Relations. Functions.
Bijection and counting.

Subsets of the Catresian product

Given two sets

$$A = \{1, 2, 3\}$$
$$B = \{1, 2, 3, 4\}$$

Their Cartesian product

$$\begin{aligned} A \times B = \{&(1,1),\ (2,1),\ (3,1),\\ &(1,2),\ (2,2),\ (3,2),\\ &(1,3),\ (2,3),\ (3,3),\\ &(1,4),\ (2,4),\ (3,4)\} \end{aligned}$$

Consider three subsets of $A \times B$:

$$R_{(id)} = \{(1,1),\ (2,2),\ (3,3)\}$$
$$R_{(less)} = \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}$$
$$R_{(inc)} = \{(1,2),\ (2,3),\ (3,4)\}$$

Relations

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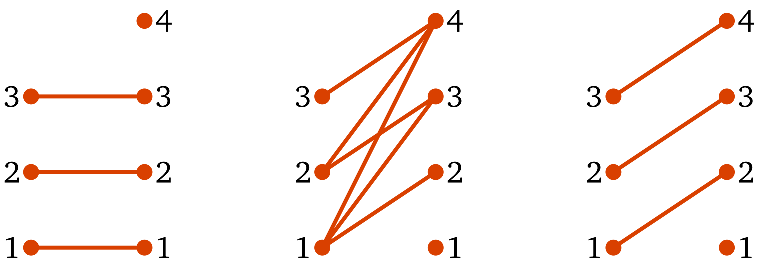
Def. A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B.

$$R_{(id)} = \{(1,1),\ (2,2),\ (3,3)\}$$
$$R_{(less)} = \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}$$
$$R_{(inc)} = \{(1,2),\ (2,3),\ (3,4)\}$$



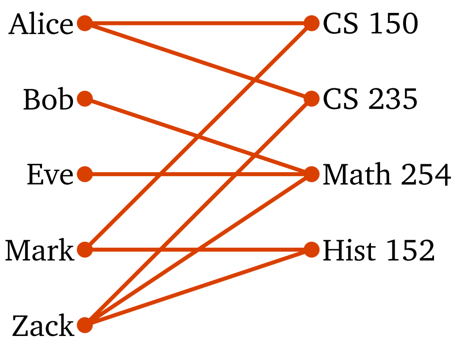
Relations

Example of a relation:

$$S = \text{set of students}$$

$$C = \text{set of classes}$$

$$R = \{(s, c) \mid \text{student } s \text{ takes class } c\}$$



Functions

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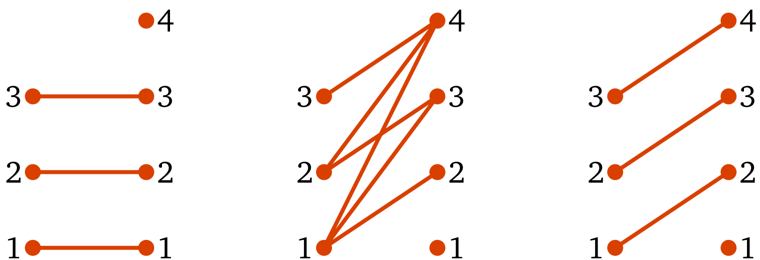
Def. A relation $R \subseteq A \times B$ is a *function* (a functional relation) if for every $a \in A$, there is at most one $b \in B$ so that $(a, b) \in R$.

$$R_{(id)} = \{(1,1),\ (2,2),\ (3,3)\}$$
$$R_{(less)} = \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}$$
$$R_{(inc)} = \{(1,2),\ (2,3),\ (3,4)\}$$



Functions

Functional relation $R \subseteq A \times B$ defines a unique way to map each element from the set A to an element from the set B.

There is a well-known and convenient notation for functions:

$$f(a) = b \qquad \text{where } a \in A \text{ and } b \in B$$

It maps elements from A to B:

$$f : A \longrightarrow B$$

$$A \xrightarrow{f} B$$

Functions

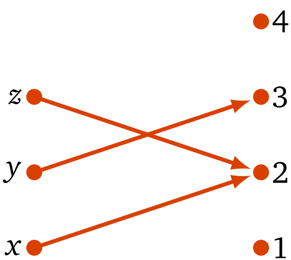
Def. For the function $f : A \to B$, set A is called *domain*, and set B is called *codomain*.

Def. $f(a)$ is the *image* of $a \in A$.

Def. The *image* of f, denoted by $f(A)$, is the set of the images $f(a)$ for all $a \in A$

$$f(A) = \{x \mid \exists a \in A\,(f(a) = x)\}.$$

The image of a function is also called *range*.



$$f : A \to B$$
$$\text{domain}(f) = A = \{x, y, z\}$$
$$\text{codomain}(f) = B = \{1, 2, 3, 4\}$$
$$f(A) = \{2, 3\}$$

Onto

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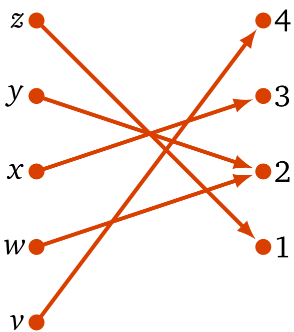
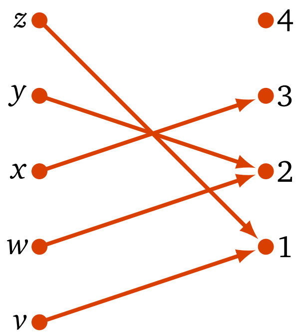
Def. A function $f : A \to B$ is called *onto* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

In other words, the image $f(A)$ is the whole codomain B.

$f : A \to B$ is onto



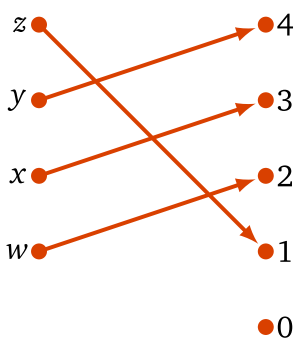
$g : A \to B$ is not onto



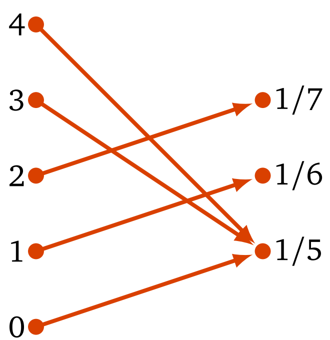
One-to-one

Def. A function $f : A \to B$ is said to be *one-to-one* if and only if $f(x) = f(y)$ implies that $x = y$ for all $x, y \in A$.

$f : A \to B$ is one-to-one



$g : B \to C$ is not one-to-one



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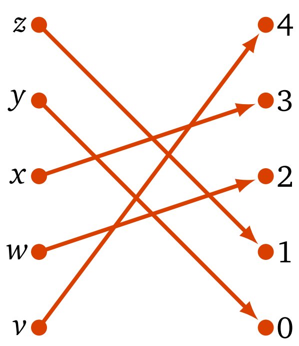
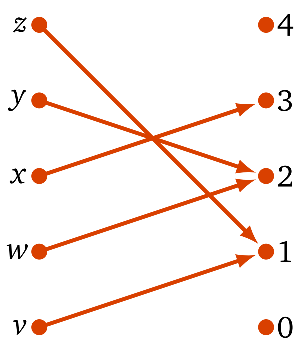
Bijection

Def. The function f is a *bijection* (also called one-to-one correspondence) if and only if it is both one-to-one and onto.

$f : A \to B$ is a bijection



$g : A \to B$ is not a bijection



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Bijection

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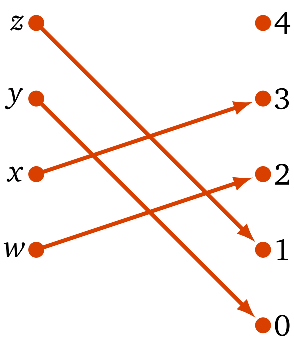
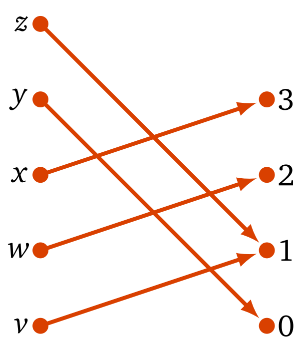
$f : A \to B$ is one-to-one, but not onto



$g : C \to D$ is onto, but not one-to-one

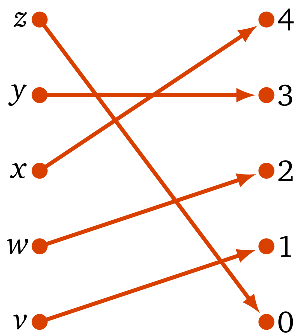


So, both functions are not bijections.

Bijection. Observation

Bijection Rule. Given two sets A and B, if there exists a bijection

$$f : A \to B, \quad \text{then } |A| = |B|.$$



We can count the size of the set A, instead of the size of B!

Bijection Rule.

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Consider two similar problems:

(a) How many bit strings contain exactly three 1s and two 0s?

$$11010$$

(b) How many strings can be composed of three 'A's and five 'b's so that an 'A' is always followed by a 'b'?

$$AbAbbAbb$$

We show that this two problems are equivalent by constructing a bijection.

Bijection Rule.

Let X be the set of bit strings

$$X = \{11010, \ldots\}$$

and Y be the set of 'A' and 'b' strings

$$Y = \{AbAbbAbb, \ldots\}$$

We can construct a bijection $f : X \to Y$:

$$1 \text{ gets replaced by } Ab$$
$$0 \text{ gets replaced by } b$$

Bijection Rule.

$$
\begin{aligned}
f: 11100 &\mapsto Ab\ Ab\ Ab\ b\ b \\
11010 &\mapsto Ab\ Ab\ b\ Ab\ b \\
11001 &\mapsto Ab\ Ab\ b\ b\ Ab \\
10110 &\mapsto Ab\ b\ Ab\ Ab\ b \\
10101 &\mapsto Ab\ b\ Ab\ b\ Ab \\
10011 &\mapsto Ab\ b\ b\ Ab\ Ab \\
01110 &\mapsto b\ Ab\ Ab\ Ab\ b \\
01101 &\mapsto b\ Ab\ Ab\ b\ Ab \\
01011 &\mapsto b\ Ab\ b\ Ab\ Ab \\
00111 &\mapsto b\ b\ Ab\ Ab\ Ab
\end{aligned}
$$

Function f is one-to-one and onto, so it is a bijection. Therefore, the cardinalities of two sets are equal: $|X| = |Y| = \binom{5}{3} = 10$.

Bijection. Observation

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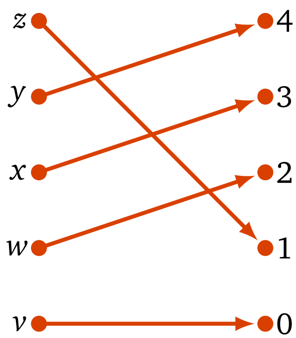
Observation For every bijection $f : A \to B$, exists an *inverse* function

$$f^{-1} : B \to A$$

$f : A \to B$



Bijection. Observation

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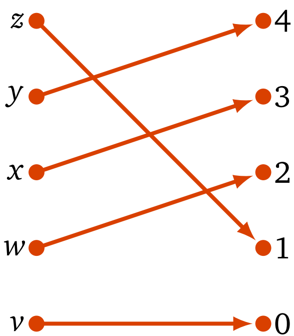
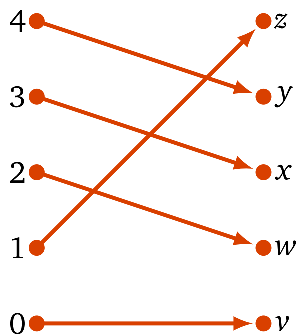


$f^{-1} : B \to A$



The inverse function is a bijection too.

Bijection. Observation

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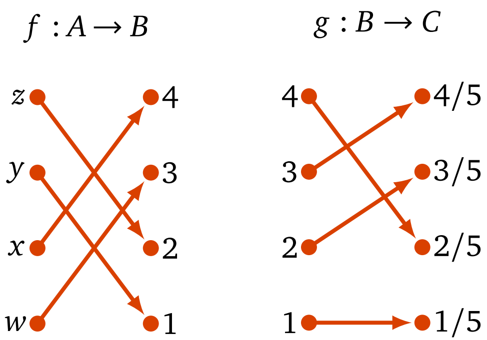
Given two bijections $f : A \to B$ and $g : B \to C$.
Consider their composition

$$h(x) = g(f(x))$$

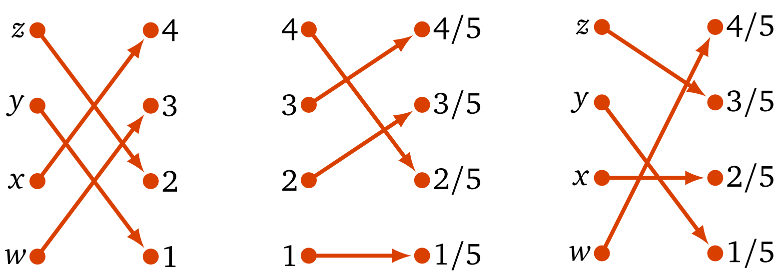


Bijection. Observation

Given two bijections $f : A \to B$ and $g : B \to C$.
Consider their composition

$$h(x) = g(f(x))$$

$f : A \to B$ $\qquad$ $g : B \to C$ $\qquad$ $h : B \to C$



$h : A \to C$ is a bijection, and therefore $|A| = |C|$.

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Bijection. Catalan numbers

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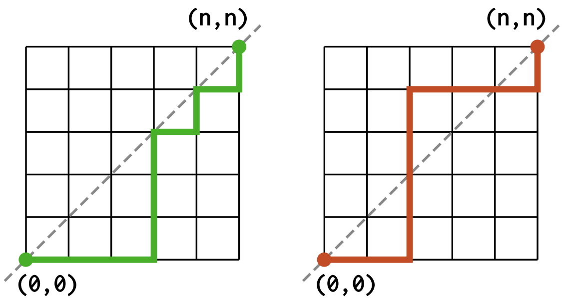
Let’s make a bijection with something more complex.

Catalan numbers provide many nice examples.



Bijection. Catalan numbers

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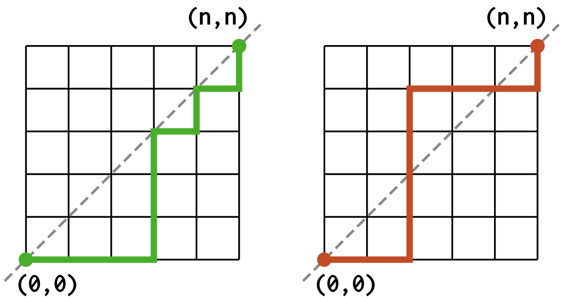

Let's find a bijection between valid paths on the grid of size n and the *strings of n pairs of correctly matching parentheses*:

(())((()()))	Ok
)())((()(()	Error

Bijection. Catalan numbers

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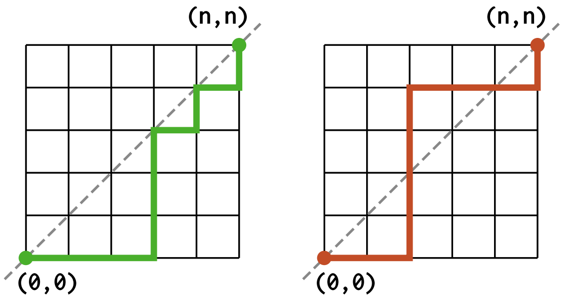


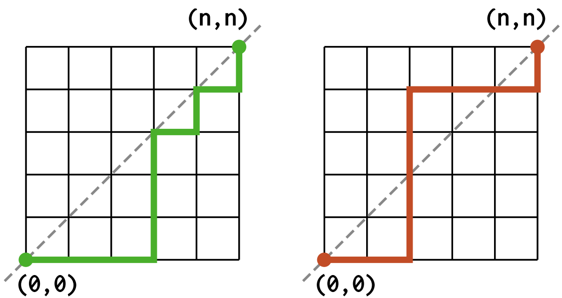
To map a path to a string from $\{`(', `)'\}^n$, we go along the path:

(a) horizontal step $\mapsto$ (

(b) vertical step $\mapsto$).

By construction, the resulting sequence produces only valid strings of n pairs of parentheses (we never close more than we open). And this is a bijection, because no two different paths map to the same string, and eventually all strings are mapped.

Bijection. Catalan numbers



((()))()()   (())))((()

Bijection. Counting subsets

Please, count the number of subsets of a set

$$A = \{a, b, c, d, e\}$$

by reducing the problem to counting bit strings.

Bijection. Counting subsets

Please, count the number of subsets of a set

$$A = \{a, b, c, d, e\}$$

by reducing the problem to counting bit strings.

Let's find a bijectioni f between the power set

$$\mathscr{P}(A) = \{\varnothing, \{a\}, \{b\}, \ldots\}$$

and the set of bit stings of length 5:

$$\{0, 1\}^5 = \{00000, 00001, 00010, 00011, \ldots\}$$

Bijection. Counting subsets

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$$f : \mathscr{P}(\{a,b,c,d,e\}) \to \{0,1\}^5$$

0s and 1s encode the membership of the five elements of $\{a,b,c,d,e\}$

$$\begin{aligned} f : \varnothing &\mapsto 00000 \\ \{a\} &\mapsto 10000 \\ \{b\} &\mapsto 01000 \\ \{a,b\} &\mapsto 11000 \\ \{c\} &\mapsto 00100 \\ \{a,c\} &\mapsto 10100 \\ \{b,c\} &\mapsto 01100 \\ \{a,b,c\} &\mapsto 11100 \\ &\ldots\text{skipping um.. 23 subsets} \\ \{a,b,c,d,e\} &\mapsto 11111 \end{aligned}$$

The cardinality

$$\left|\{0,1\}^5\right| = 2^5 = 32$$

Therefore, by the bijection rule,

$$\left|\mathscr{P}(A)\right| = 2^5$$

Inclusion-Exclusion principle

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We remember the subtraction rule for the union of two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Can it be generalized for a union of n sets

$$|A_1 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n| - \langle something \rangle ?$$

Inclusion-Exclusion principle

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Can it be generalized for a union of n sets

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Of course, it can!

Inclusion-Exclusion principle

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Union of three sets

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| = \quad & |A_1| + |A_2| + |A_3| \\ & - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| \\ & + |A_1 \cap A_2 \cap A_3|\end{aligned}$$

$$|\{1,2,3\} \cup \{2,3,4\} \cup \{3,4,1\}| = 3+3+3-2-2-2+1 = 4$$

Inclusion-Exclusion principle

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Union of n sets

$$
\begin{aligned}
|A_1 \cup \ldots \cup A_n| = \quad & \text{the sum of the sizes of the individual sets} \\
\text{minus} \quad & \text{the sizes of all two-way intersections} \\
\text{plus} \quad & \text{the sizes of all three-way intersections} \\
\text{minus} \quad & \text{the sizes of all four-way intersections} \\
\text{plus} \quad & \text{the sizes of all five-way intersections} \\
& \text{etc.}
\end{aligned}
$$

Infinity?

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We know that the set of natural numbers, $\mathbb{N}$, is infinite, so, definitely, there are sets with infinitely many elements.

How is it possible to construct such sets?

Let's definine an operation

$$A^+ = A \cup \{A\}$$

We start with $\varnothing$ and apply this operation:

$$\begin{aligned} \varnothing &= \varnothing \\ \varnothing^+ &= \{\varnothing\} \\ (\varnothing^+)^+ &= \{\varnothing, \{\varnothing\}\} \\ ((\varnothing^+)^+)^+ &= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \\ &\dots \end{aligned}$$

Infinity?

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$$\begin{aligned}
\text{'}0\text{'} &= \varnothing = \varnothing \\
\text{'}1\text{'} &= \varnothing^+ = \{\varnothing\} = \{\text{'}0\text{'}\} \\
\text{'}2\text{'} &= (\varnothing^+)^+ = \{\varnothing, \{\varnothing\}\} = \{\text{'}0\text{'}, \text{'}1\text{'}\} \\
\text{'}3\text{'} &= ((\varnothing^+)^+)^+ = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{\text{'}0\text{'}, \text{'}1\text{'}, \text{'}2\text{'}\} \\
&\dots
\end{aligned}$$

This is von Neumann's construction of natural numbers.

Infinity

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Suppose that you have infinitely many one dollar bills (numbered $1, 3, 5, \ldots$) and you come upon the Devil, who is willing to pay two dollars for each of your one-dollar bills.

The Devil is very particular, however, about the order in which the bills are exchanged. The contract stipulates that in each sub-transaction he buys from you your lowest-numbered bill and pays you with higher-numbered bills.

First sub-transaction takes $1/2$ hour, then $1/4$ hour, $1/8$, and so on, so that after one hour the entire exchange will be complete.