Discrete Structures. CSCI-150. Spring 2016.

Homework 4. Due Wed. Mar. 2, 2016.

Problem 1 (Graded)

Ellen draws 5 cards from a standard deck of 52 cards.

(a) In how many ways can her selection result in a hand with no clubs?

(b) A hand with at least one club?

Problem 2 (Graded)

A computer science professor has eleven different programming books on a bookshelf. Five of the books deal with the programming language C++, the other six with LISP. In how many ways can the professor arrange these books on the shelf

(a) if there are no restrictions?

(b) if all the C++ books must be next to each other?

(c) if all the C++ books must be next to each other and all the LISP books must be next to each other?

(d) if the languages should alternate?

Note that all the books are distinct. Not for grade, you may consider the case when two of the C++ books are identical copies. Then, what if all the C++ books and all the LISP books are identical copies?

Problem 3

How many different sets can be made out of 5 possible elements: a, b, c, d, e? Don't forget to count the empty set (that contains none of these elements).

Problem 4

A pizzeria offers 777 types of pizza and 3 types of soda. Mary goes there everyday for lunch, always buying one slice of pizza and one soda. However, she never gets exactly the same thing on two consecutive days (that is, each time, either the drink or the pizza (or both) is different from what she had yesterday).

In how many ways can she plan her lunch for the next 15 days if today she tried a different pizzeria and did not like that place at all?

Answer: approximately 3.240×10^{50} (but you should try to find the exact formula, not an approximation).

Proving identities and Double counting

Problem 5 (Graded)

In this problem, you have to find two proofs for the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(a) The first proof is algebraic. Using the fact that $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$, show that the equation is always true. It may involve some factorial manipulation, but almost everything should cancel out.

Remember: when proving the identity (or anything else in general), don't prove it "backwards", it's a logically inconsistent and faulty technique.

You may consider the left-hand side and the right-hand side separately, showing that they are equal to the same formula. *However, don't make it look like a "backwards" proof, please!*

(b) For the second part, prove the same identity using the technique called "Double counting" or "Combinatorial argument". It's a combinatorial proof technique for showing that two expressions are equal by demonstrating that they are two ways of counting the size of one set. In class, we used this technique to prove that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

To solve the problem, show that two formulas: $\binom{2n}{2}$ and $2\binom{n}{2} + n^2$ describe two counting procedures that count the same set.

A hint: We know that the first formula, $\binom{2n}{2}$, counts the number of ways to choose 2 objects out of available $2n$. Show that the second formula, $2\binom{n}{2} + n^2$, counts the same thing.

Problem 6 (Graded)

Give a "double counting' proof for the identity

$$2^5 = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}.$$

Hint: Think of the problem 3 from this homework.

Problem 7

Find "double counting" proofs for the following identities:

$$(2n)! = \binom{2n}{n} \cdot (n!)^2$$

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

$$\binom{n-1+2}{2} = n + \binom{n}{2}$$

$$\binom{n-1+3}{3} = n + \binom{n}{2} \cdot 2 + \binom{n}{3}$$

If you try proving the last two identities, think of selection with repetition.

Pigeonhole principle

Problem 8 (Graded)

(a) There are 50 white socks and 50 black socks in a drawer. How many socks do you have to take to be sure that you have at least one matching pair?

(b) At least one mismatching pair?

Problem 9

Prove that among any five points selected inside an equilateral triangle with side equal to 1, there always exists a pair at the distance not greater than $1/2$.

Problem 10 (Graded)

In a certain cubic region of our galaxy of the dimensions $4 \times 4 \times 4$ light years, there are 70 stars. Prove that among those 70 stars, there are at least two that are no more than $\sqrt{3}$ light years apart.

Comment: One light year is a unit of length equal to $\approx 9.46 \times 10^{15}$ meters.