Foulkes characters

Alexander R. Miller

Darstellungstheorietage
TU Kaiserslautern
29 Oktober 2022

Adding random numbers: Holte, Diaconis–Fulman

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$$
\begin{array}{rr}
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$
\begin{array}{rr}
 & 0 \\
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$
\begin{array}{rr}
 & 30 \\
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
 & 3
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$
\begin{array}{rr}
 & 130 \\
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
 & 43
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$
\begin{array}{rr}
 & 1130 \\
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
 & 943
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$
\begin{array}{rr}
 & 21130 \\
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
 & 3943
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$
\begin{array}{rr}
 & 1\ 21130 \\
 & 28033\ 73919 \\
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\hline
 & 23943
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$\begin{array}{rr} & 01\ 21130 \\ & 28033\ 73919 \\ + & 52682\ 17057 \\ + & 75413\ 08629 \\ + & 15890\ 24338 \\ \hline & 9\ 23943 \end{array}$$

Adding random numbers: Holte, Diaconis–Fulman

$$
\begin{array}{rr}
 & 201\ 21130 \\
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
 & 19\ 23943
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$
\begin{array}{rr}
 & 2201\ 21130 \\
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
 & 019\ 23943
\end{array}
$$

Adding random numbers: Holte, Diaconis–Fulman

$$\begin{array}{rr}
 & 22201\ 21130 \\
 & 28033\ 73919 \\
+ & 52682\ 17057 \\
+ & 75413\ 08629 \\
+ & 15890\ 24338 \\
\hline
 & 2019\ 23943
\end{array}$$

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$$
\begin{array}{rr}
 & 1\ 22201\ 21130 \\
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+ & 15890\ 24338 \\
\hline
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$$M(i,j) = \text{chance}\{\text{next carry } j \mid \text{last carry is } i\}$$

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- Diaconis–Fulman recognized Foulkes characters.

Foulkes characters (Foulkes, Kerber–Thürlings, Diaconis–Fulman, Isaacs,...)

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Foulkes characters for S_4:

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ϕ_1	11	3	−1	−1	−3
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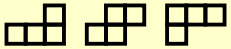	ϕ_1	11	3	-1	-1	-3
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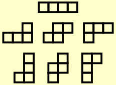

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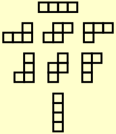

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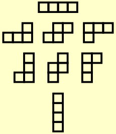

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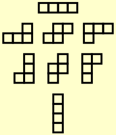

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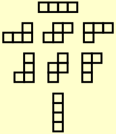

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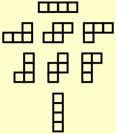

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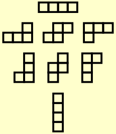

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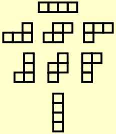

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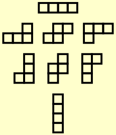

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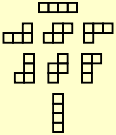

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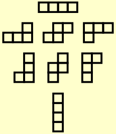

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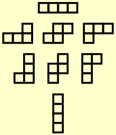

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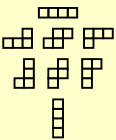

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$$\mathbf{\Phi} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 11 & 3 & -1 & -3 \\ 11 & -3 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

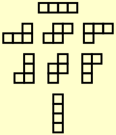
$$\det \mathbf{\Phi} = 4!3!2! = 4 \cdot 3^2 \cdot 2^3$$

Reflection $r \in \mathrm{GL}(\mathbf{C}^n)$: $\ker(r-1)$ hyp.

Foulkes characters (Foulkes, Kerber–Thürlings, Diaconis–Fulman, Isaacs,...)

Constructed from **ribbon representations** of S_n: $\mathrm{Specht}($$)$.

Foulkes characters for $\mathbf{S_4}$:



	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
ϕ_0	1	1	1	1	1
ϕ_1	11	3	−1	−1	−3
ϕ_2	11	−3	−1	−1	3
ϕ_3	1	−1	1	1	−1

- $\phi_0 + \phi_1 + \ldots + \phi_{n-1} = \mathrm{reg}_{S_n}$.
- $\phi_i(\sigma)$ depends only on $\#\mathrm{cycles}(\sigma)$.
- Interesting $\mathbf{Q}$ span.
- Nice explicit expression.
- Nice determinant.
- Degrees are Eulerian numbers.
- <u>But</u> ad hoc proofs. Seems special to S_n. How about other reflection groups?

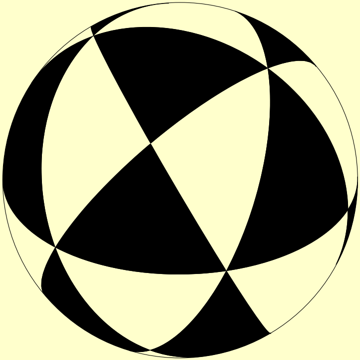
$$\Phi = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 11 & 3 & -1 & -3 \\ 11 & -3 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\det \Phi = 4!3!2! = 4 \cdot 3^2 \cdot 2^3$$

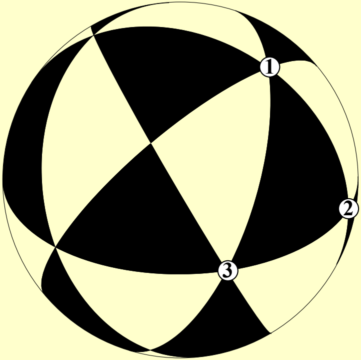
Reflection $r \in \mathrm{GL}(\mathbf{C}^n)$: $\ker(r-1)$ hyp.
(Finite) reflection group $G \leq \mathrm{GL}(\mathbf{C}^n)$

Ribbon representations for reflection groups

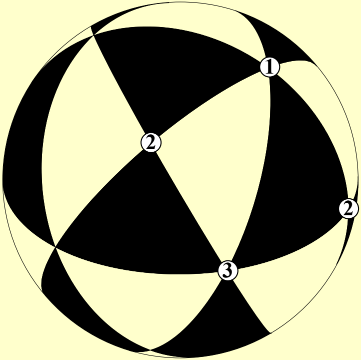
Ribbon representations for reflection groups



Ribbon representations for reflection groups



Ribbon representations for reflection groups



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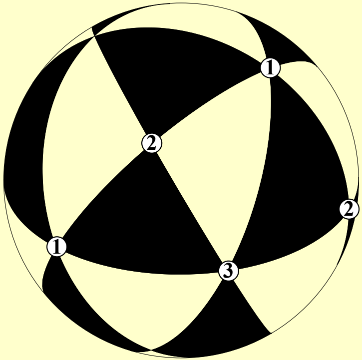
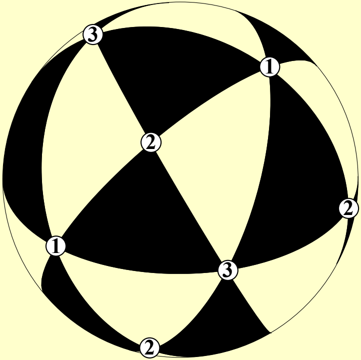


Ribbon representations for reflection groups



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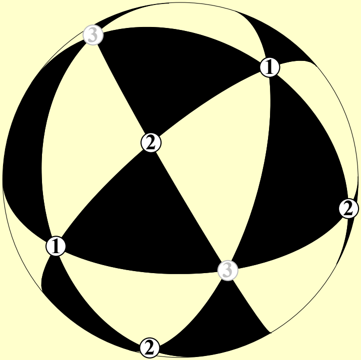
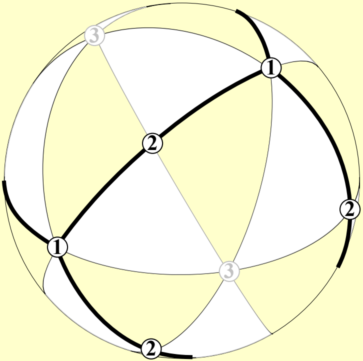
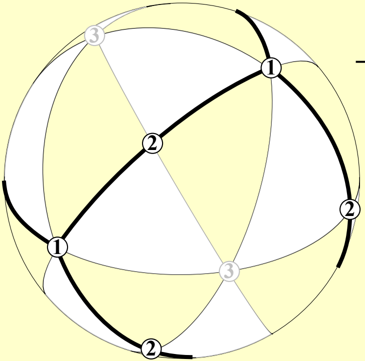


Ribbon representations for reflection groups

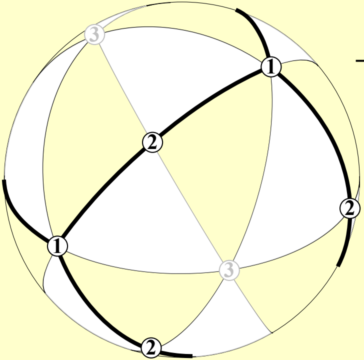


Ribbon representations for reflection groups



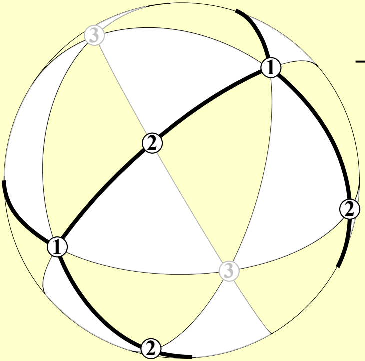
S	12

Ribbon representations for reflection groups



S	12
$\widetilde{H}_{\text{top}}(\Delta_S)$	

Ribbon representations for reflection groups



S	12
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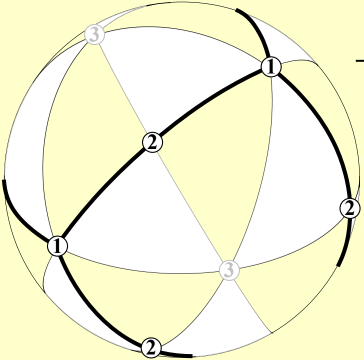
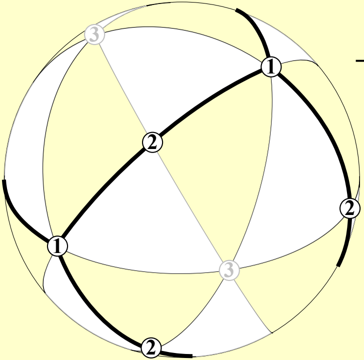


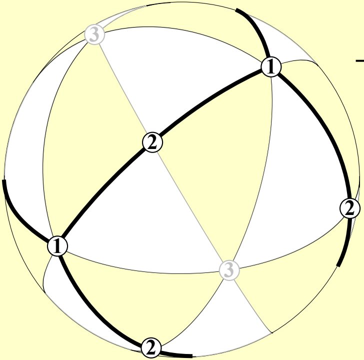
S		12	13
$\widetilde{H}_{\mathrm{top}}(\Delta_S)$			

Ribbon representations for reflection groups



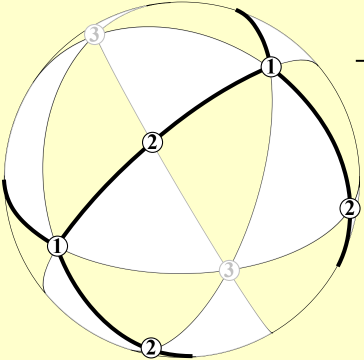
S	12	13	23
$\widetilde{H}_{\mathrm{top}}(\Delta_S)$			

Ribbon representations for reflection groups



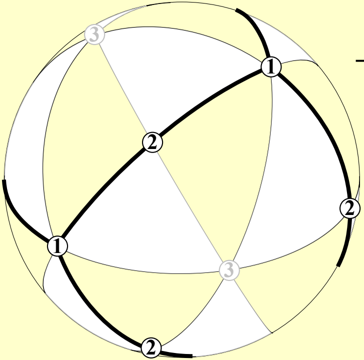
S	12	13	23	123
$\widetilde{H}_{\mathrm{top}}(\Delta_S)$				

Ribbon representations for reflection groups



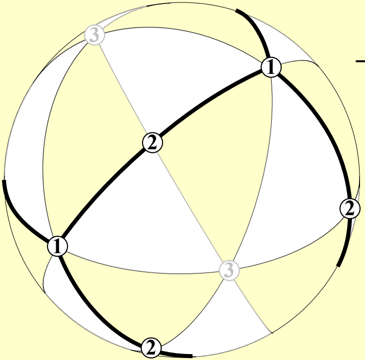
S	$\emptyset$	12	13	23	123
$\widetilde{H}_{\mathrm{top}}(\Delta_S)$					

Ribbon representations for reflection groups



S	$\emptyset$	1	2	3	12	13	23	123
$\widetilde{H}_{\mathrm{top}}(\Delta_S)$								

Ribbon representations for reflection groups



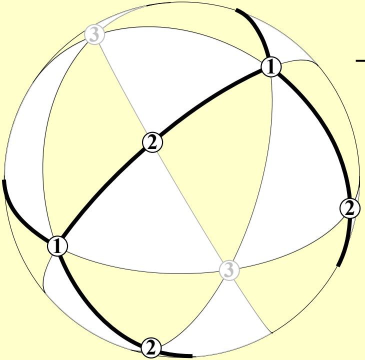
S	$\emptyset$	1	2	3	12	13	23	123
$\widetilde{H}_{\text{top}}(\Delta_S)$								

General setup: Fix G finite

$\langle r_1, r_2, \ldots, r_N \mid r_i^{p_i} = 1, \; r_i r_j r_i \ldots = r_j r_i r_j \ldots \quad i \neq j \rangle$

$m_{ij} = m_{ji}, p_i \geq 2$ and $p_i = p_j$ if m_{ij} odd.

Ribbon representations for reflection groups



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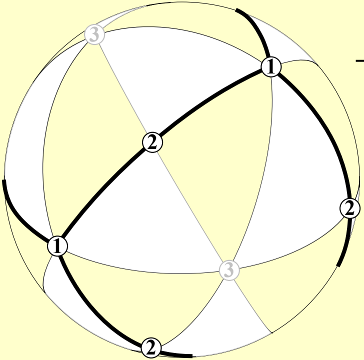

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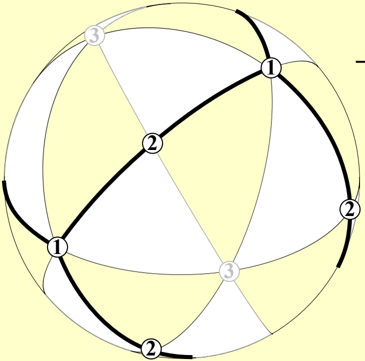
- $\Gamma(G)$: $\overset{p_i}{\bullet}\!\overset{m_{ij}}{\text{———}}\!\overset{p_j}{\bullet}$ $\quad (m_{ij} > 2).$

Ribbon representations for reflection groups



S	$\emptyset$	1	2	3	12	13	23	123
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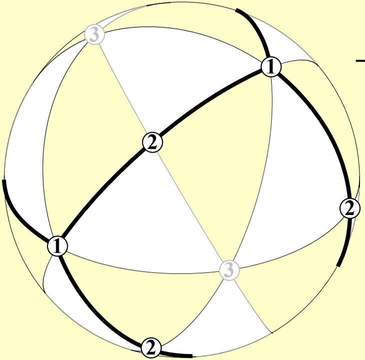
Ribbon representations for reflection groups



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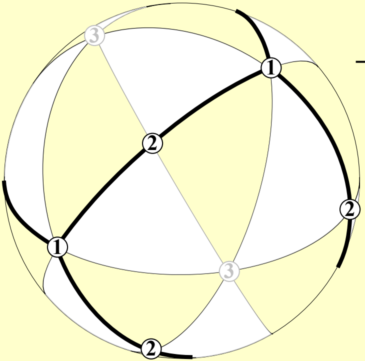
Ribbon representations for reflection groups



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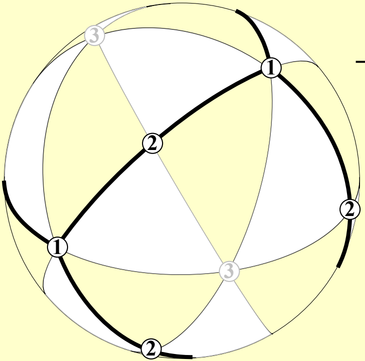

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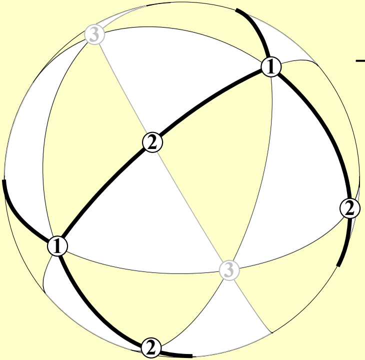
Milnor fiber complex $\boldsymbol{\Delta}(\mathbf{G}, \mathbf{R})$ (Orlik):

Ribbon representations for reflection groups



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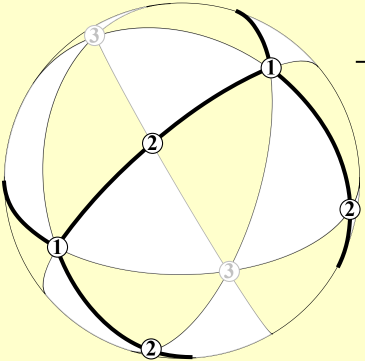
gG_J face of $hG_K \Leftrightarrow gG_J \supset hG_K \quad (G_J = \langle J \rangle, \ J \subset R)$

Ribbon representations for reflection groups



S	$\emptyset$	1	2	3	12	13	23	123
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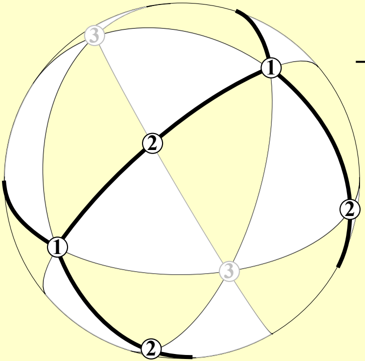
- $\mathrm{type}(gG_{R \setminus J}) = J$

Ribbon representations for reflection groups



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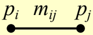
- $\Gamma(G)$:  $(m_{ij} > 2)$.
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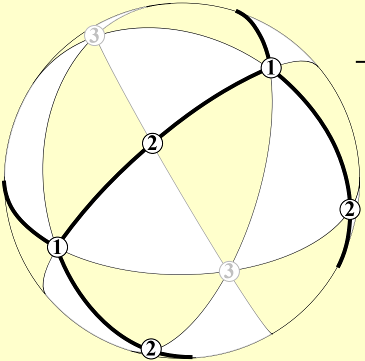
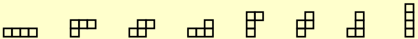
Ribbon representations for reflection groups



S	$\emptyset$	1	2	3	12	13	23	123
$\widetilde{H}_{\mathrm{top}}(\Delta_S)$								



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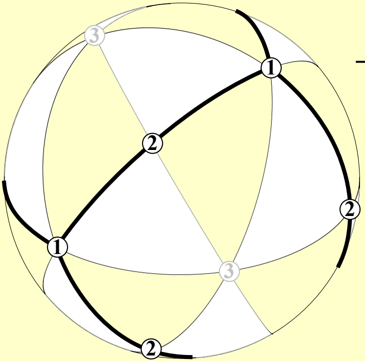
Ribbon representations (Solomon, M.):

Ribbon representations for reflection groups



S	$\emptyset$	1	2	3	12	13	23	123
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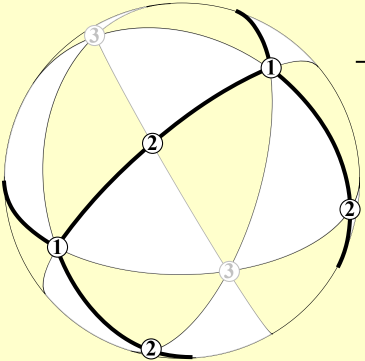
Ribbon representations (Solomon, M.): $\rho_S = \sum_{J \subset S} (-1)^{|S \setminus J|} \mathrm{Ind}_{G_{R\setminus J}}^{G} \mathbf{1} \quad (S \subset R)$.

Ribbon representations for reflection groups



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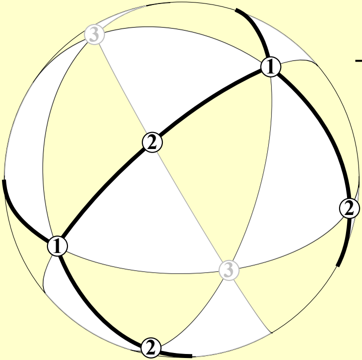
- $\mathrm{type}(gG_{R\setminus J}) = J$ • $\Delta_S = \{\sigma \mid \mathrm{type}(\sigma) \subset S\}$

Ribbon representations (Solomon, M.): $\rho_S = \sum_{J\subset S}(-1)^{|S\setminus J|}\mathrm{Ind}_{G_{R\setminus J}}^G \mathbf{1} \quad (S \subset R)$.

Generalized Foulkes characters (M.):

Ribbon representations for reflection groups



S	$\emptyset$	1	2	3	12	13	23	123
$\widetilde{H}_{\mathrm{top}}(\Delta_S)$								

General setup: Fix G finite

$\langle r_1, r_2, \ldots, r_N \mid r_i^{p_i} = 1,\ r_i r_j r_i \ldots = r_j r_i r_j \ldots \quad i \neq j \rangle$

$m_{ij} = m_{ji}, p_i \geq 2$ and $p_i = p_j$ if m_{ij} odd.

- $R = \{r_1, r_2, \ldots, r_N\}, \quad V = \mathbf{C}^N$.
- $\Gamma(G)$: $\overset{p_i}{\bullet} \overset{m_{ij}}{\text{———}} \overset{p_j}{\bullet}$ $(m_{ij} > 2)$.
- Canonical rep. $G \subset \mathrm{GL}(V)$ as reflection group.
- G irreducible $\Leftrightarrow$ $\Gamma(G)$ connected.
- Finite irreducible Coxeter and Shephard groups.

Milnor fiber complex $\mathbf{\Delta}(\mathbf{G}, \mathbf{R})$ (Orlik):

gG_J face of $hG_K \Leftrightarrow gG_J \supset hG_K \quad (G_J = \langle J \rangle,\ J \subset R)$

- $\mathrm{type}(gG_{R\setminus J}) = J$ • $\Delta_S = \{\sigma \mid \mathrm{type}(\sigma) \subset S\}$

Ribbon representations (Solomon, M.): $\rho_S = \sum_{J \subset S} (-1)^{|S \setminus J|} \mathrm{Ind}_{G_{R\setminus J}}^{G} \mathbf{1} \quad (S \subset R)$.

Generalized Foulkes characters (M.): $\phi_k = \sum_{\substack{S \subset R \\ |S| = k}} \rho_S \qquad (k = 0, 1, \ldots, N)$.

A formula for Foulkes characters

Theorem (M.)

$$\phi_k(g) = \sum_{i=0}^{k} (-1)^{k-i} \binom{N-i}{k-i} f_{i-1}(\Delta \cap X),$$

- $X = V^g$ so that $\Delta \cap X = \Delta^g = \{\sigma \in \Delta : g\sigma = \sigma\}$,
- $f_i(\Delta) = \#\{i\text{-dimensional simplices in } \Delta\}$.

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Define $\Phi = [\phi_i(g_j)]_{0 \le i \le N,\ 0 \le j \le c}$ for class representatives $g_0, g_1, \ldots, g_c$.

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Define $\Phi = [\phi_i(g_j)]_{0 \le i \le N,\ 0 \le j \le c}$ for class representatives $g_0, g_1, \ldots, g_c$.

Corollary (M.) $\Phi = L \times F$ for

$$L = \left[(-1)^{i-j}\tbinom{N-j}{i-j}\right]_{0 \le i,j \le N} \qquad F = \left[f_{i-1}(\Delta \cap X_j)\right]_{0 \le i \le N,\ 0 \le j \le c}.$$

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
ϕ^0	1	1	1	1	1
ϕ^1	11	3	-1	-1	-3
ϕ^2	11	-3	-1	-1	3
ϕ^3	1	-1	1	1	-1

f_{-1}					
f_0					
f_1					
f_2					

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
ϕ^0	1	1	1	1	1
ϕ^1	11	3	-1	-1	-3
ϕ^2	11	-3	-1	-1	3
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	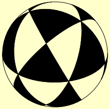				
f_{-1}					
f_0					
f_1					
f_2					

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
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	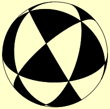
f_{-1}	1
f_0	
f_1	
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	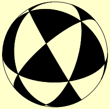				
f_{-1}	1				
f_0	14				
f_1					
f_2					

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
ϕ^0	1	1	1	1	1
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	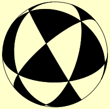				
f_{-1}	1				
f_0	14				
f_1	36				
f_2					

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
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	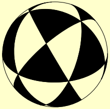
f_{-1}	1
f_0	14
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	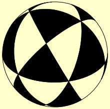	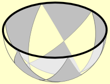			
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	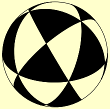	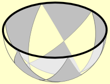			
f_{-1}	1	1			
f_0	14				
f_1	36				
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	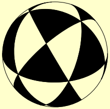	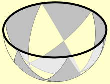			
f_{-1}	1	1			
f_0	14	6			
f_1	36				
f_2	24				

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
ϕ^0	1	1	1	1	1
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	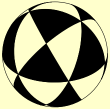	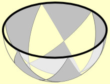			
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	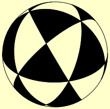	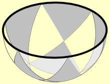	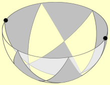		
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	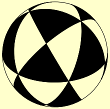	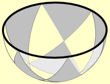	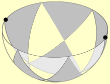		
f_{-1}	1	1	1		
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	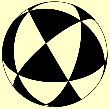	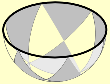	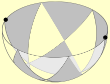		
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f_1	36	6			
f_2	24				

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
ϕ^0	1	1	1	1	1
ϕ^1	11	3	-1	-1	-3
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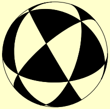
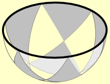
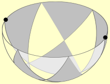
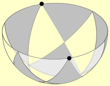
ϕ^3	1	-1	1	1	-1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	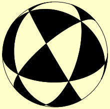	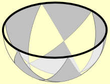	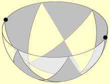	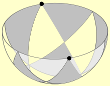	
f_{-1}	1	1	1	1	
f_0	14	6	2		
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f_2	24				

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
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	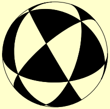	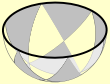	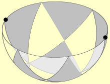	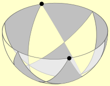	
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	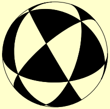	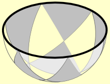	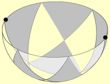	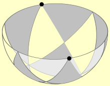	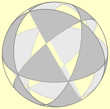
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	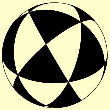	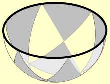	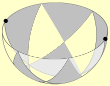	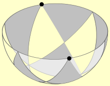	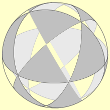
f_{-1}	1	1	1	1	1
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	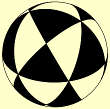	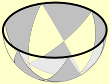	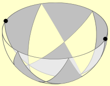	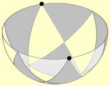	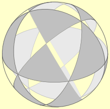
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	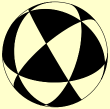	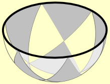	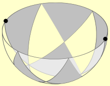	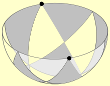	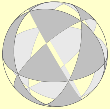
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	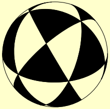	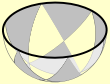	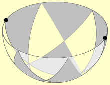	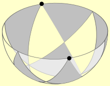	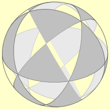
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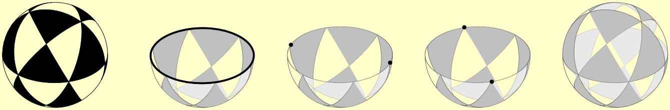

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- **Note:** When $G = S_n$, $\ell(g) := \#\mathrm{cycles}(g) = n - \mathrm{length}(g)$.

Theorem (M.) If $r > 1$, then the characters of $Z_r \wr S_n$ that depend only on length are the $\mathbf{N}$-linear combinations of Foulkes characters.

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Theorem (M.) The smallest positive integer that clears all denominators is

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where f is the second Chebyschev function.

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Theorem (M.) Let $\mathscr{D}_i = \sum_{\mathrm{des}(\pi)=i} \pi$. Then

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Third solution follows from work of Delsarte in 1976 in a context void of characters and groups, and given 4 years before the ϕ_i's were introduced by Foulkes in 1980.

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Theorem (M.) Let $\phi_0, \phi_1, \ldots, \phi_k$ be the Foulkes characters of the hyperoctahedral group of rank k. Then

$$M_B(i,j) = [\phi_i, b^{\ell} \phi_j] \times b^{-k}.$$

A curious classification

Theorem (M.) Let G be an finite irreducible Coxeter or Shephard group. Then the following are equivalent.

1. The characters $\phi_i(g)$ depend only on the dimension of the fixed space of g.
2. The characters $\phi_0, \phi_1, \ldots, \phi_N$ form a $\mathbf{Q}$ basis for the space of rational class functions χ that depend only on the dimension of the fixed space.
3. The reduced Foulkes character table Φ is square and $\det \Phi = d_1^N d_2^{N-1} \cdots d_N^1$.
4. The Smith entries $s_0, s_1, \ldots, s_N$ of the table Φ are given by $s_i = d_1 d_2 \cdots d_i$.
5. The isomorphism class of L^X depends only on the dimension of X.
6. The cell counts $f_i(\Delta \cap X)$ depend only on the dimension of X.
7. The numbers $B_X(m_1)$ depend only on the dimension of X.
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A curious classification

Theorem (M.) Let G be an finite irreducible Coxeter or Shephard group. Then the following are equivalent.

1. The characters $\phi_i(g)$ depend only on the dimension of the fixed space of g.
2. The characters $\phi_0, \phi_1, \ldots, \phi_N$ form a $\mathbf{Q}$ basis for the space of rational class functions χ that depend only on the dimension of the fixed space.
3. The reduced Foulkes character table Φ is square and $\det \Phi = d_1^N d_2^{N-1} \cdots d_N^1$.
4. The Smith entries $s_0, s_1, \ldots, s_N$ of the table Φ are given by $s_i = d_1 d_2 \cdots d_i$.
5. The isomorphism class of L^X depends only on the dimension of X.
6. The cell counts $f_i(\Delta \cap X)$ depend only on the dimension of X.
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Walls in Milnor fiber complexes: Δ^r

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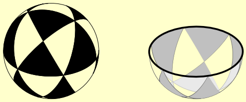

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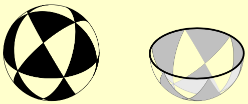

When is every wall in a Coxeter complex again a Coxeter complex?

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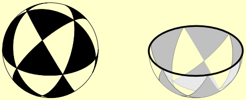

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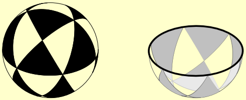

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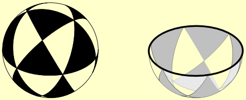

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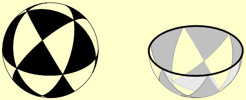

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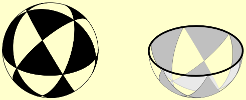

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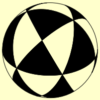

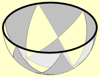

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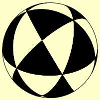
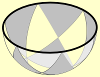

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- Both theorems imply Abramenko's result.

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