

Strong Induction.  
Catalan Numbers.

# A game

A game

Strong Induction

Another game

Catalan Numbers

The game starts with a stack of  $n$  coins. In each move, you divide one stack into two nonempty stacks.

$$\begin{aligned} |||| &\rightarrow ||| + || \\ &\rightarrow ||| + | + | \\ &\rightarrow || + | + | + | \\ &\rightarrow | + | + | + | + | \end{aligned}$$

If the new stacks have height  $a$  and  $b$ , then you score  $ab$  points for the move.

$$|||| \rightarrow ||| + || \quad \text{you get } 3 \cdot 2 = 6 \text{ points}$$

What is the maximum score you can get?

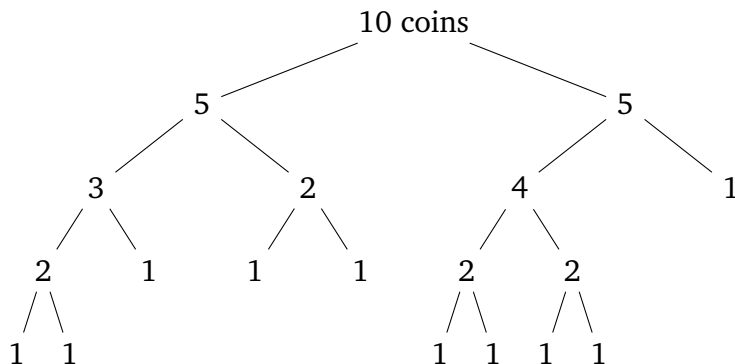
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The total score:  $25 + 6 + 4 + 2 + 1 + 4 + 1 + 1 + 1 = 45$  points.

Can we find a better strategy?

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**Theorem.** Every way of unstacking  $n$  coins gives a score of

$$\frac{n(n-1)}{2} \text{ points.}$$

And we want to prove it by induction.

Let  $P(n)$  be the proposition that every way of unstacking  $n$  coins gives a score of  $n(n-1)/2$ .

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In the inductive step we have to show that

$$P(n) = P(k) + P(n-k) + k(n-k) = \frac{n(n-1)}{2}.$$

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We would like to have something a little stronger than ordinary induction.

# Strong induction

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Principle of Strong Induction. Let  $P(n)$  be a predicate. If

- $P(0)$  is true, and
- for all  $n \in \mathbb{N}$ ,  $P(0), P(1), \dots, P(n)$  imply  $P(n+1)$ ,

then  $P(n)$  is true for all  $n \in \mathbb{N}$ .

Strong induction *allows you to assume*  $P(0), \dots, P(n)$  in the inductive step, whereas in ordinary induction, you assume  $P(n)$  only.

# Strong induction

A game

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Another game

Catalan Numbers

Strong induction is *no more powerful* than ordinary induction.

Consider a predicate

$$Q(n) = \forall k (0 \leq k \leq n \rightarrow P(k))$$

$Q(n)$  says that  $P(k)$  is true for all  $0 \leq k \leq n$ .

Ordinary induction on the predicate  $Q(n)$  is equivalent to strong induction on  $P(n)$ .

Any theorem that can be proved with strong induction can also be proved with ordinary induction. However, an appeal to the strong induction principle can make some proofs a bit simpler.



# Unstacking $n$ coins

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Catalan Numbers

**Theorem.** Every way of unstacking  $n$  coins gives a score of

$$\frac{n(n-1)}{2} \text{ points.}$$

**Proof.** By strong induction. Let  $P(n)$  be the proposition that every way of unstacking  $n$  coins gives a score of  $n(n-1)/2$ .

*The base case:*

When  $n = 1$ , no moves is possible, so the score is  $P(1) = 0$ . The formula works.

*The inductive step:*

$$\begin{aligned} P(n) &= P(k) + P(n-k) + n(n-k) \\ &= \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} + k(n-k) \\ &= \frac{1}{2}(k^2 - k + n^2 - nk - n - nk + k^2 + k + 2kn - 2k^2) = \frac{n(n-1)}{2}. \end{aligned}$$

# Another game

- A game
- Strong Induction
- Another game
- Catalan Numbers

Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches.

The player who removes the last match wins the game.

Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

# Another game

A game

Strong Induction

Another game

Catalan Numbers

Let  $P(n)$  be the proposition that the second player has a winning strategy if each pile contains  $n$  matches.

*The base case:*

$P(1)$  is true, because if each pile contains just 1 match, there is an obvious strategy for the second player.

*The inductive step:* When  $n > 1$ .

# Another game

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Let  $P(n)$  be the proposition that the second player has a winning strategy if each pile contains  $n$  matches.

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*The inductive step:* When  $n > 1$ .

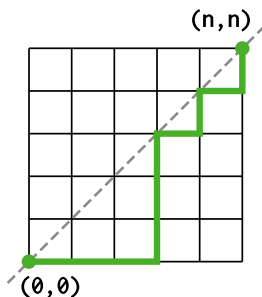
If the first player removes  $n$  matches from the first pile, the second player removes  $n$  matches from the other pile and wins.

Otherwise, if the first player removes  $k < n$  from the first pile, the second player is doing the same, removing  $k$  matches from the other pile, so there are  $n-k$  matches remain in both piles. And now, by the inductive hypothesis, for  $n-k$  matches the second player has a winning strategy.

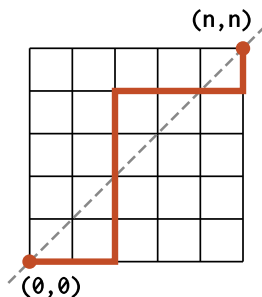
# Count the number of good paths, $C_n$

A game  
Strong Induction  
Another game  
Catalan Numbers

Paths should go entirely below the diagonal line



Good paths are  
below the diagonal



Bad paths cross it

The number of such paths,  $C_n$ , is the  $n^{\text{th}}$  Catalan number.

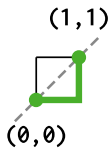
# Count the number of good paths, $C_n$

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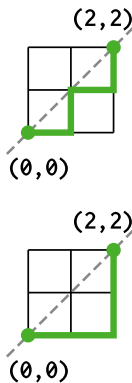
The cases when  $n$  is small: 0, 1, 2.



$$C_0 = 1$$



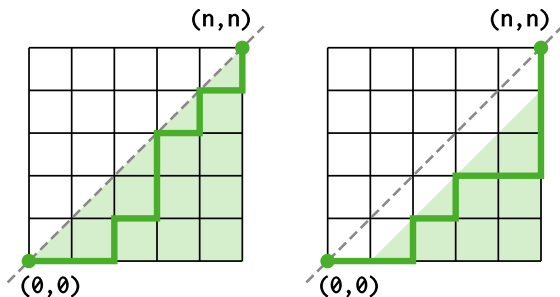
$$C_1 = 1$$



$$C_2 = 2$$

# Recurrent formula

$C_n$  is the number of paths that go below the diagonal (or touch the diagonal).

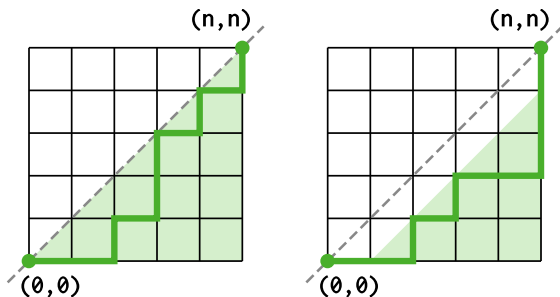


Introduce  $D_n$ , the number of paths that don't touch the diagonal in the middle points of the path.

# Recurrent formula

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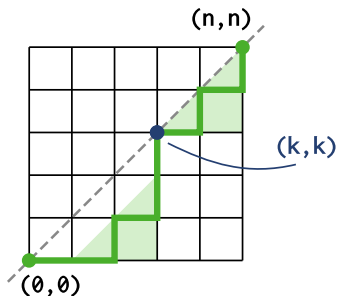
$$D_n = C_{n-1}$$



# Recurrent formula

A game  
Strong Induction  
Another game  
Catalan Numbers

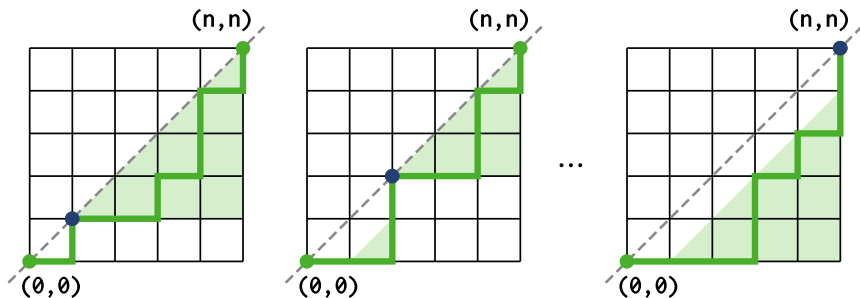
$(k, k)$  be the first point of the given path that is on the diagonal and  $k \neq 0$ .



Given  $(k, k)$ , the number of paths is  $D_k C_{n-k}$

# Recurrent formula

A game  
Strong Induction  
Another game  
Catalan Numbers



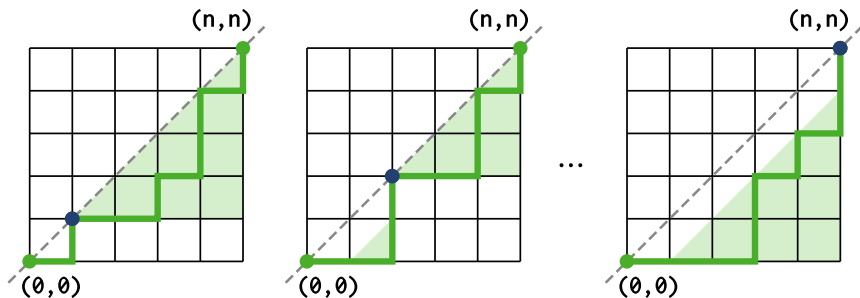
The diagonal point can be anywhere:  $(1, 1), (2, 2), \dots, (n, n)$

So, to count the total number of paths, we add up these  $n$  cases:

$$C_n = D_1 C_{n-1} + D_2 C_{n-2} + \dots + D_n C_0 = \sum_{k=1}^n D_k C_{n-k}$$

# Recurrent formula

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Strong Induction  
Another game  
Catalan Numbers



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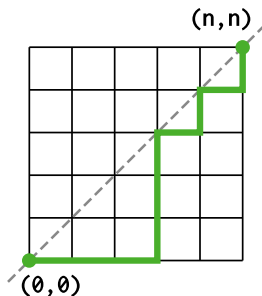
So, to count the total number of paths, we add up these  $n$  cases:

$$C_n = D_1 C_{n-1} + D_2 C_{n-2} + \dots + D_n C_0 = \sum_{k=1}^n D_k C_{n-k}$$

$$\text{since } D_k = C_{k-1}, \text{ we get } C_n = \sum_{k=1}^n C_{k-1} C_{n-k}$$

# Recurrent formula

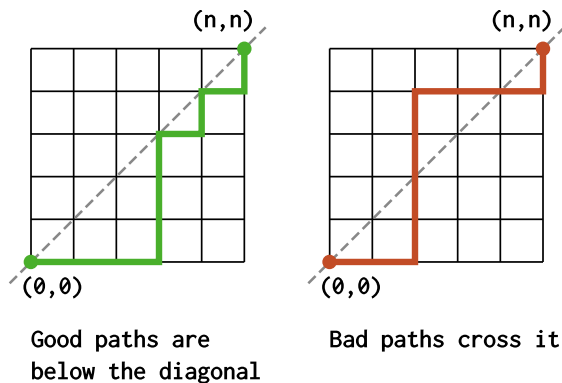
A game  
Strong Induction  
Another game  
Catalan Numbers



$$C_n = \sum_{k=1}^n C_{k-1} C_{n-k}$$

# Closed form formula

A game  
Strong Induction  
Another game  
Catalan Numbers

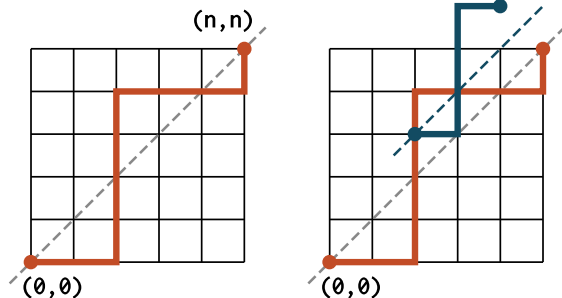


We already know that the number of paths from the bottom-left to the top-right corner is  $B_n = \binom{2n}{n}$

Let's try to count the number of paths that cross the diagonal, there is  $B_n - C_n$  of them.

# Closed form formula

A game  
Strong Induction  
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Catalan Numbers



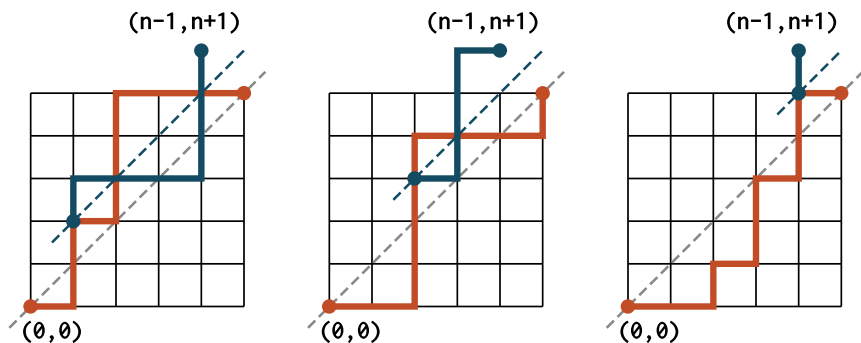
Consider a bad path that crosses the diagonal.

Lets say that the point  $P = (k, k + 1)$  is the first point above the diagonal. We mirror the remaining part of the path (shown in blue).

We can construct such new path for any invalid path.

# Closed form formula

A game  
Strong Induction  
Another game  
Catalan Numbers

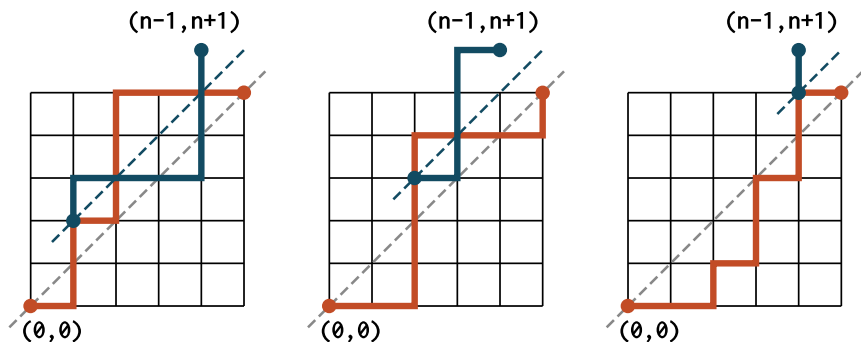


Since we mirror the path starting at  $P = (k, k + 1)$ , the remaining part of the path consisted of  $(n - k, n - k - 1)$  horizontal and vertical moves. Once reflected, it contains  $(n - k - 1, n - k)$  moves.

So, the resulting path ends up at the point  
 $Z = (k + n - k - 1, k + 1 + n - k) = (n - 1, n + 1)$ .  
It does not depend on  $k$ .

# Closed form formula

A game  
Strong Induction  
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Catalan Numbers



Every invalid paths becomes a path with  $(n-1, n+1)$  horizontal and vertical moves.

So there is

$$B_n - C_n = \binom{n-1+n+1}{n+1} = \binom{2n}{n+1} \text{ of them.}$$



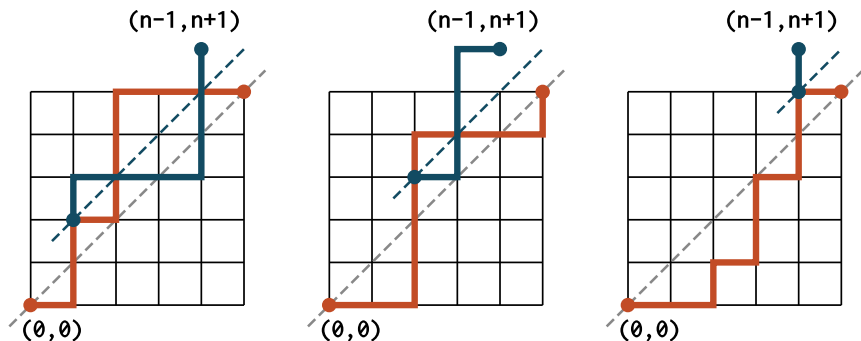
# Closed form formula

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Strong Induction

Another game

Catalan Numbers



$$C_n = B_n - \binom{2n}{n+1}$$

Therefore,

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{2} - \frac{n}{n+1} \binom{2n}{n} = \frac{1}{n+1} \binom{2n}{n}$$

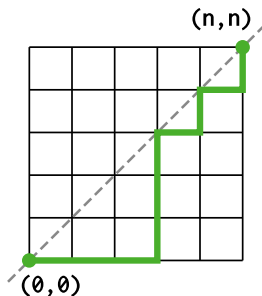
# Three formulas for $C_n$

A game

Strong Induction

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Catalan Numbers

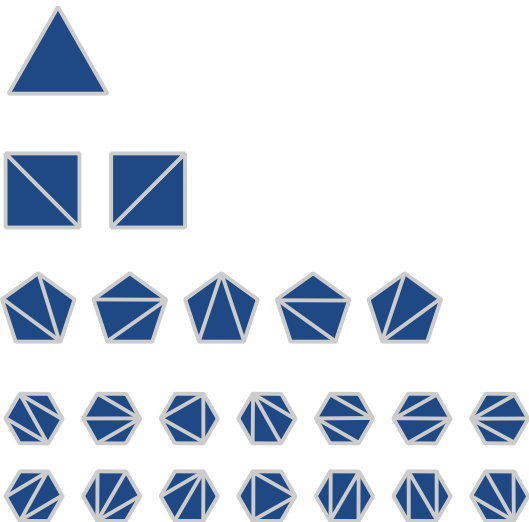


$$C_n = \sum_{k=1}^n C_{k-1} C_{n-k}$$

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} \quad C_n = \frac{1}{n+1} \binom{2n}{n}$$

# Catalan numbers are more than that

A game  
Strong Induction  
Another game  
Catalan Numbers



The number of ways to triangulate convex polygons:  
1, 2, 5, 14, ...

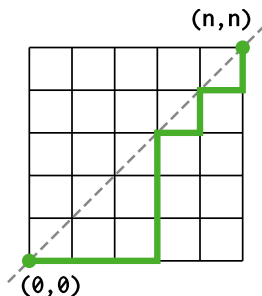
# Catalan numbers are more than that

A game

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Let's encode the path with bits,  $\{0, 1\}$ .

If every move to the right is 1, and every move up is 0:

1110001010

Well, not particularly interesting ...

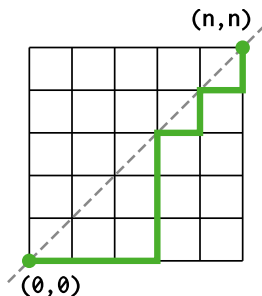
# Catalan numbers are more than that

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Let's encode the path with parentheses,  $\{ (, ) \}$ .

If every move to the right is  $($ , and every move up is  $)$ :

$(( ( ) ) ) ( ) ( )$

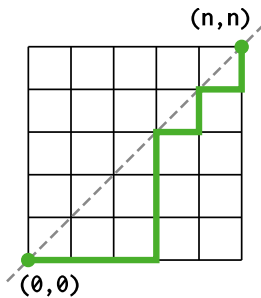
# Catalan numbers are more than that

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Another game

Catalan Numbers

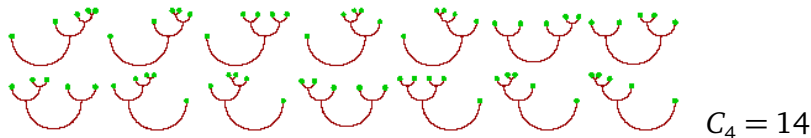
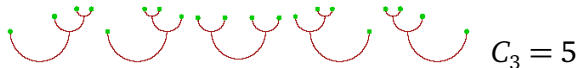


(( ( ) ) ) ( ) ( )

$C_n$  is the number of strings made of  $n$  pairs of correctly balanced parentheses.

# Catalan numbers are more than that

A game  
Strong Induction  
Another game  
Catalan Numbers



$C_n$  is the number of full binary trees with  $n + 1$  leaves:  
2, 5, 14, ...

(A rooted binary tree is full if every internal node has two children)

# Alternative “proof”

A game

Strong Induction

Another game

Catalan Numbers

All paths from the bottom-left to the top-right corner:

$$B_n = \binom{2n}{n}$$

$$B_n = \sum_{k=1}^n 2D_k B_{n-k} = \sum_{k=1}^n 2C_{k-1} B_{n-k}$$

Count the paths that cross the diagonal (not valid paths):

$$B_n - C_n = \sum_{k=0}^{n-1} C_k \cdot 1 \cdot \binom{2(n-k)-1}{(n-k)-1}$$

First  $2k$  moves were valid, and you ended up on the diagonal, you could do so in  $C_k$  ways. At the move  $2k+1$ , you cross the diagonal by going up. And after that simply finish the remaining  $2n-2k-1$  moves so that  $n-k-1$  times you are going up.



# Alternative “proof”

A game  
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Another game  
Catalan Numbers

Put it all together

valid paths: 
$$C_n = \sum_{k=1}^n C_{k-1} C_{n-k}$$

all paths: 
$$B_n = \sum_{k=1}^n 2C_{k-1} B_{n-k}$$

invalid paths: 
$$B_n - C_n = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

# Alternative “proof”

A game  
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Another game  
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Adjust the indices

$$C_n = \sum_{k=0}^{n-1} C_k C_{(n-k)-1}$$

$$B_n = \sum_{k=0}^{n-1} 2C_k B_{(n-k)-1}$$

$$B_n - C_n = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

# Alternative “proof”

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Another game

Catalan Numbers

$$C_n = \sum_{k=0}^{n-1} C_k C_{(n-k)-1}$$

$$B_n = \sum_{k=0}^{n-1} 2C_k B_{(n-k)-1}$$

$$B_n - C_n = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

Substitute  $B_n$  and  $C_n$  in the last formula

$$\sum_{k=0}^{n-1} 2C_k B_{(n-k)-1} - \sum_{k=0}^{n-1} C_k C_{(n-k)-1} = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

# Alternative “proof”

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Strong Induction

Another game

Catalan Numbers

$$\sum_{k=0}^{n-1} 2C_k B_{(n-k)-1} - \sum_{k=0}^{n-1} C_k C_{(n-k)-1} = \sum_{k=0}^{n-1} C_k \cdot \binom{2(n-k)-1}{(n-k)-1}$$

Rewrite

$$\sum_{k=0}^{n-1} C_k \left( 2B_{(n-k)-1} - C_{(n-k)-1} \right) = \sum_{k=0}^{n-1} C_k \binom{2(n-k)-1}{(n-k)-1}$$

**Warning! Handwaving here ... I failed at this point.**

These sums are equal only if their coefficients are equal:

$$2B_{(n-k)-1} - C_{(n-k)-1} = \binom{2(n-k)-1}{(n-k)-1}$$

Replace  $m = (n-k) - 1$

$$2B_m - C_m = \binom{2m+1}{m}$$

# Alternative “proof”

A game

Strong Induction

Another game

Catalan Numbers

$$2B_n - C_n = \binom{2n+1}{n}$$

Remember that  $B_n = \binom{2n}{n}$ :

$$2\binom{2n}{n} - C_n = \binom{2n+1}{n}$$

By Pascal's Identity,  $\binom{2n+1}{n} = \binom{2n}{n} + \binom{2n}{n-1}$ , so

$$2\binom{2n}{n} - C_n = \binom{2n}{n} + \binom{2n}{n-1}$$

Finally,

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$