Recursion in Mathematics and Programming

Summation

Compute summation

$$\sum_{k=1}^{n} = 1 + 2 + \ldots + n.$$

It's recursive definition:

$$\begin{aligned} \textbf{sum}(0) &= 0 \\ \textbf{sum}(n) &= \textbf{sum}(n-1) + n \qquad (\text{for } n \geq 1) \end{aligned}$$

Source code "`sum.jl`".

Factorial

Factorial function

$$\mathbf{fact}(n) = n! = 1 \cdot 2 \cdot \ldots \cdot n.$$

It's recursive definition:

$$\begin{aligned} \mathbf{fact}(0) &= 1 \\ \mathbf{fact}(n) &= \mathbf{fact}(n-1) \cdot n \qquad (\text{for } n \geq 1) \end{aligned}$$

Source code "`fact.jl`".

Factorial

Evaluation of the expression $\mathbf{fact}(3)$

$$
\begin{aligned}
\mathbf{fact}(3) \quad &\Rightarrow \quad \mathbf{fact}(2) \cdot 3 \\
&\Rightarrow \quad (\mathbf{fact}(1) \cdot 2) \cdot 3 \\
&\Rightarrow \quad ((\mathbf{fact}(0) \cdot 1) \cdot 2) \cdot 3 \\
&\Rightarrow \quad ((1 \cdot 1) \cdot 2) \cdot 3 \\
&\Rightarrow \quad (1 \cdot 2) \cdot 3 \\
&\Rightarrow \quad 2 \cdot 3 \\
&\Rightarrow \quad 6
\end{aligned}
$$

Fibonacci numbers

Recursive definition of the Fibonacci numbers:

$$\begin{aligned} \mathbf{fib}(0) &= 1 \\ \mathbf{fib}(1) &= 1 \\ \mathbf{fib}(n) &= \mathbf{fib}(n-1) + \mathbf{fib}(n-2) \qquad (\text{for } n \geq 2) \end{aligned}$$

Source code “`fib.jl`”.

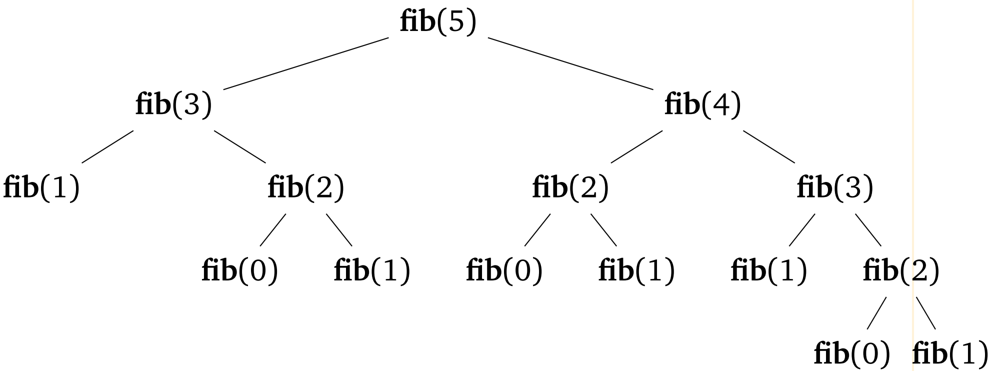
Fibonacci numbers

Evaluation of the expression $\mathbf{fib}(5)$

$$
\begin{aligned}
\mathbf{fib}(5) \quad & \Rightarrow \quad \mathbf{fib}(3) + \mathbf{fib}(4) \\
& \Rightarrow \quad (\mathbf{fib}(1) + \mathbf{fib}(2)) + (\mathbf{fib}(2) + \mathbf{fib}(3)) \\
& \Rightarrow \quad (1 + (\mathbf{fib}(0) + \mathbf{fib}(1))) + ((\mathbf{fib}(0) + \mathbf{fib}(1)) + (\mathbf{fib}(1) + \mathbf{fib}(2))) \\
& \Rightarrow \quad (1 + (1 + 1)) + ((1 + 1) + (1 + (\mathbf{fib}(0) + \mathbf{fib}(1)))) \\
& \Rightarrow \quad (1 + 2) + (2 + (1 + (1 + 1))) \\
& \Rightarrow \quad 3 + (2 + (1 + 2)) \\
& \Rightarrow \quad 3 + (2 + 3) \\
& \Rightarrow \quad 3 + 5 \\
& \Rightarrow \quad 8
\end{aligned}
$$

Exponential running time, $O(2^n)$, too slow to be practical.

Fibonacci numbers



Improved recursive Fibonacci

Evaluation of the expression **fib2**(5)

$$\begin{aligned}
\mathbf{fib2}(5) &\Rightarrow \mathbf{next}(1,1,2) \\
&\Rightarrow \mathbf{next}(1,2,3) \\
&\Rightarrow \mathbf{next}(2,3,4) \\
&\Rightarrow \mathbf{next}(3,5,5) \\
&\Rightarrow 3+5 \\
&\Rightarrow 8
\end{aligned}$$

Linear time, $O(n)$, this is a real improvement.

A simple calculator language

Syntax

$$
\begin{aligned}
< expr > \quad ::= \quad & \textit{strings of digits}, \text{ represent integer numbers} \\
| \quad & (\ < expr > \ + \ < expr > \ ) \\
| \quad & (\ < expr > \ - \ < expr > \ ) \\
| \quad & (\ < expr > \ \times \ < expr > \ )
\end{aligned}
$$

Examples:

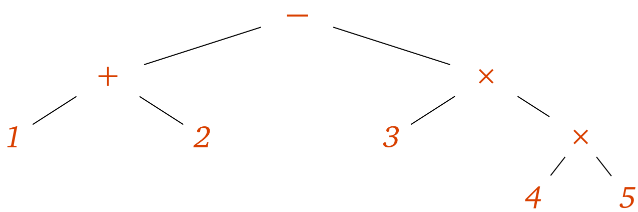
$$55$$

$$(\ 1 + 2\ )$$

$$(\ (\ 1 + 2\ )\ -\ (\ 3 \times\ (\ 4 \times 5\ )\ )\ )$$

A simple calculator language

$$(\;\; (\; 1 + 2 \;) \; - \; (\; 3 \times \; (\; 4 \times 5 \;) \;) \;)$$



The evaluating function looks at the root of the tree only, the operators are evaluated one at a time

$$eval\Big[\; (\; E_1 \; - \; E_2 \;) \; \Big]$$

$$\Rightarrow \quad eval\big[E_1\big] - eval\big[E_2\big]$$

A simple calculator language

Define the evaluation function:

Operators

$$
\begin{aligned}
eval[\ (\ E_1 + E_2\ )\ ] &= eval[E_1] + eval[E_2] \\
eval[\ (\ E_1 - E_2\ )\ ] &= eval[E_1] - eval[E_2] \\
eval[\ (\ E_1 \times E_2\ )\ ] &= eval[E_1] \times eval[E_2]
\end{aligned}
$$

Numbers

$$eval[n] = n$$

A simple calculator language

$$
\begin{aligned}
& eval\big[\,(\ (\ 1 + 2\ )\ -\ (\ 3 \times\ (\ 4 \times 5\ )\ )\ )\,\big] \\
\Rightarrow\ & eval\big[\,(\ 1 + 2\ )\,\big]\ -\ eval\big[\,(\ 3 \times\ (\ 4 \times 5\ )\ )\,\big] \\
\Rightarrow\ & (eval\big[1\big] + eval\big[2\big]) - (eval\big[3\big] \times eval\big[\,(\ 4 \times 5\ )\,\big]) \\
\Rightarrow\ & (1 + 2) - (3 \times (eval\big[4\big] \times eval\big[5\big])) \\
\Rightarrow\ & 3 - (3 \times (4 \times 5)) \\
\Rightarrow\ & 3 - (3 \times 20) \\
\Rightarrow\ & 3 - 60 \\
\Rightarrow\ & -57
\end{aligned}
$$

Recursion in Mathematics

Some prominent examples of recursion

- *Euclid's algorithm*
  a algorithm for computing the greatest common divisor.
- *Newton's method*
  a recursive method for finding successively better approximations to the roots of a real-valued function.
- Complex objects such as fractals can be defined using recursive definition.

Define set M

$c \in M$ if the sequence

$$\begin{aligned} z_0 &= 0 \\ z_n &= z_{n-1}^2 + c \quad \text{does not go to infinity.} \end{aligned}$$

Define set M

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Numbers c and z_n are *complex numbers*. So, we need to quickly learn how to add and multiply them.

Quick intro to complex numbers

The set of complex numbers $\mathbb{C}$ is an extension of the set of real numbers $\mathbb{R}$.

Any complex number can be represented by a pair of real numbers

$$(x, y) \qquad x, y \in \mathbb{R}$$

x is the *real part*, and y is the *imaginary part*.

Alternative notation. We can represent the pair as a sum of its real and imaginary part

$$(x, y) = x + yi$$

$i = \sqrt{-1}$ is the *imaginary unit*.

Quick intro to complex numbers

Addition

$$(a, b) + (c, d) = (a + c, b + d)$$

Multiplication

$$(a, b) \cdot (c, d) = (ac - bd, bc + ad)$$

Particularly,

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1$$

Define set M

A complex number $c \in M$, if the sequence

$$
\begin{aligned}
z_0 &= 0 \\
z_n &= z_{n-1}^2 + c \quad \text{does not go to infinity.}
\end{aligned}
$$

$z_n = x_n + y_n i$ will go to infinity if one of its components gets large:

$$\sqrt{x_n^2 + y_n^2} \geq 2$$

This condition is used practically to compute the membership of c in the set.

Source code "`mset.jl`".

L-systems

An L-system or Lindenmayer system is a parallel *rewriting system*.

It consists of

- an alphabet of symbols
- an initial string (called an axiom) to start construction
- a collection of production rules that expand each symbol into some larger string of symbols

Example:

Alphabet:	$\{A, B\}$
Initial string:	A
Production rules:	$A \longrightarrow AB$
	$B \longrightarrow A$

L-systems

Example:

$$\begin{aligned} \textit{Alphabet:} \quad & \{A, B\} \\ \textit{Initial string:} \quad & A \\ \textit{Production rules:} \quad & A \longrightarrow AB \\ & B \longrightarrow A \end{aligned}$$

Start rewriting:

$$\begin{aligned} n &= 0 : A \\ n &= 1 : AB \\ n &= 2 : ABA \\ n &= 3 : ABAAB \\ n &= 4 : ABAABABA \\ n &= 5 : ABAABABAABAAB \\ n &= 6 : ABAABABAABAABABAABABA \end{aligned}$$

L-systems

Koch curve:

$$\begin{aligned}
\textit{Alphabet:}\quad & \{F, L, R\} \\
\textit{Initial string:}\quad & F \\
\textit{Production rules:}\quad & F \longrightarrow F\ L\ F\ R\ F\ R\ F\ L\ F \\
& L \longrightarrow L \\
& R \longrightarrow R
\end{aligned}$$

When F stands for moving forward, and L and R are the commands for turning left and right

Source code "`lsys.jl`".