ALMOST ALL WREATH PRODUCT CHARACTER VALUES ARE DIVISIBLE BY GIVEN PRIMES

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Abstract. For a finite group G with integer-valued character table and a prime p, we show that almost every entry in the character table of $G \wr S_N$ is divisible by p as $N \to \infty$. This result generalizes the work of Peluse and Soundararajan on the character table of S_N.

1. Introduction

Let S_N be the symmetric group on N letters. The complex irreducible characters of S_N were calculated by Frobenius in 1900; in particular, Frobenius showed that the characters are integer-valued. In 2019, Alex Miller investigated the distribution of the parity of entries of the character table of S_N [Mil19]. He made the remarkable conjecture that for any prime p and exponent $\ell \geq 1$, the proportion of entries of the character table of S_N divisible by p^ℓ tends to 1 as $N \to \infty$. This conjecture was recently proved by Peluse and Soundararajan in the case $\ell = 1$ in [PS22].

This leaves the question of investigating the distribution of residues modulo p for more general finite groups with integer-valued character tables. A natural infinite family of such is the wreath product $G \wr S_N$ as $N \to \infty$. When G is a fixed group with integer-valued character table, it is known that the characters of $G \wr S_N$ are also integer-valued [Jam06, Corollary 4.4.11]. These families include the Weyl group of type B_N, when $G = \mathbb{Z}/2\mathbb{Z}$, and wreath products $S_M \wr S_N$ of symmetric groups.

Our main result is a generalization of Peluse and Soundararajan's theorem:

Theorem (see Theorem 3.8 below). *Let G be a group with integer-valued character table and let $G \wr S_N$ be the wreath product of G with S_N. For all primes p, the proportion of entries in the character table of $G \wr S_N$ which are divisible by p tends to 1 as $N \to \infty$.*

The proof relies on the combinatorics of the representations of $G \wr S_N$. If G has k conjugacy classes, then conjugacy classes and representations of $G \wr S_N$ are both naturally labelled by k-multipartitions of N. One of the key inputs is characterizing when two elements of $G \wr S_N$ have

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columns in the character table congruent modulo p. In Lemma 3.2, we give a combinatorial characterization directly generalizing the corresponding criterion for S_N.

It is known that the character tables of all Weyl groups are integer-valued. The Weyl groups of type A are the symmetric groups, where our question was answered by Peluse and Soundararajan. The Weyl groups of type B_N and C_N are both equal to $\mathbb{Z}/2\mathbb{Z} \wr S_N$, handled by our main theorem. The only remaining infinite family of Weyl groups is that of type D. In Section 4, we also show that the proportion of character values of the Weyl group of type D_N divisible by a prime p tends to 1 as $N \to \infty$.

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2. Preliminaries

2.1. **Representation Theory of the Wreath Product.** Let G be a finite group and let S_N be the symmetric group on N letters.

Definition 2.1. The *wreath product of G with S_N*, denoted $G \wr S_N$, is the group of $N \times N$ permutation matrices with nonzero entries in G.

We begin by recalling the representation theory of $G \wr S_N$. The representation theory of wreath products was first studied in Specht's dissertation [Spe32]; see also [Zel81; Jam06] for more modern treatments. If we take the representation theory of G as input data and let N vary, the representation theory has structural similarities to the representation theory of S_N, the case when $G = 1$. While representations of the symmetric group are labelled by partitions of N, representations of the wreath product are labelled by multipartitions:

Definition 2.2. A *k-multipartition* of an integer N is $\lambda = (\lambda_1, \ldots, \lambda_k)$ where λ_i is a partition for all i such that $\sum_{i=1}^{k} |\lambda_i| = N$.

Suppose that G has k conjugacy classes. Then k-multipartitions of N label the conjugacy classes of $G \wr S_N$. We will not need to use the specific form of this bijection in this paper; it is used in the proofs of character formulas in Propositions 2.4 and 2.10, which we omit.

Proposition 2.3 ([Jam06], Theorem 4.2.8)**.** *If G has k conjugacy classes, then the conjugacy classes of $G \wr S_N$ are indexed by k-multipartitions of N. Given $x \in G \wr S_N$, the multipartition λ corresponding to x is formed as follows: for each cycle in x of length ℓ, if the product of the nonzero entries in that cycle is in the ith conjugacy class of G, then add ℓ to λ_i.*

To find the complex irreducible representations of $G \wr S_N$, we need the complex irreducible representations of G as input; call the irreducible G-representations $V_1, \ldots, V_k$.

Proposition 2.4 ([Jam06], Theorem 4.4.3)**.** *If G has k conjugacy classes, then the irreducible representations of $G \wr S_N$ are in bijection with k-multipartitions of N. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ a k-multipartition of N, let $a_i = |\lambda_i|$ and $G_a = G \wr S_a$. Then the irreducible representation of $G \wr S_N$ corresponding to λ is*

$$V^\lambda = \mathrm{Ind}_{G_{a_1} \times \cdots \times G_{a_k}}^{G_N} \left(\boxtimes_{i=1}^k \left(S^{\lambda_i} \otimes V_i^{\otimes a_i} \right) \right)$$

where S^{λ_i} is the Specht module for S_N corresponding to λ_i.

Character values of wreath products can be calculated using a modified version of the Murnaghan-Nakayama rule for the symmetric group. Let χ^λ be the character of V^λ and χ^λ_μ be the value of χ^λ on the conjugacy class corresponding to μ. Then χ^λ_μ is calculated by decomposing the of Young diagrams of λ_i for all i using rimhooks:

Definition 2.5. A *rimhook* of a k-multipartition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is k adjacent boxes in the Young diagram of some λ_i such that no other boxes are remaining south or east after the rimhook has been removed and no box in the rimhook has a southeast neighbor.

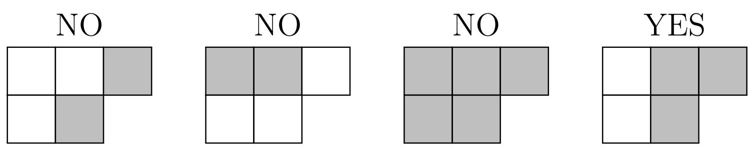


FIGURE 1. Examples of three invalid and one valid rimhooks in $\lambda = ((3^1 2^1))$.

Definition 2.6. For k-multipartitions λ and μ, a *rimhook decomposition* of λ by μ is obtained by repeatedly removing rimhooks in λ with parts of μ in a fixed ordering such that after all rimhooks have been taken, there are no boxes of λ left. All the possible ways to take rimhooks of λ with parts of μ is the set $RHD(\lambda, \mu)$.

The Murnaghan-Nakayama rule can be modified for wreath products as follows:

Proposition 2.7 ([Jam06], Theorem 4.4.10)**.** *Let λ and μ be k-multipartitions of N. Let $\chi^1, \chi^2, \ldots, \chi^k$ be the irreducible characters of G. For $\rho \in RHD(\lambda, \mu)$, let $\psi(\rho)$ be defined by*

$$\psi(\rho) = \prod_{i=1}^{k} \left(\prod_{\text{rimhooks } h \text{ in } \lambda_i} \chi^i(c_h) \right)$$

where c_h is the conjugacy class of G associated to h. Then

$$\chi^\lambda_\mu = \sum_{\rho \in RHD(\lambda, \mu)} (-1)^{ht(\rho)} \psi(\rho),$$

where $ht(\rho)$ is the height of the rimhook decomposition.

The permutation module characters of wreath products form another basis for the space of class functions of $G \wr S_N$ that is easier to work with.

Definition 2.8. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a k-multipartition of N and let $a_i = |\lambda_i|$. For each λ_i, let S_{λ_i} be the Young subgroup of S_{a_i} corresponding to λ_i and let $G_{\lambda_i} = G \wr S_{\lambda_i}$. Then the *permutation module* M^λ for $G \wr S_N$ is defined by

$$M^\lambda = \mathrm{Ind}_{G_{\lambda_1} \times \cdots \times G_{\lambda_k}}^{G \wr S_N} \left(\boxtimes_{i=1}^k V_i^{\otimes a_i} \right).$$

There is a character formula for M^λ using row decompositions instead of rimhook decompositions. It is as follows:

Definition 2.9. Let λ and μ be k-multipartitions of N. A *row decomposition* of λ by μ is a tiling of the Young diagrams λ_i together over all $1 \le i \le k$ by parts of μ_j over all $1 \le j \le k$ such that each row of a μ_j lies in a row of a λ_i. We tile the Young diagrams λ_i together over all $1 \le i \le k$ by parts of μ_j over all $1 \le j \le k$ such that each row of a μ_j lies in a row of a λ_i. These tilings are taken with a fixed ordering of the parts of μ_i, placing tiles from right to left in that order. The set of all row decompositions of λ by μ is denoted $RD(\lambda, \mu)$.

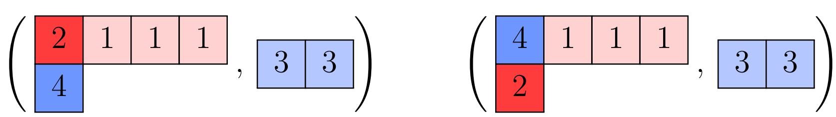


FIGURE 2. All valid row decompositions of $(\yng(4,1), \yng(2))$ by $(31, 21)$. The numbers in the boxes indicate the order in which parts of μ are placed into rows of λ, with fixed right-to-left placement.

Proposition 2.10. *Let λ and μ be k-multipartitions of N. Let $\chi^1, \chi^2, \ldots, \chi_k$ be the irreducible characters of G. For $\rho \in RD(\lambda, \mu)$, let $\alpha(\rho)$ be defined by*

$$\alpha(\rho) = \prod_{q=1}^{k} \left(\prod_{\textit{cycles } r \textit{ placed into } \lambda_q} \chi^q(c_r) \right),$$

where c_r is the conjugacy class of G associated to r. Then the character for permutation module M^λ at μ is

$$M^\lambda_\mu = \sum_{\rho \in RD(\lambda, \mu)} \alpha(\rho).$$

The proof is a straightforward consequence of the character formula for induced representations.

We now describe the change-of-basis between irreducible and permutation characters.

Definition 2.11. The *dominance order* on k-multipartitions is defined by $\lambda \succcurlyeq \eta$ if and only if λ_i dominates η_i for all i.

Lemma 2.12. *The matrix of multiplicities $[M^\lambda : V^\eta]$ of the irreducible representations of $G \wr S_N$ in permutation modules is unimodular and upper-triangular with respect to dominance order.*

Proof. Recall the Kostka numbers $K^{\beta,\gamma}$ for β, γ partitions of N are defined by

$$M^\beta = \mathrm{Ind}_{S_\beta}^{S_N} 1 = \bigoplus_\gamma (V^\gamma)^{\oplus K^{\beta,\gamma}},$$

where S_β is the Young subgroup corresponding to β and V^γ is the Specht module corresponding to γ. Note that our notation for M^β and V^γ agrees with that of wreath products $G \wr S_N$ when $G = 1$. The Kostka numbers satisfy $K^{\beta,\beta} = 1$ and $K^{\beta,\gamma} > 0$ if and only if $\beta \succcurlyeq \gamma$ in dominance order [Mac98, p. I.6].

We claim that

$$M^\lambda = \bigoplus_\eta (V^\eta)^{\oplus c(\lambda,\eta)}, \qquad c(\lambda,\eta) = \left(\prod_{i=1}^k K^{\lambda_i,\eta_i} \right). \tag{1}$$

By Definition 2.8, if $a_i = |\lambda_i|$ for all i and $H = G_{a_1} \times G_{a_2} \times \cdots \times G_{a_k}$, then

$$M^\lambda = Ind_{G_\lambda}^{G_N} \left(\boxtimes_{i=1}^k V_i^{\otimes a_i} \right) = Ind_H^{G_N} \left(\boxtimes_{i=1}^k M^{\lambda_i} \otimes V_i^{\otimes a_i} \right),$$

where we make $M^{\lambda_i} \otimes V_i^{\otimes a_i}$ a representation of G_{a_i} by having S_{a_i} act diagonally and G^{a_i} naturally on $V_i^{\otimes a_i}$. Then (1) follows from multilinearity of the tensor product and linearity of induction.

Now since the matrix of Kostka numbers is unimodular and upper-triangular with respect to dominance order, the same is true of the matrix $\{c(\lambda,\mu)\}_{\lambda,\mu}$. □

2.2. **Asymptotics of Partitions.** We recall a form of the Hardy-Ramanujan asymptotic for the number of partitions of N, denoted $p(N)$.

Proposition 2.13 ([HR18],(1.36))**.** *If $\delta > 0$, then*

$$\left(\frac{2\pi}{\sqrt{6}} - \delta \right) \sqrt{N} \leq \log p(N) \leq \left(\frac{2\pi}{\sqrt{6}} + \delta \right) \sqrt{N}$$

for sufficiently large N.

Let $p_k(N)$ denote the number of k-multipartitions of N.

Claim 2.14. *If $\delta > 0$, then*

$$\left(\frac{2\pi}{\sqrt{6}} - \delta \right) \sqrt{kN} \leq \log p_k(N) \leq \left(\frac{2\pi}{\sqrt{6}} + \delta \right) \sqrt{kN}$$

for sufficiently large N.

This formula also appears in [Mur13]. We provide an elementary inductive proof.

Proof. We proceed by induction on k. The base case $k = 1$ is Proposition 2.13.

For $\delta > 0$, let $\delta' = \frac{4}{5}\delta$. By inductive hypothesis, there exists a constant B such that if $C \geq B$, then

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right)\left(\sqrt{(k-1)C}\right)\right) \ll p_{k-1}(C) \ll \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\left(\sqrt{(k-1)C}\right)\right)$$

and

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right)\left(\sqrt{C}\right)\right) \ll p(C) \ll \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\left(\sqrt{C}\right)\right).$$

By considering the size of the first partition in a k-multipartition, it follows that

$$p_k(N) = \sum_{a=0}^{N} p(a)p_{k-1}(N-a).$$

We break up the sum for $p_k(N)$ into distinct parts: let

$$D_1 = \sum_{a=0}^{B-1} p(a)p_{k-1}(N-a),$$
$$D_2 = \sum_{a=B}^{N-B} p(a)p_{k-1}(N-a),$$
$$D_3 = \sum_{a=N-B+1}^{N} p(a)p_{k-1}(N-a).$$

In D_2, for $B \leq a \leq N - B$, we have

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta'\right)\left(\sqrt{a} + \sqrt{(k-1)(N-a)}\right)\right) \ll p(a)p_{k-1}(N-a)$$
$$\ll \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\left(\sqrt{a} + \sqrt{(k-1)(N-a)}\right)\right).$$

Note that $\sqrt{a} + \sqrt{(k-1)(N-a)} \leq \sqrt{kN}$, with equality achieved at $a = \frac{N}{k}$. Summing over $a \in [B, N-B]$, we get

$$\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta\right)\sqrt{kN}\right) \ll D_2 \ll (N-2B)\exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right). \tag{2}$$

We now consider D_1 and D_3. Note that for $a \in [0, B)$, we have $p(a)p_{k-1}(N-a) \leq p(B)p_{k-1}(N)$, and for $a \in (N-B, N]$, we have $p(a)p_{k-1}(N-a) \leq p(N)p_{k-1}(B)$. Hence

$$0 \leq D_1 \leq Bp(B)p_{k-1}(N) \ll B\exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right) \tag{3}$$

for sufficiently large N. Likewise,

$$0 \leq D_3 \leq Bp(N)p_{k-1}(B) \ll B \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right). \tag{4}$$

Combining (2), (3), and (4), we have that

$$\begin{aligned}\exp\left(\left(\frac{2\pi}{\sqrt{6}} - \delta\right)\sqrt{kN}\right) \ll p_k(N) &\ll N \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta'\right)\sqrt{kN}\right) \\ &\ll \exp\left(\left(\frac{2\pi}{\sqrt{6}} + \delta\right)\sqrt{kN}\right)\end{aligned}$$

for sufficiently large N. $\square$

The above estimate implies k-multipartitions concentrate around having close to equal-size parts:

Corollary 2.15. *For all $\delta > 0$, the proportion of k-multipartitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ such that*

$$\frac{N}{k}(1-\delta) < |\lambda_i| < \frac{N}{k}(1+\delta)$$

for all $1 \leq i \leq k$ goes to 1 as $N \to \infty$.

Proof. Pick $0 < \varepsilon < \delta$ and $1 \leq i \leq k$. The number of k-multipartitions of N where $|\lambda_i| \notin \left(\frac{N}{k}(1-\epsilon), \frac{N}{k}(1+\epsilon)\right)$ is

$$\sum_{\lambda \text{ s.t. } |\lambda_i| \notin (\frac{N}{k}(1-\varepsilon), \frac{N}{k}(1+\varepsilon))} p(|\lambda_i|)p_{k-1}(|\lambda_1|, \ldots |\hat{\lambda_i}|, \ldots, |\lambda_k|). \tag{5}$$

By Claim 2.14, the rate at which (5) approaches infinity is significantly slower than the rate at which $p_k(N)$ approaches infinity. Since $\delta > \varepsilon$, we can conclude that the number of k-multipartitions λ such that $|\lambda_i| \in \left(\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta)\right)$ for all i tends to 1 as $N \to \infty$. $\square$

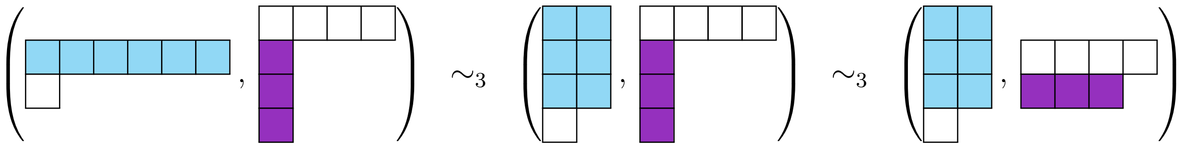
3. Main Results

3.1. Character Table Column Congruences.

Corollary 3.3 below, which we call "the mashing rule," gives a criterion for mod p congruence of two columns of the character table of $G \wr S_N$ in terms of k-multipartitions.

In this section, we must assume that G has integer-valued character table. By [Ser77, §13.1], the group G has integer-valued character table if and only if $\sigma \in G$ is conjugate to σ^j whenever j is prime to the order of σ.

Definition 3.1. Let $\sim_p$ be the equivalence relation on k-multipartitions generated by the following: $\mu \sim_p \nu$ if there is j such that $\mu_i = \nu_i$ for $i \neq j$, and ν_j is formed by replacing one part of size mp in μ_j with p parts of size m in ν_j.



$(61, 41^3) \quad \sim_3 \quad (2^31, 41^3) \quad \sim_3 \quad (2^31, 43)$

Figure 3. Example of three conjugacy classes which are congruent mod 3 in $\mathbb{Z}/2\mathbb{Z} \wr S_N$ (note that $m = 2$ in the first cycle type and $m = 1$ in the second).

Lemma 3.2. *Let p be a prime and G be a group with integer-valued character table. Let $\mu = (\mu_1, ..., \mu_k)$ and $\nu = (\nu_1, ..., \nu_k)$ be k-multipartitions of N, indexing conjugacy classes of $G \wr S_N$. If $\mu \sim_p \nu$, then $M^\lambda_\mu \equiv M^\lambda_\nu \pmod p$ for all k-multipartitions λ of N.*

Proof. It suffices to show $M^\lambda_\mu \equiv M^\lambda_\nu \pmod p$ if there exists j such that $\mu_i = \nu_i$ for all $i \neq j$, $\mu_j = (\xi, mp)$ for some ξ, and $\nu_j = (\xi, m^p)$. We break $RD(\lambda, \nu)$ into two cases. In case one, we consider the row decompositions of ν where m^p is tiled in the same row of λ. In case two we consider the row decompositions when m^p is not tiled in the same row. Recalling our formula for characters of permutation modules in Proposition 2.10, let

$$\beta = \sum_{\substack{\rho \in RD(\lambda,\nu) \text{ s.t. } m^p \text{ is tiled} \\ \text{in the same row}}} \alpha(\rho) \tag{6}$$

and

$$\gamma = \sum_{\substack{\rho \in RD(\lambda,\nu) \text{ s.t. } m^p \text{ is not tiled} \\ \text{in the same row}}} \alpha(\rho), \tag{7}$$

so that $M^\lambda_\nu = \beta + \gamma$. Case one will show $\beta \equiv M^\lambda_\mu \pmod p$. Case two shows $\gamma \equiv 0 \pmod p$. Together, these two congruences imply $M^\lambda_\mu \equiv M^\lambda_\nu \pmod p$.

In both cases, we break into subcases based on the ways to tile μ_i for $i \neq j$ and ξ. In case one, we have compatible tilings for μ and ν, and in case two, we have additional tilings for ν.

In case one, assume we have tiled all rows of μ_i for all $i \neq j$ and we have tiled ξ. We now have one row remaining. There is only one way to tile the last row for both μ and ν: put the remaining pieces into the remaining row. Let these row decompositions be denoted ρ_μ and ρ_ν respectively.

For ρ_μ, say that we place the final row r of size mp in the partition λ_q. The associated cycle product is c_j because mp comes from μ_j. Then mp contributes $\chi_q(c_j)$ to the product $\alpha(\rho_\mu)$. Then for ρ_ν, the p rows of size m are placed into λ_q. The conjugacy class of G associated with the p rows of size m is again c_j, so m^p contributes a factor of $\chi_q(c_j)^p$ to $\alpha(\rho_\nu)$.

By assumption, the character values of G are integral, so by Fermat's little theorem, $\chi_q(c_j) \equiv \chi_q(c_j)^p \pmod p$. All other factors in $\alpha(\rho_\mu)$ contributed by μ_i for $i \neq j$ and ξ are identical to the corresponding factors in $\alpha(\rho_\nu)$ Hence, $\alpha(\rho_\mu) \equiv \alpha(\rho_\nu) \pmod p$.

Summing over all the tilings in case one, we find $M^\lambda_\mu \equiv \beta \pmod p$.

In case two, assume we have tiled all rows of μ_i for $i \neq j$ and ξ, after which there are $t > 1$ remaining unfilled rows of the Young diagrams of λ. If $T \subseteq RD(\lambda, \nu)$ is the set of row decompositions extending our given tiling by μ_i for $i \neq j$ and ξ, then we will show

$$\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod p.$$

Then γ is the sum over all such T of $\sum_{\rho \in T} \alpha(\rho)$, from which it will follow $\gamma \equiv 0 \mod p$.

Call the lengths of the remaining rows $(m\ell_1, m\ell_2, \ldots, m\ell_t)$. Since the elements of T are in bijection with choices of placements of p cycles of length m into these rows,

$$|T| = \binom{p}{\ell_1, \ell_2, \ldots, \ell_t}. \tag{8}$$

Let $\rho \in T$. Note that all pieces of m^p come from μ_j, and thus have cycle product c_j, while all other cycles in μ are in the same place in T. Thus $\alpha(\rho) = \alpha(\rho')$ for all $\rho, \rho' \in T$. Hence $\sum_{\rho \in T} \alpha(\rho)$ is a sum of $|T|$ identical terms. Then $\sum_{\rho \in T} \alpha(\rho) \equiv 0 \pmod p$ because $|T|$ is divisible by p.

Case one has shown that $M^\lambda_\mu \equiv \beta \pmod p$, and case two has shown that $\gamma \equiv 0 \pmod p$. Since $M^\lambda_\nu = \beta + \gamma$, we conclude $M^\lambda_\mu \equiv M^\lambda_\nu \pmod p$. ☐

Corollary 3.3 (The mashing rule)**.** *Let G have integer-valued character table and k conjugacy classes. Let μ and ν be k-multipartitions of N. If $\mu \sim_p \nu$, then $\chi^\lambda_\mu \equiv \chi^\lambda_\nu \pmod p$ for all irreducible characters χ^λ of $G \wr S_N$.*

Proof. The set of irreducible characters and the set of characters of permutation modules form bases for the space of class functions on $G \wr S_N$. Since the change of basis matrix between these two bases is unimodular and upper-triangular, as stated in Lemma 2.12, χ^λ can be expressed as an integral linear combination of M^η for all k-multipartitions λ. It follows from Lemma 3.2 that $\mu \sim_p \nu$ implies $\chi^\lambda_\mu \equiv \chi^\lambda_\nu \pmod p$. ☐

3.2. **Proof of Main Theorem.** Using Corollary 3.3, the existence of one zero in the character table implies many more entries are divisible by p. We proceed, following Peluse and Soundararajan in [PS22], by using Proposition 2.7 to show sufficiently many entries of the character table are zero.

Definition 3.4. A partition is called a *t-core* if none of the hook lengths of its Young diagram are divisible by t where $t \in \mathbb{Z}$. For example, from Figure 4 one can see that $(4, 2, 1)$ is a 5-core.

$$\begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

FIGURE 4. Hook-lengths for $\lambda_i = (4, 2, 1)$

Peluse and Soundararajan proved the following estimate of the number of t-cores when t is slightly larger than the typical longest cycle in a random conjugacy class:

Proposition 3.5 ([PS22], Proposition 1)**.** *Let L be a positive integer, and let A be a real number with $1 \leq A \leq \log L / \log\log L$. Additionally suppose that t is a positive integer with*

$$t \geq \frac{\sqrt{6}}{2\pi}\sqrt{L}(\log L)\left(1 + \frac{1}{A}\right). \tag{9}$$

Then the number of partitions λ of L which are not t-cores is at most

$$O\left(p(L)\frac{\log L}{L^{\frac{1}{2A}}}\right),$$

independent of t satisfying (9).

Complementing the estimate in Proposition 3.5, Peluse and Soundararajan also estimated how many columns of the character table are congruent to a column corresponding to a partition with a large first part:

Proposition 3.6 ([PS22], Proposition 2)**.** *Let $p \leq \frac{(\log L)}{(\log\log L)^2}$ be a prime. Starting with a partition μ of L, we repeatedly replace every occurrence of p parts of the same size m by one part of size mp until we arrive at a partition $\tilde{\mu}$ where no part appears more than $p-1$ times. Then the largest part of $\tilde{\mu}$ exceeds*

$$\frac{\sqrt{6}}{2\pi}\sqrt{L}\,(\log L)\left(1 + \frac{1}{5p}\right),$$

except for at most

$$O\left(p(L)\exp\left(-L^{\frac{1}{15p}}\right)\right)$$

partitions μ.

We now extend Peluse and Soundarajan's estimate in Proposition 3.6 to k-multipartitions.

Proposition 3.7. *Let $p \ll N$ be a prime. Given a k-multipartition $\mu = (\mu_1, \ldots \mu_k)$ of N, for all μ_i with $1 \leq i \leq k$, we repeatedly replace every occurrence of p parts of the same size m by one part of size mp until we arrive at a k-multipartition $\tilde{\mu}$ where no part in any $\tilde{\mu}_i$ appears more than $p-1$ times.*

Then the largest part of $\tilde{\mu}$ is of size at least

$$\frac{\sqrt{6}}{2\pi}\sqrt{\frac{N}{k}}\left(\log\frac{N}{k}\right)\left(1+\frac{1}{5p}\right) \tag{10}$$

except for a number of multipartitions μ which is at most

$$O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right)p_k(N)\right).$$

Proof. For a k-multipartition $\mu = (\mu_1, \mu_2, \dots \mu_k)$ of N, let $\tilde{\mu}$ be as above. We will bound above the number of k-multipartitions μ such that $\tilde{\mu}$ has largest part less than (10).

For any μ, we know that for some $1 \leq i \leq k$, $|\mu_i| \geq \frac{N}{k}$. Fix i such that μ_i has size $|\mu_i| = a \geq \frac{N}{k}$. Then Proposition 3.6 tells us that the largest part of $\tilde{\mu}_i$ exceeds

$$\frac{\sqrt{6}}{2\pi}\sqrt{a}\,(\log a)\left(1+\frac{1}{5p}\right) \geq \frac{\sqrt{6}}{2\pi}\sqrt{\frac{N}{k}}\left(\log\frac{N}{k}\right)\left(1+\frac{1}{5p}\right)$$

except for at most

$$O\left(p(a)\exp\left(-a^{\frac{1}{15p}}\right)\right)$$

partitions μ_i of size a and therefore at most

$$O\left(p(a)\exp\left(-a^{\frac{1}{15p}}\right)p_{k-1}(N-a)\right)$$

total k-multipartitions μ with $|\mu_i| = a$. Furthermore, since $a \geq \frac{N}{k}$,

$$\exp\left(-a^{\frac{1}{15p}}\right) \leq \exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right),$$

and therefore summing over all $a \geq \frac{N}{k}$ we have that the number of multipartitions μ such that $|\mu_i| \geq \frac{N}{k}$ with no part in $\tilde{\mu}_i$ exceeding (10) is at most

$$\begin{aligned}
O\left(\sum_{a=\frac{N}{k}}^{N}\exp\left(-a^{\frac{1}{15p}}\right)p(a)p_{k-1}(N-a)\right) &\leq O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right)\sum_{a=\frac{N}{k}}^{N}p(a)p_{k-1}(N-a)\right)\\
&\leq O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right)\sum_{a=0}^{N}p(a)p_{k-1}(N-a)\right)\\
&= O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right)p_k(N)\right).
\end{aligned}$$

Since this bound is identical for each i, the number of k-multipartitions μ such that $\tilde{\mu}$ does not have a part of size greater than (10) is at most a factor of k greater than the bound above, and therefore also at most

$$O\left(\exp\left(-\left(\frac{N}{k}\right)^{\frac{1}{15p}}\right)p_k(N)\right).$$

□

Theorem 3.8. *Let G be a group with integer-valued character table, and let $G \wr S_N$ be the wreath product of G with the symmetric group S_N. For all primes p, the proportion of entries in the character table of $G \wr S_N$ divisible by p tends to 1 as $N \to \infty$.*

Proof. Let k be the number of conjugacy classes of G. Given a k-multipartition μ, let $\tilde{\mu}$ be the multipartition obtained by repeatedly replacing p parts of μ_i size m with one part of size mp until no μ_i has a part appearing more than $p-1$ times. For $A = 5p$, Proposition 3.7 implies that the largest part of $\tilde{\mu}$ has size

$$t \geq \frac{\sqrt{6}}{2\pi}\sqrt{\frac{N}{k}}\left(\log\frac{N}{k}\right)\left(1+\frac{1}{A}\right) \tag{11}$$

for a proportion of μ tending to 1 as $N \to \infty$. Now pick $A' \geq 1$ and $\delta > 0$ such that

$$\left(\log\frac{N}{k}\right)\left(1+\frac{1}{A}\right) \geq \sqrt{1+\delta}\left(\log\left(\frac{N}{k}(1+\delta)\right)\right)\left(1+\frac{1}{A'}\right).$$

By Corollary 2.15, the proportion of k-multipartitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ such that $|\lambda_i| \in \left(\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta)\right)$ for all i tends to 1 as $N \to \infty$. Thus, consider only (λ, μ) satisfying the above conditions.

Our choice of δ and A' imply that

$$\frac{\sqrt{6}}{2\pi}\sqrt{\frac{N}{k}}\left(\log\frac{N}{k}\right)\left(1+\frac{1}{A}\right) \geq \frac{\sqrt{6}}{2\pi}\sqrt{\frac{N(1+\delta)}{k}}\left(\log\left(\frac{N}{k}(1+\delta)\right)\right)\left(1+\frac{1}{A'}\right).$$

So if $(N_1, \ldots, N_k)$ is a partition of sufficiently large N, then by Proposition 3.5, the proportion of k-multipartitions λ with $|\lambda_i| = N_i$ such that some λ_i is not a t-core is

$$\sum_{i=1}^{k} O\left(\frac{\log N_i}{N_i^{\frac{1}{2A'}}}\right)$$

for all t satisfying (11), independent of t. Hence, over all k-multipartitions λ satisfying $|\lambda_i| \in \left(\frac{N}{k}(1-\delta), \frac{N}{k}(1+\delta)\right)$, the proportion of λ such that some λ_i is not a t-core is

$$O\left(\frac{\log\left(\frac{N}{k}(1+\delta)\right)}{\left(\frac{N}{k}(1-\delta)\right)^{\frac{1}{2A'}}}\right).$$

It follows that most (λ, μ) satisfy that λ_i is a t-core for t the largest part of $\tilde{\mu}$. Thus, for a proportion of (λ, μ) tending to 1 as $N \to \infty$, we have $\chi^{\lambda}_{\tilde{\mu}} = 0$ by Proposition 2.7 and therefore $\chi^{\lambda}_{\mu} \equiv 0 \mod p$ by Corollary 3.3. □

4. Weyl groups of type D

Definition 4.1. The *Weyl group of type D_N* is the group of $N \times N$ signed permutation matrices with an even number of entries equal to -1.

We will denote this group by D_N also (note that it is distinct from the dihedral group). D_N is a subgroup of $\mathbb{Z}/2\mathbb{Z} \wr S_N$ of index two. Hence, Clifford theory determines its representations:

Proposition 4.2. *The irreducible representations of the Weyl group of type D_N are as follows:*

(1) if (λ, μ) is a 2-multipartition of N such that $\lambda \neq \mu$, then

$$\operatorname{Res}_{D_N}^{B_N} V^{\lambda,\mu} = \operatorname{Res}_{D_N}^{B_N} V^{\mu,\lambda}$$

is an irreducible representation of D_N;

(2) if (λ, λ) is a 2-multipartition of N with equal parts, then

$$Res_{D_N}^{B_N} V^{\lambda,\lambda}$$

is the sum of two irreducible representations of D_N.

(3) Each irreducible representation of D_N appears exactly once in (1) or (2).

Proof. Let $\psi : B_N \to \{\pm 1\}$ be the character defined by taking the product of the nonzero entries of B_N. Then $\psi \otimes V^{\lambda,\mu} = V^{\mu,\lambda}$. Now the Proposition follows from Clifford theory (see [CR81]). $\square$

Corollary 4.3. *For all primes p, the proportion of entries in the character table of D_N which are divisible by p tends to 1 as $N \to \infty$.*

Proof. The number of irreducible representations of D_N of the form $\operatorname{Res}_{D_N}^{B_N} V^{\lambda,\mu}$ for $\lambda \neq \mu$ equals $\frac{1}{2}\left(p_2(N) - p(N/2)\right)$ when N is even, and $\frac{1}{2}p_2(N)$ when N is odd. The number of irreducible representations appearing as a summand of $\operatorname{Res}_{D_N}^{B_N} V^{\lambda,\lambda}$ is $2p(N/2)$ when N is even and 0 when N is odd. By Claim 2.14, we have $p_2(N) \gg p(N/2)$ for large enough N, so the proportion of irreducibles of the form $\operatorname{Res}_{D_N}^{B_N} V^{\lambda,\mu}$ goes to 1 as $N \to \infty$.

Since $D_N \subseteq B_N$ is of index two, at least half of the conjugacy classes in B_N intersect D_N. Since most entries in the character table of B_N are divisible by p, the same is true when we restrict to the columns which intersect D_N, since they are at least half of the columns. Hence, the proportion of entries in the character table of D_N which are divisible by p goes to 1 as $N \to \infty$. $\square$

References

[CR81] Charles W Curtis and Irving Reiner. *Methods of Representation Theory With Applications to Finite Groups and Orders, Vol. 1.* Wiley & Sons, Inc., 1981.

[HR18] G. H. Hardy and S. Ramanujan. "Asymptotic Formulaæ in Combinatory Analysis". In: *Proceedings of the London Mathematical Society* s2-17.1 (1918), pp. 75–115. DOI: https://doi.org/10.1112/plms/s2-17.1.75.

[Jam06] Gordon Douglas James. *The representation theory of the symmetric groups*. Vol. 682. Springer, 2006.

[Mac98] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.

[Mil19] Alexander R Miller. "On parity and characters of symmetric groups". In: *Journal of Combinatorial Theory, Series A* 162 (2019), pp. 231–240.

[Mur13] M Ram Murty. "The partition function revisited". In: *The Legacy of Srinivasa Ramanujan, in: Ramanujan Math. Soc. Lect. Notes Ser., Ramanujan Mathematical Society* (2013), pp. 261–279.

[PS22] Sarah Peluse and Kannan Soundararajan. "Almost all entries in the character table of the symmetric group are multiples of any given prime". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* (2022).

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Vol. 42. Springer, 1977.

[Spe32] Wilhelm Specht. "Eine Verallgemeinerung der symmetrischen Gruppe". PhD thesis. 1932.

[Zel81] Andrey V Zelevinsky. *Representations of finite classical groups: a Hopf algebra approach*. Vol. 869. Lecture Notes in Mathematics. Springer, 1981.

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