

# CONGRUENCES IN CHARACTER TABLES OF SYMMETRIC GROUPS

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ABSTRACT. If  $\lambda$  and  $\mu$  are two non-empty Young diagrams with the same number of squares, and  $\lambda$  and  $\mu$  are obtained by dividing each square into  $d^2$  congruent squares, then the corresponding character value  $\chi_\lambda(\mu)$  is divisible by  $d!$ .

## 1. Introduction

For any partition  $\lambda = 1^{m_1} 2^{m_2} \dots n^{m_n}$  of an integer  $n$ , let  $\chi_\lambda$  be the corresponding irreducible character of the symmetric group  $S_n$ , let  $\chi_\lambda(\mu)$  be the value at any  $\sigma \in S_n$  of cycle type  $\mu$ , and, fixing once and for all a positive integer  $d$ , define partitions

$$d.\lambda = d^{m_1} (2d)^{m_2} \dots (nd)^{m_n}, \quad \lambda = d^{dm_1} (2d)^{dm_2} \dots (nd)^{dm_n},$$

so  $d.\lambda$  is obtained by scaling the parts of  $\lambda$ , and  $\lambda$  is obtained by subdividing the squares of the Young diagram of  $\lambda$ . The purpose of this paper is to prove:

**Theorem 1.** *For any two partitions  $\lambda$  and  $\mu$  of a positive integer,*

$$(1.1) \quad \chi_\lambda(\mu) \equiv 0 \pmod{d!}.$$

*More generally, for any partition  $\lambda$  of a positive integer  $n$ , and any partition  $\mu$  of  $dn$ ,*

$$(1.2) \quad \chi_\lambda(d.\mu) \equiv 0 \pmod{d!}.$$

*For any two partitions  $\lambda$  and  $\mu$  of a positive integer not divisible by  $d$ ,*

$$(1.3) \quad \chi_\lambda(d^2.\mu) = 0.$$

Explicit results like these are rare. Previous results include J. McKay's characterization of partitions  $\lambda$  of  $n$  satisfying  $\chi_\lambda(1^n) \equiv 0 \pmod{2}$  [11], I. G. Macdonald's generalization for  $\chi_\lambda(1^n) \equiv 0 \pmod{p}$  [9], the corollary of Murnaghan–Nakayama that  $\chi_\lambda(\mu) = 0$  under certain conditions involving hook lengths [10], and the relation between ordinary and modular vanishing given by the fact that Frobenius' formula for  $\chi_\lambda(\mu)$  [3] implies, for any prime  $p$ , that  $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \pmod{p}$  whenever  $\nu$  can be obtained from  $\mu$  by breaking some part into  $p$  equal parts.

There are also general results of Burnside, J. G. Thompson, and P. X. Gallagher, with Burnside proving that zeros exist for nonlinear irreducible characters of a finite group [1], Thompson modifying Burnside's argument with a result of C. L. Siegel [19] to show that each irreducible character evaluates to zero or a root of unity on more than a third of the group elements [7], and Gallagher proving similarly that more than a third of the irreducible characters are zero or a root of unity on any larger than average class [4].

A few years ago, it was shown that if  $\chi \in \text{Irr}(S_n)$  and  $\sigma \in S_n$  are chosen at random, then  $\chi(\sigma) = 0$  with probability  $\rightarrow 1$  as  $n \rightarrow \infty$  [12]. The analogous result for  $\text{GL}(n, q)$  was established in joint work of the author with Gallagher and Larsen [5]. Larsen and the author subsequently showed that the proportion of zeros in the character table of a finite simple group of Lie type goes to 1 as the rank goes to infinity [8]. The limiting behavior for the proportion of zeros in the character table of  $S_n$  is not yet known, but

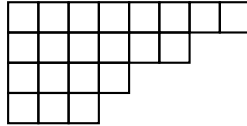
it was conjectured in [13] that, for any prime  $p$ , the proportion of  $p$ -divisible entries in the character table of  $S_n$  goes to 1 as  $n \rightarrow \infty$ .<sup>1</sup> The classical results of McKay [11] and Macdonald [9] imply that the proportion of  $p$ -divisible entries in the column of degrees  $\chi_\lambda(1)$  goes to 1 as  $n \rightarrow \infty$ , and recent results of Gluck [6] and Morotti [14] deal with certain other columns. Very recently, Peluse [17] established the conjecture for primes  $\leq 13$ , and then Peluse and Soundararajan [18] together established the full conjecture for all primes. So for each prime  $p$  we have  $\chi_\lambda(\mu) \equiv 0 \pmod{p}$  with probability  $\rightarrow 1$  as  $n \rightarrow \infty$ . Theorem 1 gives an unexpected stability result that answers the natural question of what happens if the shapes  $\lambda$  and  $\mu$  are naturally dilated with large scale factor: for any prime  $p$ , (1.1) with  $d \geq p$  implies that  $\chi_\lambda(\mu) \equiv 0 \pmod{p}$  for *all* partitions  $\lambda, \mu$  of a positive integer.

We prove (1.1) and (1.2) by showing that in the Murnaghan–Nakayama formula for computing  $\chi_\lambda(d, \mu)$  as a weighted sum over certain rim hook tableaux, the relevant rim hook tableaux admit an action of  $S_d$  that is both free and weight-preserving. This is done by first translating from rim hook tableaux to some new objects we call *cascades*, which are a matrix analogue of Com  t’s classical one-line binary notation for partitions, and which can be viewed as collections of lattice paths with weight defined in terms of crossings. As a benefit of independent interest, we obtain a lattice-path version of Murnaghan–Nakayama in Proposition 1. Then in Theorem 2 we establish an explicit weight-preserving free action of  $S_d$  on cascades. As a corollary we obtain (1.1) and (1.2), while (1.3) will come from Proposition 1.

## 2. Preliminaries

By *partition* of an integer  $n \geq 0$  we mean an integer sequence  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$  satisfying  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 1$  and  $\lambda_1 + \lambda_2 + \dots + \lambda_l = n$ . We say  $\lambda$  has *size*  $n$  with  $l$  *parts*, writing  $|\lambda| = n$  and  $\ell(\lambda) = l$ . The alternative shorthand  $\lambda = 1^{m_1} 2^{m_2} \dots n^{m_n}$  means  $\lambda$  is the partition with  $m_1$  1’s,  $m_2$  2’s, and so on, e.g.  $(4, 2, 1, 1) = 1^2 2^1 4^1$ .

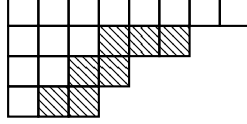
We identify  $\lambda$  with its *shape* or *Young diagram*, i.e. the left-justified array with  $\lambda_1$  squares in the first row,  $\lambda_2$  squares in the second row, and so on, e.g. the partition  $(8, 6, 4, 3)$  is identified with the following shape:



By *rim hook*  $\rho$  of  $\lambda$  we mean the union of a non-empty sequence of squares in  $\lambda$  such that each square is directly to the left or directly below the previous square and  $\lambda \setminus \rho$  is

<sup>1</sup>As suggested by the computations in [13], the author suspects that the same is true for arbitrary prime powers: if  $\lambda$  and  $\mu$  are chosen uniformly at random from the partitions of  $n$ , then for any prime power  $q$ ,  $\chi_\lambda(\mu) \equiv 0 \pmod{q}$  with probability  $\rightarrow 1$  as  $n \rightarrow \infty$ .

a Young diagram, e.g. the following is a rim hook of size 7 in  $(8, 6, 4, 3)$ :



By *rim hook tableau*  $T$  we mean a labeling of the squares of a non-empty Young diagram  $\lambda$  with integers  $1, 2, \dots, m$  such that the squares with label  $\geq i$  form a Young diagram  $T_i$  and, for  $1 \leq i \leq m$ , the squares labeled  $i$  form a (non-empty) rim hook of size  $\alpha_i$  in  $T_i$ . We say  $T$  has *shape*  $\lambda$  and *content*  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ , we write

$$T = \text{Tab}(T_1, T_2, \dots, T_{m+1}),$$

so  $T_1 = \lambda$  and  $T_{m+1} = \emptyset$ , and we define the *weight* of  $T$  by

$$(2.1) \quad \text{wt}(T) = \prod_{i=1}^m (-1)^{\#\{\text{rows of } T \text{ occupied by } i\} - 1}.$$

An example rim hook tableau of shape  $(8, 6, 4, 3)$  and content  $(4, 4, 6, 3, 2, 2)$  is

6	5	3	3	2	1	1	1
6	5	3	2	2	1		
4	4	3	2				
4	3	3					

which has weight  $(-1)^{2-1+3-1+4-1+2-1+2-1+2-1}$ .

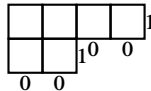
Denoting by  $\mathcal{T}(\lambda, \alpha)$  the set of all rim hook tableaux of shape  $\lambda$  and content  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ , the mapping  $T \mapsto (T_1, T_2, \dots, T_{m+1})$  takes  $\mathcal{T}(\lambda, \alpha)$  bijectively onto the set of all partition sequences  $\lambda = \lambda^1, \lambda^2, \dots, \lambda^{m+1} = \emptyset$  in which each succeeding  $\lambda^i$  is obtained from the previous partition  $\lambda^{i-1}$  by removing a rim hook of size  $\alpha_{i-1}$ , so in this way rim hook tableaux serve as shorthand for the various ways of going from  $\lambda$  to  $\emptyset$  by successively removing rim hooks of prescribed size.

The *Murnaghan–Nakayama formula* [15, 16] gives, for any two partitions  $\lambda$  and  $\mu$  of a positive integer, and any sequence  $\alpha$  that can be rearranged to  $\mu$ ,

$$(2.2) \quad \chi_\lambda(\mu) = \sum_{T \in \mathcal{T}(\lambda, \alpha)} \text{wt}(T).$$

### 3. Cascades

By the *word* of a partition  $\lambda$  we mean the binary sequence  $w(\lambda)$  obtained from  $\lambda$  by writing 0 under each column, 1 alongside each row, and reading clockwise, e.g. the word of  $(4, 2)$  is 001001:



By the *shape* of a binary sequence  $\beta = (\beta_1, \beta_2, \dots, \beta_k)$  we mean the partition  $\text{sh}(\beta) = 1^{m_1} 2^{m_2} \dots$  where  $m_i$  is the number of 1's in  $\beta$  with exactly  $i$  0's to the left, e.g. both

001001 and 10010010 have shape  $(4, 2)$ . The word of a non-empty partition  $\lambda$  is the unique binary sequence of shape  $\lambda$  that starts with 0 and ends with 1; the word of the empty partition is the empty sequence.

The standard fact that we require goes back to Com  t in the 1950's (cf. [2]) and can be stated as follows:

**Lemma 1.** *For any finite binary sequence  $\beta$  and integer  $k$ , the mapping  $\beta' \mapsto \text{sh}(\beta')$  takes  $\mathcal{B}$ , the set of  $\beta'$  obtainable from  $\beta$  by swapping a 0 with a right-lying 1 exactly  $k$  positions away, bijectively onto the set of shapes obtainable from  $\text{sh}(\beta)$  by removing a rim hook of size  $k$ , and moreover, the number of rows occupied by the rim hook  $\text{sh}(\beta) \setminus \text{sh}(\beta')$  equals the number of 1's lying weakly between the swapped 0-1 pair.  $\square$*

For example, if  $\lambda$  is the partition  $(8, 6, 4, 3)$  and  $\rho$  is the rim hook of  $\lambda$  shown in §2, and if  $\beta = 11000101001001$ , so that  $\text{sh}(\beta) = \lambda$ , then the shape  $\lambda \setminus \rho$  corresponds to  $\beta' = 11010101000001$ .

**3.1.** Our main tool is the following:

**Definition 1.** A *cascade* is a binary matrix  $C$  with rows  $C_i = (C_{i1}, C_{i2}, \dots, C_{il})$ ,  $1 \leq i \leq m$ , such that

1)  $C_{11} = 0$  and  $C_{1l} = 1$ ,

2) for each row  $C_i$  with  $1 \leq i \leq m - 1$ , there is a unique pair  $a_i < b_i$  such that

$$C_{ia_i} = 0, \quad C_{ib_i} = 1, \quad C_{i+1} = (C_{i\tau(1)}, C_{i\tau(2)}, \dots, C_{i\tau(l)}) \quad \text{for } \tau = \tau_{C,i} = (a_i \ b_i),$$

3)  $C_m = (1, 1, \dots, 1, 0, 0, \dots, 0)$ .

The *shape* of  $C$  is the shape of  $C_1$ .

The *content* of  $C$  is the sequence

$$(b_1 - a_1, b_2 - a_2, \dots, b_{m-1} - a_{m-1}).$$

A *crossing* in  $C$  is a pair  $(i, j)$  such that

$$1 \leq i \leq m - 1, \quad C_{ij} = 1, \quad \text{and} \quad a_i < j < b_i.$$

The *weight* of  $C$  is defined by

$$\text{wt}(C) = (-1)^{\text{cr}(C)}, \quad \text{where} \quad \text{cr}(C) = \#\{\text{crossings in } C\}.$$

The *permutation* associated to  $C$  is

$$\pi_C = \begin{pmatrix} 1 & 2 & \dots & k \\ \sigma_C(i_1) & \sigma_C(i_2) & \dots & \sigma_C(i_k) \end{pmatrix},$$

where  $i_1 < i_2 < \dots < i_k$  are the positions of the 1's in the first row of  $C$ , and

$$\sigma_C = \tau_{C,m-1} \tau_{C,m-2} \dots \tau_{C,1}.$$

We denote by  $\mathcal{C}(\lambda, \alpha)$  the set of cascades of shape  $\lambda$  and content  $\alpha$ .

**Lemma 2.** *The mapping*

$$(3.1) \quad \Theta : C \mapsto \text{Tab}(\text{sh}(C_1), \text{sh}(C_2), \dots, \text{sh}(C_{\# \text{rows}(C)}))$$

*takes the set of cascades bijectively onto the set of rim hook tableaux, and it preserves shape, content, and weight.*

*Proof.* This follows from Com  t's observation in Lemma 1, the standard facts in  2 about rim hook tableaux, and the fact that there is a unique binary sequence  $\beta$  of a given non-empty shape such that  $\beta$  starts with 0 and ends with 1. In particular,

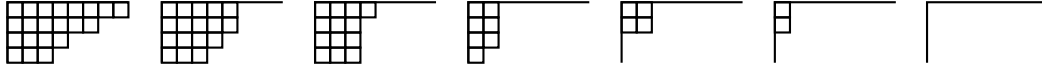
$$(3.2) \quad \Theta^{-1} : T \mapsto \text{Mat}(w_\lambda(T_1), w_\lambda(T_2), w_\lambda(T_3), \dots, w_\lambda(T_{m+1})),$$

where  $\lambda = \text{sh}(T_1)$ ,  $m$  is the largest label in  $T$ ,  $w_\lambda(T_i)$  is the sequence obtained from  $w(T_i)$  by appending to the start  $\ell(\lambda) - \ell(T_i)$  many 1's and to the end  $\lambda_1 - T_{i1}$  many 0's, and where  $\text{Mat}(r_1, r_2, \dots, r_k)$  with  $r_i = (r_{i1}, r_{i2}, \dots)$  means the matrix  $(r_{ij})$ .  $\square$

*Example.* Consider the following cascade  $C$ :

$$(3.3) \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The shape is  $(8, 6, 4, 3)$ , the content is  $(4, 4, 6, 3, 2, 2)$ , the weight is  $(-1)^{1+2+3+1+1+1}$ . The row shapes  $\text{sh}(C_k)$  are:



The corresponding rim hook tableau  $\text{Tab}(\text{sh}(C_1), \text{sh}(C_2), \dots, \text{sh}(C_7))$  is:

6	5	3	3	2	1	1	1
6	5	3	2	2	1		
4	4	3	2				
4	3	3					

The associated permutation  $\pi_C$  is the transposition  $(2\ 4)$  in  $S_4$ .

**3.2.** We define a *path* in a cascade  $C = (C_1, C_2, \dots, C_m)$  to be a sequence of column positions  $p = (p_1, p_2, \dots, p_m)$ , one position  $p_i$  for each row  $C_i$ , such that

$$C_{1p_1} = 1 \quad \text{and} \quad p_{i+1} = \tau_{C,i}(p_i) \quad \text{for} \quad 1 \leq i \leq m-1.$$

We say  $p$  *starts* at  $p_1$  and *ends* at  $p_m$ . There is exactly one path for each 1 in the first row of  $C$ , and we agree to always number the paths  $p^1, p^2, p^3, \dots$  according to relative start position, so that  $p_1^1 < p_1^2 < p_1^3 < \dots$ . With this convention,

$$(3.4) \quad \pi_C(i) = p_m^i, \quad i = 1, 2, \dots$$

By a *crossing* of paths  $p, p'$  in  $C$  we mean a pair  $(i, j)$  with  $1 \leq i \leq m - 1$  such that

$$p_i = j, \quad p_i < p'_i, \quad \text{and} \quad p'_{i+1} < p_{i+1}.$$

**Lemma 3.** For a cascade  $C$  with paths  $p^1, p^2, \dots, p^k$ ,

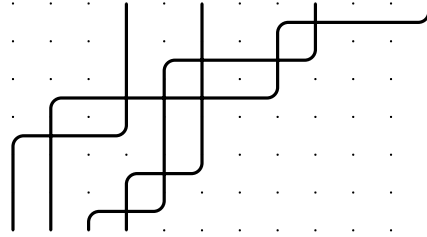
$$(3.5) \quad \{\text{crossings in } C\} = \bigcup_{1 \leq i < j \leq k} \{\text{crossings of } p^i \text{ and } p^j\}.$$

*Proof.* By comparing definitions. □

**3.3.** It is often convenient to visualize a cascade by constructing an associated graph.

**Definition 2.** The *diagram* or *graph* of a cascade is obtained by replacing each 1 by a node, each 0 by an empty space “ $\cdot$ ”, and then connecting any two nodes  $x, y$  that occupy adjacent rows and either share a single column or occupy the two columns where the two rows differ.

*Example.* The diagram of the cascade in (3.3) is:



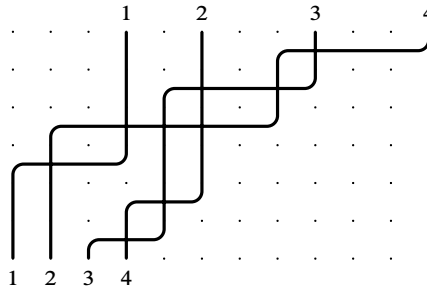
The paths of the cascade are

$$\begin{aligned} p^1 &= (4, 4, 4, 4, 1, 1, 1), & p^2 &= (6, 6, 6, 6, 6, 4, 4), \\ p^3 &= (9, 9, 5, 5, 5, 5, 3), & p^4 &= (12, 8, 8, 2, 2, 2, 2). \end{aligned}$$

There are 9 crossings in total, e.g.  $p^3$  and  $p^4$  cross 3 times. And the permutation

$$\pi_C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

can be read off from the diagram by numbering the nodes in the top row, from left to right, 1, 2,  $\dots$ , doing the same in the bottom row, and then chasing through the diagram from top to bottom:



**3.4.** Denote by  $\text{sgn}(\sigma)$  the *sign* of a permutation  $\sigma$ , so that

$$\text{sgn}(\sigma) = (-1)^{\iota(\sigma)}, \quad \iota(\sigma) = \#\{\text{pairs } i < j \text{ with } \sigma(j) < \sigma(i)\}.$$

**Lemma 4.** For any cascade  $C$ , we have

$$(3.6) \quad \text{wt}(C) = \text{sgn}(\pi_C).$$

*Proof.* Consider the paths  $p^1, p^2, \dots, p^k$  in  $C = (C_1, C_2, \dots, C_m)$ , numbered so  $\pi_C(i) = p_m^i$ , and let  $\text{cr}(p^i, p^j)$  be the number of crossings of  $p^i$  and  $p^j$ , so by (3.5),

$$(3.7) \quad \text{cr}(C) = \sum_{1 \leq i < j \leq k} \text{cr}(p^i, p^j).$$

Fix a pair  $i < j$ , so  $p^i$  starts left of  $p^j$ . If  $\pi_C(j) < \pi_C(i)$ , then  $p^i$  ends to the right of  $p^j$ , so  $p^i$  and  $p^j$  must have an odd number of crossings; if  $\pi_C(i) < \pi_C(j)$ , then  $p^i$  ends to the left of  $p^j$ , so  $p^i$  and  $p^j$  must have an even number of crossings. Hence

$$(3.8) \quad \iota(\pi_C) \equiv \sum_{1 \leq i < j \leq k} \text{cr}(p^i, p^j) \pmod{2}.$$

By (3.7) and (3.8), we have  $\text{cr}(C) \equiv \iota(\pi_C) \pmod{2}$ , so  $\text{wt}(C) = \text{sgn}(\pi_C)$ .  $\square$

As a corollary, we have the following useful reformulation of Murnaghan–Nakayama:

**Proposition 1.** For any two partitions  $\lambda$  and  $\mu$  of a positive integer, and any sequence  $\alpha$  that can be rearranged to  $\mu$ , we have

$$(3.9) \quad \chi_\lambda(\mu) = \sum_{C \in \mathcal{C}(\lambda, \alpha)} \text{wt}(C), \quad \text{wt}(C) = (-1)^{\text{cr}(C)} = \text{sgn}(\pi_C),$$

where  $\mathcal{C}(\lambda, \alpha)$  is the set of cascades of shape  $\lambda$  and content  $\alpha$ .

*Proof.* By Lemmas 2 and 4.  $\square$

#### 4. Proof of Theorem 1

**An action on cascades.** The main object of this section is to prove the following:

**Theorem 2.** Let  $\lambda$  be a partition of a positive integer  $n$ , so  $\lambda$  is a partition of  $d^2n$ , and let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$  be a sequence of positive  $d$ -divisible integers summing to  $d^2n$ .

Define a pairing  $\sigma.C$  on  $S_d \times \mathcal{C}(\lambda, \alpha)$  by

$$(4.1) \quad (\sigma, C) \mapsto C\Phi(\sigma)^{-1},$$

where  $\Phi(\sigma)$  is the block-diagonal matrix

$$\Phi(\sigma) = \begin{pmatrix} \phi(\sigma) & & & \\ & \phi(\sigma) & & \\ & & \ddots & \\ & & & \phi(\sigma) \end{pmatrix}$$

with  $\lambda_1 + \ell(\lambda)$  copies of the  $d$ -by- $d$  permutation matrix  $\phi(\sigma) = (\delta_{i\sigma(j)})$  on the diagonal.

- (i) *The pairing  $\sigma.C$  is an action of  $S_d$  on  $\mathcal{C}(\lambda, \alpha)$ ,*
- (ii) *the action is free,*
- (iii) *the action is weight-preserving, i.e.  $\text{wt}(\sigma.C) = \text{wt}(C)$  for all  $\sigma$  and  $C$ .*

*Proof.* Assume  $\mathcal{C}(\lambda, \alpha) \neq \emptyset$ . Let  $l = \lambda_1 + \ell(\lambda)$  and  $L = dn + d\ell(\lambda)$ .

The word of  $\lambda$  starts with 0, ends with 1, and consists of  $\lambda_1$  0's and  $\ell(\lambda)$  1's, so the sequence  $w(\lambda)$  has length  $l$ . The word of  $\lambda$  is obtained by replacing in  $w(\lambda)$  each 0 by  $d$  consecutive 0's and each 1 by  $d$  consecutive 1's, so  $w(\lambda)$  starts with  $d$  0's, ends with  $d$  1's, has length  $L$ , and writing  $w(\lambda) = (w_1, w_2, \dots, w_L)$ ,

$$(4.2) \quad w_{1+dk} = w_{2+dk} = \dots = w_{d+dk}, \quad 0 \leq k \leq L/d - 1.$$

In particular, each  $C \in \mathcal{C}(\lambda, \alpha)$  has  $L$  columns, so the matrix multiplication on the right-hand side of (4.1) makes sense, and multiplying  $C$  on the right by  $\Phi(\sigma)^{-1}$  permutes the first  $d$  columns of  $C$ , the next  $d$  columns of  $C$ , and so on: denoting by  $\text{Col}_i(C)$  the  $i$ -th column of  $C$ , we have

$$(4.3) \quad \text{Col}_{i+dk}(C) = \text{Col}_{\sigma(i)+dk}(\sigma.C)$$

for  $1 \leq i \leq d$  and  $0 \leq k \leq L/d - 1$ .

- (i). Fix  $C \in \mathcal{C}(\lambda, \alpha)$  and  $\sigma \in S_d$ . Let  $C' = \sigma.C$ . By (4.2) and (4.3),

$$(4.4) \quad C'_1 = C_1.$$

The last row of  $C$  is  $C_m = (1, \dots, 1, 0, \dots, 0)$ , with  $d\ell(\lambda)$  1's, so by (4.3),

$$(4.5) \quad C'_m = C_m.$$

By (4.4), (4.5), and  $C$  being a cascade,  $C'$  satisfies the first and third cascade conditions.

Let  $C'_i$  and  $C'_{i+1}$  be two consecutive rows in  $C'$ . Since  $C$  is a cascade, the rows  $C_i$  and  $C_{i+1}$  differ in exactly two positions,  $a_i$  and  $b_i$  with  $a_i < b_i$ , and

$$C_{i,a_i} = 0, \quad C_{i,b_i} = 1, \quad C_{i+1,a_i} = 1, \quad C_{i+1,b_i} = 0.$$

Since the difference  $\alpha_i = b_i - a_i$  is positive and divisible by  $d$ ,

$$(4.6) \quad a_i = r_i + ds_i \quad \text{and} \quad b_i = r_i + dt_i$$

for some non-negative integers  $r_i, s_i, t_i$  with  $1 \leq r_i \leq d$  and  $s_i < t_i$ . Setting

$$(4.7) \quad a'_i = \sigma(r_i) + ds_i \quad \text{and} \quad b'_i = \sigma(r_i) + dt_i,$$

and using (4.3), we have that  $C'_i$  and  $C'_{i+1}$  differ in exactly positions  $a'_i$  and  $b'_i$ , and

$$C'_{i,a'_i} = 0, \quad C'_{i,b'_i} = 1, \quad C'_{i+1,a'_i} = 1, \quad C'_{i+1,b'_i} = 0.$$

Since  $s_i < t_i$ , we also have that  $a'_i < b'_i$ . So  $C'$  satisfies the second condition of a cascade. Hence  $C'$  is a cascade.

By (4.4), the shape of the cascade  $C'$  is  $\lambda$ . The content of  $C'$  is  $(b'_1 - a'_1, b'_2 - a'_2, \dots)$ , which by (4.6) and (4.7) equals  $\alpha$ . So  $C' \in \mathcal{C}(\lambda, \alpha)$ . This concludes the proof of (i).



(ii). Let  $z_i(C)$  be the number of 0's in the  $i$ -th column of a cascade  $C \in \mathcal{C}(\lambda, \alpha)$ . Let

$$(4.8) \quad z(C) = (z_1(C), z_2(C), \dots, z_d(C)).$$

By the cascade conditions, and the positivity and  $d$ -divisibility of the  $\alpha_i$ 's, we have

$$(4.9) \quad z_i(C) \neq z_j(C) \quad \text{for } 1 \leq i < j \leq d.$$

By (4.3),

$$(4.10) \quad z(\sigma.C) = (z_{\sigma^{-1}(1)}(C), z_{\sigma^{-1}(2)}(C), \dots, z_{\sigma^{-1}(d)}(C)).$$

From (4.9) and (4.10), for each  $C \in \mathcal{C}(\lambda, \alpha)$ , we have

$$(4.11) \quad \sigma.C = C \text{ if and only if } \sigma = 1.$$

This concludes the proof of (ii).

(iii). Fix a cascade  $C \in \mathcal{C}(\lambda, \alpha)$  and a permutation  $\sigma \in S_d$ , so  $\sigma.C \in \mathcal{C}(\lambda, \alpha)$  by (i). Let  $p^1, p^2, \dots, p^{d\ell(\lambda)}$  be the paths in  $C$ , so  $p_1^1 < p_1^2 < \dots$  and

$$(4.12) \quad \pi_C(i) = p_m^i,$$

and let  $q^1, q^2, \dots, q^{d\ell(\lambda)}$  be the paths in  $\sigma.C$ , so  $q_1^1 < q_1^2 < \dots$  and

$$(4.13) \quad \pi_{\sigma.C}(i) = q_m^i.$$

Let  $\gamma$  be the permutation in  $S_L$  given by

$$(4.14) \quad \gamma(i + dk) = \sigma(i) + dk, \quad 1 \leq i \leq d, \quad 0 \leq k \leq L/d - 1.$$

By (4.3), the sequences

$$(4.15) \quad \sigma.p^i = (\gamma(p_1^i), \gamma(p_2^i), \dots, \gamma(p_m^i)), \quad 1 \leq i \leq d\ell(\lambda),$$

are the paths of  $\sigma.C$ , in some order. Let  $\omega$  be the permutation in  $S_{d\ell(\lambda)}$  given by

$$(4.16) \quad \omega(i + dk) = \sigma(i) + dk, \quad 1 \leq i \leq d, \quad 0 \leq k \leq \ell(\lambda) - 1.$$

Then by (4.2), for each  $i$ ,

$$(4.17) \quad q^{\omega(i)} = \sigma.p^i.$$

Since  $C_m = (1, \dots, 1, 0, \dots, 0)$  with  $d\ell(\lambda)$  1's, we also have  $\gamma(p_m^i) = \omega(p_m^i)$ , so

$$(4.18) \quad q_m^{\omega(i)} = \omega(p_m^i).$$

By (4.12), (4.13), and (4.18), the permutation  $\pi_{\sigma.C}$  takes  $\omega(i)$  to  $\omega(\pi_C(i))$  for each  $i$ , i.e.

$$(4.19) \quad \pi_{\sigma.C} = \omega \pi_C \omega^{-1}.$$

So  $\pi_{\sigma.C}$  and  $\pi_C$  have the same sign. By Lemma 4, we conclude that

$$(4.20) \quad \text{wt}(\sigma.C) = \text{wt}(C)$$

for all  $\sigma \in S_d$  and  $C \in \mathcal{C}(\lambda, \alpha)$ . This concludes the proof of (iii) and Theorem 2.  $\square$

It is worth remarking that Theorem 2 and Lemma 2 together give a weight-preserving free action on rim hook tableaux:

**Corollary 1.** *For any partition  $\lambda$  of a positive integer  $n$ , and any sequence  $\alpha$  of positive  $d$ -divisible integers summing to  $d^2n$ , there is a well-defined action of  $S_d$  on  $\mathcal{T}(\lambda, \alpha)$  given by  $\sigma.T = \Theta(\sigma.\Theta^{-1}(T))$ , and this action is both free and weight-preserving.  $\square$*

*Example.* With  $d = 3$  and  $\lambda = (3, 2)$ , the following shows an  $S_d$ -orbit of a cascade  $C$  and corresponding rim hook tableau  $T$  of shape  $\lambda$  and content  $(3, 3, 6, 6, 3, 3, 6, 9, 3)$ .

$\sigma$	diagram of $\sigma.C$	$\sigma.T$																																																						
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**Proof of Theorem 1.** For (1.2), let  $\lambda$  be a partition of a positive integer  $n$ , and let  $\mu$  be a partition of  $dn$ . By Proposition 1, we have

$$\chi_\lambda(d.\mu) = \sum_{C \in \mathcal{C}(\lambda, d.\mu)} \text{wt}(C),$$

and by Theorem 2 there exists a weight-preserving free action of  $S_d$  on  $\mathcal{C}(\lambda, d.\mu)$ . So  $\chi_\lambda(d.\mu)$  is divisible by  $d!$ . This completes the proof of (1.2).

(1.1) is a special case of (1.2): let  $\lambda$  and  $\mu$  be partitions of a positive integer  $n$ , write  $\mu = 1^{m_1} 2^{m_2} \dots n^{m_n}$ , and define  $\nu = 1^{dm_1} 2^{dm_2} \dots n^{dm_n}$ , so that  $\nu$  is a partition of  $dn$  with  $d.\nu = \mu$ , and hence by (1.2),  $\chi_\lambda(\mu)$  is divisible by  $d!$ .

For (1.3), let  $\lambda$  and  $\mu$  be partitions of an integer  $n$  not divisible by  $d$ . Suppose that there exists a cascade  $C \in \mathcal{C}(\lambda, d^2.\mu)$ , let  $D$  be the matrix with columns

$$\text{Col}_1(C), \text{Col}_{d+1}(C), \text{Col}_{2d+1}(C), \dots,$$

occurring in that order, and let  $C'$  be the matrix obtained from  $D$  by deleting redundant rows. Then  $C' \in \mathcal{C}(\lambda, d.\mu')$  for some partition  $\mu'$ , hence  $n = d|\mu'|$ . So  $\mathcal{C}(\lambda, d^2.\mu) = \emptyset$ , hence by Proposition 1,  $\chi_\lambda(d^2.\mu)$  equals 0.  $\square$

## References

- [1] W. Burnside, On an arithmetical theorem connected with roots of unity, and its application to group-characteristics. *Proc. Lond. Math. Soc.* **1** (1904) 112–116.
- [2] S. Com  t, Notations for partitions. *Math. Tables Aids Comput.* **9** (1955) 143–146.
- [3] G. Frobenius,   ber die Charaktere der symmetrischen Gruppe. *Sitzungsberichte der K  niglich Preu  ischen Akademie der Wissenschaften zu Berlin*, 1900, 516–534.
- [4] P. X. Gallagher, Degrees, class sizes and divisors of character values. *J. Group Theory* **15** (2012) 455–467.
- [5] P. X. Gallagher, M. J. Larsen, and A. R. Miller, Many zeros of many characters of  $\text{GL}(n, q)$ . *Int. Math. Res. Not. IMRN*, in press.
- [6] D. Gluck, Parity in columns of the character table of  $S_n$ . *Proc. Amer. Math. Soc.* **147** (2019) 1005–1011.
- [7] I. M. Isaacs, *Character theory of finite groups*. Dover, New York, 1994.
- [8] M. J. Larsen and A. R. Miller, The sparsity of character tables of high rank groups of Lie type. Preprint arXiv:2006.00847.
- [9] I. G. Macdonald, On the degrees of the irreducible representations of symmetric groups, *Bull. London Math. Soc.* **3** (1971) 189–192.
- [10] I. G. Macdonald, *Symmetric Functions and Hall polynomials*, 2nd ed. Oxford University Press, 1995.
- [11] J. McKay, Irreducible representations of odd degree. *J. Algebra* **20** (1972) 416–418.
- [12] A. R. Miller, The probability that a character value is zero for the symmetric group. *Math. Z.* **277** (2014) 1011–1015.
- [13] A. R. Miller, On parity and characters of symmetric groups. *J. Combin. Theory Ser. A* **162** (2019) 231–240.
- [14] L. Morotti, On divisibility by primes in columns of character tables of symmetric groups. *Arch. Math. (Basel)* **114** (2020) 361–365.
- [15] F. D. Murnaghan, The characters of the symmetric group. *Amer. J. Math.* **59** (1937) 739–753.

- [16] T. Nakayama, On some modular properties of the irreducible representations of a symmetric group I, II. Jap. J. Math. **17** (1941) 165–184 and 411–423.
- [17] S. Peluse, On even entries in the character table of the symmetric group. Preprint arXiv:2007.06652.
- [18] S. Peluse and K. Soundararajan, Almost all entries in the character table of the symmetric group are multiples of any given prime. Preprint arXiv:2010.12410.
- [19] C. L. Siegel, The trace of totally positive and real algebraic integers. Ann. of Math. **46** (1945) 302–312.

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