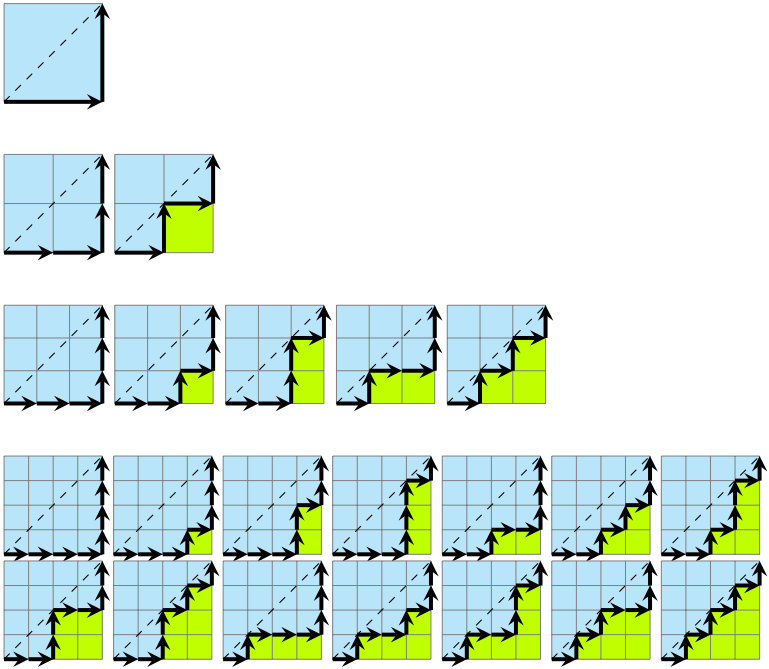
Counting

1, 2, 5, 14, …



Product Rule

Sum Rule

Finite Sets

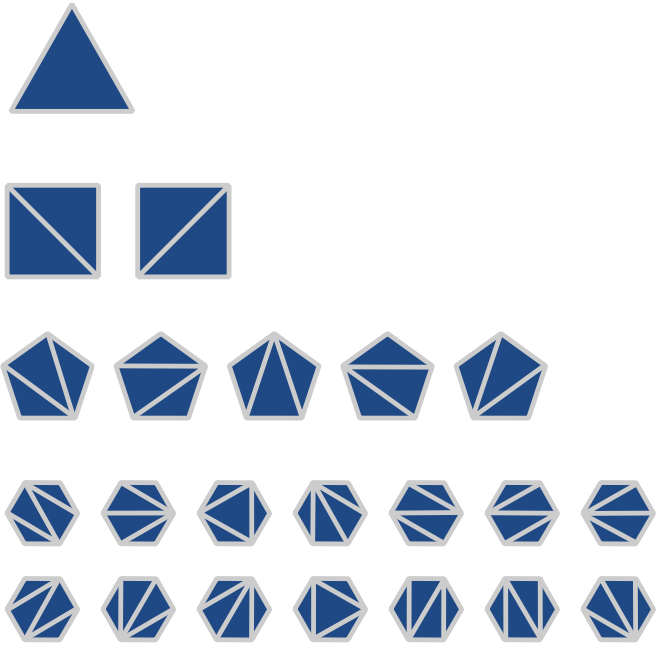
Overcounting

Subtraction Rule

Counting

Tree Diagrams

1, 2, 5, 14, . . .



Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

Problem

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

A-1 ... Z-100

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

Problem

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

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There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

$$26 \cdot 100$$

The Product Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

$$26 \cdot 100$$

The Product Rule. Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Problem 2

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

There are 32 computers in a computer center.

Each computer has 24 ports.

How many different ports to a computer in the center are there?

Problem 2

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

There are 32 computers in a computer center.

Each computer has 24 ports.

How many different ports to a computer in the center are there?

$$32 \cdot 24 = 768$$

There are 768 ways to choose one port in the center:

1. First, you choose a computer (can be done in 32 ways).
2. Then choose a port in that computer (in 24 ways).

Problem 3

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

There are 32 computers in a computer center.

Each computer has 24 ports.

How to scan all the ports in the computer center?

Problem 3

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

There are 32 computers in a computer center.

Each computer has 24 ports.

How to scan all the ports in the computer center?

```
for c := 1 to 32 do
    for p := 1 to 24 do
        scan(c, p)
```

Generalized Product Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

If a procedure consists of *k sub-tasks*, and the sub-tasks can be performed in $n_1, \ldots, n_k$ ways, then the procedure can be performed

$$\text{in } (n_1 \cdot n_2 \cdot \ldots \cdot n_k) \text{ ways.}$$

Example:
Count the number of different bit strings of length seven.

The value for each bit can be chosen in two ways (0 or 1). Therefore:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$$

Problem 4

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

How many different license plates of this format can be made?

Problem 4

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

How many different license plates of this format can be made?

$$26^3 \cdot 10^4 = 175,760,000$$

Problem 5

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

Problem 5

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

$$40 + 50 = 90 \quad \text{this is called the rule of sum}$$

The Sum Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

The Sum Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

```
for b := 1 to 40 do
    read(Sociology, b)
for b := 1 to 50 do
    read(Anthropology, b)
```

The Sum Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

The Sum Rule. If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Note that it's important that the two groups don't have common elements (We say that the sets are disjoint).

A new object

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

Def. A *set* is an unordered collection of objects. The objects are called elements.

If e is an element of the set A, we write $a \in A$.

Otherwise, if it's not in A, we write $a \notin A$.

Example:

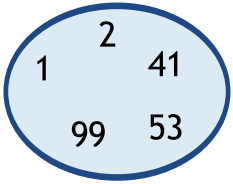
$$A = \{1, 2, 97, 3, 15\}.$$

$$1 \in A.$$

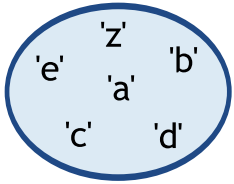
$$4 \notin A.$$

$$\{1, 2, 3, 3, 2, 1, 1, 1, 1\} = \{1, 2, 3\}$$

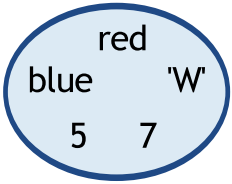
Sets



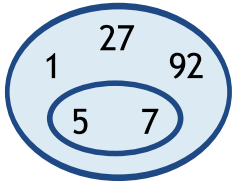
A = {1, 2, 41, 53, 99}



B = {'a', 'z', 'e', 'd', 'c', 'b'}



C = {'W', blue, 5, red, 7}



D = {27, 1, {5,7}, 92}

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

Some important sets

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

Natural numbers

$$\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ \ldots\}$$

Integer numbers

$$\mathbb{Z} = \{\ldots,\ -3,\ -2,\ -1,\quad 0,\quad 1,\quad 2,\quad 3,\ \ldots\}$$

Empty set

$$\varnothing = \{\ \}$$

Set Builder Notation

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

We can describe sets using predicates:

$$A = \{x \mid P(x)\}$$

"Set A is such that $x \in A$ if and only if $P(x)$."

Example. Positive integers:

$$Z^+ = \{n \in \mathbb{Z} \mid n > 0\} = \{1, 2, 3, \ldots\}$$

More complex predicates are fine too. Odd and even numbers:

$$Even = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}\, (n = 2k)\}$$
$$Odd = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}\, (n = 2k + 1)\}$$

Union, $\cup$

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

$A \cup B$ denotes all things that are *members of either A or B*:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Equivalently:

x belongs to $A \cup B$ if and only if $x \in A$ or $x \in B$.

Examples:

$$\{1, 2\} \cup \{\text{‘a’}, \text{‘b’}\} = \{1, 2, \text{‘a’}, \text{‘b’}\}$$
$$\{1, 2, 3\} \cup \{2, 3, 5\} = \{1, 2, 3, 5\}$$

Intersection, $\cap$

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

$A \cap B$ denotes all things that are *members of both A and B*:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Equivalently:

x belongs to $A \cap B$ if and only if $x \in A$ and $x \in B$.

Examples:

$$\{1, 2\} \cap \{\text{‘a’}, \text{‘b’}\} = \varnothing$$
$$\{1, 2, 3\} \cap \{2, 3, 5\} = \{2, 3\}$$

Sets A and B are called *disjoint* if their intersection is empty: $A \cap B = \varnothing$.

Number of elements of a finite set

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

Def. If set A is finite, and there are exactly n elements in S, then n is the *cardinality* of the set A. We write

$$|A| = n.$$

Examples:

$$\begin{aligned} A &= \{3,\ 4,\ 5,\ 6\} \\ |A| &= 4 \\ B &= \{\{3,\ 4\},\ \{5,\ 6\},\ 7\} \\ |B| &= 3 \\ |\varnothing| &= 0 \end{aligned}$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

$$A = \{1,\ 2,\ 4,\ 5\}$$

$$B = \{20,\ 21,\ 22,\ 23,\ 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| =$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

$$A = \{1,\ 2,\ 4,\ 5\}$$

$$B = \{20,\ 21,\ 22,\ 23,\ 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| = |A| + |B| = 4 + 5 = 9.$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

You are given k disjoint sets $A_1, \ldots A_k$:

It means that $A_i \cap A_j = \varnothing$ when $i \neq j$.

What is the cardinality if their union?

$$|A_1 \cup \ldots \cup A_k| =$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

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What is the cardinality if their union?

$$|A_1 \cup \ldots \cup A_k| = |A_1| + \ldots + |A_k| = \sum_{i=1}^{k} |A_i|.$$

Question

Product Rule
Sum Rule
Finite Sets
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Counting
Tree Diagrams

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What is the cardinality if their union?

$$|A_1 \cup \ldots \cup A_k| = |A_1| + \ldots + |A_k| = \sum_{i=1}^{k} |A_i|.$$

This is the same *rule of sum*, right?

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

Why do we insist on the sets being disjoint?

Really, who cares?

Because

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

$$A = \{1,\ 2,\ 3\}$$
$$B = \{3,\ 4\}$$

$$|A| + |B| = 3 + 2 = 5$$

$$|A \cup B| = \Big|\{1,\ 2,\ 3,\ 4\}\Big| = 4$$

So, in general, $|A \cup B| \neq |A| + |B|$, and if we try to use the sum rule when the sets are not disjoint, we *overcount*, and this is really bad.

Overcounting

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

$$A = \{1,\ 2,\ 3\}$$
$$B = \{3,\ 4\}$$

We were overcounting, because the common elements of A and B were counted twice:

$$|A| + |B| = \left|\{1,\ 2,\ 3\}\right| + \left|\{3,\ 4\}\right| = 5,$$
$$|A \cup B| = \left|\{1,\ 2,\ 3,\ 4\}\right| = 4.$$

Overcounting

Product Rule
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Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

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$$|A \cup B| = \left|\{1,\ 2,\ 3,\ 4\}\right| = 4.$$

Therefore, to get the correct value of $|A \cup B|$, the number of common elements must be subtracted:

$$|A \cup B| = \left|\{1,\ 2,\ 3\}\right| + \left|\{3,\ 4\}\right| - \left|\{3\}\right| = 3 + 2 - 1 = 4.$$

The Subtraction Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

$$A = \{1,\ 2,\ 3\}$$

$$B = \{3,\ 4\}$$

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$$|A \cup B| = \left|\{1,\ 2,\ 3\}\right| + \left|\{3,\ 4\}\right| - \left|\{3\}\right| = 3 + 2 - 1 = 4.$$

The general rule for arbitrary sets A and B:

$$|A \cup B| =$$

The Subtraction Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

$$A = \{1,\ 2,\ 3\}$$

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The general rule for arbitrary sets A and B:

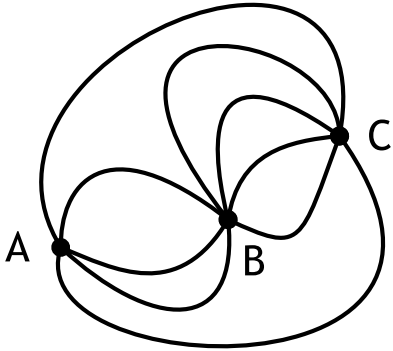
$$|A \cup B| = |A| + |B| - |A \cap B|$$

Countuing paths

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

This is a map with three cities, connected by roads.

Count the number of paths from city A to city C, such that each city is visited not more than once.



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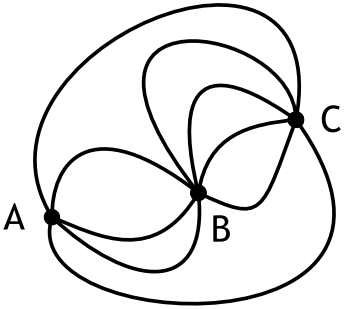

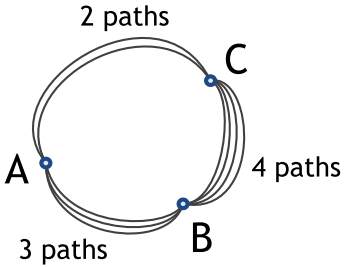

Countuing paths

Product Rule
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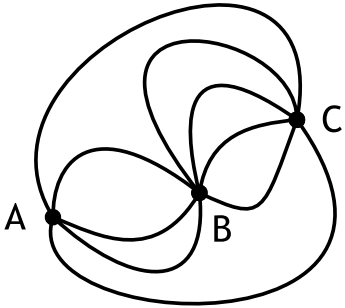
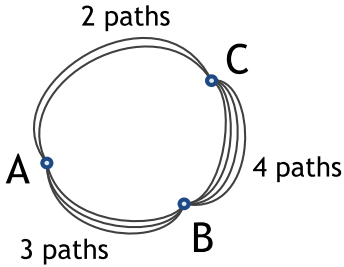




$A \to C$ or $A \to B \to C$:

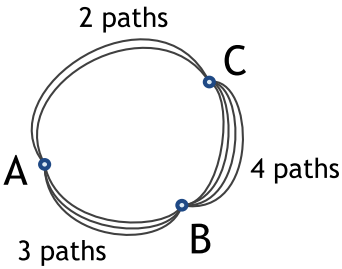
$$2 + 3 \cdot 4 = 14$$

Countuing round trips

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that cities A and C are visited not more than once.



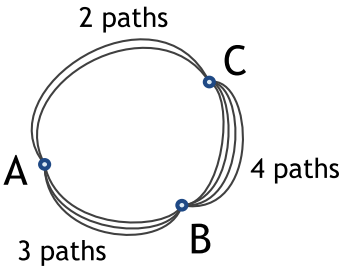
Countuing round trips

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that cities A and C are visited not more than once.

$B \to A \to B$	$3 \cdot 3 = 9$
$B \to C \to B$	$4 \cdot 4 = 16$
$B \to A \to C \to B$	$3 \cdot 2 \cdot 4 = 24$
$B \to C \to A \to B$	$4 \cdot 2 \cdot 3 = 24$
Total	73

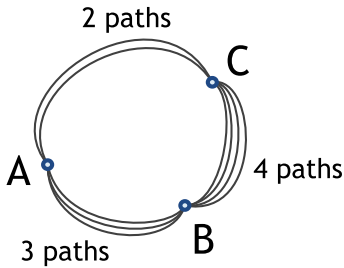


Countuing round trips II

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that
1) cities A and C are visited not more than once, and
2) each road is used not more than once during a trip.



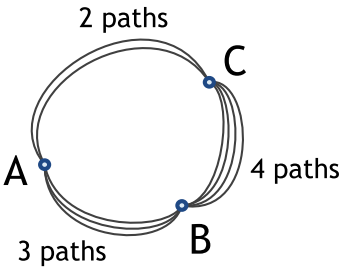
Countuing round trips II

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that
1) cities A and C are visited not more than once, and
2) each road is used not more than once during a trip.

$B \to A \to B$	$3 \cdot 2 = 6$
$B \to C \to B$	$4 \cdot 3 = 12$
$B \to A \to C \to B$	$3 \cdot 2 \cdot 4 = 24$
$B \to C \to A \to B$	$4 \cdot 2 \cdot 3 = 24$
Total	66



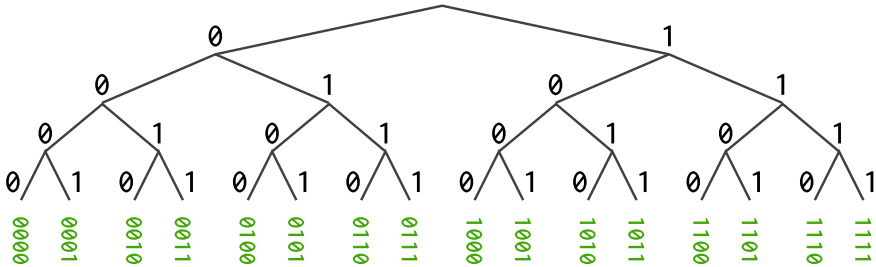
Tree Diagrams

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.
Each possible outcome is a leaf of the tree (an endpoint that does not branch).

Example: Count all bit strings of length four.



16 strings.

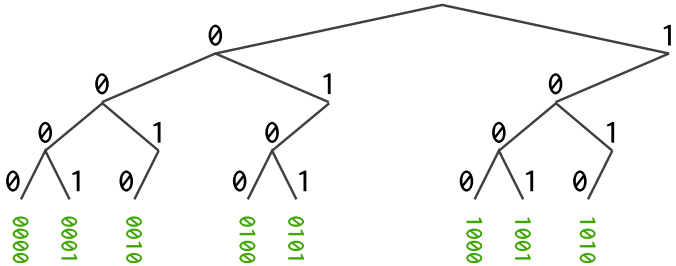
Tree Diagrams

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams

Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.
Each possible outcome is a leaf of the tree (an endpoint that does not branch).

Example 2: Count all bit strings of length four that *do not have two consecutive 1s*.



8 strings.