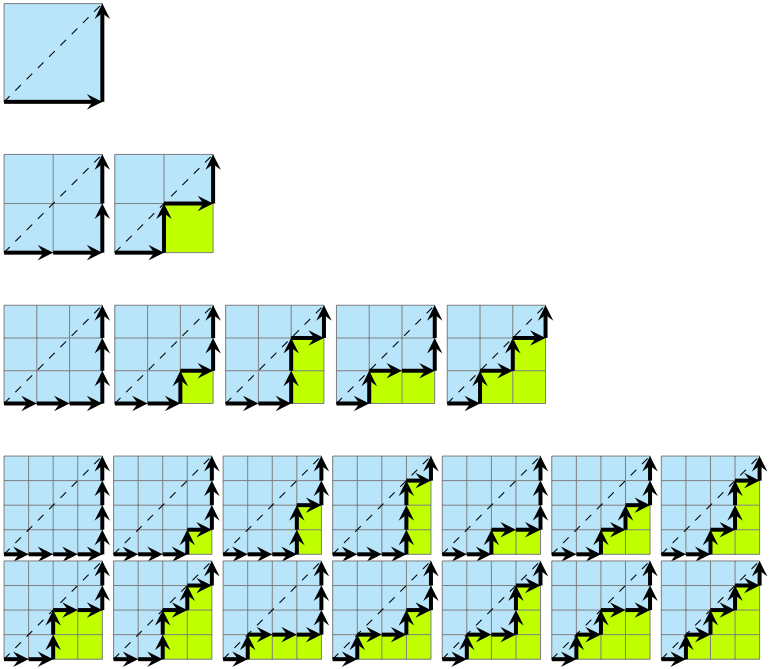
Counting

1, 2, 5, 14, …



Product Rule

Sum Rule

Finite Sets

Overcounting

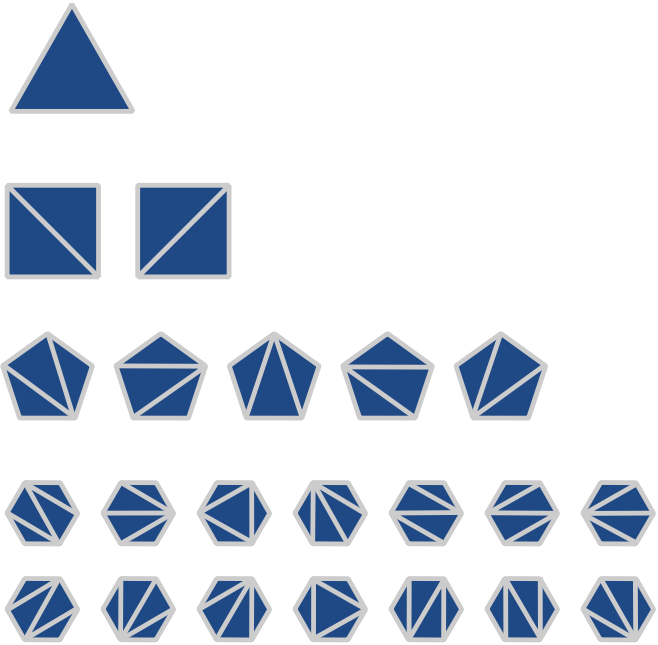
Subtraction Rule

Counting

Tree Diagrams

Factorial

1, 2, 5, 14, . . .



Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Problem

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

A-1 ... Z-100

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

Problem

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

A-1 ... Z-100

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

$$26 \cdot 100$$

The Product Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

There are

1. 26 ways to assign a letter and
2. 100 ways to assign a number.

$$26 \cdot 100$$

The Product Rule. Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Chairs again

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Consider the same problem about the labels for chairs

1. 26 ways to choose a letter and
2. 100 ways to choose a number.

Write a program that prints all 2600 labels?

Chairs again

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Consider the same problem about the labels for chairs

1. 26 ways to choose a letter and
2. 100 ways to choose a number.

Write a program that prints all 2600 labels?

```
for a := A to Z do
    for n := 1 to 100 do
        print_label(a, n)
```

Generalized Product Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

If a procedure consists of *k sub-tasks*, and the sub-tasks can be performed in $n_1, \ldots, n_k$ ways, then the procedure can be performed

$$\text{in } (n_1 \cdot n_2 \cdot \ldots \cdot n_k) \text{ ways.}$$

Example:
Count the number of different bit strings of length seven.

The value for each bit can be chosen in two ways (0 or 1). Therefore:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$$

License plates

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

How many different license plates of this format can be made?

License plates

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

How many different license plates of this format can be made?

$$26^3 \cdot 10^4 = 175,760,000$$

Another counting problem

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

Another counting problem

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

A college library has 40 books on sociology and 50 books on anthropology.

You have to *choose only one book* from the library. In how many ways can it be done?

$$40 + 50 = 90 \quad \text{this is called the rule of sum}$$

The Sum Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

The Sum Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

Write an algorithm for a robot to read all the books in the library:

```
for b := 1 to 40 do
    read(Sociology, b)
for b := 1 to 50 do
    read(Anthropology, b)
```

The Sum Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

40 books on sociology, and 50 books on anthropology.
There are $40 + 50 = 90$ ways to choose a book.

The Sum Rule. If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Note that it's important that the two groups don't have common elements (We say that the sets are disjoint).

A new object

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Def. A *set* is an unordered collection of objects. The objects are called elements.

If e is an element of the set A, we write $a \in A$.

Otherwise, if it's not in A, we write $a \notin A$.

Example:

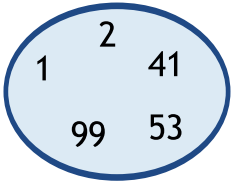
$$A = \{1, 2, 97, 3, 15\}.$$

$$1 \in A.$$

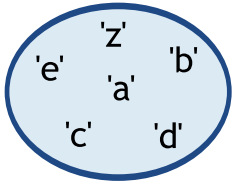
$$4 \notin A.$$

$$\{1, 2, 3, 3, 2, 1, 1, 1, 1\} = \{1, 2, 3\}$$

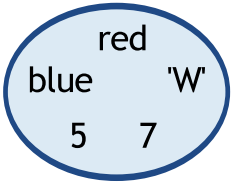
Sets



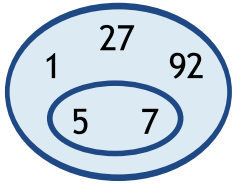
A = {1, 2, 41, 53, 99}



B = {'a', 'z', 'e', 'd', 'c', 'b'}



C = {'W', blue, 5, red, 7}



D = {27, 1, {5,7}, 92}

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Some important sets

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Natural numbers

$$\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ \ldots\}$$

Integer numbers

$$\mathbb{Z} = \{\ldots,\ -3,\ -2,\ -1,\quad 0,\quad 1,\quad 2,\quad 3,\ \ldots\}$$

Empty set

$$\varnothing = \{\ \}$$

Set Builder Notation

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

We can describe sets using predicates:

$$A = \{x \mid P(x)\}$$

"Set A is such that $x \in A$ if and only if $P(x)$."

Example. Positive integers:

$$Z^+ = \{n \in \mathbb{Z} \mid n > 0\} = \{1, 2, 3, \ldots\}$$

More complex predicates are fine too. Odd and even numbers:

$$Even = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}\, (n = 2k)\}$$
$$Odd = \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z}\, (n = 2k + 1)\}$$

Union, $\cup$

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$A \cup B$ denotes all things that are *members of either A or B*:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Equivalently:

x belongs to $A \cup B$ if and only if $x \in A$ or $x \in B$.

Examples:

$$\{1, 2\} \cup \{\text{‘a’}, \text{‘b’}\} = \{1, 2, \text{‘a’}, \text{‘b’}\}$$
$$\{1, 2, 3\} \cup \{2, 3, 5\} = \{1, 2, 3, 5\}$$

Intersection, $\cap$

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$A \cap B$ denotes all things that are *members of both A and B*:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Equivalently:

x belongs to $A \cap B$ if and only if $x \in A$ and $x \in B$.

Examples:

$$\{1, 2\} \cap \{\text{‘a’}, \text{‘b’}\} = \varnothing$$
$$\{1, 2, 3\} \cap \{2, 3, 5\} = \{2, 3\}$$

Sets A and B are called *disjoint* if their intersection is empty: $A \cap B = \varnothing$.

Number of the elements of a finite set

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Def. If set A is finite, and there are exactly n elements in S, then n is the *cardinality* of the set A. We write

$$|A| = n.$$

Examples:

$$\begin{aligned} A &= \{3,\ 4,\ 5,\ 6\} \\ |A| &= 4 \\ B &= \{\{3,\ 4\},\ \{5,\ 6\},\ 7\} \\ |B| &= 3 \\ |\varnothing| &= 0 \end{aligned}$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$$A = \{1,\ 2,\ 4,\ 5\}$$

$$B = \{20,\ 21,\ 22,\ 23,\ 24\}$$

$$A \cup B = \{1,\ 2,\ 4,\ 5,\ 20,\ 21,\ 22,\ 23,\ 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| =$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$$A = \{1,\ 2,\ 4,\ 5\}$$

$$B = \{20,\ 21,\ 22,\ 23,\ 24\}$$

$$A \cup B = \{1,\ 2,\ 4,\ 5,\ 20,\ 21,\ 22,\ 23,\ 24\}$$

If two sets are disjoint (their intersection is empty), what is the cardinality if their union?

$$|A \cup B| = 9, \quad \text{and} \quad |A| + |B| = 4 + 5 = 9.$$

$$|A \cup B| = |A| + |B| = 4 + 5 = 9.$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

You are given k disjoint sets $A_1, \ldots A_k$:

It means that $A_i \cap A_j = \varnothing$ when $i \neq j$.

What is the cardinality if their union $A_1 \cup \ldots \cup A_k$?

$$|A_1 \cup \ldots \cup A_k| =$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

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It means that $A_i \cap A_j = \varnothing$ when $i \neq j$.

What is the cardinality if their union $A_1 \cup \ldots \cup A_k$?

$$|A_1 \cup \ldots \cup A_k| = |A_1| + \ldots + |A_k| = \sum_{i=1}^{k} |A_i|.$$

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

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$$|A_1 \cup \ldots \cup A_k| = |A_1| + \ldots + |A_k| = \sum_{i=1}^{k} |A_i|.$$

This is the same *rule of sum*, right?

Question

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Why do we insist on the sets being disjoint?

Really, who cares?

Because

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$$A = \{1,\ 2,\ 3\}$$

$$B = \{3,\ 4\}$$

Their union: $A \cup B = \{1,\ 2,\ 3,\ 4\}$

$$|A| + |B| = \Big|\{1,\ 2\ ,3\}\Big| + \Big|\{3,\ 4\}\Big| = 3 + 2 = 5$$

$$|A \cup B| = \Big|\{1,\ 2,\ 3,\ 4\}\Big| = 4$$

So, in general, $|A \cup B| \neq |A| + |B|$, and if we try to use the sum rule when the sets are not disjoint, we *overcount*, and this is really bad.

Overcounting

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$$A = \{1,\ 2,\ 3\}$$

$$B = \{3,\ 4\}$$

$$A \cup B = \{1,\ 2,\ 3,\ 4\}$$

We were overcounting, because the common elements of A and B were counted twice:

$$|A| + |B| = \left|\{1,\ 2,\ 3\}\right| + \left|\{3,\ 4\}\right| = 5,$$
$$|A \cup B| = \left|\{1,\ 2,\ 3,\ 4\}\right| = 4.$$

Overcounting

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$$A = \{1,\ 2,\ 3\}$$
$$B = \{3,\ 4\}$$
$$A \cup B = \{1,\ 2,\ 3,\ 4\}$$

We were overcounting, because the common elements of A and B were counted twice:

$$|A| + |B| = \left|\{1,\ 2,\ 3\}\right| + \left|\{3,\ 4\}\right| = 5,$$
$$|A \cup B| = \left|\{1,\ 2,\ 3,\ 4\}\right| = 4.$$

Therefore, to get the correct value of $|A \cup B|$, the number of common elements must be subtracted:

$$|A \cup B| = |A| + |B| - \left|\{3\}\right| = 3 + 2 - 1 = 4.$$

The Subtraction Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$$A = \{1,\ 2,\ 3\}$$

$$B = \{3,\ 4\}$$

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Therefore, to get the correct value of $|A \cup B|$, the number of common elements must be subtracted:

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The *Subtraction Rule* for two arbitrary sets A and B:

$$|A \cup B| =$$

The Subtraction Rule

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

$$A = \{1,\ 2,\ 3\}$$

$$B = \{3,\ 4\}$$

$$A \cup B = \{1,\ 2,\ 3,\ 4\}$$

Therefore, to get the correct value of $|A \cup B|$, the number of common elements must be subtracted:

$$|A \cup B| = |A| + |B| - \Big|\{3\}\Big| = 3 + 2 - 1 = 4.$$

The *Subtraction Rule* for two arbitrary sets A and B:

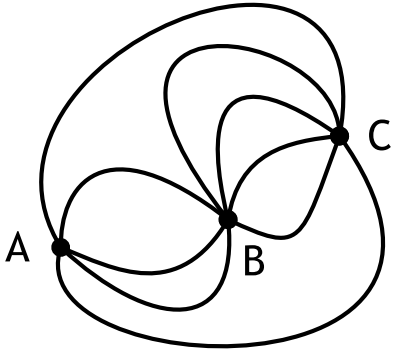
$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting paths

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

This is a map with three cities, connected by roads.

Count the number of paths from city A to city C, such that each city is visited not more than once.



Counting paths

Product Rule
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Finite Sets
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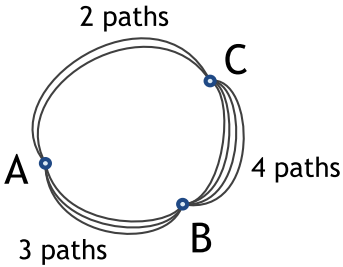
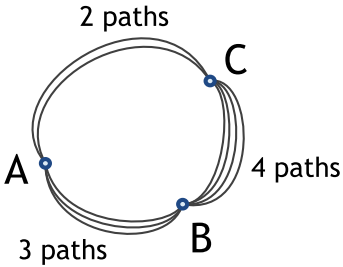


Counting paths

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

This is a map with three cities, connected by roads.

Count the number of paths from city A to city C, such that each city is visited not more than once.



$A \to C$ or $A \to B \to C$:

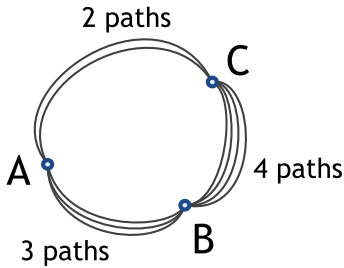
$$2 + 3 \cdot 4 = 14$$

Counting round trips

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that cities A and C are visited not more than once.



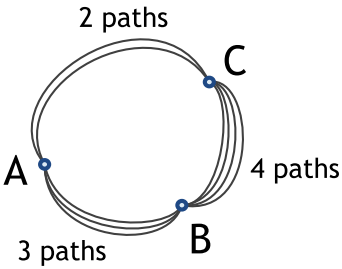
Counting round trips

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that cities A and C are visited not more than once.

$B \to A \to B$	$3 \cdot 3 = 9$
$B \to C \to B$	$4 \cdot 4 = 16$
$B \to A \to C \to B$	$3 \cdot 2 \cdot 4 = 24$
$B \to C \to A \to B$	$4 \cdot 2 \cdot 3 = 24$
Total	73

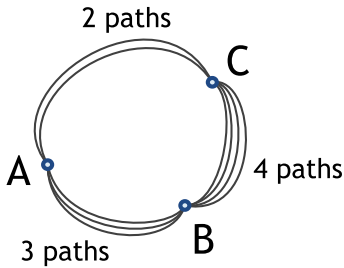


Counting round trips II

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city *B*, such that
1) cities *A* and *C* are visited not more than once, and
2) each road is used not more than once during a trip.



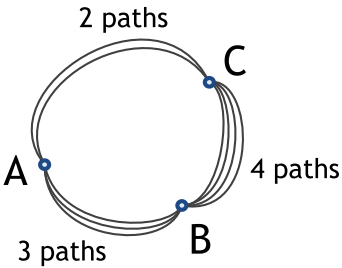
Counting round trips II

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

This is a map with three cities, connected by roads.

Count the number of round trips starting in city B, such that
1) cities A and C are visited not more than once, and
2) each road is used not more than once during a trip.

$B \to A \to B$	$3 \cdot 2 = 6$
$B \to C \to B$	$4 \cdot 3 = 12$
$B \to A \to C \to B$	$3 \cdot 2 \cdot 4 = 24$
$B \to C \to A \to B$	$4 \cdot 2 \cdot 3 = 24$
Total	66



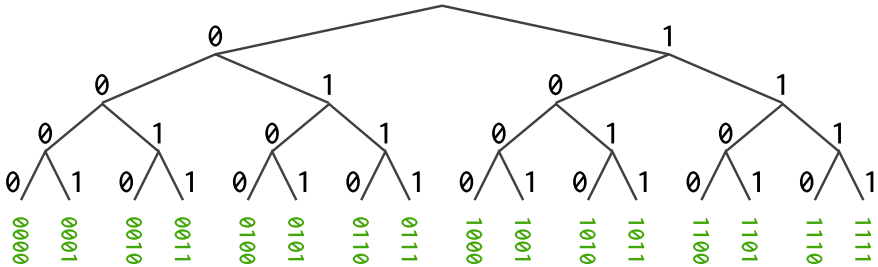
Tree Diagrams

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Counting problems can be solved using *tree diagrams*.

Each branch represent one possible choice.
Each possible outcome is a leaf of the tree (an endpoint that does not branch).

Example: Count all bit strings of length four.



16 strings.

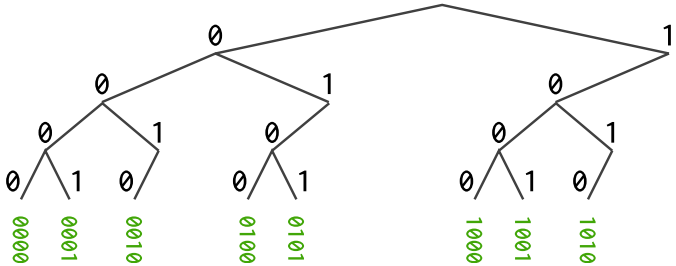
Tree Diagrams

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Counting problems can be solved using *tree diagrams*.

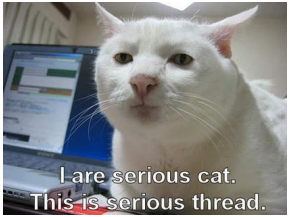
Each branch represent one possible choice.
Each possible outcome is a leaf of the tree (an endpoint that does not branch).

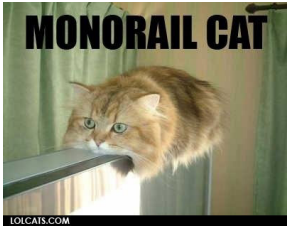
Example 2: Count all bit strings of length four that *do not have two consecutive 1s*.



8 strings.

Ranking cats





Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Ranking cats

In how many different ways can you rank a set of 6 cats:

$$\{a,\ b,\ c,\ d,\ e,\ f\}$$

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

Ranking cats

Product Rule
Sum Rule
Finite Sets
Overcounting
Subtraction Rule
Counting
Tree Diagrams
Factorial

In how many different ways can you rank a set of 6 cats:

$$\{a,\ b,\ c,\ d,\ e,\ f\}$$

- There are *6* ways to select the first cat,
- *5* ways to select the second cat among the remaining five,
- *4* ways to select the third cat . . .
- . . . continue the process
- In the end, the only remaining cat takes the last position in the rank.

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$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{ways!}$$

Factorial

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How large this number is?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This function is called *factorial* and denoted by $n!$:

$$1! = 1$$
$$2! = 2 \cdot 1$$
$$3! = 3 \cdot 2 \cdot 1$$
$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$
$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1$$

and by convention,

$$0! = 1$$