Discrete Structures. CSCI-150. Fall 2013.

Problem 1
How many binary string of length 12 contain exactly: (a) 2 zeroes, (b) 6 zeroes, (c) 12 zeroes?

Problem 2
Ternary strings are strings composed of three symbols: $\{0,\ 1,\ 2\}$. How many ternary string of length 12 contain exactly: (a) 2 zeroes, (b) 6 zeroes, (c) 12 zeroes?

Problem 3
How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

Problem 4
In how many ways 20 coins can be selected from four large containers filled with pennies, nickels, dimes, and quarters?

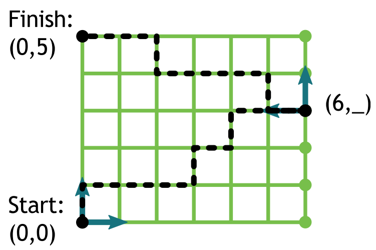
Problem 5
How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 20$, where x_1, x_2, x_3, and x_4 are nonnegative integers?

Problem 6
Count the number of ways to select 5 coins from a collection of 10 consisting of 1 penny, 1 nickel, 1 dime, 1 quarter, 1 half-collar, and 5 identical one-dollar coins.

Problem 7
In how many ways can 15 identical candy bars be distributed among five children so that the youngest gets only one or two of them?

Problem 8
In how many ways can we distribute eight identical white balls into four distinct containers so that (a) no container is left empty, (b) the fourth container has an odd number of balls in it?

Problem 9
You are in the south-western corner of a city, $(0, 0)$.
1) First, by moving **east** and **north**, you have to get to the eastern boundary of the city, $(6, y)$, for any $0 \le y \le 5$.
2) After that, by moving **west** and **north**, you need to go to the north-western corner, $(0, 5)$.
How many routes are possible.



Problem 10
Alice has an infinite supply of beads of n different colors. What is the value of n, if she can select 20 beads (with repetition of colors) in 230,230 ways.

Problem 11
How many natural numbers between 100 and 1000 are not divisible by 10? What changes if we also want all digits in the numbers to be different?

Problem 12
Let n be a positive integer. Show that

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = \binom{2n+2}{n+1}/2$$

Answers

1. (a) $\binom{12}{2}$
   (b) $\binom{12}{6}$
   (c) $\binom{12}{12}$
2. (a) $\binom{12}{2} \cdot 2^{10}$
   (b) $\binom{12}{6} \cdot 2^{6}$
   (c) $\binom{12}{12} \cdot 2^{0}$
3. $\binom{10}{2}$
4. $r = 20$, $n = 4$.
$$\binom{n+r-1}{r} = \binom{23}{20} = 1771$$
5. Same as problem 4.
6. $$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5 = 32$$
7. Case 1. The youngest gets one candy bar: $r = 14$, $n = 4$. There are $\binom{17}{14}$ ways to distribute.

   Case 2. The youngest gets two cand bars: $r = 13$, $n = 4$. $\binom{16}{13}$ ways.

   The two cases are disjoint, so by the rule of sum $\binom{17}{14} + \binom{16}{13} = 1600$.
8. (a) $r = 4$, $n = 4$. $\binom{7}{4} = 35$.
   (b) Four disjoint cases:
   $r = 7$, $n = 3$. $\binom{9}{7}$ ways.
   $r = 5$, $n = 3$. $\binom{7}{5}$ ways.
   $r = 3$, $n = 3$. $\binom{5}{3}$ ways.
   $r = 1$, $n = 3$. $\binom{3}{1}$ ways.
   In total, $\binom{9}{7} + \binom{7}{5} + \binom{5}{3} + \binom{3}{1} = 70$.
9. We assume that the paths are different, if they have different stopping points on the eastern edge of the city.

   There are 6 ways to select the stopping point on the eastern edge. This makes six disjoint cases ($k = 0,\ 1,\ 2,\ \ldots 5$).

   For each stopping point, we apply the product rule: The first subtask is to get to that point, and the second subtask is to get to the north-western corner. Therefore, there answer is
$$\sum_{k=0}^{5} \binom{6+k}{k}\binom{6+(5-k)}{5-k} = \binom{6}{0}\binom{11}{5} + \binom{7}{1}\binom{10}{4} + \binom{8}{2}\binom{9}{3} + \binom{9}{3}\binom{8}{2} + \binom{10}{4}\binom{7}{1} + \binom{11}{5}\binom{6}{0}$$

   In principle, this is just a combination of a sum rule and product rule.
10. We know that $r = 20$, and
$$\binom{n+r-1}{20} = \binom{n+19}{20} = 230230$$

You can try all different n, one by one, until you get the answer.

Eventually, you get that $n = 7$ is the correct answer.

There is one simplification to the solution. Because $230230 = 230 \cdot 1001 = 23 \cdot 10 \cdot 7 \cdot 7 \cdot 20$, we could deduce that $n \geq 4$, so we did not really have to check all the numbers, and could start with $n = 4$, and go up.

11. Observations:
    1) A number is divisible by 10, if it ends with a zero.
    2) All integers have 3 digits
    Thus the first and the third digits are not zeroes.

    Question one: $9 \cdot 10 \cdot 9 = 810$

    Question two: $9 \cdot 1 \cdot 8 + 9 \cdot 8 \cdot 7 = 576$

    Notice that the second digit can be a zero. If it is indeed a zero, then there are $9 \cdot 8$ ways to select other two digits (2-permutation formula). If it is not a zero, then ther are $9 \cdot 8 \cdot 7$ ways (this is the number of 3-permutation).

    You can write a program to check the numbers (it can be interesting to see if we are correct or not, let me know if you try that).

12. Use Pascal's identity.