Infinity. Cardinality.
Pairing function. Diagonalization.

Infinity?

We know that the set of natural numbers, $\mathbb{N}$, is infinite, so, definitely, there are sets with infinitely many elements.

How is it possible to construct such sets?

Let's definine an operation

$$A^+ = A \cup \{A\}$$

We start with $\varnothing$ and apply this operation:

$$\begin{aligned}
\varnothing &= \varnothing \\
\varnothing^+ &= \{\varnothing\} \\
(\varnothing^+)^+ &= \{\varnothing, \{\varnothing\}\} \\
((\varnothing^+)^+)^+ &= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \\
&\dots
\end{aligned}$$

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$$\begin{aligned}
\text{‘}0\text{’} &= \varnothing = \varnothing \\
\text{‘}1\text{’} &= \varnothing^+ = \{\varnothing\} = \{\text{‘}0\text{’}\} \\
\text{‘}2\text{’} &= (\varnothing^+)^+ = \{\varnothing, \{\varnothing\}\} = \{\text{‘}0\text{’}, \text{‘}1\text{’}\} \\
\text{‘}3\text{’} &= ((\varnothing^+)^+)^+ = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{\text{‘}0\text{’}, \text{‘}1\text{’}, \text{‘}2\text{’}\} \\
&\ldots
\end{aligned}$$

This is von Neumann's construction of natural numbers.

Infinity

Suppose that you have infinitely many one dollar bills (numbered 1, 3, 5, . . .) and you come upon the Devil, who is willing to pay two dollars for each of your one-dollar bills.

The Devil is very particular, however, about the order in which the bills are exchanged. The contract stipulates that in each sub-transaction he buys from you your lowest-numbered bill and pays you with higher-numbered bills.

First sub-transaction takes 1/2 hour, then 1/4 hour, 1/8, and so on, so that after one hour the entire exchange will be complete.

Infinite sets

Consider four sets:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

$$\begin{aligned} Even_N &= \{0, 2, 4, 6, 8, \ldots\} \\ Odd_N &= \{1, 3, 5, 7, 9, \ldots\} \\ \mathbb{Z}^- &= \{-1, -2, -3, -4, \ldots\} \end{aligned}$$

Can we compare their cardinalities?

Infinite sets

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Can we compare their cardinalities?

We need a definition for the cardinality of an infinite set.

Cardinality of an infinite set

Def. The sets A and B have the same cardinality if and only if there is a bijection from A to B.

When A and B have the same cardinality, we write $|A| = |B|$.

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

$$Even_N = \{0, 2, 4, 6, 8, \ldots\}$$

$$Odd_N = \{1, 3, 5, 7, 9, \ldots\}$$

$$\mathbb{Z}^- = \{-1, -2, -3, -4, \ldots\}$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Even_N = \{0, 2, 4, 6, 8, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \rightarrow Even_N$$

$$0\bullet \quad 1\bullet \quad 2\bullet \quad 3\bullet \quad \cdots$$

$$\bullet 0 \quad \bullet 2 \quad \bullet 4 \quad \bullet 6 \quad \cdots$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Even_N = \{0, 2, 4, 6, 8, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to Even_N$$

$$\begin{tikzcd} 0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & \cdots \\ 0 & 2 & 4 & 6 & \cdots \end{tikzcd}$$

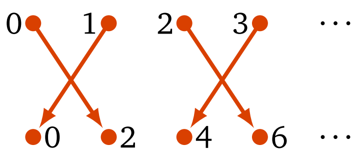
$$f(x) = 2x$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Even_N = \{0, 2, 4, 6, 8, \ldots\}$$

Alternatively

$$f : \mathbb{N} \rightarrow Even_N$$



$$\ldots$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Odd_N = \{1, 3, 5, 7, 9, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to Odd_N$$

$$0\bullet \quad 1\bullet \quad 2\bullet \quad 3\bullet \quad \cdots$$

$$\bullet 1 \quad \bullet 3 \quad \bullet 5 \quad \bullet 7 \quad \cdots$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$Odd_N = \{1, 3, 5, 7, 9, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to Odd_N$$

$$\begin{tikzcd} 0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[d] & \cdots \\ 1 & 3 & 5 & 7 & \cdots \end{tikzcd}$$

$$f(x) = 2x + 1$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$\mathbb{Z}^- = \{-1, -2, -3, -4, -5, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to \mathbb{Z}^-$$

$$0\bullet \quad 1\bullet \quad 2\bullet \quad 3\bullet \quad \cdots$$

$$\bullet{-1} \quad \bullet{-2} \quad \bullet{-3} \quad \bullet{-4} \; \cdots$$

Cardinality of an infinite set

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$\mathbb{Z}^- = \{-1, -2, -3, -4, -5, \ldots\}$$

Find a bijection

$$f : \mathbb{N} \to \mathbb{Z}^-$$

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ -1 & -2 & -3 & -4 & \cdots \end{array}$$

$$f(x) = -x - 1$$

Cardinality of an infinite set

Therefore, all these sets have the same cardinality

$$|\mathbb{N}| = |Even_N| = |Odd_N| = |\mathbb{Z}^-|$$

Countable sets

Therefore, all these sets have the same cardinality

$$|\mathbb{N}| = |Even_N| = |Odd_N| = |\mathbb{Z}^-|$$

Def. A set S is called *countable* if $|S| = |\mathbb{N}|$ or if S is a finite set.

Countable sets

Since $\mathbb{N}$ is an infinite set, the cardinality $|\mathbb{N}|$ is greater than any natural number. We need a way to denote the cardinality of this set.

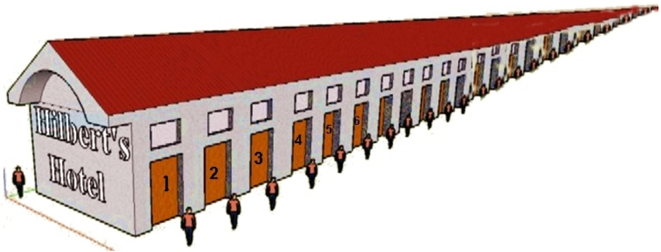
The following symbol is used

$$|\mathbb{N}| = \aleph_0$$

It reads as “aleph naught”, “aleph null”, “aleph zero”.

All infinite countable sets have the same cardinality $\aleph_0$.

Hilbert’s Hotel

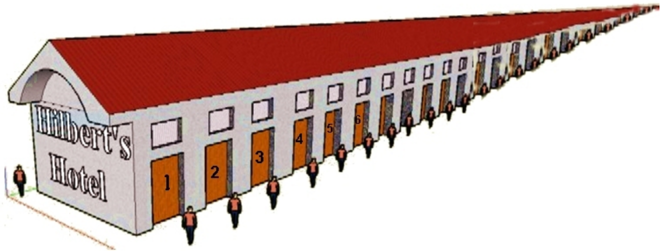


Imagine a hotel with a countably infinite number of rooms.

Each room is occupied by a guest.

Question: Can it accomodate one more guest?

Hilbert's Hotel

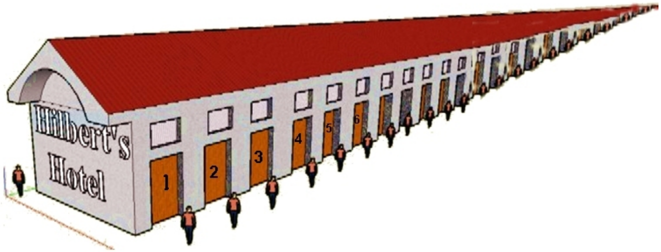


There is a bijection between $\{x\} \cup \mathbb{N}$ (guests) and $\mathbb{N}$ (rooms)

$$x\bullet \quad 0\bullet \quad 1\bullet \quad 2\bullet \quad 3\bullet \quad \cdots$$

$$\bullet 0 \quad \bullet 1 \quad \bullet 2 \quad \bullet 3 \quad \cdots$$

Hilbert’s Hotel



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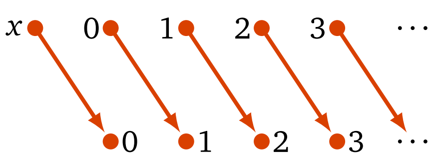

More complex cases

We want to prove that $B = \mathbb{N} \times \{T, F\}$ is countable.

Can we find a bijection between $\mathbb{N}$ and $B = \mathbb{N} \times \{T, F\}$?

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

$$B = \{(0, T), (1, T), (2, T), \dots (0, F), (1, F), (2, F), \dots\}$$

More complex cases

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$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

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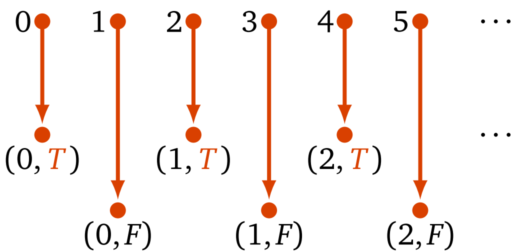

$$(0,T), (0,F), (1,T), (1,F), (2,T), (2,F), \ldots$$

More complex cases

Similarly, there is a bijection between $\mathbb{N}$ and $\mathbb{Z}$

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

We just rearrange the order of integers:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

In general, if there is a way *to list* the elements of a given set in linear order, then it is *countable* (i.e. there is a bijection between this set and $\mathbb{N}$).

More complex cases

Find a bijection $h : A \to B$, where

$$A = \mathbb{N} \times \{T, F\}$$

$$B = \mathbb{Z}$$

More complex cases

Find a bijection $h : A \longrightarrow B$, where

$$A = \mathbb{N} \times \{T, F\}$$

$$B = \mathbb{Z}$$

A and B are countable, and we know how to construct the following two bijections

$$f : \mathbb{N} \longrightarrow A$$
$$g : \mathbb{N} \longrightarrow B$$

Since f is a bijection, there exist an inverse function $f^{-1} : A \longrightarrow \mathbb{N}$, which is a bijection too, and we can find it, so

$$h(x) = g(f^{-1}(x))$$

$A \times \mathbb{N}$

We have shown that $\mathbb{Z}$ is countable, $\mathbb{N} \times \{T, F\}$ is countable.

Similarly, it's not hard to show that for any *finite* set A, its Cartesian products

$$A \times \mathbb{N} \text{ and } \mathbb{N} \times A \text{ are countable.}$$

$\mathbb{N} \times A$ and $A \times \mathbb{N}$ when A is finite

Similarly, it's not hard to show that for any *finite* set A, its Cartesian products

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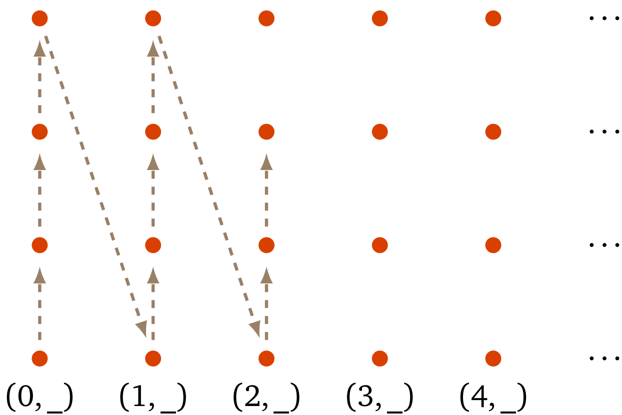

Is the set $\mathbb{N} \times \mathbb{N}$ countable?

Can we find a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$? If yes, then the set of ordered pairs of natural numbers, $\mathbb{N} \times \mathbb{N}$, is a countable set.

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & \cdots \\
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & \cdots
\end{array}
$$

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(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & \cdots
\end{array}
$$

Pairing function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$P(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y$$

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & \cdots \\
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & \cdots
\end{array}$$

The set of rational numbers, $\mathbb{Q}$

We can define the set of rational numbers as the set of all quotients p/q such that $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$:

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z} \wedge q \in \mathbb{Z}^+ \right\}$$

We can prove that $\mathbb{Q}$ is countable. The argument is similar to the proof for $\mathbb{N} \times \mathbb{N}$.

Power set

Is the power set $\mathscr{P}(\mathbb{N})$ countable?

Power set

Theorem. The power set $\mathscr{P}(\mathbb{N})$ is not countable.

Power set

Proof. (by contradiction)

Assume that $\mathscr{P}(\mathbb{N})$ is countable, so all subsets of $\mathbb{N}$ can be listed:

$$A_0, A_1, A_2, \ldots$$

We know that subsets can be encoded by sitrings of 1s and 0s.

Subset	0	1	2	3	4	5	...
A_0	0	0	0	1	0	0	...
A_1	1	1	1	0	0	1	...
A_2	1	1	1	1	1	1	...
A_3	0	0	0	0	0	1	...
A_4	1	0	0	0	0	1	...
A_5	1	1	0	0	1	1	...

Now, we want to construct a counter-example subset $C \subseteq \mathbb{N}$ that is different from each A_i.

Power set

Subset	0	1	2	3	4	5	...
A_0	*0*	0	0	1	0	0	...
A_1	1	*1*	1	0	0	1	...
A_2	1	1	*1*	1	1	1	...
A_3	0	0	0	*0*	0	1	...
A_4	1	0	0	0	*0*	1	...
A_5	1	1	0	0	1	*1*	...
...							
C	1	0	0	1	1	0	...

We construct a counter-example set C that is different from each subset A_i. How can we do it?

For all $i = 0, 1, 2, 3 \ldots$: Whenever $i \in A_i$, we choose $i \notin C$, and vice versa, when $i \notin A_i$, we choose $i \in C$. Thus, by construction, C is different from each A_i. Effectively, the set C inverts the diagonal.

Power set

Since $C \neq A_i$ for all i, and C is obviously a subset of $\mathbb{N}$ by construction, the list of subsets A_i does not contain all subsets of $\mathbb{N}$ (it does not contain C, for example), therefore, our assumption was incorrect: the subsets of $\mathbb{N}$ are not countable.

That is, *the power set $\mathscr{P}(\mathbb{N})$ is uncountable*. $\square$

This proof strategy is called diagonalization.

Similarly, we can show that the *unit interval* $0 \leq x \leq 1$ of real numbers is uncountable. (Also, see Rosen's book for the proof). And because you can make a bijection between this interval, $[0, 1]$, and $\mathbb{R}$, the set of all real number is uncountable.

More results about cardinality

Theorem. If A and B are countable sets, then their union $A \cup B$ is also countable.

Proof. Wihtout loss of generality, we can assume that A and B are disjoint. (If they are not, we continue the proof with A and $B \setminus A$)

If at least one of the sets is finite, we first list this set, then the other set.

Otherwise, if both are infinite countable sets, we list both sets by alternating elements:

$$a_0, b_0, a_1, b_1, a_2, b_2, \ldots$$

where $a_i \in A$ and $b_i \in B$. $\square$

Cardinality, one-to-one and onto

Mapping rules

If there is a *one-to-one* function $f : A \longrightarrow B$ then

$$|A| \leq |B|.$$

If there is an *onto* function $g : A \longrightarrow B$ then

$$|A| \geq |B|.$$

If there is a *bijection* $h : A \longrightarrow B$ then

$$|A| = |B|.$$

Schröder-Bernstein Theorem

Theorem (Schröder-Bernstein). Given two sets A and B, if there exist one-to-one functions $f : A \to B$ and $g : B \to A$, then there is a bijection between A and B.

In other words, to prove existence of a bijection, it's enough to prove existence of two one-to-one functions:

Once you have found a one-to-one function $f : A \to B$, instead of proving that f is onto, you can prove that there exists another one-to-one function that maps B to A.