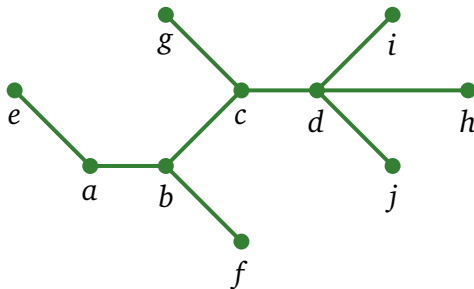


Trees

Tree



Def. A *tree* is a connected undirected graph with no simple cycles.

It does not need to have a root node, or directed edges. So, it is not hierarchical, and there are no parent/child nodes.

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Theorem. There is *exactly one simple path* between each pair of vertices in a tree.



Assume that there are two different simple paths from x to y .

Can we prove that it cannot happen in a tree?

Uniqueness of a simple path

Theorem. There is *exactly one simple path* between each pair of vertices in a tree.

Proof. Trees are connected graphs, so, there is a simple path between each pair of vertices. Are they unique?



Assume that for vertices x and y there are two simple paths between them. There must exist a vertex w , where the paths separate, and a vertex z , where they meet again for the first time. But they form a simple cycle $w \rightarrow z \rightarrow w$!

This is a contradiction, b/c trees don't have simple cycles, therefore, our assumption was incorrect, and for any x and y , there is no two different simple paths in a tree. \square

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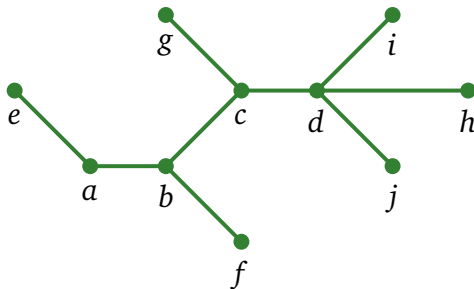
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Lemma. Removing one edge from the edge set of a tree gives a graph with two connected components, each of which is a tree.

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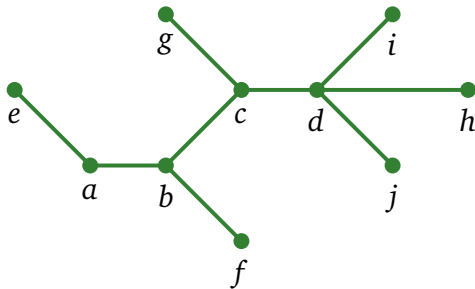
Lemma. Removing one edge from the edge set of a tree gives a graph with two connected components, each of which is a tree.

Proof. By the previous theorem, a removed edge makes the tree disconnected.

When edge (u, v) is removed, each vertex is either connected to u , or to v . So, there are two connected components.

And each connected component is a tree, because we could not introduce cycles by removing the edge (u, v) . \square

Properties of trees



Theorem. A tree with $n > 0$ vertices has $n - 1$ edges.

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Theorem. A tree with $n > 0$ vertices has $n - 1$ edges.

Proof by (strong) induction using the previous lemma.

Base case: A tree with one vertex has 0 edges, fine.

Inductive step: IH: Assume that for all trees with $0 < m < n$ vertices, there are $m - 1$ edges.

Given a tree with n vertices, remove one edge, getting two connected components C_1 and C_2 with k and $n - k$ vertices respectively. By the IH, C_1 and C_2 have $k - 1$ and $n - k - 1$ edges. Thus the original tree contained $(k - 1) + (n - k - 1) + 1 = n - 1$ edges.

Corollary. A finite tree with more than one vertex has at least one vertex of degree 1.

Therefore, trees have leaves (terminal vertices).

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Theorem (Euler formula). Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of faces in a planar representation of G . Then

$$v - e + f = 2.$$

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Proof. If there is more than one face, there is an edge separating two faces. Remove it, merging the faces. f and e are decreased by one, but $v - e + f$ remains the same.

Continue, until there is only one face. Then, since there is only one face, there is no simple cycles, so this is a tree.

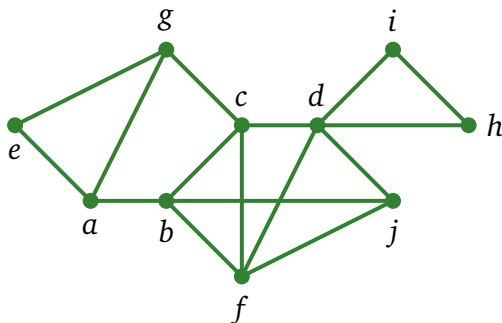
In the tree, there is at least one vertex with degree 1 (a leaf). Remove it from the tree. e and v are both decreased by 1. Therefore, $v - e + f$ remains the same.

Continue, until there is only one vertex, $v = 1$, $e = 0$, $f = 1$:

$$v - e + f = 2$$

In the original graph, $v - e + f$ was the same. □

Spanning trees



Def. Let G be a simple graph. A *spanning tree* of G is a subgraph of G that is a tree containing every vertex of G .

Tree

Properties

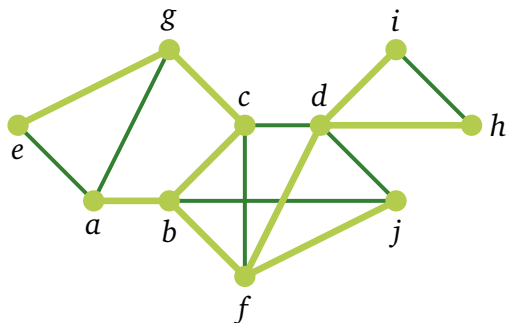
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Spanning trees



Def. Let G be a simple graph. A *spanning tree* of G is a subgraph of G that is a tree containing every vertex of G .

Tree

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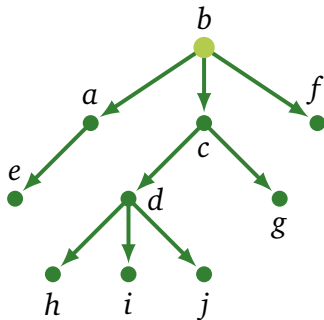
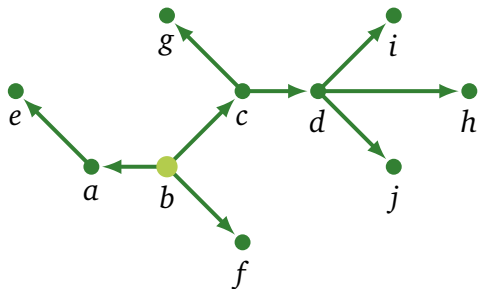
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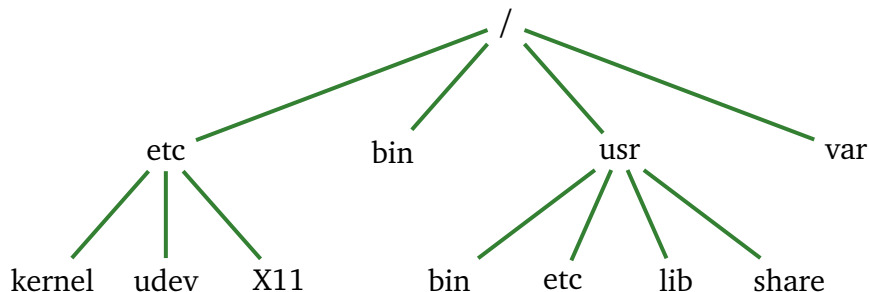
Def. A *rooted tree* is a tree in which one vertex has been designated as *the root* and every edge is directed away from the root.

A node has: the parent node, children, siblings, ancestors, descendants.

A vertex is called a *leaf* if it has no children, otherwise it is called an *internal vertex*.

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- Properties
- Euler formula for planar graphs
- Spanning trees
- Rooted trees**
- Huffman coding

Examples: File system



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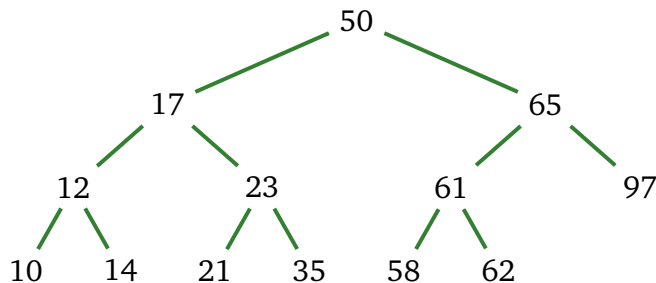
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Examples: Binary search tree



Def. A rooted tree is called a *binary* tree if every internal vertex has no more than 2 children.

(It's called “full”, if every internal node has exactly 2 children)

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Examples: Programs are trees

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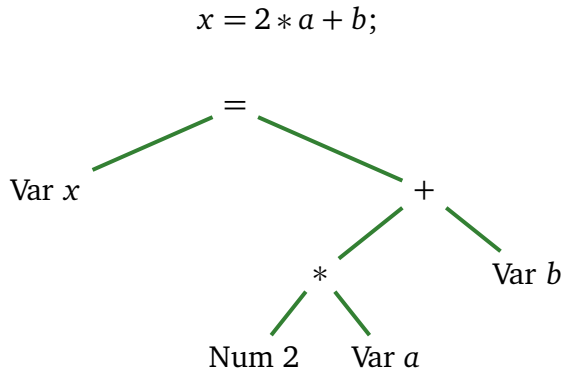
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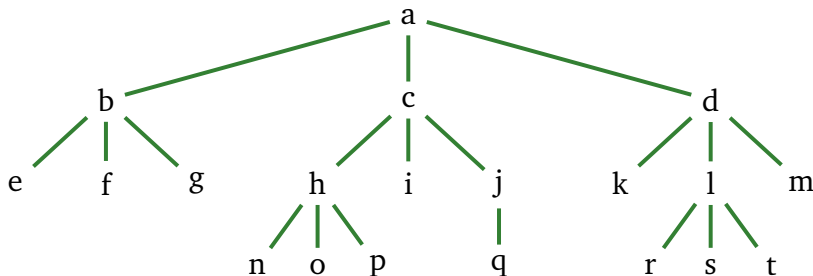
Huffman coding



Rooted trees

Def. A rooted tree is called an *m*-ary tree if every internal vertex has no more than m children. (when $m = 2$, the tree is called binary).

Def. An *m*-ary tree is called *full* if every internal vertex has exactly m children.



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Theorem. A full m -ary tree with i internal vertices contains

$$n = m \cdot i + 1 \quad \text{vertices.}$$

In particular, in binary trees, $n = 2i + 1$.

Rooted trees

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Theorem. A full m -ary tree with i internal vertices contains

$$n = m \cdot i + 1 \quad \text{vertices.}$$

In particular, in binary trees, $n = 2i + 1$.

Proof. Every vertex, except the root, is the child of an internal vertex. Because each of the i internal vertices has m children, there are mi vertices in the tree other than the root. Therefore, the tree contains $n = mi + 1$ vertices. \square

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Theorem. A full m -ary tree with i internal vertices contains

$$n = m \cdot i + 1 \quad \text{vertices.}$$

In particular, in binary trees, $n = 2i + 1$.

How can we use the theorem?

Note that the number of leaves is $l = n - i$.

Question: What is the number of internal nodes and the number of leaves in a full binary tree with n vertices?

Rooted trees

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Question: What is the number of internal nodes and the number of leaves in a full binary tree with n vertices?

Solution: $n = 2i + 1$, so the number of internal nodes is

$$i = (n - 1)/2,$$

and the number of leaves is

$$l = n - i = n - (n - 1)/2 = (n + 1)/2.$$

Thus in a large full binary tree, the number of internal nodes is almost the same as the number of leaves.

Rooted trees

Question: Find the least $n > 0$ such that there exist two trees:
a full 19-ary tree with n vertices, and
a full 32-ary tree with the same number of vertices.

Use the same theorem:

Theorem. A full m -ary tree with i internal vertices contains

$$n = m \cdot i + 1 \quad \text{vertices.}$$

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Encode a message

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Consider a situation when we are sending messages like this:

THISISAMESSAGETESTTESTTEST

Conventional **char** type takes 8 bits. So, every letter is encoded as a bit-string of length 8.

We would like to find a way to efficiently encode the letters of the English alphabet with shorter bit-strings.

Possible solution

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We encode 26 letters as bit-strings of length 5. Since $2^5 = 32 > 26$, this is enough to represent each letter.

$A = 00000$, $B = 00001$, $C = 00010$, ...

Can we do better if we know how frequently each letter occurs in the message?

Consider using bit strings of different lengths to encode letters.

Frequencies of letters

Letter	Frequency
A	8.167%
B	1.492%
C	2.782%
D	4.253%
E	12.702%
F	2.228%
G	2.015%
H	6.094%
...	

We can try

$$E = 0$$

$$A = 1$$

$$H = 00$$

$$D = 01$$

$$C = 10$$

$$F = 11$$

$$G = 000$$

$$B = 001$$

Try to decode:

011000

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Prefix codes

How to resolve the problem? Make a code so that there is no such collisions.

Encode letters so that the bit string for a letter never occurs as the prefix (first part) of the bit string for another letter.

$$E = 0$$

$$A = 10$$

$$H = 110$$

$$D = 1110$$

$$C = 11110$$

...

Codes with this property are called *prefix codes*.

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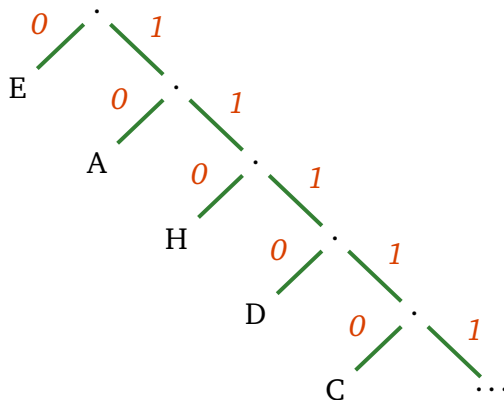
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Prefix codes

$E = 0$
 $A = 10$
 $H = 110$
 $D = 1110$
 $C = 11110$
...

Try to decode:

1100101110



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Input:

Letter	Frequency
A	8.167%
B	1.492%
C	2.782%
D	4.253%
E	12.702%
F	2.228%
G	2.015%
H	6.094%
...	

Given symbols and their frequencies, our goal is to construct a rooted binary tree where the symbols are the labels of the leaves.

Huffman coding is an algorithm that takes as input the frequencies of symbols in a string and produces as output a prefix code that encodes the string using the *fewest possible bits*.

Output: prefix code.

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Letter	Frequency
A	8%
B	10%
C	12%
D	15%
E	20%
F	35%

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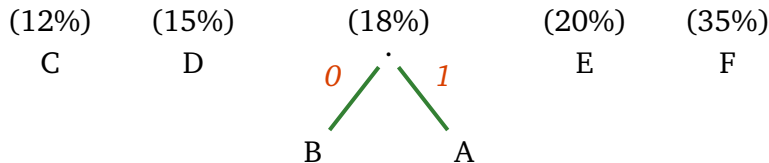
Rooted trees

Huffman coding

Initial state. Start with disjoint trees:

(8%)	(10%)	(12%)	(15%)	(20%)	(35%)
A	B	C	D	E	F

Step 1. Take two trees with the least frequencies: A and B here, and combine them in a single tree: The tree that has the higher frequency becomes the left branch.



Huffman coding

Tree

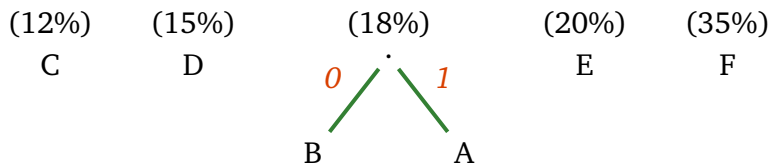
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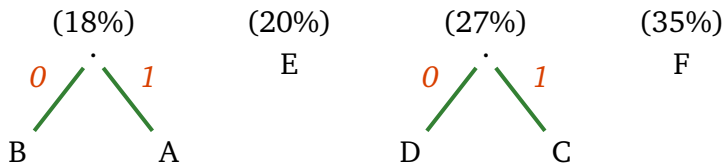
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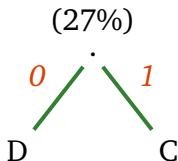
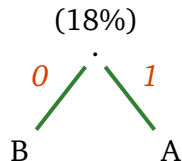
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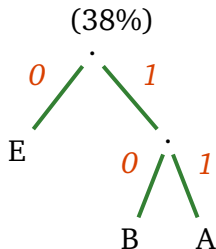
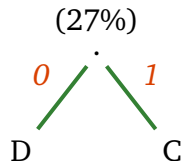
Step 2.



Huffman coding



Step 3.



Tree

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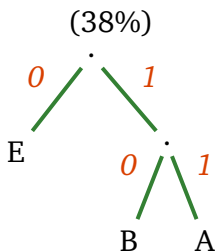
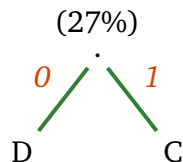
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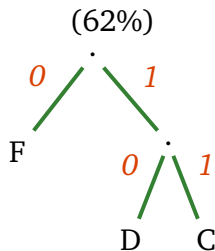
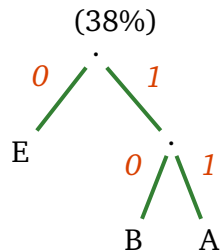
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Step 4.



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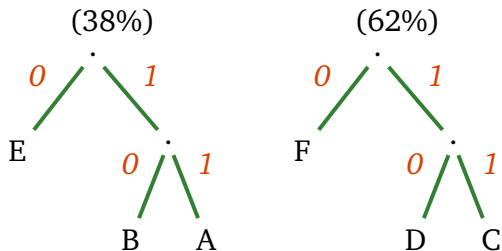
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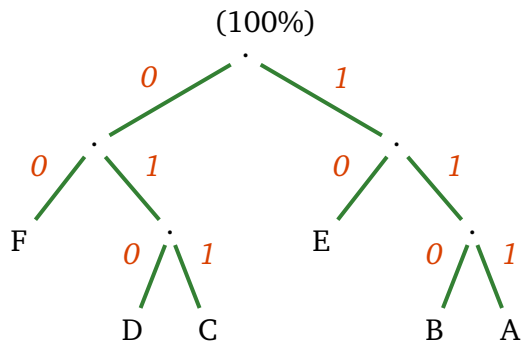
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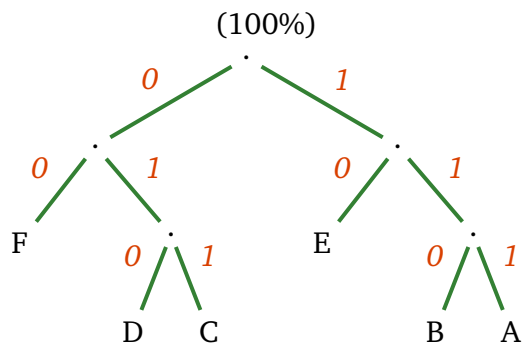
Huffman coding



Step 5.



Huffman coding



Letter	String	Freq.
A	111	8%
B	110	10%
C	011	12%
D	010	15%
E	10	20%
F	00	35%

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What is the average number of bits used to encode a character?

$$3 \cdot (0.08 + 0.10 + 0.12 + 0.15) + 2 \cdot (0.20 + 0.35) = 2.45.$$

Huffman coding is optimal code in the sense that no binary prefix code for these symbols can encode these symbols using fewer bits.