TECHNISCHE UNIVERSITÄT MÜNCHEN

Institut für Informatik

Informatik V
Chair for
Scientific Computing

Scientific Computing Lab

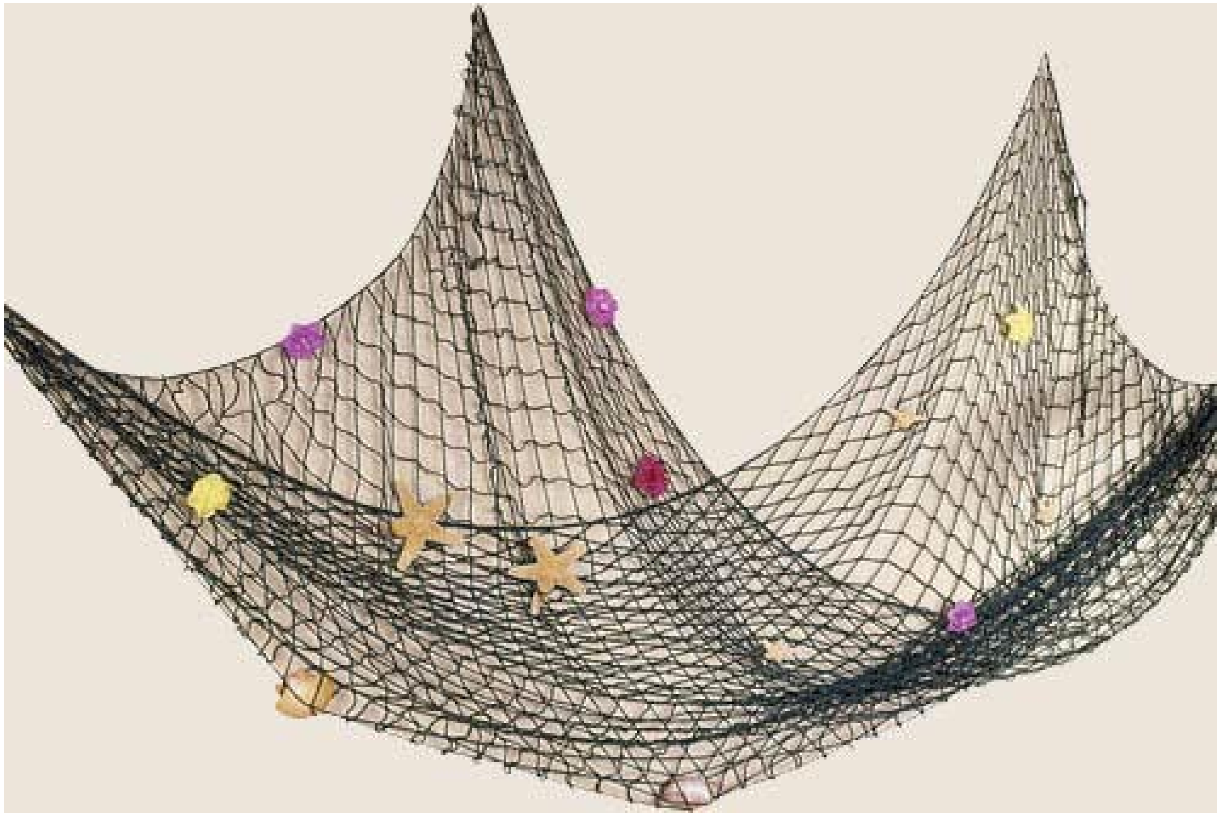
Stationary

Partial Differential Equations

Tobias Neckel

Stationary Partial Differential Equations

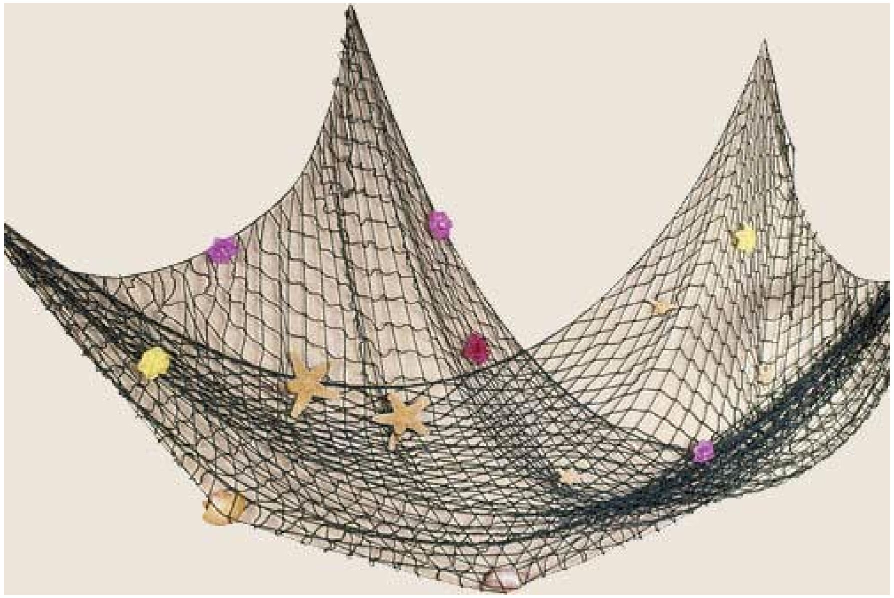
How to compute this?



http://orionwell.files.wordpress.com/2007/05/fish_net.jpg

Stationary Partial Differential Equations

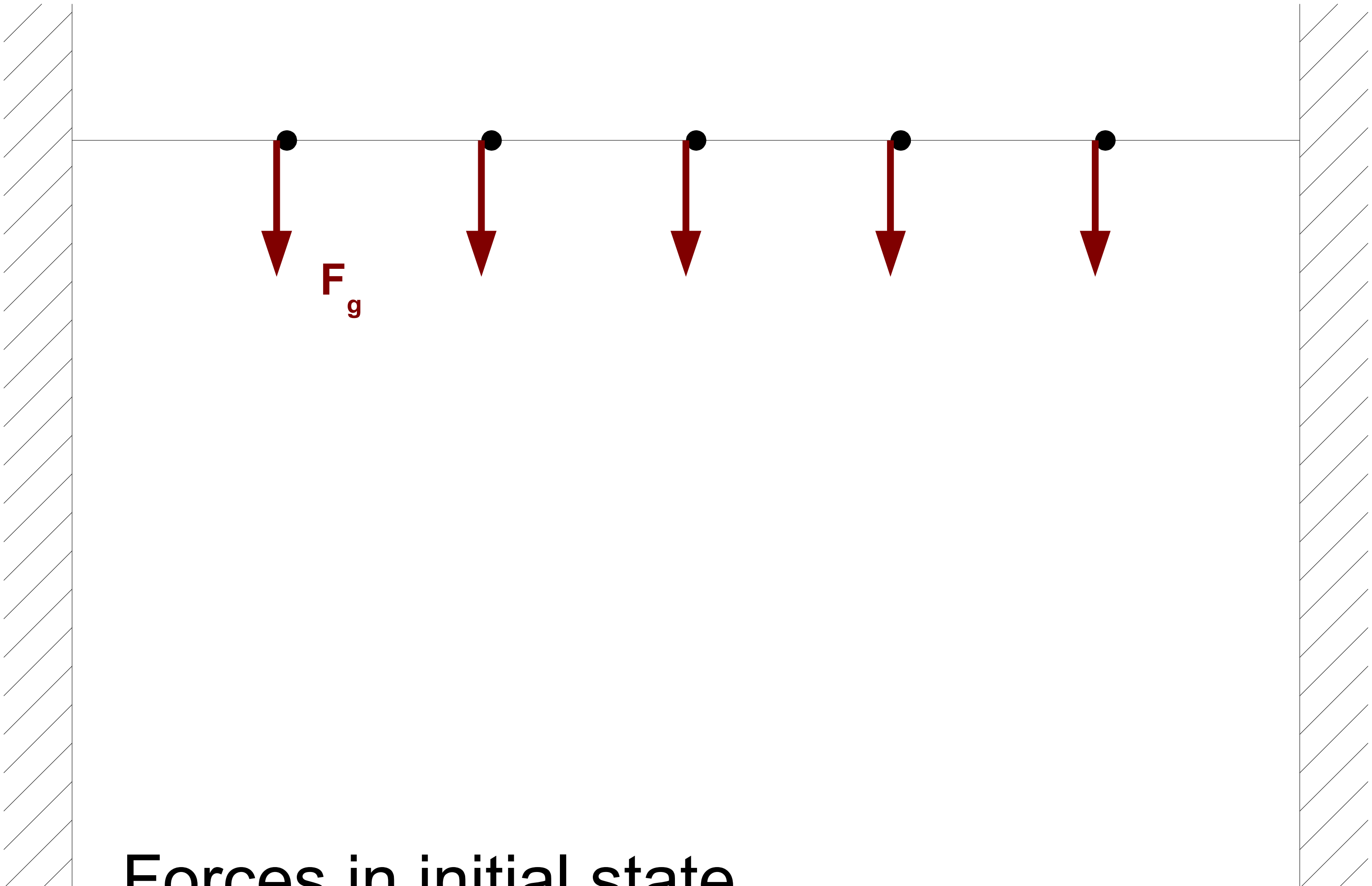
What do we want to have?



http://orionwell.files.wordpress.com/2007/05/fish_net.jpg

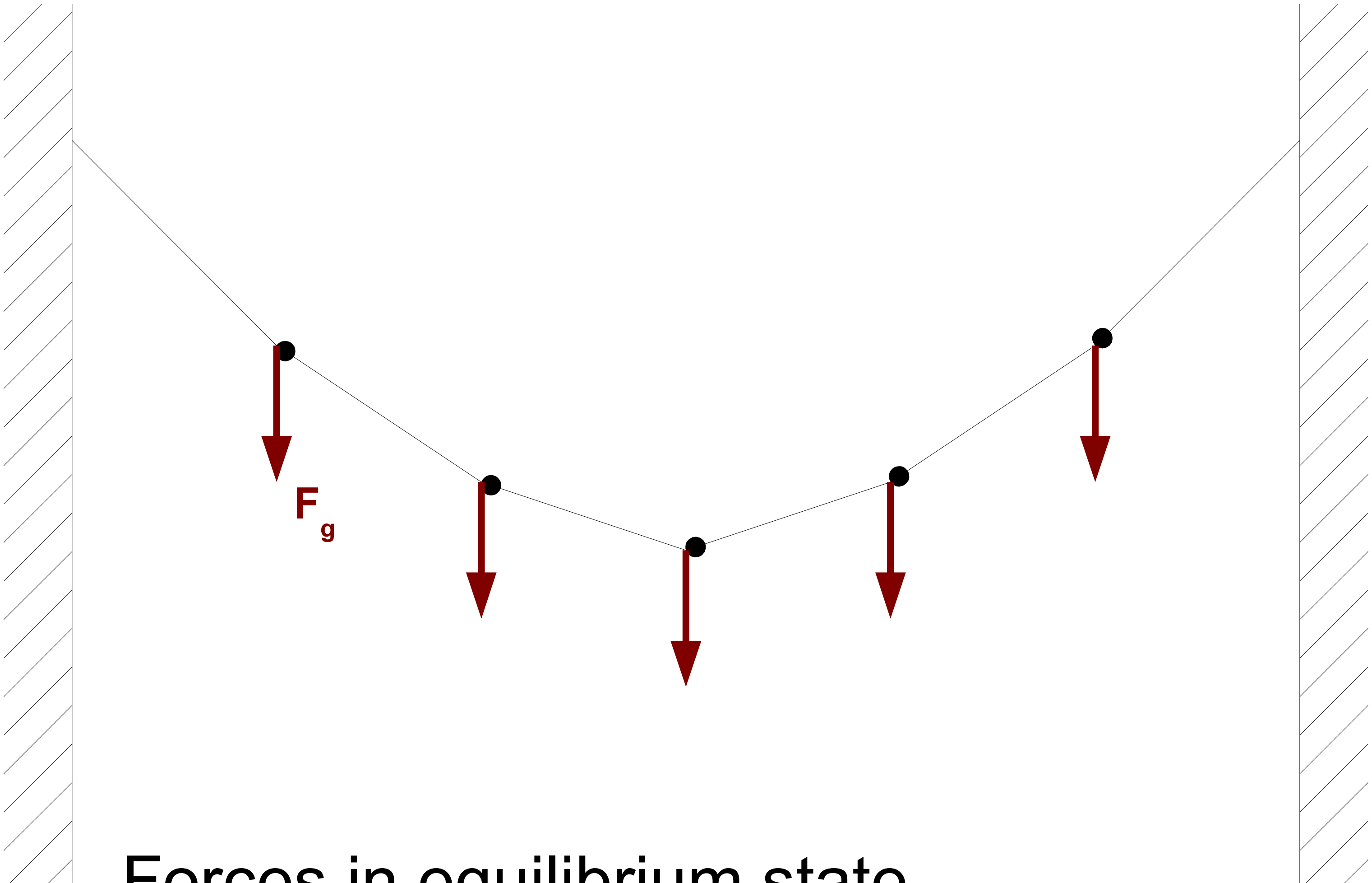
Function ***f(x,y)*** that describes the displacement for each joint ***(x,y)***.

Stationary Partial Differential Equations



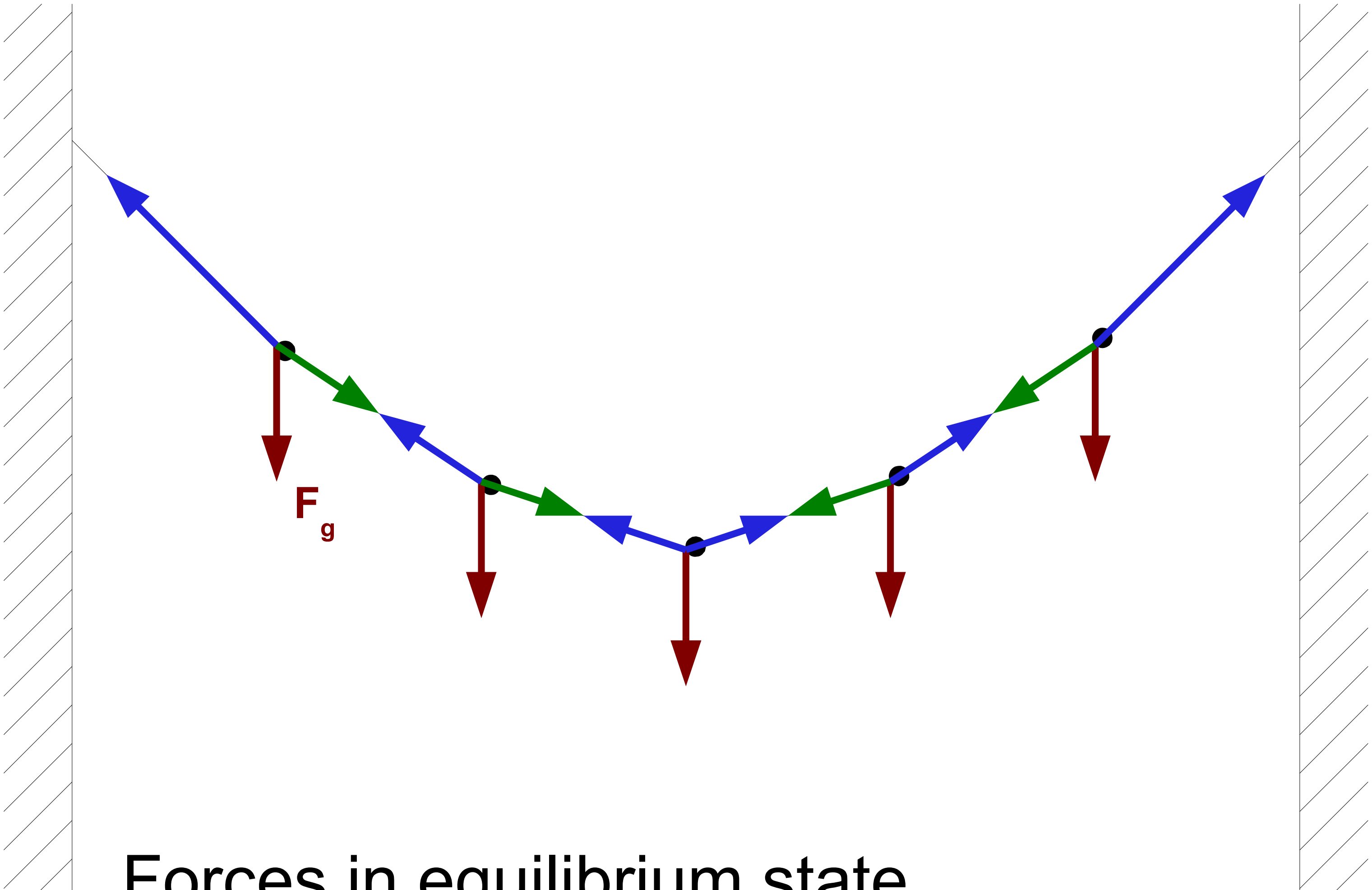
Forces in initial state

Stationary Partial Differential Equations



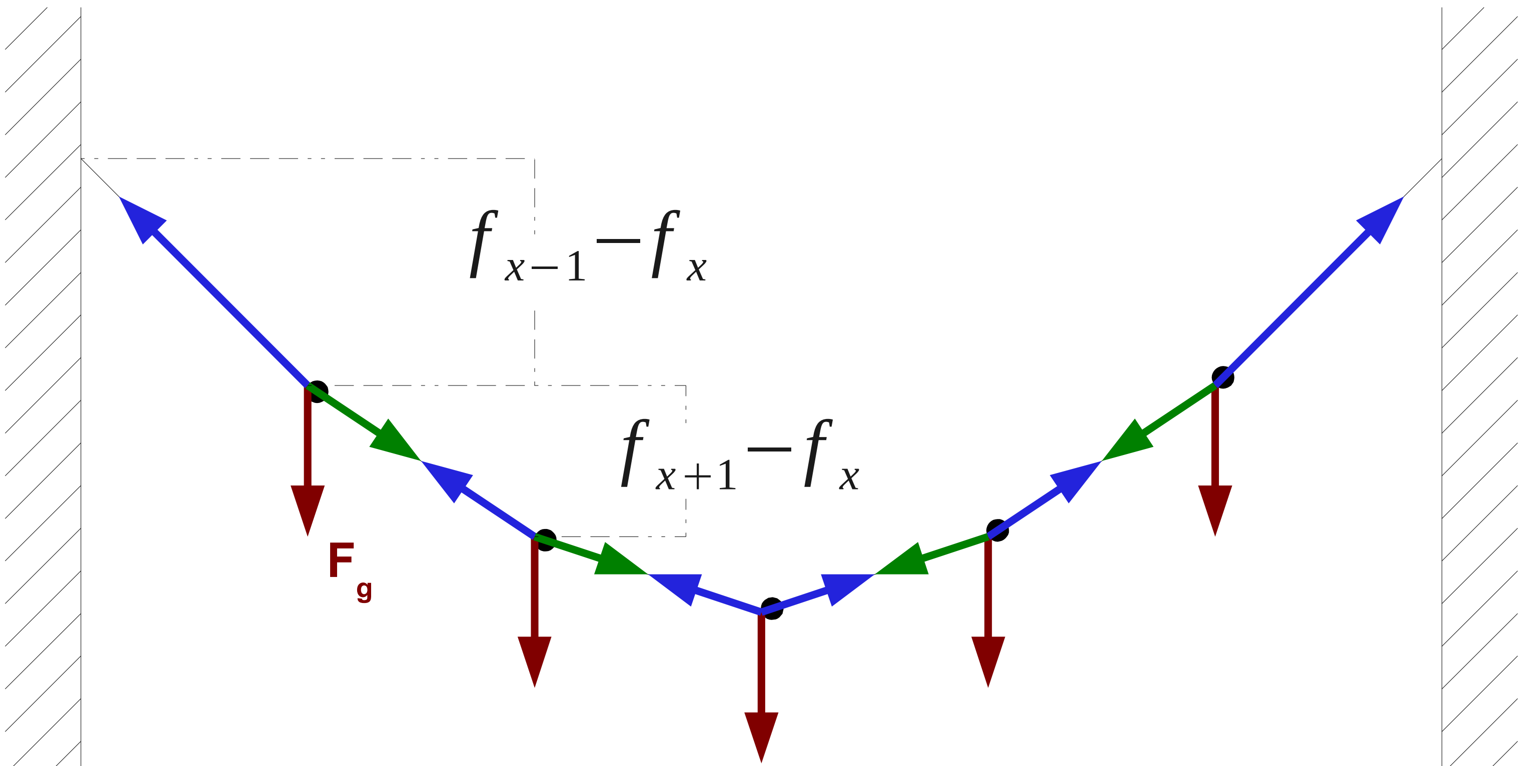
Forces in equilibrium state

Stationary Partial Differential Equations



Forces in equilibrium state

Stationary Partial Differential Equations

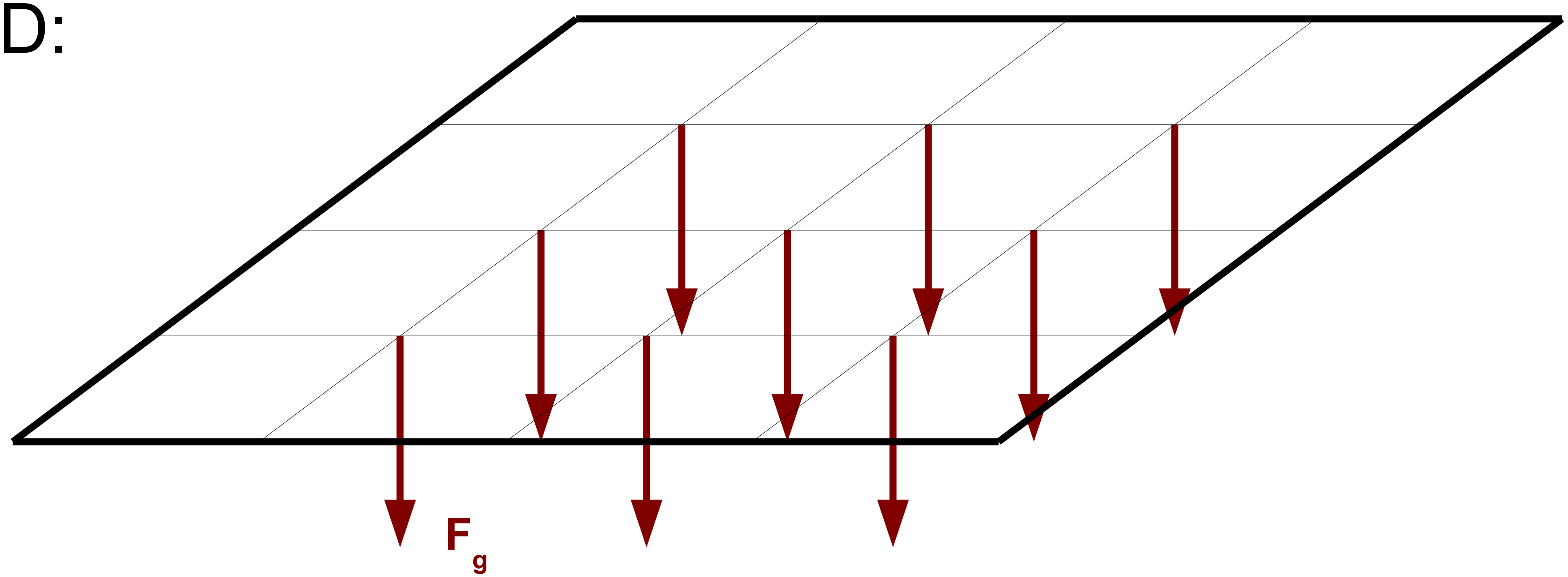


$$F_x = -F_g + f_{x-1} - f_x + f_{x+1} - f_x$$
$$F_x = -F_g + f_{x-1} - 2 \cdot f_x + f_{x+1}$$

Stationary Partial Differential Equations

1D: $F_x = -F_g + f_{x-1} - 2 \cdot f_x + f_{x+1}$

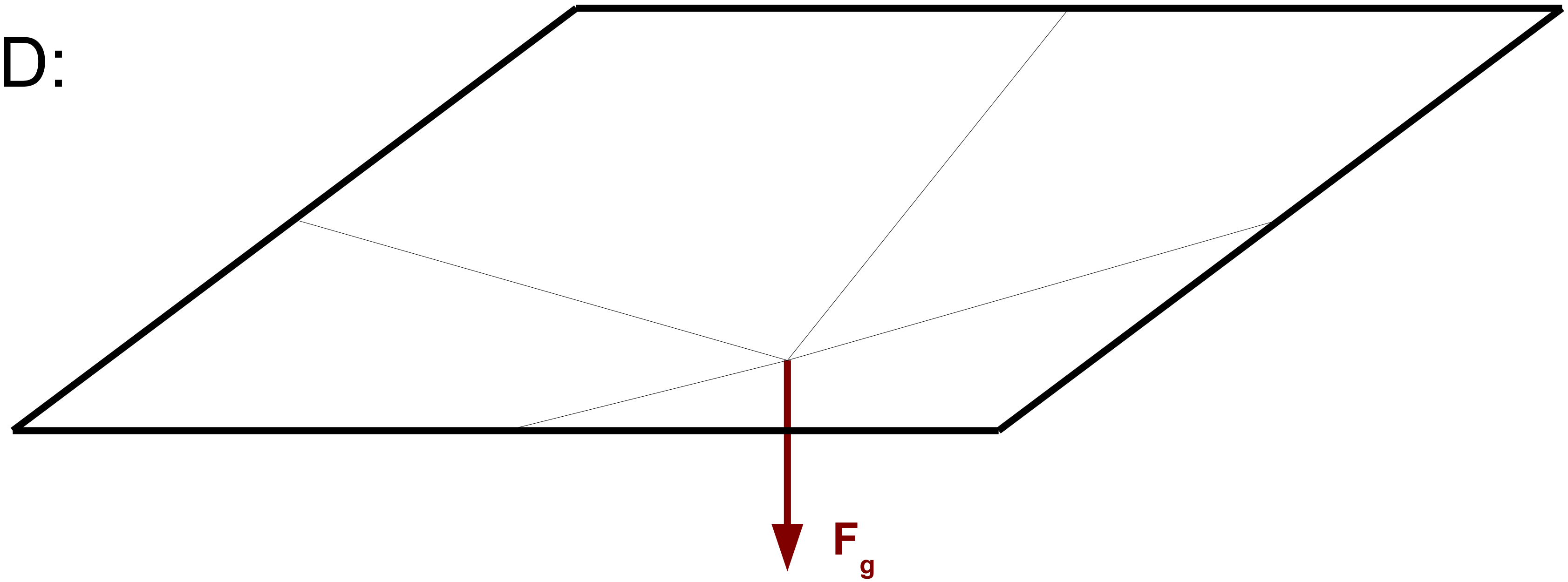
2D:



What is $F_{x,y}$? What is $f_{x,y}$?

Stationary Partial Differential Equations

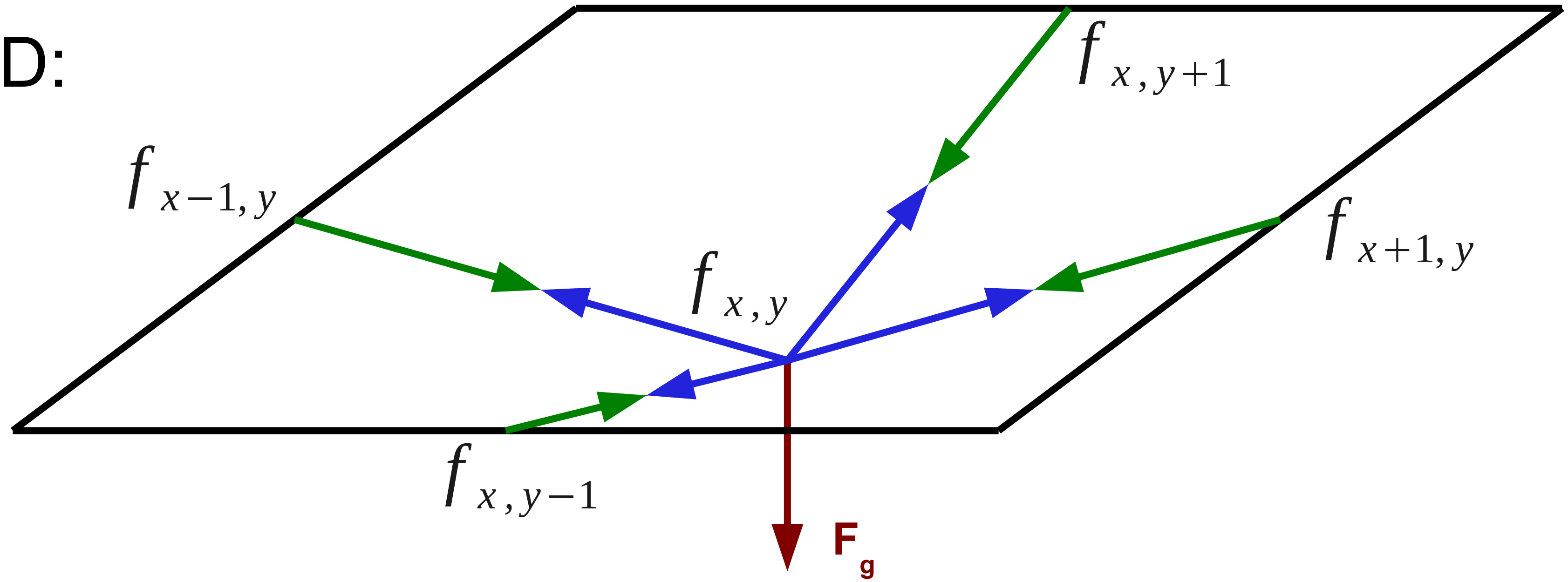
2D:



What is $F_{x,y}$? What is $f_{x,y}$?

Stationary Partial Differential Equations

2D:



$$F_{x,y} = -F_g + f_{x-1,y} + f_{x,y-1} - 4 \cdot f_{x,y} + f_{x+1,y} + f_{x,y+1}$$

Stationary Partial Differential Equations

What are Partial Differential Equations (PDEs)?

- Dependent on several variables
- Containing partial derivatives
- boundary value problems
  - no start and end (for stationary problems)

Discretization

What is discretization?
→ Solving the function just on some points

Why do we discretize?

Discretization

What is discretization?
→ Solving the function just on some points

Why do we discretize?
→ Continous form hard to store in the computer
→ Analytical solving often not possible

Discretization

- space $\Longrightarrow$ grid
- functions
- operators

finite difference/volume/element

Discretization

- space $\Longrightarrow$ grid
- functions
- operators

$\left.\begin{array}{l}\text{functions}\\ \text{operators}\end{array}\right\}$ **finite difference**/volume/element

- How to solve this?
- $F_{x,y} = -F_g + f_{x-1,y} + f_{x,y-1} - 4 \cdot f_{x,y} + f_{x+1,y} + f_{x,y+1}$
- In the equilibrium:

$$f_{x-1,y} + f_{x,y-1} - 4 \cdot f_{x,y} + f_{x+1,y} + f_{x,y+1} = F_g$$

$$-F_g + f_{x-1,y} + f_{x,y-1} - 4 \cdot f_{x,y} + f_{x+1,y} + f_{x,y+1} = 0$$

Discretization

$$f_{x-1,y}+f_{x,y-1}-4\cdot f_{x,y}+f_{x+1,y}+f_{x,y+1}=F_g$$

How to transform this to the following form?

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\cdot\begin{pmatrix} f_{0,0} \\ f_{1,0} \\ \vdots \\ f_{N,N} \end{pmatrix}=\begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Discretization

$$f_{x-1,y}+f_{x,y-1}-4\cdot f_{x,y}+f_{x+1,y}+f_{x,y+1}=F_g$$

Applied on each point (x,y) this gives

$$\begin{pmatrix} & & & \cdots & & & & \\ 1 & \cdots & 1 & -4 & 1 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & \cdots & & & & \end{pmatrix} . \begin{pmatrix} f_{0,0} \\ f_{1,0} \\ \vdots \\ f_{N,N} \end{pmatrix} = \begin{pmatrix} F_g \\ F_g \\ \vdots \\ F_g \end{pmatrix}$$

→ Linear System of Equations

Solvers for Systems of Linear Equations

What solving strategies do you know?

Solvers for Systems of Linear Equations

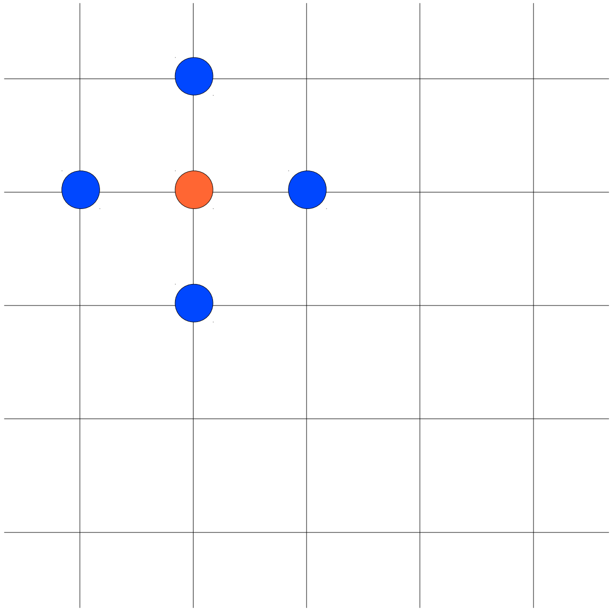
What solving strategies do you know?

- Direct solvers
  - Full matrix
  - Sparse matrix
- Iterative solvers
  - Jacobi
  - Gauß-Seidel
  - ...

Gauß-Seidel Solver

- point-by-point processing
- eliminate local residual

$$f_{x-1,y}+f_{x,y-1}-4\cdot f_{x,y}+f_{x+1,y}+f_{x,y+1}=F_g$$



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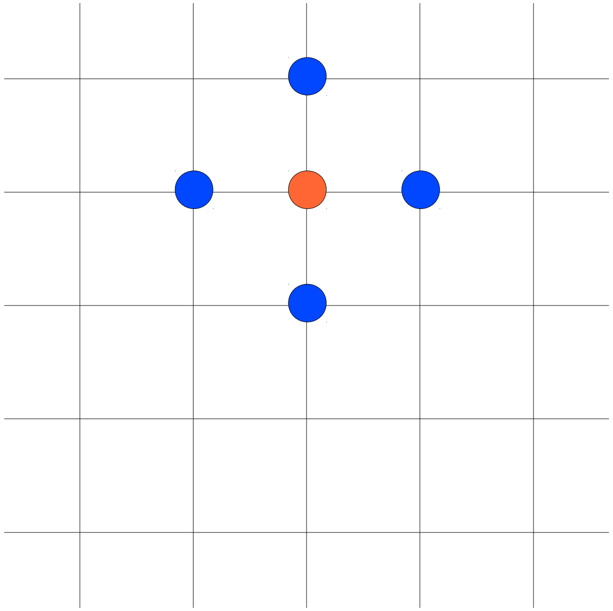

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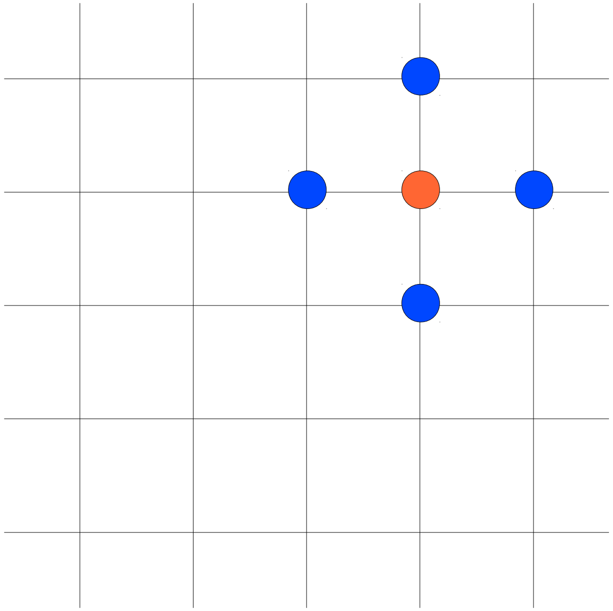

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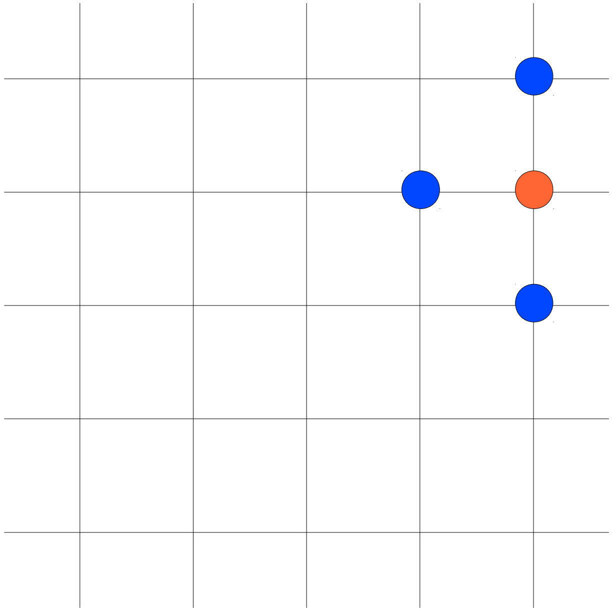

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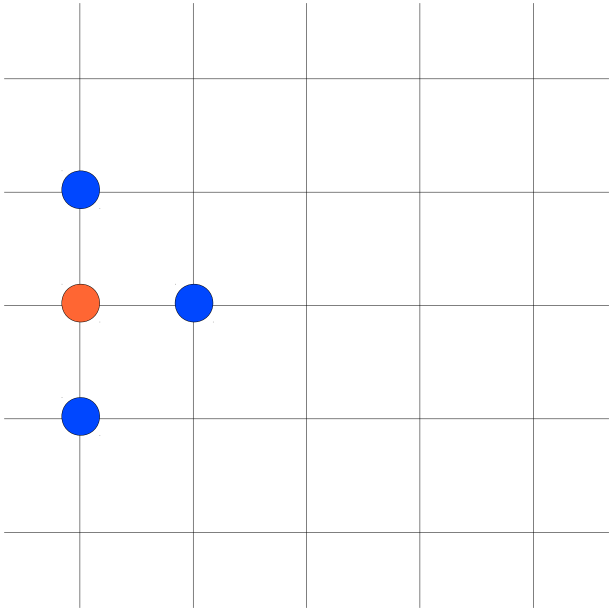

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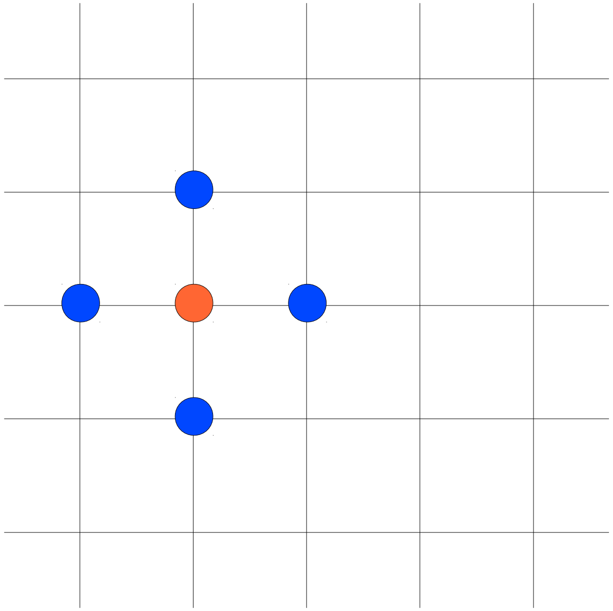

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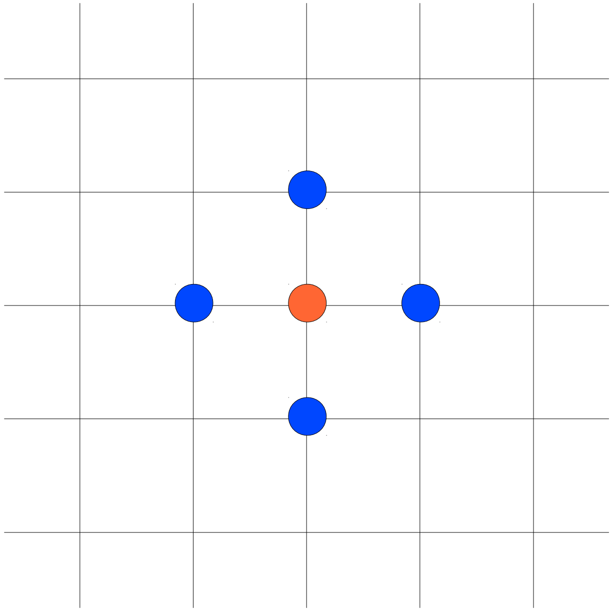

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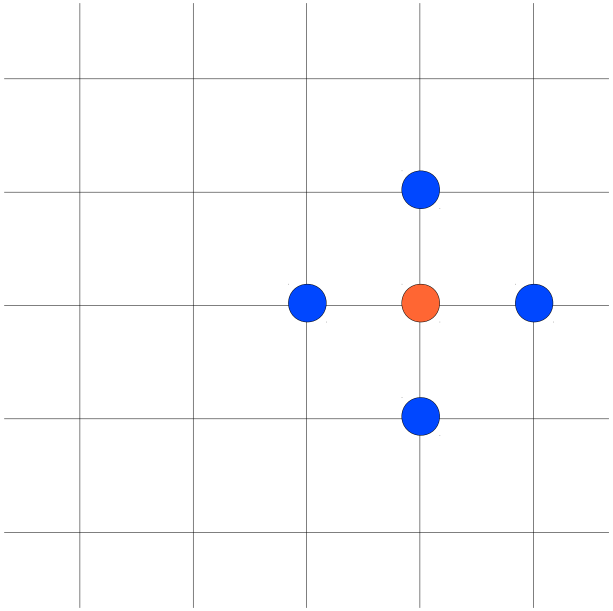

Gauß-Seidel Solver

- point-by-point processing
- eliminate local residual
- elimination for one point increases residual of neighbouring points

$\longrightarrow$ Overall residual is smaller than before

$\longrightarrow$ Iterate until desired accuracy

Stationary Partial Differential Equations

What did we learn today?

- Formulation of Partial Differential Equations
- Discretization by Finite Differences
- Transforming to an LSE
- Solving the LSE by direct or iterative solvers