Recursion in Mathematics and Programming

Summation

Compute summation

$$\sum_{k=1}^{n} = 1 + 2 + \ldots + n.$$

It's recursive definition:

$$\begin{aligned} \mathbf{sum}(0) &= 0 \\ \mathbf{sum}(n) &= \mathbf{sum}(n-1) + n \qquad (\text{for } n \geq 1) \end{aligned}$$

Source code "`sum.jl`".

Summation

Evaluation of the expression **sum**(4)

$$
\begin{aligned}
\mathbf{sum}(4) &\Rightarrow \mathbf{sum}(3) + 4 \\
&\Rightarrow (\mathbf{sum}(2) + 3) + 4 \\
&\Rightarrow ((\mathbf{sum}(1) + 2) + 3) + 4 \\
&\Rightarrow (((\mathbf{sum}(0) + 1) + 2) + 3) + 4 \\
&\Rightarrow (((0 + 1) + 2) + 3 + 4) \\
&\Rightarrow ((1 + 2) + 3) + 4 \\
&\Rightarrow (3 + 3) + 4 \\
&\Rightarrow 6 + 4 \\
&\Rightarrow 10
\end{aligned}
$$

Summation

$$
\begin{array}{c}
\mathbf{sum}(4) \\
| \\
\mathbf{sum}(3) \\
| \\
\mathbf{sum}(2) \\
| \\
\mathbf{sum}(1) \\
| \\
\mathbf{sum}(0)
\end{array}
$$

To compute **sum**(4), the function **sum** was called 5 times.

And to compute **sum**(n), the function **sum** will be called $n+1$ times.

Fibonacci numbers

Recursive definition of the Fibonacci numbers:

$$\begin{aligned} \mathbf{fib}(0) &= 0 \\ \mathbf{fib}(1) &= 1 \\ \mathbf{fib}(n) &= \mathbf{fib}(n-1) + \mathbf{fib}(n-2) \qquad (\text{for } n \geq 2) \end{aligned}$$

Source code “`fib.jl`”.

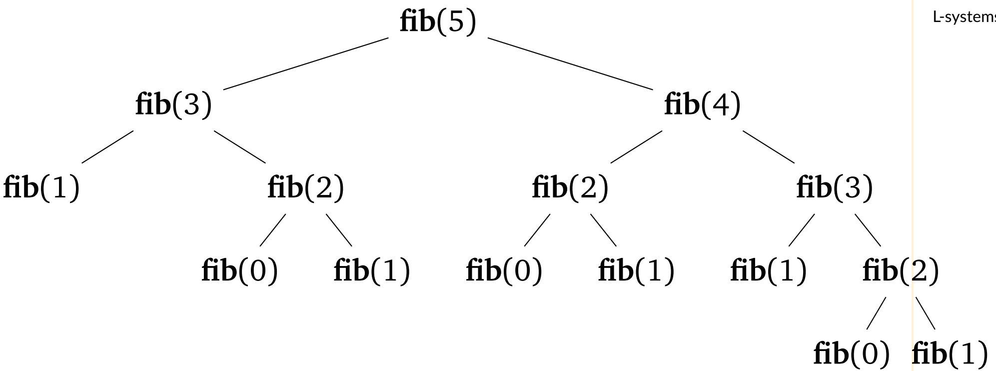
Fibonacci numbers

Evaluation of the expression **fib**(5)

$$
\begin{aligned}
\mathbf{fib}(5) &\Rightarrow \mathbf{fib}(3) + \mathbf{fib}(4) \\
&\Rightarrow (\mathbf{fib}(1) + \mathbf{fib}(2)) + (\mathbf{fib}(2) + \mathbf{fib}(3)) \\
&\Rightarrow (1 + (\mathbf{fib}(0) + \mathbf{fib}(1))) + ((\mathbf{fib}(0) + \mathbf{fib}(1)) + (\mathbf{fib}(1) + \mathbf{fib}(2))) \\
&\Rightarrow (1 + (0 + 1)) + ((0 + 1) + (1 + (\mathbf{fib}(0) + \mathbf{fib}(1)))) \\
&\Rightarrow (1 + 1) + (1 + (1 + (0 + 1))) \\
&\Rightarrow 2 + (1 + (1 + 1)) \\
&\Rightarrow 2 + (1 + 2) \\
&\Rightarrow 2 + 3 \\
&\Rightarrow 5
\end{aligned}
$$

How many times did we call function **fib** to compute **fib**(5)?

Fibonacci numbers



To compute **fib**(n), we need to do approximately 2^n function calls.

Exponential running time, $O(2^n)$, too slow to be practical.

Improved recursive Fibonacci

Evaluation of the expression **fib_loop_rec**(5)

$$\begin{aligned}
\textbf{fib_loop_rec}(5) &\Rightarrow \textbf{loop}(1,0,1)\\
&\Rightarrow \textbf{loop}(2,1,1)\\
&\Rightarrow \textbf{loop}(3,1,2)\\
&\Rightarrow \textbf{loop}(4,2,3)\\
&\Rightarrow \textbf{loop}(5,3,5)\\
&\Rightarrow 5
\end{aligned}$$

Just $n+1$ function calls to compute **fib_loop_rec**(n).

This is a linear time $O(n)$ algorithm, we get a real improvement!

A simple calculator language

We want to describe a computer that can do the following:

Input		Output
55	$\Rightarrow$	55
$(\; 1 \; + \; 2 \;)$	$\Rightarrow$	3
$(\; 12 \; + \; (\; (\; 5 \; \times \; 2 \;) \; \times \; 7 \;) \;) \;)$	$\Rightarrow$	82

A simple calculator language

$$100$$

$$(\; 5 + (\; 74 - 15 \;) \;)$$

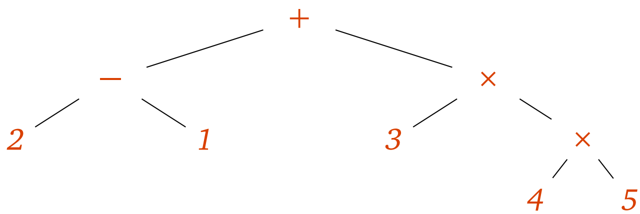
$$(\; (\; 12 + 5 \;) \times (\; 5 - 7 \;) \;) \;)$$

The syntax of the language:

$$\begin{aligned} < expr > \quad ::= \quad & \textit{strings of digits}, \text{ represent integer numbers} \\ \mid \quad & (\; < expr > + < expr > \;) \\ \mid \quad & (\; < expr > - < expr > \;) \\ \mid \quad & (\; < expr > \times < expr > \;) \end{aligned}$$

A simple calculator language

$$(\ (\ 2 - 1 \ ) \ + \ (\ 3 \times \ (\ 4 \times 5 \ ) \ ) \ )$$



The evaluating function looks at the root of the tree only and desides what to do

$$eval\big[\ (\ E_1 \ + \ E_2 \ ) \ \big]$$
$$\Rightarrow \quad eval\big[E_1\big] + eval\big[E_2\big]$$

A simple calculator language

Simple examples

Evaluating expressions

Recursion in Mathematics

L-systems

Evaluation function:

$$
\begin{aligned}
eval\left[\ (\ E_1\ +\ E_2\ )\ \right] &= eval\left[E_1\right] + eval\left[E_2\right] \\
eval\left[\ (\ E_1\ -\ E_2\ )\ \right] &= eval\left[E_1\right] - eval\left[E_2\right] \\
eval\left[\ (\ E_1\ \times\ E_2\ )\ \right] &= eval\left[E_1\right] \times eval\left[E_2\right] \\
eval\left[n\right] &= n
\end{aligned}
$$

A simple calculator language

$$
\begin{aligned}
& eval[\ (\ (\ 2 - 1\ )\ +\ (\ 3 \times\ (\ 4 \times 5\ )\ )\ )\ ] \\
\Rightarrow\ & eval[\ (\ 2 - 1\ )\ ]\ +\ eval[\ (\ 3 \times\ (\ 4 \times 5\ )\ )\ ] \\
\Rightarrow\ & (eval[2] - eval[1]) + (eval[3] \times eval[\ (\ 4 \times 5\ )\ ]) \\
\Rightarrow\ & (2 - 1) + (3 \times (eval[4] \times eval[5])) \\
\Rightarrow\ & 1 + (3 \times (4 \times 5)) \\
\Rightarrow\ & 1 + (3 \times 20) \\
\Rightarrow\ & 1 + 60 \\
\Rightarrow\ & 61
\end{aligned}
$$

Recursion in Mathematics

Some prominent examples of recursion

- *Euclid's algorithm*
  a algorithm for computing the greatest common divisor.
- *Newton's method*
  a recursive method for finding successively better approximations to the roots of a real-valued function.
- Complex objects such as fractals can be defined using recursive definition.

Consider a recurrence

Given a number c,

$$z_0 = 0$$
$$z_n = z_{n-1}^2 + c$$

What sequences of numbers can be generated with this recurrence? The result, of course, depends on the value of c.

Consider a recurrence

$$z_0 = 0$$
$$z_n = z_{n-1}^2 + c$$

Let us test different values for c.

$$\begin{aligned}
\text{when } c = 0: &\quad 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \\
\text{when } c = -1: &\quad 0 \to (-1) \to 0 \to (-1) \to \cdots \\
\text{when } c = -2: &\quad 0 \to (-2) \to 2 \to 2 \to 2 \to \cdots
\end{aligned}$$

$$\begin{aligned}
\text{when } c = 1: &\quad 0 \to 1 \to 2 \to 5 \to 26 \to \cdots \infty \\
\text{when } c = -3: &\quad 0 \to (-3) \to 6 \to 33 \to \cdots \infty
\end{aligned}$$

Define set M

$$
\begin{aligned}
\text{when } c = 0: &\quad 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \\
\text{when } c = -1: &\quad 0 \to (-1) \to 0 \to (-1) \to \cdots \\
\text{when } c = -2: &\quad 0 \to (-2) \to 2 \to 2 \to 2 \to \cdots
\end{aligned}
$$

$$
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$$

We are interested in all c, such that the corresponding sequence z_n **does not go to infinity**.

Let's say that all such numbers c belong to the set M.

Define set M

$$\begin{aligned}
\text{when } c = 0: &\quad 0 \to 0 \to 0 \to 0 \to 0 \to \cdots \\
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\text{when } c = -2: &\quad 0 \to (-2) \to 2 \to 2 \to 2 \to \cdots
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We are interested in all c, such that the corresponding sequence z_n **does not go to infinity**.

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Numbers c and z_n are *complex numbers*. So, we need to quickly learn how to add and multiply them.

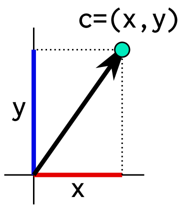
Quick intro to complex numbers

The set of complex numbers $\mathbb{C}$ is an extension of the set of real numbers $\mathbb{R}$.

Any complex number can be represented by a pair of real numbers

$$(x, y) \qquad x, y \in \mathbb{R}$$

x is the *real part*,
y is the *imaginary part*.



Alternative notation. We can represent the pair as a sum of its real and imaginary part

$$(x, y) = x + yi$$

$i = \sqrt{-1}$ is the *imaginary unit*.

Quick intro to complex numbers

Addition

$$(a, b) + (c, d) = (a + c, b + d)$$

Multiplication

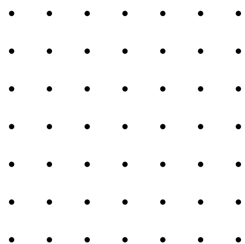
$$(a, b) \cdot (c, d) = (ac - bd, bc + ad)$$

Particularly,

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1$$

How to draw the set M?

It's impossible to check all numbers c. But we can take a rectangular grid of points around the origin, $(0,0)$. We are going to mark all those points that belong to the set M.



$z_n = (x_n, y_n)$ will go to infinity if one of its components get large, $\sqrt{x_n^2 + y_n^2} \geq 2$. This condition is used practically to compute the membership of c in the set.

Source code "`mset.jl`".

L-systems

An L-system or Lindenmayer system is a parallel *rewriting system*.

It consists of

- an alphabet of symbols
- an initial string (called an axiom) to start construction
- a collection of production rules that expand each symbol into some larger string of symbols

Example:

Alphabet:	$\{A, B\}$
Initial string:	A
Production rules:	$A \longrightarrow AB$
	$B \longrightarrow A$

L-systems

Example:

Alphabet:	$\{A, B\}$
Initial string:	A
Production rules:	$A \longrightarrow AB$
	$B \longrightarrow A$

Start rewriting:

$$n = 0 : A$$
$$n = 1 : AB$$
$$n = 2 : ABA$$
$$n = 3 : ABAAB$$
$$n = 4 : ABAABABA$$
$$n = 5 : ABAABABAABAAB$$
$$n = 6 : ABAABABAABAABABAABABA$$

L-systems

Koch curve:

Alphabet:	$\{F, L, R\}$
Initial string:	F
Production rules:	$F \longrightarrow F\ L\ F\ R\ F\ R\ F\ L\ F$
	$L \longrightarrow L$
	$R \longrightarrow R$

When F stands for moving forward, and L and R are the commands for turning left and right

Source code "`lsys.jl`".