot 1\\
&\ldots\\
&n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1\\
&\quad \text{and by convention,}\\
&0! = 1
\end{aligned}
$$

Permutations

Def. A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

The number of permutations of n objects is

$$P(n) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$$

So, for example, given 6 pictures of cats, there are $6! = 720$ ways to arrange them in a row.

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While waiting, you skimmed through each of them. In how many different orders it was possible to do?

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$$15! = 1307674368000.$$

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$$15 \cdot 14 \cdot 13 = 2730.$$

Permutations

An arrangement of 3 objects out of n is called a 3-permutation.

Def. An ordered arrangement of r elements from a set of n is called an *r-permutation*.

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

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Permutations

Let's prove the last formula.

The number of r-permutations of the set of n elements:

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Multiply and divide by $(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1$,

$$P(n,r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$$

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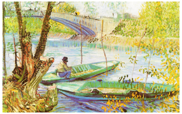
$$P(100, 2) = 100 \cdot 99 = 9900.$$

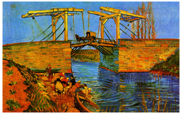
Alternatively,

$$P(100, 2) = \frac{100!}{(100-2)!} = \frac{1 \cdot 2 \cdot \ldots \cdot 100}{1 \cdot 2 \cdot \ldots \cdot 98} = 99 \cdot 100 = 9900.$$

Another problem

There are 4 paintings in a collection. In how many ways can we hang 3 of them on a wall?





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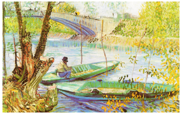
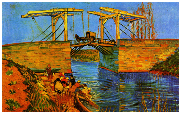
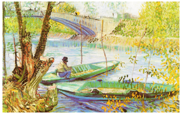
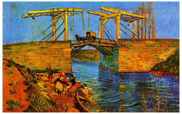
   

We can compute the number of 3-permutations of 4 objects, which is $P(4,3) = 4 \cdot 3 \cdot 2 = 24$.

$$\begin{array}{cccccccc}
abc & acb & bac & bca & cab & cba & & \{a,\ b,\ c\} \\
abd & adb & bad & bda & dab & dba & & \{a,\ b,\ d\} \\
acd & acb & cad & cda & dac & dca & & \{a,\ c,\ d\} \\
bcd & bcb & cbd & cdb & dbc & dcb & & \{b,\ c,\ d\}
\end{array}$$

Observe that there are 4 ways to pick a set of 3 paintings, and each of them can be arranged in $3! = 6$ many ways, and $4 \cdot 3! = 24$ too.

Another problem

$$P(4,3) = 24 = 4 \cdot 3!$$

When we arrange r objects out of n:

$$P(n,r) = \frac{n!}{(n-r)!}$$

But also,

$$P(n,r) = X \cdot r!$$

Where X is the number of ways to *choose r objects out of n without assigning any order* to them. This is exactly what we did when we were selecting 3 paintings.

Combinations

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Def. An *r-combinations* is an unordered selection of r objects from a set of n objects.

We write $$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

(Note that $\binom{n}{r}$ reads as "n choose r").

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

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$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

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More examples:

$$\binom{52}{2} = \frac{51 \cdot 52}{2} = 1326. \qquad \binom{52}{1} = 52.$$

Summary

Given a set with n elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

The number of *r-permutations* of the set:

$$P(n,r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *r-combinations* (“n choose r”):

$$\binom{n}{r} = \frac{P(n,r)}{P(r)} = \frac{n!}{(n-r)!\,r!}$$

Solve

Problem 1.

How many permutations of the letters ABCDEFG contain

(a) the string BCD?

(b) the string CFGA?

(c) the strings BA and GF?

(d) the strings BAC and CED?

Solve

Problem 2.

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Problem 3.

How many bit strings contain exactly 8 zeros and 10 ones?

Problem 4.

How many bit strings contain exactly 8 zeros and 10 ones if every zero must be immediately followed by a one?

Solve: “Knights of the Round Table”

Def. A *circular permutation* of n people is their seating around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

Def. Similarly, a *circular r-permutation* of n people is a seating of r of these n people around a circular table, where, again, the seatings that can be obtained by rotation are considered to be the same.

Problem 5.

In how many ways can King Arthur seat n different knights at his round table?

Problem 6.

Count the number of circular r-permutations of n people.

“Knights of the Round Table”

Problem 5. In how many ways can King Arthur seat n different knights at his round table?

Answer: Because n normal permutations result in a single circular permutation, $\frac{n!}{n} = (n-1)!$

Problem 6.

Count the number of circular r-permutations of n people.

Answer: r normal r-permutations result in one circular r-permutation, so we get $\frac{n!}{(n-r)!\,r}$. Transform this formula:

$$\frac{n!}{(n-r)!\,r} = \frac{n!}{(n-r)!\,r!} \cdot \frac{r!}{r} = \binom{n}{r} \cdot \frac{r!}{r}$$

So, equivalently, we first choose r knights out of n, and then count their circular permutations.

A typical situation

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Four is enough.

The Pigeonhole Principle

If you have 10 pigeons in 9 boxes, at least one box contains two birds.

The pigeonhole principle. If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

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What is the minimum number of students to be invited for a survey to make sure that at least two are from the same college?

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$$7 + 1 = 8 \text{ students.}$$

Generalized Pigeonhole Principle

The ceiling function:

$$\lceil x \rceil = \text{the smallest integer not less that that } x$$

So, for example,

$$\lceil 2.0 \rceil = 2$$

$$\lceil 0.5 \rceil = 1$$

$$\lceil -3.5 \rceil = -3$$

The Generalized Pigeonhole Principle.

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

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Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

“Birds” are cards. “Boxes” are suits, $k = 4$.
How many cards, N, should we take to guarantee that at least three of them fall in the same “box” (suit):

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{N}{4} \right\rceil \geq 3 > 2.$$

The smallest possible $N = 2 \cdot 4 + 1 = 9$. $\clubsuit\clubsuit\diamondsuit\diamondsuit\heartsuit\heartsuit\spadesuit\spadesuit$

Solve

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.