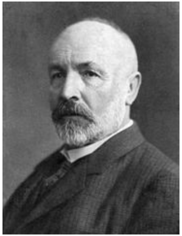
Sets. Ordered pairs.

Sets

The set theory is a branch of mathematical logic that was created by Georg Cantor in 1870s.



Def. A *set* is a unordered collection of objects being regarded as a single object.

Examples:

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{x \in A \mid (x \geq 3) \wedge (x \text{ is odd})\}$$

$$C = \{\varnothing, \{A\}, \{B\}, \{A, B\}\}$$

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Sets

If x belongs to a set A, we say that it is a *member* (or an element) of A and write

$$x \in A.$$

If x is not a member of A, we write

$$x \notin A.$$

Empty set, denoted by $\varnothing$ or $\emptyset$, has no members:

$$\forall x \; (x \notin \varnothing).$$

Sets

Two sets A and B are *equal*, $A = B$, iff they have exactly the same elements:

$$\forall x\, (x \in A \longleftrightarrow x \in B)$$

For any two objects x and y, we can make a set containing exactly these two objects

$$\{x, y\}$$

If those two objects are identical, $x = y$, we get a singleton set,

$$\{x, x\} = \{x\}.$$

Notice that these sets are not equal:

$$\varnothing, \quad \{\varnothing\}, \quad \{\varnothing, \{\varnothing\}\}$$

Sets

Set-builder notation.

A set of all objects that satisfy the property P:

$$A = \{x \mid P(x)\}$$

Examples:

$$B = \{x \in \mathbb{Z} \mid x \text{ is even}\}$$

$$C = \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}\,(x = 2k)\}$$

$$D = \{x \mid (x \in \mathbb{Z}) \wedge (\exists k \in \mathbb{Z}\,(x = 2k))\}$$

$$E = \{0, 2, -2, 4, -4, 6, -6, \ldots\}$$

In naive set theory, any definable collection is a valid set. And usually it works fine.

However, it leads to contradictions, such as Russell's paradox.

Russell's paradox

Let R be the set of all sets that are not members of themselves:

$$R = \{x \mid x \notin x\}$$

It is legal to ask, is R a member of itself or not.

There are only two cases, either $R \in R$, or $R \notin R$.

So, where is the paradox?

Russell's paradox

$$R = \{x \mid x \notin x\}$$

(a) If R is a member of itself, then by its definition, $R \notin R$,

$$R \in R \to R \notin R.$$

(b) Otherwise, if R is not a member of itself, then $R \in R$,

$$R \notin R \to R \in R.$$

Therefore, we get a contradiction, $R \in R \leftrightarrow R \notin R$.

There exist several axiomatic systems that rule out such pathological cases. For example, Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC).

Union and Intersection

Union of two sets A and B,

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

$$\{1,2\} \cup \{2,3\} = \{1,2,3\}$$

Intersection of two sets A and B,

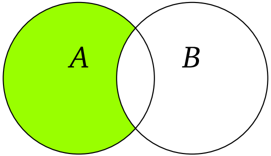
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

$$\{1,2\} \cap \{2,3\} = \{2\}$$

Difference

Difference between two sets A and B,

$$A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\}$$

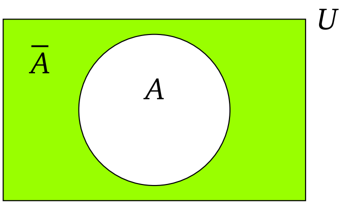


$$\{1,2\} \setminus \{2,3\} = \{1\}$$

Complement

If there is a *universal set* U of all possible objects, then the *complement* of a set A is

$$\overline{A} = U \setminus A = \{x \in U \mid x \notin A\} = \{x \in U \mid \neg(x \in A)\}$$



Exmaple: When $U = \mathbb{Z}$:

$$Odd = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$$

$$Even = \overline{Odd} = \mathbb{Z} \setminus Odd$$

Set identities

Given the universal set U, sets with respect to union, intersection, and complement satisfy the same identities as propositions with respect to $\wedge$, $\vee$, and $\neg$

Sets:	$A \cap B$	$A \cup B$	$\overline{A}$	U	$\varnothing$
Propositions:	$p \wedge q$	$p \vee q$	$\neg p$	T	F

$$\overline{\overline{A}} = A$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

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See the full list in Rosen's book.

Set identities

Let's prove De Morgan's law for sets:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\begin{aligned}(x \in \overline{A \cup B}) &= \neg(x \in A \cup B) \\ &= \neg((x \in A) \vee (x \in B))\end{aligned}$$

$$\begin{aligned}(x \in \overline{A} \cap \overline{B}) &= (x \in \overline{A}) \wedge (x \in \overline{B}) \\ &= \neg(x \in A) \wedge \neg(x \in B) \\ &= \neg((x \in A) \vee (x \in B))\end{aligned}$$

The propositions in the right hand sides of the equations are equal, therefore, the left hand sides are equal too.

Subset

A is a *subset* of B iff every element of A is an element of B:

$$A \subseteq B \quad \leftrightarrow \quad \Big(\ \forall x((x \in A) \rightarrow (x \in B))\ \Big)$$

$$\{1,2\} \subseteq \{1,2,3\}$$

$$\{1,2,3\} \subseteq \{1,2,3\}$$

$$\varnothing \subseteq \{1,2,3\}$$

A is a *proper subset* of B iff A is a subset of B, but it's not equal to B

$$A \subsetneq B \quad \leftrightarrow \quad \Big(\ \forall x((x \in A) \rightarrow (x \in B))\ \wedge\ \exists x((x \in B) \wedge (x \notin A))\ \Big)$$

$$\{1,2\} \subsetneq \{1,2,3\}$$

$$\varnothing \subsetneq \{1,2,3\}$$

Proper subset A is strictly "smaller" than B.

Power set

Def. The set of all subsets of A is called a *power set* of A, denoted by $\mathscr{P}(A)$.

Examples:

$$\mathscr{P}(\{0,1\}) = \big\{\varnothing,\ \{0\},\ \{1\},\ \{0,1\}\big\}$$

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Cardinality

Def. The *cardinality* of a finite set A is equal to the number of elements in A. It's denoted by $|A|$.

$$|\varnothing| = 0$$
$$|\{0, 1, 2, 3, 4\}| = 5$$

We already know the subtraction rule for the cardinality of a union:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Question

Compute the cardinality of the power set $\mathscr{P}(A)$ if $|A| = n$.

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$$|\mathscr{P}(A)| = 2^{|A|} = 2^n$$

Generalized union and intersection

Union of a finite set of sets:

$$\bigcup\{A_0\} = A_0$$

$$\bigcup\{A_0, A_1, \ldots A_n\} = \bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \ldots \cup A_n$$

Intersection of a finite set of sets:

$$\bigcap\{A_0\} = A_0$$

$$\bigcap\{A_0, A_1, \ldots A_n\} = \bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap \ldots \cap A_n$$

Ordered pair

Def. The *ordered pair* of $a \in A$ and $b \in B$ is an ordered collection (a, b).

Two ordered pairs are equal

$$(a, b) = (c, d) \quad \text{if and only if} \quad (a = c) \wedge (b = d).$$

Observe that this property implies that

$$(a, b) \neq (b, a)$$

unless $a = b$. So, the order matters. This is why it is called the ordered pair, and (a, b) is not equivalent to a set $\{a, b\}$.

$$\begin{aligned} (1, 2) &= (1, 2) \\ (1, 2) &\neq (1, 3) \\ (1, 2) &\neq (2, 1) \end{aligned}$$

Ordered pair

More examples of ordered pairs:

$$(1, 2)$$
$$(\{1\}, \{2\})$$
$$(1, \{2, 3, 4, 5\})$$
$$((1, 2), \varnothing)$$

If $a \in A$ and $b \in B$, what is the set of all ordered pairs (a, b)?

Cartesian product

Def. Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

(Named after Rene Decartes)

Question. Given two sets $A = \{1, 2, 3\}$ and $B = \{C, D\}$, what is their Cartesian product $A \times B$?

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$$\begin{aligned} A \times B = \{&(1, C),\ (2, C),\ (3, C), \\ &(1, D),\ (2, D),\ (3, D)\} \end{aligned}$$

Cartesian product

Question. If the $|A| = n$, and $|B| = m$, what is the cardinality of $A \times B$?

Building a list

Ordered pair is fundamental for defining data types.
Question. How to implement the list data type using only ordered pairs?

Example of a list:

$$[1, 2, 3, 4]$$

Interface:

$$\begin{aligned} \mathrm{construct}\,(\,1, [2, 3, 4]\,) &= [1, 2, 3, 4] \\ \mathrm{head}\,(\,[1, 2, 3, 4]\,) &= 1 \\ \mathrm{tail}\,(\,[1, 2, 3, 4]\,) &= [2, 3, 4] \end{aligned}$$

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Possible implementation:

$$(1, (2, (3, 4)))$$
$$\text{construct} \,(\, h, t \,) = (h, t)$$
$$\text{head} \,(\, (h, t) \,) = h$$
$$\text{tail} \,(\, (h, t) \,) = t$$

Ordered n-tuple

Def. The *ordered n-tuple* of is an ordered collection $(a_1, a_2, \ldots, a_n)$.

It is just an extension of an ordered pair for joining n elements together.

Def. The *Cartesian product* of the sets A_1, $\ldots A_n$, is the set of all n-tuples such that

$$A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}$$

If all sets A_i are equal, that is, $A_1 = \ldots = A_n = A$, then their Cartesian product is denoted by A^n

$$A_1 \times \ldots \times A_n = \underbrace{A \times \ldots \times A}_{n \text{ times}} = A^n$$