

Predicates and Quantifiers.

Predicates

Propositional logic, studied previously, cannot adequately express the meaning of all statements in mathematics and in natural language.

Examples?

“ n is a prime number.”

“ x is greater than y .”

“ $user$ is waiting.”

Def. A **predicate** is a proposition whose truth depends on the value of one or more variables.

Predicates

For convenience, we can give every predicate a name:

$$P(n) = "n \text{ is a prime number.}"$$

$$Q(x, y) = "x \text{ is greater than } y."$$

When a predicate is applied to a , the result is a simple proposition:
Depending on the, the predicates are either true or false:

$$P(3) = T$$

$$P(4) = F$$

$$P(99) = F$$

$$Q(2, 1) = T$$

How can we use predicates?

Predicates

Quantifiers

Let $C(x)$ = “ x is playing chess.”

There are three persons in the room: *Ed*, *Paul*, and *Tom*.

Ed and *Paul* are playing chess, and *Tom* is sleeping.

Formally: $C(Ed) = T$, $C(Paul) = T$, $C(Tom) = F$.

How can we use predicates?

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Consider a proposition “Someone is playing chess.” Is it true?

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Formally: $C(\text{Ed}) = T$, $C(\text{Paul}) = T$, $C(\text{Tom}) = F$.

Consider a proposition “Someone is playing chess.” Is it true?

Yes, because, for example, $C(\text{Ed}) = T$.

How can we use predicates?

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Quantifiers

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Another proposition “Everyone is playing chess in the room.”

How can we use predicates?

Predicates

Quantifiers

Let $C(x)$ = “ x is playing chess.”

There are three persons in the room: *Ed*, *Paul*, and *Tom*.

Ed and *Paul* are playing chess, and *Tom* is sleeping.

Formally: $C(Ed) = T$, $C(Paul) = T$, $C(Tom) = F$.

Another proposition “Everyone is playing chess in the room.”

It's false, because $C(Tom) = F$.

Always true / Sometimes true

Predicates

Quantifiers

Let $P(x) = "x^2 \geq 0."$

Always true:

"For all n , $P(n)$ is true."

"For all x , $x^2 \geq 0."$

" $P(n)$ is true for every n ."

" $x^2 \geq 0$ for every x ."

Always true / Sometimes true

Predicates

Quantifiers

“For all x , $x^2 \geq 0$.”

An assertion that a predicate is always true is called
a *universal quantification*.

Always true / Sometimes true

Predicates

Quantifiers

Let $P(x) = "5x^2 - 7 = 0."$

It's true only when $x = \pm\sqrt{7/5}$.

Sometimes true:

"There exist an n such that
 $P(n)$ is true."

"There exist an x such that
 $5x^2 - 7 = 0."$

" $P(n)$ is true for some n ."

" $5x^2 - 7 = 0$ for some x ."

" $P(n)$ is true for at least one n ."

" $5x^2 - 7 = 0$ for at least one x ."

Always true / Sometimes true

Predicates

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“Exists an x such that $5x^2 - 7 = 0$ is true.”

An assertion that a predicate is true for some values of the variable is called

an *existential quantification*.

Sentences can be ambiguous

Predicates

Quantifiers

“If you can solve any problem we come up with, then you get an A for the course.”

Is it a universal (for all), or an existential (for some) quantification?

Sentences can be ambiguous

Predicates

Quantifiers

Universal:

“You can solve every problem we come up with.”

Existential:

“You can solve at least one problem we come up with.”

Notation

Predicates

Quantifiers

Universal:

$\forall x P(x)$ means that *for all* x , $P(x)$ is true.

Existential:

$\exists x P(x)$ means that there *exists* an x such that $P(x)$ is true.

We consider only those values of the variable x that belong to a given set called the *domain of discourse*, or the *universe of discourse*.

To prove or disprove a quantification

Predicates

Quantifiers

Statement	When true?	When false?
$\forall x P(x)$	$P(x)$ is true for every x .	There is at least one <i>counterexample</i> x such that $P(x)$ is false.
$\exists x P(x)$	There is at least one x such that $P(x)$ is true.	$P(x)$ is false for every x .

Example

Predicates

Quantifiers

Let $P(x) = "x^2 > 0."$

The universe of discourse are all integer numbers.

Is it true or false that $\forall x P(x)$?

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To disprove, we have to find a counterexample for which it's false.

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The universe of discourse are all integer numbers.

Is it true or false that $\forall x P(x)$?

To prove it, we must show that $P(x)$ is true for all integers.

To disprove, we have to find a counterexample for which it's false.

Counterexample:

$P(x)$ is false for $x = 0$. So, the quantified statement $\forall x P(x)$ is false.

Scope, variable binding

Predicates

Quantifiers

$$\forall x (P(x) \wedge Q(x))$$

Variable x is said to be bound by a quantifier. The scope of the quantifier is the expression to which it's applied. Here, the scope is $P(x) \wedge Q(x)$.

Scope, variable binding

Predicates

Quantifiers

Why should we care?

$$\forall x (P(x) \wedge Q(x)) \wedge \exists y (R(y))$$

is equivalent to

$$\forall x (\underline{P(x) \wedge Q(x)}) \wedge \exists x (\underline{R(x)})$$

“Every dog has four legs and has a tail; and there exists a dog that barks.”

Scope, variable binding

Predicates

Quantifiers

If all variables are bound by a quantifier or set equal to a particular value then a statement is a proposition:

$\forall x (P(x) \wedge Q(z)) \wedge \exists y (R(y))$, and it's given that $z = 3.1415$.

x is bound by \forall , y is bound by \exists , and z is specified by the given equation.

So, this is a proposition.

Scope, variable binding

Predicates

Quantifiers

If a variable is not bound, it's called *free*.

$$\forall x (P(y) \wedge Q(x)) \wedge \exists y (R(x)).$$

Scope, variable binding

Predicates

Quantifiers

If a variable is not bound, it's called *free*.

$$\forall \underline{x} (P(\underline{y}) \wedge Q(\underline{x})) \wedge \exists y (R(\underline{x})).$$

Only the first x is bound by \forall .

This is not a proposition.

Mixing quantifiers

“For every even integer n greater than 2, there exist prime numbers p and q such that $n = p + q$.”

(Prime numbers are integers > 1 , divisible only by itself and 1.
They are 2, 3, 5, 7, 11, 13, 17, ...)

The universe of discourse:

n is an even integer, $n > 2$.

p and q are prime numbers.

$$\forall n (\exists p (\exists q (n = p + q)))$$

Order of quantifiers

Predicates

Quantifiers

Swapping the order of different kinds of quantifiers (existential or universal) changes the meaning of a proposition.

$$\forall n (\exists p (\exists q (n = p + q)))$$

$$4 = 2 + 2$$

$$6 = 3 + 3$$

$$8 = 3 + 5$$

$$10 = 5 + 5$$

$$\exists p (\exists q (\forall n (n = p + q)))$$

$$4 \neq 3 + 5$$

$$6 \neq 3 + 5$$

$$8 = 3 + 5$$

$$10 \neq 3 + 5$$

Negating quantifiers

Predicates

Quantifiers

“It is not the case that everyone likes to snowboard.”

$$\neg(\forall x S(x))$$

“There exists someone who does not like to snowboard.”

$$\exists x (\neg S(x))$$

Negating quantifiers

“It is not the case that everyone likes to snowboard.”

$$\neg(\forall x S(x))$$

“There exists someone who does not like to snowboard.”

$$\exists x (\neg S(x))$$

To negate “ \forall ”:

$$\neg(\forall x S(x)) \equiv \exists x (\neg S(x))$$

Negating quantifiers

Predicates

Quantifiers

“There does not exist anyone who likes skiing over magma.”

$$\neg(\exists x M(x))$$

“Everyone dislikes skiing over magma.”

$$\forall x (\neg M(x))$$

Negating quantifiers

Predicates

Quantifiers

“There does not exist anyone who likes skiing over magma.”

$$\neg(\exists x M(x))$$

“Everyone dislikes skiing over magma.”

$$\forall x (\neg M(x))$$

To negate “ \exists ”:

$$\neg(\exists x M(x)) \equiv \forall x (\neg M(x))$$

Negating quantifiers

When negating more complex expressions with quantifiers, you “flip” the quantifier, and negate the expression to which the quantifier was applied.

$$\neg(\forall z (\exists y (\forall x (P(x) \wedge Q(y, z)))))$$

$$= \exists z \neg(\exists y (\forall x (P(x) \wedge Q(y, z))))$$

$$= \exists z \forall y \neg(\forall x (P(x) \wedge Q(y, z)))$$

$$= \exists z \forall y \exists x \neg(P(x) \wedge Q(y, z))$$

$$= \exists z (\forall y (\exists x (\neg P(x) \vee \neg Q(y, z))))$$

Example

Predicates

Quantifiers

“Everyone at Hunter College is smart.”

$$\forall x (AtHunter(x) \wedge Smart(x))$$

$$\forall x (AtHunter(x) \rightarrow Smart(x))$$

Example

Predicates

Quantifiers

“Everyone at Hunter College is smart.”

$\forall x (AtHunter(x) \wedge Smart(x))$ **Wrong!**

“Everyone is at Hunter College and is smart. No one is elsewhere.”

$\forall x (AtHunter(x) \rightarrow Smart(x))$

Example

Predicates

Quantifiers

“Someone at City College is smart.”

$$\exists x (AtCCNY(x) \wedge Smart(x))$$

$$\exists x (AtCCNY(x) \rightarrow Smart(x))$$

Example

Predicates

Quantifiers

“Someone at City College is smart.”

$$\exists x (AtCCNY(x) \wedge Smart(x))$$

$$\exists x (AtCCNY(x) \rightarrow Smart(x)) \textbf{Wrong!}$$

“There is someone, who is smart if he(she) is at City College.”

It is true if there is anyone who is not at City College, say in Boston.

Example

Predicates

Quantifiers

“All lions are fierce.”

“Some lions do not drink coffee.”

“Some fierce creatures do not drink coffee.”

Predicates:

$L(x)$ = x is a lion.

$C(x)$ = x drinks coffee.

$F(x)$ = x is fierce.

Example

“All lions are fierce.”

“Some lions do not drink coffee.”

“Some fierce creatures do not drink coffee.”

Predicates:

$L(x)$ = x is a lion.

$C(x)$ = x drinks coffee.

$F(x)$ = x is fierce.

$$\forall x (L(x) \rightarrow F(x))$$

$$\exists x (L(x) \wedge \neg C(x))$$

$$\exists x (F(x) \wedge \neg C(x))$$