Vanishing results

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Part 1. Character vanishing

Burnside Nonlinear irreducible characters have zeros.

Question What is the chance that $\chi(g)$ equals 0?

Two natural ways to pick a random character value $\chi(g)$

S_4

	1 id	6 (12)	3 (12)(34)	8 (123)	6 (1234)
χ_1	1	1	1	1	1
χ_2	3	1	-1	0	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	1	-1	1	1	-1

1. Choose $\chi \in \mathrm{Irr}(G)$ and $g \in G$, and then evaluate $\chi(g)$.

$$\mathrm{Prob}(\chi(g) = 0) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|} = \frac{28}{120} \approx 0.194$$

2. Choose $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, and then evaluate $\chi(K)$.

$$\mathrm{Prob}(\chi(K) = 0) = \frac{|\{(\chi, K) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(K) = 0\}|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|} = \frac{4}{5^2} = 0.16$$

$\mathrm{Prob}(\chi(g) = 0)$ for S_n

Theorem (M.) If $\chi \in \mathrm{Irr}(S_n)$ and $g \in S_n$ are chosen uniformly at random, then $\chi(g) = 0$ with probability $\to 1$ as $n \to \infty$.

One reason A vanishingly small proportion of classes covers almost all of S_n.

Lemma (M.) For any $\mathcal{K} \subseteq \mathrm{Cl}(G)$,

$$\mathrm{Prob}(\chi(g) = 0) \geq \frac{|\{g \in G : g^G \in \mathcal{K}\}|}{|G|} - \frac{|\mathcal{K}|}{|\mathrm{Cl}(G)|}.$$

$\mathrm{Prob}(\chi(g) = 0)$ and $\mathrm{Prob}(\chi(K) = 0)$ for some other groups

Lemma (Gallagher–Larsen–M.) For each finite group G and $\epsilon > 0$,

$$\mathrm{Prob}(\chi(g) \neq 0) \leq \frac{\left|\{(\chi, g) : \mathrm{gcd}(\chi(1), |g^G|) \geq \epsilon\chi(1)\}\right|}{|\mathrm{Irr}(G) \times G|} + \epsilon^2.$$

Theorem (Gallagher–Larsen–M.) For $G = \mathrm{GL}(n, q)$, the proportion $P_{n,q}$ of pairs $(\chi, g) \in \mathrm{Irr}(G) \times G$ with $\chi(g) = 0$ satisfies

$$\inf_q P_{n,q} \to 1 \text{ as } n \to \infty.$$

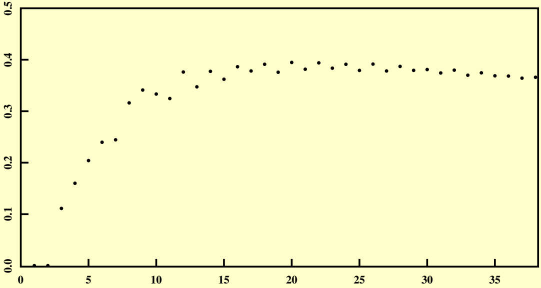
So for any sequence of prime powers $q_1, q_2, \ldots$, we have $P_{n,q_n} \to 1$ as $n \to \infty$.

Theorem (Larsen–M.) If G_n is any sequence of finite simple groups of Lie type with rank tending to ∞, then almost every entry in the character table of G_n is zero as n tends to ∞.

$\mathrm{Prob}(\chi(K) = 0)$ for S_n

Question What can be said about the limiting behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$?

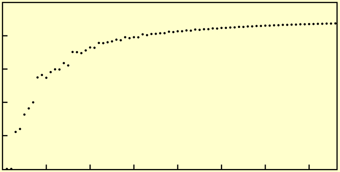
n	$\mathrm{Prob}(\chi_\lambda(\mu) = 0)$
2	0.0000
3	0.1111
4	0.1600
5	0.2041
...	...
37	0.3642
38	0.3659



Best known bound $\mathrm{Prob}(\chi_\lambda(\mu) = 0) \geq C/\log n$

Part 2. Modular vanishing for S_n

n	4	5	6	7	8	9	10	...	76
$\mathrm{Prob}(\chi_\lambda(\mu) = 0 \text{ mod } 2)$	.24	.33	.36	.40	.55	.56	.55		.87



- The character table of S_{76} has about 86 trillion entries.
- Several months of computations on super computers used for earth science.
- Similar computations for other primes and prime powers.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime p as $n \to \infty$. In other words, $\mathrm{Prob}(\chi_\lambda(\mu) = 0 \mod p) \to 1$ as $n \to \infty$.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed m as $n \to \infty$.

Part 2. Modular zeros for S_n

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Conjecture (M.)
Almost every entry in the character table of S_n is divisible by any fixed integer.

Lemma $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \mod p$ if ν is obtained by joining p equal parts in μ.

Example For $p = 2$, $\chi_\lambda(42111) \equiv \chi_\lambda(4221) \equiv \chi_\lambda(441) \equiv \chi_\lambda(81) \mod 2$.

Corollary $\chi_\lambda(\mu) \equiv 0 \mod p$ if λ is a $\mathfrak{l}_p(\mu)$-core.

n	30	40	50	60	70	80	90	100	110	120
$\mathrm{Prob}(\lambda$ is an $\mathfrak{l}_2(\mu)$-core$)$	.45	.44	.45	.47	.48	.48	.48	.47	.47	.46

Partial results M., Gluck, Morotti, Ono, McSpirit, Harman, Peluse, Soundararajan,...

Lemma (Morotti) $\#\{$partitions of n that are not t-cores$\} \leq (t+1)p(n-t)$.

Theorem (Peluse–Soundararajan)
Almost every entry in the character table of S_n is divisible by any fixed prime.

Part 3. Two variations

Let $P(G)$ be either $\mathrm{Prob}(\chi(g)=0)$ or $\mathrm{Prob}(\chi(K)=0)$.

1. Treat $P(G)$ as a random variable itself.

Theorem (M.) The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.

2.
$$\begin{array}{ccc} G_1 < G_2 < \dots & \longrightarrow & P(G_1), P(G_2), \dots \in [0,1] \\ ? & \longleftarrow & a_1\ ,\ a_2\ ,\ \dots \end{array}$$

Theorem (M.) If $a_1, a_2, \dots \in [0,1]$ and $\epsilon_1, \epsilon_2, \dots \in (0,\infty)$, then for each prime p there exists a chain of p-groups $G_1 < G_2 < \dots$ such that, for each i,

$$|P(G_i) - a_i| < \epsilon_i.$$

In particular, the set $\{P(G) : |G| < \infty\}$ is dense in $[0,1]$.

Part 4. Zeros and roots of unity

	1	6	3	8	6	
	id	(12)	(12)(34)	(123)	(1234)	
χ_1	1	1	1	1	1	24/24
χ_2	3	1	-1	0	-1	23/24
χ_3	2	0	2	-1	0	20/24
χ_4	3	-1	-1	0	1	23/24
χ_5	1	-1	1	1	-1	24/24
		5/5		5/5	5/5	

Definition

$$\theta(G) = \min_{\chi \in \mathrm{Irr}(G)} \frac{\#\{g \in G : \chi(g) \text{ is 0 or a root of unity}\}}{|G|}$$

$$\theta'(G) = \min_{\text{L.T.A. } K} \frac{\#\{\chi \in \mathrm{Irr}(G) : \chi(K) \text{ is 0 or a root of unity}\}}{|\mathrm{Cl}(G)|}$$

Example $\theta(S_4) = 20/24$ and $\theta'(S_4) = 1$.

Theorem (J.G. Thompson) $\theta(G) > 1/3$.

Theorem (P.X. Gallagher) $\theta'(G) > 1/3$.

Theorem (C.L. Siegel) For totally positive $\alpha \neq 0, 1$, $\tilde{\mathrm{Tr}}(\alpha) \geq 3/2$.
(Siegel gets applied to $\alpha = |\chi(g)|^2$. Burnside used $\tilde{\mathrm{Tr}}(\alpha) \geq 1$ for $\alpha \neq 0$.)

Theorem (Thompson) $\theta(G) > 1/3$.

Theorem (Gallagher) $\theta'(G) > 1/3$.

Question What are the greatest lower bounds: $\inf_G \theta(G)$, $\inf_G \theta'(G)$?

Conjecture (M.) $\inf \theta(G), \inf \theta'(G) = 1/2$.

Proposition (M.) For $G_n = Suz(2^{2n+1})$, we have $\theta(G_n), \theta'(G_n) \to 1/2$.

Theorem (M.) The conjecture holds for:

- Finite groups of order $< 2^9$.
- Simple groups of order $\leq 10^9$.
- All sporadic groups.
- A_n, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$.
- All finite nilpotent groups.

Theorem (M.) Each nonlinear irreducible character of a finite nilpotent group is zero on more than half the group.

Theorem (M.) More than half the irreducible characters of a finite nilpotent group are zero on any given larger-than-average class.

Theorem (M.) Let G be finite nilpotent, let $\chi \in \mathrm{Irr}(G)$, and let $g \in G$. Then

$$\chi(g) = 0 \quad \text{or} \quad \tilde{\mathrm{Tr}}\left(|\chi(g)|^2\right) \geq 2^{\#\{\text{primes dividing } \chi(1)\}}$$