

The Pigeonhole Principle. Permutations and Combinations

A typical situation

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?



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Pascal's Triangle

The Binomial Theorem

A typical situation

A drawer in a dark room contains red socks, green socks, and blue socks. How many socks must you withdraw to be sure that you have a matching pair?



Four is enough.

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The Pigeonhole Principle



If you have 10 pigeons in 9 boxes, at least one box contains two birds.

The pigeonhole principle. If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

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What is the minimum number of students required in a class to be sure that at least two will receive the same grade, if there are five possible grades, A, B, C, D, and F?

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Theorem

What is the minimum number of students required in a class to be sure that at least two will receive the same grade, if there are five possible grades, A, B, C, D, and F?

$$5 + 1 = 6 \text{ students.}$$

Generalized Pigeonhole Principle

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The Generalized Pigeonhole Principle. If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Ceiling function:

$\lceil x \rceil$ = the smallest integer not less than x

So, for example,

$$\lceil 2.0 \rceil = 2$$

$$\lceil 0.5 \rceil = 1$$

$$\lceil -3.5 \rceil = -3$$

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The Generalized Pigeonhole Principle. If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

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The Generalized Pigeonhole Principle. If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

“Birds” are cards. “Boxes” are suits, $k = 4$.

How many cards, N , should we take to guarantee that at least three of them fall in the same “box” (suit):

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{N}{4} \right\rceil \geq 3 > 2.$$

The smallest possible $N = 2 \cdot 4 + 1 = 9$. ♣♣♦♦♥♥♠♠

Permutations

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Theorem

Count the number of ways to arrange the elements of this set:

$$\{a, b, c, d, e, f\}$$

- There are 6 ways to select the first element,
- 5 ways to select the second element,
- 4 ways to select the third ...
- ...continue the process
- In the end, the only remaining element takes the last position.

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{ways!}$$

Factorial

How large this number is?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This function is called *factorial* and denoted by $n!$:

$$1! = 1$$

$$2! = 2 \cdot 1$$

$$3! = 3 \cdot 2 \cdot 1$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 1$$

and by convention,

$$0! = 1$$

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Def. A *permutation* of a set of distinct objects is an ordered arrangement of these objects.

A finite set of 6 elements has

$$P(6) = 6! = 720 \text{ permutations.}$$

A finite set A with cardinality $|A| = n$ has

$$P(n) = n! \text{ permutations.}$$

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What if we want to arrange only r elements?

Def. An ordered arrangement of r elements of a set is called an *r -permutation*.

Can we get the formula the the number of r -permutations?

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Count the number of ways to arrange 4 elements of the set:

$$\{a, b, c, d, e, f\}$$

- There are 6 ways to select the first element,
- 5 ways to select the second element,
- 4 ways to select the third ...
- 3 ways to select the fourth ...

$$6 \cdot 5 \cdot 4 \cdot 3 \quad \text{ways!}$$

The number of r -permutations of the set of n elements:

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

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The number of r -permutations of the set of n elements:

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

Multiply and divide by $(n-r) \cdot (n-r-1) \cdot \dots \cdot 2 \cdot 1$,

$$P(n, r) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \dots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$$

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The number of r -permutations of the set of n elements:

$$P(n, r) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

$$P(n, r) = \frac{n!}{(n-r)!}$$

The formula makes sense only for $0 \leq r \leq n$, otherwise the notion of r -permutation does not make sense.

Permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

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Theorem

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(100, 3) = 100 \cdot 99 \cdot 98 = 970200.$$

Alternatively,

$$P(100, 3) = \frac{100!}{(100-3)!} = \frac{1 \cdot 2 \cdot \dots \cdot 100}{1 \cdot 2 \cdot \dots \cdot 97} = 98 \cdot 99 \cdot 100 = 970200.$$

What if the order does not matter?

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You are given all r -permutations of a set.

Now, let's say that you don't really care about the ordering in each r -permutation.

Concrete example. You are given a group of 4 students, $\{a, b, c, d\}$. How many groups of 3 students can be formed?

The total number of 3-permutations: $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$.

All possible groups:

$abc, acb, bac, bca, cab, cba$ the same subset $\{a, b, c\}$

$abd, adb, bad, bda, dab, dba$ the same subset $\{a, b, d\}$

$acd, acb, cad, cda, dac, dca$ the same subset $\{a, c, d\}$

$bcd, bcb, cbd, cdb, dbc, dc b$ the same subset $\{b, c, d\}$

What if the order does not matter?

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Concrete example. You are given a group of 4 students, $\{a, b, c, d\}$. How many groups of 3 students can be formed?

The total number of 3-permutations: $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$.

Let x be the number of unordered selections of 3 students, such as, for example: $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

Each such selection can be realized in $P(3) = 3! = 6$ permutations, e.g. $\{a, b, c\}$ has the following 6 permutations: $abc, acb, bac, bca, cab, cba$.

$$P(4, 3) = x \cdot P(3).$$

Thus, there are only $x = \frac{P(4, 3)}{P(3)} = \frac{24}{6} = 4$ unordered selections of 3 students.

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Theorem

Def. An ***r -combination*** of elements of a set is an unordered selection of r elements from the set.

The number of r -combinations is

$$\binom{n}{r} = \frac{P(n, r)}{P(r)}$$

This notation reads as “ n choose r ”.

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Def. An *r-combination* of elements of a set is an unordered selection of r elements from the set.

The number of r -combinations is

$$\binom{n}{r} = \frac{P(n, r)}{P(r)}$$

Let's express it in terms of n , r , and their factorials:

$$P(n, r) = \frac{n!}{(n-r)!} \text{ and } P(r) = r!, \text{ therefore}$$

$$\binom{n}{r} = \frac{n!}{(n-r)! r!}$$

Example with cards

$$\binom{n}{r} = \frac{n!}{(n-r)! r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

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$$\binom{n}{r} = \frac{n!}{(n-r)! r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)! 5!} = \frac{52!}{47! 5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

Example with cards

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$$\binom{n}{r} = \frac{n!}{(n-r)! r!}$$

Counting hands of 5 cards from the deck of 52.

Count the number of ways 5 cards can be dealt from the deck of 52 if their order does not matter.

$$\binom{52}{5} = \frac{52!}{(52-5)! 5!} = \frac{52!}{47! 5!} = \frac{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52}{5!} = 2598960.$$

More examples:

$$\binom{52}{2} = \frac{51 \cdot 52}{2} = 1326. \quad \binom{52}{1} = 52.$$

Summary

The Pigeonhole Principle

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Given a set with n elements.

The number of *permutations* of the elements the set:

$$P(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1$$

The number of *r-permutations* of the set:

$$P(n, r) = \frac{n!}{(n-r)!} = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-r+1)}_{\text{product of } r \text{ numbers}}$$

The number of unordered *r-combinations* (“ n choose r ”):

$$\binom{n}{r} = \frac{P(n, r)}{P(r)} = \frac{n!}{(n-r)! r!}$$

Let's try to be more systematic

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How does $\binom{n}{r}$ change with r ?

$$\binom{0}{0} = \frac{0!}{0! 0!} = 1.$$

$$\binom{1}{0} = \frac{1!}{1! 0!} = 1, \quad \binom{1}{1} = \frac{1!}{0! 1!} = 1.$$

$$\binom{2}{0} = \frac{2!}{2! 0!} = 1, \quad \binom{2}{1} = \frac{2!}{1! 1!} = 2, \quad \binom{2}{2} = \frac{2!}{0! 2!} = 1.$$

$$\binom{3}{0} = \frac{3!}{3! 0!} = 1, \quad \binom{3}{1} = \frac{3!}{2! 1!} = 3, \quad \binom{3}{2} = \frac{3!}{1! 2!} = 3, \quad \binom{3}{3} = \frac{3!}{0! 3!} = 1.$$

Pascal's Triangle

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$$\binom{0}{0}$$

1

$$\binom{1}{0} \quad \binom{1}{1}$$

1 1

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

1 2 1

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

1 3 3 1

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

1 4 6 4 1

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1 5 10 10 5 1

Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x + y)^n$:

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

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Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x + y)^n$:

$$(x + y)^1 = 1x + 1y$$

$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

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Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x + y)^n$:

$$(x + y)^1 = 1x + 1y$$

$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The general formula is

$$(x + y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \dots + \binom{n}{n} \cdot y^n.$$

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Pascal's Triangle

The numbers in Pascal's Triangle are the coefficients of the polynomials of the form $(x + y)^n$:

$$(x + y)^1 = 1x + 1y$$

$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The general formula is

$$(x + y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \dots + \binom{n}{n} \cdot y^n.$$

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

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Coefficients of $(x + y)^n$

Let's prove

$$(x + y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \dots + \binom{n}{n} \cdot y^n.$$

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$$\begin{aligned}(x + y)^n &= \underbrace{(x + y) \cdot (x + y) \cdot \dots \cdot (x + y)}_{n \text{ times}} = \\&\quad \underbrace{x \cdot x \cdot \dots \cdot x}_{=x^n} + \\&\quad \underbrace{(x \cdot \dots \cdot x)}_{=x^{n-1}} \cdot y + \dots + y \cdot \underbrace{(x \cdot \dots \cdot x)}_{=x^{n-1}} + \\&\quad \underbrace{(x \cdot \dots \cdot x)}_{=x^{n-2}} \cdot \underbrace{(y \cdot y)}_{=y^2} + \dots + \underbrace{(y \cdot y)}_{=y^2} \cdot \underbrace{(x \cdot \dots \cdot x)}_{=x^{n-2}} + \\&\quad \dots + \\&\quad \underbrace{y \cdot y \cdot \dots \cdot y}_{=y^n}\end{aligned}$$

Coefficients of $(x + y)^n$

Let's prove

$$(x + y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \dots + \binom{n}{n} \cdot y^n.$$

$$(x + y)^n = \underbrace{(x + y) \cdot (x + y) \cdot \dots \cdot (x + y)}_{n \text{ times}} =$$

$$\begin{aligned} &1 \cdot x^n + \\ &n \cdot x^{n-1}y + \\ &\binom{n}{2} \cdot x^{n-2}y^2 + \\ &\dots + \\ &1 \cdot y^n \end{aligned}$$

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Coefficients of $(x + y)^n$

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Let's prove

$$(x + y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \dots + \binom{n}{n} \cdot y^n.$$

$$(x + y)^n = \underbrace{(x + y) \cdot (x + y) \cdot \dots \cdot (x + y)}_{n \text{ times}} = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \dots + \binom{n}{n} \cdot y^n.$$

Shorter notation for the same thing:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

The Binomial Theorem

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Theorem

$$(x + y)^n = \binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1}y + \binom{n}{2} \cdot x^{n-2}y^2 + \dots + \binom{n}{n} \cdot y^n.$$

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

This result is called *The Binomial Theorem*, and this is why the coefficients, $\binom{n}{k}$, are also called the *binomial coefficients*.

The Binomial Theorem

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The Binomial
Theorem

Let's prove that

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n.$$

The Binomial Theorem

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The Binomial
Theorem

Let's prove that

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n.$$

$$(1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^n \binom{n}{k} \cdot 1 = \sum_{k=0}^n \binom{n}{k}.$$

Therefore,

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

The Binomial Theorem

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Theorem

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n.$$

Results like this are not so obvious.

Recall that

$$\binom{n}{k} = \frac{n!}{(n-k)! k!}$$

So,

$$\frac{n!}{n! 0!} + \frac{n!}{(n-1)! 1!} + \frac{n!}{(n-2)! 2!} + \dots + \frac{n!}{0! n!} = 2^n.$$

In this form, the result seems to be much harder to prove.

Pascal's Triangle Again

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The Binomial
Theorem

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal's Identity

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Theorem

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Every number in Pascal's triangle is equal to the sum of the two numbers that are immediately above.