Discrete Structures. CSCI-150. Summer 2016.

Homework 4. Due Thr. Jun. 16, 2016.

Problem 1 (Graded)

Ellen draws 5 cards from a standard deck of 52 cards.

(a) In how many ways can her selection result in a hand with no clubs?

(b) A hand with at least one club?

Problem 2

Count the number of bit strings that start with 4 zeroes or end with 3 ones if the length of the bit string is

(a) 4, (b) 7, (c) 8, (d) 6.

Hint:
Don't forget that, if you have to count the number of objects that belong to at least one of the two given sets, then the simple summation rule does not work if the sets are not disjoint. Then, you have to also subtract the cardinality of the intersection of the two sets.

Problem 3 (Graded)

A computer science professor has eleven different programming books on a bookshelf. Five of the books deal with the programming language C++, the other six with LISP. In how many ways can the professor arrange these books on the shelf

(a) if there are no restrictions?

(b) if all the C++ books must be next to each other?

(c) if all the C++ books must be next to each other and all the LISP books must be next to each other?

(d) if the languages should alternate?

Note that all the books are distinct. Not for grade, you may consider the case when two of the C++ books are identical copies. Then, what if all the C++ books and all the LISP books are identical copies?

Problem 4 (Graded)

A pizzeria offers 777 types of pizza and 3 types of soda. Mary goes there everyday for lunch, always buying one slice of pizza and one soda. However, she never gets exactly the same thing on two consecutive days (that is, each time, either the drink or the pizza (or both) is different from what she had yesterday).

In how many ways can she plan her lunch for the next 15 days if today she tried a different pizzeria and did not like that place at all?

Answer: approximately 3.240×10^{50} (but you should try to find the exact formula, not an approximation).

Proving identities and Double counting

Problem 5 (Graded)

In this problem, you have to find two proofs for the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(a) The first proof is algebraic. Using the fact that $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$, show that the equation is always true. It may involve some factorial manipulation, but almost everything should cancel out.

Remember: when proving the identity (or anything else in general), don't prove it "backwards", it's a logically inconsistent and faulty technique.

You may consider the left-hand side and the right-hand side separately, showing that they are equal to the same formula. *However, don't make it look like a "backwards" proof, please!*

(b) For the second part, prove the same identity using the technique called "Double counting" or "Combinatorial argument". It's a combinatorial proof technique for showing that two expressions are equal by demonstrating that they are two ways of counting the size of one set. In class, we used this technique to prove that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

To solve the problem, show that two formulas: $\binom{2n}{2}$ and $2\binom{n}{2} + n^2$ describe two counting procedures that count the same set.

A hint: We know that the first formula, $\binom{2n}{2}$, counts the number of ways to choose 2 objects out of available $2n$. Show that the second formula, $2\binom{n}{2} + n^2$, counts the same thing.

Problem 6

Give a "double counting' proof for the identity

$$2^5 = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}.$$

Problem 7

Find "double counting" proofs for the following identities:

$$(2n)! = \binom{2n}{n} \cdot (n!)^2$$

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

$$\binom{n-1+2}{2} = n + \binom{n}{2}$$

$$\binom{n-1+3}{3} = n + \binom{n}{2} \cdot 2 + \binom{n}{3}$$

If you try proving the last two identities, think of selection with repetition.

Selection with repetition

Problem 8 (Graded)

In how many ways can 21 identical computers be distributed among 5 computer stores if

(a) there are no restrictions?

(b) each store gets at least two?

(c) the largest store gets no less than half?

(d) each store gets at least four?

Problem 9

Find the number of integer solutions to the equation

$$w + x + y + z = 19,$$

where the variables are positive integers.

Pigeonhole principle

Problem 10 (Graded)

(a) There are 50 white socks and 50 black socks in a drawer. How many socks do you have to take to be sure that you have at least one matching pair?

(b) At least one mismatching pair?

Problem 11

Prove that among any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair at the distance not greater than $1/2$.

Problem 12

In a certain cubic region of our galaxy of the dimensions $4 \times 4 \times 4$ light years, there are 70 stars. Prove that among those 70 stars, there are at least two that are no more than $\sqrt{3}$ light years apart.

Comment: One light year is a unit of length equal to $\approx 9.46 \times 10^{15}$ meters.

Problem 13 (Grade)

In a room there are 10 people, none of them is younger than 1 or older than 100 years. Prove that among them, one can always find two groups of people (possibly intersecting, but different) the sums of whose ages are the same.

Hint: Use the pigeonhole principle. How many different groups of people can be picked? How large the sum of their ages can be?