Induction

Consider a problem

Let's prove that the sum of the first n positive integers

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all positive integers $n = 1, 2, 3, 4, \ldots$

Consider a problem

Let

$$P(n):\ 1+2+\ldots+n\ =\ \frac{n(n+1)}{2}.$$

Prove that

$$P(n)$$

for all positive natural numbers $n = 1,\, 2,\, 3,\, 4\ldots$

The idea

If we can prove that

$$
\begin{aligned}
&P(1)\\
&P(1) \rightarrow P(2)\\
&P(2) \rightarrow P(3)\\
&\dots\\
&P(n) \rightarrow P(n+1)\\
&\dots
\end{aligned}
$$

Then it follows that

$$P(n) \text{ for all } n \geq 1$$

The idea

The implications can be grouped together. Thus it is enough to prove that

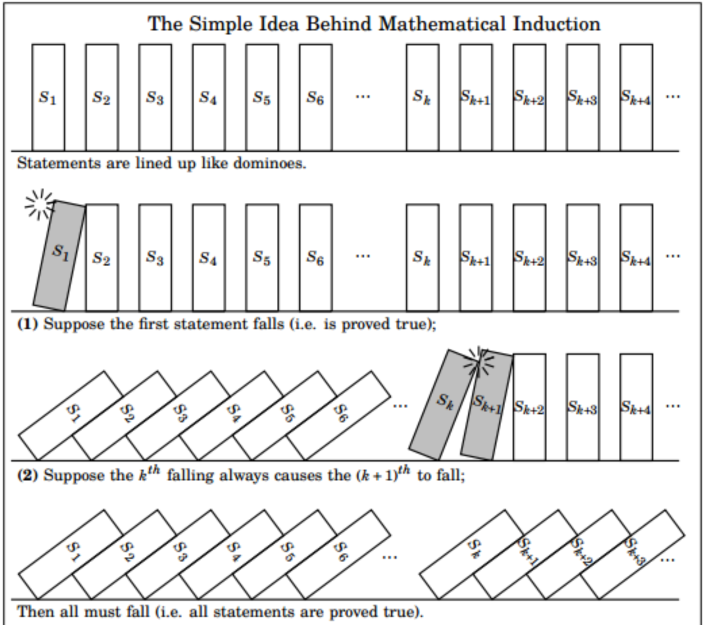
$$1) \quad P(1)$$

$$2) \quad \forall n \geq 1 \ : \ P(n) \rightarrow P(n+1)$$

Then it follows that

$$P(n) \ \text{for all } n \geq 1$$

Principle of induction



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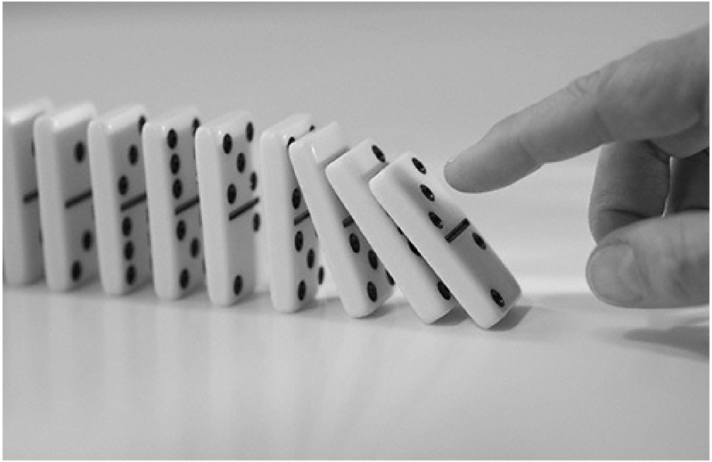

Principle of induction

If we can prove

1. *"The basis step"*

$$P(1) \text{ is true, and}$$

2. *"The inductive step"*

$$\text{for all } n \geq 1, \;\; P(n) \quad \text{implies} \quad P(n+1).$$

then $P(n)$ is true for every integer $n \geq 1$.

Example proof

Let's use induction to prove the formula of the sum of positive integers from 1 to n:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

(We have to show that both, the *basis step* and the *inductive step* are correct)

Example proof

Prove that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Part 1. *The basis step.* <u>Consider $n = 1$</u>.

The left hand side is just

$$1$$

The right-hand side:

$$\frac{1 \cdot (1+1)}{2} = 1$$

They are equal, so it is true.

Example proof

Part 2. *The inductive step.*

Assume that the formula is true for an arbitrary $n \geq 1$:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

We have to prove that

$$1 + 2 + 3 + \ldots + (n+1) = \frac{(n+1)((n+1)+1)}{2}$$

Example proof

Part 2. *The inductive step.*

Assume that the formula is true for an arbitrary $n \geq 1$:

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

We have to prove that

$$1+2+3+\ldots+(n+1)=\frac{(n+1)((n+1)+1)}{2}$$

Consider the left-hand side:

$$\underbrace{1+2+3+\ldots+n}_{=\frac{n(n+1)}{2}}+(n+1)=\frac{n(n+1)}{2}+n+1=\frac{n(n+1)+2(n+1)}{n+1}$$

$$=\frac{(n+1)(n+2)}{2}=\frac{(n+1)((n+1)+1)}{2}$$

Example proof

The base case and the inductive step are true.

Therefore, *by induction*, the formula is correct for every positive integer n.

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

This is a very useful formula, by the way.

Rewrite it using the sigma-notation

$$\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

Prove by induction

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \qquad \text{for all } n \geq k$$

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That is, for all $n \geq k$,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Example proof 2

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \qquad \text{for all } n \geq k$$

Choose some integer k. We have to show that then, for all $n \geq k$, the identity holds.

The base case. $n = k$.

The left hand side:

$$\sum_{i=k}^{k} \binom{i}{k} = \binom{k}{k} = 1.$$

The right hand side: $\binom{k+1}{k+1} = 1$. Both are equal.

Example proof 2

The inductive step.

Assume that the equality holds for some $n \geq k$:

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Prove that it also holds for $n+1$:

$$\sum_{i=k}^{n+1} \binom{i}{k} = \binom{(n+1)+1}{k+1}$$

To prove that, take the left hand side, and show that it is equal to the right hand side.

Example proof 2

Prove that

$$\sum_{i=k}^{n+1} \binom{i}{k} = \binom{(n+1)+1}{k+1}$$

The left hand side:

$$\sum_{i=k}^{n+1} \binom{i}{k} = \sum_{i=k}^{n} \binom{i}{k} + \binom{n+1}{k}$$

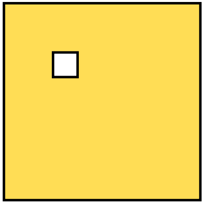
By the inductive hypothesis, the sum $\sum_{i=k}^{n} = \binom{n+1}{k+1}$, so

$$\sum_{i=k}^{n+1} \binom{i}{k} = \binom{n+1}{k+1} + \binom{n+1}{k} = \binom{n+2}{k+1},$$

where the last equality holds because of Pascal's identuty.

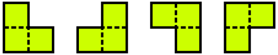
Therefore, the statement is true by induction.

Tiling $2^n \times 2^n$ with 1 square removed



For all n>0, a checkerboard 2^n x 2^n with one square removed

can be tiled by L-shaped tiles



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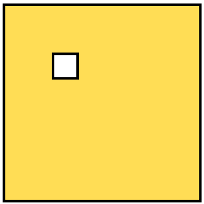

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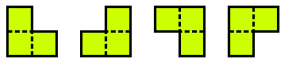
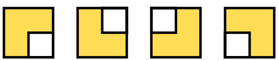
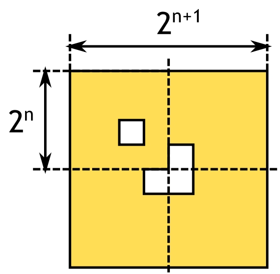


The base case for n=1



Inductive step.
Assuming that we can tile 2^n x 2^n with one removed, prove that it's possible to tile 2^{n+1} x 2^{n+1} with one removed



Another example proof

Let $P(n)$ be the predicate, "*I can lift n grains of sand*."

- I can lift one grain of sand, so $P(1)$ is true. This is my basis step.

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$$P(n) \rightarrow P(n+1)$$

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  $$P(n) \rightarrow P(n+1)$$

  This is my inductive step.

So, by induction, *I can lift any amount of sand*. Right?

Where is a mistake?

Of course, the proof is wrong. But should we blame induction for that?

Well, we made an error in the proof of $P(n) \to P(n+1)$.

It is hard to say for exactly which n it is false, but certainly there is some value!

Sum of b^k

Prove by induction that

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

First we can do the base case, then the inductive step.

Sum of b^k

Prove that

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

Basis step ($n = 0$):

$$b^0 = 1, \quad \text{and} \quad \frac{b^1 - 1}{b - 1} = 1$$

.

Sum of b^k

Prove that

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

Inductive step:

As always, we make a hypothesis that

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1} \quad \text{is true for } n \geq 0$$

And we have to prove that the formula is correct for $n + 1$:

$$b^0 + b^1 + b^2 + \ldots + b^{n+1} = \frac{b^{n+2} - 1}{b - 1}$$

Sum of b^k

Inductive step:

We have to prove that the formula is correct for $n+1$:

$$b^0 + b^1 + b^2 + \ldots + b^{n+1} = \frac{b^{n+2}-1}{b-1}$$

$$\underbrace{b^0 + b^1 + b^2 + \ldots + b^n}_{=\frac{b^{n+1}-1}{b-1} \text{ by the hypothesis}} + b^{n+1} = \frac{b^{n+1}-1}{b-1} + b^{n+1}$$

$$= \frac{b^{n+1}-1+b^{n+2}-b^{n+1}}{b-1} = \frac{b^{n+2}-1}{b-1}.$$

Sum of b^k

So, this formula for the sum is correct

$$b^0 + b^1 + b^2 + \ldots + b^n = \frac{b^{n+1} - 1}{b - 1}$$

In the sigma-notation:

$$\sum_{k=0}^{n} b^k = \frac{b^{n+1} - 1}{b - 1}$$

Sum of b^k

We can multiply both sides by a constant a:

$$\sum_{k=0}^{n} ab^k = ab^0 + ab^1 + ab^2 + \ldots + ab^n = \frac{a(b^{n+1}-1)}{b-1},$$

The sequence of numbers

$$a,\ ab,\ ab^2,\ ab^3,\ \ldots\ ab^n,\ \ldots$$

is called a *Geometric progression*.

So, we proved the formula for the partial sum of a geometic progression.

Sum of kb^{k-1}

The partial sum of the geometric progression:

$$\sum_{k=0}^{n} ab^k = ab^0 + ab^1 + ab^2 + \ldots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1},$$

Can we compute

$$\sum_{k=0}^{n} kb^{k-1}$$

$$= 0 + 1 + 2b + 3b^2 + 4b^3 \ldots + nb^{n-1}$$

So, instead of the constant a, we have an increasing sequence of coefficients now.

Sum of kb^{k-1}

There is a cheap trick:

$$\frac{d}{db}\sum_{k=0}^{n} b^k = \frac{d}{db}(1 + b + b^2 + b^3 + b^4 + \ldots + b^n)$$

$$= 0 + 1 + 2b + 3b^2 + 4b^3 + \ldots + nb^{n-1} = \sum_{k=0}^{n} kb^{k-1}$$

On the other hand,

$$\frac{d}{db}\sum_{k=0}^{n} b^k = \frac{d}{db}\Big(\frac{b^{n+1}-1}{b-1}\Big) = \frac{(n+1)b^n(b-1) - b^{n+1} + 1}{(b-1)^2}$$

$$= \frac{nb^{n+1} - (n+1)b^n + 1}{(b-1)^2}$$

Therefore, $\displaystyle\sum_{k=0}^{n} kb^{k-1} = \frac{nb^{n+1} - (n+1)b^n + 1}{(b-1)^2}$.

Geometric progression again

The partial sum of the geometric progression:

$$\sum_{k=0}^{n} ab^k = ab^0 + ab^1 + ab^2 + \ldots + ab^n = \frac{a(b^{n+1} - 1)}{b - 1}$$

Well, what if we want to add up an infinite sequence?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Infinite geometric progression

The partial sum of the geometric progression is

$$\sum_{k=0}^{n} ab^k = ab^0 + ab^1 + ab^2 + \ldots + ab^n = \frac{a(b^{n+1}-1)}{b-1}$$

If b is a small real number, specifically, if the absolute value $|b| < 1$, then

$$|b| > |b^2| > |b^3| > \ldots$$

In the limit, $\lim_{n\to\infty} b^n = 0$

$$\sum_{n=0}^{\infty} ab^n = \lim_{n\to\infty} ab^n = \lim_{n\to\infty} \frac{a(b^{n+1}-1)}{b-1} = \frac{a(-1)}{b-1} = \frac{a}{1-b}$$

Proving inequalities

Using mathematical induction, prove that for $n \geq 1$:

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The inductive step:
Assume that $2^n > n$ for $n \geq 1$. Prove that $2^{n+1} > (n+1)$.

Equivalently, we have to prove that

$$2^{n+1} - (n+1) > 0.$$

Proving inequalities

We assumed that $2^n > n$ for $n \geq 1$.

We have to prove that $2^{n+1} - (n+1) > 0$.

$$\begin{aligned} 2^{n+1} - (n+1) &= 2 \cdot 2^n - n - 1 \\ &> 2 \cdot n - n - 1 \qquad \text{(by the I.H.)} \\ &= n - 1 \\ &\geq 1 - 1 = 0. \qquad (\text{b/c } n \geq 1) \end{aligned}$$

Therefore, *by induction*, $2^n > n$ is true for $n \geq 1$.

One more proof

Theorem. All horses are the same color.

We can prove this by induction on the number of horses in a given set.

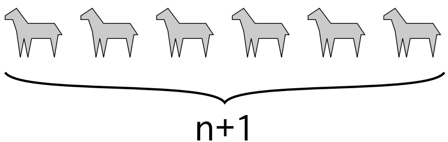
All horses are the same color

Principle
Examples
Summations
Inequalities

The basis step. If there's just one horse then it's the same color as itself.

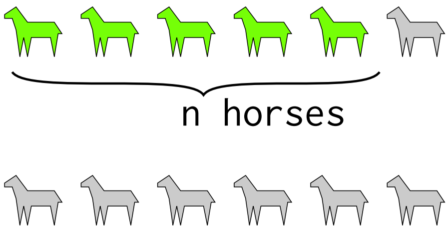
For the *inductive step*, assume that n horses are of the same color.

Assume that there are $n+1$ horses numbered 1 to $n+1$.



All horses are the same color

By the induction hypothesis, horses 1 through n are the same color, and similarly horses 2 through $n+1$ are the same color.



But the middle horses, 2 through n , can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and $n+1$ must be the same color as well. Thus all $n+1$ horses are the same color.

What, if anything, is wrong with this reasoning?

All horses are the same color

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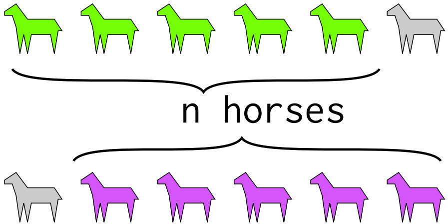

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