Discrete Structures. CSCI-150. Fall 2015.

Homework 10. Due Wed. Nov. 11, 2015.

Proving identities and Double counting

Problem 1 (Graded)

In this problem, you have to find two proofs for the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(a) The first proof is algebraic. Using the fact that $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$, show that the equation is always true. It may involve some factorial manipulation, but almost everything should cancel out.

Remember: when proving the identity (or anything else in general), don't prove it "backwards", it's a logically inconsistent and faulty technique.

You may consider the left-hand side and the right-hand side separately, showing that they are equal to the same formula. *However, don't make it look like a "backwards" proof!*

(b) For the second part, prove the same identity using the technique called "Double counting" or "Combinatorial argument". It's a combinatorial proof technique for showing that two expressions are equal by demonstrating that they are two ways of counting the size of one set.

In this particular case, show that two formulas: $\binom{2n}{2}$ and $2\binom{n}{2} + n^2$ describe two counting procedures that count the same set.

A hint: We know that the first formula, $\binom{2n}{2}$, counts the number of ways to choose 2 objects out of available $2n$. Show that the second formula, $2\binom{n}{2} + n^2$, counts the same thing.

Problem 2

Find "double counting" proofs for the following identities:

$$(2n)! = \binom{2n}{n} \cdot (n!)^2$$

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

$$\binom{n-1+2}{2} = n + \binom{n}{2}$$

$$\binom{n-1+3}{3} = n + \binom{n}{2} \cdot 2 + \binom{n}{3}$$

If you try proving the last two identities, think of selection with repetition.

Pigeonhole principle

Problem 3 (Graded)

(a) There are 50 white socks and 50 black socks in a drawer. How many socks do you have to take to be sure that you have at least one matching pair?

(b) At least one mismatching pair?

(c) There are 50 identical pairs of shoes. When taking the shoes one by one at random, how many shoes must be taken to guarantee that you have got at least one matching pair?

Problem 4 (Graded)

Prove that among any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair at the distance not greater than 1/2.

Problem 5

In a room there are 10 people, none of them is younger than 1 or older than 100 years. Prove that among them, one can always find two groups of people (possibly intersecting, but different) the sums of whose ages are the same.

Hint: Use the pigeonhole principle. How many different groups of people can be picked? How large the sum of their ages can be?

Selection with repetition

Problem 6 (Graded)

In how many ways can 21 identical computers be distributed among 5 computer stores if

(a) there are no restrictions?

(b) each store gets at least two?

(c) the largest store gets no less than half?

(d) each store gets at least four?

Problem 7

Find the number of integer solutions to the equation

$$w + x + y + z = 19,$$

where the variables are positive integers.