Predicates and Quantifiers.

Predicates

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Propositional logic, studied previously, cannot adequately express the meaning of all statements in mathematics and in natural language.

Examples?

“x is greater than 5.”

“x is greater than y.”

“n is a prime number.”

“$user$ is waiting.”

Def. *A predicate is a proposition whose truth depends on the value of one or more variables.*

Predicates

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

For convenience, we can give every predicate a name:

$$P(x) = \text{"}x \text{ is greater than 5."}$$

$$Q(x, y) = \text{“}x \text{ is greater than } y\text{.”}$$

When the values of the variables are specified, the result is a simple proposition: Depending on the, the predicats are either true or false:

$$P(4) = F$$
$$P(10) = T$$

$$Q(2, 1) = T$$

How can we use predicates?

Let $C(x) =$ “x is playing chess.”

There are three persons in the room: Ed, $Paul$, and Tom.

Ed and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Ed) = T$, $C(Paul) = T$, $C(Tom) = F$.

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

How can we use predicates?

Let $C(x) =$ "x is playing chess."

There are three persons in the room: Ed, $Paul$, and Tom.

Ed and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Ed) = T$, $C(Paul) = T$, $C(Tom) = F$.

Consider a proposition "Someone is playing chess in the room."
Is it true?

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

How can we use predicates?

Let $C(x) =$ "x is playing chess."

There are three persons in the room: Ed, $Paul$, and Tom.

Ed and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Ed) = T$, $C(Paul) = T$, $C(Tom) = F$.

Consider a proposition "Someone is playing chess in the room."
Is it true?

Yes, because, for example, $C(Ed) = T$.

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

How can we use predicates?

Let $C(x) =$ "x is playing chess."

There are three persons in the room: Ed, $Paul$, and Tom.

Ed and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Ed) = T$, $C(Paul) = T$, $C(Tom) = F$.

Another proposition "Everyone is playing chess in the room."

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

How can we use predicates?

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Let $C(x) =$ "x is playing chess."

There are three persons in the room: Ed, $Paul$, and Tom.

Ed and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Ed) = T$, $C(Paul) = T$, $C(Tom) = F$.

Another proposition "Everyone is playing chess in the room."

It's false, because $C(Tom) = F$.

Always true or sometimes true ?

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Let $P(x) =$ "$x^2 \geq 0$."

Always true

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Let $P(x) =$ "$x^2 \geq 0$."

Always true:

"For all n, $P(n)$ is true."

"$P(n)$ is true for every n."

"For all x, $x^2 \geq 0$."

"$x^2 \geq 0$ for every x."

Always true

"For all x, $x^2 \geq 0$."

An assertion that a predicate is always true is called a *universal quantification*.

Always true or sometimes true?

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Another predicate:

$$\text{Let } Q(x) : \text{“}5x^2 - 7 = 0.\text{”}$$

Always true or sometimes true?

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Another predicate:

$$\text{Let } Q(x) : \text{“}5x^2 - 7 = 0.\text{”}$$

It’s true only when $x = \pm\sqrt{7/5}$.

Sometimes true

Let $Q(x)$: "$5x^2 - 7 = 0$."

It's true only when $x = \pm\sqrt{7/5}$.

Sometimes true:

"There exist an n such that $Q(n)$ is true."

"$Q(n)$ is true for some n."

"$Q(n)$ is true for at least one n."

"There exist an x such that $5x^2 - 7 = 0$."

"$5x^2 - 7 = 0$ for some x."

"$5x^2 - 7 = 0$ for at least one x."

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Sometimes true

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

“Exists an x such that $5x^2 - 7 = 0$ is true.”

An assertion that a predicate is true for some values of the variable is called

an *existential quantification*.

Sentences can be ambiguous

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

"If you can solve *any* problem we come up with, then you get an A for the course."

Is it a universal (for all), or an existential (for some) quantification?

Sentences can be ambiguous

The last sentence was ambiguous.
The right way to say it in math class:

Universal (always):

"You can solve *every* problem we come up with."

Existential (sometimes):

"You can solve *at least one* problem we come up with."

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Mathematical notation

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

"For every x, it's true that $x + 1 > x$"

becomes

$$\forall x \, (x + 1 > x)$$

Mathematical notation

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

“For every x, $P(x)$”

becomes

$$\forall x\,(P(x))$$

Mathematical notation

"Exists x such that $x^2 = 4$"

becomes

$$\exists x \; (x^2 = 4)$$

Mathematical notation

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

"Exists x such that $P(x)$"

becomes

$$\exists x \, (P(x))$$

Mathematical notation

Universal:

$$\forall x\ P(x) \qquad \text{means that } \textit{for every } x,\ P(x) \text{ is true.}$$

Existential:

$\exists x\ P(x)$ means that there *exists* an x such that $P(x)$ is true.

We consider only those values of the variable x that belong to a given set called the *domain of discourse*, or the *universe of discourse*.

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Example of the universe of discourse

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

For the predicate $Odd(x)$ and $Even(x)$, the universe of discourse is the set of all integers:

$$\ldots, \text{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5}, \ldots$$

$Odd(x)$ is true for -5, -3, -1, 1, 3, 5, etc.

$Even(x)$ is true for -4, -2, 0, 2, 4, etc.

We consider only those values of the variable x that belong to a given set called the *domain of discourse*, or the *universe of discourse*.

Translating sentences

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

$$
\begin{aligned}
White(c) &\quad : \quad c \text{ is white} \\
Black(c) &\quad : \quad c \text{ is black} \\
Red(c) &\quad : \quad c \text{ is red} \\
Stronger(x, y) &\quad : \quad \text{cat } x \text{ is stronger than cat } y
\end{aligned}
$$

We want to say that:

(a) There exist white, black, and red cats.

(b) White cats are always stronger than black cats.

(c) There is one red cat that is stronger than any other cat.

(d) Not all cats are black, white, or red.

To prove or disprove a quantification

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Statement	When true?	When false?
$\forall x\ P(x)$	$P(x)$ is true for every x.	There is at least one *counterexample* x such that $P(x)$ is false.
$\exists x\ P(x)$	There is at least one x such that $P(x)$ is true.	$P(x)$ is false for every x.

Example

Let $P(x) =$ "$x^2 > 0$."

The universe of discourse are all integer numbers.

Is it true or false that $\forall x \; P(x)$?

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Example

Let $P(x) =$ "$x^2 > 0$."

The universe of discourse are all integer numbers.

Is it true or false that $\forall x \; P(x)$?

To prove it, we must show that $P(x)$ is true for all integers.
To disprove, we have to find a counterexample for which it's false.

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Example

Let $P(x) =$ "$x^2 > 0$."

The universe of discourse are all integer numbers.

Is it true or false that $\forall x\ P(x)$?

To prove it, we must show that $P(x)$ is true for all integers.
To disprove, we have to find a counterexample for which it's false.

Counterexample:
$P(x)$ is false for $x = 0$. So, the quantified statement $\forall x\ P(x)$ is false.

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Scope, variable binding

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

$$\forall x\ (P(x) \wedge Q(x))$$

The *scope* of the quantifier is the expression to which it's applied. Here, the scope is $P(x) \wedge Q(x)$. Quantifiers *bind* variables inside their scope.

$\forall$ binds x in the logical expression $(P(x) \wedge Q(x))$.

Scope, variable binding

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Why should we care?

$$\forall x\,(P(x) \land Q(x)) \;\land\; \exists y\,(R(y))$$

is equivalent to

$$\forall x\,(\underline{P(x) \land Q(x)}) \;\land\; \exists x\,(\underline{R(x)})$$

"Every dog has four legs and has a tail; and there exists a dog that barks."

Scope, variable binding

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

If all variables are bound by a quantifier or set equal to a particular value then a statement is a proposition:

$$\forall x\,(P(x) \land Q(z)) \;\land\; \exists y(R(y)),\ \text{and it's given that } z = 3.1415.$$

x is bound by $\forall x$, y is bound by $\exists y$, and z is specified by the given equation.

So, this is a proposition.

Scope, variable binding

If a variable is not bound, it's called *free*.

$$\forall x\,\big(A(y) \wedge B(x)\big) \;\wedge\; \exists y\,\big(C(x,y) \to D(y)\big).$$

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Scope, variable binding

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

If a variable is not bound, it's called *free*.

$$\underline{\forall x}\,\big(A(y) \wedge B(\underline{x})\big) \;\wedge\; \underline{\exists y}\,\big(C(x, \underline{y}) \to D(\underline{y})\big).$$

Only the first x is bound by $\forall x$.

This is not a proposition.

Nested quantifiers

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Quantifiers are nested if one is within the scope of the other:

$$\forall x \left(\exists y \ (x + y = 0) \right)$$

It reads as follows

"For every x exists y such that $x + y = 0$."

We usually drop the external parentheses. Equivalent expression:

$$\forall x \ \exists y \ (x + y = 0)$$

Nested quantifiers

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

A few more examples (you know this formulas):

$$\forall x\ \forall y\ \forall z\ \big(x + (y + z) = (x + y) + z\big)$$

$$\forall x\ \forall y\ \big(x + y = y + x\big)$$

$$\forall w\ \forall x\ \forall y\ \forall z\ \left((y \neq 0 \wedge w \neq 0) \rightarrow \frac{x}{y} + \frac{z}{w} = \frac{xw + zy}{yw}\right)$$

Order of quantifiers

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Does the order of quantifiers matter?

$$\forall x \left(\exists y \; (x + y = 0) \right)$$

"For every x exists y such that $x + y = 0$."

$$\exists y \left(\forall x \; (x + y = 0) \right)$$

"Exists y such that for every x: $x + y = 0$."

Order of quantifiers

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Does the order of quantifiers matter?

$$\forall x \left(\exists y \; (x + y = 0)\right)$$

"For every x exists y such that $x + y = 0$."

$$\exists y \left(\forall x \; (x + y = 0)\right)$$

"Exists y such that for every x: $x + y = 0$."

The meaning of the two expressions is different. You cannot swap nested $\forall x$ and $\exists y$.

Order of quantifiers

However, we can swap two (or more) nested quantifiers of the same kind:

$$\forall x \, \forall y \, (x + y = y + x)$$
$$\forall y \, \forall x \, (x + y = y + x)$$

$$\exists x \, \exists y \, (x^y = 4)$$
$$\exists y \, \exists x \, (x^y = 4)$$

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Order of quantifiers

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

These two statements are equivalent:

$$\forall w\ \exists x\ \forall y\ \forall z\ \big(P(w,x,y,z)\big)$$

$$\forall w\ \exists x\ \underline{\forall z\ \forall y}\ \big(P(w,x,y,z)\big)$$

But, these two statements are not equivalent:

$$\forall w\ \exists x\ \forall y\ \underline{\forall z}\ \big(P(w,x,y,z)\big)$$

$$\forall w\ \underline{\forall z}\ \exists x\ \forall y\ \big(P(w,x,y,z)\big)$$

Order of quantifiers

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

"For every even integer n greater than 2, there exist prime numbers p and q such that $n = p + q$."

(Prime numbers are integers > 1, divisible only by itself and 1. They are 2, 3, 5, 7, 11, 13, 17, ...)

The universe of discourse:

n is an *even* integer, $n > 2$.

p and q are *prime* numbers.

$$\forall n\ \exists p\ \exists q\ (n = p + q)$$

Order of quantifiers

Swapping the order of different kinds of quantifiers (existential or universal) changes the meaning of a proposition.

$$\forall n\,(\exists p\,(\exists q\,(n = p + q)))$$

$$4 = 2 + 2$$
$$6 = 3 + 3$$
$$8 = 3 + 5$$
$$\dots$$

$$\exists p\,(\exists q\,(\forall n\,(n = p + q)))$$

$$4 \stackrel{?}{=} p + q$$
$$6 \stackrel{?}{=} p + q$$
$$\dots$$

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Distributivity

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

$$\forall x\, (P(x) \wedge Q(x)) \equiv (\forall x\, P(x)) \wedge (\forall x\, Q(x))$$

we can distribute a universal quantifier $\forall$ over a conjunction.

$$\exists x\, (P(x) \vee Q(x)) \equiv (\exists x\, P(x)) \vee (\exists x\, Q(x))$$

and we can distribute an existential quantifier $\exists$ over a disjunction.

We **cannot** distribute $\forall$ over a disjunction, or $\exists$ over a conjunction.

Negating quantifiers

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

"It is not the case that everyone likes to snowboard."

$$\neg(\forall x\ S(x))$$

"There exists someone who does not like to snowboard."

$$\exists x\ (\neg S(x))$$

Negating quantifiers

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

"It is not the case that everyone likes to snowboard."

$$\neg(\forall x\, S(x))$$

"There exists someone who does not like to snowboard."

$$\exists x\, (\neg S(x))$$

To negate "$\forall$":

$$\neg(\forall x\, S(x)) \equiv \exists x\, (\neg S(x))$$

Negating quantifiers

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

"There does not exist anyone who likes skiing over magma."

$$\neg(\exists x\ M(x))$$

"Everyone dislikes skiing over magma."

$$\forall x\ (\neg M(x))$$

Negating quantifiers

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

"There does not exist anyone who likes skiing over magma."

$$\neg(\exists x \; M(x))$$

"Everyone dislikes skiing over magma."

$$\forall x \; (\neg M(x))$$

To negate "∃":

$$\neg(\exists x \; M(x)) \equiv \forall x \; (\neg M(x))$$

Negating quantifiers

When negating more complex expressions with quantifiers, you "flip" the quantifier, and negate the expression to which the quantifier was applied.

$$\neg\left(\forall z\ \left(\exists y\ \left(\forall x\ (P(x) \wedge Q(y,z))\right)\right)\right)$$

$$\begin{aligned}
&= \exists z\ \neg\left(\exists y\ \left(\forall x\ (P(x) \wedge Q(y,z))\right)\right) \\
&= \exists z\ \forall y\ \neg\left(\forall x\ (P(x) \wedge Q(y,z))\right) \\
&= \exists z\ \forall y\ \exists x\ \neg(P(x) \wedge Q(y,z))
\end{aligned}$$

$$= \exists z\ \left(\forall y\ \left(\exists x\ (\neg P(x) \vee \neg Q(y,z))\right)\right)$$

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

Example

"Everyone at Hunter College is smart."

$$\forall x\ \big(AtHunter(x) \wedge Smart(x)\big)$$

$$\forall x\ \big(AtHunter(x) \rightarrow Smart(x)\big)$$

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Example

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

"Everyone at Hunter College is smart."

$$\forall x\,\big(AtHunter(x) \wedge Smart(x)\big)$$ **Wrong!**

"Everyone is at Hunter College and is smart. No one is elsewhere."

$$\forall x\,\big(AtHunter(x) \rightarrow Smart(x)\big)$$

Example

"Someone at City College is smart."

$$\exists x\,\big(AtCCNY(x) \wedge Smart(x)\big)$$

$$\exists x\,\big(AtCCNY(x) \rightarrow Smart(x)\big)$$

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Example

"Someone at City College is smart."

$$\exists x \left(AtCCNY(x) \wedge Smart(x) \right)$$

$$\exists x \left(AtCCNY(x) \rightarrow Smart(x) \right) \textbf{Wrong!}$$

"There is someone, who is smart if he(she) is at City College."
It is true if there is anyone who is not at City College, say in Boston.

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Example

Predicates

Quantifiers

Scope, bound variables

Nested quantifiers

Distributivity

Negation

Examples

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

Predicates:
$L(x) = x$ is a lion.
$C(x) = x$ drinks coffee.
$F(x) = x$ is fierce.

Example

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

Predicates:
$L(x) = x$ is a lion.
$C(x) = x$ drinks coffee.
$F(x) = x$ is fierce.

$$\forall x\ (L(x) \rightarrow F(x))$$
$$\exists x\ (L(x) \wedge \neg C(x))$$
$$\exists x\ (F(x) \wedge \neg C(x))$$

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples