

Recurrences

A side note about induction

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We have seen many examples with induction used to prove summation formulas. But we have to make it clear: *Induction does not have to be applied to summations* or any arithmetic expressions. It is a much more general approach:

$$\frac{P(0) \\ P(n) \rightarrow P(n+1) \text{ for all } n \geq 0}{P(k) \text{ for all } k \geq 0}$$

For example, the problem, where we were tiling checkerboards. There were no summation, but it was a good example of induction.

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Recall that we used induction to prove statements like

$$\sum_{k=0}^n k = 0 + 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

In problems like this, we used a common pattern:

$$\sum_{k=0}^0 k = 0$$

$$\sum_{k=0}^n k = \left(\sum_{k=0}^{n-1} k \right) + n, \quad \text{when } n > 0$$

That is, we can express the sum of natural numbers recursively in terms of a smaller sum.

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Let $S(n)$ be the sum of all natural numbers not greater than n :

$$S(n) = \sum_{k=0}^n k,$$

It can be convenient to redefine the sum $S(n)$ as a *recurrence*:

$$S(0) = 0$$

$$S(n) = S(n-1) + n \quad (\forall n > 0)$$

This is just another way to express the same function S .

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Exponentiation:

$$E(a, n) = a^n$$

Recursively:

$$E(a, 0) = 1$$

$$E(a, n) = E(a, n-1) \cdot a \quad (\forall n > 0)$$

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Factorial:

$$n! = 1 \cdot 2 \cdot \dots \cdot n$$

Recursively:

$$0! = 1$$

$$n! = (n-1)! \cdot n \quad (\forall n > 0)$$

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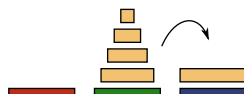
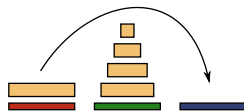
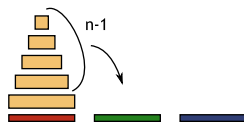


<http://www.mathsisfun.com/games/towerofhanoi.html>

The towers of Hanoi

Our recursive algorithm to move a tower of height n from #1 to #3:

1. Move an $(n-1)$ -tower from #1 to #2.
2. Move an 1-tower from #1 to #3.
3. Move an $(n-1)$ -tower from #2 to #3.



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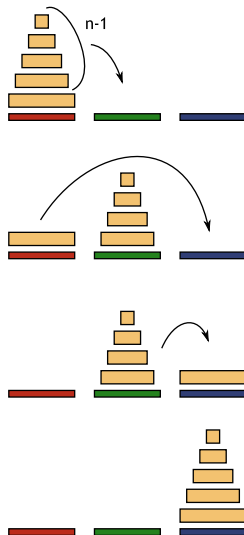
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The towers of Hanoi

Our recursive algorithm to move a tower of height n from #1 to #3:

1. Move an $(n-1)$ -tower from #1 to #2.
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There is a way to find a recurrent formula for T_n , the total number of steps to move the tower from the peg 1 to the peg 3.



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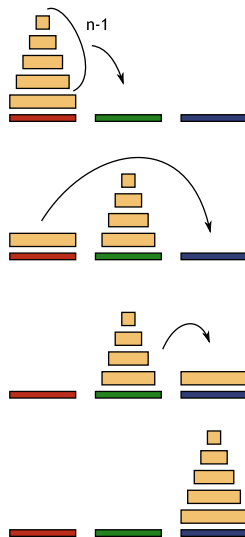
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T_n , the time to move a tower of height n :

$$T_1 = 1$$

$$T_n = T_{n-1} + 1 + T_{n-1} \quad (\forall n > 1)$$



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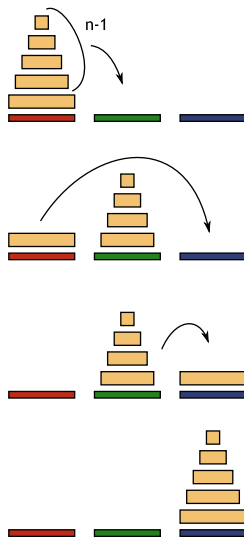
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T_n , the time to move a tower of height n :

$$T_1 = 1$$

$$T_n = T_{n-1} + 1 + T_{n-1} \quad (\forall n > 1)$$

There is a proof by induction that this time is optimal for any algorithm.



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$$T_1 = 1$$

$$T_n = 2T_{n-1} + 1 \quad (\forall n > 1)$$

Our goal is to find a closed form expression for T_n as a function of n , without any recurrence.

The towers of Hanoi

Before we get a closed form formula for T_n ,
what are the numbers?

$$T_1 = 1$$

$$T_n = 2T_{n-1} + 1 \quad (\forall n > 1)$$

We can compute a list like this:

$$T_1 = 1$$

$$T_2 = 3$$

$$T_3 = 7$$

$$T_4 = 15$$

$$T_5 = 31$$

$$T_6 = 63$$

...

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$$T_1 = 1$$

$$T_n = 2T_{n-1} + 1 \quad (\forall n > 1)$$

$$T_1 = 1$$

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$$T_6 = 63$$

...

Guess and verify method... Let's try $T_n = 2^n - 1$?

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$$T_1 = 1$$

$$T_n = 2T_{n-1} + 1 \quad (\forall n > 1)$$

Guess and verify method... Let's try $T_n = 2^n - 1$? We can show by induction that this formula is correct.

The base case, $n = 1$:

$$T_1 = 2^1 - 1 = 1.$$

Ok, the base case is true.

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$$T_1 = 1$$

$$T_n = 2T_{n-1} + 1 \quad (\forall n > 1)$$

We want to prove the closed form formula $T_n = 2^n - 1$.

The inductive step, $n > 1$:

Assume that $T_n = 2^n - 1$, and show that then $T_{n+1} = 2^{n+1} - 1$.

Proof. From the recurrence:

$$T_{n+1} = 2T_n + 1$$

By the inductive hypothesis:

$$2T_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$$

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Why is it useful to know that the recurrence

$$T_1 = 1$$

$$T_n = 2T_{n-1} + 1 \quad (\forall n > 1)$$

is equivalent to the closed form formula $T_n = 2^n - 1$?

The 7-disk puzzle will require $T_7 = 2^7 - 1 = 127$ moves to complete.

And the 100-disk puzzle will require

$$T_{100} = 2^{100} - 1 = 1267650600228229401496703205375 \text{ moves.}$$

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function Merge:

Given two sorted lists, combine them into a single sorted list:

$$[1, 2, 4, 5] + [3, 4, 5, 6] \mapsto [1, 2, 3, 4, 4, 5, 5, 6]$$

function Sort:

Given a list: if it contains a single element, return it. Otherwise, split it in two halves sort them separately and merge the results:

$$S[5] \mapsto [5]$$

$$S[6, 7, 1, 8, 9, 7, 4, 3] \mapsto S[6, 7, 1, 8] + S[9, 7, 4, 3]$$

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$S[1, 8, 3, 6, 5, 4, 7, 2] \mapsto$

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$$\begin{aligned} S[1, 8, 3, 6, 5, 4, 7, 2] &\mapsto \\ S[1, 8, 3, 6] + S[5, 4, 7, 2] &\mapsto \end{aligned}$$

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$$\begin{aligned} &S[1, 8, 3, 6, 5, 4, 7, 2] \mapsto \\ &S[1, 8, 3, 6] + S[5, 4, 7, 2] \mapsto \\ &(S[1, 8] + S[3, 6]) + (S[5, 4] + S[7, 2]) \mapsto \end{aligned}$$

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$$S[1, 8, 3, 6, 5, 4, 7, 2] \mapsto$$

$$S[1, 8, 3, 6] + S[5, 4, 7, 2] \mapsto$$

$$(S[1, 8] + S[3, 6]) + (S[5, 4] + S[7, 2]) \mapsto$$

$$((S[1] + S[8]) + (S[3] + S[6])) + ((S[5] + S[4]) + (S[7] + S[2])) \mapsto$$

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$$S[1, 8, 3, 6, 5, 4, 7, 2] \mapsto$$

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$$(((1] + [8]) + ([3] + [6])) + (((5] + [4]) + ([7] + [2]))) \mapsto$$

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$$S[1, 8, 3, 6, 5, 4, 7, 2] \mapsto$$

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$$((([1] + [8]) + ([3] + [6]))) + ((([5] + [4]) + ([7] + [2]))) \mapsto$$

$$([1, 8] + [3, 6]) + ([4, 5] + [2, 7]) \mapsto$$

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$$S[1, 8, 3, 6, 5, 4, 7, 2] \mapsto$$

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$$([1, 8] + [3, 6]) + ([4, 5] + [2, 7]) \mapsto$$

$$[1, 3, 6, 8] + [2, 4, 5, 7] \mapsto$$

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$$S[1, 8, 3, 6, 5, 4, 7, 2] \mapsto$$

$$S[1, 8, 3, 6] + S[5, 4, 7, 2] \mapsto$$

$$(S[1, 8] + S[3, 6]) + (S[5, 4] + S[7, 2]) \mapsto$$

$$((S[1] + S[8]) + (S[3] + S[6])) + ((S[5] + S[4]) + (S[7] + S[2])) \mapsto$$

$$((([1] + [8]) + ([3] + [6])) + (([5] + [4]) + ([7] + [2]))) \mapsto$$

$$([1, 8] + [3, 6]) + ([4, 5] + [2, 7]) \mapsto$$

$$[1, 3, 6, 8] + [2, 4, 5, 7] \mapsto$$

$$[1, 2, 3, 4, 5, 6, 7, 8]$$

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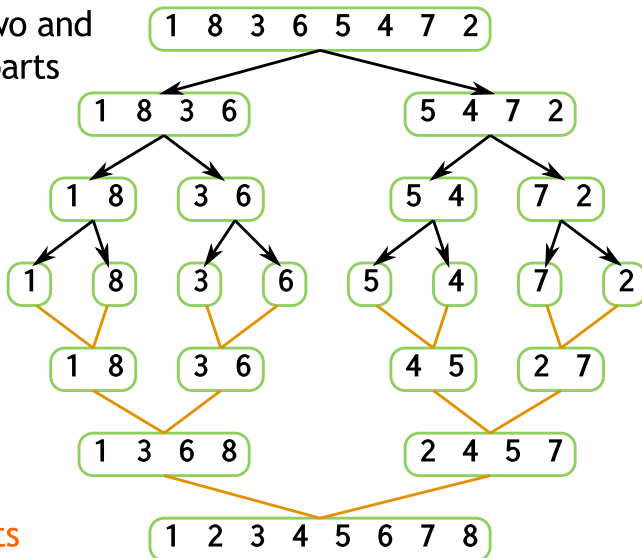
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Split in two and
sort the parts



Merge
sorted lists

Merge sort

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How much time does it take to sort a list of n elements?

To estimate the time complexity, we are going to *count the number of comparisons* between the elements.

We assume that the size of the given list is a power of 2. It makes the analysis easier, but does not affect the result.

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- (a) To merge two lists of size $n/2$, we need to do at most $n - 1$ comparisons.
- (b) To sort a list, we have to split it in two, sort both halves, and merge them.

Therefore,

$$T(1) = 0$$

$$T(n) = 2T(n/2) + n - 1 \quad (\forall n > 1)$$

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Given

$$T(n) = 2T(n/2) + n - 1 \quad (\forall n > 1)$$

Since $n = 2^k$,

$$\begin{aligned} T(n) &= T(2^k) = 2T(2^{k-1}) + (2^k - 1) \\ &= 2(2T(2^{k-2}) + 2^{k-1} - 1) + (2^k - 1) \\ &= 2^2 T(2^{k-2}) + (2^k - 2) + (2^k - 1) \\ &= 2^2 (2T(2^{k-3}) + 2^{k-2} - 1) + (2^k - 2) + (2^k - 1) \\ &= 2^3 T(2^{k-3}) + (2^k - 4) + (2^k - 2) + (2^k - 1) \\ &= 2^3 (2T(2^{k-4}) + 2^{k-3} - 1) + (2^k - 4) + (2^k - 2) + (2^k - 1) \\ &= 2^4 T(2^{k-4}) + (2^k - 8) + (2^k - 4) + (2^k - 2) + (2^k - 1) \\ &= \dots = 2^k \underbrace{T(2^{k-k})}_{T(1)=0} + \sum_{i=0}^{k-1} (2^k - 2^i) = \sum_{i=0}^{k-1} (2^k - 2^i). \end{aligned}$$

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$$T(n) = T(2^k) = \sum_{i=0}^{k-1} (2^k - 2^i) = \sum_{i=0}^{k-1} (n - 2^i) = n \cdot k - \sum_{i=0}^{k-1} 2^i.$$

The sum of the geometric progression is

$$\sum_{i=0}^{k-1} 2^i = \frac{2^k - 1}{2 - 1} = 2^k - 1 = n - 1.$$

Thus $T(n) = n \cdot k - n + 1$. And since $n = 2^k$, $k = \log_2 n$, so

$$T(n) = n \log_2 n - n + 1.$$

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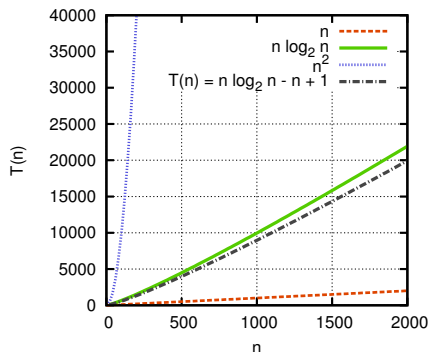
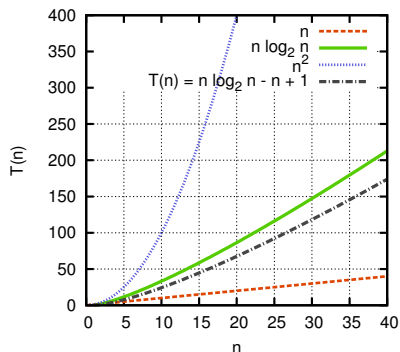
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To sort a list of length n , takes time (the number of comparisons)

$$T(n) = n \log_2 n - n + 1 \approx n \log_2 n.$$



Merge sort

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In merge sort, we had a recurrence:

$$T(n) = 2T(n/2) + n - 1$$

In general, if the time complexity of an algorithm is expressed by a recurrence:

$$T(n) = a \cdot T(n/b) + f(n)$$

To solve such recurrences, there is a so called *Master theorem*:

https://en.wikipedia.org/wiki/Master_theorem

It covers different forms of the function f , as well as difference values of the constants a and b .

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