

# Predicates and Quantifiers.

# Predicates

Propositional logic, studied previously, cannot adequately express the meaning of all statements in mathematics and in natural language.

Examples?

“ $x$  is greater than 5.”

“ $x$  is greater than  $y$ .”

“ $n$  is a prime number.”

“*user* is waiting.”

**Def.** A **predicate** is a proposition whose truth depends on the value of one or more variables.

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

# Predicates

## Predicates

### Quantifiers

### Scope, bound variables

### Nested quantifiers

### Distributivity

### Negation

### Examples

For convenience, we can give every predicate a name:

$$P(x) = "x \text{ is greater than } 5."$$

$$Q(x, y) = "x \text{ is greater than } y."$$

When the values of the variables are specified, the result is a simple proposition: Depending on the, the predicates are either true or false:

$$P(4) = F$$

$$P(10) = T$$

$$Q(2, 1) = T$$

# How can we use predicates?

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Let  $C(x)$  = “ $x$  is playing chess.”

There are three persons in the room: *Ed*, *Paul*, and *Tom*.

*Ed* and *Paul* are playing chess, and *Tom* is sleeping.

Formally:  $C(Ed) = T$ ,  $C(Paul) = T$ ,  $C(Tom) = F$ .

# How can we use predicates?

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

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Let  $C(x) = \text{"}x \text{ is playing chess."}$

There are three persons in the room: *Ed*, *Paul*, and *Tom*.

*Ed* and *Paul* are playing chess, and *Tom* is sleeping.

Formally:  $C(\textit{Ed}) = T$ ,  $C(\textit{Paul}) = T$ ,  $C(\textit{Tom}) = F$ .

Consider a proposition "Someone is playing chess in the room."

Is it true?

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Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Let  $C(x)$  = “ $x$  is playing chess.”

There are three persons in the room: *Ed*, *Paul*, and *Tom*.

*Ed* and *Paul* are playing chess, and *Tom* is sleeping.

Formally:  $C(\textit{Ed}) = T$ ,  $C(\textit{Paul}) = T$ ,  $C(\textit{Tom}) = F$ .

Consider a proposition “Someone is playing chess in the room.”

Is it true?

Yes, because, for example,  $C(\textit{Ed}) = T$ .

# How can we use predicates?

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

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Let  $C(x)$  = “ $x$  is playing chess.”

There are three persons in the room: *Ed*, *Paul*, and *Tom*.

*Ed* and *Paul* are playing chess, and *Tom* is sleeping.

Formally:  $C(Ed) = T$ ,  $C(Paul) = T$ ,  $C(Tom) = F$ .

Another proposition “Everyone is playing chess in the room.”

# How can we use predicates?

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Let  $C(x) = \text{"}x \text{ is playing chess."}$

There are three persons in the room: *Ed*, *Paul*, and *Tom*.

*Ed* and *Paul* are playing chess, and *Tom* is sleeping.

Formally:  $C(Ed) = T$ ,  $C(Paul) = T$ ,  $C(Tom) = F$ .

Another proposition "Everyone is playing chess in the room."

It's false, because  $C(Tom) = F$ .



# Always true or sometimes true ?

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Let  $P(x) = "x^2 \geq 0."$

# Always true

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Let  $P(x) = "x^2 \geq 0."$

*Always true:*

"For all  $n$ ,  $P(n)$  is true."

"For all  $x$ ,  $x^2 \geq 0."$

" $P(n)$  is true for every  $n$ ."

" $x^2 \geq 0$  for every  $x$ ."

# Always true

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“For all  $x$ ,  $x^2 \geq 0$ .”

An assertion that a predicate is always true is called  
a *universal quantification*.

# Always true or sometimes true?

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Another predicate:

Let  $Q(x) : "5x^2 - 7 = 0."$

It's true only when  $x = \pm\sqrt{7/5}$ .

# Sometimes true

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Let  $Q(x) : "5x^2 - 7 = 0."$

It's true only when  $x = \pm\sqrt{7/5}$ .

*Sometimes true:*

"There exist an  $n$  such that  
 $P(n)$  is true."

"There exist an  $x$  such that  
 $5x^2 - 7 = 0."$

" $P(n)$  is true for some  $n$ ."

" $5x^2 - 7 = 0$  for some  $x$ ."

" $P(n)$  is true for at least one  $n$ ."

" $5x^2 - 7 = 0$  for at least one  $x$ ."

# Sometimes true

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“Exists an  $x$  such that  $5x^2 - 7 = 0$  is true.”

An assertion that a predicate is true for some values of the  
variable is called

an *existential quantification*.

# Sentences can be ambiguous

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“If you can solve *any* problem we come up with, then you get an A for the course.”

Is it a universal (for all), or an existential (for some) quantification?

# Sentences can be ambiguous

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

The last sentence was ambiguous.  
The right way to say it in math class:

*Universal (always):*

“You can solve *every* problem we come up with.”

*Existential (sometimes):*

“You can solve *at least one* problem we come up with.”



# Mathematical notation

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“For every  $x$ , it's true that  $x + 1 > x$ ”

*becomes*

$$\forall x (x + 1 > x)$$

# Mathematical notation

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“For every  $x$ ,  $P(x)$ ”

*becomes*

$$\forall x (P(x))$$

# Mathematical notation

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“Exists  $x$  such that  $x^2 = 4$ ”

*becomes*

$$\exists x (x^2 = 4)$$

# Mathematical notation

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“Exists  $x$  such that  $P(x)$ ”

*becomes*

$$\exists x (P(x))$$

# Mathematical notation

## *Universal:*

$\forall x P(x)$  means that *for every*  $x$ ,  $P(x)$  is true.

## *Existential:*

$\exists x P(x)$  means that there *exists* an  $x$  such that  $P(x)$  is true.

We consider only those values of the variable  $x$  that belong to a given set called the *domain of discourse*, or the *universe of discourse*.

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

# Example of the universe of discourse

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

For the predicate  $Odd(x)$  and  $Even(x)$ , the universe of discourse is the set of all integers:

$\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots$

$Odd(x)$  is true for -5, -3, -1, 1, 3, 5, etc.

$Even(x)$  is true for -4, -2, 0, 2, 4, etc.

We consider only those values of the variable  $x$  that belong to a given set called the *domain of discourse*, or the *universe of discourse*.

# To prove or disprove a quantification

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Statement	When true?	When false?
$\forall x P(x)$	$P(x)$ is true for every $x$ .	There is at least one <i>counterexample</i> $x$ such that $P(x)$ is false.
$\exists x P(x)$	There is at least one $x$ such that $P(x)$ is true.	$P(x)$ is false for every $x$ .

# Example

Let  $P(x) = "x^2 > 0."$

The universe of discourse are all integer numbers.

Is it true or false that  $\forall x P(x)$ ?

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples



# Example

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Let  $P(x) = "x^2 > 0."$

The universe of discourse are all integer numbers.

Is it true or false that  $\forall x P(x)$ ?

To prove it, we must show that  $P(x)$  is true for all integers.

To disprove, we have to find a counterexample for which it's false.

# Example

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Let  $P(x) = "x^2 > 0."$

The universe of discourse are all integer numbers.

Is it true or false that  $\forall x P(x)$ ?

To prove it, we must show that  $P(x)$  is true for all integers.

To disprove, we have to find a counterexample for which it's false.

*Counterexample:*

$P(x)$  is false for  $x = 0$ . So, the quantified statement  $\forall x P(x)$  is false.

# Scope, variable binding

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

$$\forall x (P(x) \wedge Q(x))$$

The *scope* of the quantifier is the expression to which it's applied. Here, the scope is  $P(x) \wedge Q(x)$ . Quantifiers *bind* variables inside their scope.

$\forall$  binds  $x$  in the logical expression  $(P(x) \wedge Q(x))$ .

# Scope, variable binding

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Why should we care?

$$\forall x (P(x) \wedge Q(x)) \wedge \exists y (R(y))$$

is equivalent to

$$\forall x (\underline{P(x) \wedge Q(x)}) \wedge \exists x (\underline{R(x)})$$

“Every dog has four legs and has a tail; and there exists a dog that barks.”

# Scope, variable binding

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

If all variables are bound by a quantifier or set equal to a particular value then a statement is a proposition:

$\forall x (P(x) \wedge Q(z)) \wedge \exists y (R(y))$ , and it's given that  $z = 3.1415$ .

$x$  is bound by  $\forall$ ,  $y$  is bound by  $\exists$ , and  $z$  is specified by the given equation.

So, this is a proposition.

# Scope, variable binding

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

If a variable is not bound, it's called *free*.

$$\forall x (A(y) \wedge B(x)) \wedge \exists y (C(x, y) \rightarrow D(x)).$$

# Scope, variable binding

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

If a variable is not bound, it's called *free*.

$$\forall \underline{x} (A(\underline{y}) \wedge B(\underline{x})) \wedge \exists \underline{y} (C(\underline{x}, \underline{y}) \rightarrow D(\underline{y})).$$

Only the first  $x$  is bound by  $\forall$ .

This is not a proposition.

# Nested quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Quantifiers are nested if one is within the scope of the other:

$$\forall x (\exists y (x + y = 0))$$

It reads as follows

“For every  $x$  exists  $y$  such that  $x + y = 0$ .”

We usually drop the external parentheses. Equivalent expression:

$$\forall x \exists y (x + y = 0)$$



# Nested quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

A few more examples (you know this formulas):

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

$$\forall x \forall y (x + y = y + x)$$

$$\forall w \forall x \forall y \forall z \left( (y \neq 0 \wedge w \neq 0) \rightarrow \frac{x}{y} + \frac{z}{w} = \frac{xw + zy}{yw} \right)$$

# Order of quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Does the order of quantifiers matter?

$$\forall x (\exists y (x + y = 0))$$

“For every  $x$  exists  $y$  such that  $x + y = 0$ .”

$$\exists x (\forall y (x + y = 0))$$

“Exists  $x$  such that for every  $y$ :  $x + y = 0$ .”

# Order of quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

Does the order of quantifiers matter?

$$\forall x (\exists y (x + y = 0))$$

“For every  $x$  exists  $y$  such that  $x + y = 0$ .”

$$\exists x (\forall y (x + y = 0))$$

“Exists  $x$  such that for every  $y$ :  $x + y = 0$ .”

The meaning of the two expressions is different. You cannot exchange nested  $\forall$  and  $\exists$ .

# Order of quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

However, we can exchange two (or more) nested quantifiers of the same kind:

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

$$\forall y \forall x \forall z (x + (y + z) = (x + y) + z)$$

$$\forall z \forall x \forall y (x + (y + z) = (x + y) + z)$$

$$\exists x \exists y (x^y = 4)$$

$$\exists y \exists x (x^y = 4)$$

# Order of quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

These two statements are equivalent:

$$\forall w \exists x \forall y \forall z (P(w, x, y, z))$$

$$\forall w \exists x \underline{\forall z \forall y} (P(w, x, y, z))$$

But, these two statements are not equivalent:

$$\forall w \exists x \forall y \underline{\forall z} (P(w, x, y, z))$$

$$\forall w \underline{\forall z} \exists x \forall y (P(w, x, y, z))$$

# Order of quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“For every even integer  $n$  greater than 2, there exist prime numbers  $p$  and  $q$  such that  $n = p + q$ .”

(Prime numbers are integers  $> 1$ , divisible only by itself and 1.  
They are 2, 3, 5, 7, 11, 13, 17, ...)

The universe of discourse:

$n$  is an even integer,  $n > 2$ .

$p$  and  $q$  are prime numbers.

$$\forall n (\exists p (\exists q (n = p + q)))$$

# Order of quantifiers

Swapping the order of different kinds of quantifiers (existential or universal) changes the meaning of a proposition.

$$\forall n (\exists p (\exists q (n = p + q)))$$

$$4 = 2 + 2$$

$$6 = 3 + 3$$

$$8 = 3 + 5$$

$$10 = 5 + 5$$

$$\exists p (\exists q (\forall n (n = p + q)))$$

$$4 \neq 3 + 5$$

$$6 \neq 3 + 5$$

$$8 = 3 + 5$$

$$10 \neq 3 + 5$$

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

# Distributivity

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

$$\forall x (P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$$

we can distribute a universal quantifier  $\forall$  over a conjunction.

$$\exists x (P(x) \vee Q(x)) \equiv (\exists x P(x)) \vee (\exists x Q(x))$$

and we can distribute an existential quantifier  $\exists$  over a disjunction.

We **cannot** distribute  $\forall$  over disjunction, or  $\exists$  over conjunction.



# Negating quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“It is not the case that everyone likes to snowboard.”

$$\neg(\forall x S(x))$$

“There exists someone who does not like to snowboard.”

$$\exists x (\neg S(x))$$

# Negating quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“It is not the case that everyone likes to snowboard.”

$$\neg(\forall x S(x))$$

“There exists someone who does not like to snowboard.”

$$\exists x (\neg S(x))$$

To negate “ $\forall$ ”:

$$\neg(\forall x S(x)) \equiv \exists x (\neg S(x))$$

# Negating quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“There does not exist anyone who likes skiing over magma.”

$$\neg(\exists x M(x))$$

“Everyone dislikes skiing over magma.”

$$\forall x (\neg M(x))$$

# Negating quantifiers

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“There does not exist anyone who likes skiing over magma.”

$$\neg(\exists x M(x))$$

“Everyone dislikes skiing over magma.”

$$\forall x (\neg M(x))$$

To negate “ $\exists$ ”:

$$\neg(\exists x M(x)) \equiv \forall x (\neg M(x))$$

# Negating quantifiers

When negating more complex expressions with quantifiers, you “flip” the quantifier, and negate the expression to which the quantifier was applied.

$$\neg(\forall z (\exists y (\forall x (P(x) \wedge Q(y, z)))))$$

$$= \exists z \neg(\exists y (\forall x (P(x) \wedge Q(y, z))))$$

$$= \exists z \forall y \neg(\forall x (P(x) \wedge Q(y, z)))$$

$$= \exists z \forall y \exists x \neg(P(x) \wedge Q(y, z))$$

$$= \exists z (\forall y (\exists x (\neg P(x) \vee \neg Q(y, z))))$$

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

# Example

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“Everyone at Hunter College is smart.”

$$\forall x (AtHunter(x) \wedge Smart(x))$$

$$\forall x (AtHunter(x) \rightarrow Smart(x))$$

# Example

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“Everyone at Hunter College is smart.”

$\forall x (AtHunter(x) \wedge Smart(x))$  **Wrong!**

“Everyone is at Hunter College and is smart. No one is elsewhere.”

$\forall x (AtHunter(x) \rightarrow Smart(x))$

# Example

“Someone at City College is smart.”

$$\exists x (AtCCNY(x) \wedge Smart(x))$$

$$\exists x (AtCCNY(x) \rightarrow Smart(x))$$

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples



# Example

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“Someone at City College is smart.”

$$\exists x (AtCCNY(x) \wedge Smart(x))$$

$$\exists x (AtCCNY(x) \rightarrow Smart(x)) \textbf{Wrong!}$$

“There is someone, who is smart if he(he) is at City College.”

It is true if there is anyone who is not at City College, say in Boston.

# Example

“All lions are fierce.”

“Some lions do not drink coffee.”

“Some fierce creatures do not drink coffee.”

Predicates:

$L(x)$  =  $x$  is a lion.

$C(x)$  =  $x$  drinks coffee.

$F(x)$  =  $x$  is fierce.

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

# Example

Predicates

Quantifiers

Scope, bound  
variables

Nested quantifiers

Distributivity

Negation

Examples

“All lions are fierce.”

“Some lions do not drink coffee.”

“Some fierce creatures do not drink coffee.”

Predicates:

$L(x)$  =  $x$  is a lion.

$C(x)$  =  $x$  drinks coffee.

$F(x)$  =  $x$  is fierce.

$$\forall x (L(x) \rightarrow F(x))$$

$$\exists x (L(x) \wedge \neg C(x))$$

$$\exists x (F(x) \wedge \neg C(x))$$