

Sets. Ordered pairs.

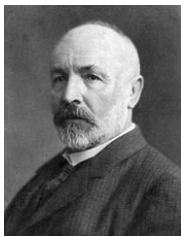
Sets

Sets

Ordered pair

The set theory is a branch of mathematical logic that was created by Georg Cantor in 1870s.

Def. A *set* is a unordered collection of objects being regarded as a single object.



Examples:

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{x \in A \mid (x \geq 3) \wedge (x \text{ is odd})\}$$

$$C = \{\emptyset, \{A\}, \{B\}, \{A, B\}\}$$

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

Sets

If x belongs to a set A , we say that it is a *member* (or an element) of A and write

$$x \in A.$$

If x is not a member of A , we write

$$x \notin A.$$

Empty set, denoted by \emptyset or \varnothing , has no members:

$$\forall x (x \notin \emptyset).$$

Sets

Two sets A and B are *equal*, $A = B$, iff they have exactly the same elements:

$$\forall x (x \in A \leftrightarrow x \in B)$$

For any two objects x and y , we can make a set containing exactly these two objects

$$\{x, y\}$$

If those two objects are identical, $x = y$, we get a singleton set,

$$\{x, x\} = \{x\}.$$

Notice that these sets are not equal:

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}$$

Sets

Set-builder notation.

A set of all objects that satisfy the property P :

$$A = \{x \mid P(x)\}$$

Examples:

$$B = \{x \in \mathbb{Z} \mid x \text{ is even}\}$$

$$C = \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z} (x = 2k)\}$$

$$D = \{x \mid (x \in \mathbb{Z}) \wedge (\exists k \in \mathbb{Z} (x = 2k))\}$$

$$E = \{0, 2, -2, 4, -4, 6, -6, \dots\}$$

In naive set theory, any definable collection is a valid set. And usually it works fine.

However, it leads to contradictions, such as Russell's paradox.

Russell's paradox

Sets

Ordered pair

Let R be the set of all sets that are not members of themselves:

$$R = \{x \mid x \notin x\}$$

It is legal to ask, is R a member of itself or not.

There are only two cases, either $R \in R$, or $R \notin R$.

So, where is the paradox?

Russell's paradox

Sets

Ordered pair

$$R = \{x \mid x \notin x\}$$

(a) If R is a member of itself, then by its definition, $R \notin R$,

$$R \in R \rightarrow R \notin R.$$

(b) Otherwise, if R is not a member of itself, then $R \in R$,

$$R \notin R \rightarrow R \in R.$$

Therefore, we get a contradiction, $R \in R \leftrightarrow R \notin R$.

There exist several axiomatic systems that rule out such pathological cases. For example, Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC).

Union and Intersection

Sets

Ordered pair

Union of two sets A and B ,

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

$$\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$$

Intersection of two sets A and B ,

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

$$\{1, 2\} \cap \{2, 3\} = \{2\}$$

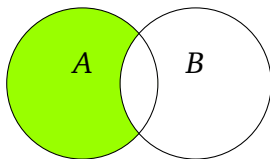
Difference

Sets

Ordered pair

Difference between two sets A and B ,

$$A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\}$$



$$\{1, 2\} \setminus \{2, 3\} = \{1\}$$

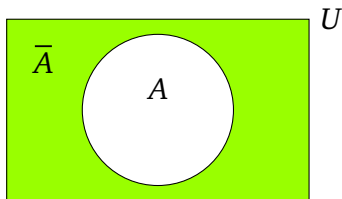
Complement

Sets

Ordered pair

If there is a *universal set* U of all possible objects, then the *complement* of a set A is

$$\bar{A} = U \setminus A = \{x \in U \mid x \notin A\} = \{x \in U \mid \neg(x \in A)\}$$



Exmample: When $U = \mathbb{Z}$:

$$Odd = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$$

$$Even = \overline{Odd} = \mathbb{Z} \setminus Odd$$

Set identities

Sets

Ordered pair

Given the universal set U , sets with respect to union, intersection, and complement satisfy the same identities as propositions with respect to \wedge , \vee , and \neg

Sets:	$A \cap B$	$A \cup B$	\bar{A}	U	\emptyset
Propositions:	$p \wedge q$	$p \vee q$	$\neg p$	T	F

$$\overline{\bar{A}} = A$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

See the full list in Rosen's book.

Set identities

Sets

Ordered pair

Let's prove De Morgan's law for sets:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cup B} = \{x \mid \neg((x \in A) \vee (x \in B))\}$$

$$\bar{A} \cap \bar{B} = \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$$

The propositions in the right hand sides of the equations are equal, therefore, the left hand sides are equal too.

Subset

A is a *subset* of B iff every element of A is an element of B :

$$A \subseteq B \iff \left(\forall x((x \in A) \rightarrow (x \in B)) \right)$$

$$\{1, 2\} \subseteq \{1, 2, 3\}$$

$$\{1, 2, 3\} \subseteq \{1, 2, 3\}$$

$$\emptyset \subseteq \{1, 2, 3\}$$

A is a *proper subset* of B iff A is a subset of B , but it's not equal to B

$$A \subsetneq B \iff \left(\forall x((x \in A) \rightarrow (x \in B)) \wedge \exists x((x \in B) \wedge (x \notin A)) \right)$$

$$\{1, 2\} \subsetneq \{1, 2, 3\}$$

$$\emptyset \subsetneq \{1, 2, 3\}$$

Proper subset A is strictly “smaller” than B .

Power set

Sets

Ordered pair

Def. The set of all subsets of A is called a *power set* of A , denoted by $\mathcal{P}(A)$.

Examples:

$$\mathcal{P}(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$$

$$\mathcal{P}(\{0, 1, 2\}) =$$

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$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

Cardinality

Sets

Ordered pair

Def. The *cardinality* of a finite set A is equal to the number of elements in A . It's denoted by $|A|$.

$$|\emptyset| = 0$$

$$|\{0, 1, 2, 3, 4\}| = 5$$

We already know the subtraction rule for the cardinality of a union:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Question

Sets

Ordered pair

Compute the cardinality of the power set $\mathcal{P}(A)$ if $|A| = n$.

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In other words, since the power set is the set of all subsets, the task is to count the number of subsets of a set with n elements.

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$$|\mathcal{P}(A)| = 2^{|A|} = 2^n$$

Generalized union and intersection

Union of a finite set of sets:

$$\bigcup \{A_0\} = A_0$$

$$\bigcup \{A_0, A_1, \dots, A_n\} = \bigcup_{i=0}^n A_i = A_0 \cup A_1 \cup \dots \cup A_n$$

Intersection of a finite set of sets:

$$\bigcap \{A_0\} = A_0$$

$$\bigcap \{A_0, A_1, \dots, A_n\} = \bigcap_{i=0}^n A_i = A_0 \cap A_1 \cap \dots \cap A_n$$

Ordered pair

Def. The *ordered pair* of $a \in A$ and $b \in B$ is an ordered collection (a, b) .

Two ordered pairs are equal

$$(a, b) = (c, d) \quad \text{if and only if} \quad (a = c) \wedge (b = d).$$

Observe that this property implies that

$$(a, b) \neq (b, a)$$

unless $a = b$. So, the order matters. This is why it is called the ordered pair, and (a, b) is not equivalent to a set $\{a, b\}$.

$$(1, 2) = (1, 2)$$

$$(1, 2) \neq (1, 3)$$

$$(1, 2) \neq (2, 1)$$

Ordered pair

Sets

Ordered pair

More examples of ordered pairs:

$$(1, 2)$$

$$(\{1\}, \{2\})$$

$$(1, \{2, 3, 4, 5\})$$

$$((1, 2), \emptyset)$$

If $a \in A$ and $b \in B$, what is the set of all ordered pairs (a, b) ?

Cartesian product

Sets

Ordered pair

Def. Let A and B be sets. The *Cartesian product* of A and B , denoted by $A \times B$, is the set of all ordered pairs (a, b) , where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

(Named after Rene Decartes)

Question. Given two sets $A = \{1, 2, 3\}$ and $B = \{C, D\}$, what is their Cartesian product $A \times B$?

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$$A \times B = \{(1, C), (2, C), (3, C), \\ (1, D), (2, D), (3, D)\}$$

Cartesian product

Sets

Ordered pair

Question. If the $|A| = n$, and $|B| = m$, what is the cardinality of $A \times B$?

Building a list

Sets

Ordered pair

Ordered pair is fundamental for defining data types.

Question. How to implement the list data type using only ordered pairs?

Example of a list:

$[1, 2, 3, 4]$

Interface:

$\text{construct} (1, [2, 3, 4]) = [1, 2, 3, 4]$

$\text{head} ([1, 2, 3, 4]) = 1$

$\text{tail} ([1, 2, 3, 4]) = [2, 3, 4]$

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Possible implementation:

$(1, (2, (3, 4)))$

$\text{construct} (h, t) = (h, t)$

$\text{head} ((h, t)) = h$

$\text{tail} ((h, t)) = t$

Ordered n -tuple

Sets

Ordered pair

Def. The *ordered n -tuple* of is an ordered collection (a_1, a_2, \dots, a_n) .

It is just an extension of an ordered pair for joining n elements together.

Def. The *Cartesian product* of the sets A_1, \dots, A_n , is the set of all n -tuples such that

$$A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, \dots, n\}$$

If all sets A_i are equal, that is, $A_1 = \dots = A_n = A$, then their Cartesian product is denoted by A^n

$$A_1 \times \dots \times A_n = \underbrace{A \times \dots \times A}_{n \text{ times}} = A^n$$