TECHNISCHE UNIVERSITÄT MÜNCHEN

Institut für Informatik

Informatik V
Chair for
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Scientific Computing Lab

Partial Differential Equations

Instationary Equations

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Instationary PDEs

- Stationary PDEs (time-independent):
  - Only spatial derivatives

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=rhs(x,y)$$

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- Stationary PDEs (time-independent):
  - Only spatial derivatives

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=rhs(x,y)$$

- Instationary PDEs (time-dependent):
  - Derivatives in space and time

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}$$

Discretization

- Spatial discretization
  - Finite differences
  - See worksheet 4
- Time discretization
  - Compare ODE
  - See worksheet 2 & 3

Explicit/Implicit

- explicit time steps
  - restricted time step sizes
  - dependent on spatial step
  - no equations to solve
- implicit time step
  - unrestricted time step sizes
  - large systems of equations

Accuracy Issues

- Discretization error in time and space

$$e = e_{space} + e_{time}$$

- High-order space disc. → e_{time} dominates
- High-order time disc. → e_{space} dominates

Accuracy Issues

- Take into account:
  - Order of discretization
  - Region of stability
  - Maximum allowed timestep size

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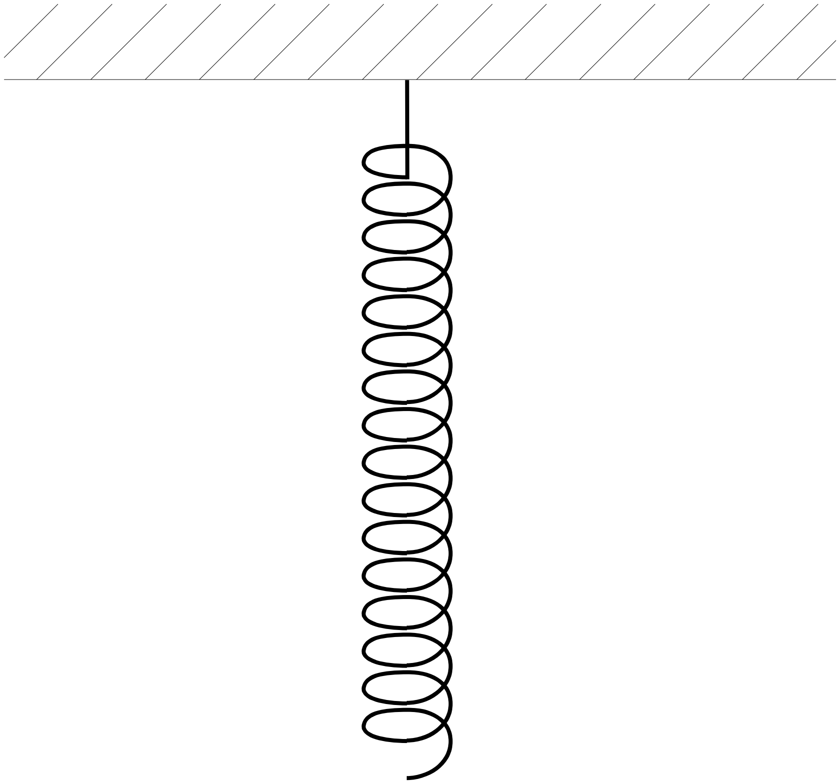
- Take into account:
  - Order of discretization
  - Region of stability
  - Maximum allowed timestep size
- Higher-order methods do not harm (in terms of accuracy; but be aware of comput. costs!)

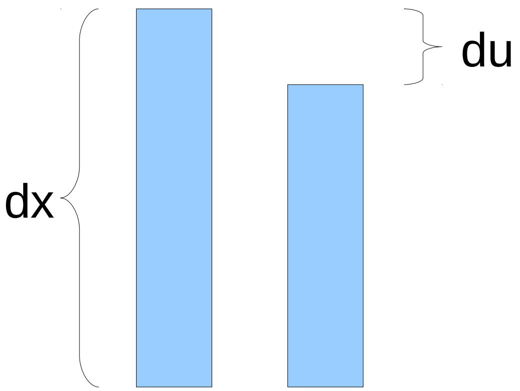
Explicit/Implicit

- Separation in space/time-discretization
  - Compute spatial derivatives
  - Consider function value on each grid-point as an ODE
- Sketch: 2.5 ways to a full discretization

Example



Example - Continuous



Hook's Law:

$$\sigma = E\,\varepsilon$$

Normal tension:

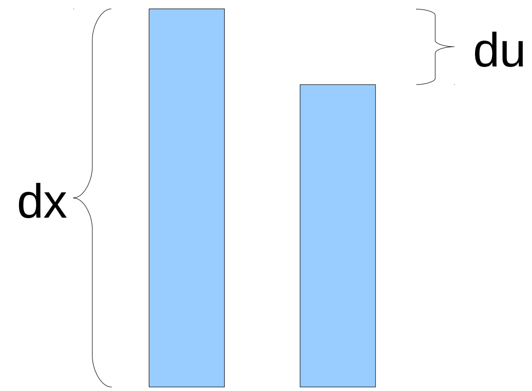
$$\sigma = \frac{F_u}{A} \quad \text{with} \quad \varepsilon = \frac{du}{dx} \rightarrow F_u = E\,A\,\frac{du}{dx}$$

Newton's Law:

$$dF_u = dm\,\frac{d^2u}{dt^2} \quad \text{with} \quad dm = \rho\,A\,dx \rightarrow \frac{dF_u}{dx} = \rho\,A\,\frac{d^2u}{dt^2}$$

Example - Continuous

$$\frac{dF_u}{dx} = \rho A \frac{d^2 u}{dt^2}$$ and $$F_u = E A \frac{du}{dx}$$



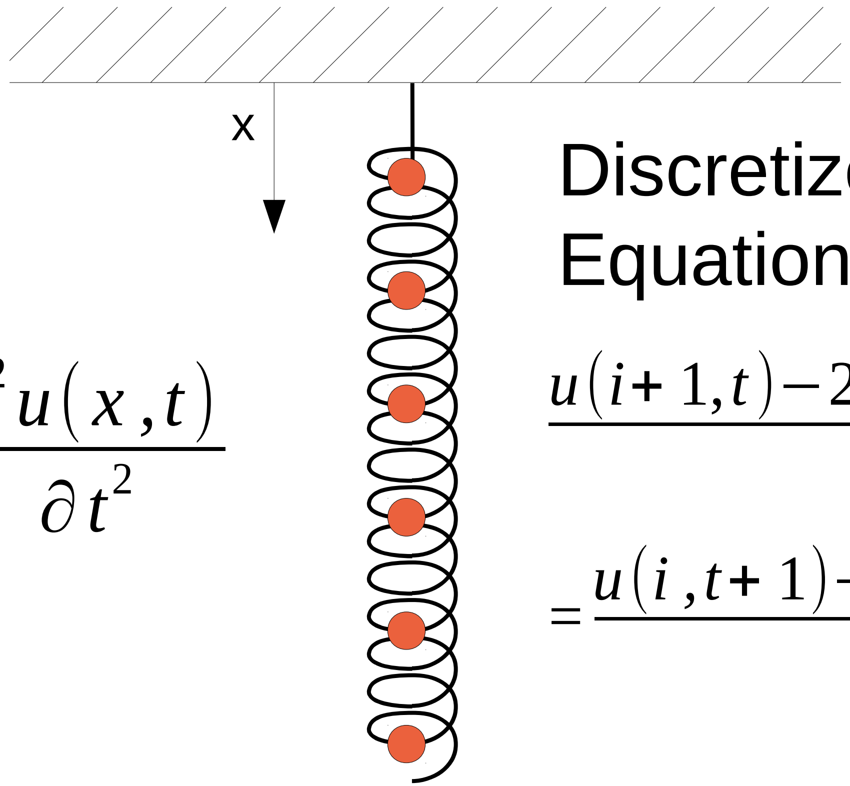
Differentiate with regard to x:

$$E A \frac{\partial^2 u}{dx^2} = \rho A \frac{\partial^2 u}{\partial t^2}$$

$$\rightarrow \quad E \frac{\partial^2 u}{dx^2} = \rho \frac{\partial^2 u}{\partial t^2}$$

In the following, we set $E = 1, \rho = 1$

Example - Discretization



Continuous Equation:

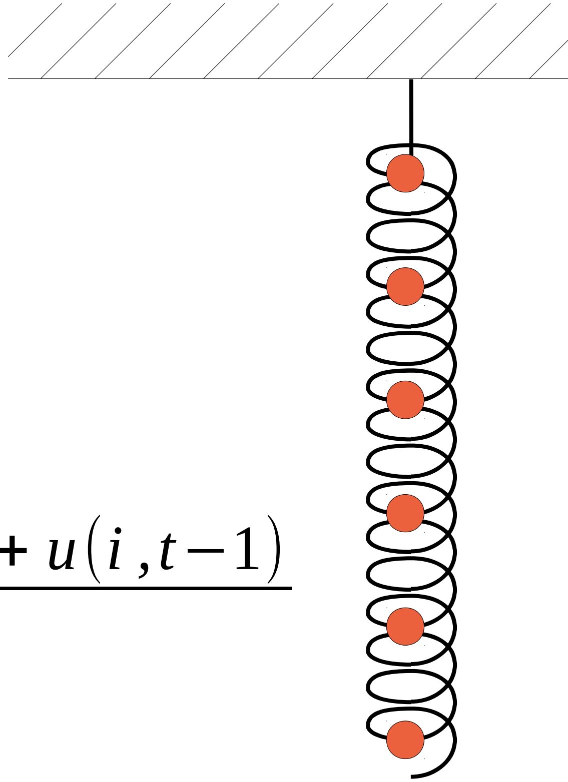
$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

Discretized Equation:

$$\frac{u(i+1,t) - 2u(i,t) + u(i-1,t)}{h_0 \cdot h_1}$$

$$= \frac{u(i,t+1) - 2\mathrm{u}(i,t) + u(i,t-1)}{\delta t^2}$$

Example - Discretization

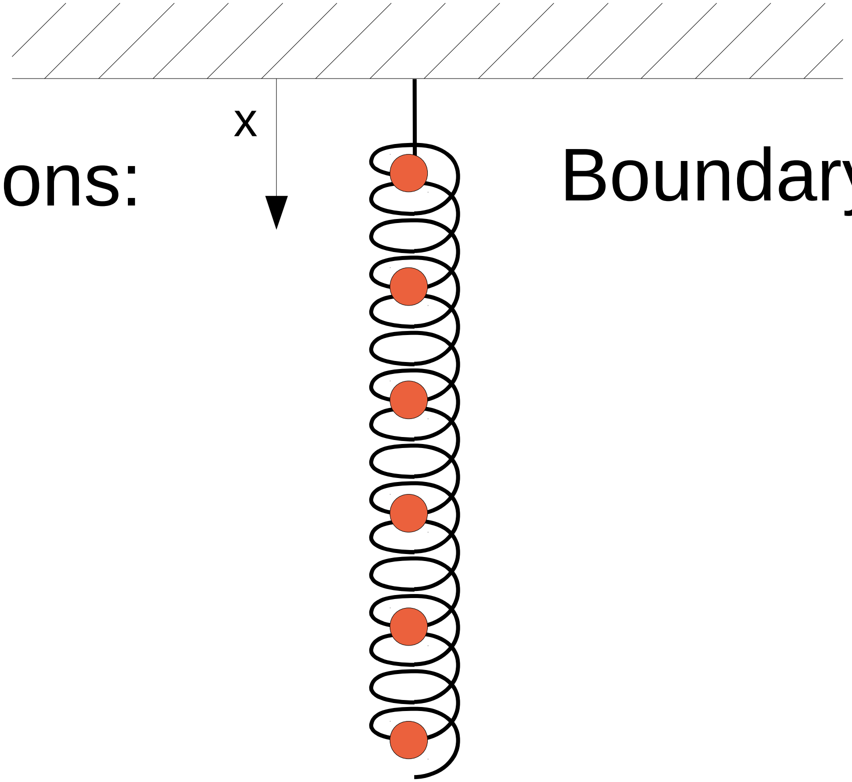


Discretized Equation:

$$\frac{u(i+1,t)-2u(i,t)+u(i-1,t)}{h_0 \cdot h_1}=\frac{u(i,t+1)-2\mathrm{u}(i,t)+u(i,t-1)}{\delta t^2}$$

$$u(i,t+1)=\frac{\delta t^2}{h_0 \cdot h_1}\cdot\left(u(i+1,t)-2u(i,t)+u(i-1,t)\right)+2\mathrm{u}(i,t)-u(i,t-1)$$

Example – Initial/Boundary Conditions



Initial Conditions:

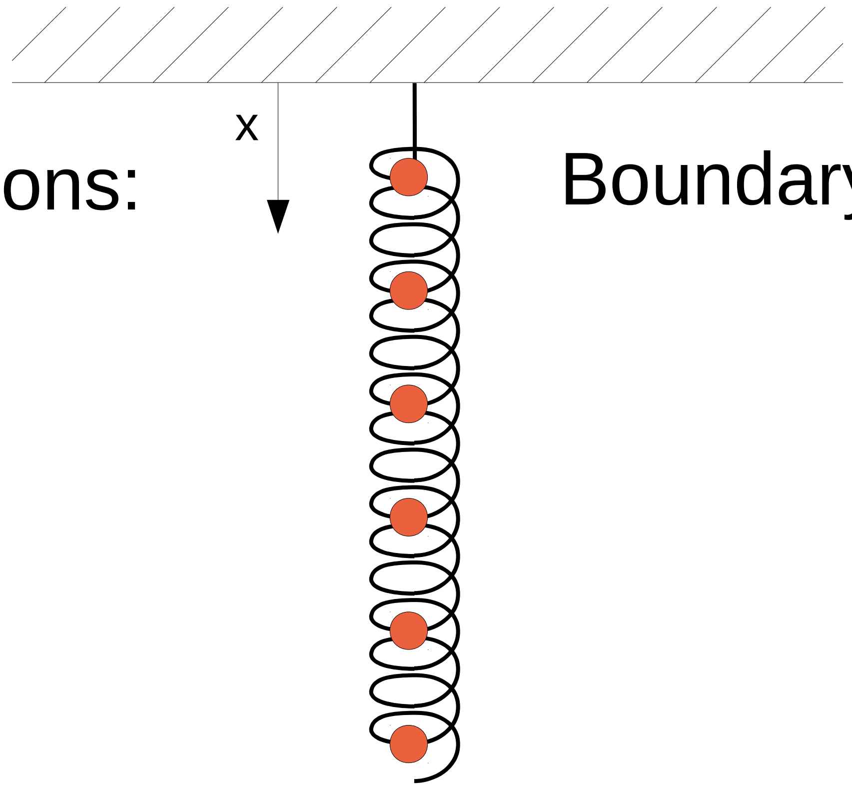
Boundary Conditions:

Example – Initial/Boundary Conditions

Initial Conditions:

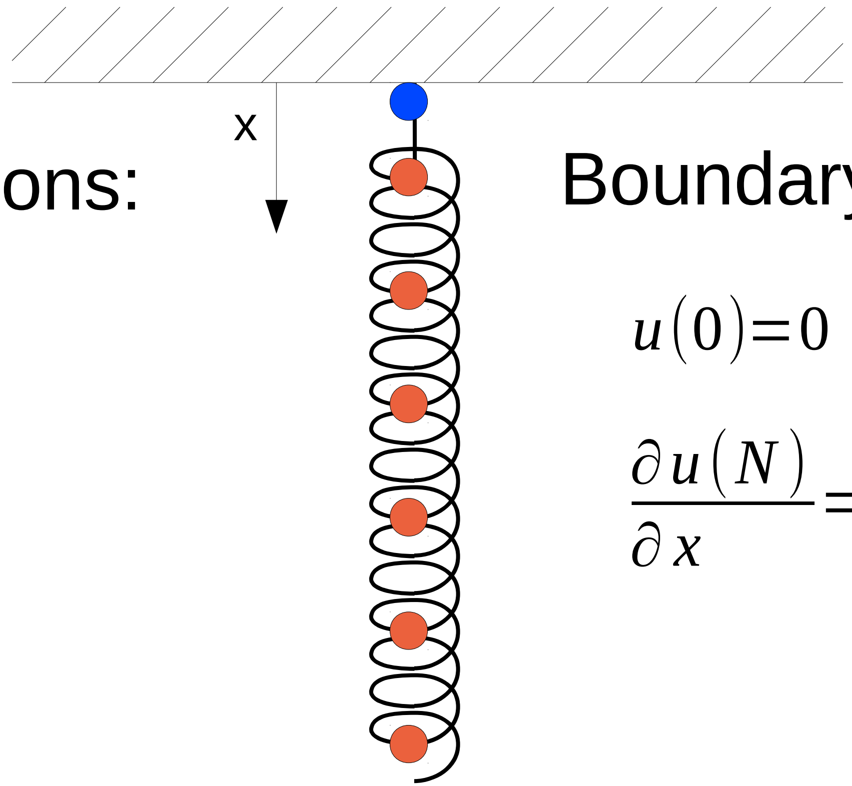
$$\begin{pmatrix} u(1) \\ \vdots \\ u(N) \end{pmatrix}$$

$$\begin{pmatrix} v(1) \\ \vdots \\ v(N) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$



Boundary Conditions:

Example – Initial/Boundary Conditions



Initial Conditions:

$$\begin{pmatrix} u(1) \\ \vdots \\ u(N) \end{pmatrix}$$

$$\begin{pmatrix} v(1) \\ \vdots \\ v(N) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Boundary Conditions:

$$u(0) = 0$$

$$\frac{\partial u(N)}{\partial x} = 0$$