# Discrete Structures. CSCI-150. Summer 2016.

## Homework 5. Due Mon. Jun 20, 2016.

### Problem 1 (Graded)

We are going to prove that the following summation formula is correct for integer $n \geq 1$:

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

First, check that it is correct for $n = 1$, $n = 2$, and $n = 3$.

After that, prove this formula by induction for all $n \geq 1$.

> Always write inductive proofs in full:
>
> - First, write what the base case is and give its proof.
> - Then the inductive case:
>   1) write the assumtion (the inductive hypothesis), then
>   2) write the formula you have to prove that follows from the hypothesis,
>   3) then write a proof for it.

### Problem 2 (Graded)

Prove by induction that for all $n \geq 0$:

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

In the inductive step, use Pascal's identity, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Hint: So, we want to prove the following infinite sequence of identities by induction:

$$\binom{0}{0} = 2^0$$

$$\binom{1}{0} + \binom{1}{1} = 2^1$$

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 2^2$$

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$$

$$\vdots$$

It turns out each of the equations (except the very first one) can be proven from the previous if you correctly use Pascal's Identity. That's the essense of this task.

## Problem 3 (Graded)

Find a mistake in the inductive "proof" that all horses are the same color. The proof was given at the very end of the lecture 9.

## Problem 4 (Graded)

Prove by induction that

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

Hint: Be careful, correctly identify what the base case should be.

## Problem 5

Consider $n$ lines in the infinite plane such that no two lines are parallel, and no three lines intersect at the same point (so all intersections are only pairwise). Therefore, each line intersects with each of the other lines in exactly one distinct point. See an example below for 6 lines.

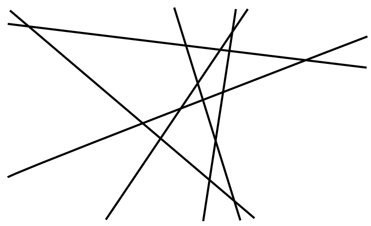

**Figure 1:** *An example with 6 lines*

Prove by induction that if there are $n$ lines, then the total number of intersections is equal to $n(n-1)/2$.

## Problem 6

Prove by induction that $\forall n \geq 3$:

$$n^2+1 \geq 3n$$