Discrete Structures. CSCI-150. Spring 2016.

Homework 10. Due Wed. Apr 20, 2016.

Problem 1 (Graded)

Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ that is

(a) one-to-one, but not onto,

(b) onto, but not one-to-one,

(c) neither one-to-one, nor onto.

(d) onto and one-to-one (bijection), which is not the identity function $f(x) = x$.

(e) Function $h : \mathbb{N} \to \mathbb{N} \times \{0, 1\}$ that is onto and one-to-one (bijection).

(f) Bijection $g : \mathbb{N} \to \mathbb{Z}$. (Hint: When solving this problem, if you want, you may assume that you know the function h from the previous question. Would that help to make the function g?)

When constructing the functions, try to define them by formulas. (Feel free to use such operations as *absolute value*, *floor*, *ceiling*, *remainder*, in addition to normal arithmetical operations). Definition by cases is another option.

By definition, $\mathbb{N}$ is the set of all non-negative integers: $\mathbb{N} = \{0, 1, 2, \ldots\}$.

For each function, explain (in the best way you can) why they satisfy the required conditions. (One or two sentences should be enough)

Problem 2 (Graded)

Draw the diagrams (as we did in class) for all bijections $f : A \to A$ when the set A is

(a) $A = \{a\}$

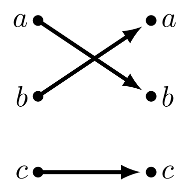
(b) $A = \{a, b\}$

(c) $A = \{a, b, c\}$

(d) For this question, either repeat the task for $A = \{a, b, c, d\}$, or derive a formula for the total number of bijections from A to A, when $|A| = n$. (Explain your answer).

Example of the diagram for a function

$f : \{a, b, c\} \to \{a, b, c\}$



Problem 3

Given 8 letters: A B C D E F G H, find a bijection between the set of all permutations of these letters and the set of all 7-permutations of the same letters.

(The existence of the bijection means that the sets have the same cardinality, which means that the number of permutations of the 8 letters is equal to the number of 7-permutations of the same letters, and this is a known result from counting, the number is equal to 8!).

Problem 4

(a) Please count how many functions

$$f \, : \, D \to \{0, 1\}$$

can be defined if the domain D is a finite set with the cardinality $|D| = n$.

(b) Can you find a bijection between the set of all such functions and the powerset $\mathcal{P}(D)$?

Problem 5 (Graded)

Let Σ be the set of letters $\{a, b, \ldots z\}$. Also, let Σ^* (notice the superscript star in its name) be the set of all finite string made of the letters Σ, for example

$$\text{“cat”} \in \Sigma^*, \qquad \text{“dog”} \in \Sigma^*, \qquad \text{“mathematics”} \in \Sigma^*, \qquad \text{“”} \in \Sigma^* \text{ (the empty string).}$$

We define a “permutation” relation P as follows

$$P = \{(x, y) \in \Sigma^* \times \Sigma^* \mid x \text{ is a permutation of } y\}.$$

It contains all the pairs of strings that are permutations of each other, for example

$$(\text{“flow”}, \text{“wolf”}) \in P, \qquad (\text{“teenager”}, \text{“generate”}) \in P, \qquad (\text{“player”}, \text{“replay”}) \in P.$$

Question: Is this relation P an equivalence relation or a partial order relation? (An equivalence relation is reflexive, symmetric, not antisymmetric, and transitive, and a partial order relation is reflexive, antisymmetric, not symmetric, and transitive).

Prove your claim. Your argument does not have to be very formal, a good explanation in English should be sufficient.

Problem 6 (Graded)

Draw the Hasse diagram for divisibility on the set:

(a) $\{1, 2, 3, 4, 5, 6\}$, (b) $\{3, 4, 7, 12, 28, 42\}$, (c) $\{3, 5, 6, 9, 25, 27\}$, (d) $\{3, 5, 7, 11, 13, 16, 17\}$,

(e) $\{6, 10, 14, 15, 21, 22, 26, 33, 35, 39, 55, 65, 77, 91, 143\}$, (f) $\{1, 3, 9, 27, 81, 243\}$.

Bonus (Extra credit). Submodular functions and diminishing returns.

Introduction. Set functions.

Given a set A, consider a function from the powerset $\mathcal{P}(A)$ to reals, $\mathbb{R}$:

$$f : \mathcal{P}(A) \to \mathbb{R}.$$

This function assigns a numerical value to any subset of A. Why is it useful? Let's say that A is a set of items you want to buy. Then the function f may estimate, for example, the happiness you gain from having a subset of those items.

For example, let

$$A = \{\text{ phone, charger, tea, coffee, cups, Rosen's book, } \dots \} \qquad \text{– a set of various items.}$$

Now consider a function f that assigns value to any subset of items $S \subseteq A$, and may be simply equal to the total price of the items in S:

$$f_1(S) = \sum_{item \in S} \text{Price}(item)$$

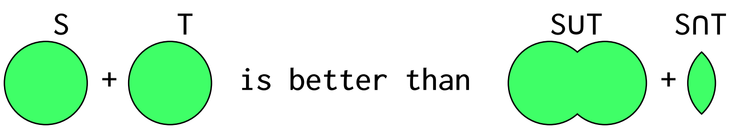
However, do you value things solely by their price? Let's think. Some objects may be only valuable together with certain other objects but not by themselves. Also, having too many items may even reduce their individual value. For example:

$$\begin{aligned}
f_2(\{\text{phone}\}) &= 50 \\
f_2(\{\text{phone, phone}_2\}) &= 70 \qquad \leftarrow \text{you don't gain much by having a second phone} \\
f_2(\{\text{phone, phone}_2\text{, phone}_3\}) &= 75 \\
f_2(\{\text{phone, phone}_2\text{, phone}_3\text{, phone}_4\}) &= 76 \\
f_2(\{\text{charger}\}) &= 0 \\
f_2(\{\text{phone, charger}\}) &= 100 \qquad \leftarrow \text{charger adds value only when you have a phone} \\
&\dots
\end{aligned}$$

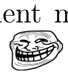
Submodularity.

Def. A function $f : \mathcal{P}(A) \to \mathbb{R}$ is called submodular if for any two subsets $S, T \subseteq A$:

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$



Submodular functions have a natural diminishing returns property which makes them suitable for many applications in Computer Science, Game theory, and Economics.

The diminishing returns property, in the context of submodular function, means that the difference in the value of the function that a single element makes when added to an input set decreases as the size of the input set increases. (Clear? )

Let's consider an example, if you don't know how to play piano, a single lesson may make a significant improvement to your skill. However, a professional pianist will not learn anything new from the same lesson.

Another example, Bill Gates' happiness does not increase much if he earns extra 100 dollars. However, a person with no money will become much happier in the same situation.

The returns diminish if you already have (invested) a lot. Let's prove that this property holds for submodular functions.

We are going to prove that if $X \subseteq Y$ and $a \notin Y$ then for a submodular function f the following inequality holds:

$$f(X \cup \{a\}) - f(X) \geq f(Y \cup \{a\}) - f(Y).$$

In English, if the set X is "smaller" than the set Y (i.e. $X \subseteq Y$), then the addition of a new element $\{a\}$ to X makes a bigger difference than the addition of the same element to the bigger set Y. (Here, X is somebody with no money, and Y is Bill Gates).

To prove this property, use the definition of submodular function (see the previous page). Assume that $S = X \cup \{a\}$ and $T = Y$.