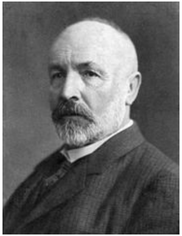
Sets. Ordered pairs.
Functions and relations.

Sets

The set theory is a branch of mathematical logic that was created by Georg Cantor in 1870s.



Def. A *set* is a unordered collection of objects being regarded as a single object.

Examples:

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{x \in A \mid (x \geq 3) \land (x \text{ is odd})\}$$

$$C = \{\varnothing, \{A\}, \{B\}, \{A, B\}\}$$

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Sets

If x belongs to a set A, we say that it is a *member* (or an element) of A and write

$$x \in A.$$

If x is not a member of A, we write

$$x \notin A.$$

Empty set, denoted by $\varnothing$ or $\emptyset$, has no members:

$$\forall x\ (x \notin \varnothing).$$

Sets

Two sets A and B are *equal*, $A = B$, iff they have exactly the same elements:

$$\forall x\,(x \in A \longleftrightarrow x \in B)$$

For any two objects x and y, we can make a set containing exactly these two objects

$$\{x, y\}$$

If those two objects are identical, $x = y$, we get a singleton set,

$$\{x, x\} = \{x\}.$$

Notice that these sets are not equal:

$$\varnothing, \quad \{\varnothing\}, \quad \{\varnothing, \{\varnothing\}\}$$

Sets

Set-builder notation.

A set of all objects that satisfy the property P:

$$A = \{x \mid P(x)\}$$

Examples:

$$B = \{x \in \mathbb{Z} \mid x \text{ is even}\}$$

$$C = \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}\,(x = 2k)\}$$

$$D = \{x \mid (x \in \mathbb{Z}) \wedge (\exists k \in \mathbb{Z}\,(x = 2k))\}$$

$$E = \{0, 2, -2, 4, -4, 6, -6, \ldots\}$$

In naive set theory, any definable collection is a valid set. And usually it works fine.

However, it leads to contradictions, such as Russell's paradox.

Russell's paradox

Let R be the set of all sets that are not members of themselves:

$$R = \{x \mid x \notin x\}$$

It is legal to ask, is R a member of itself or not.

There are only two cases, either $R \in R$, or $R \notin R$.

So, where is the paradox?

Russell's paradox

$$R = \{x \mid x \notin x\}$$

(a) If R is a member of itself, then by its definition, $R \notin R$,

$$R \in R \rightarrow R \notin R.$$

(b) Otherwise, if R is not a member of itself, then $R \in R$,

$$R \notin R \rightarrow R \in R.$$

Therefore, we get a contradiction, $R \in R \longleftrightarrow R \notin R$.

There exist several axiomatic systems that rule out such pathological cases. For example, Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC).

Union and Intersection

Union of two sets A and B,

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

$$\{1,2\} \cup \{2,3\} = \{1,2,3\}$$

Intersection of two sets A and B,

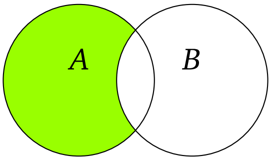
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

$$\{1,2\} \cap \{2,3\} = \{2\}$$

Difference

Difference between two sets A and B,

$$A \setminus B = \{x \mid (x \in A) \land (x \notin B)\}$$

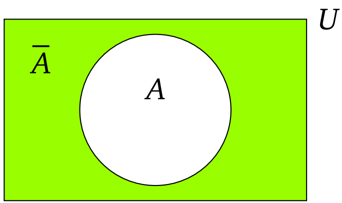


$$\{1,2\} \setminus \{2,3\} = \{1\}$$

Complement

If there is a *universal set* U of all possible objects, then the *complement* of a set A is

$$\overline{A} = U \setminus A = \{x \in U \mid x \notin A\} = \{x \in U \mid \neg(x \in A)\}$$



Exmaple: When $U = \mathbb{Z}$:

$$Odd = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$$

$$Even = \overline{Odd} = \mathbb{Z} \setminus A$$

Set identities

Given the universal set U, sets with respect to union, intersection, and complement satisfy the same identities as propositions with respect to $\wedge$, $\vee$, and $\neg$

Sets:	$A \cap B$	$A \cup B$	$\overline{A}$	U	$\emptyset$
Propositions:	$p \wedge q$	$p \vee q$	$\neg p$	T	F

$$\overline{\overline{A}} = A$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$$
$$A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$$

See the full list in Rosen's book.

Set identities

Let's prove De Morgan's law for sets:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cup B} = \{x \mid \neg((x \in A) \vee (x \in B))\}$$
$$\overline{A} \cap \overline{B} = \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$$

The propositions in the right hand sides of the equations are equal, therefore, the left hand sides are equal too.

Subset

A is a *subset* of B iff every element of A is an element of B:

$$A \subseteq B \quad \longleftrightarrow \quad \forall x((x \in A) \rightarrow (x \in B))$$

$$\{1,2\} \subseteq \{1,2,3\}$$

$$\{1,2,3\} \subseteq \{1,2,3\}$$

$$\varnothing \subseteq \{1,2,3\}$$

A is a *proper subset* of B iff A is a subset of B, but it's not equal to B

$$A \subsetneq B \quad \longleftrightarrow \quad \forall x((x \in A) \rightarrow (x \in B)) \;\wedge\; \exists x((x \in B) \wedge (x \notin A))$$

$$\{1,2\} \subsetneq \{1,2,3\}$$

$$\varnothing \subsetneq \{1,2,3\}$$

Proper subset A is strictly "smaller" than B.

Power set

Def. The set of all subsets of A is called a *power set* of A, denoted by $\mathscr{P}(A)$.

Examples:

$$\mathscr{P}(\{0,1\}) = \big\{\varnothing,\ \{0\},\ \{1\},\ \{0,1\}\big\}$$

$$\mathscr{P}(\{0,1,2\}) =$$

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$$\mathscr{P}(\{0,1,2\}) = \big\{\varnothing,\ \{0\},\ \{1\},\ \{2\},\ \{0,1\},\ \{0,2\},\ \{1,2\},\ \{0,1,2\}\big\}$$

Cardinality

Def. The *cardinality* of a finite set A is equal to the number of elements in A. It's denoted by $|A|$.

$$|\varnothing| = 0$$
$$|\{0, 1, 2, 3, 4\}| = 5$$

We already know the subtraction rule for the cardinality of a union:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Question

Compute the cardinality of the power set $\mathscr{P}(A)$ if $|A| = n$.

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$$|\mathscr{P}(A)| = 2^{|A|} = 2^n$$

Generalized union and intersection

Union of a finite set of sets:

$$\bigcup\{A_0\} = A_0$$

$$\bigcup\{A_0, A_1, \ldots A_n\} = \bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \ldots \cup A_n$$

Intersection of a finite set of sets:

$$\bigcap\{A_0\} = A_0$$

$$\bigcap\{A_0, A_1, \ldots A_n\} = \bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap \ldots \cap A_n$$

Ordered pair

Def. The *ordered pair* of $a \in A$ and $b \in B$ is an ordered collection (a, b).

Two ordered pairs are equal

$$(a, b) = (c, d) \quad \text{if and only if} \quad (a = c) \wedge (b = d).$$

Observe that this property implies that

$$(a, b) \neq (b, a)$$

unless $a = b$. So, the order matters. This is why it is called the ordered pair, and (a, b) is not equivalent to a set $\{a, b\}$.

$$(1, 2) = (1, 2)$$
$$(1, 2) \neq (1, 3)$$
$$(1, 2) \neq (2, 1)$$

Ordered pair

More examples of ordered pairs:

$$
\begin{aligned}
&(1, 2) \\
&(\{1\}, \{2\}) \\
&(1, \{2, 3, 4, 5\}) \\
&((1, 2), \varnothing)
\end{aligned}
$$

If $a \in A$ and $b \in B$, what is the set of all ordered pairs (a, b)?

Cartesian product

Def. Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

(Named after Rene Decartes)

Question. Given two sets $A = \{1, 2, 3\}$ and $B = \{C, D\}$, what is their Cartesian product $A \times B$?

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$$\begin{aligned} A \times B = \{&(1, C),\ (2, C),\ (3, C), \\ &(1, D),\ (2, D),\ (3, D)\} \end{aligned}$$

Cartesian product

Question. If the $|A| = n$, and $|B| = m$, what is the cardinality of $A \times B$?

Building a list

Ordered pair is fundamental for defining data types.
Question. How to implement the list data type with ordered pairs?

Example of a list:

$$[1,2,3,4]$$

Interface:

$$\begin{aligned} \text{construct}\,(\,1,[2,3,4]\,) &= [1,2,3,4] \\ \text{head}\,(\,[1,2,3,4]\,) &= 1 \\ \text{tail}\,(\,[1,2,3,4]\,) &= [2,3,4] \end{aligned}$$

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$$\text{construct}\,(\,1, [2, 3, 4]\,) = [1, 2, 3, 4]$$
$$\text{head}\,(\,[1, 2, 3, 4]\,) = 1$$
$$\text{tail}\,(\,[1, 2, 3, 4]\,) = [2, 3, 4]$$

Possible implementation:

$$(1, (2, (3, 4)))$$
$$\text{construct}\,(\,h, t\,) = (h, t)$$
$$\text{head}\,(\,(h, t)\,) = h$$
$$\text{tail}\,(\,(h, t)\,) = t$$

Ordered n-tuple

Def. The *ordered n-tuple* of is an ordered collection $(a_1, a_2, \ldots, a_n)$.

It is just an extension of an ordered pair for joining n elements together.

Def. The *Cartesian product* of the sets A_1, $\ldots A_n$, is the set of all n-tuples such that

$$A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}$$

If all sets A_i are equal, that is, $A_1 = \ldots = A_n = A$, then their Cartesian product is denoted by A^n

$$A_1 \times \ldots \times A_n = \underbrace{A \times \ldots \times A}_{n \text{ times}} = A^n$$

Subsets of the Catresian product

Given two sets

$$A = \{1, 2, 3\}$$
$$B = \{1, 2, 3, 4\}$$

Their Cartesian product

$$\begin{aligned} A \times B = \{&(1,1),\ (2,1),\ (3,1),\\ &(1,2),\ (2,2),\ (3,2),\\ &(1,3),\ (2,3),\ (3,3),\\ &(1,4),\ (2,4),\ (3,4)\} \end{aligned}$$

Consider three subsets of $A \times B$:

$$\begin{aligned} R_{(id)} &= \{(1,1),\ (2,2),\ (3,3)\}\\ R_{(less)} &= \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}\\ R_{(inc)} &= \{(1,2),\ (2,3),\ (3,4)\} \end{aligned}$$

Relations

Def. A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B.

$$R_{(id)} = \{(1,1),\ (2,2),\ (3,3)\}$$
$$R_{(less)} = \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}$$
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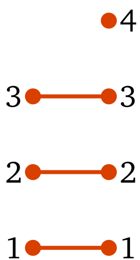
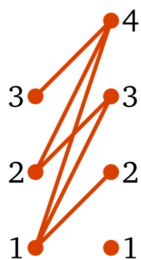
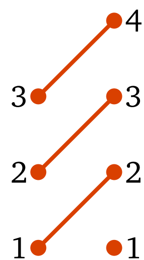






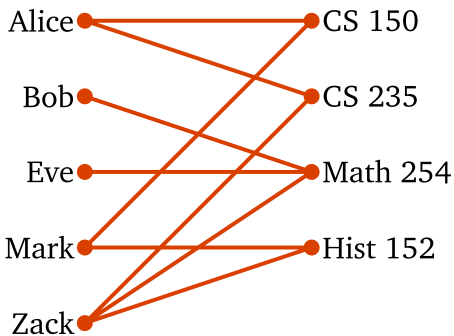
Relations

Example of a relation:

$$S = \text{set of students}$$

$$C = \text{set of classes}$$

$$R = \{(s, c) \mid \text{student } s \text{ takes class } c\}$$



Functions

Def. A relation $R \subseteq A \times B$ is a *function* (a functional relation) if for every $a \in A$, there is at most one $b \in B$ so that $(a, b) \in R$.

$$R_{(id)} = \{(1,1),\ (2,2),\ (3,3)\}$$
$$R_{(less)} = \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}$$
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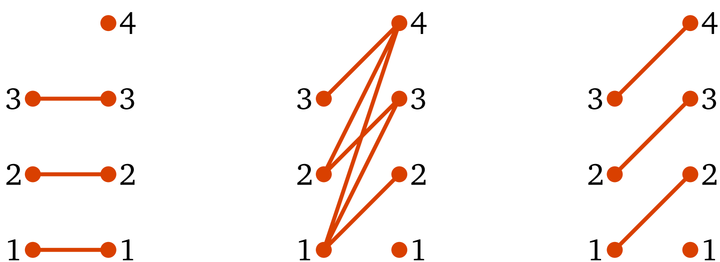


Functions

Functional relation $R \subseteq A \times B$ defines a unique way to map each element from the set A to an element from the set B.

There is a well-known and convenient notation for functions:

$$f(a) = b \qquad \text{where } a \in A \text{ and } b \in B$$

It maps elements from A to B:

$$f : A \longrightarrow B$$

$$A \xrightarrow{f} B$$

Functions

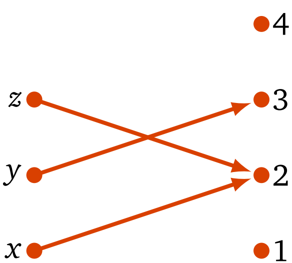
Def. For the function $f : A \to B$, set A is called *domain*, and set B is called *codomain*.

Def. $f(a)$ is the *image* of $a \in A$.

Def. The *image* of f, denoted by $f(A)$, is the set of the images $f(a)$ for all $a \in A$

$$f(A) = \{x \mid \exists a \in A\,(f(a) = x)\}.$$

The image of a function is also called *range*.



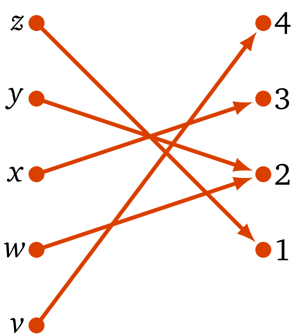
$$\begin{gathered} f : A \to B \\ \mathrm{domain}(f) = A = \{x, y, z\} \\ \mathrm{codomain}(f) = B = \{1, 2, 3, 4\} \\ f(A) = \{2, 3\} \end{gathered}$$

Onto

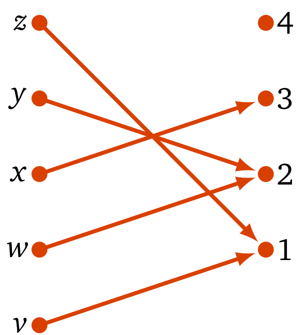
Def. A function $f : A \to B$ is called *onto* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

In other words, the image $f(A)$ is the whole codomain B.

$f : A \to B$ is onto



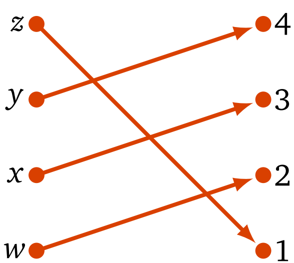
$g : A \to B$ is not onto



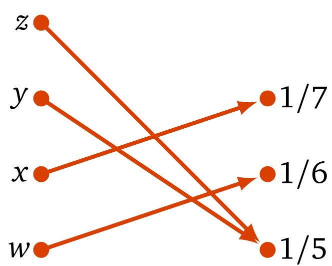
One-to-one

Def. A function $f : A \rightarrow B$ is said to be *one-to-one* if and only if $f(x) = f(y)$ implies that $x = y$ for all $x, y \in A$.

$f : A \rightarrow B$ is one-to-one



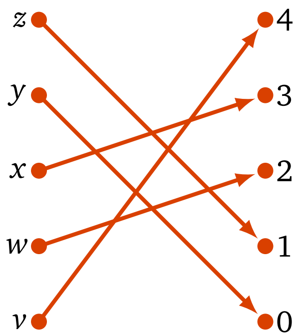
$g : A \rightarrow C$ is not one-to-one



Bijection

Def. The function f is a *bijection* (also called one-to-one correspondence) if and only if it is both one-to-one and onto.

$f : A \to B$ is a bijection



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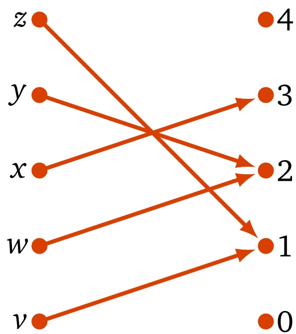

Infinity?

We know that the set of natural numbers, $\mathbb{N}$, is infinite, so, definitely, there are sets with infinitely many elements.

How is it possible to construct such sets?

Let's definine an operation

$$A^+ = A \cup \{A\}$$

We start with $\varnothing$ and apply this operation:

$$\begin{aligned}
\varnothing &= \varnothing \\
\varnothing^+ &= \{\varnothing\} \\
(\varnothing^+)^+ &= \{\varnothing, \{\varnothing\}\} \\
((\varnothing^+)^+)^+ &= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \\
&\dots
\end{aligned}$$

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$$\begin{aligned}
0 &= \varnothing = \varnothing \\
1 &= \varnothing^+ = \{\varnothing\} = \{0\} \\
2 &= (\varnothing^+)^+ = \{\varnothing, \{\varnothing\}\} = \{0, 1\} \\
3 &= ((\varnothing^+)^+)^+ = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{0, 1, 2\} \\
&\dots
\end{aligned}$$

This is von Neumann's construction of natural numbers.

Infinity

Suppose that you have infinitely many one dollar bills (numbered 1, 3, 5, ...) and you come upon the Devil, who is willing to pay two dollars for each of your one-dollar bills.

The Devil is very particular, however, about the order in which the bills are exchanged. The contract stipulates that in each sub-transaction he buys from you your lowest-numbered bill and pays you with higher-numbered bills.

First sub-transaction takes 1/2 hour, then 1/4 hour, 1/8, and so on, so that after one hour the entire exchange will be complete.