Vanishing results

Alexander R. Miller

Part 1. Character vanishing

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	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
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Theorem (M.) If $\chi \in \mathrm{Irr}(S_n)$ and $g \in S_n$ are chosen uniformly at random, then $\chi(g) = 0$ with probability $\to 1$ as $n \to \infty$.

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Lemma (M.) For any $\mathcal{K} \subseteq \mathrm{Cl}(G)$,

$$\mathrm{Prob}(\chi(g) = 0) \geq \frac{|\{g \in G : g^G \in \mathcal{K}\}|}{|G|} - \frac{|\mathcal{K}|}{|\mathrm{Cl}(G)|}.$$

$\mathrm{Prob}(\chi(g) = 0)$ and $\mathrm{Prob}(\chi(K) = 0)$ for some other groups

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Lemma (Gallagher–Larsen–M.) For each finite group G and $\epsilon > 0$,

$$\mathrm{Prob}(\chi(g) \neq 0) \leq \frac{\left|\left\{(\chi, g) : \mathsf{gcd}(\chi(1), |g^G|) \geq \epsilon\chi(1)\right\}\right|}{|\mathrm{Irr}(G) \times G|} + \epsilon^2.$$

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Theorem (Gallagher–Larsen–M.) For $G = \mathrm{GL}(n, q)$, the proportion $P_{n,q}$ of pairs $(\chi, g) \in \mathrm{Irr}(G) \times G$ with $\chi(g) = 0$ satisfies

$$\inf_q P_{n,q} \to 1 \text{ as } n \to \infty.$$

So for any sequence of prime powers $q_1, q_2, \ldots$, we have $P_{n,q_n} \to 1$ as $n \to \infty$.

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So for any sequence of prime powers $q_1, q_2, \ldots$, we have $P_{n,q_n} \to 1$ as $n \to \infty$.

Theorem (Larsen–M.) If G_n is any sequence of finite simple groups of Lie type with rank tending to ∞, then almost every entry in the character table of G_n is zero as n tends to ∞.

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Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

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3	0.1111
4	

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n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
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3	0.1111
4	0.1600

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4	0.1600
5	0.2041

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...	...
37	0.3642
38	0.3659

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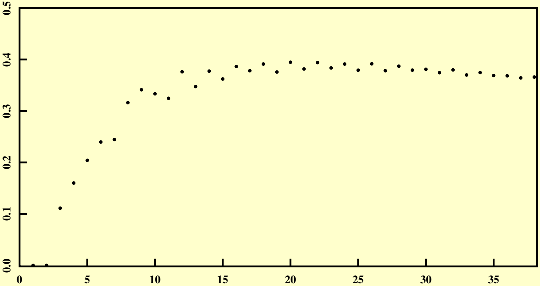

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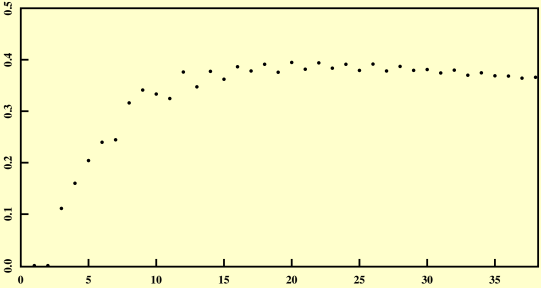

Best known bound $\mathrm{Prob}(\chi_\lambda(\mu) = 0) \geq C/\log n$

Part 2.

Part 2. Modular vanishing for S_n

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n
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$

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n	4
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24

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n	4	5
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n	4	5	6
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n	4	5	6	7
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40

Part 2. Modular vanishing for S_n

n	4	5	6	7	8
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55

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$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55	

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$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55		.87

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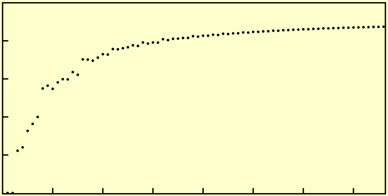

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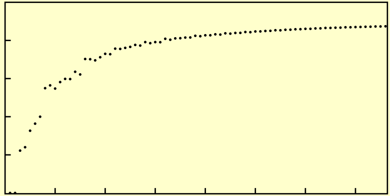

- The character table of S_{76} has about 86 trillion entries.

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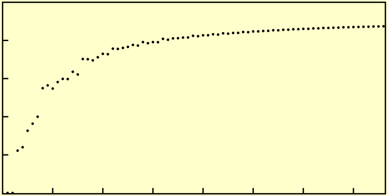

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- Several months of computations on super computers used for earth science.

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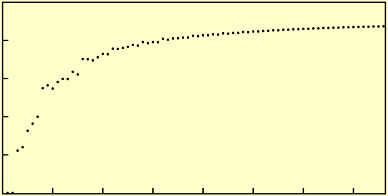
n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \text{ mod } 2)$	.24	.33	.36	.40	.55	.56	.55		.87



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- Similar computations for other primes and prime powers.

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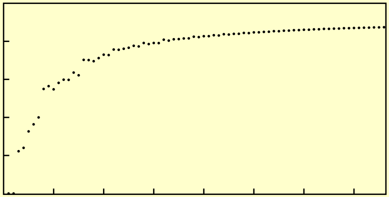

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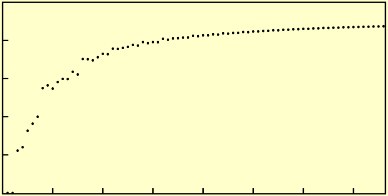
Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55		.87



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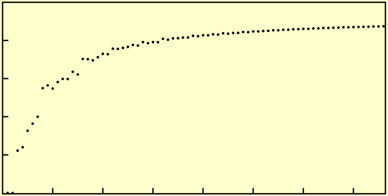

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$\mathrm{Prob}(\lambda$ is an $\mathfrak{l}_2(\mu)$-core$)$	.45	.44	.45	.47	.48	.48	.48	.47	.47	.46

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- Progress is being made on prime powers by various groups.

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Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

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In particular, the set $\{P(G) : |G| < \infty\}$ is dense in $[0,1]$.

Part 4. Zeros and roots of unity

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	1	6	3	8	6
	id	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

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		5/5				

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		5/5		5/5	5/5	

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		5/5		5/5	5/5	

Definition

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{\#\{g \in G : \chi(g) \text{ is 0 or a root of unity}\}}{|G|}$$

$$\theta'(G) = \min_{\text{L.T.A. } K} \frac{\#\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is 0 or a root of unity}\}}{|\mathrm{Cl}(G)|}$$

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Example $\theta(S_4) = 20/24$

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Theorem (J.G. Thompson) $\theta(G) > 1/3$.

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Theorem (P.X. Gallagher) $\theta'(G) > 1/3$.

Theorem (C.L. Siegel) For totally positive $\alpha \neq 0, 1$, $\tilde{\mathrm{Tr}}(\alpha) \geq 3/2$.
(Siegel gets applied to $\alpha = |\chi(g)|^2$. Burnside used $\tilde{\mathrm{Tr}}(\alpha) \geq 1$ for $\alpha \neq 0$.)

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Theorem (M.) Let G be finite nilpotent, let $\chi \in \mathrm{Irr}(G)$, and let $g \in G$. Then

$$\chi(g) = 0 \quad \text{or} \quad \tilde{\mathrm{Tr}}\left(|\chi(g)|^2\right) \geq 2^{\#\{\text{primes dividing } \chi(1)\}}$$

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Conjecture (M.) For $G_{p,n} \in \mathrm{Syl}_p(S_n)$, we have $P(G_{p,n}) \to 1$ as $n \to \infty$.