TECHNISCHE UNIVERSITÄT MÜNCHEN

Institut für Informatik

Informatik V
Chair for
Scientific Computing

Scientific Computing Lab

Ordinary Differential Equations

Explicit Discretization

Ordinary Differential Equations

- Equation with a function y(t) and derivatives of y(t)

$$y(t) + \dot{y}(t) = 0$$

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = 0$$

- typical: development of a variable over time

$$\dot{y}(t) = f(t, y(t))$$

Ordinary Differential Equations

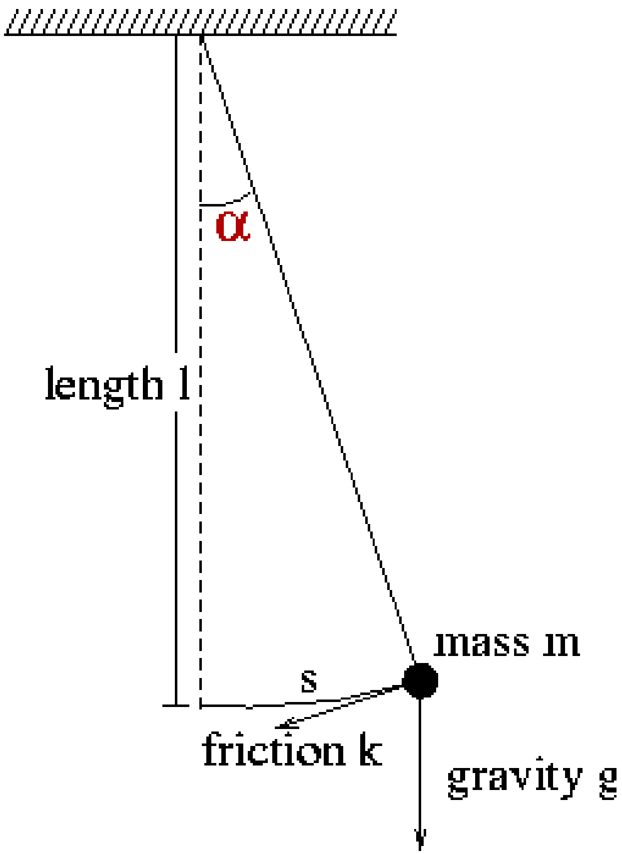
- Example: radioactive decay
  - Half-life: Period of time in which half of the atoms decay
  - Decay constant k: Describes rate of decay and thus half-life

$$\frac{dr(t)}{dt} = -k \cdot r(t)$$

Ordinary Differential Equations

- Example: pendulum

$$\frac{d^2\alpha}{dt^2} + \frac{k}{m} \cdot \frac{d\alpha}{dt} + \frac{g}{l} \cdot \sin(\alpha) = 0$$



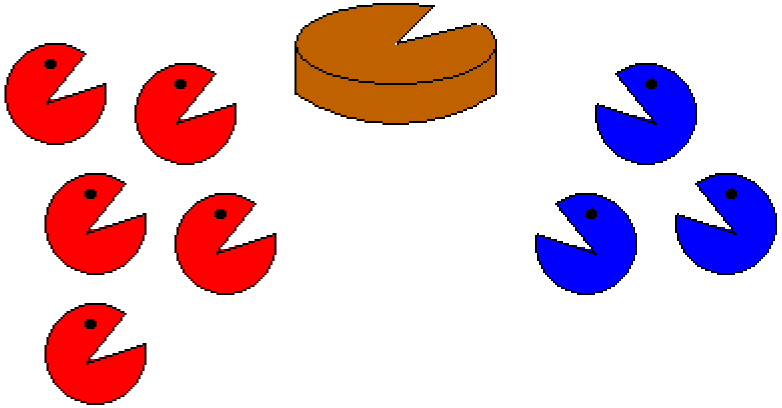
Ordinary Differential Equations

- Example: population growth
- populations P and Q

$$\frac{dp(t)}{dt} = (2 - p - q) \cdot p$$

$$\frac{dq(t)}{dt} = (2 - p - q) \cdot q$$

competition



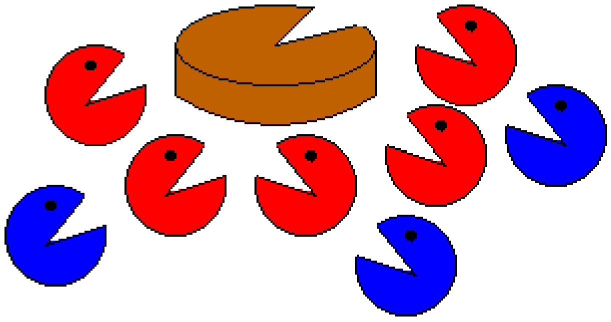
Ordinary Differential Equations

Example: population growth

populations P and Q – predator-prey

$$\frac{dp(t)}{dt} = (2 - p + q) \cdot p$$

$$\frac{dq(t)}{dt} = (2 - 10\mathrm{p} - q) \cdot q$$



Solving
Ordinary Differential Equations

- Solving = finding y(t)
- Two ways to solve the ODE:
  - ~~Analytically~~
  - Numerically

→ Needs discretization (timestepping)

Solving Ordinary Differential Equations

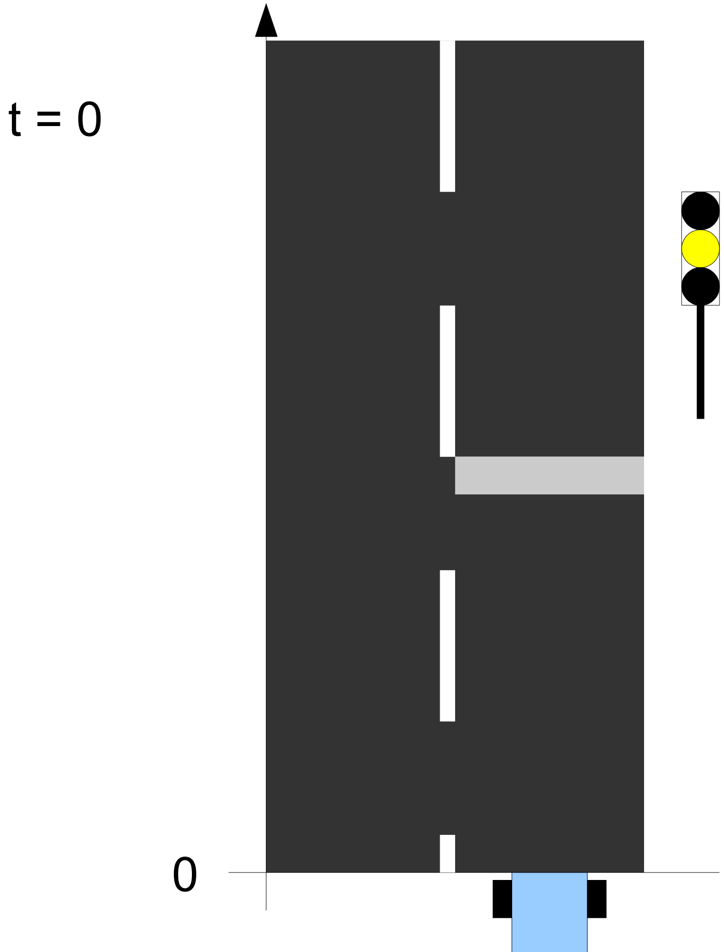
- Your turn: Try to compute y(t)

Time t	0	5	10	15	20	25	30	35	40	45
$\dot{y}(t)$	5	1	0	-1	0	0	0	0	2	5

- Time in seconds, $\dot{y}$ in m/s
- Any ideas what y describes?

Solving
Ordinary Differential Equations

- Your turn: Try to compute y(t)



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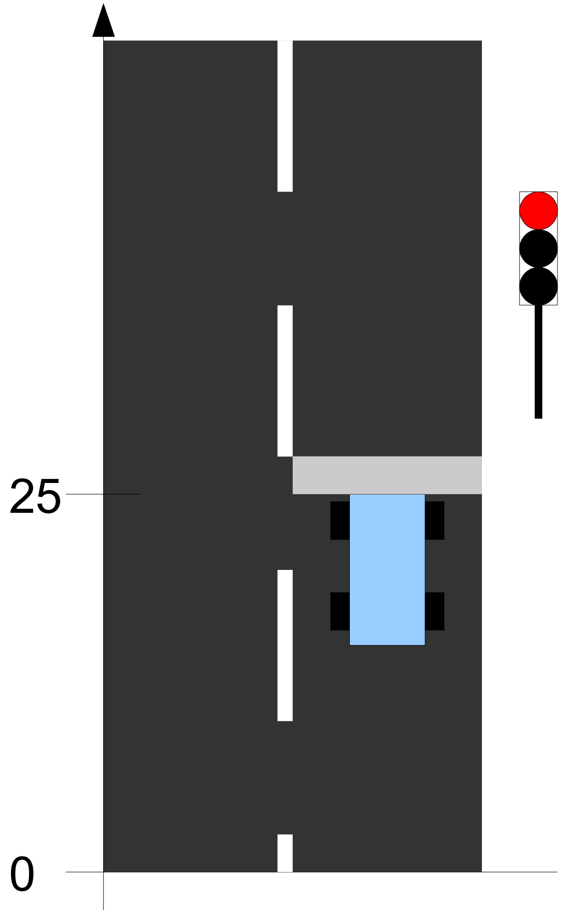
Solving
Ordinary Differential Equations

- Your turn: Try to compute y(t)

t = 5



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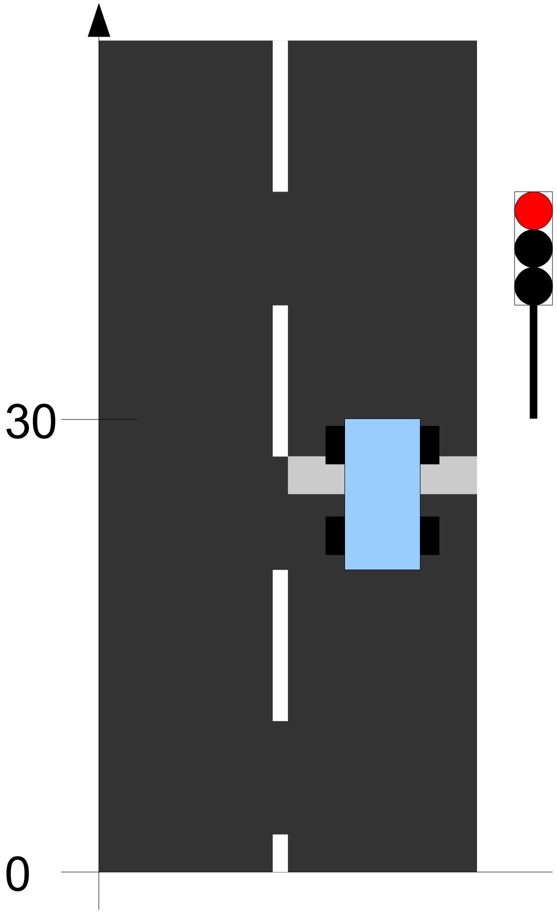
Solving Ordinary Differential Equations

- Your turn: Try to compute y(t)

t = 10



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Solving
Ordinary Differential Equations

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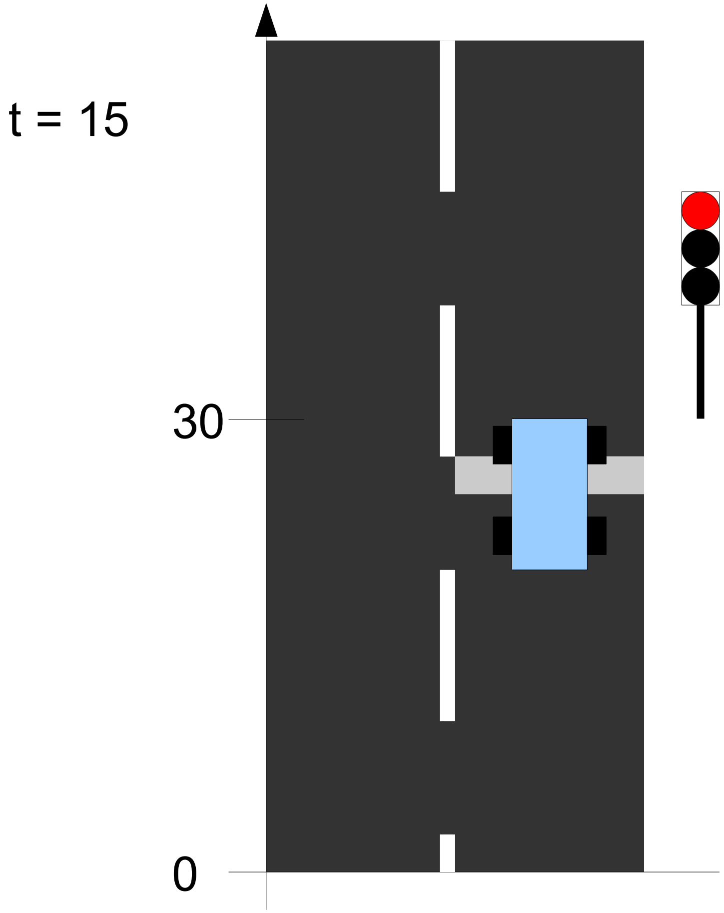

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Solving
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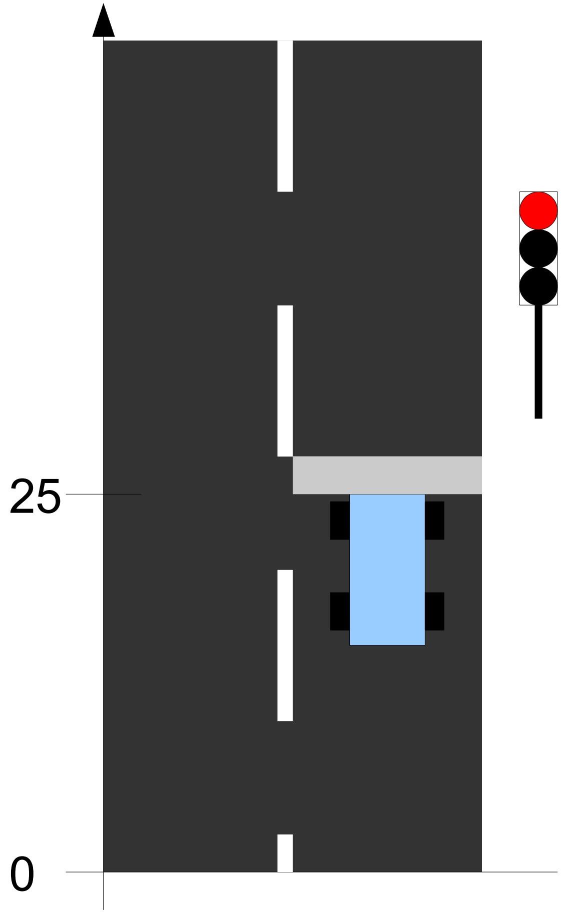
t = 20



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Solving
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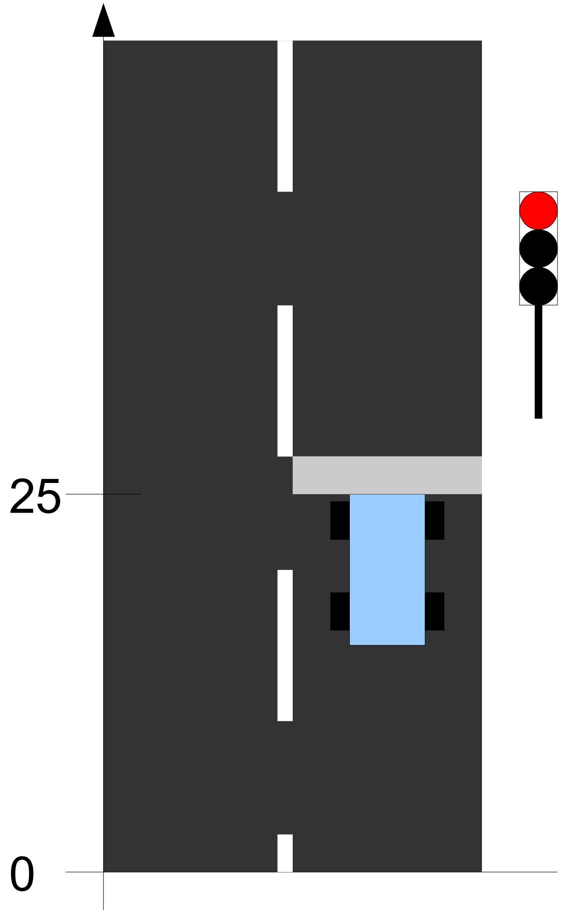
t = 25



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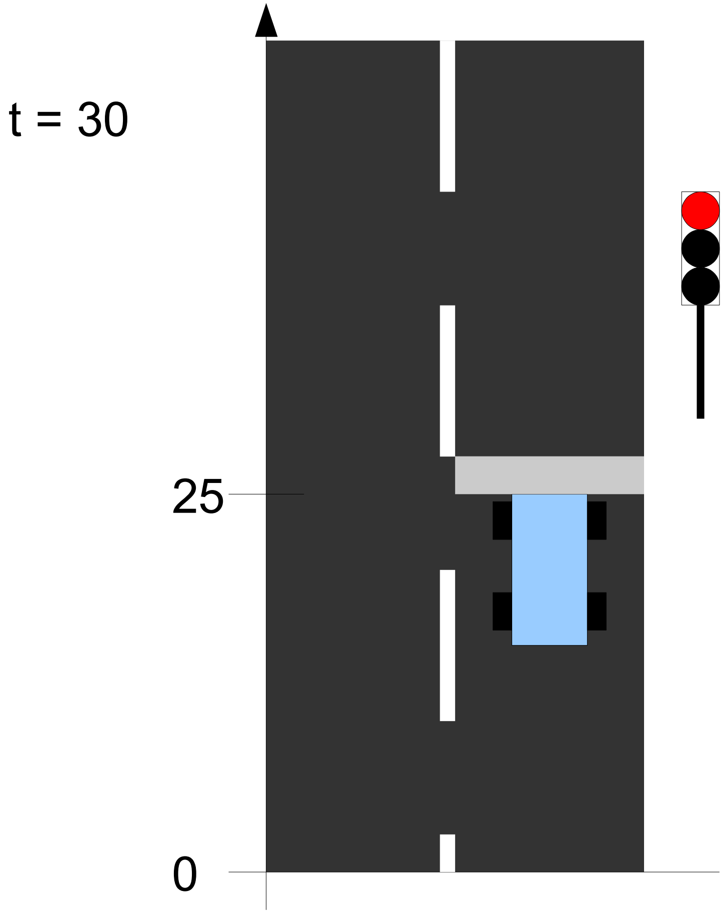

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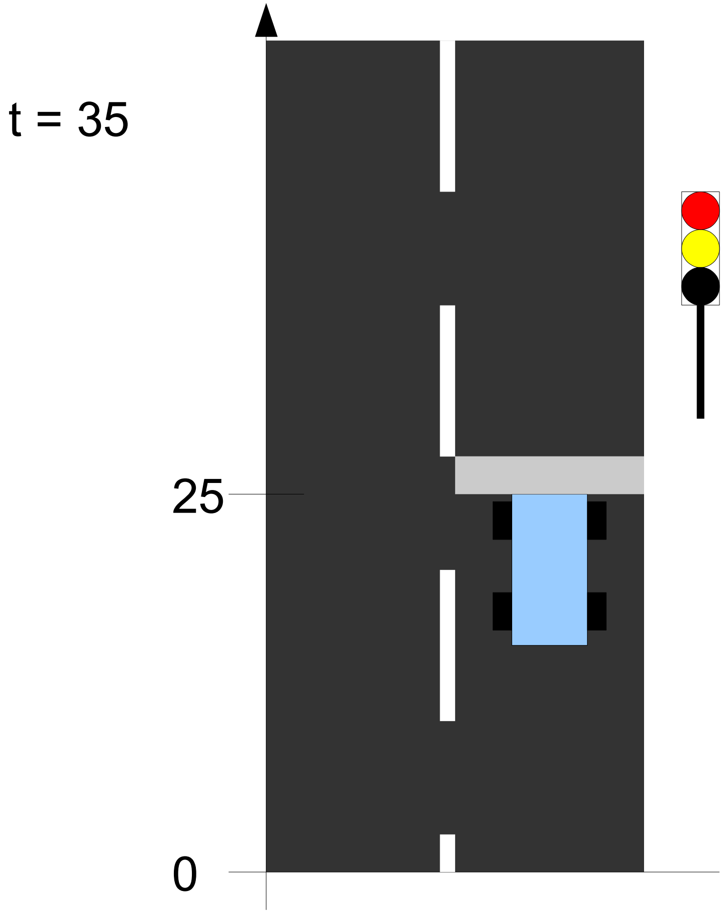

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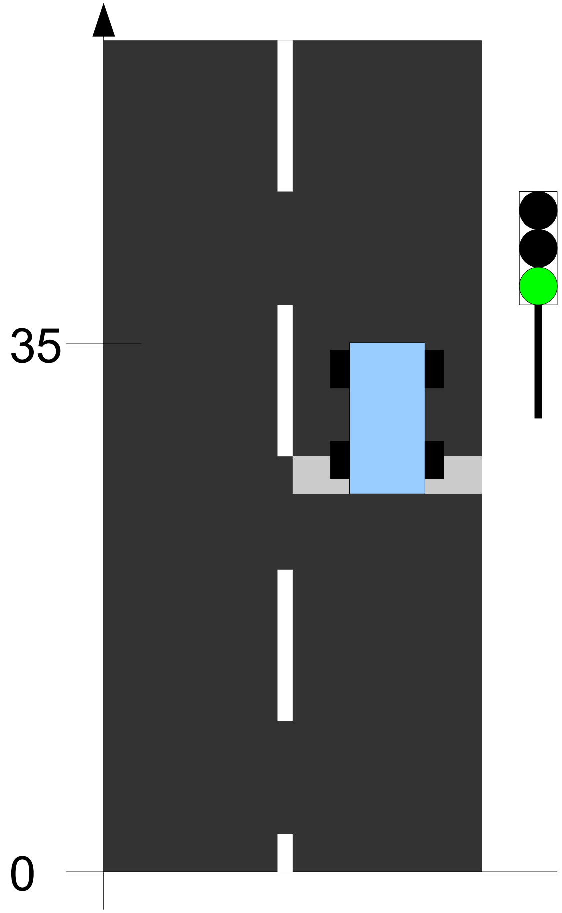
Solving
Ordinary Differential Equations

- Your turn: Try to compute y(t)

t = 40



Time t	0	5	10	15	20	25	30	35	40	45
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Solving Ordinary Differential Equations

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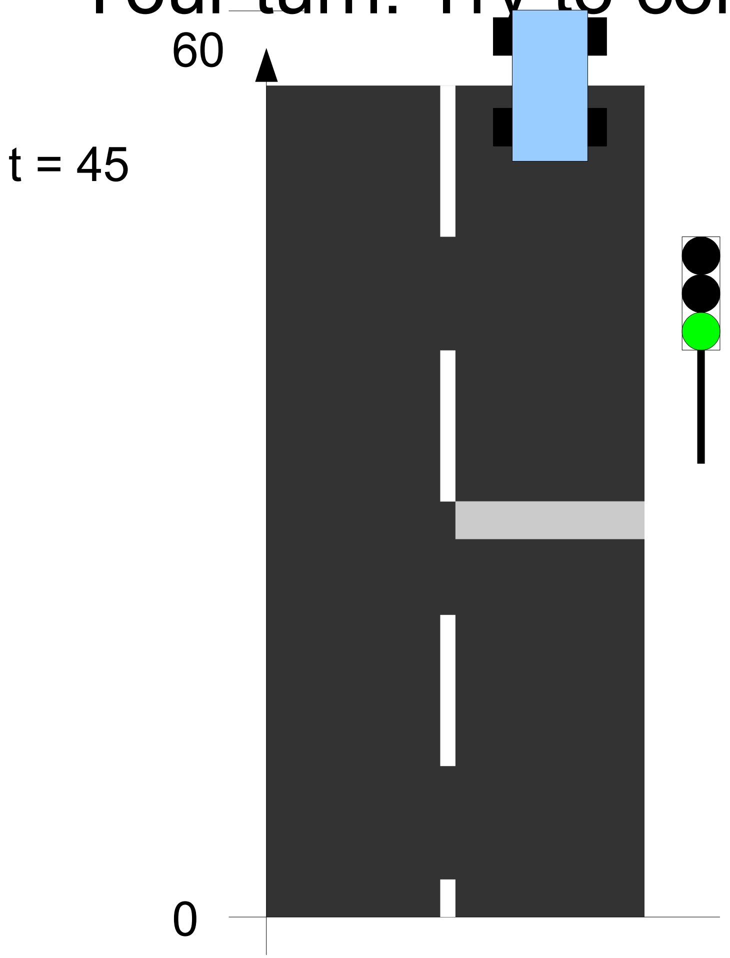

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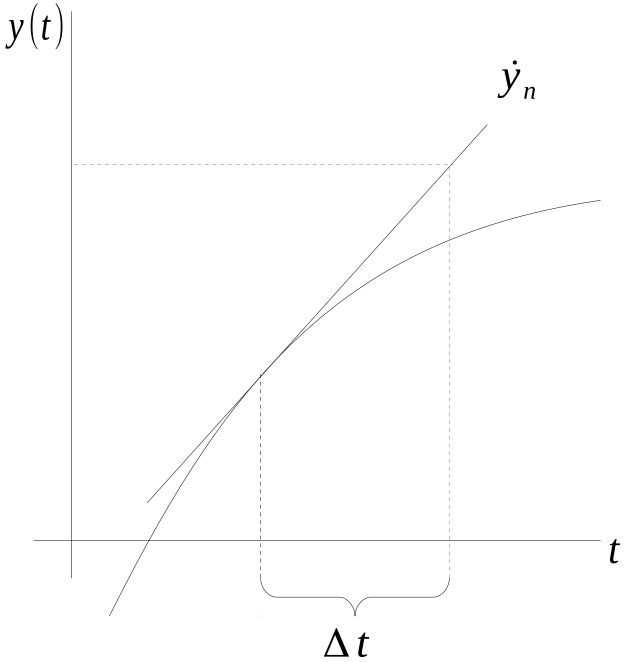
Explicit Discretization

- explicit Euler
- method of Heun
- Runge-Kutta

➢ $$y_{n+1} = F(y_n, t_n, \Delta t)$$

Explicit Discretization

- explicit Euler



$$y_{n+1} = y_n + \Delta t \cdot \dot{y}_n$$

$$= y_n + \Delta t \cdot f(t_n, y_n)$$

Explicit Discretization

- method of Heun

$$y_{n+1} = y_n + \Delta t \cdot \frac{1}{2}\left(\dot{y}_n + \dot{y}_{n+1}\right)$$

with

$$\dot{y}_{n+1} = f\left(t_{n+1}, y_n + \Delta t \cdot f\left(t_n, y_n\right)\right)$$

Explicit Discretization

- Runge-Kutta

$$y_{n+1} = y_n + \Delta t \cdot \frac{1}{6}\left(Y_1' + 2Y_2' + 2Y_3' + Y_4'\right)$$

with

$$Y_1' = f\left(t_n, y_n\right)$$

$$Y_2' = f\left(t_{n+\frac{1}{2}}, y_n + \frac{\Delta t}{2} \cdot Y_1'\right)$$

$$Y_3' = f\left(t_{n+\frac{1}{2}}, y_n + \frac{\Delta t}{2} \cdot Y_2'\right)$$

$$Y_4' = f\left(t_{n+1}, y_n + \Delta t \cdot Y_3'\right)$$

What is Efficiency?

- number of operations?
- runtime?
- accuracy?

What is Efficiency?

- number of operations?
- runtime?
- accuracy?

➢ relation accuracy/cost !!!!!

Accuracy

- definition of convergence:

$$\left\| u_{exact} - u_{approx} \right\| = O\left(dt^{p}\right)$$

- experimental computation?