

# Lab Course Scientific Computing

## Worksheet 4

distributed: 14.12.2016

due to: 08.01.2017, midnight (submitted on the Moodle page)

oral examination: 10.01.2017 (register for slots in Moodle)

In this worksheet, we switch to differential equations for functions with more than one independent variable, for example functions depending on several space dimensions. Such differential equations are called partial differential equations. We start with a first simple example, the two-dimensional stationary heat equation

$$T_{xx} + T_{yy} = -2\pi^2 \sin(\pi x) \sin(\pi y) \quad (1)$$

on the unit square  $]0; 1[^2$  with the temperature  $T(x, y)$ , the two-dimensional coordinates  $x$  and  $y$ , and homogeneous Dirichlet boundary conditions

$$T(x, y) = 0 \text{ for all } (x, y) \text{ in } \partial]0; 1[^2. \quad (2)$$

The boundary value problem (1),(2) has the analytical solution

$$T(x, y) = \sin(\pi x) \sin(\pi y). \quad (3)$$

- a) The discretization of (1) and (2) leads to a large system of linear equations for the values  $T_{i,j}, i \in \{1, \dots, N_x\}, j \in \{1, \dots, N_y\}$  denoting the approximate values of the temperature  $T$  at the discrete grid points  $\left(i \cdot \frac{1}{N_x+1}, j \cdot \frac{1}{N_y+1}\right)$ .

Use the finite difference approximation of the second derivatives

$$T_{xx}|_{i,j} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h_x^2},$$

$$T_{yy}|_{i,j} \approx \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{h_y^2}$$

with  $h_x = \frac{1}{N_x+1}$  and  $h_y = \frac{1}{N_y+1}$ .

Sketch a few lines of the resulting system in the following scheme

$$\begin{pmatrix} \vdots \\ T_{1,1} \\ T_{2,1} \\ \vdots \\ \vdots \\ \vdots \\ T_{N_x-1,N_y} \\ T_{N_x,N_y} \end{pmatrix} \begin{pmatrix} T_{1,1} \\ T_{2,1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T_{N_x-1,N_y} \\ T_{N_x,N_y} \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}.$$

- b) Implement a function creating the matrix from **a)** as a function of  $N_x$  and  $N_y$ .
- c) Implement a Gauss-Seidel solver for the system in **a)** as a function of the right hand side  $b$ ,  $N_x$ , and  $N_y$ . The output of the solver is a matrix with dimensions  $N_x + 2$  and  $N_y + 2$  containing the computed approximate values of  $T$  at the grid

points  $(i \cdot h_x, j \cdot h_y)$  in entry  $(i, j)$ . As a termination criterion for the iteration use an accuracy limit of  $10^{-4}$  for the residual norm.

**Remark 1:** Do not(!!!) use the explicit system matrix from **b)** in your Gauss-Seidel solver. Use your knowledge about the particular, constant form of the lines of the linear system to completely avoid any storage of matrix entries!

**Remark 2:** The residual norm for a general system  $Ax = b$  is defined as

$$R = \sqrt{\frac{1}{N} \sum_k \left( b_k - \sum_m a_{k,m} x_m \right)^2},$$

where  $N$  denotes the number of unknowns. In the Gauss-Seidel solver,  $N = N_x * N_y$ ,  $a_{i,j}$  are not stored (but known), and the entries of  $x$  are the temperature values  $T_{i,j}$  at the grid points  $(i \cdot h_x, j \cdot h_y)$ .

**d)** Solve the system from **a)**

- 1) storing the system matrix as a normal (full)  $N \times N$  matrix and using the MATLAB direct solver,
- 2) storing the system matrix as a sparse matrix and using the MATLAB direct solver,
- 3) without storing the system matrix (use Gauss-Seidel with zero as initial guess for  $T$ !).

**e)** Visualize the solutions as a

- 1) a coloured surface where the temperature represents the height of the surface,
- 2) a contour plot.

for  $N_x = N_y = 7, 15, 31, 63$ . Set the range of the temperature values so that the surface doesn't look flat. Think about a reasonable range.

**f)** Compare the runtimes and the storage requirements (**measured by the number of entries of the arrays and/or vectors needed**) for 1) – 3) in **d)** and for  $N_x = N_y = 7, 15, 31, 63$ :

direct solution with full matrix				
$N_x, N_y$	7	15	31	63
runtime				
storage				

direct solution with sparse matrix				
$N_x, N_y$	7	15	31	63
runtime				
storage				

iterative solution with Gauss-Seidel				
$N_x, N_y$	7	15	31	63
runtime				
storage				

g) Compute the solutions for  $N_x = N_y = 7, 15, 31, 63, 127$  with the Gauss-Seidel

solver and fill in the resulting errors

$$e = \sqrt{\frac{1}{N_x \cdot N_y} \sum_{j=1}^{N_y} \sum_{i=1}^{N_x} (T_{i,j} - T(x_i, y_j))^2}$$

in the following tabular:

$N_x = N_y$	7	15	31	63	127
error					
error red.	—				

### Questions:

- 1) How many non-zero entries do you achieve in the system matrix from **a)**?
- 2) Compare the number of non-zero entries to the number of entries of a full matrix with the same size. What conclusion concerning the methods to be used for the storage of the system matrix can you draw for increasing  $N_x$  and  $N_y$ ?
- 3) Which solver would you suggest to use for very big  $N_x$  and  $N_y$ ?
- 4) Use the results of **g)** to guess the convergence order of the discretization used for the Laplacian operator.