

# Fibonacci Numbers. Solving Linear Recurrences

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0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...      Fibonacci numbers

# Consider a recurrence

An example:

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 0$$

$$f(1) = 1$$

When solving a linear recurrence like this, first, we are looking for a solution in the form

$$f(n) = x^n$$

We get

$$x^n = x^{n-1} + x^{n-2}$$

# Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 0$$

$$f(1) = 1$$

$$x^n = x^{n-1} + x^{n-2} \quad \text{divide by } x^{n-2}$$

$$x^2 = x + 1$$

So, we have to solve the quadratic equation

$$x^2 - x - 1 = 0$$

It is called the *characteristic equation* of the recurrence.

# Quadratic equations

Recall that to solve a quadratic equation

$$ax^2 + bx + c = 0$$

We compute the discriminant

$$\Delta = b^2 - 4ac.$$

If  $\Delta \geq 0$  there are two solutions (roots):

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

If  $\Delta < 0$ , there is no solutions.

Note that if  $\Delta = 0$ ,  $x_1 = x_2$ .



# Consider a recurrence

Solve the characteristic equation

$$x^2 - x - 1 = 0$$

The discriminant:

$$\Delta = (-1)^2 - 4 \cdot 1 \cdot (-1) = 1 + 4 = 5 \geq 0$$

So, the solutions (roots) are

$$x_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad x_2 = \frac{1 - \sqrt{5}}{2}$$

# Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 0$$

$$f(1) = 1$$

$$x_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad x_2 = \frac{1 - \sqrt{5}}{2}$$

So were were looking for the solution of the first equation of the recurrence in the form  $f(n) = x^n$ . We found two:

$$f(n) = x_1^n = \left(\frac{1 + \sqrt{5}}{2}\right)^n \quad f(n) = x_2^n = \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

Good, but this is not the end. We have to satisfy the boundary conditions too.

# Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 0$$

$$f(1) = 1$$

$$x_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad x_2 = \frac{1 - \sqrt{5}}{2}$$

Consider a linear combination of  $x_1^n$  and  $x_2^n$  with yet unknown coefficients  $b$  and  $c$ :

$$f(n) = bx_1^n + cx_2^n$$

# Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 0$$

$$f(1) = 1$$

$$f(n) = bx_1^n + cx_2^n$$

$$f(n) = b\left(\frac{1+\sqrt{5}}{2}\right)^n + c\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This  $f(n)$  satisfies the first equation of the recurrence too. Let's show that.

# Consider a recurrence

$$f(n) = bx_1^n + cx_2^n$$

$x_1$  and  $x_2$  are the roots of the characteristic equation:

$$x_1^n = x_1^{n-1} + x_1^{n-2}$$

$$x_2^n = x_2^{n-1} + x_2^{n-2}$$

Multiply the equations by  $b$  and  $c$ , respectively, and add them up:

$$\underbrace{bx_1^n + cx_2^n}_{=f(n)} = \underbrace{bx_1^{n-1} + cx_2^{n-1}}_{=f(n-1)} + \underbrace{bx_1^{n-2} + cx_2^{n-2}}_{=f(n-2)}$$

Therefore, the linear combination  $f(n) = bx_1^n + cx_2^n$  satisfies the first equation of the recurrence too:

$$f(n) = f(n-1) + f(n-2)$$

# Consider a recurrence

$$f(n) = f(n-1) + f(n-2)$$

$$f(0) = 0$$

$$f(1) = 1$$

The proposed solution

$$f(n) = bx_1^n + cx_2^n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + c\left(\frac{1-\sqrt{5}}{2}\right)^n$$

has to satisfy the boundary conditions

$$f(0) = b\left(\frac{1+\sqrt{5}}{2}\right)^0 + c\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$$

$$f(1) = b\left(\frac{1+\sqrt{5}}{2}\right)^1 + c\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

# Consider a recurrence

$$f(0) = b\left(\frac{1+\sqrt{5}}{2}\right)^0 + c\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$$

$$f(1) = b\left(\frac{1+\sqrt{5}}{2}\right)^1 + c\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

So, this is a system of two equations and two unknowns  $b$  and  $c$

$$\begin{cases} b + c = 0 \\ b\frac{1+\sqrt{5}}{2} + c\frac{1-\sqrt{5}}{2} = 1 \end{cases}$$

From the first equation,  $c = -b$ . Therefore,

$$b\frac{1+\sqrt{5}}{2} + (-b)\frac{1-\sqrt{5}}{2} = 1; \quad b\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

# Consider a recurrence

$$b\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

$$b\frac{2\sqrt{5}}{2} = 1$$

$$b = \frac{1}{\sqrt{5}}$$

$$c = -\frac{1}{\sqrt{5}}$$

$$f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$



# General Linear Recurrence

A general homogeneous linear recurrence

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \dots + a_d f(n-d)$$

with boundary conditions:

$$f(0) = z_1,$$

$$f(1) = z_2,$$

...

$$f(d-1) = z_d$$

$f(n)$  is a linear combinations of  $f(n-1), \dots f(n-d)$ .

$a_1, \dots a_d$  and  $z_1 \dots z_d$  are constants (numbers, in fact).

# General Linear Recurrence

**Step 1. Find the roots,  $x_i$ , of the characteristic equation.**

Take the recurrence,

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \dots + a_d f(n-d)$$

First, assume  $f(n) = x^n$ :

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_d x^{n-d}$$

Divide by  $x^{n-d}$  to obtain the characteristic equation:

$$x^d = a_1 x^{d-1} + a_2 x^{d-2} + \dots + a_d$$

After solving the equation, we get its roots  $x_1, x_2, \dots, x_d$ .

# General Linear Recurrence

## Step 2A. If all roots are distinct, then

The solution of the recurrence is a linear combination of  $x_i^n$ :

$$f(n) = b_1x_1^n + b_2x_2^n + \dots + b_dx_d^n$$

We find the unknown coefficients  $b_1, \dots, b_d$  from the boundary conditions.

## Step 2B. If not all roots are distinct:

If a root  $x_i$  has multiplicity two, then instead of  $b_ix_i^n$ , it contributes

$$b_ix_i^n + c_inx_i^n \quad \text{to the sum.}$$

If a root  $x_i$  has multiplicity three, it contributes

$$b_ix_i^n + c_inx_i^n + d_in^2x_i^n.$$

$b_i, c_i, d_i$  are constants, we find them from the boundary conditions.