The Pigeonhole Principle. Permutations and Combinations

A typical situation

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

A typical situation

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?

Four is enough.

The Pigeonhole Principle

If you have 10 pigeons in 9 boxes, at least one box contains two birds.

The pigeonhole principle. If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle. If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Ceiling function:

$$\lceil x \rceil = \text{the smallest integer not less that that } x$$

So, for example,

$$\lceil 2.0 \rceil = 2$$

$$\lceil 0.5 \rceil = 1$$

$$\lceil -3.5 \rceil = -3$$

Generalized Pigeonhole Principle

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Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

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Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

“Birds” are cards. “Boxes” are suits, $k = 4$.
How many cards, N, should we take to guarantee that at least three of them fall in the same “box” (suit):

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{N}{4} \right\rceil \geq 3 > 2.$$

The smallest possible $N = 2 \cdot 4 + 1 = 9$. $\clubsuit\clubsuit\diamondsuit\diamondsuit\heartsuit\heartsuit\spadesuit\spadesuit$

Solve

Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Permutations

Count the number of ways to arrange the elements of this set:

$$\{a, b, c, d, e, f\}$$

- There are *6* ways to select the first element,
- *5* ways to select the second element,
- *4* ways to select the third ...
- ... continue the process
- In the end, the only remaining element takes the last position.

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{ways!}$$

Factorial

How large this number is?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This function is called *factorial* and denoted by $n!$:

$$\begin{aligned}
1! &= 1 \\
2! &= 2 \cdot 1 \\
3! &= 3 \cdot 2 \cdot 1 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 \\
5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
n! &= n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1 \\
&\quad \text{and by convention,} \\
0! &= 1
\end{aligned}$$

Permutations

Def. A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

A finite set of 6 elements has

$$P(6) = 6! = 720 \text{ permutations.}$$

A finite set A with cardinality $|A| = n$ has

$$P(n) = n! \text{ permutations.}$$

Permutations

What if we want to arrange only r elements?

Def. An ordered arrangement of r elements of a set is called an *r-permutation*.

Can we get the formula the the number of r-permutations?

Permutations

Count the number of ways to arrange 4 elements of the set:

$$\{a,\ b,\ c,\ d,\ e,\ f\}$$

- There are *6* ways to select the first element,
- *5* ways to select the second element,
- *4* ways to select the third ...
- *3* ways to select the fourth ...

$$6 \cdot 5 \cdot 4 \cdot 3 \quad \text{ways!}$$

The number of r-permutations of the set of n elements:

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Permutations

The number of r-permutations of the set of n elements:

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Multiply and divide by $(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1$,

$$P(n,r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$$

Permutations

The number of r-permutations of the set of n elements:

$$P(n,r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

The formula makes sense only for $0 \leq r \leq n$, otherwise the notion of r-permutation does not make sense.

Permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970200.$$

Alternatively,

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{1 \cdot 2 \cdot \ldots \cdot 100}{1 \cdot 2 \cdot \ldots \cdot 97} = 98 \cdot 99 \cdot 100 = 970200.$$

What if the order does not matter?

Given a group of 4 students, $\{a,\ b,\ c,\ d\}$.

The number of 3-permutations: $P(4,3) = \dfrac{4!}{(4-1)!} = 4 \cdot 3 \cdot 2 = 24.$

$$
\begin{array}{ll}
abc,\ acb,\ bac,\ bca,\ cab,\ cba & \{a,\ b,\ c\} \\
abd,\ adb,\ bad,\ bda,\ dab,\ dba & \{a,\ b,\ d\} \\
acd,\ acb,\ cad,\ cda,\ dac,\ dca & \{a,\ c,\ d\} \\
bcd,\ bcb,\ cbd,\ cdb,\ dbc,\ dcb & \{b,\ c,\ d\}
\end{array}
$$

Alternatively, we can pick 3 stidents out of 4, and multiply by the number of their permutations: $P(4,3) = \binom{4}{3} \cdot 3! = 4 \cdot 3! = 4 \cdot 6 = 24.$

Where $\binom{n}{r}$, reads as "n choose r", denotes the number of ways to choose r elements out of n without specifying the order.

$$P(n,r) = \frac{n!}{(n-r)!}, \quad \text{or} \quad P(n,r) = \binom{n}{r} \cdot r!$$

Combinations

Def. An *r-combination* of elements of a set is an unordered selection of r elements from the set.

The number of r-combinations is

$$\binom{n}{r} = \frac{P(n,r)}{P(r)} = \frac{P(n,r)}{r!}$$

Let's express it in terms of n, r, and their factorials:

$$P(n,r) = \frac{n!}{(n-r)!} \text{ and } P(r) = r!, \text{ therefore}$$

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)!\, 5!} = \frac{52!}{47!\, 5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

More examples:

$$\binom{52}{2} = \frac{51 \cdot 52}{2} = 1326. \qquad \binom{52}{1} = 52.$$

Summary

Given a set with n elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

The number of *r-permutations* of the set:

$$P(n,r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *r-combinations* ("n choose r"):

$$\binom{n}{r} = \frac{P(n,r)}{P(r)} = \frac{n!}{(n-r)!\, r!}$$

Solve

Problem 1.

How many permutations of the letters ABCDEFG contain

(a) the string BCD?

(b) the string CFGA?

(c) the strings BA and GF?

(d) the strings BAC and CED?

Solve

Problem 2.

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Problem 3.

How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

Solve: “Knights of the Round Table”

Def. A *circular permutation* of n people is their seating around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

Def. Similarly, a *circular r-permutation* of n people is a seating of r of these n people around a circular table, where, again, the seatings that can be obtained by rotation are considered to be the same.

Problem 4.

In how many ways can King Arthur seat n different knights at his round table?

Problem 5.

Count the number of circular r-permutations of n people.

“Knights of the Round Table”

Problem 4. In how many ways can King Arthur seat n different knights at his round table?

Answer: Because n normal permutations result in a single circular permutation, $\frac{n!}{n} = (n-1)!$

Problem 5.

Count the number of circular r-permutations of n people.

Answer: r normal r-permutations result in one circular r-permutation, so we get $\frac{n!}{(n-r)!r}$. Transform this formula:

$$\frac{n!}{(n-r)!\,r} = \frac{n!}{(n-r)!\,r!} \cdot \frac{r!}{r} = \binom{n}{r} \cdot \frac{r!}{r}$$

So, equivalently, we first choose r knights out of n, and then count their circular permutations.

Let's try to be more systematic

How does $\binom{n}{r}$ change with r?

$$\binom{0}{0} = \frac{0!}{0!\,0!} = 1.$$

$$\binom{1}{0} = \frac{1!}{1!\,0!} = 1, \quad \binom{1}{1} = \frac{1!}{0!\,1!} = 1.$$

$$\binom{2}{0} = \frac{2!}{2!\,0!} = 1, \quad \binom{2}{1} = \frac{2!}{1!\,1!} = 2, \quad \binom{2}{2} = \frac{2!}{0!\,2!} = 1.$$

$$\binom{3}{0} = \frac{3!}{3!\,0!} = 1, \quad \binom{3}{1} = \frac{3!}{2!\,1!} = 3, \quad \binom{3}{2} = \frac{3!}{1!\,2!} = 3, \quad \binom{3}{3} = \frac{3!}{0!\,3!} = 1.$$

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x+y)^n$:

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x+y)^n$:

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

Pascal's Triangle

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$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The general formula is

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

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$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The general formula is

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Coefficients of $(x + y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \binom{n}{2} \cdot x^{n-2} y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

Example:

$$\begin{aligned}(x+y)^3 =& (x+y)(x+y)(x+y) = \\ & xxx + \\ & xxy + xyx + yxx + \\ & xyy + yxy + yyx + \\ & yyy\end{aligned}$$

$2^n = 2^3 = 8$ terms in total. The same as the number of the bit strings of length 3.

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n \text{ times}} =$$

$$\underbrace{x \cdot x \cdot \ldots \cdot x}_{=x^n} +$$

$$(\underbrace{x \cdot \ldots \cdot x}_{=x^{n-1}}) \cdot y + \ldots + y \cdot (x \cdot \ldots \cdot x) +$$

$$(\underbrace{x \cdot \ldots \cdot x}_{=x^{n-2}}) \cdot (y \cdot y) + \ldots + (y \cdot y) \cdot (\underbrace{x \cdot \ldots \cdot x}_{=x^{n-2}}) +$$

$$\ldots +$$

$$\underbrace{y \cdot y \cdot \ldots \cdot y}_{=y^n}$$

Coefficients of $(x+y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)}_{n \text{ times}} =$$

$$1 \cdot x^n +$$

$$n \cdot x^{n-1}y +$$

$$\binom{n}{2} \cdot x^{n-2}y^2 +$$

$$\ldots +$$

$$1 \cdot y^n$$

Coefficients of $(x + y)^n$

Let's prove

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \binom{n}{2} \cdot x^{n-2} y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \underbrace{(x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)}_{n \text{ times}} =$$
$$\binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} y + \binom{n}{2} \cdot x^{n-2} y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

Shorter notation for the same thing:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

The Binomial Theorem

$$(x+y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \ldots + \binom{n}{n} \cdot y^n.$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This result is called *The Binomial Theorem*, and this is why the coefficients, $\binom{n}{k}$, are also called the *binomial coefficients*.

The Binomial Theorem

Let's prove that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

The Binomial Theorem

Let's prove that

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} \cdot 1 = \sum_{k=0}^{n} \binom{n}{k}.$$

Therefore,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

The Binomial Theorem

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.$$

Results like this are not so obvious.

Recall that

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

So,

$$\frac{n!}{n!\,0!} + \frac{n!}{(n-1)!\,1!} + \frac{n!}{(n-2)!\,2!} + \ldots + \frac{n!}{0!\,n!} = 2^n.$$

In this form, the result seems to be much harder to prove.

The Binomial Theorem

Using the binomial theorem

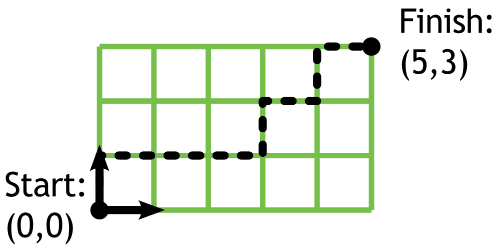
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} =$$

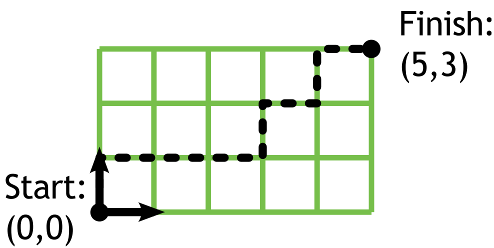
$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n} = 3^n.$$

Counting routes



You can go only North and East.
Count the number of paths from $(0,0)$ to $(5,3)$.

Counting routes



You can go only North and East.
Count the number of paths from $(0,0)$ to $(5,3)$. Answer: $\binom{5+3}{3}$.

Pascal's Triangle Again

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Every number in Pascal's triangle is equal to the sum of the two numbers that are immediately above.

Another Identity

Prove that for $r \leq n$ and $r \leq m$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Vandermonde's Identity

Prove that for $r \leq n$ and $r \leq m$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

We split the initial set of $m+n$ objsects into two arbitrary subsets of m and n objects. After that, we can choose r objects from the two subsets in the following ways:

k	subset of size m	subset of size n
0	choose r	choose none
1	$r-1$	1
2	$r-2$	2
$\ldots$		
r	0	r

$\binom{m}{r}\binom{n}{0} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r-2}\binom{n}{2} + \ldots + \binom{m}{0}\binom{n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$

r-combinations without repetition

Recall that without repetitions, this is $\binom{n}{r}$.

For example, you have n books, but don't have time to read all of them, and have to select only r books to read.

In how many ways can you do so?

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

There are $\binom{n}{r}$ ways to make the choice.

r-combinations with repetition

There is a vending machine with 3 types of drinks, \$1 each drink.

You have to spend \$5.

$$\$ \quad \$ \quad \$ \quad \$ \quad \$$$

r-combinations with repetition

There is a vending machine with 3 types of drinks, \$1 each drink.

You have to spend \$5.

$$\$ \mid \$ \quad \$ \mid \$ \quad \$$$

r-combinations with repetition

There is a vending machine with 3 types of drinks, \$1 each drink.

You have to spend \$5.

$$\underbrace{\$}_{\text{1 drink}} \mid \underbrace{\$ \quad \$}_{\text{2 drinks}} \mid \underbrace{\$ \quad \$}_{\text{2 drinks}}$$

r-combinations with repetition

There is a vending machine with 3 types of drinks, \$1 each drink.

You have to spend \$5.

$$\$ \quad \$ \mid \$ \quad \$ \quad \$ \mid$$

r-combinations with repetition

The Pigeonhole Principle
Permutations
Combinations
Pascal's Triangls
The Binomial Theorem
Combinations with repetition
Permutations with repetition

There is a vending machine with 3 types of drinks, \$1 each drink.

You have to spend \$5.

$$\underbrace{\$\quad\$}_{\text{2 drink}} \mid \underbrace{\$\quad\$\quad\$}_{\text{3 drinks}} \mid \underbrace{\qquad}_{\text{none}}$$

r-combinations with repetition

So, there are $5+(3-1)$ places that stand for 5 dollars and $(3-1)$ separators between the drinks' types.

$$\_\ \_\ \_\ \_\ \_\ \_\ \_$$

$$\$\ |\ \$\ \$\ |\ \$\ \$$$

$$\$\ \$\ |\ \$\ \$\ \$\ |$$

$$\$\ \$\ \$\ \$\ \$\ |\ |$$

r-combinations with repetition

So, there are $5+(3-1)$ places that stand for 5 dollars and $(3-1)$ separators between the drinks' types.

$$\_\ \_\ \_\ \_\ \_\ \_\ \_$$

$$\$ \mid \$\ \$ \mid \$\ \$$$

$$\$\ \$ \mid \$\ \$\ \$ \mid$$

$$\$\ \$\ \$\ \$\ \$ \mid \mid$$

$$\binom{7}{5} = \binom{7}{2} = 21 \text{ ways to buy 5 drinks}$$

r-combinations with repetition

To select r objects out of n with repetitions, there are

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1} \text{ ways.}$$

$$\_\;\_\;\_\;\_\;\_\;\_\;\_\;\_\;\_\;\_$$

$$*\;*\;*\;*\;*\;*\;|\;|\;|\;|$$

(r objects and $n-1$ separator)

In other words, this is the number of r-combinations with repetition from the set of n objects.

r-permutations with repetition

We know that the number of r-permutations of n objects *without repetition* is

$$n(n-1)(n-2)\cdot\ldots\cdot(n-r+1) = \frac{n!}{(n-r)!}$$

But, *if repetitions are allowed*, it is even easier, the simple product rule works just fine!

$$n\cdot n\cdot\ldots\cdot n = n^r$$

Summary

	with repetitions?	
r-combination	No	$\binom{n}{r}$
r-combination	Yes	$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$
r-permutation	No	$P(n,r) = \frac{n!}{(n-r)!}$
r-permutation	Yes	n^r