Discrete Structures. CSCI-150. Summer 2015.

Homework 12. Due Mon. Jul 20, 2015.

Problem 1

For which values of n, does the complete graph K_n have an Euler cycle?
For which values of n and m, does the complete bipartite graph $K_{n,m}$ have an Euler cycle?

Problem 2

Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 faces, how many vertices does this graph have?

Problem 3 (Graded)

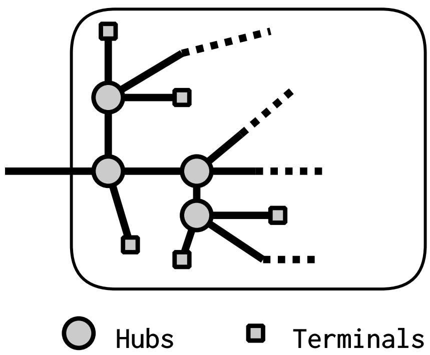
How many edges does a full binary tree with 10000 internal vertices have?

Problem 4

Suppose 1000 people enter a chess tournament. Use a rooted tree model of the tournament to determine how many games must be played to determine a champion, if a player is eliminated after one loss and games are played until only one entrant has not lost. (Assume there are no ties.)

Problem 5 (Graded)

Amtrak plans to extend their railroad network to a big island, which is connected to the continent by a bridge.



According to the plans, there will be M stations on the island. There are two types of stations: the first type are hubs that connect 4 railroads, the second type are dead-end (terminal) stations, with only one railway line. To reduce the costs, the railroads don't make loops, that is, there are *no simple cycles* in the network, so the system is cheaper, although all stations are connected. Only one of the hubs is directly connected to the outside world.

How many hubs, and how many terminals will be built? (The total number of stations is M).

Problem 6

Show that a simple graph is a tree if and only if it is connected but the deletion of any of its edges produces a graph that is not connected.

Problem 7 (Graded)

Use Huffman coding to encode these symbols with given frequencies:
A: 0.05, B: 0.07, C: 0.08, D: 0.10, E: 0.15, F: 0.25, G: 0.30.

Show all intermediate steps. What is the average number of bits required to encode a symbol?

Probability

A bit of theory first

The complement of an event A, is the event $\overline{A} = \Omega \setminus A$, thus the following properties hold: $A \cap \overline{A} = \varnothing$ and $A \cup \overline{A} = \Omega$.

$$P(\overline{A}) \equiv P(\Omega \setminus A) = 1 - P(A)$$

The formula can be very useful, because sometimes it is much easier to compute $P(A)$ rather than $P(\overline{A})$, or the other way around.

Example of A and $\overline{A}$: Five cards are drawn from a standard deck. What is the probability that there is at least one ace among them?

A: there is at least one ace. $\overline{A}$: there are no aces.

(This is not needed for the problems 1 and 2, by the way).

Problem 8 (Graded)

Given six cards:

$$A\spadesuit, J\spadesuit, 2\spadesuit, A\heartsuit, 2\heartsuit, 2\diamondsuit,$$

you pick one card at random.

Consider two events:

$$\begin{aligned} A &: \quad \text{the chosen card is an ace} \\ S &: \quad \text{the chosen card is a spade} \end{aligned}$$

(a) What is the sample space Ω?

(b) Compute the probabilities $P(A)$ and $P(S)$.

(c) Are the events A and S independent?

(d) Can you find any (other?) pair of independent events for the given set of cards?

Problem 9

Three cards are drawn from a standard 52-card deck.

Each combination of three cards was equally likely, find the probability that the following hand is obtained: $\{K\spadesuit, Q\heartsuit, J\diamondsuit\}$ (this is a set, the order does not matter).

Problem 10 (Graded)

A project was implemented by three developers: Alice, Bob, and Carol. They used four languages: C, C++, Python, and JavaScript. The table summarizes what fraction of the code was written by each person in each language.

	C	C++	Python	JavaScript
Alice	5/24	1/8	1/6	0
Bob	1/24	1/8	1/12	0
Carol	0	0	1/12	1/6

You pick a piece of code at random.

(a) Who is most likely to be the author of that piece of code?

(b) Who is most likely to be the author given that it was written in JS?

(c) Who is most likely to be the author given that it was written in C or C++?

(d) What is the probability that it was written by Bob? Does the probability change if we know that the code is in Python? Are the events *Python* and *Bob* independent or not?

(e) Are the events *Alice* and C independent?

(f) The same question for $Carol$ and JS.

Problem 11

A fair six-sided die is rolled twice. What is the probability that the outcome of the second roll is the same as the outcome of the first roll?

Problem 12 (Graded)

Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

(a) the probability of no successes

(b) the probability of at least one success

(c) the probability of at most one success

(d) the probability of at least two successes

Problem 13

By rolling a six-sided die 6 times, a strictly increasing sequence of numbers was obtained, what is the probability of such an event?

Additional problems

Problem 14. Birthdays

(See the discussion in Rosen and LL).

What is the minimum number of people who need to be in a room so that the probability that at least two of them have the same birthday is greater than 1/2?

Assume that there are $n = 366$ days in a year, and all birthdays are independent and equally likely.

Problem 15

You are playing a game, in which at every stage you can either win a dollar or lose one, with probabilities p and $1 - p$, respectively. The game is going until you don't have any money. You start with $N_0 = \$1$ in the beginning. What is the probability that after the stage n you have again $N_n = \$1$ in your bank?

Problem 16

Assume that it's observed that in each episode of The Simpsons, the probability that Homer will shout "Doh!" k times is $\dfrac{1}{2^{k+1}}$.

Today, you are going to watch a new episode:

(a) What is the probability that Homer will express his annoyance at least twice?

(b) What is the expected number of times he will do that during the episode?