

Counting

What can we count?

- In how many ways can we paint 6 rooms, choosing from 15 available colors?
- What if we want all rooms painted with different colors?
- In how many different ways 10 books can be arranged on a shelf?
- What if 2 of those 10 books are identical copies?

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

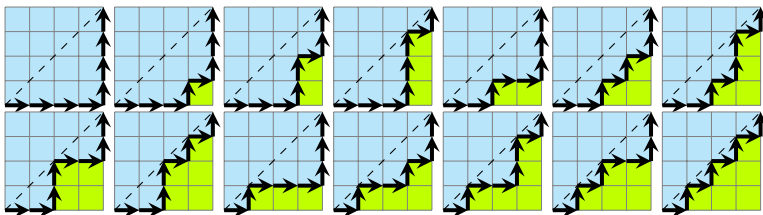
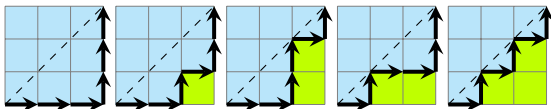
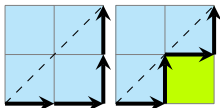
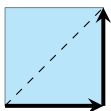
Subtraction Rule

Counting

Tree Diagrams

Factorial

1, 2, 5, 14, ...



The Rule of Product

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1, 2, 5, 14, ...



The Rule of Product

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Problem

The Rule of Product

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Factorial

The chairs of an auditorium are to be labeled with an uppercase letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

A-1, A-2, ... Z-100

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

Problem

The Rule of Product

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The chairs of an auditorium are to be labeled with an uppercase letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

A-1, A-2, ... Z-100

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

$$26 \cdot 100$$

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There are

1. 26 ways to assign a letter and

$$26 \cdot 100$$

2. 100 ways to assign a number.

The Rule of Product. Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Chairs again

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Consider the same problem about the labels for chairs

1. 26 ways to choose a letter and
2. 100 ways to choose a number.

Write a program that prints all 2600 labels?

Chairs again

The Rule of Product

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Factorial

Consider the same problem about the labels for chairs

1. 26 ways to choose a letter and
2. 100 ways to choose a number.

Write a program that prints all 2600 labels?

```
for  $a := A$  to  $Z$  do  
  for  $n := 1$  to  $100$  do  
    print_label(a,n)
```

Generalized Product Rule

If a procedure consists of *k sub-tasks*, and the sub-tasks can be performed in n_1, \dots, n_k ways, then the procedure can be performed in $(n_1 \cdot n_2 \cdot \dots \cdot n_k)$ ways.

Example:

Count the number of different bit strings of length seven.

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Generalized Product Rule

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Factorial

If a procedure consists of *k sub-tasks*, and the sub-tasks can be performed in n_1, \dots, n_k ways, then the procedure can be performed

in $(n_1 \cdot n_2 \cdot \dots \cdot n_k)$ ways.

Example:

Count the number of different bit strings of length seven.

The value for each bit can be chosen in two ways (0 or 1). Therefore:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$$

License plates



How many different license plates of this format can be made?

The Rule of Product

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License plates



How many different license plates of this format can be made?

$$26^3 \cdot 10^4 = 175,760,000$$

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Another counting problem

The Rule of Product

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A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

Another counting problem

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Factorial

A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

$40 + 50 = 90$ this is called the rule of sum

The Rule of Sum

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

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The Rule of Sum

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

Subtraction Rule

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Factorial

40 books on sociology, and 50 books on anthropology.

There are $40 + 50 = 90$ ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

```
for  $b := 1$  to 40 do  
  read(Sociology,  $b$ )  
for  $b := 1$  to 50 do  
  read(Anthropology,  $b$ )
```

The Rule of Sum

The Rule of Product

The Rule of Sum

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Factorial

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

The Rule of Sum. If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Note that it's important that the two groups don't have common elements (We say that they are disjoint sets).

Problem

You have 3 textbooks, 5 novels, 4 magazines, and 2 comic books.

You want to pick only one book to read in the subway. How many options do you have?

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Problem

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You have 3 textbooks, 5 novels, 4 magazines, and 2 comic books.

You want to pick only one book to read in the subway. How many options do you have?

$$3 + 5 + 4 + 2 = 14.$$

Problem

NYS wants to change the license plates format, allowing
3 letters + 3 digits; 2 letters + 2 digits; and 1 letter + 1 digit.

AAA 111

AA 11

A 1

How many license plates can be made?

The Rule of Product

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Problem

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NYS wants to change the license plates format, allowing up to 3 letters followed by up to 3 digits.

A 1	A 11	A 111
AA 1	AA 11	AA 111
AAA 1	AAA 11	AAA 111

How many license plates can be made?

A new object

Def. A *set* is an unordered collection of objects. The objects are called elements.

If e is an element of the set A , we write $e \in A$.

Otherwise, if it's not in A , we write $e \notin A$.

Example:

$$A = \{1, 2, 97, 3, 15\}.$$

$$1 \in A.$$

$$4 \notin A.$$

$$\{1, 2, 3, 3, 2, 1, 1, 1, 1\} = \{1, 2, 3\}$$

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

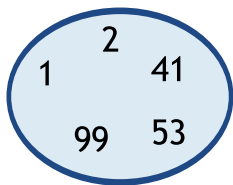
Subtraction Rule

Counting

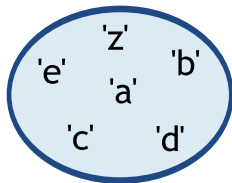
Tree Diagrams

Factorial

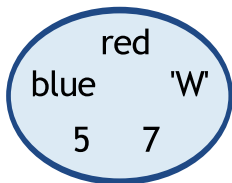
Sets



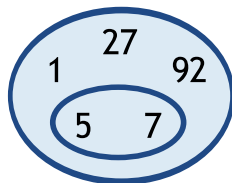
$$A = \{1, 2, 41, 53, 99\}$$



$$B = \{'a', 'z', 'e', 'd', 'c', 'b'\}$$



$$C = \{'W', \text{blue}, 5, \text{red}, 7\}$$



$$D = \{27, 1, \{5, 7\}, 92\}$$

The Rule of Product

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Some important sets

The Rule of Product

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Natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Integer numbers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Empty set

$$\emptyset = \{ \}$$

Set Builder Notation

We can describe sets using predicates:

$$A = \{x \mid P(x)\}$$

“Set A is such that $x \in A$ if and only if $P(x)$.”

Example. Positive integers:

$$\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n > 0\} = \{1, 2, 3, \dots\}$$

More complex predicates are fine too. Odd and even numbers:

$$Even = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z} (n = 2k)\}$$

$$Odd = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z} (n = 2k + 1)\}$$

The Rule of Product

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Union, \cup

The Rule of Product

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$A \cup B$ denotes all things that are *members of either A or B*:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Equivalently:

x belongs to $A \cup B$ if and only if $x \in A$ or $x \in B$.

Examples:

$$\{1, 2\} \cup \{a, b\} = \{1, 2, a, b\}$$

$$\{1, 2, 3\} \cup \{2, 3, 5\} = \{1, 2, 3, 5\}$$

Intersection, \cap

$A \cap B$ denotes all things that are *members of both A and B*:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Equivalently:

x belongs to $A \cap B$ if and only if $x \in A$ and $x \in B$.

Examples:

$$\{1, 2\} \cap \{ 'a', 'b' \} = \emptyset$$

$$\{1, 2, 3\} \cap \{2, 3, 5\} = \{2, 3\}$$

Sets A and B are called *disjoint* if their intersection is empty:
 $A \cap B = \emptyset$.

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

Subtraction Rule

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Factorial

Number of the elements of a finite set

The Rule of Product

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Factorial

Def. If set A is finite, and there are exactly n elements in S , then n is the *cardinality* of the set A . We write

$$|A| = n.$$

Examples:

$$A = \{3, 4, 5, 6\}$$

$$|A| = 4$$

$$B = \{\{3, 4\}, \{5, 6\}, 7\}$$

$$|B| = 3$$

$$|\emptyset| = 0$$

Question

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

Subtraction Rule

Counting

Tree Diagrams

Factorial

$$A = \{1, 2, 4, 5\}$$

$$B = \{20, 21, 22, 23, 24\}$$

$$A \cup B = \{1, 2, 4, 5, 20, 21, 22, 23, 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| =$$

Question

The Rule of Product

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Factorial

$$A = \{1, 2, 4, 5\}$$

$$B = \{20, 21, 22, 23, 24\}$$

$$A \cup B = \{1, 2, 4, 5, 20, 21, 22, 23, 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| = 9, \quad \text{and} \quad |A| + |B| = 4 + 5 = 9.$$

$$|A \cup B| = |A| + |B| = 4 + 5 = 9.$$

Question

The Rule of Product

The Rule of Sum

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You are given k disjoint sets A_1, \dots, A_k :

It means that $A_i \cap A_j = \emptyset$ when $i \neq j$.

What is the cardinality of their union $A_1 \cup \dots \cup A_k$?

$$|A_1 \cup \dots \cup A_k| =$$

Question

The Rule of Product

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Factorial

You are given k disjoint sets A_1, \dots, A_k :

It means that $A_i \cap A_j = \emptyset$ when $i \neq j$.

What is the cardinality of their union $A_1 \cup \dots \cup A_k$?

$$|A_1 \cup \dots \cup A_k| = |A_1| + \dots + |A_k| = \sum_{i=1}^k |A_i|.$$

Question

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The Rule of Sum

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Tree Diagrams

Factorial

You are given k disjoint sets A_1, \dots, A_k :

It means that $A_i \cap A_j = \emptyset$ when $i \neq j$.

What is the cardinality of their union $A_1 \cup \dots \cup A_k$?

$$|A_1 \cup \dots \cup A_k| = |A_1| + \dots + |A_k| = \sum_{i=1}^k |A_i|.$$

This is the same *rule of sum*, right?

Question

The Rule of Product

The Rule of Sum

Finite Sets

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Factorial

Why do we insist on the sets being disjoint?

Really, who cares?

Because

$$A = \{1, 2, 3\}$$

$$B = \{3, 4\}$$

Their union: $A \cup B = \{1, 2, 3, 4\}$

$$|A| + |B| = |\{1, 2, 3\}| + |\{3, 4\}| = 3 + 2 = 5$$

$$|A \cup B| = |\{1, 2, 3, 4\}| = 4$$

So, in general, $|A \cup B| \neq |A| + |B|$, and if we try to use the sum rule when the sets are not disjoint, we *overcount*, and this is really bad.

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

Subtraction Rule

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Tree Diagrams

Factorial

Overcounting

$$A = \{1, 2, 3\}$$

$$B = \{3, 4\}$$

$$A \cup B = \{1, 2, 3, 4\}$$

We were overcounting, because the common elements of A and B were counted twice:

$$|A| + |B| = |\{1, 2, 3\}| + |\{3, 4\}| = 5,$$

$$|A \cup B| = |\{1, 2, 3, 4\}| = 4.$$

The Rule of Product

The Rule of Sum

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$$A = \{1, 2, 3\}$$

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$$A \cup B = \{1, 2, 3, 4\}$$

We were overcounting, because the common elements of A and B were counted twice:

$$|A| + |B| = |\{1, 2, 3\}| + |\{3, 4\}| = 5,$$

$$|A \cup B| = |\{1, 2, 3, 4\}| = 4.$$

Therefore, to get the correct value of $|A \cup B|$, the number of common elements must be subtracted:

$$|A \cup B| = |A| + |B| - |\{3\}| = 3 + 2 - 1 = 4.$$

The Subtraction Rule

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

Subtraction Rule

Counting

Tree Diagrams

Factorial

$$A = \{1, 2, 3\}$$

$$B = \{3, 4\}$$

$$A \cup B = \{1, 2, 3, 4\}$$

Therefore, to get the correct value of $|A \cup B|$, the number of common elements must be subtracted:

$$|A \cup B| = |A| + |B| - |\{3\}| = 3 + 2 - 1 = 4.$$

The *Subtraction Rule* for two arbitrary sets A and B :

$$|A \cup B| =$$

The Subtraction Rule

The Rule of Product

The Rule of Sum

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Subtraction Rule

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Tree Diagrams

Factorial

$$A = \{1, 2, 3\}$$

$$B = \{3, 4\}$$

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Therefore, to get the correct value of $|A \cup B|$, the number of common elements must be subtracted:

$$|A \cup B| = |A| + |B| - |\{3\}| = 3 + 2 - 1 = 4.$$

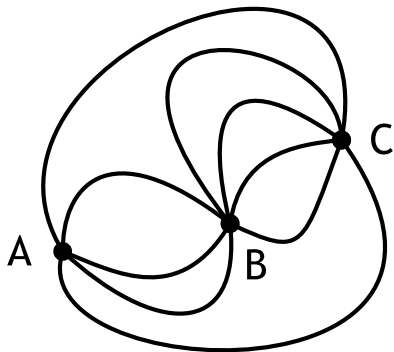
The *Subtraction Rule* for two arbitrary sets A and B :

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting paths

This is a map with three cities, connected by roads.

Count the number of paths from city *A* to city *C*, such that each city is visited not more than once.



The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

Subtraction Rule

Counting

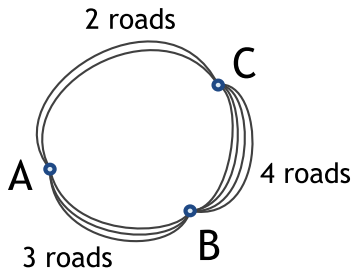
Tree Diagrams

Factorial

Counting paths

This is a map with three cities, connected by roads.

Count the number of paths from city **A** to city **C**, such that each city is visited not more than once.



The Rule of Product

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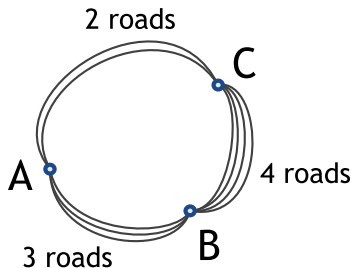
Tree Diagrams

Factorial

Counting paths

This is a map with three cities, connected by roads.

Count the number of paths from city **A** to city **C**, such that each city is visited not more than once.



$A \rightarrow C$ or $A \rightarrow B \rightarrow C$:

$$2 + 3 \cdot 4 = 14$$

The Rule of Product

The Rule of Sum

Finite Sets

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Counting

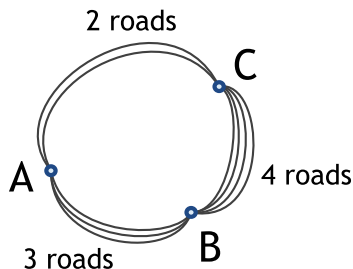
Tree Diagrams

Factorial

Counting round trips

This is a map with three cities, connected by roads.

Count the number of round trips starting in city *B*, such that cities *A* and *C* are visited not more than once.



The Rule of Product

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Tree Diagrams

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Counting round trips

The Rule of Product

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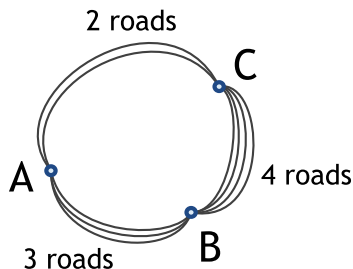
Tree Diagrams

Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city *B*, such that cities *A* and *C* are visited not more than once.

$B \rightarrow A \rightarrow B$	$3 \cdot 3 = 9$
$B \rightarrow C \rightarrow B$	$4 \cdot 4 = 16$
$B \rightarrow A \rightarrow C \rightarrow B$	$3 \cdot 2 \cdot 4 = 24$
$B \rightarrow C \rightarrow A \rightarrow B$	$4 \cdot 2 \cdot 3 = 24$
<hr/>	
Total	73



Counting round trips II

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

Subtraction Rule

Counting

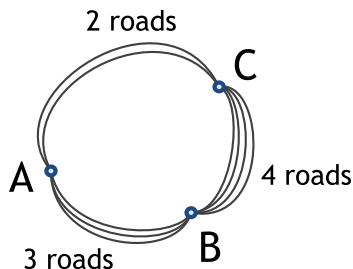
Tree Diagrams

Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city **B**, such that

- 1) cities **A** and **C** are visited not more than once, and
- 2) each road is used not more than once during a trip.



Counting round trips II

The Rule of Product

The Rule of Sum

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Overcounting

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Tree Diagrams

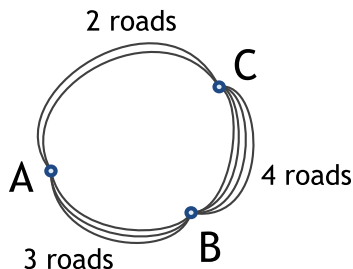
Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city **B**, such that

- 1) cities **A** and **C** are visited not more than once, and
- 2) each road is used not more than once during a trip.

$B \rightarrow A \rightarrow B$	$3 \cdot 2 = 6$
$B \rightarrow C \rightarrow B$	$4 \cdot 3 = 12$
$B \rightarrow A \rightarrow C \rightarrow B$	$3 \cdot 2 \cdot 4 = 24$
$B \rightarrow C \rightarrow A \rightarrow B$	$4 \cdot 2 \cdot 3 = 24$
<hr/>	
Total	66



Tree Diagrams

The Rule of Product

The Rule of Sum

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Tree Diagrams

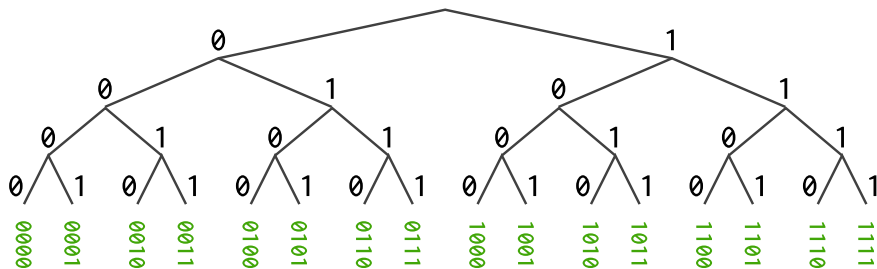
Factorial

Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.

Each possible outcome is a leaf of the tree (an endpoint that does not branch).

Example: Count all bit strings of length four.



16 strings.

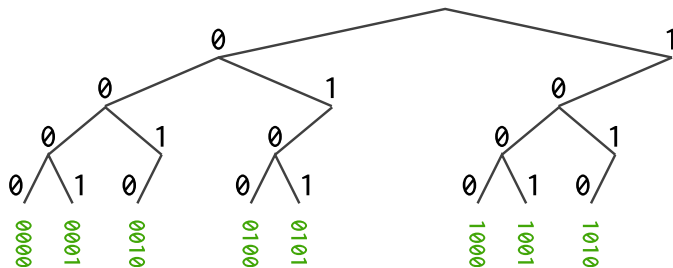
Tree Diagrams

Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.

Each possible outcome is a leaf of the tree (an endpoint that does not branch).

Example 2: Count all bit strings of length four that *do not have two consecutive 1s*.



8 strings.

The Rule of Product

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Finite Sets

Overcounting

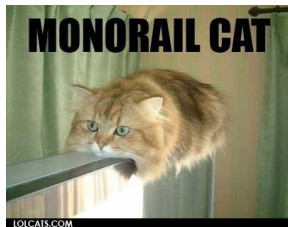
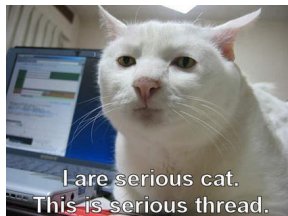
Subtraction Rule

Counting

Tree Diagrams

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Ranking cats



The Rule of Product

The Rule of Sum

Finite Sets

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Subtraction Rule

Counting

Tree Diagrams

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Ranking cats

In how many different ways can you rank
a set of 6 cats:

$$\{a, b, c, d, e, f\}$$

The Rule of Product

The Rule of Sum

Finite Sets

Overcounting

Subtraction Rule

Counting

Tree Diagrams

Factorial

Ranking cats

In how many different ways can you rank
a set of 6 cats:

$$\{a, b, c, d, e, f\}$$

- There are 6 ways to select the first cat,
- 5 ways to select the second cat among the remaining five,
- 4 ways to select the third cat ...
- ...continue the process
- In the end, the only remaining cat takes the last position in the rank.

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$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{ways!}$$

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Factorial

How large this number is?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This function is called *factorial* and denoted by $n!$:

$$1! = 1$$

$$2! = 2 \cdot 1$$

$$3! = 3 \cdot 2 \cdot 1$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 1$$

and by convention,

$$0! = 1$$

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