Vanishing results

Alexander R. Miller

Part 1. Character vanishing

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

***S*$_4$**

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1.

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0)$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

***S*₄**

S_4	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) =$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|}$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} =$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

2.

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

2. Choose $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, and then evaluate $\chi(K)$.

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

2. Choose $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, and then evaluate $\chi(K)$.

$$\mathrm{Prob}(\chi(K) = 0)$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

2. Choose $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, and then evaluate $\chi(K)$.

$$\mathrm{Prob}(\chi(K) = 0) =$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

2. Choose $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, and then evaluate $\chi(K)$.

$$\mathrm{Prob}(\chi(K) = 0) = \frac{|\{(\chi, K) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(K) = 0\}|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|}$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

2. Choose $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, and then evaluate $\chi(K)$.

$$\mathrm{Prob}(\chi(K) = 0) = \frac{|\{(\chi, K) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(K) = 0\}|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|} =$$

Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

2. Choose $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, and then evaluate $\chi(K)$.

$$\mathrm{Prob}(\chi(K) = 0) = \frac{|\{(\chi, K) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(K) = 0\}|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|} = \frac{4}{5^2} = 0.16$$

$\mathrm{Prob}(\chi(g) = 0)$ for S_n

$\mathrm{Prob}(\chi(g) = 0)$ for S_n

Theorem (M.) If $\chi \in \mathrm{Irr}(S_n)$ and $g \in S_n$ are chosen uniformly at random, then $\chi(g) = 0$ with probability $\to 1$ as $n \to \infty$.

$\mathrm{Prob}(\chi(g) = 0)$ for S_n

Theorem (M.) If $\chi \in \mathrm{Irr}(S_n)$ and $g \in S_n$ are chosen uniformly at random, then $\chi(g) = 0$ with probability $\to 1$ as $n \to \infty$.

One reason A vanishingly small proportion of classes covers almost all of S_n.

$\mathrm{Prob}(\chi(g) = 0)$ for S_n

Theorem (M.) If $\chi \in \mathrm{Irr}(S_n)$ and $g \in S_n$ are chosen uniformly at random, then $\chi(g) = 0$ with probability $\to 1$ as $n \to \infty$.

One reason A vanishingly small proportion of classes covers almost all of S_n.

Lemma (M.) For any $\mathcal{K} \subseteq \mathrm{Cl}(G)$,

$$\mathrm{Prob}(\chi(g) = 0) \geq \frac{|\{g \in G : g^G \in \mathcal{K}\}|}{|G|} - \frac{|\mathcal{K}|}{|\mathrm{Cl}(G)|}.$$

$\mathrm{Prob}(\chi(g) = 0)$ and $\mathrm{Prob}(\chi(K) = 0)$ for some other groups

$\mathrm{Prob}(\chi(g) = 0)$ and $\mathrm{Prob}(\chi(K) = 0)$ for some other groups

Lemma (Gallagher–Larsen–M.) For each finite group G and $\epsilon > 0$,

$$\mathrm{Prob}(\chi(g) \neq 0) \leq \frac{\left|\left\{(\chi, g) : \mathsf{gcd}(\chi(1), |g^G|) \geq \epsilon\chi(1)\right\}\right|}{|\mathrm{Irr}(G) \times G|} + \epsilon^2.$$

$\mathrm{Prob}(\chi(g) = 0)$ and $\mathrm{Prob}(\chi(K) = 0)$ for some other groups

Lemma (Gallagher–Larsen–M.) For each finite group G and $\epsilon > 0$,

$$\mathrm{Prob}(\chi(g) \neq 0) \leq \frac{\left|\left\{(\chi, g) : \gcd(\chi(1), |g^G|) \geq \epsilon\chi(1)\right\}\right|}{|\mathrm{Irr}(G) \times G|} + \epsilon^2.$$

Theorem (Gallagher–Larsen–M.) For $G = \mathrm{GL}(n, q)$, the proportion $P_{n,q}$ of pairs $(\chi, g) \in \mathrm{Irr}(G) \times G$ with $\chi(g) = 0$ satisfies

$$\inf_q P_{n,q} \to 1 \text{ as } n \to \infty.$$

So for any sequence of prime powers $q_1, q_2, \ldots$, we have $P_{n,q_n} \to 1$ as $n \to \infty$.

$\mathrm{Prob}(\chi(g) = 0)$ and $\mathrm{Prob}(\chi(K) = 0)$ for some other groups

Lemma (Gallagher–Larsen–M.) For each finite group G and $\epsilon > 0$,

$$\mathrm{Prob}(\chi(g) \neq 0) \leq \frac{\left|\{(\chi, g) : \mathrm{gcd}(\chi(1), |g^G|) \geq \epsilon\chi(1)\}\right|}{|\mathrm{Irr}(G) \times G|} + \epsilon^2.$$

Theorem (Gallagher–Larsen–M.) For $G = \mathrm{GL}(n, q)$, the proportion $P_{n,q}$ of pairs $(\chi, g) \in \mathrm{Irr}(G) \times G$ with $\chi(g) = 0$ satisfies

$$\inf_q P_{n,q} \to 1 \text{ as } n \to \infty.$$

So for any sequence of prime powers $q_1, q_2, \ldots$, we have $P_{n,q_n} \to 1$ as $n \to \infty$.

Theorem (Larsen–M.) If G_n is any sequence of finite simple groups of Lie type with rank tending to ∞, then almost every entry in the character table of G_n is zero as n tends to ∞.

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	0.1111

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	0.1111
4	

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	0.1111
4	0.1600

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	0.1111
4	0.1600
5	0.2041

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

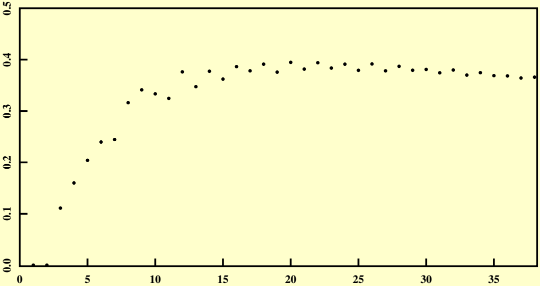
Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	0.1111
4	0.1600
5	0.2041
...	...
37	0.3642
38	0.3659

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

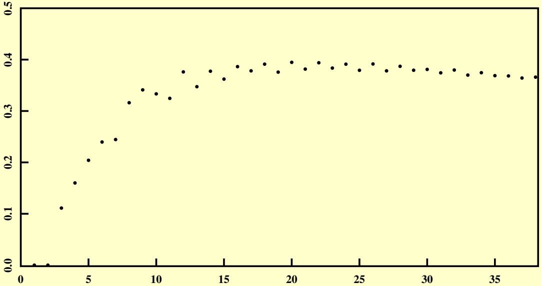
n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	0.1111
4	0.1600
5	0.2041
...	...
37	0.3642
38	0.3659



$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	0.1111
4	0.1600
5	0.2041
...	...
37	0.3642
38	0.3659



Best known bound $\mathrm{Prob}(\chi_\lambda(\mu) = 0) \geq C/\log n$

Part 2.

Part 2. Modular vanishing for S_n

Part 2. Modular vanishing for S_n

n
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$

Part 2. Modular vanishing for S_n

n	4
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \text{ mod } 2)$	.24

Part 2. Modular vanishing for S_n

n	4	5
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33

Part 2. Modular vanishing for S_n

n	4	5	6
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36

Part 2. Modular vanishing for S_n

n	4	5	6	7
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40

Part 2. Modular vanishing for S_n

n	4	5	6	7	8
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55

Part 2. Modular vanishing for S_n

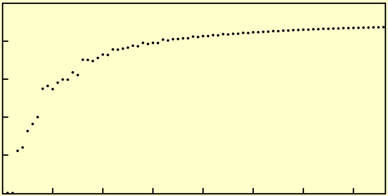
n	4	5	6	7	8	9	10	...
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55	

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55		.87

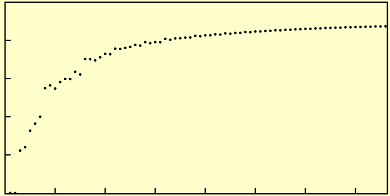
Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \text{ mod } 2)$	.24	.33	.36	.40	.55	.56	.55		.87



Part 2. Modular vanishing for S_n

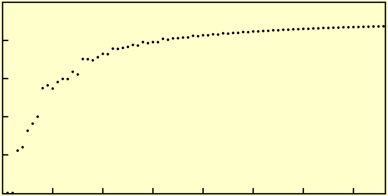
n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55		.87



- The character table of S_{76} has about 86 trillion entries.

Part 2. Modular vanishing for S_n

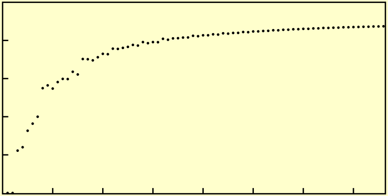
n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \text{ mod } 2)$	.24	.33	.36	.40	.55	.56	.55		.87



- The character table of S_{76} has about 86 trillion entries.
- Several months of computations on super computers used for earth science.

Part 2. Modular vanishing for S_n

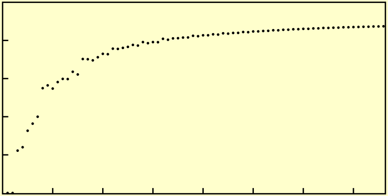
n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55		.87



- The character table of S_{76} has about 86 trillion entries.
- Several months of computations on super computers used for earth science.
- Similar computations for other primes and prime powers.

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55		.87

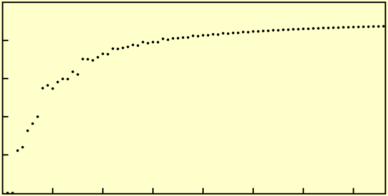


- The character table of S_{76} has about 86 trillion entries.
- Several months of computations on super computers used for earth science.
- Similar computations for other primes and prime powers.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime p as $n \to \infty$.

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \bmod 2)$	.24	.33	.36	.40	.55	.56	.55		.87

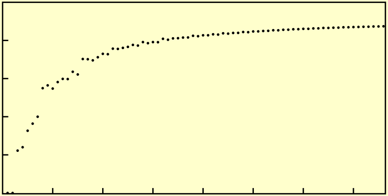


- The character table of S_{76} has about 86 trillion entries.
- Several months of computations on super computers used for earth science.
- Similar computations for other primes and prime powers.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime p as $n \to \infty$. In other words, $\mathrm{Prob}(\chi_\lambda(\mu) = 0 \mod p) \to 1$ as $n \to \infty$.

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \text{ mod } 2)$	.24	.33	.36	.40	.55	.56	.55		.87



- The character table of S_{76} has about 86 trillion entries.
- Several months of computations on super computers used for earth science.
- Similar computations for other primes and prime powers.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime p as $n \to \infty$. In other words, $\mathrm{Prob}(\chi_\lambda(\mu) = 0 \mod p) \to 1$ as $n \to \infty$.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed m as $n \to \infty$.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu)$ mod p if ν is obtained by joining p equal parts in μ.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111)$.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu)$ mod p if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221)$.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu)$ mod p if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441)$.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu)$ mod p if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81)$.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81) \mod 2$.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81) \mod 2$.

Corollary $\chi_\lambda(\mu) \equiv 0 \mod p$ if λ is a $\mathfrak{l}_p(\mu)$-core.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81) \mod 2$.

Corollary $\chi_\lambda(\mu) \equiv 0 \mod p$ if λ is a $\mathfrak{l}_p(\mu)$-core.

n	30	40	50	60	70	80	90	100	110	120
$\mathrm{Prob}(\lambda$ is an $\mathfrak{l}_2(\mu)$-core)	.45	.44	.45	.47	.48	.48	.48	.47	.47	.46

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81) \mod 2$.

Corollary $\chi_\lambda(\mu) \equiv 0 \mod p$ if λ is a $\mathfrak{l}_p(\mu)$-core.

n	30	40	50	60	70	80	90	100	110	120
$\mathrm{Prob}(\lambda$ is an $\mathfrak{l}_2(\mu)$-core$)$	.45	.44	.45	.47	.48	.48	.48	.47	.47	.46

Partial results M., Gluck, Morotti, Ono, McSpirit, Harman, Peluse, Soundararajan,. . .

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81) \mod 2$.

Corollary $\chi_\lambda(\mu) \equiv 0 \mod p$ if λ is a $\mathfrak{l}_p(\mu)$-core.

n	30	40	50	60	70	80	90	100	110	120
$\mathrm{Prob}(\lambda$ is an $\mathfrak{l}_2(\mu)$-core)	.45	.44	.45	.47	.48	.48	.48	.47	.47	.46

Partial results M., Gluck, Morotti, Ono, McSpirit, Harman, Peluse, Soundararajan, . . .

Lemma (Morotti) $\#\{$partitions of n that are not t-cores$\} \leq (t+1)p(n-t)$.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81) \mod 2$.

Corollary $\chi_\lambda(\mu) \equiv 0 \mod p$ if λ is a $\mathfrak{l}_p(\mu)$-core.

n	30	40	50	60	70	80	90	100	110	120
$\mathrm{Prob}(\lambda$ is an $\mathfrak{l}_2(\mu)$-core$)$	.45	.44	.45	.47	.48	.48	.48	.47	.47	.46

Partial results M., Gluck, Morotti, Ono, McSpirit, Harman, Peluse, Soundararajan,. . .

Lemma (Morotti) $\#\{\text{partitions of } n \text{ that are not } t\text{-cores}\} \leq (t+1)p(n-t)$.

Theorem (Peluse–Soundararajan)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Part 2. Modular vanishing for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81) \mod 2$.

Corollary $\chi_\lambda(\mu) \equiv 0 \mod p$ if λ is a $\mathfrak{l}_p(\mu)$-core.

n	30	40	50	60	70	80	90	100	110	120
$\mathrm{Prob}(\lambda$ is an $\mathfrak{l}_2(\mu)$-core$)$	.45	.44	.45	.47	.48	.48	.48	.47	.47	.46

Partial results M., Gluck, Morotti, Ono, McSpirit, Harman, Peluse, Soundararajan,. . .

Lemma (Morotti) $\#\{$partitions of n that are not t-cores$\} \leq (t+1)p(n-t)$.

Theorem (Peluse–Soundararajan)
Almost every entry in the character table of S_n is divisible by any fixed prime.

- Progress is being made on prime powers by various groups.

Part 3. Two variations

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

1. Treat $P(G)$ as a random variable itself.

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

2.

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

2. $G_1 < G_2 < \dots$

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

2. $G_1 < G_2 < \ldots \longrightarrow P(G_1), P(G_2), \ldots \in [0, 1]$

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

2. $G_1 < G_2 < \ldots \longrightarrow P(G_1), P(G_2), \ldots \in [0, 1]$

$$a_1 \ , \ a_2 \ , \ \ldots$$

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

$$\begin{array}{llcl} \textbf{2.} & G_1 < G_2 < \dots & \longrightarrow & P(G_1), P(G_2), \dots \in [0,1] \\ & \quad ? & \longleftarrow & \ a_1\ ,\ a_2\ ,\ \dots \end{array}$$

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g) = 0)$ or $\mathrm{Prob}(\chi(K) = 0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

$$\begin{array}{llll} \textbf{2.} & G_1 < G_2 < \dots & \longrightarrow & P(G_1), P(G_2), \dots \in [0,1] \\ & \quad ? & \longleftarrow & \quad a_1 \;,\; a_2 \;,\; \dots \end{array}$$

Theorem (M.) If $a_1, a_2, \dots \in [0,1]$ and $\epsilon_1, \epsilon_2, \dots \in (0, \infty)$, then for each prime p there exists a chain of p-groups $G_1 < G_2 < \dots$ such that, for each i,

$$|P(G_i) - a_i| < \epsilon_i.$$

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g)=0)$ or $\mathrm{Prob}(\chi(K)=0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

2. $$\begin{array}{ccc} G_1 < G_2 < \dots & \longrightarrow & P(G_1), P(G_2), \dots \in [0,1] \\ ? & \longleftarrow & a_1\ ,\ a_2\ ,\ \dots \end{array}$$

Theorem (M.) If $a_1, a_2, \ldots \in [0,1]$ and $\epsilon_1, \epsilon_2, \ldots \in (0,\infty)$, then for each prime p there exists a chain of p-groups $G_1 < G_2 < \ldots$ such that, for each i,

$$|P(G_i) - a_i| < \epsilon_i.$$

In particular, the set $\{P(G) : |G| < \infty\}$ is dense in $[0,1]$.

Part 4. Zeros and roots of unity

Part 4. Zeros and roots of unity

	1	6	3	8	6
	id	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	
χ_3	2	0	2	-1	0	
χ_4	3	-1	-1	0	1	
χ_5	1	-1	1	1	-1	

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	
χ_4	3	-1	-1	0	1	
χ_5	1	-1	1	1	-1	

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	
χ_5	1	-1	1	1	-1	

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5				

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5		

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5	5/5	

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5	5/5	

Definition

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{\#\{g \in G : \chi(g) \text{ is 0 or a root of unity}\}}{|G|}$$

$$\theta'(G) = \min_{\text{L.T.A. } K} \frac{\#\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is 0 or a root of unity}\}}{|\mathrm{Cl}(G)|}$$

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5	5/5	

Definition

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{\#\{g \in G : \chi(g) \text{ is 0 or a root of unity}\}}{|G|}$$

$$\theta'(G) = \min_{\text{L.T.A. } K} \frac{\#\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is 0 or a root of unity}\}}{|\mathrm{Cl}(G)|}$$

Example $\theta(S_4) = 20/24$

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5	5/5	

Definition

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{\#\{g \in G : \chi(g) \text{ is 0 or a root of unity}\}}{|G|}$$

$$\theta'(G) = \min_{\text{L.T.A. } K} \frac{\#\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is 0 or a root of unity}\}}{|\mathrm{Cl}(G)|}$$

Example $\theta(S_4) = 20/24$ and $\theta'(S_4) = 1$.

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5	5/5	

Definition

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{\#\{g \in G : \chi(g) \text{ is 0 or a root of unity}\}}{|G|}$$

$$\theta'(G) = \min_{\text{L.T.A. } K} \frac{\#\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is 0 or a root of unity}\}}{|\mathrm{Cl}(G)|}$$

Example $\theta(S_4) = 20/24$ and $\theta'(S_4) = 1$.

Theorem (J.G. Thompson) $\theta(G) > 1/3$.

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5	5/5	

Definition

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{\#\{g \in G : \chi(g) \text{ is 0 or a root of unity}\}}{|G|}$$

$$\theta'(G) = \min_{\text{L.T.A. } K} \frac{\#\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is 0 or a root of unity}\}}{|\mathrm{Cl}(G)|}$$

Example $\theta(S_4) = 20/24$ and $\theta'(S_4) = 1$.

Theorem (J.G. Thompson) $\theta(G) > 1/3$.

Theorem (P.X. Gallagher) $\theta'(G) > 1/3$.

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5	5/5	

Definition

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{\#\{g \in G : \chi(g) \text{ is 0 or a root of unity}\}}{|G|}$$

$$\theta'(G) = \min_{\text{L.T.A. } K} \frac{\#\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is 0 or a root of unity}\}}{|\mathrm{Cl}(G)|}$$

Example $\theta(S_4) = 20/24$ and $\theta'(S_4) = 1$.

Theorem (J.G. Thompson) $\theta(G) > 1/3$.

Theorem (P.X. Gallagher) $\theta'(G) > 1/3$.

Theorem (C.L. Siegel) For totally positive $\alpha \neq 0, 1$, $\tilde{\mathrm{Tr}}(\alpha) \geq 3/2$.
(Siegel gets applied to $\alpha = |\chi(g)|^2$. Burnside used $\tilde{\mathrm{Tr}}(\alpha) \geq 1$ for $\alpha \neq 0$.)

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.
- Simple groups of order $\leq 10^9$.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.
- Simple groups of order $\leq 10^9$.
- All sporadic groups.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf\limits_G \theta(G)$, $\inf\limits_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.
- Simple groups of order $\leq 10^9$.
- All sporadic groups.
- A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.
- Simple groups of order $\leq 10^9$.
- All sporadic groups.
- A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$.
- All finite nilpotent groups.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.
- Simple groups of order $\leq 10^9$.
- All sporadic groups.
- A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$.
- All finite nilpotent groups.

Theorem (M.) Each nonlinear irreducible character of a finite nilpotent group is zero on more than half the group.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.
- Simple groups of order $\leq 10^9$.
- All sporadic groups.
- A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$.
- All finite nilpotent groups.

Theorem (M.) Each nonlinear irreducible character of a finite nilpotent group is zero on more than half the group.

Theorem (M.) More than half the irreducible characters of a finite nilpotent group are zero on any given larger-than-average class.

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.
- Simple groups of order $\leq 10^9$.
- All sporadic groups.
- A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$.
- All finite nilpotent groups.

Theorem (M.) Each nonlinear irreducible character of a finite nilpotent group is zero on more than half the group.

Theorem (M.) More than half the irreducible characters of a finite nilpotent group are zero on any given larger-than-average class.

Theorem (M.) Let G be finite nilpotent, let $\chi \in \mathrm{Irr}(G)$, and let $g \in G$. Then

$$\chi(g) = 0 \quad \text{or} \quad \tilde{\mathrm{Tr}}\left(|\chi(g)|^2\right) \geq 2^{\#\{\text{primes dividing } \chi(1)\}}$$