

# Induction



# Consider a problem

Principle

Examples

Summations

Inequalities

Let's prove that the sum of the first  $n$  positive integers

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all positive integers  $n = 1, 2, 3, 4, \dots$

# Consider a problem

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Let

$$P(n) : 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Prove that

$$P(n)$$

for all positive natural numbers  $n = 1, 2, 3, 4 \dots$

# The idea

Principle

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If we can prove that

$$P(1)$$

$$P(1) \rightarrow P(2)$$

$$P(2) \rightarrow P(3)$$

...

$$P(n) \rightarrow P(n+1)$$

...

Then it follows that

$$P(n) \text{ for all } n \geq 1$$

# The idea

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The implications can be grouped together. Thus it is enough to prove that

$$1) \quad P(1)$$

$$2) \quad \forall n \geq 1 : P(n) \rightarrow P(n+1)$$

Then it follows that

$$P(n) \text{ for all } n \geq 1$$

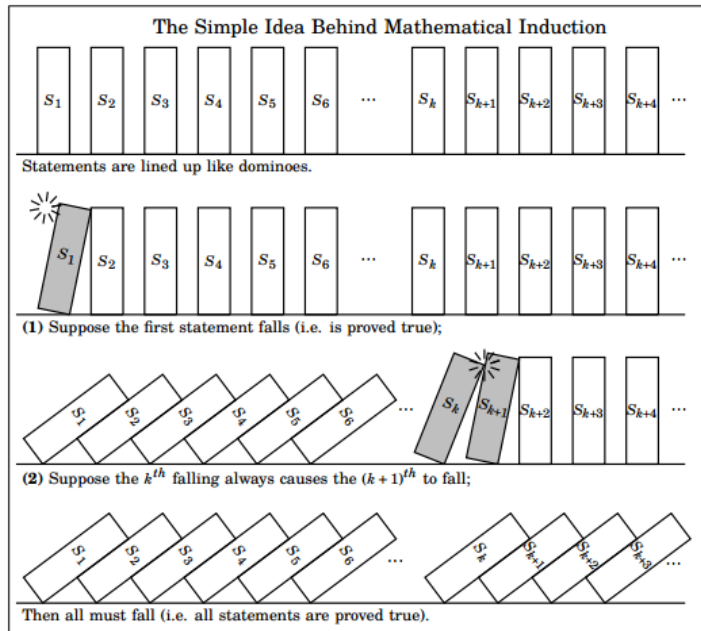
# Principle of induction

Principle

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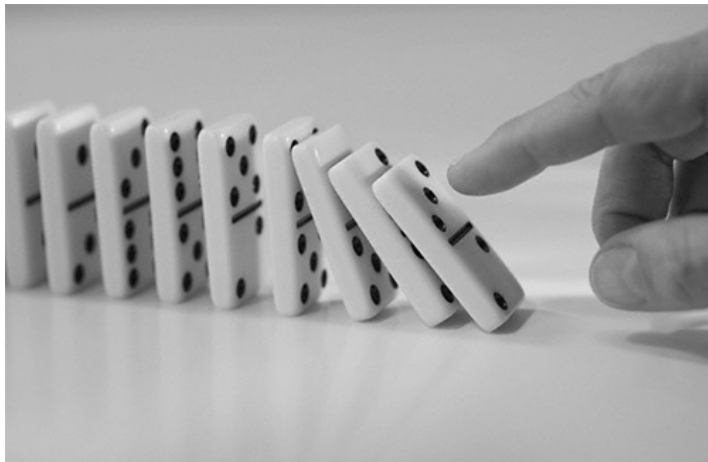
# Principle of induction

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# Principle of induction

Principle

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If we can prove

1. *“The basis step”*

$P(1)$  is true, and

2. *“The inductive step”*

for all  $n \geq 1$ ,  $P(n)$  implies  $P(n + 1)$ .

then  $P(n)$  is true for every integer  $n \geq 1$ .

# Example proof

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Let's use induction to prove the formula of the sum of positive integers from 1 to  $n$ :

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

(We have to show that both, the *basis step* and the *inductive step* are correct)

# Example proof

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Prove that

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Part 1. *The basis step.* Consider  $n = 1$ .

The left hand side is just

$$1$$

The right-hand side:

$$\frac{1 \cdot (1 + 1)}{2} = 1$$

They are equal, so it is true.

# Example proof

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Part 2. *The inductive step.*

Assume that the formula is true for an arbitrary  $n \geq 1$ :

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

We have to prove that

$$1 + 2 + 3 + \dots + (n+1) = \frac{(n+1)((n+1)+1)}{2}$$

# Example proof

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Part 2. *The inductive step.*

Assume that the formula is true for an arbitrary  $n \geq 1$ :

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We have to prove that

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Consider the left-hand side:

$$\begin{aligned} \underbrace{1 + 2 + 3 + \dots + n}_{= \frac{n(n+1)}{2}} + (n+1) &= \frac{n(n+1)}{2} + n+1 = \frac{n(n+1) + 2(n+1)}{n+1} \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2} \end{aligned}$$

# Example proof

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The base case and the inductive step are true.

Therefore, *by induction*, the formula is correct for every positive integer  $n$ .

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

This is a very useful formula, by the way.

Rewrite it using the sigma-notation

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

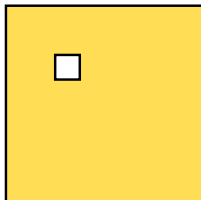
# Tiling $2^n \times 2^n$ with 1 square removed

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For all  $n > 0$ , a checkerboard  $2^n \times 2^n$  with one square removed

can be tiled by L-shaped tiles



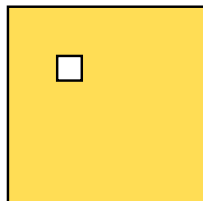
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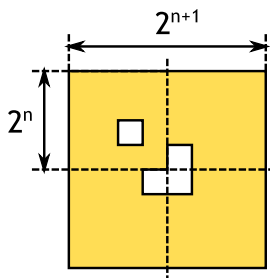


The base case for  $n=1$



Inductive step.

Assuming that we can tile  $2^n \times 2^n$  with one removed, prove that it's possible to tile  $2^{n+1} \times 2^{n+1}$  with one removed





# Another example proof

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Inequalities

Let  $P(n)$  be the predicate, "*I can lift  $n$  grains of sand.*"

- I can lift one grain of sand, so  $P(1)$  is true. This is my basis step.

# Another example proof

Principle

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Inequalities

Let  $P(n)$  be the predicate, “I can lift  $n$  grains of sand.”

- I can lift one grain of sand, so  $P(1)$  is true. This is my basis step.
- Then, surely, if I can lift  $n$  grains, then I can lift  $n + 1$ , it does not make any difference!

$$P(n) \rightarrow P(n + 1)$$

This is my inductive step.

# Another example proof

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Let  $P(n)$  be the predicate, “I can lift  $n$  grains of sand.”

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- Then, surely, if I can lift  $n$  grains, then I can lift  $n + 1$ , it does not make any difference!

$$P(n) \rightarrow P(n + 1)$$

This is my inductive step.

So, by induction, *I can lift any amount of sand*. Right?

# Where is a mistake?

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Of course, the proof is wrong. But should we blame induction for that?

Well, we made an error in the proof of  $P(n) \rightarrow P(n + 1)$ .

It is hard to say for exactly which  $n$  it is false, but certainly there is some value!

# Sum of $b^k$

Principle

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Prove by induction that

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

First we can do the base case, then the inductive step.

# Sum of $b^k$

Principle

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Prove that

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

*Basis step ( $n = 0$ ):*

$$b^0 = 1, \quad \text{and} \quad \frac{b^1 - 1}{b - 1} = 1$$

.

# Sum of $b^k$

Principle

Examples

Summations

Inequalities

Prove that

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

*Inductive step:*

As always, we make a hypothesis that

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1} \quad \text{is true for } n \geq 0$$

And we have to prove that the formula is correct for  $n + 1$ :

$$b^0 + b^1 + b^2 + \dots + b^{n+1} = \frac{b^{n+2} - 1}{b - 1}$$

# Sum of $b^k$

Principle

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*Inductive step:*

We have to prove that the formula is correct for  $n + 1$ :

$$b^0 + b^1 + b^2 + \dots + b^{n+1} = \frac{b^{n+2} - 1}{b - 1}$$

$$\begin{aligned} \underbrace{b^0 + b^1 + b^2 + \dots + b^n}_{= \frac{b^{n+1} - 1}{b - 1} \text{ by the hypothesis}} + b^{n+1} &= \frac{b^{n+1} - 1}{b - 1} + b^{n+1} \\ &= \frac{b^{n+1} - 1 + b^{n+2} - b^{n+1}}{b - 1} = \frac{b^{n+2} - 1}{b - 1}. \end{aligned}$$



# Sum of $b^k$

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So, this formula for the sum is correct

$$b^0 + b^1 + b^2 + \dots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

In the sigma-notation:

$$\sum_{k=0}^n b^k = \frac{b^{n+1} - 1}{b - 1}$$

# Sum of $b^k$

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We can multiply both sides by a constant  $a$ :

$$\sum_{k=0}^n ab^k = ab^0 + ab^1 + ab^2 + \dots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1},$$

The sequence of numbers

$$a, ab, ab^2, ab^3, \dots, ab^n, \dots$$

is called a *Geometric progression*.

So, we proved the formula for the partial sum of a geometric progression.

# Sum of $kb^{k-1}$

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The partial sum of the geometric progression:

$$\sum_{k=0}^n ab^k = ab^0 + ab^1 + ab^2 + \dots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1},$$

Can we compute

$$\begin{aligned} \sum_{k=0}^n kb^{k-1} \\ = 0 + 1 + 2b + 3b^2 + 4b^3 \dots + nb^{n-1} \end{aligned}$$

So, instead of the constant  $a$ , we have an increasing sequence of coefficients now.

# Sum of $kb^{k-1}$

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There is a cheap trick:

$$\begin{aligned}\frac{d}{db} \sum_{k=0}^n b^k &= \frac{d}{db} (1 + b + b^2 + b^3 + b^4 + \dots + b^n) \\ &= 0 + 1 + 2b + 3b^2 + 4b^3 + \dots + nb^{n-1} = \sum_{k=0}^n kb^{k-1}\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{d}{db} \sum_{k=0}^n b^k &= \frac{d}{db} \left( \frac{b^{n+1} - 1}{b - 1} \right) = \frac{(n+1)b^n(b-1) - b^{n+1} + 1}{(b-1)^2} \\ &= \frac{nb^{n+1} - (n+1)b^n + 1}{(b-1)^2}\end{aligned}$$

$$\text{Therefore, } \sum_{k=0}^n kb^{k-1} = \frac{nb^{n+1} - (n+1)b^n + 1}{(b-1)^2}.$$

# Geometric progression again

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The partial sum of the geometric progression:

$$\sum_{k=0}^n ab^k = ab^0 + ab^1 + ab^2 + \dots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1}$$

Well, what if we want to add up an infinite sequence?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

# Infinite geometric progression

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The partial sum of the geometric progression is

$$\sum_{k=0}^n ab^k = ab^0 + ab^1 + ab^2 + \dots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1}$$

If  $b$  is a small real number, specifically, if the absolute value  $|b| < 1$ , then

$$|b| > |b^2| > |b^3| > \dots$$

In the limit,  $\lim_{n \rightarrow \infty} b^n = 0$

$$\sum_{n=0}^{\infty} ab^n = \lim_{n \rightarrow \infty} ab^n = \lim_{n \rightarrow \infty} \frac{a(b^{n+1} - 1)}{b - 1} = \frac{a(-1)}{b - 1} = \frac{a}{1 - b}$$

# Proving inequalities

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Using mathematical induction, prove that for  $n \geq 1$ :

$$2^n > n$$

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Using mathematical induction, prove that for  $n \geq 1$ :

$$2^n > n$$

*The basis step:*

$n = 1$ .  $2 > 1$  is true.



# Proving inequalities

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Inequalities

Using mathematical induction, prove that for  $n \geq 1$ :

$$2^n > n$$

*The basis step:*

$n = 1$ .  $2 > 1$  is true.

*The inductive step:*

Assume that  $2^n > n$  for  $n \geq 1$ . Prove that  $2^{n+1} > (n + 1)$ .

Equivalently, we have to prove that

$$2^{n+1} - (n + 1) > 0.$$

# Proving inequalities

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Inequalities

We assumed that  $2^n > n$  for  $n \geq 1$ .

We have to prove that  $2^{n+1} - (n + 1) > 0$ .

$$\begin{aligned}2^{n+1} - (n + 1) &= 2 \cdot 2^n - n - 1 \\&> 2 \cdot n - n - 1 \quad (\text{by the I.H.}) \\&= n - 1 \\&\geq 1 - 1 = 0. \quad (\text{b/c } n \geq 1)\end{aligned}$$

Therefore, *by induction*,  $2^n > n$  is true for  $n \geq 1$ .

# One more proof

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**Theorem.** All horses are the same color.

We can prove this by induction on the number of horses in a given set.

# All horses are the same color

Principle

Examples

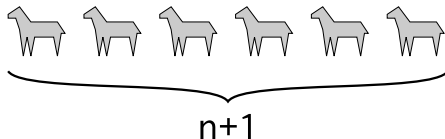
Summations

Inequalities

*The basis step.* If there's just one horse then it's the same color as itself.

For the *inductive step*, assume that  $n$  horses are of the same color.

Assume that there are  $n + 1$  horses numbered 1 to  $n + 1$ .



# All horses are the same color

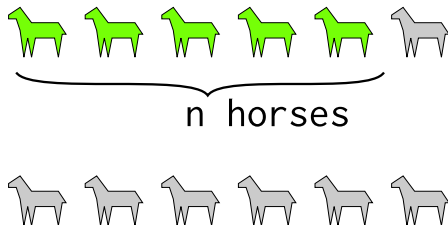
Principle

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By the induction hypothesis, horses 1 through  $n$  are the same color, and similarly horses 2 through  $n + 1$  are the same color.



But the middle horses, 2 through  $n$ , can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and  $n + 1$  must be the same color as well. Thus all  $n + 1$  horses are the same color.

What, if anything, is wrong with this reasoning?

# All horses are the same color

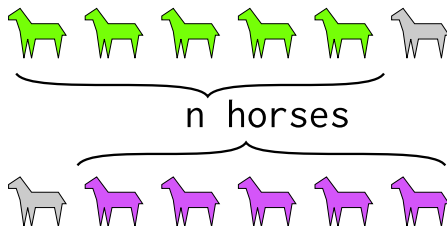
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