THE CHARACTERS OF SYMMETRIC GROUPS THAT DEPEND ONLY ON LENGTH

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ABSTRACT. The characters $\chi(\pi)$ of S_n that depend only on the number of cycles of π are completely determined.

1. INTRODUCTION

Let $\ell(\pi)$ denote the number of cycles of a permutation $\pi \in S_n$. Let $\phi_0, \phi_1, \ldots, \phi_{n-1}$ be the Foulkes characters of S_n, so ϕ_i is afforded by the sum of Specht modules V_β with β of border shape with exactly n boxes and $i+1$ rows. For history, properties, and a number of recent developments, including the Diaconis–Fulman connection with adding random numbers, see [2, 3, 6, 7, 8, 9, 10, 11].

The Foulkes characters have long been studied for their remarkable properties:

a) They depend only on length in the sense that

$$\phi_i(\pi) = \phi_i(\sigma) \text{ whenever } \ell(\pi) = \ell(\sigma).$$

b) They form a basis for the space $\mathrm{CF}_\ell(S_n)$ of class functions of S_n that depend only on ℓ, with each $\theta \in \mathrm{CF}_\ell(S_n)$ decomposing uniquely as

$$\theta = \sum_{i=0}^{n-1} \frac{\langle \theta, \varepsilon_i \rangle}{\varepsilon_i(1)} \phi_i,$$

where ε_i is the irreducible character $\chi_{(n-i,1,\ldots,1)}$.

c) They decompose the character ρ of the regular representation:

$$\phi_0 + \phi_1 + \ldots + \phi_{n-1} = \rho.$$

d) Their degrees are the Eulerian numbers:

$$\phi_i(1) = \#\{\pi \in S_n \mid \mathrm{des}(\pi) = i\}, \quad \mathrm{des}(\pi) = \#\{i \mid \pi(i) > \pi(i+1)\}.$$

e) They branch according to

$$\phi_i|_{S_{n-1}} = (n-i)\phi_{i-1} + (i+1)\phi_i, \quad \text{where} \quad \phi_{-1} = 0.$$

f) And they even admit an explicit expression:

$$\phi_i(\pi) = \sum_{j=0}^{n-1} (-1)^{i-j} \binom{n+1}{i-j} (j+1)^{\ell(\pi)},$$

with the usual convention, used throughout this paper, that $\binom{u}{v} = 0$ if v does not satisfy $u \geq v \geq 0$.

Two missing properties were recently added to the list in [11], the first being a solution to the problem of decomposing products:

g) For any two Foulkes characters ϕ_i and ϕ_j of S_n,

$$\phi_i\phi_j = \sum_{k=0}^{n-1} c_{ijk}\phi_k,$$

where

$$c_{ijk} = \#\{(x,y) \in S_n \times S_n \mid \operatorname{des}(x) = i,\ \operatorname{des}(y) = j,\ xy = z\}$$

for any fixed $z \in S_n$ with $\operatorname{des}(z) = k$, or more explicitly,

$$c_{ijk} = \sum_{\substack{0 \leq u \leq i \\ 0 \leq v \leq j}} (-1)^{i-u+j-v} \binom{n+1}{i-u}\binom{n+1}{j-v}\binom{uv+u+v+n-k}{n}.$$

In addition to the combinatorial solution and the explicit solution presented here, a recursive solution was also found by specializing results of Delsarte that predate the introduction of Foulkes characters and were discovered in a completely different context, see [11]. The combinatorial solution follows from an earlier result [7, Thm. 9] that connects Foulkes characters with Eulerian idempotents from cyclic homology, namely that

$$\mathscr{D}_i = \sum_{j=1}^{n} \phi_i(C_j)\mathscr{E}_{j-1},$$

where $\phi_i(C_k)$ denotes the value $\phi_i(\sigma)$ at any $\sigma \in C_k = \{\pi \in S_n \mid \ell(\pi) = k\}$,

$$\mathscr{D}_i = \sum_{\substack{\pi \in S_n \\ \operatorname{des}(\pi)=i}} \pi,$$

and the $\mathscr{E}_i$'s are the Eulerian idempotents, which are certain orthogonal idempotents in $\mathbb{Q}[S_n]$, see [11, (2.5)].

The second property added to the list in [11] solves the problem of finding a natural description of the unique inner product on $\mathrm{CF}_\ell(S_n)$ with respect to which the ϕ_i's form an orthonormal basis:

h) The Foulkes characters $\phi_0, \phi_1, \ldots, \phi_{n-1}$ of S_n form an orthonormal basis for the Hilbert space $\mathrm{CF}_\ell(S_n)$ with inner product $[\text{-}\,,\text{-}]$ defined by

$$[\theta, \psi] = \frac{1}{|S_n|}\sum_{i,j=1}^{n} \theta(C_i)\overline{\psi(C_j)}\mathbf{E}|\sigma C_i \cap \tau C_j|,$$

where σ and τ are n-cycles chosen uniformly at random from C_1, so $\mathbf{E}|\sigma C_i \cap \tau C_j|$ equals the expected number of ways that the product $\sigma\tau$ can be written as a product $\alpha\beta$ with $\alpha \in C_i$ and $\beta \in C_j$.

This inner product $[\text{-}\,,\text{-}]$ gives a new and natural construction of Foulkes characters by applying the Gram–Schmidt process to a natural choice of basis for $\mathrm{CF}_\ell(S_n)$ that is composed of characters, analogous to how the irreducible characters themselves can be constructed by applying the Gram–Schmidt process to the characters $(1_{S_\lambda})^{S_n}$, see [11].

The author gave a different construction of Foulkes characters in [7] using sums of modules afforded by reduced homology groups of subcomplexes of certain equivariant strong deformation retracts of wedges of spheres called Milnor fibers coming from invariant theory, which works not only for $S_n \cong G(1,1,n)$ but for all of the full monomial groups $G(r,1,n)$, including the hyperoctahedral group $G(2,1,n)$. In this more general setting, the role of ℓ is played by $n - \mathfrak{l}$, where $\mathfrak{l}$ is the most natural choice of "length",

$$\mathfrak{l}(x) = \min\{k \geq 0 \mid x = y_1 y_2 \ldots y_k \text{ for some reflections } y_i \in G(r,1,n)\}.$$

In addition to new properties and proofs in the classical case, analogues of all the properties that we have described so far have been established for $G(r,1,n)$. These generalized Foulkes characters also have connections with certain Markov chains, just as in the case of S_n. Most notably, Diaconis and Fulman [3] connected the hyperoctahedral Foulkes characters with a Markov chain for adding random numbers in balanced ternary, a number system that reduces carries and that Donald Knuth famously described as "perhaps the prettiest number system of all."

But what is missing from the list of properties presented so far is an analogue of the fundamental characterization of characters, by which we mean if $\chi_0, \chi_1, \ldots, \chi_{k-1}$ are the irreducible characters of a finite group G, then the characters of G are the non-negative integer linear combinations

$$\mathfrak{a}_0\chi_0 + \mathfrak{a}_1\chi_1 + \ldots + \mathfrak{a}_{k-1}\chi_{k-1}, \quad \mathfrak{a} = (\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{k-1})^t \in \mathbb{N}^k,$$

where by $\mathbb{N}$ we mean the set of non-negative integers. Interestingly, for the hyperoctahedral groups and all the other full monomial groups with $r \geq 2$, there is an analogue of this characterization [8]:

i′) If $r \geq 2$, then the characters of $G(r,1,n)$ that depend only on length are precisely the non-negative integer linear combinations of the Foulkes characters of the group.

So for the monomial groups $G(r,1,n)$ with $r \geq 2$, the picture is complete.

For the symmetric group, however, the problem of determining the characters that depend only on ℓ is more complicated and has remained open for a few decades. Already for $n = 3$, one finds that not all characters in $\mathrm{CF}_\ell(S_n)$ are integer linear combinations of Foulkes characters, for example $\chi_{(2,1)}$, and it was recently shown in [8], using symmetric functions and Chebyshev's theorem about primes, that the same is true for each $n \geq 3$, but little else is known about the characters in $\mathrm{CF}_\ell(S_n)$.

Given this gap in the story for S_n, a natural question to ask is, instead of the Foulkes characters, does there exist a different sequence of n characters that is better suited for the role of irreducibles in $\mathrm{CF}_\ell(S_n)$ in that their non-negative integer linear combinations are the characters that depend only on length? Our first theorem answers in the negative.

Theorem 1.1. *If $n > 3$, then there does not exist a collection of n characters $\chi_0, \chi_1, \ldots, \chi_{n-1} \in \mathrm{CF}_\ell(S_n)$ such that every character of S_n that depends only on ℓ can be written as a non-negative integer linear combination of the χ_i's.*

In our main theorem, we completely resolve the problem of determining the characters of S_n that depend only on length. Our solution is explicit and complete with a parametrization in terms of non-negative integers, just as in the case of ordinary characters. Let $\{r\} = r - \lfloor r \rfloor$ denote the fractional part of any given rational number r.

Theorem 1.2. *The characters of S_n that depend only on ℓ are the linear combinations*

$$\theta_{\mathfrak{a}} = \tilde{\mathfrak{a}}_0\phi_0 + \tilde{\mathfrak{a}}_1\phi_1 + \ldots + \tilde{\mathfrak{a}}_{n-1}\phi_{n-1}, \quad \mathfrak{a} \in \mathbb{N}^n,$$

$$\tilde{\mathfrak{a}}_k = \left\lfloor \frac{\mathfrak{a}_k}{d_{k+1}} \right\rfloor + \left\{ \sum_{j=0}^{n-1} \binom{n-k-1}{j-k} \frac{\mathfrak{a}_j}{d_{j+1}} \right\}, \quad d_k = \frac{k}{\gcd(n,k)},$$

and moreover,

$$\theta_{\mathfrak{a}} = \theta_{\mathfrak{b}} \quad \text{if and only if} \quad \mathfrak{a} = \mathfrak{b}.$$

As a consequence, we find that the non-negative integer linear combinations of Foulkes characters of S_n have exponentially decaying density among the characters of S_n that depend only on length. We also obtain an upper bound for this density and information about the denominators of the rational coefficients that occur in Theorem 1.2.

Theorem 1.3. *The number of characters of S_n that depend only on ℓ and lie in the fundamental parallelepiped $\{\sum t_i\phi_i \mid t_i \in [0,1)\}$ equals*

$$\frac{n!}{\gcd(1,n)\gcd(2,n)\ldots\gcd(n,n)}.$$

Theorem 1.4. *If σ_n denotes the smallest positive integer such that, for each character χ of S_n that depends only on ℓ, $\sigma_n\chi$ is a non-negative integer linear combination of Foulkes characters, then*

$$\sigma_n = \frac{\mathrm{lcm}(1,2,\ldots,n)}{n} = \frac{e^{f(n)}}{n},$$

where f is the second Chebyshev function.

Asymptotics of f, and in turn σ_n, have important number-theoretic significance, with the prime number theorem being equivalent to

$$f(n) \sim n$$

and more precise statements being equivalent to the Riemann Hypothesis.

2. Number theoretic preliminaries

Important for our analysis is a certain result which compliments and improves an old result of Cauchy.

Given a partition λ of n, abbreviated $\lambda \vdash n$, we shall write $m_k(\lambda)$ for the number of parts of λ that equal k,

$$\ell(\lambda) = \sum_{k=1}^{n} m_k(\lambda),$$

and

$$\mathsf{M}(\lambda) = \binom{\ell(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda)} = \frac{\ell(\lambda)!}{m_1(\lambda)! m_2(\lambda)! \ldots m_n(\lambda)!}.$$

Schönemann proved that

$$\frac{\gcd(m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda))}{\ell(\lambda)} \mathsf{M}(\lambda) \in \mathbb{Z}$$

and Cauchy independently proved that

$$\frac{n}{\ell(\lambda)} \mathsf{M}(\lambda) \in \mathbb{Z},$$

see Dickson's book [4, Chap. IX, p. 265]. We find the exact gcd of the $\mathsf{M}(\lambda)$'s where λ partitions n into exactly k non-zero parts.

Proposition 2.1. *For any positive integers $n \geq k \geq 1$,*

$$\gcd\{\mathsf{M}(\lambda) \mid \lambda \vdash n,\ \ell(\lambda) = k\} = \frac{k}{\gcd(n,k)}. \tag{2.1}$$

Proof. Fixing positive integers n and k with $n \geq k$, let

$$g = \gcd\{\mathsf{M}(\lambda) \mid \lambda \vdash n,\ \ell(\lambda) = k\}.$$

Let λ be a partition of n with $\ell(\lambda) = k$. For $1 \leq j \leq n$,

$$\frac{m_j(\lambda)}{k} \binom{k}{m_j(\lambda)} = \binom{k-1}{m_j(\lambda) - 1},$$

so

$$\frac{m_j(\lambda)}{k} \mathsf{M}(\lambda) \in \mathbb{Z}.$$

Multiplying the left-hand side by j and then summing over j gives Cauchy's result

$$\frac{n}{k} \mathsf{M}(\lambda) \in \mathbb{Z}.$$

Writing $\gcd(n,k) = un + vk$ with $u, v \in \mathbb{Z}$, we therefore have

$$\frac{\gcd(n,k)}{k} \mathsf{M}(\lambda) = u \frac{n}{k} \mathsf{M}(\lambda) + v \frac{k}{k} \mathsf{M}(\lambda) \in \mathbb{Z}.$$

Hence $\frac{k}{\gcd(n,k)}$ divides g.

To show that g divides $\frac{k}{\gcd(n,k)}$, we show that for each prime p,

$$v_p(g) \leq v_p\left(\frac{k}{\gcd(n,k)}\right),$$

where $v_p(m)$ denotes the p-adic valuation of m. Fix p and let $e = v_p(k)$, so $p^e \mid k$ and $p^{e+1} \nmid k$. Write $n = ak + r$ for non-negative integers a and r with $0 \leq r < k$. Similarly write $r = bp^e + s$ with $0 \leq s < p^e$. Writing $u = p^{v_p(s)}$, let μ be the partition with u parts of size $a + \frac{r}{u}$ and $k - u$ parts of size a. Then μ is a partition of n, $\ell(\mu) = k$, and

$$\mathsf{M}(\mu) = \binom{k}{u}.$$

Using Kummer's theorem, then

$$v_p(\mathsf{M}(\mu)) = v_p(k) - v_p(u) = v_p\left(\frac{k}{\gcd(n,k)}\right).$$

Hence $v_p(g) \leq v_p(\frac{k}{\gcd(n,k)})$. □

3. Proof of Theorem 1.1

Lemma 3.1. *The Frobenius characteristic of $(n-1)^{\ell-1} \in \mathrm{CF}_\ell(S_n)$ is a non-negative integer linear combination of the symmetric functions h_λ. In particular, the class function $(n-1)^{\ell-1}$ is a character of S_n.*

Proof. The case $n = 1$ is trivial, so assume $n > 1$. By [8, Cor. 8], the Frobenius characteristic $\mathrm{ch}(\theta)$ of the class function $\theta(\pi) = (n-1)^{\ell(\pi)-1}$ satisfies

$$\mathrm{ch}(\theta) = \frac{1}{n-1}\sum_{\lambda \vdash n} c_\lambda h_\lambda,$$

$$c_\lambda = \binom{n-1}{\ell(\lambda)}\mathsf{M}(\lambda) = \binom{n-1}{m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda), n-\ell(\lambda)-1}.$$

For any non-negative integers $k_1, k_2, \ldots, k_s$ with sum k, we have

$$\frac{k_i}{k}\binom{k}{k_1, k_2, \ldots, k_s} = \binom{k-1}{k_1, k_2, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_s} \in \mathbb{Z}.$$

So for $\lambda \vdash n$ and $1 \leq k \leq n$,

$$\frac{m_k(\lambda)}{n-1}c_\lambda \in \mathbb{Z},$$

and in turn

$$\frac{n}{n-1}c_\lambda = \sum_{k=1}^{n} k\frac{m_k(\lambda)}{n-1}c_\lambda \in \mathbb{Z}.$$

Therefore, for each partition λ of n,

$$\frac{1}{n-1}c_\lambda = \frac{n}{n-1}c_\lambda - c_\lambda \in \mathbb{Z}.$$

So $\mathrm{ch}(\theta)$ is a non-negative integer linear combination of the h_λ's. □

Proof of Theorem 1.1. Suppose that $n > 3$ and $\chi_0, \chi_1, \ldots, \chi_{n-1}$ are characters in $\mathrm{CF}_\ell(S_n)$ with the property that every character in $\mathrm{CF}_\ell(S_n)$ is a non-negative integer linear combination of the χ_i's. Then the change of basis matrices $A = (A_{ij})_{0 \le i,j \le n-1}$ and A^{-1}, where $\chi_j = \sum_i A_{ij}\phi_i$, both have all entries non-negative. If at least two entries in a row of A were non-zero, say in columns s and t, then from the non-negativity and the relation $AA^{-1} = I$, the s and t rows of A^{-1} would be linearly dependent. So A must be a positive definite diagonal matrix times a permutation matrix, meaning that for some positive scalars $s_0, s_1, \ldots, s_{n-1}$ and some permutation σ of the indices, $\chi_j = s_j \phi_{\sigma(j)}$ for $0 \le j \le n-1$.

Let θ be the class function of S_n given by

$$\theta(\pi) = (n-1)^{\ell(\pi)-1}.$$

By Lemma 3.1, θ is a character of S_n. We claim that θ can not be written as a non-negative integer linear combination of characters $\eta_0, \eta_1, \ldots, \eta_{n-1}$ with $\eta_i \in \{s\phi_i \mid s \ge 0\}$ for $0 \le i \le n-1$. Equivalently, if θ is written as

$$\theta = c_0\phi_0 + c_1\phi_1 + \ldots + c_{n-1}\phi_{n-1}, \tag{3.1}$$

then at least one of the summands $c_i\phi_i$ is not a character. We will show that in fact $c_{n-2}\phi_{n-2}$ is not a character.

Let $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ be the characters of S_n given by

$$\gamma_k(\pi) = (k+1)^{\ell(\pi)},$$

so γ_k is afforded by $(\mathbb{C}^{k+1})^{\otimes n}$ with

$$\pi.(u_1 \otimes u_2 \otimes \ldots \otimes u_n) = u_{\pi^{-1}(1)} \otimes u_{\pi^{-1}(2)} \otimes \ldots \otimes u_{\pi^{-1}(n)}.$$

Then

$$\theta = \frac{\gamma_{n-2}}{n-1}$$

and

$$\phi_i = \sum_{j=0}^{n-1} (-1)^{i-j} \binom{n+1}{i-j} \gamma_j, \quad 0 \le i \le n-1. \tag{3.2}$$

The inverse of the matrix

$$\left[(-1)^{i-j} \binom{n+1}{i-j} \right]_{0 \le i,j \le n-1}$$

equals

$$\left[\binom{n+i-j}{n} \right]_{0 \le i,j \le n-1},$$

so

$$\gamma_i = \sum_{j=0}^{n-1} \binom{n+i-j}{n} \phi_j, \quad 0 \le i \le n-1. \tag{3.3}$$

Therefore, the coefficient of ϕ_j in θ is

$$c_j = \frac{1}{n-1}\binom{2n-2-j}{n}, \quad 0 \le j \le n-1. \tag{3.4}$$

Let ν be the partition of n with $m_2(\nu) = 2$ and $m_1(\nu) = n-4$. Using that for any k,

$$\langle k^\ell, \chi_\nu \rangle = \prod_b \frac{k + c(b)}{h(b)},$$

where b runs over the boxes in the diagram of ν and where $c(b)$ and $h(b)$ denote the content and hook length of b, we have

$$\langle \gamma_j, \chi_\nu \rangle = \begin{cases} 0 & \text{if } j \le n-4, \\ \frac{(n-2)(n-3)}{2} & \text{if } j = n-3, \\ \frac{n(n-1)(n-3)}{2} & \text{if } j = n-2, \\ \frac{n^2(n+1)(n-3)}{4} & \text{if } j = n-1. \end{cases} \tag{3.5}$$

Combining (3.2), (3.4), and (3.5), we have

$$\begin{aligned} \langle c_{n-2}\phi_{n-2}, \chi_\nu \rangle &= \frac{1}{n-1}\left[-\binom{n+1}{1}\frac{(n-2)(n-3)}{2} + \frac{n(n-1)(n-3)}{2} \right] \\ &= \frac{n-3}{n-2}, \end{aligned}$$

which is not an integer for $n > 3$. So $c_{n-2}\phi_{n-2}$ is not a character. □

4. Proofs of Theorems 1.2–1.4

Fixing n, we have a chain of lattices

$$\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z},$$

where $\mathcal{Z}$ is the lattice of virtual characters of S_n,

$$\mathcal{Z} = \left\{ \textstyle\sum_{\chi \in \mathrm{Irr}(S_n)} a_\chi \chi \;\middle|\; a_\chi \in \mathbb{Z} \right\},$$

$\mathcal{Y}$ is the lattice of virtual characters of S_n that depend only on ℓ,

$$\mathcal{Y} = \{\chi \in \mathcal{Z} \mid \chi \text{ depends only on } \ell\},$$

and $\mathcal{X}$ is the sublattice of $\mathcal{Y}$ spanned by the Foulkes characters of S_n,

$$\mathcal{X} = \{a_0\phi_0 + a_1\phi_1 + \ldots + a_{n-1}\phi_{n-1} \mid a_i \in \mathbb{Z}\}.$$

That the lattice of virtual characters that depend only on length can also be characterized as the set of virtual characters in the rational span of Foulkes characters is fundamental to our approach.

Proposition 4.1.

$$\mathcal{Y} = \mathcal{Z} \cap \{r_0\phi_0 + r_1\phi_1 + \ldots + r_{n-1}\phi_{n-1} \mid r_i \in \mathbb{Q}\}.$$

Proof. If a rational linear combination of Foulkes characters is a virtual character, then it is a virtual character that depends only on ℓ. For the other inclusion, if $\theta \in \mathcal{Y}$, then $\theta \in \mathcal{Z}$ and

$$\theta = \sum_{i=0}^{n-1} \frac{\langle \theta, \varepsilon_i \rangle}{\varepsilon_i(1)} \phi_i, \quad \varepsilon_i = \chi_{(n-i,1,\ldots,1)},$$

so θ is a virtual character in the rational span of Foulkes characters. $\square$

Let $\mathcal{P}$ be the fundamental parallelepiped

$$\mathcal{P} = \{t_0\phi_0 + t_1\phi_1 + \ldots + t_{n-1}\phi_{n-1} \mid t_i \in [0,1)\}.$$

Then each $\theta \in \mathcal{Y}$ decomposes uniquely as

$$\theta = \theta_{\mathcal{X}} + \theta_{\mathcal{P}}, \qquad \theta_{\mathcal{X}} \in \mathcal{X}, \quad \theta_{\mathcal{P}} \in \mathcal{Y} \cap \mathcal{P}.$$

4.1. We are interested in the genuine characters in our lattices. Let

$$\mathcal{L} = \{\text{characters of } S_n \text{ that depend only on } \ell\}$$

and

$$\mathcal{F} = \{\text{non-negative integer linear combinations of } \phi_i\text{'s}\}.$$

So $\mathcal{F} \subset \mathcal{L}$ and both $\mathcal{F}$ and $\mathcal{L}$ are closed under addition.

Lemma 4.2.

$$\mathcal{Z} \cap \mathcal{P} = \mathcal{Y} \cap \mathcal{P} = \mathcal{L} \cap \mathcal{P} \subset \mathcal{L}.$$

Proof. Let $\theta \in \mathcal{Z} \cap \mathcal{P}$. Then θ is a linear combination of the Foulkes characters with non-negative coefficients, so θ depends only on length and satisfies $\langle \theta, \chi \rangle \geq 0$ for each $\chi \in \mathrm{Irr}(S_n)$. Since $\langle \theta, \chi \rangle$ is an integer for each $\chi \in \mathrm{Irr}(S_n)$, we conclude that θ is a character of S_n that depends only on length. $\square$

Proposition 4.3. *The characters θ of S_n that depend only on length are precisely the sums*

$$\theta = \theta_{\mathcal{F}} + \theta_{\mathcal{P}}, \qquad \theta_{\mathcal{F}} \in \mathcal{F}, \quad \theta_{\mathcal{P}} \in \mathcal{Z} \cap \mathcal{P},$$

and the components $\theta_{\mathcal{F}}$ and $\theta_{\mathcal{P}}$ are uniquely determined by θ. Equivalently, the addition map $(x, y) \mapsto x + y$ takes $\mathcal{F} \times (\mathcal{Z} \cap \mathcal{P})$ bijectively onto $\mathcal{L}$. Moreover, if

$$\theta = \sum_{i=0}^{n-1} r_i \phi_i,$$

then

$$\theta_{\mathcal{F}} = \sum_{i=0}^{n-1} \lfloor r_i \rfloor \phi_i \quad \textit{and} \quad \theta_{\mathcal{P}} = \sum_{i=0}^{n-1} \{r_i\} \phi_i.$$

Proof. By Lemma 4.2, we have a mapping

$$\alpha : \mathcal{F} \times (\mathcal{Z} \cap \mathcal{P}) \to \mathcal{L} \quad \text{given by} \quad \alpha(x, y) = x + y.$$

Each element of $\mathcal{Y}$ is uniquely expressible as an element from the sublattice $\mathcal{X}$ plus an element from the fundamental domain $\mathcal{Y} \cap \mathcal{P} = \mathcal{Z} \cap \mathcal{P}$, and we also have the inclusions $\mathcal{F} \subset \mathcal{X}$ and $\mathcal{L} \subset \mathcal{Y}$, so α is injective.

To show that α is surjective, consider a character θ that depends only on length. Then

$$\theta = \sum_{i=0}^{n-1} r_i \phi_i, \qquad r_i = \frac{\langle \theta, \chi_{(n-i,1,\ldots,1)} \rangle}{\binom{n-1}{i}} \geq 0.$$

Hence

$$\theta = \phi + \psi,$$

where

$$\phi = \sum_{i=0}^{n-1} \lfloor r_i \rfloor \phi_i \in \mathcal{F}$$

and

$$\psi = \theta - \phi = \sum_{i=0}^{n-1} \{r_i\} \phi_i \in \mathcal{Z} \cap \mathcal{P}.$$

□

4.2. We now proceed to describe a particularly good pair of bases for the lattice $\mathcal{Y}$ and the sublattice $\mathcal{X}$, from which we will be able to better understand the fundamental domain $\mathcal{Z} \cap \mathcal{P}$ for $\mathcal{X} \subset \mathcal{Y}$. Specifically, we find a basis for $\mathcal{Y}$ that extends to a basis of $\mathcal{Z}$ and has the property that certain multiples of the basis elements form a basis for the sublattice $\mathcal{X}$.

4.2.1. The new basis for $\mathcal{X}$ will be denoted by $\psi_0, \psi_1, \ldots, \psi_{n-1}$.

Definition 4.4. Define $\psi_0, \psi_1, \ldots, \psi_{n-1} \in \mathrm{CF}_\ell(S_n)$ by

$$\psi_i = \sum_{j=0}^{n-1} (-1)^{i-j} \binom{i+1}{i-j} (j+1)^\ell. \tag{4.1}$$

Denoting by γ_k the character $(k+1)^\ell$ in $\mathrm{CF}_\ell(S_n)$, we have the following relations between the ψ_i's, the ϕ_i's, and the γ_i's.

Proposition 4.5.

$$\phi_i = \sum_{j=0}^{n-1} (-1)^{i-j} \binom{n+1}{i-j} \gamma_j, \qquad \gamma_i = \sum_{j=0}^{n-1} \binom{n+i-j}{i-j} \phi_j, \tag{4.2}$$

$$\psi_i = \sum_{j=0}^{n-1} (-1)^{i-j} \binom{i+1}{i-j} \gamma_j, \qquad \gamma_i = \sum_{j=0}^{n-1} \binom{i+1}{i-j} \psi_j, \tag{4.3}$$

$$\phi_i = \sum_{j=0}^{n-1} (-1)^{i-j} \binom{n-j-1}{i-j} \psi_j, \qquad \psi_i = \sum_{j=0}^{n-1} \binom{n-j-1}{i-j} \phi_j. \tag{4.4}$$

Proof. The first equality in (4.2) is the usual well-known description of ϕ_i. The second equality in (4.2) is (3.3), which can alternatively be obtained by first writing

$$\gamma_i = \sum_{j=0}^{n-1} \frac{\langle \gamma_i, \varepsilon_j \rangle}{\binom{n-1}{j}} \phi_j,$$

where $\varepsilon_j = \chi_{(n-j,1,\ldots,1)}$, and then using that for any partition λ of n,

$$\langle \gamma_i, \chi_\lambda \rangle = \prod_b \frac{i+1+c(b)}{h(b)}, \tag{4.5}$$

where the product is taken over all boxes b in the diagram of λ and where $c(b)$ and $h(b)$ denote the content and hook length of b, which gives

$$\frac{\langle \gamma_i, \varepsilon_j \rangle}{\binom{n-1}{j}} = \binom{n+i-j}{i-j}. \tag{4.6}$$

The first equality in (4.3) is the definition of ψ_i, and the second equality in (4.3) follows from the first by using the fact that the inverse of the matrix $[\binom{j}{i}]_{1\le i,j\le n}$ equals $[(-1)^{i-j}\binom{j}{i}]_{1\le i,j\le n}$.

For the second equality in (4.4), by combining the first relation in (4.3) and the second relation in (4.2), we have

$$\psi_i = \sum_{j=0}^{n-1} \sum_{u=0}^{n-1} (-1)^{i-u} \binom{n+u-j}{u-j} \binom{i+1}{i-u} \phi_j. \tag{4.7}$$

For $1 \le s, t \le n$,

$$\sum_{u=1}^{n} \binom{n-s}{n-u} \binom{t}{u} = \binom{n-s+t}{n},$$

so, for $1 \le s, t \le n$,

$$\sum_{u=1}^{n} (-1)^{u-t} \binom{n-s+u}{n} \binom{t}{u} = \binom{n-s}{n-t}. \tag{4.8}$$

Using (4.8) in (4.7), we conclude that

$$\psi_i = \sum_{j=0}^{n-1} \binom{n-j-1}{i-j} \phi_j.$$

The first equality in (4.4) follows from the second equality in (4.4) by using both that the inverse of $[\binom{j}{i}]_{0\leq i,j\leq n-1}$ equals $[(-1)^{i-j}\binom{j}{i}]_{0\leq i,j\leq n-1}$ and that matrix inversion commutes with the operation of transposing across the anti-diagonal, i.e. $A \mapsto JA^tJ$ with $J = [\delta_{i+j,n-1}]_{0\leq i,j\leq n-1}$. □

Theorem 4.6. *The sequences* $\{\phi_i\}_{i=0}^{n-1}$, $\{\gamma_i\}_{i=0}^{n-1}$, $\{\psi_i\}_{i=0}^{n-1}$ *are bases for* $\mathcal{X}$.

Proof. The sequence of ϕ_i's is a basis for $\mathcal{X}$, so by Proposition 4.5, the sequence of γ_i's and the sequence of ψ_i's are also bases for $\mathcal{X}$. □

Proposition 4.7. *The Frobenius characteristic* $\mathrm{ch}(\psi_k)$ *of the character* ψ_k *of* S_n *satisfies*

$$\mathrm{ch}(\psi_k) = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k+1}} \mathsf{M}(\lambda) h_\lambda. \tag{4.9}$$

Proof. By [8, Cor. 8],

$$\mathrm{ch}(\gamma_j) = \sum_{\lambda\vdash n} \binom{j+1}{\ell(\lambda)} M(\lambda) h_\lambda, \tag{4.10}$$

so

$$\begin{aligned}
\mathrm{ch}(\psi_k) &= \sum_{j=0}^{n-1} (-1)^{k-j} \binom{k+1}{j+1} \sum_{\lambda\vdash n} \binom{j+1}{\ell(\lambda)} \mathsf{M}(\lambda) h_\lambda \\
&= \sum_{\lambda\vdash n} \mathsf{M}(\lambda) h_\lambda \sum_{j=1}^{n} (-1)^{k+1-j} \binom{k+1}{j} \binom{j}{\ell(\lambda)} \\
&= \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k+1}} \mathsf{M}(\lambda) h_\lambda.
\end{aligned}$$

□

4.2.2. Next, we show that certain scalar multiples of the ψ_i basis for $\mathcal{X}$ give a basis for $\mathcal{Y}$ and that this basis for $\mathcal{Y}$ can be extended to a basis for $\mathcal{Z}$.

Definition 4.8. Define $\omega_0, \omega_1, \ldots, \omega_{n-1} \in \mathrm{CF}_\ell(S_n)$ by

$$\omega_k = \frac{1}{d_{k+1}} \psi_k, \quad d_k = \frac{k}{\gcd(n,k)}. \tag{4.11}$$

Proposition 4.9. $\omega_0, \omega_1, \ldots, \omega_{n-1}$ *are linearly independent characters in* $\mathcal{L}$.

Proof. The ω_i's are linearly independent because the ψ_i's are linearly independent, and they are characters by Proposition 2.1 and Proposition 4.7. □

Theorem 4.10. $\omega_0, \omega_1, \ldots, \omega_{n-1}$ *can be extended to a basis of* $\mathcal{Z}$.

Proof. For $0 \le k \le n-1$, let

$$B_k = \{\mathrm{ch}^{-1}(h_\lambda) \mid \lambda \vdash n,\ \ell(\lambda) = k+1\},$$

so $B_0, B_1, \ldots, B_{n-1}$ are pairwise disjoint and $B = \bigcup_{k=0}^{n-1} B_k$ is a basis for $\mathcal{Z}$.
By Proposition 2.1 and Proposition 4.7, for $0 \le k \le n-1$,

$$\omega_k = \sum_{\xi \in B_k} a_\xi \xi \tag{4.12}$$

with coefficients a_ξ that are positive integers satisfying

$$\gcd\{a_\xi \mid \xi \in B_k\} = 1. \tag{4.13}$$

Now, if S is any non-empty subset of B, and if χ is any character of shape

$$\chi = \sum_{\xi \in S} a_\xi \xi$$

with positive integer coefficients a_ξ that satisfy $\gcd\{a_\xi \mid \xi \in S\} = 1$, then

$$\chi \text{ can be extended to a basis of } \textstyle\bigoplus_{\xi \in S} \mathbb{Z}\xi, \tag{4.14}$$

which we establish by induction on $|S|$ as follows. The $|S| = 1$ case is trivial. Assuming the statement for all subsets of B with exactly k elements, suppose χ is of the described shape as a sum over a subset $S \subset B$ with $|S| = k+1$. Let $T = S \smallsetminus \{\eta\}$ for some $\eta \in S$ and write

$$\chi = a\eta + b\theta,$$

where a is the coefficient of η in χ and $b = \gcd\{\text{coeff. of } \xi \text{ in } \chi \mid \xi \in T\}$, so $\gcd(a, b) = 1$. Writing $au + bv = 1$ for some $u, v \in \mathbb{Z}$, let $\psi = u\theta - v\eta$. Then

$$\eta = u\chi - b\psi \quad \text{and} \quad \theta = v\chi + a\psi,$$

so the pair χ, ψ is a basis for $\mathbb{Z}\eta \oplus \mathbb{Z}\theta$. By hypothesis, θ can be extended to a basis of $\bigoplus_{\xi \in T} \mathbb{Z}\xi$. Therefore, χ can be extended to a basis of $\bigoplus_{\xi \in S} \mathbb{Z}\xi$.
By (4.12), (4.13), and (4.14), for $0 \le k \le n-1$, ω_k can be extended to a basis of $\bigoplus_{\xi \in B_k} \mathbb{Z}\xi$. So $\omega_0, \omega_1, \ldots, \omega_{n-1}$ can be extended to a basis of $\mathcal{Z}$. □

Theorem 4.11. *$\omega_0, \omega_1, \ldots, \omega_{n-1}$ is a basis for $\mathcal{Y}$.*

Proof. By Proposition 4.9, the ω_i's are linearly independent, so it remains to show that they span $\mathcal{Y}$. By Theorem 4.6, the ϕ_i's and the ψ_i's are bases of $\mathcal{X}$, so they have the same rational span, which, by Definition 4.8, must be the rational span of the ω_i's. So by Proposition 4.1,

$$\mathcal{Y} = \mathcal{Z} \cap \{r_0\omega_0 + r_1\omega_1 + \ldots + r_{n-1}\omega_{n-1} \mid r_i \in \mathbb{Q}\}. \tag{4.15}$$

By Theorem 4.10, the right-hand side of (4.15) is the set of integer linear combinations of $\omega_0, \omega_1, \ldots, \omega_{n-1}$. So $\omega_0, \omega_1, \ldots, \omega_{n-1}$ is a basis of $\mathcal{Y}$. □

4.3. Proofs of Theorems 1.2–1.4. From Theorem 4.6 and Theorem 4.11 we will obtain an explicit description of the characters of S_n that lie in the fundamental parallelepiped $\mathcal{P}$. We start by reading off the number of these characters.

Theorem 4.12.

$$\mathcal{Y}/\mathcal{X} \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \ldots \times \mathbb{Z}/d_n\mathbb{Z}, \quad d_k = \frac{k}{\gcd(n,k)}.$$

In particular,

$$[\mathcal{Y} : \mathcal{X}] = |\mathcal{Z} \cap \mathcal{P}| = d_1 d_2 \ldots d_n = \frac{n!}{\gcd(1,n)\gcd(2,n)\ldots\gcd(n,n)}.$$

Proof of Theorem 4.12 and Theorem 1.3. By Theorem 4.6, Theorem 4.11, and the relation $\psi_k = d_{k+1}\omega_k$ for $0 \le k \le n-1$. □

Proof of Theorem 1.4. From Theorem 4.12, we have

$$\sigma_n = \mathrm{lcm}(d_1, d_2, \ldots, d_n), \quad d_k = \frac{k}{\gcd(n,k)}.$$

So

$$\begin{aligned} \sigma_n &= \frac{1}{n}\mathrm{lcm}\left(\frac{n}{\gcd(1,n)}, \frac{2n}{\gcd(2,n)}, \ldots, \frac{n^2}{\gcd(n,n)}\right) \\ &= \frac{1}{n}\mathrm{lcm}(\mathrm{lcm}(1,n), \mathrm{lcm}(2,n), \ldots, \mathrm{lcm}(n,n)) \\ &= \frac{\mathrm{lcm}(1,2,\ldots,n)}{n}. \end{aligned}$$

□

Theorem 4.13. *The elements of $\mathcal{Z} \cap \mathcal{P}$ are*

$$\theta_{\mathfrak{a}} = \tilde{\mathfrak{a}}_0\phi_0 + \tilde{\mathfrak{a}}_1\phi_1 + \ldots + \tilde{\mathfrak{a}}_{n-1}\phi_{n-1}, \quad \mathfrak{a} \in \mathbb{N}^n, \quad 0 \le \mathfrak{a}_k < d_{k+1},$$

where

$$\tilde{\mathfrak{a}}_k = \left\{ \sum_{j=0}^{n-1} \binom{n-k-1}{j-k} \frac{\mathfrak{a}_j}{d_{j+1}} \right\}, \quad d_k = \frac{k}{\gcd(n,k)}.$$

Proof. Let

$$\mathcal{A} = \{(\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{n-1})^t \in \mathbb{N}^n \mid 0 \le \mathfrak{a}_k < d_{k+1}\}.$$

Then by Theorem 4.6, Theorem 4.11, and the relation $\psi_k = d_{k+1}\omega_k$, the characters

$$\xi_{\mathfrak{a}} = \mathfrak{a}_0\omega_0 + \mathfrak{a}_1\omega_1 + \ldots + \mathfrak{a}_{n-1}\omega_{n-1}, \quad \mathfrak{a} \in \mathcal{A},$$

form a complete set of pairwise distinct representatives for the cosets in $\mathcal{Y}/\mathcal{X}$. Using the second equality in (4.4),

$$\xi_{\mathfrak{a}} = \hat{\mathfrak{a}}_0\phi_0 + \hat{\mathfrak{a}}_1\phi_1 + \ldots + \hat{\mathfrak{a}}_{n-1}\phi_{n-1}, \quad \mathfrak{a} \in \mathcal{A},$$

where

$$\hat{\mathfrak{a}}_k = \sum_{j=0}^{n-1} \binom{n-k-1}{j-k} \frac{\mathfrak{a}_j}{d_{j+1}}.$$

So

$$\xi_{\mathfrak{a}} + \mathcal{X} = \theta_{\mathfrak{a}} + \mathcal{X}, \quad \mathfrak{a} \in \mathcal{A},$$

and hence the $\theta_{\mathfrak{a}}$ with $\mathfrak{a} \in \mathcal{A}$ also form a complete set of pairwise distinct representatives for the cosets in $\mathcal{Y}/\mathcal{X}$. Since $\theta_{\mathfrak{a}} \in \mathcal{P}$ for each $\mathfrak{a} \in \mathcal{A}$, we conclude that the $\theta_{\mathfrak{a}}$'s are precisely the pairwise distinct elements of $\mathcal{Z} \cap \mathcal{P}$. □

Proof of Theorem 1.2. By Theorem 4.13 and Proposition 4.3. □

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