# Scientific Computing Lab

## Local and Global Errors

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- Distinguish two aspects of the concept of *accuracy*:
  - first, the error which results *locally* at every point $t_k$.
  - second, the error that accumulates totally *globally* during the calculation from $a$ to $b$.
- $\rightarrow$ Two different types of discretization errors:
  - **Local discretization error** (which is the error that arises newly after every time step, even if the difference quotient was generated with the exact $y(t)$).
  - **Global discretization error** (which is the maximum of how far off base the calculations over the whole time interval are at the end).

# The Local Discretization Error

- **Local discretization error:** Maximum error which arises solely from the local transition from differential to difference quotients – even if the exact solution $y(t)$ is always used.
- Local discretization error of Euler's method:

$$l(\delta t) \;:=\; \max_{a \leq t \leq b-\delta t} \left\{ \left| \frac{y(t+\delta t)-y(t)}{\delta t} - f(t, y(t)) \right| \right\} .$$

- Assume the availability of the exact local solution in every point and consider the local errors arising due to "differences instead of derivatives".
- If $l(\delta t) \to 0$ for $\delta t \to 0$: Discretization scheme is called **consistent**.
- Consistency obviously is the minimum that has to be required.

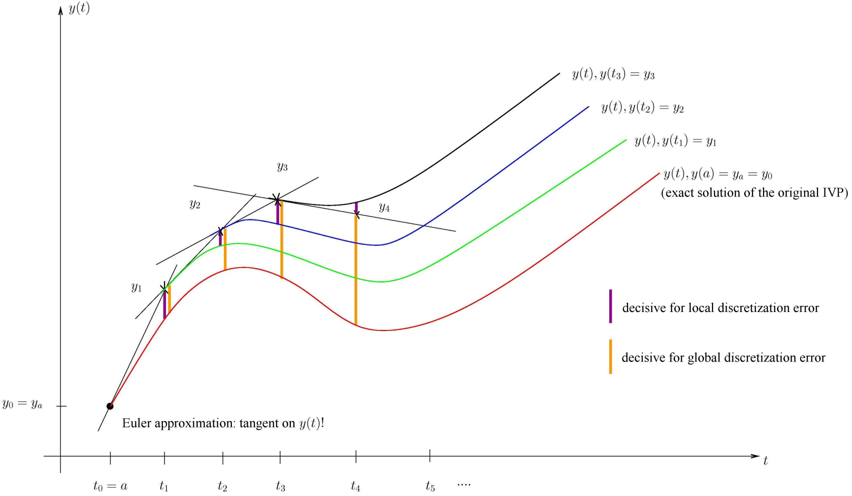
# The Global Discretization Error

- **Global discretization error:** Maximum error between the computed approximations $y_k$ and the corresponding values $y(t_k)$ of the exact solution $y(t)$ at the discrete points in time $t_k$:

$$g(\delta t) \ := \ \max_{k=0,\ldots,N} \{|y_k - y(t_k)|\} \ .$$

- If $g(\delta t) \to 0$ for $\delta t \to 0$: The discretization scheme is called **convergent**. Investing more and more computational effort (increasingly smaller steps in time $\delta t$) will then lead to increasingly better approximations for the exact solution (infinitesimal error).
- Consistency is the weaker of those two terms, of rather technical nature, and often easy to prove. Convergence, in contrast, is the stronger term (convergence implies consistency, but not the other way round!), of fundamental practical importance, and often not that trivial to show.

# Local and Global Discretization Error in Comparison



# Consistency and Convergence of the Methods of Euler, Heun, and Runge-Kutta

- All three methods introduced so far are consistent and convergent:
  - **Euler**: method of *first order*, i.e.

$$l(\delta t) = O(\delta t)\,, \qquad g(\delta t) = O(\delta t)\,;$$

  - **Heun**: method of *second order*, i.e.

$$l(\delta t) = O((\delta t)^2)\,, \qquad g(\delta t) = O((\delta t)^2)\,;$$

  - **Runge-Kutta**: method of *fourth order*, i.e.

$$l(\delta t) = O((\delta t)^4)\,, \qquad g(\delta t) = O((\delta t)^4)\,.$$