Discrete Structures. CSCI-150. Spring 2017.

Homework 6. Due Mon. Mar 13, 2017.

Problem 1 (Graded)

A zombie apocalypse TV show starts with 10 human survivors, and 1 zombie. In every episode the following events occur:

- First, each zombie turns one human into a zombie.
- After that, the population of the remaining humans doubles (for example, new people join the group or new kids are born).

Assuming the show does not get canceled, does it have a happy ending?

Problem 2 (Graded)

We are going to prove that the following summation formula is correct for integer $n \geq 1$:

$$1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

First, check that it is correct for $n = 1$, $n = 2$, and $n = 3$.

After that, prove this formula by induction for all $n \geq 1$.

> Always <u>write inductive proofs in full</u>:
>
> - First, write what the base case is and give its proof.
> - Then do the inductive case:
>   1) write the assumption (the inductive hypothesis), then
>   2) write the formula you have to prove that follows from the hypothesis,
>   3) then write a proof for it.

Problem 3

Prove by induction that

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

<u>Hint:</u> Be careful, correctly identify what the base case should be.

Problem 4 (Graded)

Prove by induction that in the zombie apocalypse problem, if the population of the remaining humans grows faster and *triples* instead of doubling, then their population by the end of the episode N will be:

$$3 \cdot 2^N + 7 \cdot 3^N$$

Hint: Before doing the proof, first find a *recurrent formula*: if there are $H(k-1)$ humans at the beginning of the episode k, then how many humans, $H(k)$, is left at the end of the episode?

Hint 2: To figure out the recurrent formula, it may help to first find how many zombies will be there at the beginning of the episode k.

Problem 5

Given the recurrence

$$S(0) = 0,$$
$$S(n+1) = 3S(n) + 1,$$

prove by induction that for all $n \geq 0$:

$$S(n) = \frac{3^n - 1}{2}.$$

Problem 6 (Graded)

Given the recurrence

$$T(1) = 2,$$
$$T(n) = T(n-1) + 2n \qquad \text{(for } n > 1\text{)},$$

first, find the closed form expression for $T(n)$. You may apply the method we used in class, where we repeatedly substitute $T(n)$ in terms of $T(n-1)$, then $T(n-1)$ in terms of $T(n-2)$, and so on, eventually identifying the pattern. This method is also described in Lehman and Leighton's book (p.147), where it is called "Plug-and-Chug" method.

After that, prove by induction that the closed form expression you've found is correct.

Problem 7

Solve another recurrence (do the same steps as in the previous problem):

$$R(1) = 1,$$
$$R(n) = 2R(n/2) + n^2 \qquad \text{(for } n > 1\text{)},$$

(You can assume that n is a power of 2, that is, $n = 2^k$).

Hint: The closed-form formula for the recurrence will be $R(n) = n(2n-1)$.

Problem 8

Prove by induction that for all $n \geq 0$:

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

In the inductive step, use Pascal's identity, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Hint: So, we want to prove the following infinite sequence of identities by induction:

$$\binom{0}{0} = 2^0$$

$$\binom{1}{0} + \binom{1}{1} = 2^1$$

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 2^2$$

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$$

$$\vdots$$

It turns out each of the equations (except the very first one) can be proven from the previous if you correctly use Pascal's Identity. That's the essence of this task.