

Induction



Consider a problem

Principle

Examples

Summations

Inequalities

Let's prove that the sum of the first n positive integers

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all positive integers $n = 1, 2, 3, 4, \dots$

Consider a problem

Principle

Examples

Summations

Inequalities

Let

$$P(n) : 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Prove that

$$P(n)$$

for all positive natural numbers $n = 1, 2, 3, 4 \dots$

The idea

Principle

Examples

Summations

Inequalities

If we can prove that

$$P(1)$$

$$P(1) \rightarrow P(2)$$

$$P(2) \rightarrow P(3)$$

...

$$P(n) \rightarrow P(n+1)$$

...

Then it follows that

$$P(n) \text{ for all } n \geq 1$$

The idea

Principle

Examples

Summations

Inequalities

The implications can be grouped together. Thus it is enough to prove that

$$1) \quad P(1)$$

$$2) \quad \forall n \geq 1 : P(n) \rightarrow P(n+1)$$

Then it follows that

$$P(n) \text{ for all } n \geq 1$$

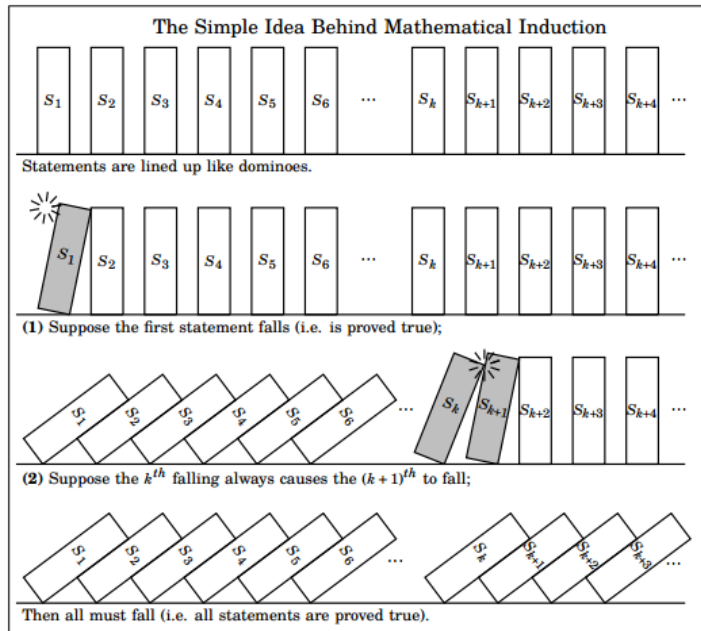
Principle of induction

Principle

Examples

Summations

Inequalities



Principle of induction

Principle

Examples

Summations

Inequalities



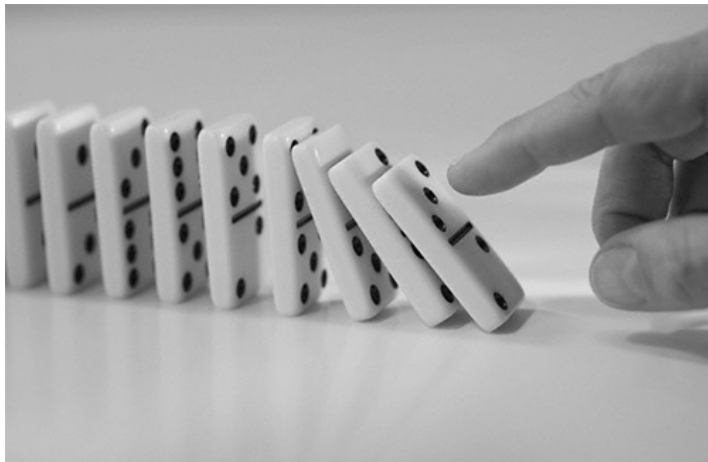
Principle of induction

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Examples

Summations

Inequalities



Principle of induction

Principle

Examples

Summations

Inequalities

If we can prove

1. *“The basis step”*

$P(1)$ is true, and

2. *“The inductive step”*

for all $n \geq 1$, $P(n)$ implies $P(n + 1)$.

then $P(n)$ is true for every integer $n \geq 1$.

Example proof

Principle

Examples

Summations

Inequalities

Let's use induction to prove the formula of the sum of positive integers from 1 to n :

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

(We have to show that both, the *basis step* and the *inductive step* are correct)

Example proof

Principle

Examples

Summations

Inequalities

Prove that

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Part 1. *The basis step.* Consider $n = 1$.

The left hand side is just

$$1$$

The right-hand side:

$$\frac{1 \cdot (1 + 1)}{2} = 1$$

They are equal, so it is true.

Example proof

Principle

Examples

Summations

Inequalities

Part 2. *The inductive step.*

Assume that the formula is true for an arbitrary $n = m \geq 1$:

$$1 + 2 + 3 + \dots + m = \frac{m(m+1)}{2}$$

We have to prove that it is also true for the next value $n = m + 1$:

$$1 + 2 + 3 + \dots + m + (m+1) = \frac{(m+1)((m+1)+1)}{2}$$

Example proof

Principle

Examples

Summations

Inequalities

Part 2. *The inductive step.*

Assume that the formula is true for an arbitrary $n = m \geq 1$:

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We have to prove that it is also true for the next value $n = m + 1$:

$$1 + 2 + 3 + \dots + m + (m+1) = \frac{(m+1)((m+1)+1)}{2}$$

Consider the left-hand side:

$$\begin{aligned} \underbrace{1 + 2 + 3 + \dots + m}_{= \frac{m(m+1)}{2}} + (m+1) &= \frac{m(m+1)}{2} + m+1 = \frac{m(m+1) + 2(m+1)}{m+1} \\ &= \frac{(m+1)(m+2)}{2} = \frac{(m+1)((m+1)+1)}{2} \end{aligned}$$

Example proof

Principle

Examples

Summations

Inequalities

The base case and the inductive step are true.

Therefore, *by induction*, the formula is correct for every positive integer n .

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

This is a very useful formula, by the way.

Rewrite it using the sigma-notation

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

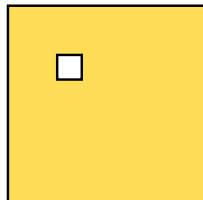
Tiling $2^n \times 2^n$ with 1 square removed

Principle

Examples

Summations

Inequalities



For all $n > 0$, a checkerboard $2^n \times 2^n$ with one square removed

can be tiled by L-shaped tiles



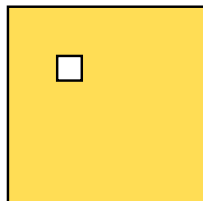
Tiling $2^n \times 2^n$ with 1 square removed

Principle

Examples

Summations

Inequalities



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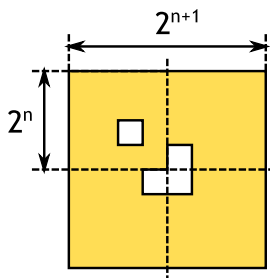


The base case for $n=1$



Inductive step.

Assuming that we can tile $2^n \times 2^n$ with one removed, prove that it's possible to tile $2^{n+1} \times 2^{n+1}$ with one removed



Another example proof

Principle

Examples

Summations

Inequalities

Let $P(n)$ be the predicate, "*I can lift n grains of sand.*"

- I can lift one grain of sand, so $P(1)$ is true. This is my basis step.

Another example proof

Principle

Examples

Summations

Inequalities

Let $P(n)$ be the predicate, “I can lift n grains of sand.”

- I can lift one grain of sand, so $P(1)$ is true. This is my basis step.
- Then, surely, if I can lift m grains, then I can lift $m + 1$, it does not make any difference!

$$P(m) \rightarrow P(m + 1)$$

This is my inductive step.

Another example proof

Principle

Examples

Summations

Inequalities

Let $P(n)$ be the predicate, “I can lift n grains of sand.”

- I can lift one grain of sand, so $P(1)$ is true. This is my basis step.
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$$P(m) \rightarrow P(m + 1)$$

This is my inductive step.

So, by induction, *I can lift any amount of sand*. Right?

Where is a mistake?

Principle

Examples

Summations

Inequalities

Of course, the proof is wrong. But should we blame induction for that?

Well, we made an error in the proof of $P(m) \rightarrow P(m + 1)$.

It is hard to say for exactly which m it is false, but certainly there is some value!

Sum of b^k

Principle

Examples

Summations

Inequalities

Prove by induction that

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

First we can do the base case, then the inductive step.

Sum of b^k

Principle

Examples

Summations

Inequalities

Prove that

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

Basis step ($n = 0$):

$$b^0 = 1, \quad \text{and} \quad \frac{b^1 - 1}{b - 1} = 1$$

.

Sum of b^k

Principle

Examples

Summations

Inequalities

Prove that

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

Inductive step:

As always, we make a hypothesis that for some $n = m \geq 0$ the formula holds:

$$b^0 + b^1 + b^2 + \dots + b^m = \frac{b^{m+1} - 1}{b - 1}$$

And we have to prove that the formula is correct for $n = m + 1$:

$$b^0 + b^1 + b^2 + \dots + b^m + b^{m+1} = \frac{b^{m+2} - 1}{b - 1}$$

Sum of b^k

Principle

Examples

Summations

Inequalities

Inductive step:

We have to prove that the formula is correct for $n = m + 1$:

$$b^0 + b^1 + b^2 + \dots + b^m + b^{m+1} = \frac{b^{m+2} - 1}{b - 1}$$

$$\begin{aligned} \underbrace{b^0 + b^1 + b^2 + \dots + b^m}_{= \frac{b^{m+1} - 1}{b - 1} \text{ by the hypothesis}} + b^{m+1} &= \frac{b^{m+1} - 1}{b - 1} + b^{m+1} \\ &= \frac{b^{m+1} - 1 + b^{m+2} - b^{m+1}}{b - 1} = \frac{b^{m+2} - 1}{b - 1}. \end{aligned}$$

Sum of b^k

Principle

Examples

Summations

Inequalities

So, this formula for the sum is correct

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

In the sigma-notation:

$$\sum_{k=0}^n b^k = \frac{b^{n+1} - 1}{b - 1}$$

Sum of b^k

Principle

Examples

Summations

Inequalities

We can multiply both sides by a constant a :

$$\sum_{k=0}^n ab^k = ab^0 + ab^1 + ab^2 + \dots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1},$$

The sequence of numbers

$$a, ab, ab^2, ab^3, \dots, ab^n, \dots$$

is called a *Geometric progression*.

So, we proved the formula for the partial sum of a geometric progression.

Sum of kb^{k-1}

Principle

Examples

Summations

Inequalities

The partial sum of the geometric progression with the coefficient $a = 1$:

$$\sum_{k=0}^n b^k = b^0 + b^1 + b^2 + \dots + b^n = \frac{(b^{n+1} - 1)}{b - 1},$$

Can we compute

$$\sum_{k=0}^n kb^{k-1} = 0 + 1 + 2b + 3b^2 + 4b^3 \dots + nb^{n-1} \quad ?$$

The terms b^k have these increasing coefficients now ...

Sum of kb^{k-1}

Principle

Examples

Summations

Inequalities

There is a cheap trick. We know that

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{(b^{n+1} - 1)}{b - 1}$$

If two functions of the same argument are equal, then their derivatives with respect to that argument are equal too

$$\frac{d}{db}(b^0 + b^1 + b^2 + \dots + b^n) = \frac{d}{db}\left(\frac{(b^{n+1} - 1)}{b - 1}\right)$$

$$0 + 1 + 2b + 3b^2 + 4b^3 + \dots + nb^{n-1} = \frac{d}{db}\left(\frac{b^{n+1} - 1}{b - 1}\right) = \frac{nb^{n+1} - (n+1)b^n + 1}{(b-1)^2}$$

$$\text{Therefore, } \sum_{k=0}^n kb^{k-1} = \frac{nb^{n+1} - (n+1)b^n + 1}{(b-1)^2}.$$

Geometric progression again

Principle

Examples

Summations

Inequalities

The partial sum of the geometric progression:

$$\sum_{k=0}^n ab^k = ab^0 + ab^1 + ab^2 + \dots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1}$$

Well, what if we want to add up an infinite sequence?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

Infinite geometric progression

Principle

Examples

Summations

Inequalities

The partial sum of the geometric progression is

$$\sum_{k=0}^n ab^k = ab^0 + ab^1 + ab^2 + \dots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1}$$

If b is a small real number, specifically, if the absolute value $|b| < 1$, then

$$|b| > |b^2| > |b^3| > \dots$$

In the limit, $\lim_{n \rightarrow \infty} b^n = 0$

$$\sum_{n=0}^{\infty} ab^n = \lim_{n \rightarrow \infty} ab^n = \lim_{n \rightarrow \infty} \frac{a(b^{n+1} - 1)}{b - 1} = \frac{a(-1)}{b - 1} = \frac{a}{1 - b}$$

Proving inequalities

Principle

Examples

Summations

Inequalities

Using mathematical induction, prove that for $n \geq 1$:

$$2^n > n$$

Proving inequalities

Principle

Examples

Summations

Inequalities

Using mathematical induction, prove that for $n \geq 1$:

$$2^n > n$$

The basis step:

$n = 1$. $2 > 1$ is true.

Proving inequalities

Principle

Examples

Summations

Inequalities

Using mathematical induction, prove that for $n \geq 1$:

$$2^n > n$$

The basis step:

$n = 1$. $2 > 1$ is true.

The inductive step:

Assume that $2^n > n$ for $n \geq 1$. Prove that $2^{n+1} > (n + 1)$.

Equivalently, we have to prove that

$$2^{n+1} - (n + 1) > 0.$$

Proving inequalities

Principle

Examples

Summations

Inequalities

We assumed that $2^n > n$ for $n \geq 1$.

We have to prove that $2^{n+1} - (n + 1) > 0$.

$$\begin{aligned}2^{n+1} - (n + 1) &= 2 \cdot 2^n - n - 1 \\&> 2 \cdot n - n - 1 \quad (\text{by the I.H.}) \\&= n - 1 \\&\geq 1 - 1 = 0. \quad (\text{b/c } n \geq 1)\end{aligned}$$

Therefore, *by induction*, $2^n > n$ is true for $n \geq 1$.

One more proof

Principle

Examples

Summations

Inequalities



Theorem. All horses are the same color.

We can prove this by induction on the number of horses in a given set.

All horses are the same color

Principle

Examples

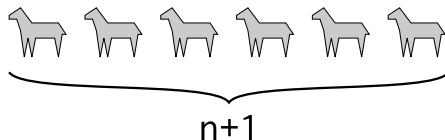
Summations

Inequalities

The basis step. If there's just one horse then it's the same color as itself.

For the *inductive step*, assume that n horses are of the same color.

Assume that there are $n + 1$ horses numbered 1 to $n + 1$.



All horses are the same color

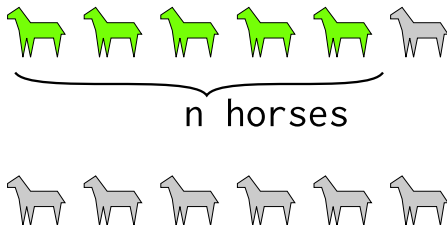
Principle

Examples

Summations

Inequalities

By the induction hypothesis, horses 1 through n are the same color, and similarly horses 2 through $n + 1$ are the same color.



But the middle horses, 2 through n , can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and $n + 1$ must be the same color as well. Thus all $n + 1$ horses are the same color.

What, if anything, is wrong with this reasoning?

All horses are the same color

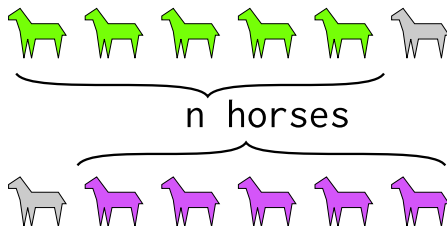
Principle

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What, if anything, is wrong with this reasoning?