ZEROS AND ROOTS OF UNITY IN CHARACTER TABLES

ALEXANDER ROSSI MILLER

Dedicated to the memory of Patrick X. Gallagher

Abstract. For any finite group G, Thompson proved that, for each $\chi \in \mathrm{Irr}(G)$, $\chi(g)$ is a root of unity or zero for more than a third of the elements $g \in G$, and Gallagher proved that, for each larger than average class g^G, $\chi(g)$ is a root of unity or zero for more than a third of the irreducible characters $\chi \in \mathrm{Irr}(G)$. We show that in many cases "more than a third" can be replaced by "more than half".

For any finite group G, let

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{|\{g \in G : \chi(g) \text{ is a root of unity or zero}\}|}{|G|}$$

and

$$\theta'(G) = \min_{|g^G| \geq \frac{|G|}{|\mathrm{Cl}(G)|}} \frac{|\{\chi \in \mathrm{Irr}(G) : \chi(g) \text{ is a root of unity or zero}\}|}{|\mathrm{Irr}(G)|}.$$

Burnside proved that each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$ has at least one zero, P. X. Gallagher proved that each $g \in G$ with $|g^G| > |G|/|\mathrm{Cl}(G)|$ is a zero of at least one $\chi \in \mathrm{Irr}(G)$, J. G. Thompson proved that

$$\theta(G) > 1/3,$$

and Gallagher proved that

$$\theta'(G) > 1/3.$$

The proofs run by taking the relations $\sum_{g \in G} |\chi(g)|^2 = |G|$ ($\chi \in \mathrm{Irr}(G)$) and $\sum_{\chi \in \mathrm{Irr}(G)} |\chi(g)|^2 = |G|/|g^G|$, applying the elements σ of the Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$, averaging over $\mathcal{G}$, and using that the average over $\mathcal{G}$ of $|\sigma(\alpha)|^2$ is ≥ 1 for any nonzero algebraic integer $\alpha \in \mathbb{Q}(e^{2\pi i/|G|})$, or using the fact, due to C. L. Siegel, that the average over $\mathcal{G}$ of $|\sigma(\alpha)|^2$ is $\geq 3/2$ for any algebraic integer $\alpha \in \mathbb{Q}(e^{2\pi i/|G|})$ which is neither a root of unity nor zero, cf. [1, 3, 4, 11]. For certain groups, there are also strong asymptotic results about zeros due to Gallagher, M. Larsen, and the author [5, 8, 10].

Are the lower bounds of 1/3 for $\{\theta(G) : |G| < \infty\}$ and 1/3 for $\{\theta'(G) : |G| < \infty\}$ given by Thompson and Gallagher the best possible?

Question 1. *What is the greatest lower bound of* $\{\theta(G) : |G| < \infty\}$*?*

Question 2. *What is the greatest lower bound of* $\{\theta'(G) : |G| < \infty\}$*?*

The author suspects that the answers to these questions are both 1/2. In particular, we propose the following:

Conjecture 1. *$\theta(G)$ and $\theta'(G)$ are $\geq 1/2$ for every finite group G.*

We establish the conjecture for all finite nilpotent groups by establishing a much stronger result about zeros for this family of groups, which includes all p-groups. The number of p-groups of order p^n was shown by G. Higman [6] and C. C. Sims [12] to equal $p^{\frac{2}{27}n^3+O(n^{8/3})}$ with $n \to \infty$, and it is a folklore conjecture that almost all finite groups are nilpotent in the sense that

$$\frac{\text{the number of nilpotent groups of order at most } n}{\text{the number of groups of order at most } n} = 1 + o(1),$$

which, in view of our result, would mean that Conjecture 1 holds for almost all finite groups.

Conjecture 1 is readily verified for rational groups, such as Weyl groups, and all groups of order $< 2^9$, and although $\theta(G) = 1/2$ for certain dihedral groups, the second inequality is strict in all known cases. The author suspects that both inequalities are strict for all finite simple groups:

Conjecture 2. *$\theta(G)$ and $\theta'(G)$ are $> 1/2$ for every finite simple group G.*

We verify Conjecture 2 for A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$, all sporadic groups, and all simple groups of order $\leq 10^9$. We also show that both $\theta(Suz(2^{2n+1}))$ and $\theta'(Suz(2^{2n+1}))$ tend to $1/2$ as $n \to \infty$. In particular, the answers to Questions 1 and 2 must lie between 1/3 and 1/2.

§1

We begin with our results on finite nilpotent groups.

Theorem 1. *For each finite nilpotent group G, and each $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$, $\chi(g) = 0$ for more than half of the elements $g \in G$.*

Theorem 2. *Let G be a finite nilpotent group, and let $g \in G$.*
If $|g^G| > \frac{|G|}{|\mathrm{Cl}(G)|}$, then $\chi(g) = 0$ for more than half of the nonlinear $\chi \in \mathrm{Irr}(G)$.
If $|g^G| = \frac{|G|}{|\mathrm{Cl}(G)|}$, then $\chi(g) = 0$ for at least half of the nonlinear $\chi \in \mathrm{Irr}(G)$.

Corollary 3. *$\theta(G)$ and $\theta'(G)$ are $> 1/2$ whenever G is nilpotent.*

The key ingredient in the proofs of Theorems 1 and 2 is Proposition 8, which will replace the result of Siegel used by Thompson and Gallagher. Its proof relies on some auxiliary results of independent interest and is based on arithmetic in cyclotomic fields.

For each positive integer k, we denote by ζ_k a primitive k-th root of unity. For any algebraic integer α contained in some cyclotomic field, we denote by $\mathfrak{l}(\alpha)$ the least integer l such that α is a sum of l roots of unity, by $\mathfrak{f}(\alpha)$ the least positive integer k such that $\alpha \in \mathbb{Q}(\zeta_k)$, and by $\mathfrak{m}(\alpha)$ the normalized trace

$$\frac{1}{[\mathbb{Q}(|\alpha|^2) : \mathbb{Q}]} \mathrm{Tr}_{\mathbb{Q}(|\alpha|^2)/\mathbb{Q}}(|\alpha|^2),$$

so for any cyclotomic field $\mathbb{Q}(\zeta)$ containing α,

$$\mathfrak{m}(\alpha) = \frac{1}{|\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})|} \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} |\sigma(\alpha)|^2.$$

Lemma 4. *Let $a_1, a_2, \ldots, a_l$ and $b_1, b_2, \ldots, b_m$ be rational integers, and let $\alpha_1, \alpha_2, \ldots, \alpha_l$ and $\beta_1, \beta_2, \ldots, \beta_m$ be p^n-th roots of unity with p prime and n nonnegative. If*

$$\sum_{j=1}^{l} a_j \alpha_j = \sum_{k=1}^{m} b_k \beta_k,$$

then

$$\sum_{j=1}^{l} a_j \equiv \sum_{k=1}^{m} b_k \pmod{p}.$$

Proof of Lemma 4. If $n = 0$, then there is nothing to prove, so assume $n \geq 1$. Let ζ be a primitive p^n-th root of unity. For each α_j and β_k, let r_j and s_k be nonnegative integers such that $\alpha_j = \zeta^{r_j}$ and $\beta_k = \zeta^{s_k}$. Put

$$P(x) = \sum_{j=1}^{l} a_j x^{r_j} - \sum_{k=1}^{m} b_k x^{s_k}.$$

Then $P(\zeta) = 0$, so $P(x)$ is divisible in $\mathbb{Z}[x]$ by the cyclotomic polynomial

$$\Phi_{p^n}(x) = \Phi_p(x^{p^{n-1}}).$$

Hence $P(1) \equiv 0 \pmod{p}$. □

Proposition 5. *Let G be a finite group, let $\chi \in \mathrm{Irr}(G)$, and let g be an element of G with order a power of a prime p. If $p = 2$ or $\chi(1) \not\equiv \pm 2 \pmod{p}$, then either $\chi(g) = 0$, $\chi(g)$ is a root of unity, or $\mathfrak{m}(\chi(g)) \geq 2$.*

Proof of Proposition 5. Suppose that $p = 2$ or $\chi(1) \not\equiv \pm 2 \pmod{p}$. Let p^n be the order of g, and let ζ be a primitive p^n-th root of unity, so $\chi(g) \in \mathbb{Q}(\zeta)$. Let $\alpha = \zeta^m \chi(g)$ with m such that

$$\mathfrak{f}(\alpha) = \min_k \mathfrak{f}(\zeta^k \chi(g)). \tag{1}$$

We will show that either $\alpha = 0$, α is a root of unity, or $\mathfrak{m}(\alpha) \geq 2$.

Let $P = \mathfrak{f}(\alpha)$. Using $\mathbb{Q}(\zeta_k) \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}(\zeta_{(k,l)})$, then P divides p^n. If $P = 1$, then α is rational and the conclusion follows. If P is divisible by p^2, then for γ a primitive P-th root of unity, α is uniquely of the shape

$$\alpha = \sum_{k=0}^{p-1} \alpha_k \gamma^k, \quad \alpha_k \in \mathbb{Q}(\zeta_{P/p}), \tag{2}$$

the α_k are algebraic integers, and a straightforward calculation [2, p. 115] shows that $\mathfrak{m}(\alpha)$ is at least the number of nonzero α_k. By (1), at least two of the α_k are nonzero. Hence $\mathfrak{m}(\alpha) \geq 2$ if $p^2 \mid P$.

It remains to consider the case $P = p$. Since $\mathbb{Q}(\zeta_2) = \mathbb{Q}(\zeta_1)$, we must have $p > 2$. If $\mathfrak{l}(\alpha) = 0$, then $\alpha = 0$; if $\mathfrak{l}(\alpha) = 1$, then α is a root of unity; and if $\mathfrak{l}(\alpha) > 2$, then $\mathfrak{m}(\alpha) \geq 2$ by a result of Cassels [2, Lemma 3]. So assume $\mathfrak{l}(\alpha) = 2$. Then by [9, Thm. 1(i)], α can be written in the shape

$$\alpha = \epsilon_1\xi_1 + \epsilon_2\xi_2, \quad \epsilon_k^2 = 1,$$

where ξ_1 and ξ_2 are p-th roots of unity. If $\xi_1 = \xi_2$, then either $\alpha = 0$ or $\mathfrak{m}(\alpha) = 4$. So assume

$$\xi_1 \neq \xi_2.$$

By Lemma 4,

$$\epsilon_1 + \epsilon_2 \equiv \chi(1) \pmod{p}. \tag{3}$$

By (3) and the fact that $\chi(1) \not\equiv \pm 2 \pmod{p}$,

$$\epsilon_1 + \epsilon_2 = 0.$$

Hence, for some root of unity ρ and primitive p-th root of unity ξ,

$$\alpha = (\xi - 1)\rho.$$

Hence

$$\mathfrak{m}(\alpha) = \mathfrak{m}(\xi - 1) = 2 - \frac{1}{p-1}\sum_{k=1}^{p-1}(\xi^k + \xi^{-k}) = 2 + \frac{2}{p-1} > 2. \qquad \square$$

Lemma 6. *Let G be a finite group, let $\chi \in \mathrm{Irr}(G)$, and let g be an element of G with order a power of a prime p. If $\chi(1) \not\equiv \pm 1 \pmod{p}$, then $\chi(g)$ is not a root of unity.*

Proof of Lemma 6. Let p^n be the order of g, so $\chi(g) \in \mathbb{Q}(\zeta_{p^n})$, and suppose that $\chi(g)$ is a root of unity. Since the roots of unity in a given cyclotomic field $\mathbb{Q}(\zeta_k)$ are the l-th roots of unity for l the least common multiple of 2 and k, we then have

$$\chi(g) = \epsilon\xi$$

for some $\epsilon \in \{1, -1\}$ and p^n-th root of unity ξ. So by Lemma 4, either $\chi(1) \equiv 1 \pmod{p}$ or $\chi(1) \equiv -1 \pmod{p}$. $\square$

Lemma 7. *Let G be a finite group of prime-power order, let $g \in G$, and let $\chi \in \mathrm{Irr}(G)$. If $\chi(1) > 1$, then either $\chi(g) = 0$ or $\mathfrak{m}(\chi(g)) \geq 2$.*

Proof of Lemma 7. If $|G| = p^n$ with p prime, then each $g \in G$ has order a power of p, and each $\chi \in \mathrm{Irr}(G)$ has degree a power of p. So if $\chi(1) > 1$, then by Proposition 5 and Lemma 6, for each $g \in G$, $\chi(g) = 0$ or $\mathfrak{m}(\chi(g)) \geq 2$. $\square$

For any character χ of a finite group, let

$$\omega(\chi) = |\{\text{primes dividing } \chi(1)\}|.$$

Proposition 8. *Let G be a finite nilpotent group, let $\chi \in \mathrm{Irr}(G)$, and let $g \in G$. Then*

$$\chi(g) = 0 \quad \textit{or} \quad \mathfrak{m}(\chi(g)) \geq 2^{\omega(\chi)}. \tag{4}$$

Proof of Proposition 8. If $|G| = 1$, then $\chi(g) = \chi(1) = 1$, so assume $|G| > 1$. Since G is nilpotent, it is the direct product of its nontrivial Sylow subgroups $P_1, P_2, \ldots, P_n$. Let $g_1, g_2, \ldots, g_n$ be the unique sequence with $g_k \in P_k$ and

$$g = g_1 g_2 \ldots g_n.$$

For each P_k, let $\chi_k \in \mathrm{Irr}(P_k)$ be the unique irreducible constituent of the restriction of χ to P_k. Then

$$\chi(g) = \chi_1(g_1)\chi_2(g_2)\ldots\chi_n(g_n), \quad \chi(1) = \chi_1(1)\chi_2(1)\ldots\chi_n(1), \tag{5}$$

$$\chi_k(1) \text{ divides } |P_k|, \tag{6}$$

$$(|P_j|, |P_k|) = 1 \quad \text{for} \quad j \neq k, \tag{7}$$

and

$$\chi_k(g_k) \in \mathbb{Q}(\zeta_{|P_k|}). \tag{8}$$

For any algebraic integers $\alpha \in \mathbb{Q}(\zeta_l)$ and $\beta \in \mathbb{Q}(\zeta_m)$ with $(l, m) = 1$, we have $\mathbb{Q}(\zeta_{lm}) = \mathbb{Q}(\zeta_l)\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_l) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}$, and hence

$$\mathfrak{m}(\alpha\beta) = \mathfrak{m}(\alpha)\mathfrak{m}(\beta). \tag{9}$$

By (5), (7), (8), and (9),

$$\mathfrak{m}(\chi(g)) = \mathfrak{m}(\chi_1(g_1))\mathfrak{m}(\chi_2(g_2))\ldots\mathfrak{m}(\chi_n(g_n)). \tag{10}$$

By (10) and Lemma 7,

$$\chi(g) = 0 \quad \text{or} \quad \mathfrak{m}(\chi(g)) \geq 2^w,$$

where w denotes the number of characters χ_k with $\chi_k(1) > 1$. From (5), (6), and (7), w is equal to the number of prime divisors of $\chi(1)$. □

Proposition 9. *For each finite nilpotent group G, and each $\chi \in \mathrm{Irr}(G)$,*

$$\frac{|\{g \in G : \chi(g) = 0\}|}{|G|} \geq 1 - \frac{1}{2^{\omega(\chi)}}\left(\frac{|G| - \chi(1)^2 + 2^{\omega(\chi)}}{|G|}\right). \tag{11}$$

Proof of Proposition 9. Let G be a finite nilpotent group, and let $\chi \in \mathrm{Irr}(G)$. By Proposition 8, for each $g \in G$,

$$\chi(g) = 0 \quad \text{or} \quad \mathfrak{m}(\chi(g)) \geq 2^{\omega(\chi)}. \tag{12}$$

Now take the relation

$$|G| = \sum_{g \in G} |\chi(g)|^2,$$

apply the elements σ of the Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\zeta_{|G|})/\mathbb{Q})$, and average over $\mathcal{G}$. This gives

$$|G| = \sum_{g \in G} \mathfrak{m}(\chi(g)). \tag{13}$$

From (12) and (13),

$$|G| \geq \chi(1)^2 + 2^{\omega(\chi)}|\{g \in G : \chi(g) \neq 0\}| - 2^{\omega(\chi)}. \tag{14}$$

By (14), we have (11). □

Proof of Theorem 1. By Proposition 9. □

Proof of Theorem 2. Taking the relation

$$\frac{|G|}{|g^G|} = \sum_{\chi \in \mathrm{Irr}(G)} |\chi(g)|^2,$$

applying the elements σ of the Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\zeta_{|G|})/\mathbb{Q})$, and averaging over $\mathcal{G}$, we have

$$\frac{|G|}{|g^G|} = \sum_{\chi \in \mathrm{Irr}(G)} \mathfrak{m}(\chi(g)). \tag{15}$$

So for $\mathcal{L} = \{\chi \in \mathrm{Irr}(G) : \chi(1) = 1\}$ and $\mathcal{N} = \mathrm{Irr}(G) - \mathcal{L}$,

$$\frac{|G|}{|g^G|} = |\mathcal{L}| + \sum_{\chi \in \mathcal{N}} \mathfrak{m}(\chi(g)). \tag{16}$$

By Proposition 8, for each $\chi \in \mathcal{N}$,

$$\chi(g) = 0 \quad \text{or} \quad \mathfrak{m}(\chi(g)) \geq 2. \tag{17}$$

From (16) and (17),

$$\frac{|G|}{|g^G|} \geq |\mathcal{L}| + 2|\{\chi \in \mathcal{N} : \chi(g) \neq 0\}|. \tag{18}$$

By (18), if $|\mathrm{Cl}(G)| = |G|/|g^G|$, then $|\{\chi \in \mathcal{N} : \chi(g) = 0\}| \geq |\mathcal{N}|/2$, and if $|\mathrm{Cl}(G)| > |G|/|g^G|$, then $|\{\chi \in \mathcal{N} : \chi(g) = 0\}| > |\mathcal{N}|/2$. □

Proof of Corollary 3. By Theorem 1 and Theorem 2. □

§2

We now establish Conjecture 2 for several families of simple groups.

Theorem 10. *Let $n > 0$.*

I. *For $G = A_n$, we have*

$$\theta(G), \theta'(G) > \begin{cases} 1/2 & \text{if } n < 9, \\ 3/4 & \text{if } n \geq 9. \end{cases} \tag{19}$$

II. *For $G = Suz(q)$ with $q = 2^{2n+1}$, we have*

$$\theta(G) = \frac{1}{2} + \frac{(q+1)(q^2+2)}{2q^2(q^2+1)} \tag{20}$$

and

$$\theta'(G) = \frac{1}{2} + \frac{5}{2(q+3)}, \tag{21}$$

so $\theta(G), \theta'(G) > 1/2$ and

$$\theta(G), \theta'(G) \to 1/2 \text{ as } q \to \infty. \tag{22}$$

III. *For $G = L_2(q)$ with $q = p^n$ a prime power, we have $\theta(G), \theta'(G) > 1/2$.*

IV. *For $G = Ree(3^{2n+1})$, we have $\theta(G), \theta'(G) > 1/2$.*
V. *For each sporadic group G, we have $\theta(G), \theta'(G) > 1/2$.*
VI. *For each finite simple group G of order $\leq 10^9$, we have $\theta(G), \theta'(G) > 1/2$.*

Corollary 11. $\inf\{\theta(G) : |G| < \infty\}, \inf\{\theta'(G) : |G| < \infty\} \in [1/3, 1/2]$.

Proof of Corollary 11. Thompson and Gallagher give the lower bound of 1/3. The upper bound of 1/2 follows from part II of Theorem 10. □

Verification of I. (19) holds up to $n = 14$, so assume $n \geq 15$. In the character table of A_n, the values are rational integers, except some values $\chi(g)$ with

$$|\chi(g)|^2 = \frac{1 + \lambda_1\lambda_2\dots}{4}$$

for some partition λ of n into distinct odd parts $\lambda_1 > \lambda_2 > \dots$. Since $n \geq 15$, it follows that each pair $(\chi, g) \in \mathrm{Irr}(G) \times G$ satisfies

$$\chi(g) = 0,\ |\chi(g)| = 1,\ \text{or } |\chi(g)|^2 \geq 4. \tag{23}$$

Using (23) and the fact that simple groups do not have irreducible characters of degree 2, we get that each nonprincipal $\chi \in \mathrm{Irr}(G)$ satisfies

$$|G| > |\{g \in G : |\chi(g)| = 1\}| + 4|\{g \in G : |\chi(g)| \neq 0, 1\}|, \tag{24}$$

and from (24) it follows that $\theta(G) > 3/4$. Similarly, for any class g^G with $|g^G| \geq |G|/|\mathrm{Cl}(G)|$, we have

$$|\mathrm{Cl}(G)| \geq |\{\chi \in \mathrm{Irr}(G) : |\chi(g)| = 1\}| + 4|\{\chi \in \mathrm{Irr}(G) : |\chi(g)| \neq 0, 1\}|,$$

and hence $\theta'(G) > 3/4$. □

Verification of II. Let $n \geq 2$, $r = 2^n$, $q = 2^{2n-1}$, and $G = Suz(q)$, so

$$|G| = q^2(q-1)(q^2+1) = q^2(q-1)(q-r+1)(q+r+1).$$

Maintaining the notation of Suzuki [13], there are elements $\sigma, \rho, \xi_0, \xi_1, \xi_2$ such that each element of G can be conjugated into exactly one of the sets

$$1^G, \sigma^G, \rho^G, (\rho^{-1})^G, A_0 - \{1\}, A_1 - \{1\}, A_2 - \{1\},$$

where $A_i = \langle \xi_i \rangle$ $(i = 1, 2, 3)$, and the irreducible characters of G are given by the following table [13, Theorem 13]:

	1	σ	ρ, ρ^{-1}	$\xi_0^t \neq 1$	$\xi_1^t \neq 1$	$\xi_2^t \neq 1$
1	1	1	1	1	1	1
X	q^2	0	0	1	-1	-1
X_i	q^2+1	1	1	$\epsilon_0^i(\xi_0^t)$	0	0
Y_j	$(q-r+1)(q-1)$	$r-1$	-1	0	$-\epsilon_1^j(\xi_1^t)$	0
Z_k	$(q+r+1)(q-1)$	$-r-1$	-1	0	0	$-\epsilon_2^k(\xi_2^t)$
W_l	$\frac{r(q-1)}{2}$	$-\frac{r}{2}$	$\pm\frac{r\sqrt{-1}}{2}$	0	1	-1

Here, $1 \leq i \leq \frac{q}{2} - 1$, $1 \leq j \leq \frac{q+r}{4}$, $1 \leq k \leq \frac{q-r}{4}$, $1 \leq l \leq 2$,

$$\epsilon_0^i(\xi_0^t) = \zeta^{it} + \zeta^{-it}, \quad \zeta = e^{2\pi\sqrt{-1}/(q-1)}, \tag{25}$$

and ϵ_1^j and ϵ_2^k are certain characters on A_1 and A_2. The A_i's satisfy

$$|A_0| = q-1, \quad |A_1| = q+r+1, \quad |A_2| = q-r+1, \tag{26}$$

and denoting by G_i $(i = 0, 1, 2)$ the set of elements $g \in G$ that can be conjugated into $A_i - \{1\}$, we have

$$|G_i| = \frac{|A_i|-1}{l_i}\frac{|G|}{|A_i|}, \tag{27}$$

where $l_0 = 2$ and $l_1 = l_2 = 4$.

Let $\gamma_s = \zeta^s + \zeta^{-s}$ with $\zeta = e^{2\pi\sqrt{-1}/(q-1)}$ and $s \in \mathbb{Z}$. Then

$$|\gamma_s| = 1 \Leftrightarrow 6s \pm (q-1) \equiv 0 \pmod{3(q-1)}$$

and

$$\gamma_s = 0 \Leftrightarrow 4s \pm (q-1) \equiv 0 \pmod{2(q-1)}.$$

Since $q-1 \equiv 1 \pmod 3$, and $q-1 \equiv 1 \pmod 2$, it follows that

$$|\gamma_s| \notin \{0,1\} \text{ for all } s \in \mathbb{Z}. \tag{28}$$

So for any X_i,

$$|\{g \in G : X_i(g) \text{ is a root of unity or zero}\}| = |G| - |G_0| - 1, \tag{29}$$

and for any $g \in G_0$,

$$|\{\chi \in \mathrm{Irr}(G) : \chi(g) \text{ is a root of unity or zero}\}| = \frac{q}{2} + 4. \tag{30}$$

By (29) and (26)–(27),

$$\theta(G) \leq \frac{1}{2} + \frac{(q+1)(q^2+2)}{2q^2(q^2+1)}. \tag{31}$$

Equality must hold in (31) because

$$|\{g \in G : W_l(g) \in \{0,1,-1\}| = |G_0| + |G_1| + |G_2| > |G| - |G_0| - 1$$

and for any $\chi \in \mathrm{Irr}(G) - \{X_i\} - \{W_l\}$,

$$|\{g \in G : \chi(g) \in \{0,1,-1\}\}| \geq 2|\rho^G| + |G_0| + |G_2| > |G| - |G_0| - 1.$$

By (30) and the fact that, for any $g \in G_0$, $|C_G(g)| = q-1 < q+3 = |\mathrm{Cl}(G)|$, we have

$$\theta'(G) \leq \frac{1}{2} + \frac{5}{2(q+3)}. \tag{32}$$

Equality must hold in (32) because 1^G, σ^G, and ρ^G have size $< |G|/|\mathrm{Cl}(G)|$, and for any $g \in G_1 \cup G_2$,

$$|\{\chi \in \mathrm{Irr}(G) : \chi(g) \in \{0,1,-1\}\}| \geq \frac{3q-r+12}{4} \geq \frac{q}{2} + 4. \qquad \square$$

Verification of III. Let $q = p^n$ with p prime, $G = L_2(q)$, let R and S be as in [7, pp. 402–403], and let G_0 (resp. G_1) be the set of nonidentity elements $g \in G$ that can be conjugated into $\langle R \rangle$ (resp. $\langle S \rangle$).

Assuming first $p \neq 2$, then

$$|G| = \frac{q(q^2-1)}{2}, \qquad |\mathrm{Cl}(G)| = \frac{q+5}{2}, \tag{33}$$

$$|G_0| = \frac{q(q+1)(q-3)}{4}, \qquad |G_1| = |G_0| + q = \frac{q(q-1)^2}{4}, \tag{34}$$

and $G - G_0 \cup G_1$ consists of 3 classes: 1^G, a^G, b^G, with $|C_G(a)| = |C_G(b)| = q$. Inspecting Jordan's table [7, p. 402], each $\chi \in \mathrm{Irr}(G)$ satisfies either

(i) $\chi(g) \in \{0, 1, -1\}$ on $G_0 \cup G_1$, or
(ii) $\chi(g) \in \{1, -1\}$ on $a^G \cup b^G$ and $\chi(g) = 0$ on G_0 or G_1.

If $q > 3$, then

$$|G_0| + |G_1| > |G|/2 \quad \text{and} \quad |a^G| + |b^G| + |G_0| > |G|/2, \tag{35}$$

and if $q = 3$, then $G \cong A_4$. So $\theta(G) > 1/2$. Similarly,

$$|\{\chi \in \mathrm{Irr}(G) : \chi(a), \chi(b) \in \{0, 1, -1\}\}| = \frac{q+1}{2},$$

and for $g \in G_0$ (resp. $g \in G_1$) and $\chi \in \mathrm{Irr}(G)$, we have $\chi(g) \in \{0, 1, -1\}$ away from the $\leq (q-3)/4$ irreducible characters of degree $q+1$ (resp. the $\leq (q-1)/4$ characters of degree $q-1$), from which it follows that $\theta'(G) > 1/2$.

For $p = 2$, we have $|G| = q(q^2-1)$, $|\mathrm{Cl}(G)| = q + 1$,

$$|G_0| = \frac{q(q+1)(q-2)}{2}, \quad |G_1| = \frac{q^2(q-1)}{2}, \tag{36}$$

and $G - G_0 \cup G_1$ consists of 2 classes: 1^G and a^G with $|C_G(a)| = q$. The irreducible characters of G are given by Jordan [7, p. 403]. There is the principal character, 1 character of degree q, $q/2$ characters of degree $q-1$, and $q/2 - 1$ characters of degree $q+1$. All the characters satisfy $\chi(g) \in \{0, 1, -1\}$ on a^G, the character of degree q is ± 1 on G_0 and G_1, the characters of degree $q-1$ vanish on G_0, and the characters of degree $q+1$ vanish on G_1. From this, it follows that $\theta(G)$ and $\theta'(G)$ are $> 1/2$. □

Verification of IV. Let n be a positive integer, $m = 3^n$, $q = 3^{2n+1}$, and $G = Ree(q)$, so

$$|G| = q^3(q-1)(q+1)(q^2-q+1), \quad |\mathrm{Cl}(G)| = q + 8.$$

The irreducible characters of G are given by Ward [14] in a 16-by-16 table, with the last 6 rows being occupied by 6 families of exceptional characters, the sizes of which are, from top to bottom,

$$\frac{q-3}{4}, \ \frac{q-3}{4}, \ \frac{q-3}{24}, \ \frac{q-3}{8}, \ \frac{q-3m}{6}, \ \frac{q+3m}{6}.$$

From Ward's table, we find that for any class $g^G \notin \{1^G, X^G, J^G\}$, $\chi(g) \in \{0, 1, -1\}$ for more than half of the irreducible characters χ of G. Since the classes $1^G, X^G, J^G$ all have size $< |G|/|\mathrm{Cl}(G)|$, we conclude that $\theta'(G) > 1/2$.

The first step in verifying $\theta(G) > 1/2$ is to compute the following table:

TABLE 1

$\Omega \subset G$	$\lvert \cup_{g\in G}\, g\Omega g^{-1} \rvert$
$\{1\}$	1
$\langle R\rangle - \{1\}$	$\frac{q^3(q-3)(q^3+1)}{4}$
$\langle S\rangle - \{1\}$	$\frac{q^3(q-1)(q-3)(q^2-q+1)}{24}$
$M^- - \{1\}$	$\frac{q^3(q-1)(q+1)(q^2-2q-3m)}{6}$
$M^+ - \{1\}$	$\frac{q^3(q-1)(q+1)(q^2-2q+3m)}{6}$
$\{X\}$	$\lvert G\rvert/q^3$
$\{Y\}$	$\lvert G\rvert/3q$
$\{T\}$	$\lvert G\rvert/2q^2$
$\{T^{-1}\}$	$\lvert G\rvert/2q^2$
$\{YT\}$	$\lvert G\rvert/3q$
$\{YT^{-1}\}$	$\lvert G\rvert/3q$
$\{JT\}$	$\lvert G\rvert/2q$
$\{JT^{-1}\}$	$\lvert G\rvert/2q$
$J\langle R\rangle - \{J\}$	$\frac{q^3(q-3)(q^3+1)}{4}$
$J\langle S\rangle - \{J\}$	$\frac{q^3(q-1)(q-3)(q^2-q+1)}{8}$
$\{J\}$	$\lvert G\rvert/q(q^2-1)$

Then with Table 1 and Ward's table in hand, a straightforward check establishes that, for each $\chi \in \mathrm{Irr}(G)$,

$$|\{g \in G : \chi(g) \in \{0, 1, -1\}\}| > |G|/2.$$

Hence $\theta(G) > 1/2$. $\square$

Verification of V and VI. Here, in Tables 2 and 3, we report the values of θ and θ' for sporadic groups and simple groups of order $\leq 10^9$. All values are rounded to the number of digits shown.

TABLE 2. The sporadic groups.

G	$\theta(G)$	$\theta'(G)$
M_{11}	0.7290	0.8000
M_{12}	0.7955	0.8667
M_{22}	0.7117	0.8333
M_{23}	0.6827	0.7647
M_{24}	0.6913	0.7692
J_1	0.5583	0.6000
J_2	0.6373	0.6190
J_3	0.5840	0.7143
J_4	0.6925	0.7903
Co_1	0.8739	0.8515
Co_2	0.8347	0.8333
Co_3	0.7528	0.8333
Fi_{22}	0.8029	0.8769

G	$\theta(G)$	$\theta'(G)$
Fi_{23}	0.8328	0.8469
Fi'_{24}	0.8808	0.8056
HS	0.7853	0.8750
McL	0.6722	0.8333
He	0.7088	0.7576
Ru	0.8517	0.8333
Suz	0.8141	0.8372
$O'N$	0.6830	0.8667
HN	0.6362	0.7593
Ly	0.7879	0.8491
Th	0.7978	0.8750
B	0.8812	0.8587
M	0.8855	0.8711

TABLE 3. The simple groups of order $\leq 10^9$ that are not cyclic, A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$, or sporadic.

G	$\theta(G)$	$\theta'(G)$
$L_3(3)$	0.6736	0.8333
$U_3(3)$	0.7049	0.8571
$L_3(4)$	0.6000	0.8000
$S_4(3)$	0.8713	0.9000
$U_3(4)$	0.6892	0.7273
$U_3(5)$	0.7103	0.8571
$L_3(5)$	0.6754	0.8667
$S_4(4)$	0.6433	0.7037
$S_6(2)$	0.8867	0.8333
$L_3(7)$	0.6235	0.7273
$U_4(3)$	0.7121	0.9000
$G_2(3)$	0.8321	0.9130
$S_4(5)$	0.6501	0.6471
$U_3(8)$	0.5701	0.6786
$U_3(7)$	0.6741	0.7586
$L_4(3)$	0.6911	0.8621

G	$\theta(G)$	$\theta'(G)$
$L_5(2)$	0.7038	0.7778
$U_5(2)$	0.8041	0.9149
$L_3(8)$	0.6650	0.7083
${}^2F_4(2)'$	0.7006	0.8182
$L_3(9)$	0.5488	0.6000
$U_3(9)$	0.6237	0.6739
$U_3(11)$	0.5494	0.6250
$S_4(7)$	0.7341	0.7308
$O_8^+(2)$	0.8555	0.9245
$O_8^-(2)$	0.7578	0.8462
${}^3D_4(2)$	0.6920	0.6571
$L_3(11)$	0.6660	0.6970
$G_2(4)$	0.6449	0.7500
$L_3(13)$	0.5354	0.5938
$U_3(13)$	0.6662	0.6957
$L_4(4)$	0.6020	0.5714

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Vienna, Austria
Email address: `alexander.r.miller@univie.ac.at`