

Sets. Ordered pairs.

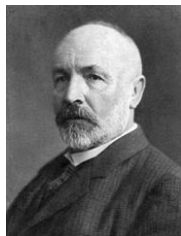
# Sets

## Sets

Ordered pair

The set theory is a branch of mathematical logic that was created by Georg Cantor in 1870s.

**Def.** A *set* is a unordered collection of objects being regarded as a single object.



Examples:

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{x \in A \mid (x \geq 3) \wedge (x \text{ is odd})\}$$

$$C = \{\emptyset, \{A\}, \{B\}, \{A, B\}\}$$

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

# Sets

If  $x$  belongs to a set  $A$ , we say that it is a *member* (or an element) of  $A$  and write

$$x \in A.$$

If  $x$  is not a member of  $A$ , we write

$$x \notin A.$$

*Empty set*, denoted by  $\emptyset$  or  $\varnothing$ , has no members:

$$\forall x (x \notin \emptyset).$$

# Sets

Two sets  $A$  and  $B$  are *equal*,  $A = B$ , iff they have exactly the same elements:

$$\forall x (x \in A \leftrightarrow x \in B)$$

For any two objects  $x$  and  $y$ , we can make a set containing exactly these two objects

$$\{x, y\}$$

If those two objects are identical,  $x = y$ , we get a singleton set,

$$\{x, x\} = \{x\}.$$

Notice that these sets are not equal:

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}$$

# Sets

*Set-builder notation.*

A set of all objects that satisfy the property  $P$ :

$$A = \{x \mid P(x)\}$$

Examples:

$$B = \{x \in \mathbb{Z} \mid x \text{ is even}\}$$

$$C = \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z} (x = 2k)\}$$

$$D = \{x \mid (x \in \mathbb{Z}) \wedge (\exists k \in \mathbb{Z} (x = 2k))\}$$

$$E = \{0, 2, -2, 4, -4, 6, -6, \dots\}$$

In naive set theory, any definable collection is a valid set. And usually it works fine.

However, it leads to contradictions, such as Russell's paradox.

# Russell's paradox

Sets

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Let  $R$  be the set of all sets that are not members of themselves:

$$R = \{x \mid x \notin x\}$$

It is legal to ask, is  $R$  a member of itself or not.

There are only two cases, either  $R \in R$ , or  $R \notin R$ .

So, where is the paradox?

# Russell's paradox

$$R = \{x \mid x \notin x\}$$

(a) If  $R$  is a member of itself, then by its definition,  $R \notin R$ ,

$$R \in R \rightarrow R \notin R.$$

(b) Otherwise, if  $R$  is not a member of itself, then  $R \in R$ ,

$$R \notin R \rightarrow R \in R.$$

Therefore, we get a contradiction,  $R \in R \leftrightarrow R \notin R$ .

There exist several axiomatic systems that rule out such pathological cases. For example, Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC).

# Union and Intersection

Sets

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*Union* of two sets  $A$  and  $B$ ,

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

$$\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$$

*Intersection* of two sets  $A$  and  $B$ ,

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

$$\{1, 2\} \cap \{2, 3\} = \{2\}$$



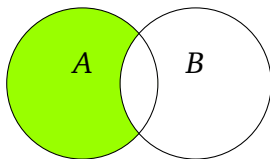
# Difference

Sets

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*Difference* between two sets  $A$  and  $B$ ,

$$A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\}$$



$$\{1, 2\} \setminus \{2, 3\} = \{1\}$$

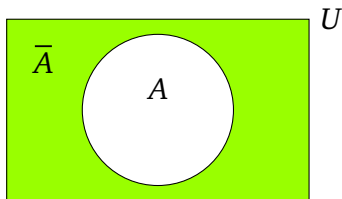
# Complement

Sets

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If there is a *universal set*  $U$  of all possible objects, then the *complement* of a set  $A$  is

$$\bar{A} = U \setminus A = \{x \in U \mid x \notin A\} = \{x \in U \mid \neg(x \in A)\}$$



Exmaple: When  $U = \mathbb{Z}$ :

$$Odd = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$$

$$Even = \overline{Odd} = \mathbb{Z} \setminus A$$

# Set identities

## Sets

Ordered pair

Given the universal set  $U$ , sets with respect to union, intersection, and complement satisfy the same identities as propositions with respect to  $\wedge$ ,  $\vee$ , and  $\neg$

Sets:	$A \cap B$	$A \cup B$	$\bar{A}$	$U$	$\emptyset$
Propositions:	$p \wedge q$	$p \vee q$	$\neg p$	$T$	$F$

$$\overline{\bar{A}} = A$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

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$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

See the full list in Rosen's book.

# Set identities

Sets

Ordered pair

Let's prove De Morgan's law for sets:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cup B} = \{x \mid \neg((x \in A) \vee (x \in B))\}$$

$$\bar{A} \cap \bar{B} = \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$$

The propositions in the right hand sides of the equations are equal, therefore, the left hand sides are equal too.

# Subset

$A$  is a *subset* of  $B$  iff every element of  $A$  is an element of  $B$ :

$$A \subseteq B \iff \forall x((x \in A) \rightarrow (x \in B))$$

$$\{1, 2\} \subseteq \{1, 2, 3\}$$

$$\{1, 2, 3\} \subseteq \{1, 2, 3\}$$

$$\emptyset \subseteq \{1, 2, 3\}$$

$A$  is a *proper subset* of  $B$  iff  $A$  is a subset of  $B$ , but it's not equal to  $B$

$$A \subsetneq B \iff \forall x((x \in A) \rightarrow (x \in B)) \wedge \exists x((x \in B) \wedge (x \notin A))$$

$$\{1, 2\} \subsetneq \{1, 2, 3\}$$

$$\emptyset \subsetneq \{1, 2, 3\}$$

Proper subset  $A$  is strictly “smaller” than  $B$ .

# Power set

Sets

Ordered pair

**Def.** The set of all subsets of  $A$  is called a *power set* of  $A$ , denoted by  $\mathcal{P}(A)$ .

Examples:

$$\mathcal{P}(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$$

$$\mathcal{P}(\{0, 1, 2\}) =$$

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# Cardinality

Sets

Ordered pair

**Def.** The *cardinality* of a finite set  $A$  is equal to the number of elements in  $A$ . It's denoted by  $|A|$ .

$$|\emptyset| = 0$$

$$|\{0, 1, 2, 3, 4\}| = 5$$

We already know the subtraction rule for the cardinality of a union:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



# Question

Sets

Ordered pair

Compute the cardinality of the power set  $\mathcal{P}(A)$  if  $|A| = n$ .

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$$|\mathcal{P}(A)| = 2^{|A|} = 2^n$$

# Generalized union and intersection

Sets

Ordered pair

Union of a finite set of sets:

$$\bigcup \{A_0\} = A_0$$

$$\bigcup \{A_0, A_1, \dots, A_n\} = \bigcup_{i=0}^n A_i = A_0 \cup A_1 \cup \dots \cup A_n$$

Intersection of a finite set of sets:

$$\bigcap \{A_0\} = A_0$$

$$\bigcap \{A_0, A_1, \dots, A_n\} = \bigcap_{i=0}^n A_i = A_0 \cap A_1 \cap \dots \cap A_n$$

# Ordered pair

**Def.** The *ordered pair* of  $a \in A$  and  $b \in B$  is an ordered collection  $(a, b)$ .

Two ordered pairs are equal

$$(a, b) = (c, d) \quad \text{if and only if} \quad (a = c) \wedge (b = d).$$

Observe that this property implies that

$$(a, b) \neq (b, a)$$

unless  $a = b$ . So, the order matters. This is why it is called the ordered pair, and  $(a, b)$  is not equivalent to a set  $\{a, b\}$ .

$$(1, 2) = (1, 2)$$

$$(1, 2) \neq (1, 3)$$

$$(1, 2) \neq (2, 1)$$

# Ordered pair

Sets

Ordered pair

More examples of ordered pairs:

$$(1, 2)$$

$$(\{1\}, \{2\})$$

$$(1, \{2, 3, 4, 5\})$$

$$((1, 2), \emptyset)$$

If  $a \in A$  and  $b \in B$ , what is the set of all ordered pairs  $(a, b)$ ?

# Cartesian product

Sets

Ordered pair

**Def.** Let  $A$  and  $B$  be sets. The *Cartesian product* of  $A$  and  $B$ , denoted by  $A \times B$ , is the set of all ordered pairs  $(a, b)$ , where  $a \in A$  and  $b \in B$ . Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

(Named after Rene Decartes)

*Question.* Given two sets  $A = \{1, 2, 3\}$  and  $B = \{C, D\}$ , what is their Cartesian product  $A \times B$ ?

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$$A \times B = \{(1, C), (2, C), (3, C), \\ (1, D), (2, D), (3, D)\}$$



# Cartesian product

Sets

Ordered pair

*Question.* If the  $|A| = n$ , and  $|B| = m$ , what is the cardinality of  $A \times B$ ?

# Building a list

Sets

Ordered pair

Ordered pair is fundamental for defining data types.

*Question.* How to implement the list data type with ordered pairs?

Example of a list:

$[1, 2, 3, 4]$

Interface:

$\text{construct} ( 1, [2, 3, 4] ) = [1, 2, 3, 4]$

$\text{head} ( [1, 2, 3, 4] ) = 1$

$\text{tail} ( [1, 2, 3, 4] ) = [2, 3, 4]$

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Possible implementation:

$(1, (2, (3, 4)))$

$\text{construct} ( h, t ) = (h, t)$

$\text{head} ( (h, t) ) = h$

$\text{tail} ( (h, t) ) = t$

# Ordered $n$ -tuple

Sets

Ordered pair

**Def.** The *ordered  $n$ -tuple* of is an ordered collection  $(a_1, a_2, \dots, a_n)$ .

It is just an extension of an ordered pair for joining  $n$  elements together.

**Def.** The *Cartesian product* of the sets  $A_1, \dots, A_n$ , is the set of all  $n$ -tuples such that

$$A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, \dots, n\}$$

If all sets  $A_i$  are equal, that is,  $A_1 = \dots = A_n = A$ , then their Cartesian product is denoted by  $A^n$

$$A_1 \times \dots \times A_n = \underbrace{A \times \dots \times A}_{n \text{ times}} = A^n$$