STOCHASTIC LIQUIDITY MODELS AND APPLICATION TO NITFY 50

Independent Research Project

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Abstract

This research investigates the influence of liquidity on option pricing using data from the NIFTY 50 index in India. While initial attempts employed a data-driven approach using artificial neural networks, limited data availability and challenges with liquidity metrics led to the adoption of a stochastic liquidity model. This model incorporates liquidity as a stochastic factor, allowing for analytical pricing of European options. By calibrating the model using real-world data from the NIFTY 50 during a period of liquidity shock, the study explores the model's effectiveness in capturing liquidity's impact on option prices. Additionally, the paper examines an extension of this model with regime switching to account for changing economic conditions. Monte Carlo simulations and closed-form solutions were implemented to evaluate model performance, with particular focus on in-sample and out-of-sample errors. Results indicate that while the stochastic liquidity model shows strong performance, the regime-switching variant faced calibration challenges, highlighting the complexity of capturing real-world liquidity dynamics. The study concludes by emphasizing the need for further refinement and faster calibration techniques to improve model utility in practical applications.

1. INTRODUCTION

The pricing of financial derivatives, particularly options, has long relied on models that assume idealized market conditions, such as the Black-Scholes model, which assumes perfect liquidity in the underlying asset market. However, real-world markets exhibit liquidity constraints, which can significantly impact asset prices and derivative valuation. In particular, liquidity risk—the risk that an asset cannot be traded quickly without affecting its price—is a key factor in modern financial markets, especially during periods of market stress. Numerous studies have demonstrated that liquidity risk plays a crucial role in asset pricing, yet its influence on option pricing remains an area of active research. This study aims to investigate the effect of liquidity on option pricing using data from the NIFTY 50 index, a benchmark stock index in India that includes the 50 largest companies by free-float market capitalization.

Our initial attempt at a data-driven approach to analyse the impact of liquidity on option prices was hindered by limited data availability, particularly on liquidity measures specific to the Indian market. As a result, we turned to a theoretical framework based on a stochastic liquidity model, which extends the classical option pricing models by incorporating liquidity as a stochastic factor. This model provides an analytical solution for European option pricing, allowing for rapid and accurate calculations without the need for extensive simulations. However, the calibration of this model is computationally intensive, given the high-dimensional nature of the pricing function and the complexity of the analytical solution.

Despite these computational challenges, the model offers significant insights into the interplay between liquidity and option prices. By applying this stochastic liquidity model to the NIFTY 50 options data, we aim to quantify the effect of liquidity risk on option pricing in the Indian market, applying new models from the growing body of literature that emphasizes the importance of liquidity in financial markets.

2. NIFTY 50 INDEX OVERVIEW

The NIFTY 50 is a flagship stock market index of the National Stock Exchange (NSE) of India. It represents the performance of 50 of the largest and most liquid companies listed on the NSE, making it a key indicator of the Indian equity market's overall performance. Constituted in 1996, the index covers approximately 65% of the total free-float market capitalization of the NSE.

The NSE offers a broad array of derivatives on the NIFTY 50, including futures and options contracts. These derivatives are widely used for hedging and speculative purposes, with European-style options being the most traded. The options expire on the last Thursday of every month, New contracts are quoted using **the Black-Scholes model,** where volatility is typically implied by the NIFTY VIX (Volatility Index).

The data utilized in this study is the historical price and volume data for NIFTY 50 index options, which was sourced from the National Stock Exchange (NSE) of India website. Below is a summary of the attributes available in the dataset:

- **Symbol**: The ticker symbol representing the NIFTY 50 index options contract.
- **Date**: The trading date on which the data was recorded.
- **Expiry**: The expiration date of the options contract. NIFTY 50 options are European-style and expire on the last Thursday of every month.
- **Option Type**: Specifies whether the option is a **Call (CE)** or **Put (PE)** option.
- **Strike Price**: The fixed price at which the underlying asset (NIFTY 50 index) can be bought or sold upon expiration of the option.
- **Open**: The first traded price of the option on a given trading day.
- **High**: The highest traded price of the option during the trading day.
- **Low**: The lowest traded price of the option during the trading day.
- **Close**: The average of the last 30 minutes of trading of the option. If no trades are made in that time, the last traded price (LTP) of the option is used instead.
- **Settle Price**: A specially adjusted price which represents the final price of the option adjusted to be consistent with put call parity.
- **No. of Contracts**: The total number of option contracts traded during the trading day.
- **Turnover (₹ Lakhs)**: The total value of contracts traded during the day, measured in Indian rupees (₹), with turnover expressed in **lakhs** (1 lakh = 100,000).
- **Premium Turnover (₹ Lakhs)**: The total premium paid or received for the options contracts traded during the day, also expressed in **lakhs**.
- **Open Interest (OI)**: The total number of outstanding option contracts that have not been settled.
- **Change in OI**: The daily change in the open interest of the option, representing the difference in the number of outstanding contracts from the previous trading day.
- **Underlying Value**: The value of the NIFTY 50 index, which is the underlying asset for the options contracts.

While not as rich in information as the option chain data that is available for current expiries, this historical dataset provides us with a plethora of information that can be used. To study specifically the period where the Indian market experienced a large liquidity shock due to demonetization, we will consider the option prices in the period from Nov 2016 to Mar 2017 with option expiries up to Jun 2017.

3. OPTION PRICING WITH AN ARTIFICIAL NEURAL NETWORK

3.1 DEFINING LIQUIDITY

Liquidity is a critical factor in financial markets, influencing asset prices, trading strategies, and the efficiency of market operations. However, there is no universally accepted definition or metric for liquidity. The complexity of financial markets means that liquidity can be characterized in various ways, depending on the specific context and the available data. The literature presents multiple[1] methods for quantifying[2] liquidity, each with its strengths and limitations. The following section explains the different possible liquidity measures and how they apply in our case with the data available.

1. **Bid-Ask Spread as an Indicator of Liquidity**
   The bid-ask spread is one of the most commonly used measures of liquidity. A narrow spread typically indicates high liquidity, as there is little difference between the prices at which buyers and sellers are willing to transact. Conversely, a wide bid-ask spread suggests lower liquidity, as participants demand a premium for executing trades.
   Unfortunately, the NIFTY 50 options data available for this study does not include bid-ask spread information. Therefore, we are unable to use this metric as a proxy for liquidity.

2. **Trading Speed as a Discounting Factor for Option Prices**
   Another approach described involves using the trading speed of an asset, defined as the rate at which trades occur, as a discounting factor in option pricing models. Faster trading implies greater liquidity, as it indicates that the market can absorb larger volumes of trades without substantial price fluctuations. However, trading speed is not directly available in the historical price volume data for NIFTY 50 options. As such, this method could not be implemented in our study.

3. **Number of Trades and Open Interest as a Measure of Liquidity**
   Given the limitations of our dataset, we focus on two available metrics: number of trades and open interest. Both are widely recognized as important indicators of market liquidity. The number of trades reflects the frequency of transactions within a given period, with higher trade counts suggesting a more liquid market. Open interest **(OI)** measures the total number of outstanding derivative contracts (in this case, options) that have not been settled. A higher OI indicates greater market participation and a more liquid market, as more participants are engaged in trading the options.

In the absence of more granular data like bid-ask spreads or trading speed, we use the number of trades and open interest. By incorporating these variables into our model, we aim to capture the effect of liquidity on option pricing. Finally, empirical studies have consistently demonstrated that liquidity is a key factor in explaining asset price movements and returns. In liquid markets, large trades can be executed with minimal impact on price, whereas illiquid markets are prone to price slippage, where the act of trading itself affects prices. Moreover, market-wide liquidity conditions have been shown to impact the returns of stocks and options, especially during periods of market stress when liquidity tends to dry up.

3.2 ARTIFICIAL NEURAL NETWORK

The initial approach for this study was to employ a purely data-driven pricing model using an artificial neural network (ANN). The goal was to leverage the historical data available for NIFTY 50 options to predict the pricing of options based on liquidity and other market factors, without relying on traditional analytical models. The chosen target variable for the ANN model was the relative error between the settle price and the close price of the options. The settle price represents the final price at which the options are settled upon expiry, while the close price is the last traded price during the trading session. This metric was used to quantify the difference between the expected price at expiration (settle price) and the actual trading price at the market close (close price). A successful prediction of the relative error would allow the model to effectively forecast discrepancies between market expectations and actual trading behaviour.

To capture the impact of liquidity on option pricing, the ratio of number of contracts traded to open interest was chosen as a proxy for liquidity in this data-driven approach. This ratio reflects how actively an option is traded relative to how much it is being held by market participants. Higher values of this ratio indicate that options are trading more actively relative to how much they are being held, suggesting greater liquidity.

3.3 Challenges and Limitations

Despite initial efforts, this purely data-driven approach did not yield satisfying results. A key limitation was the insufficient quantity of available data. The historical NIFTY 50 options data lacked sufficient depth and breadth to robustly train the artificial neural network. As a result, the model struggled to make accurate predictions about the relative error between settle and close prices, particularly when factoring in the complex role of liquidity.

Furthermore, the inherent structure of the underlying data was not fully understood, making it challenging to generate synthetic or simulated data to augment the dataset. Simulating additional data without a clear understanding of the relationships between key variables could introduce significant bias and reduce the reliability of the model's predictions. Thus, while a purely data-driven approach holds promise, the available dataset was not rich enough to support meaningful conclusions using this method alone.

In light of these challenges, the focus of the research shifted towards incorporating stochastic liquidity models, which provide a theoretical framework for understanding liquidity's role in option pricing. This approach offers a more structured and analytically sound method for tackling the liquidity problem.

4. Stochastic Liquidity Based Modelling

To fully comprehend stochastic liquidity-based modelling, it is essential to first establish a basic understanding of key foundational concepts and models in asset pricing, notably the Black-Scholes model, stochastic volatility models, and risk-neutral pricing. These concepts provide the framework upon which stochastic liquidity models can be built. Although the details of these topics will not be discussed in depth, a general familiarity with stochastic calculus, geometric Brownian motion, and their applications to financial models is assumed.

4.1 Summary of Prerequisite Knowledge

4.1.1 Black-Scholes Model Dynamics and PDE

The Black-Scholes model is the cornerstone of modern option pricing theory, providing a closed-form solution for the price of European-style options. It rests on the assumption that the price of the underlying asset (e.g., a stock) follows a geometric Brownian motion (GBM) with constant volatility.

This stochastic process governs how the price of the stock evolves over time. Other key assumptions of the model that we are concerned with include no arbitrage opportunities in the market and efficient markets with perfect liquidity and no slippage or transaction costs.

The dynamics of the stock price S_t under the Black-Scholes model are described by the following stochastic differential equation (SDE):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

where:

- S_t is the price of the stock at time t,
- μ is the constant drift rate, representing the expected return of the stock,
- σ is the constant volatility of the stock's returns, which measures the uncertainty or risk associated with the stock,
- dW_t is a Wiener process (also known as Brownian motion) that introduces randomness into the stock price dynamics.

To derive the price of an option on this stock, the Black-Scholes model introduces a partial differential equation (PDE) that the price of the option must satisfy. Consider a European call option, which gives the holder the right to buy the stock at a fixed price (strike price) at a future time (expiration). Let $V(S,t)$ be a function of the stock price S and time t. Using Ito's Lemma and applying no-arbitrage conditions, the dynamics of the function are described by the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Using the function for the price of a European call as $V(S,t)$ we arrive at the Black-Scholes formula.

$$C(S,t) = S_t N(d_1) - Xe^{-r(T-t)}N(d_2)$$

where:

- $C(S,t)$ is the price of the call option,
- X is the strike price of the option,
- $N(d_1)$ and $N(d_2)$ are the cumulative normal distribution functions, and

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

Once again, volatility is assumed to be a constant value, which is not true in practice.

4.1.2 Stochastic Volatility Models

One of the main limitations of the Black-Scholes model is the assumption of constant volatility, which does not accurately reflect the behaviour of financial markets. Stochastic volatility models address this by allowing volatility to vary randomly over time. We will not be discussing the specifics of stochastic volatility models in detail, but it is important to introduce here as the stochastic liquidity models used are very similar to stochastic volatility models. In these models, the stock dynamics can be described by:

$$dS_t = \mu S_t\,dt + \sigma(t, S_t) S_t\,dW_t^S$$

Where volatility itself is a stochastic process with its own governing dynamics described by a stochastic differential equation.

4.1.3 Risk Neutral Pricing and Change of Measure

Under the risk-neutral measure, it is assumed at all investors are risk-neutral, meaning they do not require any additional return for taking on risk beyond the risk-free rate. This assumption allows for a simplified and consistent valuation of financial assets by pricing them as though all future cash flows are discounted at the risk-free rate. While it is true that in reality investors tend to be more risk averse, in practice risk-neutral pricing is used throughout the financial services industry.

The Radon-Nikodym derivative provides the formal link between the real-world (or physical) probability measure *P* the risk-neutral measure *Q*. The change of measure can be written as:

$$\frac{dP}{dQ} = \mathcal{Z}_T$$

Where Z_t is the Radon-Nikodym derivative, and it represents the density that transforms the physical probability measure *P* into the risk-neutral measure *Q*. This transformation allows for a consistent redefinition of the asset price dynamics under the risk-neutral world.

Under the risk-neutral measure *Q*, the expected growth rate of an asset's price is adjusted so that it equals the risk-free rate, rather than the asset's actual expected return. Specifically, under the risk-neutral measure, the drift term of the asset price dynamics is replaced with the risk-free rate *r*. This adjustment enables the use of the risk-free rate for discounting future cash flows, leading to a consistent and theoretically sound method of pricing derivatives, such as options. In a complete market with no arbitrage opportunities, the fundamental theorem of asset pricing tells us that there is a unique risk neutral measure for all asset prices.

The price $V(S,t)$ of a derivative at time *t* with a general payoff function $g(S, t)$, under the risk-neutral measure *Q*, is given by:

$$V(S,t) = e^{-r(T-t)}\mathbb{E}_Q[g(S_T)]$$

4.2 Stochastic Liquidity Model

For the stochastic liquidity models, we will be using a model that incorporates liquidity risk through the use of a liquidity discounting factor that considers stochastic market-wide liquidity. This type of model was first proposed by Feng et al[3] and then expanded upon by Puneet Pasricha et al. The model described and implemented below is based on the expanded work of Puneet Pasricha et al[4].

4.2.1 Dynamics of Stochastic Liquidity Model

To incorporate the effects of liquidity, we assume that the market liquidity risk acts as a discounting factor on the stock price. Furthermore we assume that the Black-Scholes stock price is the price without discounting. This means that the price under liquidity can be written as:

$$S_t = \frac{1}{\gamma_t} S_t^{BS}$$

Now that we have introduced a market liquidity discount factor, we define it as a stochastic process dependant on market liquidity level L_t and the sensitivity of the price to market liquidity $\beta>0$. Following Brunetti and Caldarera[5] we define the discount factor as:

$$\frac{d\gamma_t}{\gamma_t} = \left(-\beta L_t + \frac{1}{2}\beta^2 L_t^2\right) dt - \beta L_t dB_t^\gamma$$

With $dB_t^\gamma \, dB_t^S = \rho_1 dt$. Substituting the illiquid stock price with market liquidity discounting factor into the Black-Scholes dynamics equation gives us the formula for the dynamics of the illiquid stock S_t:

$$\frac{dS_t}{S_t} = \left(\mu_S + \beta L_t + \frac{1}{2}\beta^2 L_t^2 + \rho_1 \sigma_S \beta L_t\right) dt + \beta L_t dB_t^\gamma + \sigma_s dB_t^S.$$

Finally, the liquidity level L_t also needs to be a stochastic process to complete the definition of the stochastic liquidity model. Based on empirical evidence[6] and following the work of Feng et al[3], we assume that the liquidity level follows a mean reverting (Ornstein-Uhlenbeck) process:

$$dL_t = \tilde{\alpha}\left(\tilde{\theta} - L_t\right) dt + \xi \, dB_t^L,$$

where ã,õ, and ξ are respectively the mean-reversion speed, equilibrium level and volatility of market liquidity, and B_t^L, is a Wiener process under P. The general correlation structure for the three Brownian motions is:

$$\begin{aligned} dB_t^\gamma dB_t^S &= \rho_1 dt, \\ dB_t^L dB_t^S &= \rho_2 dt, \\ dB_t^\gamma dB_t^L &= \rho_3 dt. \end{aligned}$$

The proposed model is similar to a stochastic volatility model as it can be describes as saying that the stochastic nature of the volatility of the asset is being driven by the market liquidity. However, unlike stochastic volatility, L_t can take both positive and negative values depending on if the asset in question is in surplus or shortage. Now, we change the model to a risk neutral measure so as to simulate prices. Since the model is meant to capture the dynamics of an illiquid market, the market is not complete and there is no unique risk neutral measure. The Radon-Nikodym derivative used is:

$$\frac{dQ}{dP} = \exp\left\{ -\int_0^t \lambda_1(s)dB_s^S - \int_0^t \lambda_2(s)dB_s^\gamma - \frac{1}{2}\int_0^t \lambda_1^2(s)ds - \frac{1}{2}\int_0^t \lambda_2^2(s)ds - \rho_1 \int_0^t \lambda_1(s)\lambda_2(s)ds \right\}$$

Finally, assuming that the price of liquidity risk is proportional to liquidity as suggested by Heston, we can arrive at the following risk neutral form of the model:

$$\frac{dS_t}{S_t} = rdt + \beta L_t dW_t^\gamma + \sigma_s dW_t^S, \qquad dL_t = \alpha(\theta - L_t)dt + \xi dW_t^L$$

4.2.2 MONTE CARLO SIMULATION OF MODEL

Monte Carlo simulation is a computational technique used to estimate the outcomes of complex mathematical models. It involves running repeated random simulations to approximate the distribution of possible outcomes for a given process. In finance, Monte Carlo simulation is widely used for pricing derivatives, such as options, by simulating the possible future paths of the underlying asset prices. For each simulated path, the payoff of the derivative is calculated, and the average payoff is discounted back to the present using the risk-free rate. This approach allows for the valuation of financial instruments in models that involve uncertainty, such as stochastic volatility or stochastic liquidity.

Key Steps:

1. Generate Random Variables: Simulate multiple scenarios of the underlying asset's price, typically using random numbers from a normal distribution.

2. Simulate Price Paths: Use the stochastic differential equation (SDE) governing the asset's price to simulate many possible price paths over time.

3. Calculate Payoffs: For each simulated path, compute the derivative's payoff at expiration.

4. Average and Discount: Average the payoffs from all simulations and discount them to the present value using the risk-free rate to obtain an estimate of the derivative's price.

The correlation structure between the three Brownian motions implemented using the Cholesky decomposition of the correlation matrix and multiplying the independent random variables by it. This method is very computationally expensive as compared to other ways to approximate the correlation, but it provides exact results. This is important in this case as the monte carlo simulated option price will be the benchmark used to test the accuracy of the closed form solution of the model which will be introduced in the next part. For all the graphs below, unless stated otherwise the parameter values are:

r = 0.01, sigma = 0.2 , beta = 0.5, alpha = 0.2, xi = 0.9 , theta = 0.3, rho_gamma_s = 0.2, rho_l_s = 0.2, rho_gamma_l = 0.2, tau = 1.0, $liquidity_0$ *= 0.3 S = 100, K = 110.* The number of steps is 100.

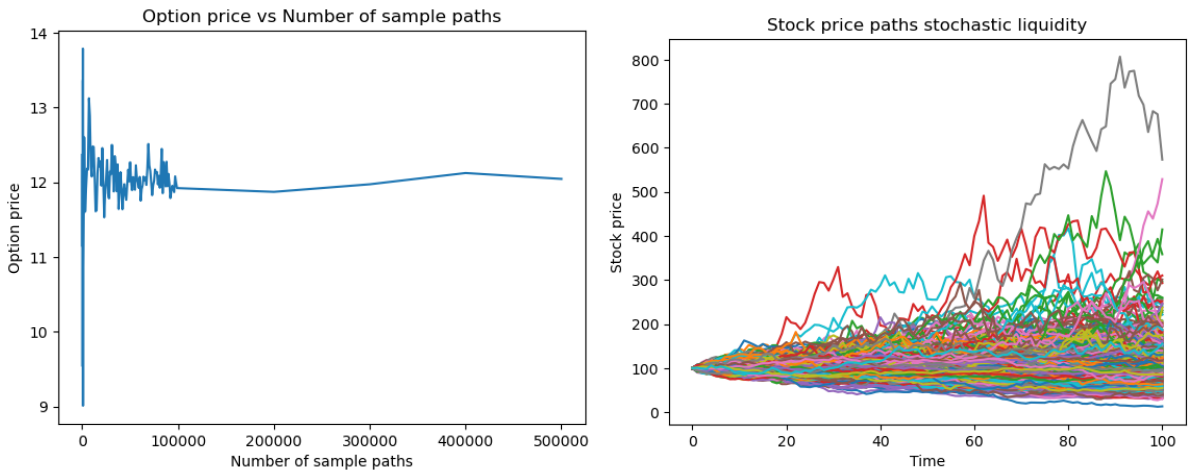


Fig. 1 Shows the convergence of the predicted option price and the paths of 1000 of the random walks

We can observe that the simulated option price seems to stabilize and converge after 100,000 paths. As such, we will simulate 150,000 paths for obtaining the price to compare the closed form formula with.

4.2.3 CLOSED FORM SOLUTION

In their paper, Puneet Pasricha et al[4] derived a closed form solution to the model which can be used to analytically price a European call option for an asset that follows the previously defined stochastic liquidity dynamics. The exact details of this derivation are extremely technical and beyond the scope of this paper, in addition only the final closed form solution is necessary to be able to numerically implement the model and generate prices. In the following section, we will see the overall procedure for the derivation without elaborating much on the specific details of each derivation step.

Under a risk neutral measure Q, the definition of the of a call option at maturity can be written as:

$$\begin{aligned} C =& E_t^Q\left(e^{-r(T-t)}(S_T-K)^+\right) \\ =& e^{-r(T-t)}E_t^Q\left(S_T I_{\{S_T\geq K\}}\right) - Ke^{-r(T-t)}E_t^Q\left(I_{\{S_T\geq K\}}\right). \end{aligned}$$

Where I_b is an indicator function that has value 1 when condition b true and 0 when it is false. We now perform a change of numeraire and introduce a new measure Q_1 with the following Radon-Nikodym derivative with respect to the risk neutral measure Q:

$$\begin{aligned} C =& S_t E_t^{Q_1}\left(I_{\{S_T\geq K\}}\right) - Ke^{-r(T-t)}E_t^Q\left(I_{\{S_T\geq K\}}\right) \\ =& S_t F_1 - Ke^{-r(T-t)}F_2, \end{aligned}$$

Here, we have rewritten the equation in terms of F_1 and F_2 to make the notation more compact. Notice how this equation is analogous to the famous Black-Scholes equation for calls but with N_1 and N_2 being replaced with F_1 and F_2. To solve for the value of F_1 and F_2 we can first decompose them into their characteristic function (Fourier transform):

$$F_1 = E_t^{Q_1}\left(I_{\{S_T \geq K\}}\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Real\left(\frac{e^{-j\eta\ln(K)}f_1(\eta, T)}{j\eta}\right)d\eta,$$

$$F_2 = E_t^{Q}\left(I_{\{S_T \geq K\}}\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Real\left(\frac{e^{-j\eta\ln(K)}f_2(\eta, T)}{j\eta}\right)d\eta.$$

with $j = \sqrt{-1}$ and,

$$f_1(\eta, T) = E_t^{Q_1}\left(e^{j\eta X_T}\right),\ f_2(\eta, T) = E_t^{Q}\left(e^{j\eta X_T}\right),$$

Where $X_t = \ln(S_t)$. It should be noted that since Q and Q_1 are equivalent martingale measures. It is possible to express f_1 in terms of f_2 using our numeraire:

$$\begin{aligned}
f_1(\eta, T) &= E_t^{Q_1}\left(e^{j\eta X_T}\right) \\
&= E_t^{Q}\left(e^{-r(T-t)+X_T-X_t}e^{j\eta X_T}\right) \\
&= e^{-r(T-t)-X_t}E_t^{Q}\left(e^{(1+j\eta)X_T}\right) \\
&= e^{-r(T-t)-X_t}f_2(\eta - j, T).
\end{aligned}$$

The only remaining equation to solve is f_2, which is dependent on the random variable X_T. Therefore, we must now determine the characteristic function of X_T. However, derivation of this function is not trivial and involves concepts that are beyond the scope. The steps for the derivation are provided below for reference but it should be understood that the focus of this research and paper was not on the derivation of the close form but the application to real world data.

Define:

$$h(\eta, t, T, x_t, l_t) = E^{Q}(e^{i\eta X_T} \mid x_t, l_t)$$ where l_t is the liquidity at time t

Then, from Feynman-Kac theorem, the partial differential equation (PDE) of h is:

$$\frac{\partial h}{\partial t} + \left(r - \frac{1}{2}\left(\beta^2 l^2 + \sigma^2\right) - \rho_1\sigma\beta l\right)\frac{\partial h}{\partial x} + \alpha(\theta - l)\frac{\partial h}{\partial l} + \left(\frac{1}{2}(\beta^2 l^2 + \sigma^2) + \rho_1\sigma\beta l\right)\frac{\partial^2 h}{\partial x^2}$$
$$+ \frac{1}{2}\xi^2\frac{\partial^2 h}{\partial l^2} + (\rho_2\sigma\xi + \rho_3\xi\beta l)\frac{\partial^2 h}{\partial x\partial l} = 0$$

Notice the similarity to the general form of the Black Scholes PDE with added partial derivatives with respect to liquidity. Assume that the function h is of the affine form:

$$h(\eta, t, T, x_t, l_t) = e^{A(t,T;\eta)+B(t,T;\eta)l_t+C(t,T;\eta)l_t^2+i\eta x_t}$$

Substitute into the PDE to obtain a system of ordinary differential equations (ODEs):

$$\frac{\partial C}{\partial t} = -2\xi^2C^2 + 2(\alpha - i\eta\rho_3\xi\beta)C + \frac{1}{2}\beta^2\left(i\eta + \eta^2\right),$$

$$\frac{\partial B}{\partial t} = \left(\alpha - 2\xi^2C(t, T, \eta) - i\eta\rho_3\xi\beta\right)B - (2\alpha\theta + 2i\rho_2\xi\sigma\eta)C(t, T) + \rho_1\sigma\beta\left(i\eta + \eta^2\right),$$

$$\frac{\partial A}{\partial t} = \frac{1}{2}\sigma^2\left(i\eta + \eta^2\right) - ir\eta - (\alpha\theta + \rho_2 i\xi\sigma\eta)B(t, T) - \frac{1}{2}\xi^2B(t, T)^2 - \xi^2C(t, T),$$

Given the terminal condition of h that when t = T, $h = f_2$, it is clear that A(T,T) = B(T,T) = C(T,T) = 0. With this information it is now possible to write an explicit form for C(t,T):

$$C(t, T) = \frac{1}{2\xi^2}\left((\alpha - i\eta\rho_3\xi\beta) - \delta_1\left(\frac{\sinh(\delta_1(T-t)) + \delta_2\cosh(\delta_1(T-t))}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))}\right)\right),$$

$$\delta_1 = \sqrt{(\alpha - i\eta\rho_3\xi\beta)^2 + \xi^2\beta^2(i\eta + \eta^2)},\ \delta_2 = \frac{\alpha - i\eta\rho_3\xi\beta}{\delta_1}$$

Now, by substituting the equation for C(t,T) into the partial derivative of B, we can determine B(t,T):

$$\begin{aligned} B(t, T) = &\frac{1}{2\delta_1\xi^2}\left(\frac{\delta_2\delta_3 - \delta_4\delta_1}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))} + \delta_4\delta_1\right) \\ &\frac{-\delta_3}{2\delta_1\xi^2}\left(\frac{\sinh(\delta_1(T-t)) + \delta_2\cosh(\delta_1(T-t))}{\cosh(\delta_1(T-t)) + \delta_2\sinh(\delta_1(T-t))}\right) \end{aligned}$$

Finally, A(t,T) can be find by substituting B and C into the partial equation for A. However, there is no explicit form of A(t,T) that we can write down. Instead, for each slice of the integral in the characteristic function of F_2, we must find the equations C(t,T) and B(t,T) and then integrate from t to T the partial derivative of A with respect to time:

$$\int_t^T \left[\frac{1}{2}\sigma^2(i\eta + \eta^2) - ir\eta - (\alpha\theta + \rho_2 i\xi\sigma\eta)B(s, T) - \frac{1}{2}\xi^2B(s, T)^2 - \xi^2C(s, T)\right] ds$$

Now that we know each of the terms of the closed form solution, we can implement it in python and compare the results with those obtained from monte carlo simulation. The parameters for the monte carlo simulation and closed form solution are the same as the ones used in section 4.2.2.

τ	K/S_0	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20
	Analytical	20.382	15.778	11.619	8.123	5.419	3.488	2.199	1.376	0.863
$\tau = 0.2$	Simulated	20.363	15.762	11.604	8.107	5.403	3.476	2.189	1.367	0.857
	Relative Error (%)	0.094	0.104	0.125	0.193	0.288	0.364	0.435	0.598	0.674
	Analytical	21.498	17.443	13.857	10.808	8.313	6.338	4.817	3.666	2.805
$\tau = 0.4$	Simulated	21.558	17.499	13.906	10.852	8.353	6.375	4.849	3.693	2.825
	Relative Error (%)	0.282	0.318	0.356	0.404	0.486	0.584	0.667	0.732	0.710
	Analytical	22.965	19.300	16.055	13.252	10.884	8.919	7.312	6.012	4.965
$\tau = 0.6$	Simulated	22.898	19.236	15.997	13.196	10.829	8.864	7.256	5.957	4.914
	Relative Error (%)	0.292	0.329	0.363	0.423	0.499	0.619	0.767	0.909	1.024
	Analytical	24.575	21.196	18.190	15.559	13.289	11.351	9.712	8.334	7.180
$\tau = 0.8$	Simulated	24.497	21.120	18.113	15.480	13.205	11.265	9.627	8.252	7.102
	Relative Error (%)	0.316	0.360	0.423	0.507	0.626	0.755	0.882	0.988	1.087
	Analytical	26.240	23.086	20.263	17.766	15.579	13.676	12.031	10.613	9.394
$\tau = 1.0$	Simulated	26.205	23.053	20.231	17.729	15.538	13.629	11.977	10.555	9.333
	Relative Error (%)	0.133	0.140	0.160	0.209	0.261	0.347	0.451	0.546	0.646

Table 1: Comparison of Analytical, Simulated Values and Relative Error for Different τ Values and strikes with stochastic liquidity

As we can see, the analytical prices from the closed form solution are extremely close to the simulated prices with a relative error less than one percent across all the tested strikes and maturities. Despite being a closed form solution, the difference in time to calculate the option price between the implementation formula and the monte carlo method is only about one order of magnitude. This is because of the fact that there is no explicit formulation of $A(t,T)$ and instead we must compute the complex number integral of the PDE of A at every step of the integration of the characteristic function. This incurs a significant performance penalty that when coupled with the generally slow execution time of python results in each calculation of the closed form price talking approximately 10 seconds to complete.

4.3 STOCHASTIC LIQUIDITY WITH REGIME SWITCHING

Building on previous work, Xin-Jiang He et al [7] developed a model which aims to account for not only liquidity risks, but also economic cycles. This is meant to capture additional dynamics information. This makes sense as economic cycles can have a major effect on asset prices, volatility, as well as liquidity. This model is meant to capture more cyclic and systematic changes in liquidity that the stochastic liquidity model cannot. This means this model should be better at predicting prices over a longer time horizon as opposed to the stochastic liquidity model. The key mathematical difference is that this model introduces a fourth source of stochasticity. Namely, a process U_t is added which represents a continuous time Markov chain with finite states $\{U_t\}_{t>0}$ belonging to the set $\{e_1, e_2, e_3, \dots e_N\}$ which represents the state of the economy. Each e_i belongs to R^N and is the i-th unit vector. We then let A represent the generator of the Markov chain (transition matrix). The different economic states all represent different regimes and have different values for the volatility parameter σ.

4.3.1 DYNAMICS OF LIQUIDITY MODEL WITH REGIME SWITCHING

The equations for the dynamics of the stock price with regime switching are very similar to the dynamics of the price in the pure stochastic liquidity model. As such, we will not explain in much detail the derivation of the model and instead focus more on the differences when compared to the previous model. Indeed, looking directly at the risk neutral formulation of this model:

$$dL_t = \alpha(\theta - L_t)dt + \xi dW_t^3 \qquad \frac{dS_t}{S_t} = r\,dt + \sigma_t\,dW_t^1 + \beta L_t\,dW_t^2$$

We can see that the only difference is that now σ is no longer a constant and changes with time, this represents how at time *t* the model can be in any of its economic cycle states and will have the corresponding σ. One additional assumption of this model is that dW^1 and dW^3 are independent and that $dW^2dW^3= \rho$. In the original stochastic liquidity model there were very few assumptions made about the correlation terms between the Brownian motions, however, to increase tractability here where there is an additional source of stochasticity we make these assumptions.

4.3.2 MONTE CARLO SIMULATION OF MODEL

Due to the added complexity of the model, the monte carlo simulation needs to be modified. Now, to simulate regime changing, the term t is treated as a stochastic variable, where the entire system can be in in one state for a random amount of time before switching to another state with probabilities determined by the transition matrix A. For the sake of further reducing complexity, we limit the number of economic states to just two- normal, and stressed (where the value of σ is higher). We will call these two values of the volatility parameter σ_1 and σ_2 The transition matrix for the two state MC is a 2x2 matrix that can be entirely described by two values; λ_{12} represents the likelihood of going from state. 1 to 2 and λ_{21} represents the likelihood of going from state 2 to 1.

The values of the parameters used for the monte carlo simulation are *r = 0.03, sigma1 = 0.1, sigma2 = 0.2, lambda12 = 1.0, lambda21 = 1.0, beta = 0.8, alpha = 1.0, theta = 0.5, xi = 0.2, rho = -0.5, tau = 1.0, liquidity = 0.3 S = 100, K = 100*. The number of steps is 100. The simulations and numerical results from this point forward will be using these parameters unless otherwise specified.

Once again, we can see that the option price converges around 100,000 simulations, as such we will use 150,000 paths for evaluation of the closed form formula. Comparing the paths of this model to the paths of the base stochastic liquidity model, we can see that the variance of the paths is much lower and more symmetrical. This is because it is not possible for the stock price to go below zero, hence if the cone of variance extends below zero, then the paths will appear to show a general upwards trend. This indicates that the simulation scheme for the base model may benefit from variance reduction methods. However, since the simulation is only being used to validate the closed form solution, it is not a pertinent issue at this time.

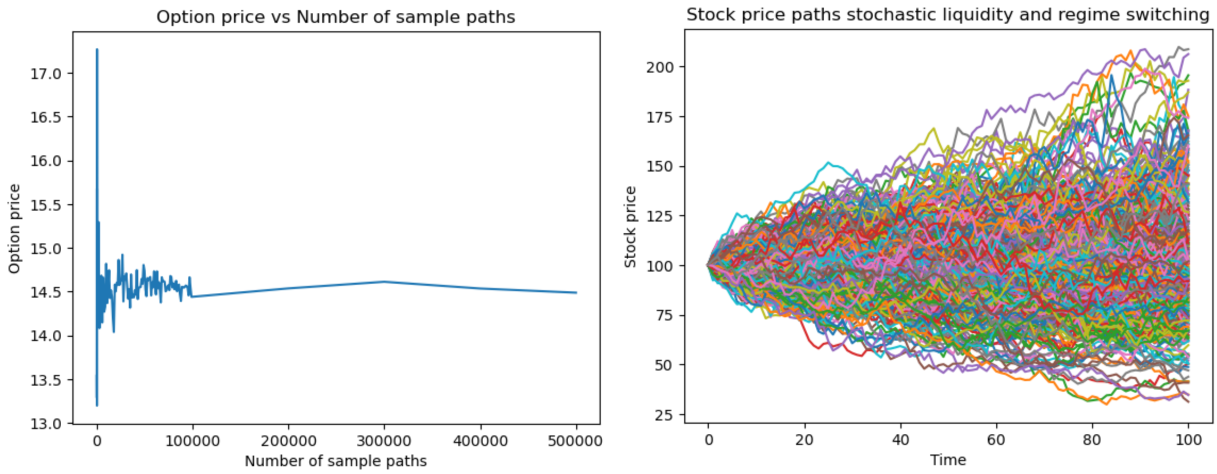


Fig. 2 Shows the convergence of the predicted option price and the paths of 1000 of random walks of the stochastic liquidity with regime switching model

4.3.3 Closed Form Solution

Because the added stochasticity was in the form of a Markov chain, it is still possible to find a closed form representation of the option price similarly to what was done in the base stochastic liquidity model. The process of deriving the closed form solution here is very similar to the one for the base liquidity model with some added complexity due to the Markov chain. Since we are mainly concerned with the implementation of the closed form and the derivation includes some new unfamiliar concepts, the steps to derive it will not be elaborated on and only the final solution is shown:

$$C(S_t, L_t, U_t, t) = e^{-r(T-t)} I_1 - K e^{-r(T-t)} I_2,$$

with

$$I_1 = f(-j; U_t, L_t, X_t, t, T) \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Real} \left[\frac{e^{-j\phi \ln K}}{j\phi} \frac{f(\phi - j; U_t, L_t, X_t, t, T)}{f(-j; U_t, L_t, X_t, t, T)} \right] d\phi \right\}$$

$$I_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Real} \left[\frac{e^{-j\phi \ln K}}{j\phi} f(\phi; U_t, L_t, X_t, t, T) \right] d\phi$$

Here $j = \sqrt{-1}$ and $f(\phi; U_t, Y_t, X_t, t, T)$ is the characteristic function of $X_T = \ln(S_T)$ just like the base model. The form of f is once again assumed to be affine:

$$f = e^{A(\phi;t,T) + B(\phi;t,T)L + C(\phi;t,T)L^2 + j\phi X} \left\langle e^{\Lambda U_t, I} \right\rangle$$

with $\Lambda = A^T \tau - diag\left[\frac{1}{2}(j\phi + \phi^2)\tilde{\sigma}^2 \tau\right], \ \tau = T - t,$

Here we can see an additional term $\left\langle e^{\Lambda U_t, I} \right\rangle$ corresponding to the state of the Markov chain.

Finally, the equations of the functions in the exponential are:

$$C = \frac{a - (2j\rho\xi\beta\phi - 2\bar{\alpha})}{4\xi^2}\frac{1 - e^{a\tau}}{1 - be^{a\tau}},$$

$$B = -\frac{\bar{\alpha}\theta\left[a - (2j\rho\xi\beta\phi - 2\bar{\alpha})\right]}{\xi^2 a}\frac{(1 - e^{\frac{1}{2}a\tau})^2}{1 - be^{a\tau}},$$

$$A = \int_0^\tau \frac{1}{2}\xi^2 B^2(s) + \bar{\alpha}\theta B(s)ds + \frac{1}{4}\left\{\left[a - (2j\rho\xi\beta\phi - 2\bar{\alpha})\right]\tau - 2\ln\left(\frac{1 - be^{a\tau}}{1 - b}\right)\right\}$$

With

$$a = 2\sqrt{(j\rho\xi\beta\phi - \bar{\alpha})^2 + \xi^2\beta^2(j\phi + \phi^2)} \qquad b = \frac{(2j\rho\xi\beta\phi - 2\bar{\alpha}) - a}{(2j\rho\xi\beta\phi - 2\bar{\alpha}) + a},$$

Now that all parts necessary for the closed form solution are known, we can implement it in python and compare the option price with those generated from monte carlo simulation.

τ	K/S_0	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20
$\tau = 0.2$	Analytical	20.723	16.124	11.897	8.231	5.284	3.121	1.689	0.838	0.383
	Simulated	20.723	16.119	11.887	8.216	5.264	3.102	1.675	0.830	0.376
	Relative Error (%)	0.002	0.028	0.086	0.183	0.368	0.590	0.834	0.924	1.876
$\tau = 0.4$	Analytical	22.052	17.910	14.148	10.841	8.043	5.770	4.002	2.688	1.751
	Simulated	21.994	17.854	14.096	10.795	8.005	5.741	3.982	2.674	1.745
	Relative Error (%)	0.265	0.315	0.363	0.421	0.468	0.499	0.499	0.496	0.308
$\tau = 0.6$	Analytical	23.554	19.718	16.227	13.117	10.410	8.112	6.210	4.673	3.461
	Simulated	23.483	19.664	16.185	13.084	10.388	8.096	6.196	4.661	3.449
	Relative Error (%)	0.303	0.276	0.256	0.249	0.214	0.199	0.215	0.253	0.345
$\tau = 0.8$	Analytical	25.075	21.463	18.161	15.189	12.557	10.264	8.297	6.639	5.262
	Simulated	25.214	21.591	18.279	15.297	12.654	10.347	8.367	6.700	5.316
	Relative Error (%)	0.556	0.595	0.647	0.710	0.769	0.813	0.833	0.917	1.041
$\tau = 1.0$	Analytical	26.567	23.130	19.976	17.113	14.545	12.269	10.274	8.546	7.064
	Simulated	26.514	23.083	19.930	17.067	14.498	12.218	10.223	8.497	7.017
	Relative Error (%)	0.200	0.207	0.229	0.266	0.325	0.414	0.503	0.576	0.663

Table 2: Comparison of Analytical, Simulated Values and Relative Error for Different τ Values and strikes with stochastic liquidity and regime switching

Once again, we can see that the relative error between the simulated price and the closed form price is small with errors mostly being less than one percent. However, for contracts very close to expiry (t=0.2) and out of the money ($K > 1.1*S_0$) we can see the errors increasing. This is not due to an instability in the model but rather due to the small values of the prices. For example, at $K/S_0 = 1.2$ the actual difference in price is only 0.007, but because the value itself is so small, the relative error is almost two percent. For this model, the calculation of the closed form price took even more time (approx.25 seconds) while the monte carlo simulation took a similar time to the base stochastic liquidity model. This meant that the close form price solution was around 3-4 times faster than simulating 150,000 paths with 100 steps each. Once again, this is mainly due to having to integrate to calculate A at each step of the integration of the characteristic function of X_T. Given the increasing computational cost of calculating and the difficulty of finding closed form solutions, it is likely that any significantly more complex model with more stochastic processes is likely to be slower or the same as monte carlo simulation if a closed form can be found at all. Now that the characteristics of both models have been examined and the closed form solutions implemented in python, it is now time to calibrate the model on real world NIFTY 50 historical price volume data.

5. Calibrating on Real World Data

We calibrate both the stochastic liquidity and stochastic liquidity with regime change models on real world data from the NIFTY 50 and test if including stochastic liquidity helps to better fit market data. As a benchmark, we will use the constant liquidity model by setting θ, α, ξ, and ρ_2 & ρ_3 to 0. Following the literature[8] , we use in-sample and out-of-sample errors as the metric for evaluation of the model. A key benefit of these models is that they only require S,K, T-t, and r as well as the real world option price to calibrate. S,K, and T-t can be easily obtained from the historical price volume contract-wise option data that we have, and r was taken to be the MIFOR (Mumbai Interbank Forward Outright Rate) which was retrieved from CEIC.

5.1 Data Selection and Preparation

As previously discussed, we are interested in studying the time period between Nov 2016 and Apr 2017 where there were drastic changes in liquidity in the Indian market brought on by demonetization. The prices that we are training on and interested in predicting are the close prices which do not always have an implied vol. As such, we will drop all options that we cannot calculate implied volatility for, this should lead to more reliable results and allow us to graph the implied vol of the market against the vol of our calibrated prices. The volatility was calculated using newtons method with the vega of the option, this leads to very fast calculation of implied vol from the price.

To avoid the 'day of the week' effect[10] where trading environments tend to have a cyclic shift in activity throughout the week, we shall only look at the Tuesday data for calibrating the model and Wednesday data to measure the out-of-sample performance. Empirically, it has been found that prices of options very close to and very far from expiry display behaviours that are atypically. Thus, we discard all options with an expiry less than 2 weeks and more than 6 months. Finally, we combine the price volume data with the rates data and also fill any missing underlying index values from the historical index data of the NIFTY 50. In the end, we had between 100-160 rows of data for each date corresponding to 2-3 expiries with around 50 strikes each.

5.2 Parameter Estimation and the Objective function

The stochastic volatility model has 9 parameters to estimate, while the constant liquidity model has 4 parameters. In contrast, the stochastic volatility with regime switching has 10 parameters which need to be estimated. The goal is to find a set of parameters for each model which can best fit the real world price data, thus, we choose the mean squared error as our objective function.

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(C_i^{Market} - C_i^{Model})^2,$$

Where C_i^{Market} is the price of the i-th call option on the day and similarly C_i^{Model} is the prediction of the model of the i-th price on the day. Given the non-smooth, non-convex and non-linear nature of the closed form solution formulas, it would be difficult for traditional optimizations techniques like gradient descent to find the optimal solution as they are likely to get stuck in a local minima. As such, we use the differential evolution method which is particularly useful when the objective function is non-differentiable, non-linear, or multimodal (has multiple local optima).

5.2.1 Differential Evolution

1. Initialization

Differential Evolution starts with a population of candidate solutions (or individuals). Each individual is represented as a vector of parameters that we are trying to optimize. For example, if you are optimizing a function with three variables, each individual in the population is a vector with three elements. The initial population is generated randomly within the problem's defined bounds.

2. Mutation

Mutation is the key operation in DE. It introduces diversity into the population by creating a mutant vector for each individual using the following formula:

$$v_i = x_{r_1} + F \cdot (x_{r_2} - x_{r_3})$$

v_i is the mutant vector, r_1, r_2, r_3 are distinct random indices selected from the population. F is the mutation factor, a constant typically in the range [0, 2], controlling the amplification of the differential variation ($x_{r2} - x_{r3}$).

3. Crossover

Once the mutant vector v_i is created, a trial vector u_i is generated by combining elements from the target vector x_i (the original individual) and the mutant vector v_i. This is done using a process called crossover. For each element j in the trial vector:

$$u_{i,j} = \begin{cases} v_{i,j}, & \text{if } \mathrm{rand}(0,1) \leq CR \text{ or } j = j_{\mathrm{rand}} \\ x_{i,j}, & \text{otherwise} \end{cases}$$

Where CR is the crossover probability and j_{rnad} is one of the elements from the mutant vector that was randomly chosen to ensure that at least one element of the target vector is mutated by the mutant vector.

4. Selection

After creating the trial vector u_i, the next step is to decide whether this trial vector should replace the original target vector x_i in the next generation. To do this, evaluate the objective function for both the trial vector u_i and the target vector x_i. If the trial vector has a better objective function value, the trial vector replaces the target vector for the next generation:

$$x_i^{(t+1)} = \begin{cases} u_i, & \text{if } f(u_i) \leq f(x_i) \\ x_i, & \text{otherwise} \end{cases}$$

5. Iteration

Steps 2 to 4 (mutation, crossover, selection) are repeated until the stopping criterion is met. The stopping criterion can be a fixed number of generations (iterations), a convergence threshold where the population does not improve significantly after several generations, or a predefined computation time.

5.3 RESULTS

The estimation of parameter values is performed on a daily basis[9], and the daily average parameter values are displayed in Table 3. Following the approach of several authors, this empirical study is conducted daily. However, it is important to note that this procedure is not fully consistent with the assumption of constant parameter values, as it does not account for the connection between Y_t and Y_{t+1}. The differential evolution algorithm initially had issues with performing the optimization due to certain combinations of parameter values causing overflows in in the characteristic function, particularly the last step where the function is exponentiated. Any value with real part over 710 used an overflow. Due to the stochastic nature of the differential evolution algorithm, a quick and effective solution was so add a penalty to the objective function so that overflows would incur a large penalty of 100,000. This discouraged the algorithm from heading towards those members of the population which had parameters that caused instability. However, despite this, there were several issues with the calibration of the stochastic liquidity model with regime switching (SLRS) perhaps due to its extra convoluted nature as compared to the basic stochastic liquidity model. It appears that the algorithm is unsuitable for this task as even after running for over 100 hours, the estimated parameters of the SLRS model gave poor fit to the market prices and their implied volatility as compared to the constant liquidity and stochastic liquidity models.

The average of the daily estimated parameters for the different models are shown below in tables 3, 4, and 5. The errors for the SLRS model should be treated as a consequence of the poor calibration rather than as a statement about the model itself, it has been included to show the work done so far. Given the similarity between the monte carlo simulations and the closed form price, it is unlikely that either model has been incorrectly implemented, it is simply due to the inefficiency of the integration to find A(t,T) and the complex nonlinear nature of the closed form solution that is causing a subpar optimization for the stochastic liquidity regime switching model. Both stochastic liquidity models show a low average daily initial liquidity value suggesting that the model is indeed capturing some information about the market dynamics and that the period following demonetization had a liquidity level lower than the average of the estimated daily liquidity mean θ. Further investigation is required to better understand the meaning of these estimated parameters and how to translate the liquidity parameters into an actual measurable metric that can be observed in the market.

Parameter	SL model	CL model	SLRS
σ_1	0.1265	0.4713	0.1686
σ_2	–	–	0.1004
λ_{12}	–	–	0.5536
λ_{21}	–	–	5.9150
β	0.2004	1.3565	0.2953
$\bar{\alpha}$	9.025	–	4.9577
$\bar{\theta}$	0.2346	–	0.0234
ξ	0.1800	–	0.6656
ρ / ρ_1	0.4389	0.2130	0.2951
ρ_2	-0.3896	–	–
ρ_3	0.0550	–	–
L_0	0.1124	0.2736	0.1618

Table 3: Average of daily parameter values for the different models

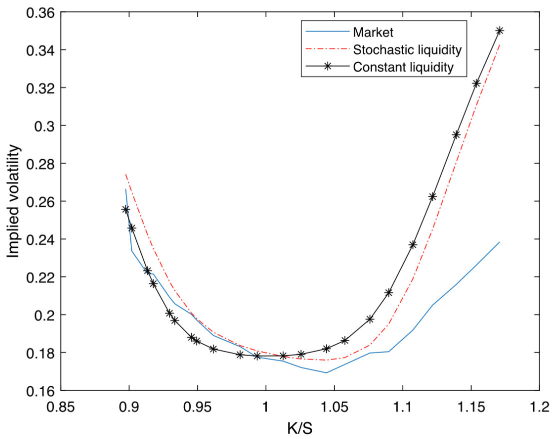
	$K/S_0 > 1.04$	$0.96 \leq K/S_0 \leq 1.04$	$K/S_0 < 0.96$
SL model	4.98E–2	4.02E–2	5.47E–2
CL model	5.37E–2	4.06E–2	8.59E–2
SLRS model	3.71	5.24	4.22

Table 5: Moneyness-based out-of-sample errors.

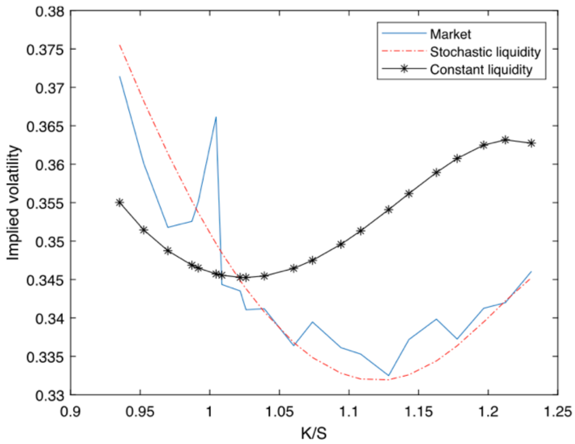
	In-sample	Out-of-sample
SL model	3.72E–2	5.27E–2
CL model	6.49E–2	7.38E–2
SLRS model	3.69	4.25

Table 4: In-sample and out-of-sample errors.

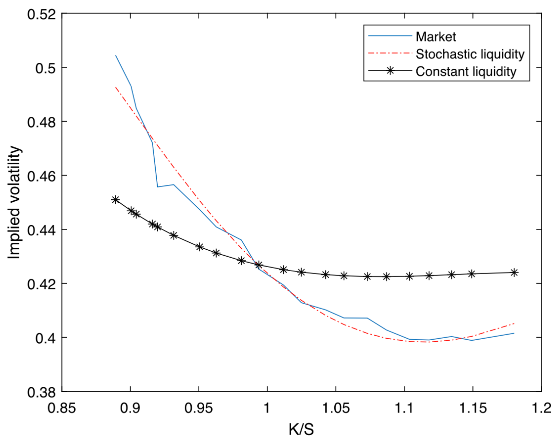
Since the models were calibrated only on the price of the options, it is interesting to see how well they caputre the underlying implied volatility structure of the market. Looking at the calibrated curves,



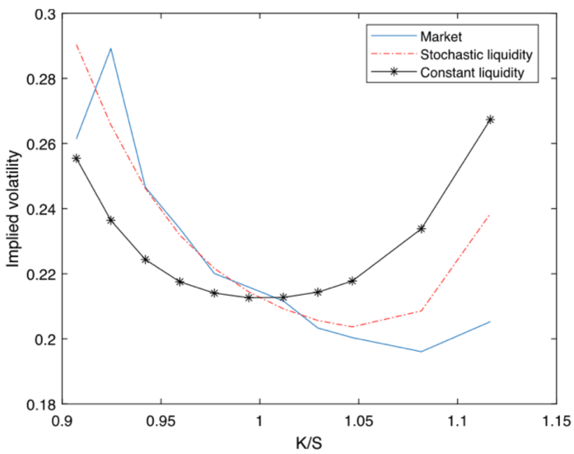
1. T = 0.137 on Jan 05, 2017



4. T = 0.233 on Jan 05, 2017



3. T = 0.310 on Jan 05, 2017

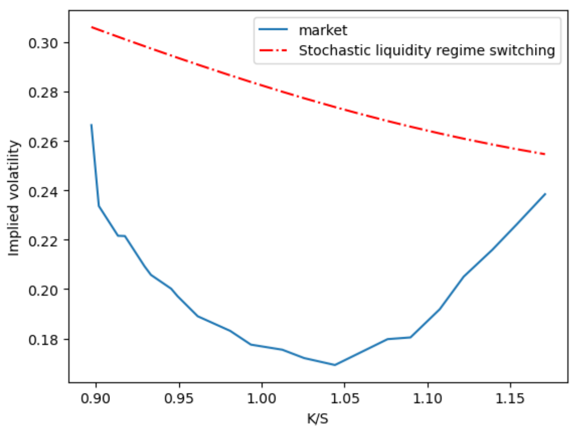


2. T = 0.115 on Feb 17, 2017

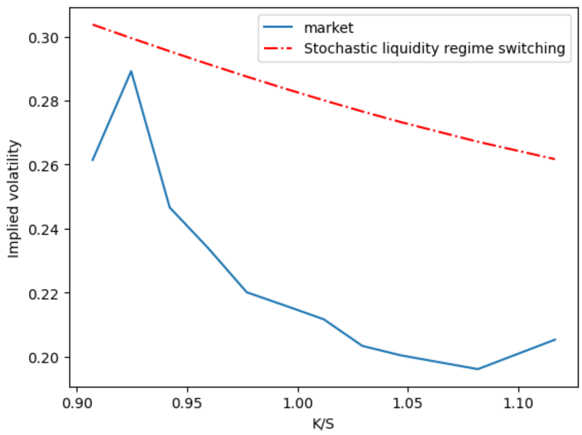
Fig. 3 shows the full calibrated curves for three maturities on Jan 05, 2017, and another curve for Feb 2017 for the stochastic and constant liquidity models

we can observe that the stochastic liquidity model does a reasonable job at capturing the dynamics on a single day for various maturities. In general, it can be observed that in all four graphs the stochastic liquidity curve does a good job of matching the market implied volatilities when the strike price is lower than S, i.e. in the money. When looking at out of the money strikes, we can see that in sub figures 1 and 4 both the constant liquidity and the stochastic liquidity models diverge wildly from the actual market data in the same manner. This implies that some characteristic of the model makes it unsuitable for precise pricing of out of the money strikes. Another observation is that the constant liquidity curve tends to be shaped more like a smile and has difficulty in dealing with more skewed curves. On the other hand, the stochastic liquidity curve shows excellent fit to skewed data. In a sense, it can be said that the constant liquidity model is better when the vols are symmetrical, but the stochastic liquidity model performs better on skew curves which are closer to what might be expect in a liquidity shock.

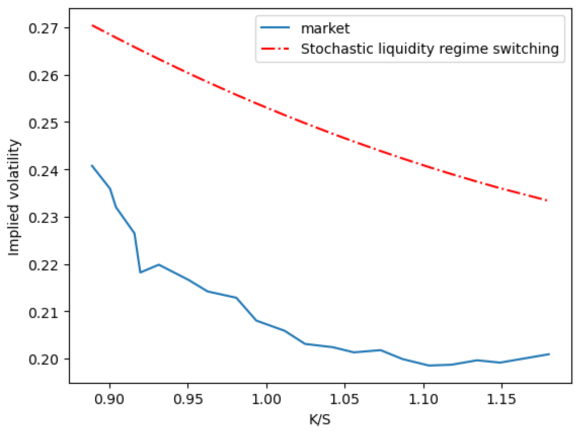
Looking at the curve for the stochastic liquidity with regime switching model for the same dates, we can clearly see that the estimated parameters are not well optimised as it performs even worse than the constant liquidity curve did. All values for implied volatility are grossly overestimated, and the curve is very close to straight line. Perhaps it is possible that the optimization by differential evolution found a set of parameters that effectively make the model into something like a Black-Scholes model which has constant volatility, but here the volatility behaves like a linear function of the moneyness. During the calibration, as a test, the differential evolution algorithm was given 100 hours of CPU time on AWS to see if it could converge to a global solution. Unfortunately, the found parameters did not perform any appreciably better than giving the algorithm less time. Seeing how this model is unsuitable for normal optimization methods, a novel solution is required.



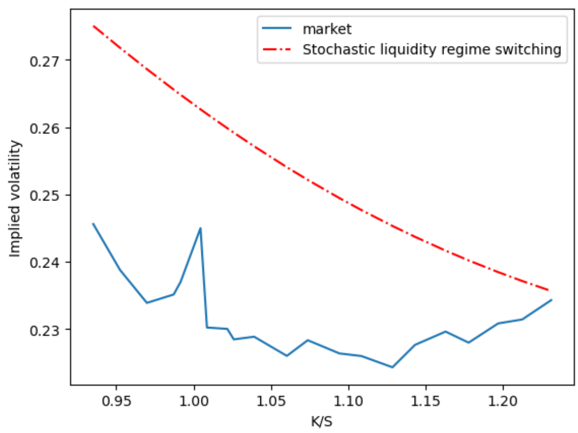
1. T = 0.137 on Jan 05, 2017



2. T = 0.233 on Jan 05, 2017



3. T = 0.310 on Jan 05, 2017



4. T = 0.115 on Feb 17, 2017

Fig. 4 shows the full calibrated curves for three maturities on Jan 05, 2017, and another curve for Feb 2017

for the stochastic liquidity with regime switching model

6. CONCLUSION

Overall, the results of the research so far have been relatively mixed, with strong performance by the stochastic liquidity model, but very poor performance from the SLRS model. The performance of the model is so poor almost surely because of the poor calibration to market data. However, as previously discussed, it is simply not feasible to give this optimization method any more resources. Especially considering that the parameters have to be calibrated every day. The stochastic liquidity model, despite performing well, still takes around 3 hours to return the parameters. While manageable, it is still far longer than we would like for the model to be practical to use. The main cause of the slowness is the need to solve the partial differential equation of A in each step of the integration of the characteristic function, for both models. The main benefit of both models is that they are analytical, which makes them far easier to compute once the parameters are known, but the catch is that the reverse, finding the parameters using the closed form, is exceedingly time and resource consuming. Finally, the fact that the model needs to be calibrated daily seems to imply that liquidity is not in fact a predictable stochastic process but probably takes some other form. This is an ongoing area of research.

6.1 FURTHER WORK

Despite the drawbacks of the model, there is still some more work I would like to do with these models. First and foremost, because the model is analytical, Puneet Pasricha et al (2022) had also managed to derive the greeks of the model. Due to their complexity to implement and some numerical instabilities, I am still in the process of implementing all the greeks. Once this is done, it may be possible to use the Jacobian of the model to try and find a traditional optimization method like gradient descent and see if it can speed up the calibration process. In addition, understanding and implementing the Greeks can also be a great diagnostic tool to better understand the dynamics of the model and why exactly the calibration of the SLRS model has been so poor.

Finally, because the model is analytical, it is relatively simple to generate new data by random sampling. This could be an opportunity to reintroduce the use of artificial neural networks to train and learn the model similar to what was done with the Bergomi model in the deep learning for finance course. If this proves, successful, it would virtually eliminate the biggest bottleneck for this model and make calibration instantaneous. Then, the closed form model can be used to obtain exact results and the greeks and be studied to gain more insight. Another possibility is using a Bayesian method for calibration to once again leverage the speed of generating random samples from the model.

Overall, I wish to and am looking forward to continuing my work on this problem and to better understand stochastic modelling as a whole as a result. I believe there is some promise in stochastic liquidity models even if they do not necessarily capture liquidity in a manner that is easily understood.

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