Containment Control of Multiagent Systems With Dynamic Leaders Based on a PI^n-Type Approach

Long Cheng, Yunpeng Wang, Wei Ren, Zeng-Guang Hou, and Min Tan

Abstract—This paper studies the containment control of multiagent systems (MASs) with multiple dynamic leaders in both continuous-time domain and discrete-time domain. The leaders' motions are described by the nth-order polynomial trajectories. This setting makes practical sense because given some critical points, the leaders' trajectories are usually planned by the polynomial interpolations. In order to drive all followers into the convex hull spanned by the leaders, a PI^n-type containment algorithm is proposed (P and I are short for proportional and integral, respectively; I^n implies that the algorithm includes up to the nth-order integral terms). It is theoretically proved that the PI^n-type containment algorithm is able to solve the containment problem of MASs where the followers are described by any order integral dynamics. Compared to the previous results on the MASs with dynamic leaders, the distinguished features of this paper are that: 1) the containment problem is studied not only in the continuous-time domain but also in the discrete-time domain while most existing results only work in the continuous-time domain; 2) to deal with the leaders with the nth-order polynomial trajectories, existing results require the follower's dynamics to be the $(n + 1)$th-order integral while the followers considered in this paper can be described by any-order integral dynamics; 3) the "sign" function is not employed in the proposed algorithm, which avoids the chattering phenomenon; and 4) both disturbance and measurement noise are taken into account. Finally, some simulation examples are given to demonstrate the effectiveness of the proposed algorithm.

Index Terms—Containment control, multiagent system (MAS), PI^n-type algorithm, polynomial trajectory.

I. INTRODUCTION

RECENTLY, the distributed coordinated control of multiagent systems (MASs) has become a research focal

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in the systems and control community. Roughly speaking, agents in concern can be divided into two categories: 1) leaders and 2) followers. Depending on whether there are leaders in MASs, the coordinated control problem becomes the consensus problem (leaderless case) [1]–[3]; the leader-following problem (single leader case) [4]; and the containment problem (multiple leaders case). This paper mainly focuses on the containment problem of MASs because, from one side, the containment problem roots in some natural phenomena such as the relationship between sheepdogs and sheep [5] and the relationship between female silkworm moths and male silkworm moths [6]; from the other side, the containment problem has many practical applications such as the mixed containment-sensing problem [7] and the coordinated control of a group of robots [8]–[10], etc.

Looking back at the history of the containment problem, the rapid development started after the publication of [11] and [12]. A leader-based containment control strategy for multiple unicycle agents was introduced in [11], where the containment problem was interpreted as a combination of the formation and agreement control problems. The leaders were convergent to a desired formation while the followers converged to the convex hull spanned by the leaders. A similar containment problem of MASs with single-integrator dynamics was studied in [13], where the consensus-like interaction rule was designed for the followers while a hybrid "stop-go" policy was applied to the leaders. Since then, a great number of results concerning the containment control have been reported. According to the type of the agent's dynamics, these results can be divided into five categories: 1) the single-integrator dynamics [14]–[17]; 2) the double-integrator dynamics [9], [18]–[25]; 3) the general linear dynamics [26]–[30]; 4) the Euler–Lagrange dynamics [31]–[33]; and 5) the nonlinear dynamics [34].

In [14], the containment problem of MASs with undirected switching communication topologies was studied. However, in practice, the communication link is usually a one-way channel. For this reason, the containment problem of MASs with directed communication topologies has been widely investigated recently. In [15], it was shown that the necessary and sufficient condition for achieving the containment of single-integrator MASs with a directed topology was that for each follower, there exists at least one leader that has a directed path to this follower. This condition was also proved to be necessary and sufficient for the containment problem of double-integrator MASs in [18] and [19]. Experimental validations on a team of mobile robots were conducted in [9]. The finite-time

containment problem of double-integrator MASs was investigated in [20]. In [21] and [22], containment control algorithms were proposed for double-integrator MASs based on only position measurements. The containment problem of double-integrator MASs with randomly switching topologies was investigated in [23], where the switching signal was described by a continuous-time irreducible Markov chain. It is proved that the containment problems could be solved if and only if for each follower, there exists at least one leader which has a directed path to this follower in the union graph of all possible communication graphs. Because the communication noise is unavoidable in practical applications, the noise effect in containment problems of single-integrator and double-integrator MASs were studied in [16] and [24], respectively. The results in [26] show that the containment control of general linear MASs can be achieved by applying a state-feedback algorithm. When the agents' states were unavailable, output-feedback based containment control algorithms were proposed for general linear MASs in [27] and [28]. In [25] and [29], the communication constraint such as the nonuniform delay was considered in studying the containment problem. In [30], the finite-time containment problem of general linear MASs is solved by using the sliding-mode based approach while the convergence rate can be further improved in [17] if the agent is described by the single-integrator dynamics. For Euler–Lagrange MASs with uncertainties, adaptive containment algorithms were proposed based on the sliding-mode estimator and the neural network in [31] and [32], respectively. Furthermore, the finite-time containment problem of Euler–Lagrange MASs was studied in [33]. Finally, in [34], the containment problem of the second-order locally Lipschitz nonlinear MASs was solved within the framework of the nonlinear input-to-state stability.

Although great effort has been made to address various factors in the containment control of MASs, there are still some limitations in the existing results. Let us first consider an application scenario shown in Fig. 1. In this application, a group of mobile robots are required to move across a partially unknown area through a narrow and safe tunnel. There are two kinds of robots: 1) master robots and 2) slave robots. Master robots are capable of self-navigation, while slave robots can only measure the relative positions with its neighbor robots. This task can be solved by the containment control strategy.

1) Master robots act as leaders. For each master robot, design a reference trajectory which is inside the safe channel. Let each master robot move along its corresponding reference trajectory.
2) Slave robots act as followers. Let all slave robots move into the area surrounded by master robots and move together with master robots.

Then the challenge is how to design the reference trajectories for master robots. For each master robot, we can select a sequence of suitable reference points inside the safe tunnel. A polynomial trajectory, which goes through these points, can be constructed by the polynomial interpolation. The obtained trajectory is checked whether it is inside the safe tunnel. If not (caused by the Runge's phenomenon), we need to reselect the reference points and construct the new polynomial trajectory.
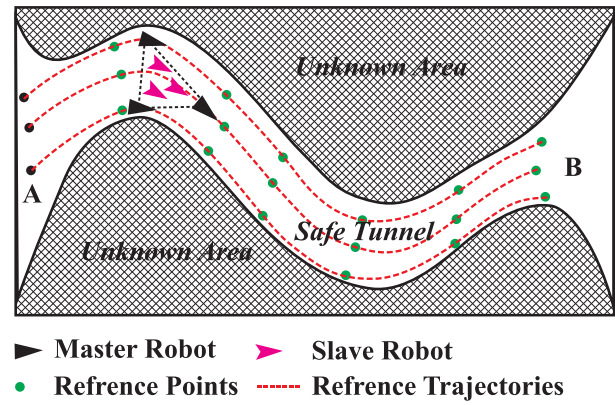

Fig. 1.   Group of mobile robots travel across a partially unknown area.

This process is repeated until the trajectory is within the safe tunnel. Then the following nth-order polynomial can be determined:

$$x(t) = a_0 + a_1 t + \cdots + a_n t^n \qquad (1)$$

whose trajectory goes through the selected $(n + 1)$ reference points ($a_0, \ldots, a_n$ are coefficients determined by these reference points). The polynomial trajectory can be used as the reference trajectory for the master robot.

In [9], [14]–[16], [18], [19], [21]–[24], [31], and [33], every leader has the following single-integrator or double-integrator dynamics:

$$\dot{x}(t) = u(t) \quad \text{or} \quad \begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = u(t) \end{cases} \qquad (2)$$

where the control input $u(t)$ is assumed to be bounded or even zero. Obviously, if the control input of (2) is bounded, then this controller cannot generate the polynomial trajectory defined by (1) with the order $n \geq 3$. If the leader has the nth-order integrator or the n-dimensional linear dynamics [26]–[28], then the polynomial trajectory can be generated by properly selecting the parameters in the system matrix of the leader's dynamics. However, by doing so, it is required that all followers have the same dynamics as the leader [26]–[28]. In reality, the follower's dynamics has no relationship with the order of the leader's polynomial trajectory. In fact, with the nth-order polynomial trajectories for leaders, how to solve the containment problem of MASs with single-integrator followers is not answered yet. In addition, from the controller design point of view, most containment algorithms include the discontinuous "sign" function to deal with the dynamic leaders [9], [15], [21], [22], [31], [33], which would cause the harmful "chattering" phenomenon. Furthermore, all results aforementioned above only study the containment problem in the continuous-time domain. The counterpart results in the discrete-time domain are not clear.

Inspired by the above observations, this paper investigates the containment problem of MASs with dynamic leaders in both continuous-time domain and discrete-time domain. It is assumed that each leader's motion is described by a corresponding polynomial trajectory. Every follower is first assumed to have the single-integrator dynamics. A so-called

PI^n-type (P and I are short for proportional and integral, respectively) algorithm is proposed to solve the containment problem, where I^n implies that the algorithm includes up-to the nth-order integral terms. It turns out that the PI^n-type algorithm is able to solve the containment problem if for each follower, there exists at least one leader which has a directed path to this follower. Then the obtained results are extended to the case where the followers are described by the high-order integral dynamics. In this case, the PI^n-type algorithm is modified as a $PI^{n-m}D^m$-type algorithm (D is short for derivative; and D^m implies that the algorithm includes up to the mth-order differential terms). Then, the counterpart results in the discrete-time domain are presented. Moreover, effects of the disturbance and the measurement noise are also taken into account in this paper. Compared to the previous results, the contributions of this paper can be summarized as follows.

1) The proposed algorithms can solve the containment problem with dynamic leaders in both continuous-time domain and discrete-time domain.
2) The follower can be described by any-order integral dynamics.
3) There is no discontinuous sign function in the proposed controllers, which avoids the chattering phenomenon.
4) Effects of the disturbance and the measurement noise are taken into account.

It is noted that there are some recent results on the "PI"-type consensus algorithms for MASs [35], [36]. Moveover, the PI-type consensus algorithm has also been applied in the distributed filter of distributed parameter systems [37]. Compared to [35] and [36], the distinguished features of this paper mainly lie in the following five aspects.

1) The control objective of [35] and [36] is to solve the leaderless consensus problem of MASs with disturbances. The aim of this paper is to solve the containment problem with dynamic leaders.
2) In [35] and [36], the agent is described by the single-integrator or double-integrator dynamics. In this paper, the leaders move along the polynomial trajectories, and the followers can be described by any-order integral dynamics.
3) In [35] and [36], the main role of the integral term is to attenuate constant disturbances. In this paper, the main purpose of employing the integral-terms is to eliminate the containment error caused by the polynomial trajectory.
4) The algorithms proposed in [35] and [36] can only deal with constant disturbances, while the ones proposed in this paper can attenuate some kinds of time-varying disturbances (polynomial-type disturbance).
5) In [35] and [36], only results in the continuous-time domain are presented. This paper studies the containment problem of MASs in both continuous-time domain and discrete-time domain.

The remainder of this paper is organized as follows. Section II gives some preliminary results on the containment problem with dynamic leaders. Section III presents a containment algorithm and the related theoretical analysis in the continuous-time domain, where the followers are described by the single-integrator dynamics. Section IV discusses how to generalize the obtained results to the case where the followers are described by the high-order integral dynamics; counterpart results in the discrete-time domain and the effect of measurement noises are discussed in Section V. Section VI concludes this paper with final remarks.

A. Notations

$\mathbf{1}_n = (1, \ldots, 1)^T \in \mathbb{R}^n$; $\mathbf{0}_n = (0, \ldots, 0)^T \in \mathbb{R}^n$; I_n denotes the $n \times n$ dimensional identity matrix; $\mathbf{0}_{m \times n} \in \mathbb{R}^{m \times n}$ denotes the $m \times n$ dimensional zero matrix; and $\otimes$ denotes the Kronecker product. $\mathbb{N}$ and $\mathbb{N}^+$ denote the set of natural numbers and the set of positive natural numbers, respectively. For a given vector $p \in \mathbb{R}^n$ and a set $\Omega \subseteq \mathbb{R}^n$, the distance between p and Ω is defined as $\mathrm{dis}(p, \Omega) = \inf_{y \in \Omega} \|p - y\|_2$. For a given matrix X, $\|X\|_2$ denotes its two-norm; $\|X\|_F$ denotes its Frobenius norm; X^T denotes its transpose; and X^H denotes its conjugate transpose. $\mathrm{diag}(\cdot)$ denotes a block diagonal matrix formed by its inputs. For a complex number c, $\Re(c)$ denotes its real part. For a given random variable or vector x, $E(x)$ denotes its mathematical expectation.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a MAS composed of $M + N$ agents. Define two sets $\mathcal{M} = \{1, 2, \ldots, M\}$ and $\mathcal{N} = \{M + 1, M + 2, \ldots, M + N\}$. Motivated by [15], [16], [18], [19], [23], [24], [26], [38], and [39], the communication topology of the MAS is modeled by a weighted digraph $\mathcal{G} = \{\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}, \mathcal{A}_\mathcal{G}\}$, where $\mathcal{V}_\mathcal{G} = \{v_1, v_2, \ldots, v_{M+N}\}$, $\mathcal{E}_\mathcal{G} \subset \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G} = \{\epsilon_{ij} | i, j \in \mathcal{M} \cup \mathcal{N}\}$, and $\mathcal{A}_\mathcal{G} = [\alpha_{ij}] \in \mathbb{R}^{(M+N) \times (M+N)}$ are the node set, the directed edge set, and the adjacency matrix, respectively. Node v_i denotes agent i; $\epsilon_{ij} \in \mathcal{E}_\mathcal{G}$ means that there is an information flow from agent i to agent j; and α_{ji} denotes the weight associated with the directed edge ϵ_{ij}. The element of $\mathcal{A}_\mathcal{G}$ satisfies that $\alpha_{ji} > 0 \Leftrightarrow \epsilon_{ij} \in \mathcal{E}_\mathcal{G}$ and $\alpha_{ji} = 0 \Leftrightarrow \epsilon_{ij} \notin \mathcal{E}_\mathcal{G}$. It is assumed that there is no self-loop in $\mathcal{V}_\mathcal{G}$ ($\epsilon_{ii} \notin \mathcal{E}_\mathcal{G}$ and $\alpha_{ii} = 0$, $i \in \mathcal{M} \cup \mathcal{N}$). If $\epsilon_{ij} \in \mathcal{E}_\mathcal{G}$, then agent i is called the parent of agent j. The neighborhood of node v_i is defined as $\mathcal{N}_i = \{v_j | \epsilon_{ji} \in \mathcal{E}_\mathcal{G}\}$. The in-degree of node v_i is defined as $\deg_{\mathrm{in}}(v_i) = \sum_{j \in \mathcal{N}_i} \alpha_{ij}$. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L}_\mathcal{G} = \mathcal{D}_\mathcal{G} - \mathcal{A}_\mathcal{G}$, where $\mathcal{D}_\mathcal{G} = \mathrm{diag}(\deg_{\mathrm{in}}(v_1), \deg_{\mathrm{in}}(v_2), \ldots, \deg_{\mathrm{in}}(v_{M+N}))$. A directed path from nodes v_{i_1} to v_{i_n} is a sequence of end-to-end directed edges $\epsilon_{i_1 i_2}, \epsilon_{i_2 i_3}, \ldots, \epsilon_{i_{n-1} i_n}$ where $\epsilon_{i_j i_{j+1}} \in \mathcal{V}_\mathcal{G}$ ($j = 1, \ldots, n-1$).

In this paper, an agent is called a leader if it has no parent; otherwise it is called a follower. Without loss of generality, it is assumed that the agents labeled from 1 to M are the leaders while the agents labeled from $M + 1$ to $M + N$ are the followers. Hence the Laplacian matrix of the communication topology graph has the following form:

$$\mathcal{L}_\mathcal{G} = \begin{bmatrix} \mathbf{0}_{M \times M} & \mathbf{0}_{M \times N} \\ L_1 & L_2 \end{bmatrix} \tag{3}$$

where $L_1 \in \mathbb{R}^{N \times M}$ and $L_2 \in \mathbb{R}^{N \times N}$.

Throughout this paper, it is assumed that the following two assumptions hold.

(A1) For each follower, there exists at least one leader that has a directed path to this follower.

(A2) Each follower can only measure the relative positions between itself and its neighbors.

Lemma 1 [33]: Under Assumption (A1):

1) all eigenvalues of L_2 defined in (3) have positive real parts;

2) each entry of $-L_2^{-1}L_1$ is nonnegative and the row sum of $-L_2^{-1}L_1$ equals to one.

The control objective is to design the control algorithms for the followers such that all followers are convergent into the convex hull spanned by the leaders (containment problem) while the leaders move along some predesigned trajectories. To this end, how to describe the motions of the leaders and followers should be given. Let us first consider the continuous-time domain case. The position of agent i at time t is denoted by $x_i(t) \in \mathbb{R}^p$ (it is assumed that the agent moves in the p-dimensional space). The ith leader's motion is assumed to move along the following nth-order polynomial trajectory:

$$x_i(t) = \mathbf{a}_0^i + \mathbf{a}_1^i t + \cdots + \mathbf{a}_n^i t^n, \quad i \in \mathcal{M} \quad (4)$$

where $\mathbf{a}_j^i \in \mathbb{R}^p$, $j = 0, \ldots, n$. The reason of employing the polynomial trajectory is that the robot's trajectory is usually planned by the polynomial interpolation.

In this paper, we do not care about the dynamics of the leaders. The leaders can be considered as the reference signals. The motion of the $(i - M)$th follower ($i \in \mathcal{N}$) is described by the following first-order differential equation:

$$\mathfrak{D}x_i(t) = u_i(t) + \delta_i(t) \quad (5)$$

where $u_i(t) \in \mathbb{R}^p$ and $x_i(t) \in \mathbb{R}^p$. In (5), the symbol $\mathfrak{D}$ denotes the differential operator [namely, $\mathfrak{D}x_i(t) = \dot{x}_i(t)$ and $\mathfrak{D}^n x(t) = \mathfrak{D}(\mathfrak{D}^{n-1}x(t)) = x^{(n)}(t)$]. And $\delta_i(t) = \mathbf{b}_0^i + \mathbf{b}_1^i t + \cdots + \mathbf{b}_r^i t^r$ is the polynomial-type disturbance, where $r \in \mathbb{N}$ and $\mathbf{b}_j^i \in \mathbb{R}^p$ ($j = 0, \ldots, r$). The inversion of this differential operator $\mathfrak{D}$ is defined as $\mathfrak{D}^{-1}x(t) = \int_0^t x(s)ds$. It is easy to see that $\mathfrak{D}(\mathfrak{D}^{-1}x(t)) = x(t)$.

By the above terminologies, the containment problem can be formally defined as follows.

Definition 1: The containment problem of MASs is solved if all followers' positions are convergent into the convex hull spanned by the leaders' positions. That is

$$\lim_{t \to \infty} \text{dis}(x_i(t), \text{co}_L(t)) = 0, \ i \in \mathcal{N}$$

where $\text{co}_L(t) = \text{co}\{x_1(t), \ldots, x_M(t)\} = \{\sum_{i \in \mathcal{M}} \mu_i x_i(t) | \sum_{i \in \mathcal{M}} \mu_i = 1, \ \mu_i \geq 0\}$ is the convex hull spanned by the leaders' positions at time t.

Definition 2: The containment error of MASs as time t is defined as $E_r(t) = \sum_{i \in \mathcal{N}} \text{dis}(x_i(t), \text{co}_L(t))$.

It is easy to see that the containment problem of MASs is solved if and only if $\lim_{t \to \infty} E_r(t) = 0$.

III. CONTAINMENT CONTROL OF MASs IN CONTINUOUS-TIME DOMAIN

Let $e_{ji}(t) = x_j(t) - x_i(t)$ denote the relative state between agent j and agent i. The following containment controller is proposed for the ith agent:

$$u_i(t) = \sum_{l=0}^{n} \kappa_l \mathfrak{D}^{-l} \left(\sum_{j \in \mathcal{M}} \alpha_{ij}e_{ji}(t) + \sum_{j \in \mathcal{N}} \alpha_{ij}e_{ji}(t) \right) \quad (6)$$

where $\{\kappa_l; \ l = 0, \ldots, n\}$ are parameters to be determined. Because (6) includes the proportional term $\sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij}e_{ji}(t)$ and up to the nth-order integral terms $\{\mathfrak{D}^{-l} \sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij}e_{ji}(t), \ l = 1, \ldots, n\}$, (6) is called PI^n-type algorithm. The main purpose of employing the integral terms is to eliminate the containment error caused by the polynomial trajectory.

Let $\xi_i(t) = (x_i^T(t), \mathfrak{D}x_i^T(t), \ldots, \mathfrak{D}^n x_i^T(t))^T$. Then the ith agent's dynamical behavior can be described by the following differential equation:

$$\begin{cases} \mathfrak{D}\xi_i(t) = (A \otimes I_p)\xi_i(t), & i \in \mathcal{M} \\ \mathfrak{D}\xi_i(t) = (A \otimes I_p)\xi_i(t) + (B \otimes I_p)\bar{u}_i(t) \\ \quad + \mathfrak{D}^n \delta_i(t), & i \in \mathcal{N} \end{cases} \quad (7)$$

where $\bar{u}_i(t) \triangleq \mathfrak{D}^n u_i(t) = \sum_{j \in \mathcal{M} \cup \mathcal{N}} K(\xi_j(t) - \xi_i(t))$, $K = (\kappa_n, \ldots, \kappa_0)$, $B = (0, \ldots, 0, 1)^T \in \mathbb{R}^{n+1}$, and

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}.$$

Let $\Xi_L(t) = (\xi_1^T(t), \ldots, \xi_M^T(t))^T$ and $\Xi_F(t) = (\xi_{M+1}^T(t), \ldots, \xi_{M+N}^T(t))^T$. Then the closed-loop dynamics of MASs can be rewritten in the following compact form:

$$\begin{cases} \mathfrak{D}\Xi_L(t) = (I_M \otimes A \otimes I_p)\Xi_L(t) \\ \mathfrak{D}\Xi_F(t) = (I_N \otimes A \otimes I_p - L_2 \otimes BK \otimes I_p)\Xi_F(t) \\ \quad - (L_1 \otimes BK \otimes I_p)\Xi_L(t) + (I_N \otimes B \otimes I_p)\Delta(t) \end{cases} \quad (8)$$

where $\Delta(t) = \text{diag}(\mathfrak{D}^n \delta_{M+1}^T(t), \ldots, \mathfrak{D}^n \delta_{M+N}^T(t))$. This leads to that

$$\mathfrak{D}\hat{\Xi}_F(t) = (I_N \otimes A \otimes I_p - L_2 \otimes BK \otimes I_p)\hat{\Xi}_F(t) + (I_N \otimes B \otimes I_p)\Delta(t) \quad (9)$$

where $\hat{\Xi}_F(t) = \Xi_F(t) + (L_2^{-1}L_1 \otimes I_{(n+1)p})\Xi_L(t)$. By Lemma 1 and Definition 1, if $\hat{\Xi}_F(t)$ is convergent to zero, then the containment problem is solved.

Theorem 1: Assume all leaders move along their polynomial trajectories described by (4) and all followers have single-integrator dynamics described by (5). Let P denote the positive definite solution to the following matrix inequality:

$$A^T P + PA + I_{n+1} - PBB^T P \leq 0. \quad (10)$$

If the order of the polynomial disturbance $\delta_i(t)$ in (5) is not greater than $n - 1$ (namely, $r \leq n - 1$), then the containment problem of MASs can be solved by (6) with $K = \varepsilon B^T P$, where $\varepsilon \geq 0.5 \max\{1, \sigma_{\min}^{-1}\}$, $\sigma_{\min} \in (0, \lambda_{\min})$, and $\lambda_{\min} = \min\{\Re(\lambda_i) | \lambda_i (i = 1, \ldots, N)$ is the eigenvalue of $L_2\}$.

Proof: First, it is proved that $(I_N \otimes A - L_2 \otimes BK) \otimes I_p$ is a Hurwitz matrix. By Schur decomposition, there must exist a

transformation matrix T_c such that

$$\Lambda_c \triangleq T_c L_2 T_c^{-1} = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

which leads to $(I_N \otimes A - L_2 \otimes BK) = (T_c \otimes I_n)(I_N \otimes A - \Lambda_c \otimes BK)(T_c^{-1} \otimes I_n)$. Hence, the diagonal blocks of $I_N \otimes A - \Lambda_c \otimes BK$ are $A - \lambda_i BK$ $(i = 1, \ldots, N)$.

By Lemma 1, all eigenvalues $\{\lambda_i; i = 1, \ldots, N\}$ have positive real parts. Hence, the set $(0, \lambda_{\min})$ is not empty. It is easy to see that $\varepsilon \Re(\lambda_i) \geq 0.5 \Re(\lambda_i) \max\{1, \sigma_{\min}^{-1}\} \geq 0.5$. This together with (10) leads to that

$$\begin{aligned} (A - \lambda_i BK)^H P &+ P(A - \lambda_i BK) \\ &= (A - \varepsilon \lambda_i BB^T P)^H P + P(A - \varepsilon \lambda_i BB^T P) \\ &= A^T P + PA - 2\varepsilon \Re(\lambda_i) PBB^T P \\ &\leq -I_{n+1} + (1 - 2\varepsilon \Re(\lambda_i)) PBB^T P < 0. \end{aligned}$$

By Lyapunov stability theory, $A - \lambda_i BK$ $(i = 1, \ldots, N)$ are all Hurwitz matrices, which implies that $(I_N \otimes A - L_2 \otimes BK) \otimes I_p$ is a Hurwitz matrix.

Next, it is proved that $\|\hat{\Xi}_F(t)\|_2$ is convergent to zero. Since $r \leq n - 1$, it is obtained that $\mathfrak{D}^n \delta_i(t) = \mathbf{0}_p$. Therefore, the solution to (9) is $\hat{\Xi}_F(t) = e^{((I_N \otimes A - L_2 \otimes BK) \otimes I_p)t} \hat{\Xi}_F(0)$. Since $(I_N \otimes A - L_2 \otimes BK) \otimes I_p$ is a Hurwitz matrix, it is obtained that

$$\begin{aligned} \lim_{t \to \infty} & \left\| \hat{\Xi}_F(t) \right\|_2 \\ & \leq \lim_{t \to \infty} \left\| e^{((I_N \otimes A - L_2 \otimes BK) \otimes I_p)t} \right\|_2 \left\| \hat{\Xi}_F(0) \right\|_2 = 0. \quad (11) \end{aligned}$$

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IV. EXTENSIONS TO FOLLOWERS WITH HIGH-ORDER INTEGRAL DYNAMICS

In Section III, the followers are described by the first-order integral dynamics. However, due to the diversity of control plants in practice, it is more interesting to study the follower described by the high-order integral dynamics. In this section, the dynamics of the $(i-M)$th follower $(i \in \mathcal{N})$ is described by

$$\mathfrak{D}^m x_i(t) = u_i(t) \quad (12)$$

where $x_i(t) \in \mathbb{R}^p$ is the position vector of the $(i - M)$th follower; $m \in \mathbb{N}^+$ denotes the relative degree; and $u_i(t) \in \mathbb{R}^p$ is the control input of the $(i - M)$th follower.

Motivated by the PI^n-type algorithm (6), we propose the following containment algorithm:

$$u_i(t) = \sum_{l=0}^{l_m-1} \kappa_l \mathfrak{D}^{m-l-1} \sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij} e_{ji}(t) \quad (13)$$

where $l_m = \max\{m, n+1\}$.

Theorem 2: Assume all leaders move along their polynomial trajectories described by (4); and all followers are described by (12). Let P denote the positive definite solution

to the following matrix inequality:

$$E^T P + PE + I_{l_m} - PFF^T P \leq 0$$

where

$$E = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{l_m \times l_m}, \text{ and } F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{l_m}.$$

The containment problem of MASs can be solved by (13) with $K = (\kappa_{l_m-1}, \ldots, \kappa_0) = \varepsilon F^T P$, where ε is defined in Theorem 1.

Proof: Let $\xi_i(t) = (x_i^T(t), x_i^{(1)^T}(t), \ldots, x_i^{(l_m-1)^T}(t))^T$, $\Xi_L(t) = (\xi_1^T(t), \ldots, \xi_M^T(t)^T)^T$, $\Xi_F(t) = (\xi_{M+1}^T(t), \ldots, \xi_{M+N}^T(t))^T$, and $\hat{\Xi}_F(t) = \Xi_F(t) + (L_2^{-1} L_1 \otimes I_{l_m p}) \Xi_L(t)$.

Following the same procedure of the proof of Theorem 1, it can be proved that there must exist two positive constants $M_1 < \infty$ and β_1, such that $\|e^{(I_N \otimes E \otimes I_p - L_2 \otimes FK \otimes I_p)t}\|_2 \leq M_1 e^{-\beta_1 t}$ and $\|\hat{\Xi}_F(t)\|_2 \leq M_1 e^{-\beta_1 t} \|\hat{\Xi}_F(0)\|_2 \to 0$ $(t \to \infty)$. By Lemma 1, the algorithm defined by (13) solves the containment problem. ∎

Remark 1: It can be seen from (13) that besides the proportional term $\sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij} e_{ji}(t)$ and the integral terms $\{\mathfrak{D}^{-l} \sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij} e_{ji}(t), \ l = 1, 2, \ldots, l_m - m\}$, (13) also includes the "differential terms" $\{\mathfrak{D}^l \sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij} e_{ji}(t), \ l = 1, 2, \ldots, m - 1\}$. Therefore, the proposed algorithm defined by (13) is essentially a generalized "*PID*" algorithm (we can call it the "$PI^{l_m-m} D^{m-1}$"-type algorithm).

It is well known that the proportional term depends on the present information; the integral term represents the accumulation of past information; and the differential term is the future information, which might be more expensive to be measured. Hence the differential terms $\{\mathfrak{D}^l \sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij} e_{ji}(t), \ l = 1, 2, \ldots, m - 1\}$ in (13) might be difficult to obtain. Motivated by [28] and [40]–[42], one way to handle this challenge is to design the state estimator for the $(i - M)$th follower agent to estimate its own state $\{x_i(t), \mathfrak{D} x_i(t), \ldots, \mathfrak{D}^{m-1} x_i(t), \ i \in \mathcal{N}\}$. Let $z_i(t) = (z_i^T(t), \ldots, z_{im}^T(t)^T)^T \in \mathbb{R}^{mp}$ denote the estimated state of the $(i - M)$th follower agent. Then the followers exchange their estimated states with their neighbor agents via the communication network $\mathcal{G}$ to obtain the differential terms. Since the leaders are essentially the reference signals [the polynomial trajectories defined by (4)], each leader should know its current position and any-order derivatives of the current position accurately. Therefore, there is no need to design the estimators for leaders. The ith leader $(i \in \mathcal{M})$ directly sends its state $z_i(t) = (x_i^T(t), \mathfrak{D} x_i^T(t), \ldots, \mathfrak{D}^{m-1} x_i^T(t))^T$ to the connected neighbors.

By the above discussion, the state estimator of the $(i-M)$th follower $(i \in \mathcal{N})$ is designed as

$$\begin{aligned} \mathfrak{D} z_i(t) = (\bar{E} \otimes I_p) z_i(t) &+ (\bar{F} \otimes I_p) u_i(t) + (K_e \otimes I_p) \\ &\times \sum_{j=1}^{M+N} \alpha_{ij} \big((G \otimes I_p)(z_j(t) - z_i(t)) - e_{ji}(t) \big) \quad (14) \end{aligned}$$

where $K_e \in \mathbb{R}^m$, $\bar{F} = (0,\ldots,0,1)^T \in \mathbb{R}^m$, $G = (1,0,\ldots,0) \in \mathbb{R}^{1 \times m}$, and

$$\bar{E} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{m \times m}.$$

Lemma 2: Let $K_e = \varepsilon P G^T$, where ε is defined in Theorem 1 and P is the solution to the following matrix inequality:

$$\bar{E}P + P\bar{E}^T + I_{l_m} - PG^T GP \leq 0.$$

Then there exist two positive constants $M_2 < \infty$ and β_2 such that $\|\mathfrak{D}^{l_m - m}\hat{Z}(t)\|_2 \leq M_2 e^{-\beta_2 t}$, where $\hat{Z}(t) = (\hat{z}_{M+1}^T(t),\ldots,\hat{z}_{M+N}^T(t))^T$, $\hat{z}_i(t) = z_i(t) - \zeta_i(t)$ and $\zeta_i(t) = (x_i^T(t), \mathfrak{D}x_i^T(t),\ldots,\mathfrak{D}^{m-1}x_i^T(t))^T$.

Proof: From (12) and (14), it is obtained that

$$\mathfrak{D}^{l_m - m + 1}\hat{Z}(t) = (I_N \otimes E \otimes I_p - L_2 \otimes K_e G \otimes I_p)\mathfrak{D}^{(l_m - m)}\hat{Z}(t).$$

Following the same procedure of the proof of Theorem 1, it can be easily proved that $(I_N \otimes E \otimes I_p - L_2 \otimes K_e G \otimes I_p)$ is a Hurwitz matrix. Therefore, there must exist two positive constants $M_2 < \infty$ and β_2 such that $\|\mathfrak{D}^{(l_m - m)}\hat{Z}(t)\|_2 = \|e^{(I_N \otimes E \otimes I_p - L_2 \otimes K_e G \otimes I_p)t}\|_2\|\hat{Z}^{(l_m - m)}(0)\|_2 \leq M_2 e^{-\beta_2 t}$. ∎

Replacing $\mathfrak{D}^l e_{ji}(t)$ with $(z_{jl}(t) - z_{il}(t))$, $l = 1,\ldots,m-1$, the containment algorithm (13) is modified as

$$u_i(t) = \sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij}\left(\sum_{l=m-1}^{l_m-1} \kappa_l \mathfrak{D}^{m-l-1}e_{ji}(t),\right.$$
$$\left. + \sum_{l=0}^{m-2} \kappa_l\big(z_{j(m-l)}(t) - z_{i(m-l)}(t)\big)\right). \quad (15)$$

Theorem 3: Assume all leaders move along their polynomial trajectories described by (4); and all followers are described by high-order integral dynamics (12). The containment problem can be solved by (15) with $K = (\kappa_{l_m-1},\ldots,\kappa_0) = \varepsilon \bar{F}^T P$, where ε is defined in Theorem 1 and P is defined in Theorem 2.

Proof: By applying the containment algorithm defined by (15), it can be obtained that

$$\mathfrak{D}\hat{\Xi}_F(t) = R_1\hat{\Xi}_F(t) - R_2\mathfrak{D}^{(l_m - m)}\hat{Z}(t) \quad (16)$$

where $\hat{\Xi}_F(t)$ is defined in the proof of Theorem 2; $R_1 = I_N \otimes E \otimes I_p - L_2 \otimes FK \otimes I_p$; $R_2 = L_2 \otimes FK_2 \otimes I_p$; and $K_2 = (0, \kappa_{m-2},\ldots,\kappa_0)$.

The solution to (16) is $\hat{\Xi}_F(t) = e^{R_1 t}\hat{\Xi}_F(0) + \int_0^t e^{R_1(t-\tau)}R_2\mathfrak{D}^{(l_m-m)}\hat{Z}(\tau)d\tau$, which together with Theorem 2 and Lemma 2 leads to

$$\left\|\hat{\Xi}_F(t)\right\|_2 \leq \left\|e^{R_1 t}\right\|_2\left\|\hat{\Xi}_F(0)\right\|_2$$
$$+ \int_0^t \left\|e^{R_1(t-\tau)}\right\|_2\|R_2\|_2\left\|\mathfrak{D}^{(l_m-m)}\hat{Z}(\tau)\right\|_2 d\tau$$
$$\leq \Sigma_1(t) + \Sigma_2(t)$$

where $\Sigma_1(t) = M_1 e^{-\beta_1 t}\|\hat{\Xi}_F[0]\|_2$ and $\Sigma_2(t) = M_1 M_2\|R_2\|_2 \int_0^t e^{-\beta_1(t-\tau)}e^{-\beta_2 \tau}d\tau$. It is easy to see that
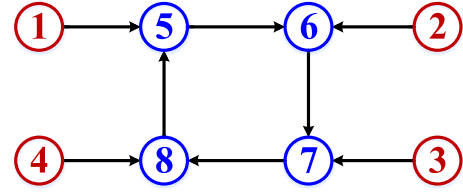

Fig. 2. Communication topology of the MAS in the simulation example of Section IV.

$\lim_{t \to \infty} \Sigma_1(t) = 0$ and

$$\lim_{t \to \infty} \Sigma_2(t)$$
$$= \begin{cases} M_1 M_2\|R_2\|_2 \lim_{t \to \infty} \dfrac{e^{-\beta_1 t} - e^{-\beta_2 t}}{\beta_1 - \beta_2} = 0, & \text{if } \beta_2 \neq \beta_1 \\ M_1 M_2\|R_2\|_2 \lim_{t \to \infty} t e^{-\beta_1 t} = 0, & \text{if } \beta_1 = \beta_2. \end{cases}$$
$$(17)$$

Therefore, $\lim_{t \to \infty}\|\hat{\Xi}_F(t)\|_2 = 0$. By Lemma 1, the containment problem is solved. ∎

Remark 2: Since the leader sends its absolute position to the connected followers, one may wonder whether these followers can calculate their own absolute positions by their relative positions with the leader and the leader's absolute position. However, this idea does not work because the follower can receive the position information not only from the leader (accurate position) but also other followers (estimated positions, not accurate). Due to the nature of distributed control of MASs, the follower cannot distinct the leader from other neighbor agents. Hence the follower can only randomly pick up one agent in its neighborhood to calculate its absolute position. If the selected agent is another follower, the calculated absolute position is obviously inaccurate.

Next, a simulation example is provided to demonstrate the effectiveness of the proposed algorithm.

A. Simulation Example

Consider a MAS composed of eight agents, whose communication topology is shown in Fig. 2. It is easy to see that agents 1–4 are leaders and agents 5–8 are followers. The ith row and the jth column entry of the adjacency matrix satisfies that $\alpha_{ij} = 1$ if there is a directed edge from agent j to agent i, otherwise $\alpha_{ij} = 0$.

The ith leader's move along the trajectory defined by the following polynomial:

$$x_i(t) = \mathbf{a}_0^i + \mathbf{a}_1^i t + \mathbf{a}_2^i t^2 + \mathbf{a}_3^i t^3 \quad (18)$$

where $x_i(t) \in \mathbb{R}^2$, and the coefficients $\mathbf{a}_j^i \in \mathbb{R}^2$ are given in Table I. The $(i-4)$th follower ($i = 5, 6, 7, 8$) has the third-order integral dynamics, i.e., $x_i^{(3)}(t) = u_i(t)$.

The control algorithm (15) is applied to solve this containment problem. By Theorem 2, the parameters in algorithm (15) are set to be $K = (\kappa_3, \kappa_2, \kappa_1, \kappa_0) = (2, 6.1554, 8.4721, 6.1554)$. The simulation result is given in Fig. 3. As shown in Fig. 3, all followers are convergent into the convex hull spanned by the leaders and move along with them.

TABLE I
VALUES OF COEFFICIENTS a_j^i IN (18)

| agent	1	2	3	4
a_0^i	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}2\\5\end{pmatrix}$	$\begin{pmatrix}-5\\10\end{pmatrix}$	$\begin{pmatrix}-10\\0\end{pmatrix}$
a_1^i	$\begin{pmatrix}0.23\\3.43\end{pmatrix}$	$\begin{pmatrix}0.3\\3.43\end{pmatrix}$	$\begin{pmatrix}0.2\\3.43\end{pmatrix}$	$\begin{pmatrix}0.2\\3.43\end{pmatrix}$
a_2^i	$\begin{pmatrix}0.0095\\-0.75\end{pmatrix}$	$\begin{pmatrix}0.0095\\-0.075\end{pmatrix}$	$\begin{pmatrix}0.01\\-0.075\end{pmatrix}$	$\begin{pmatrix}0.01\\-0.075\end{pmatrix}$
a_3^i	$\begin{pmatrix}0\\0.0005\end{pmatrix}$	$\begin{pmatrix}0\\0.0005\end{pmatrix}$	$\begin{pmatrix}0\\0.0005\end{pmatrix}$	$\begin{pmatrix}0\\0.0005\end{pmatrix}$
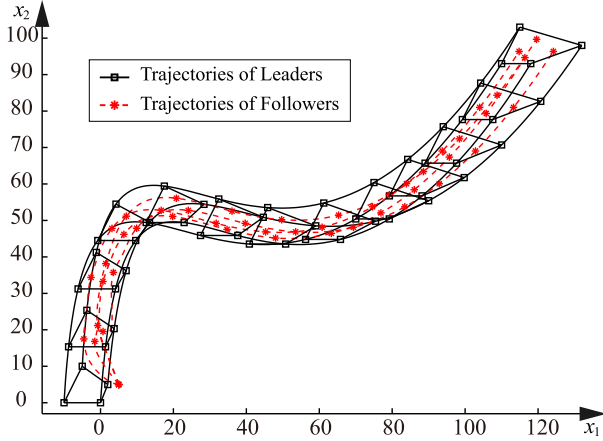
Fig. 3. Moving profiles of all agents in the simulation example of Section IV.

Therefore, the proposed algorithm is able to effectively solve the containment problem with dynamic leaders.

V. EXTENSIONS TO MASS IN DISCRETE-TIME DOMAIN

In this section, the containment problem is studied in the discrete-time domain. The ith leader ($i \in \mathcal{M}$) is assumed to move along the following polynomial trajectory:

$$x_i[k] = \mathbf{a}_0^i + \mathbf{a}_1^i k + \cdots + \mathbf{a}_n^i k^n \qquad (19)$$

where $x_i[k] \in \mathbb{R}^p$ and $\mathbf{a}_j^i \in \mathbb{R}^p$.

In this section, the symbol $\mathfrak{D}$ denotes the difference operator (namely, $\mathfrak{D}x[k] = x[k+1] - x[k]$). The inversion of the difference operator $\mathfrak{D}$ is defined as $\mathfrak{D}^{-1}x[k] = \sum_{i=0}^{k-1} x[k]$. It is easy to see that $\mathfrak{D}(\mathfrak{D}^{-1}x[k]) = x[k]$.

A. Followers With Single-Integrator Dynamics

In this section, the $(i-M)$th follower ($i \in \mathcal{N}$) is described by the following single-integrator dynamics:

$$\mathfrak{D}x_i[k] = u_i[k] + \delta_i[k] \qquad (20)$$

where $x_i[k] \in \mathbb{R}^p$ denotes the position of the $(i-M)$th follower; $u_i[k] \in \mathbb{R}^p$ is the control input, $\delta_i[k] = \mathbf{b}_0^i + \mathbf{b}_1^i k + \cdots + \mathbf{b}_r^i k^r \in \mathbb{R}^p$ is the polynomial disturbance, and $\mathbf{b}_j^i \in \mathbb{R}^p$ ($j = 0, \ldots, r$). Motivated by (6), the following discrete-time PI^n-type algorithm is proposed:

$$u_i[k] = \frac{1}{1+d_i} \sum_{l=0}^n \kappa_l \mathfrak{D}^{-l}\left(\sum_{j\in\mathcal{M}}\alpha_{ij}e_{ji}[k] + \sum_{j\in\mathcal{N}}\alpha_{ij}e_{ji}[k]\right) \qquad (21)$$

where $e_{ji}[k] = x_j[k] - x_i[k]$.

Let $\xi_i[k] = (x_i^T[k], \mathfrak{D}x_i^T[k], \ldots, \mathfrak{D}^n x_i^T[k])^T$, $\Xi_L[k] = (\xi_1^T[k], \ldots, \xi_M^T[k])^T$, and $\Xi_F[k] = (\xi_{M+1}^T[k], \ldots, \xi_{M+N}^T[k])^T$. Substituting (21) into (20) obtains the following closed-loop dynamics:

$$\begin{cases}\mathfrak{D}\Xi_L[k] = (I_M \otimes A \otimes I_p)\Xi_L[k]\\\mathfrak{D}\Xi_F[k] = \big(I_N \otimes A \otimes I_p\\\qquad\quad - (I_N + D)^{-1}L_2 \otimes BK \otimes I_p\big)\Xi_F[k]\\\qquad\quad - \big((I_N + D)^{-1}L_1 \otimes BK \otimes I_p\big)\Xi_L[k]\\\qquad\quad + (I_N \otimes B \otimes I_p)\Delta[k]\end{cases} \quad (22)$$

where $D = \text{diag}(d_{M+1}, \ldots, d_{M+N})$, $\Delta[k] = \text{diag}(\mathfrak{D}^n \delta_{M+1}^T[k], \ldots, \mathfrak{D}^n \delta_{M+N}^T[k])$, and $K = (\kappa_{n-1}, \ldots, \kappa_0)$; A and B are defined in (7). It follows from (22) that:

$$\hat{\Xi}_F[k+1] = \left(I_N \otimes \hat{A} \otimes I_p - \hat{L}_2 \otimes BK \otimes I_p\right)\hat{\Xi}_F[k] + (I_N \otimes B \otimes I_p)\Delta[k] \quad (23)$$

where $\hat{\Xi}_F[k] = \Xi_F[k] + (L_2^{-1}L_1 \otimes I_{(n+1)p})\Xi_L[k]$, $\hat{A} = A + I_{n+1}$, and $\hat{L}_2 = (I_N + D)^{-1}L_2$. By Lemma 1, if every element in $\hat{\Xi}_F[k]$ is convergent to zero, then the containment problem is solved.

Theorem 4: Assume all leaders move along their polynomial trajectories described by (19); and all followers are described by the single-integrator dynamics (20). Let P denote the positive definite solution to the following matrix inequality:

$$P > \hat{A}^T P\hat{A} - \left(1-\varepsilon^2\right)\hat{A}^T PB(B^T PB)^{-1}B^T P\hat{A} \quad (24)$$

where $\varepsilon \in (\max_{i\in\{1,2,\ldots,N\}}|1-\hat{\lambda}_i|, 1)$, and $\{\hat{\lambda}_i; i = 1, \ldots, N\}$ are the eigenvalues of $\hat{L}_2$. If the order of the polynomial disturbance $\delta_i[k]$ in (20) is not greater than $n-1$ (namely, $r \leq n-1$), then the containment problem in the discrete-time domain can be solved by the PI^n-type algorithm defined by (21) with $K = (B^T PB)^{-1}B^T P\hat{A}$.

Proof: Since $r \leq n-1$, it can be obtained that $\mathfrak{D}^n \delta_i[k] = \mathbf{0}_p$. This together with (23) leads to that

$$\hat{\Xi}_F[k] = \left(I_N \otimes \hat{A} \otimes I_p - \hat{L}_2 \otimes BK \otimes I_p\right)^k \hat{\Xi}_F[0].$$

By Gerschgorin circle theorem and Lemma 1, all eigenvalues of $\hat{L}_2$ are inside the open circle $U(1,1) \triangleq \{a + bj | a, b \in \mathbb{R}$ and $(a-1)^2 + b^2 < 1\}$. Therefore, the set $(\max_{i\in\{1,2,\ldots,N\}}|1-\lambda_i|, 1)$ is not empty. By [43, Lemma 5], we know that the matrix inequality (24) has a positive definite solution P as long as $|\varepsilon| < 1/\prod_i |\lambda_i^u(\hat{A})|$, where $\{\lambda_i^u\}$ are the unstable eigenvalues of $\hat{A}$. Since all unstable eigenvalues of $\hat{A}$ are 1, (24) has a positive definite solution P.

Next, it is proved that all eigenvalues of $\hat{A} - \hat{\lambda}_i BK$ are inside the unit circle. From (24), it can be calculated that

$$\left(\hat{A} - \hat{\lambda}_i BK\right)^H P\left(\hat{A} - \hat{\lambda}_i BK\right) - P$$
$$= \left(\hat{A}^T P\hat{A} - P\right) - \left(\hat{\lambda}_i + \bar{\hat{\lambda}}_i - \hat{\lambda}_i\bar{\hat{\lambda}}_i\right)\hat{A}^T PB(B^T PB)^{-1}B^T P\hat{A}$$
$$< \left(1-\delta^2\right)\hat{A}^T PB(B^T PB)^{-1}B^T P\hat{A}$$
$$\quad - \left(1 - \left(1-\hat{\lambda}_i\right)\left(1-\bar{\hat{\lambda}}_i\right)\right)\hat{A}^T PB(B^T PB)^{-1}B^T P\hat{A}$$
$$= \left(|1-\hat{\lambda}_i|^2 - \varepsilon^2\right)\hat{A}^T PB(B^T PB)^{-1}B^T P\hat{A} \leq 0.$$

By Lyapunov stability theory, all eigenvalues of $\hat{A} - \hat{\lambda}_i BK$ are inside the unit circle.

By Schur decomposition, there must exist a transformation matrix T such that

$$\Lambda \triangleq T\hat{L}_2 T^{-1} = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}.$$

Hence

$$I_N \otimes \hat{A} \otimes I_p - \hat{L}_2 \otimes BK \otimes I_p$$
$$= \left(\left(T^{-1} \otimes I_n \right) \left(I_N \otimes \hat{A} - \Lambda \otimes BK \right) (T \otimes I_n) \right) \otimes I_p. \tag{25}$$

The diagonal blocks of $I_N \otimes \hat{A} - \Lambda \otimes BK$ are $\hat{A} - \hat{\lambda}_i BK$ ($i = 1, \ldots, N$). Therefore, all eigenvalues of $I_N \otimes \hat{A} - \Lambda \otimes BK$ are inside the unit circle. Hence, there must exist two positive constants $M_3 < \infty$ and $\beta_3 \in (0, 1)$ such that $\|(I_N \otimes \hat{A} \otimes I_p - \hat{L}_2 \otimes BK \otimes I_p)^k\|_2 \le M_3 \beta_3^k$ and $\|\hat{\Xi}_F[k]\|_2 \le M_3 \beta_3^k \|\Xi_F(0)\|_2 \to 0$ ($k \to \infty$). ∎

In (21), the coefficient $1/(1 + d_i)$ is used to normalize the Laplacian matrix. If we replace $1/(1 + d_i)$ with a uniform constant μ, then the algorithm (21) is modified as

$$u_i[k] = \mu \sum_{l=0}^{n} \kappa_l \mathfrak{D}^{-l} \left(\sum_{j \in \mathcal{M}} \alpha_{ij} e_{ji}[k] + \sum_{j \in \mathcal{N}} \alpha_{ij} e_{ji}[k] \right). \tag{26}$$

Let $\{\lambda_1, \ldots, \lambda_N\}$ denote the eigenvalues of L_2. Then by Lemma 1, we know $\Re(\lambda_i) > 0$. Hence, there must exist a positive constant $\mu_{\min} \in (0, 1)$ such that $\max_i |1 - \mu_{\min}\lambda_i| < 1$. Following the same procedure of Theorem 4, the following corollary can be easily obtained.

Corollary 1: Let P denote the positive definite solution to the following matrix inequality:

$$P > \hat{A}^T P \hat{A} - \left(1 - \gamma^2 \right) \hat{A}^T PB \left(B^T PB \right)^{-1} B^T P \hat{A} \tag{27}$$

where $\gamma \in (\varepsilon_{\max}, 1)$ and $\varepsilon_{\max} = \max_i\{|1 - \mu_{\min}\lambda_i|\} < 1$. The containment problem of MASs in the discrete time domain can be solved by the algorithm (26) with $K = (B^T PB)^{-1} B^T P \hat{A}$ and $\mu = \mu_{\min}$.

Remark 3: Compared to [9], [15], [21], [22], [31], and [33], one distinguished feature of this paper is that the proposed algorithms do not employ the sign function. One limitation of the sign function is that it can cause the harmful chattering phenomenon. Therefore, the proposed algorithm can avoid the high-frequency control switches occurred in the chattering phenomenon. Moreover, the sign function can hardly be used in the discrete-time domain. The current literature rarely discusses the containment problem with dynamic leaders in the discrete-time domain, and this paper gives a possible solution.

B. Containment Control of MASs With Measurement Noises

In the above sections, it is assumed that all followers can accurately measure the relative states between themselves and their neighbors. However, the measurement noise is unavoidable in practice. In this section, it is assumed that the relative state $e_{ji}[k]$ is corrupted by the measurement noise $\rho_{ji}\eta_{ji}[k]$, where $\rho_{ji} = \text{diag}(\rho_{ji1}, \ldots, \rho_{jip})$ and $\rho_{ijs} < \infty$ ($s = 1, \ldots, p$) denotes the noise intensity; and $\eta_{ij}[k] \in \mathbb{R}^p$ is the standard white noise vector. Moreover, it is assumed that $\{\eta_{ji}[k] | j \in \mathcal{M} \cup \mathcal{N}; i \in \mathcal{N}\}$ are mutually independent.

Denote $\bar{e}_{ji}[k] = e_{ji}[k] + \rho_{ji}\eta_{ji}[k]$. The containment PI^n-type algorithm (21) is modified as

$$u_i[k] = \frac{1}{1 + d_i} \sum_{l=0}^{n} \kappa_l \mathfrak{D}^{-l} \left(\sum_{j \in \mathcal{M}} \alpha_{ij} \bar{e}_{ji}[k] + \sum_{j \in \mathcal{N}} \alpha_{ij} \bar{e}_{ji}[k] \right). \tag{28}$$

The definition of the containment problem should also be modified to take the noise effect into account.

Definition 3: The containment problem of MASs with measurement noises is solved in the stochastic sense if $\lim_{k \to \infty} \text{dis}(E(x_i[k]), \text{co}_L[k]) = 0$ and $E(\text{dis}(x_i[k], \text{co}_L[k]))^2 < \infty$, $i \in \mathcal{N}$.

Theorem 5: Assume all leaders move along their polynomial trajectories described by (19); and all followers are described by the single-integrator dynamics (20). If the order of the polynomial disturbance $\delta_i[k]$ in (20) is not greater than $n - 1$ (namely, $r \le n - 1$), then the containment algorithm (28) with $K = (B^T PB)^{-1} B^T P \hat{A}$ (P is defined in Theorem 4) can solve the containment problem of MASs with measurement noises in the stochastic sense.

Proof: Since $r \le n - 1$, the closed-loop dynamics (23) can be rewritten as

$$\hat{\Xi}_F[k + 1] = \left(I_N \otimes \hat{A} \otimes I_p - \hat{L}_2 \otimes BK \otimes I_p \right) \hat{\Xi}_F[k]$$
$$+ \left((I_N + D)^{-1} \Gamma \otimes BK \otimes I_p \right) V[k] \tag{29}$$

where $\Gamma = \text{diag}(\Gamma_{M+1}, \ldots, \Gamma_{M+N})$, $V[k] = (V_{M+1}^T[k], \ldots, V_{M+N}^T[k])^T$, $\Gamma_i = (\alpha_{i1}\rho_{1i}, \ldots, \alpha_{i(M+N)}\rho_{(M+N)i})$, $V_i[k] = (v_{1i}^T[k], \ldots, v_{(M+N)i}^T[k])^T$, and $v_{ji}[k] = (\eta_{ji}[k], \mathfrak{D}\eta_{ji}[k], \ldots, \mathfrak{D}^n\eta_{ji}[k])^T$.

Let $R_3 = I_N \otimes \hat{A} \otimes I_p - \hat{L}_2 \otimes BK \otimes I_p$ and $R_4 = (I_N + D)^{-1} \Gamma \otimes BK \otimes I_p$. By (29), it is obtained that

$$\hat{\Xi}_F[k] = R_3^k \hat{\Xi}_F[0] + \sum_{i=0}^{k-1} R_3^{k-i-1} R_4 V[i]. \tag{30}$$

From the proof of Theorem 4, we know that there exist two finite positive constants M_3 and $\beta_3 \in (0, 1)$ such that $\|R_3^k\|_F \le M_3 \beta_3^k$. Hence $\lim_{k \to \infty} R_3^k \hat{\Xi}_F[0] = \mathbf{0}_{p(n+1)N}$. By Lemma 6, we know that $E(\sum_{i=0}^{k-1} R_3^{k-i-1} R_4 V[i]) = \sum_{i=0}^{k-1} R_3^{k-i-1} R_4 E(V[i]) = \mathbf{0}_{p(n+1)N}$. Therefore

$$\lim_{k \to \infty} E\left(\hat{\Xi}_F[k] \right) = \lim_{k \to \infty} R_3^k \hat{\Xi}_F[0] = \mathbf{0}_{p(n+1)N} \tag{31}$$

which together with Lemma 1 leads to that $\lim_{k \to \infty} \text{dis}(E(x_i[k]), \text{co}_L[k]) = 0$, $i \in \mathcal{N}$.

Let $Y_1[k] = R_3^k \hat{\Xi}_F[0]$ and $Y_2[k] = \sum_{i=0}^{k-1} R_3^{k-i-1} R_4 V[i]$. Since $\lim_{k \to \infty} \|Y_1[k]\|_2 = 0$, there must exist a finite positive constant $\hat{M}_3$ such that $\|Y_1[k]\|_2 < \hat{M}_3$. By Lemma 5,

we know that $\forall m \in \{0, 1, \ldots, n\}$, there exists a finite positive constant $\bar{M}_3$ such that $\|\mathrm{E}(V[k]V^T[k+m])\|_F < \bar{M}_3$.

And for $\forall m \geq n + 1$, $\mathrm{E}(V[k]V^T[k+m]) = \mathbf{0}_{N(M+N)(n+1)p \times N(M+N)(n+1)}$ holds. Therefore

$$\left\| \mathrm{E}\left(Y_2[k]Y_2^T[k]\right) \right\|_F$$

$$= \left\| \mathrm{E} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} R_3^{k-i-1} R_4 V[i] V^T[j] R_4^T R_3^{k-j-1}{}^T \right\|_F$$

$$= \left\| \mathrm{E} \sum_{i=0}^{k-1} \sum_{j=\max\{0, i-n\}}^{\min\{k-1, i+n\}} R_3^{k-i-1} R_4 V[i] V^T[j] R_4^T R_3^{k-j-1}{}^T \right\|_F$$

$$\leq \sum_{i=0}^{k-1} \sum_{j=i-n}^{i+n} \left\| R_3^{k-i-1} \right\|_F \|R_4\|_F^2 \left\| \mathrm{E}\left(V[i]V^T[j]\right) \right\|_F \left\| R_3^{k-j-1} \right\|_F$$

$$\leq M_3^2 \bar{M}_3 \|R_4\|_F^2 \sum_{i=0}^{k-1} \sum_{j=i-n}^{i+n} \beta_3^{2k-i-j-2}$$

$$= M_3^2 \bar{M}_3 \|R_4\|_F^2 \frac{\left(1 - \beta_3^{2k}\right)\left(1 - \beta_3^{2n+1}\right)}{\beta_3^n (1 - \beta_3)\left(1 - \beta_3^2\right)} < \infty$$

which leads to that $\mathrm{E}\|Y_2[k]\|_2^2 \leq \|\mathrm{E}(Y_2[k]Y_2^T[k])\|_F^2 < \infty$. Therefore

$$\mathrm{E}\left\| \hat{\Xi}_F[k] \right\|_2^2 \leq 2\mathrm{E}\|Y_1[k]\|_2^2 + 2\mathrm{E}\|Y_1[k]\|_2^2 < \infty$$

which together with Lemma 1 leads to that $\mathrm{E}(\mathrm{dis}(x_i[k], \mathrm{co}_L[k]))^2 < \infty$, $i \in \mathcal{N}$. ∎

C. Followers With High-Order Integral Dynamics

In this section, the dynamics of the $(i - M)$th follower $(i \in \mathcal{N})$ is described by the following high-order difference equation:

$$\mathfrak{D}^m x_i[k] = u_i[k] \tag{32}$$

where $x_i[k] \in \mathbb{R}^p$ is the position vector of the $(i - M)$th follower; $m \in \mathbb{N}^+$ denotes the relative degree; and $u_i[k] \in \mathbb{R}^p$ is the control input.

Motivated by the algorithm (13), the following $PI^{l_m-m}D^{m-1}$-type containment algorithm is proposed:

$$u_i[k] = \frac{1}{1 + d_i} \sum_{l=0}^{l_m-1} \kappa_l \mathfrak{D}^{m-l-1} \left(\sum_{j \in \mathcal{M}} \alpha_{ij} e_{ji}[k] + \sum_{j \in \mathcal{N}} \alpha_{ij} e_{ji}[k] \right) \tag{33}$$

where $l_m = \max\{n + 1, m\}$.

Theorem 6: Assume all leaders move along their polynomial trajectories described by (19); and all followers are described by the high-order integral dynamics (32). Let P denote the positive definite solution to the following matrix inequality:

$$P > \hat{E}^T P \hat{E} - \left(1 - \varepsilon^2\right) \hat{E}^T P F \left(F^T P F\right)^{-1} F^T P \hat{E}$$

where $\varepsilon \in (\max_{i \in \{1,2,\ldots,N\}} |1 - \hat{\lambda}_i|, 1)$, $\hat{E} = I_{l_m} + E$, E and F are defined in Theorem 2. The containment problem can be solved by (33) with $K = (\kappa_{l_m-1}, \ldots, \kappa_0) = (F^T P F)^{-1} F^T P \hat{E}$.

Proof: Let $\xi_i[k] = (x_i^T[k], \mathfrak{D}x_i^T[k], \ldots, \mathfrak{D}^{l_m-1}x_i^T[k])^T$, $\Xi_L[k] = (\xi_1^T[k], \ldots, \xi_M^T[k])^T$, $\Xi_F[k] = (\xi_{M+1}^T[k], \ldots, \xi_{M+N}^T[k])^T$ and $\hat{\Xi}[k] = \Xi_F[k] + (L_2^{-1}L_1 \otimes I_{l_mp}) \Xi_L[k]$. Following the same procedure of the proof of Theorem 4, it can be proved that there exist two positive constants $M_4 < \infty$ and $\beta_4 \in (0, 1)$ such that $\|(I_N \otimes \hat{E} - \hat{L}_2 \otimes FK)^k\|_2 \leq M_4 \beta_4^k$ and $\lim_{k \to \infty} \|\hat{\Xi}[k]\|_2 \leq \lim_{k \to \infty} M_4 \beta_4^k \|\hat{\Xi}[0]\|_2 = 0$. ∎

If $\{\mathfrak{D}e_{ji}[k], \ldots, \mathfrak{D}^{m-1}e_{ji}[k]\}$ in (33) are not available for the algorithm design, motivated by (14), the following estimator is designed to estimate the $(i - M)$th follower's position and this position's differences up to the $(m - 1)$th-order, $i \in \mathcal{N}$:

$$\mathfrak{D}z_i[k] = \left(\bar{E} \otimes I_p\right)z_i[k] + \left(\bar{F} \otimes I_p\right)u_i[k] + \frac{(K_e \otimes I_p)}{1 + d_i}$$
$$\times \sum_{j=1}^{M+N} \alpha_{ij}\left((G \otimes I_p)\left(z_j[k] - z_i[k]\right) - e_{ji}[k]\right) \tag{34}$$

where $z_i[k] = (z_{i1}^T[k], \ldots, z_{im}^T[k]^T)^T$; K_e, $\bar{E}$, $\bar{F}$, and G are defined in (14).

Lemma 3: Let $K_e = \tilde{E}PG^T(GPG^T)^{-1}$, where P is the positive definite solution to the following matrix inequality:

$$P > \tilde{E}P\tilde{E}^T - \left(1 - \varepsilon^2\right)\tilde{E}PG^T\left(GPG^T\right)^{-1}GP\tilde{E}^T$$

where $\varepsilon \in (\max_{i \in \{1,2,\ldots,N\}} |1 - \hat{\lambda}_i|, 1)$ and $\tilde{E} = I_m + \bar{E}$. Then there exist two positive constants $M_5 < \infty$ and $\beta_5 \in (0, 1)$ such that $\|\mathfrak{D}^{l_m-m}\hat{Z}[k]\|_2 \leq M_5 \beta_5^k$, where $\hat{Z}[k] = (\hat{z}_{M+1}^T[k], \ldots, \hat{z}_{M+N}^T[k])^T$, $\hat{z}_i[k] = z_i[k] - \zeta_i[k]$ and $\zeta_i[k] = (x_i^T[k], \mathfrak{D}x_i^T[k], \ldots, \mathfrak{D}^{m-1}x_i^T[k])^T$.

Proof: From (32) and (34), it can be obtained that

$$\hat{Z}[k + 1] = \left((I_N \otimes \tilde{E} - L_2 \otimes K_eG) \otimes I_p\right)\hat{Z}[k]$$

which implies that

$$\mathfrak{D}^{l_m-m}\hat{Z}[k + 1] = \left((I_N \otimes \tilde{E} - L_2 \otimes K_eG) \otimes I_p\right)\mathfrak{D}^{l_m-m}\hat{Z}[k].$$

Following the same procedure of the proof of Theorem 4, it can be easily proved that all eigenvalues of $I_N \otimes \tilde{E} \otimes I_p - L_2 \otimes K_eG \otimes I_p$ are inside the unit circle. Hence, there exist two positive constants $M_5 < \infty$ and $\beta_5 \in (0, 1)$ such that $\|\mathfrak{D}^{l_m-m}\hat{Z}[k]\|_2 \leq \|(I_N \otimes \tilde{E} - L_2 \otimes K_eG)^k\|_2 \|\mathfrak{D}^{l_m-m}\hat{Z}[0]\|_2 \leq M_5 \beta_5^k$. ∎

Replacing $\mathfrak{D}^l e_{ji}[k]$ with $z_{jl}[k] - z_{il}[k]$ $(l = 1, \ldots, m - 1)$ in (33) obtains the following modified containment algorithm:

$$u_i[k] = \frac{1}{1 + d_i} \sum_{j \in \mathcal{M} \cup \mathcal{N}} \alpha_{ij} \left(\sum_{l=m-1}^{l_m-1} \kappa_l \mathfrak{D}^{m-l-1} e_{ji}[k], \right.$$
$$\left. + \sum_{l=0}^{m-2} \kappa_{m-l-1}\left(z_{j(m-l)}[k] - z_{i(m-l)}[k]\right) \right). \tag{35}$$

Theorem 7: Assume all leaders move along their polynomial trajectories described by (19); and all followers are described by the high-order integral dynamics (32). The containment problem of MASs can be solved by (35) with

$K = (\kappa_{l_m-1}, \ldots, \kappa_0) = (F^T PF)^{-1} F^T P\hat{E}$, where P is the solution to the following matrix inequality:

$$P > \hat{E}^T P\hat{E} - \left(1 - \varepsilon^2\right)\hat{E}^T PF\left(F^T PF\right)^{-1} F^T P\hat{E}$$

where $\varepsilon \in (\max_{i \in \{1,2,\ldots,N\}} |1 - \hat{\lambda}_i|, 1)$.

Proof: By applying the containment algorithm defined by (15), the closed-loop dynamics of the MAS can be rewritten in the following compact form:

$$\begin{cases} \mathfrak{D}\,\Xi_L[k] = \left(I_M \otimes E \otimes I_p\right)\Xi_L[k] \\ \mathfrak{D}\,\Xi_F[k] = \left(I_N \otimes E \otimes I_p - \hat{L}_2 \otimes FK \otimes I_p\right)\Xi_F[k] \\ \qquad - \left((I_N + D)^{-1}L_1 \otimes FK \otimes I_p\right)\Xi_L[k] \\ \qquad - \left(\hat{L}_2 \otimes FK_2 \otimes I_p\right)\mathfrak{D}^{l_m-m}\hat{Z}[k] \end{cases} \quad (36)$$

where $K_2 = (0, \kappa_{m-2}, \ldots, \kappa_0)$; $\Xi_L[k]$ and $\Xi_F[k]$ are defined in Theorem 6. Let $\hat{\Xi}_F[k] = \Xi_F[k] + (L_2^{-1}L_1 \otimes I_{lmp})\Xi_L[k]$. Then it is derived from (36) that

$$\hat{\Xi}_F[k+1] = R_5 \hat{\Xi}_F[k] - R_6 \mathfrak{D}^{l_m-m}\hat{Z}[k] \quad (37)$$

where $R_5 = I_N \otimes \hat{E} \otimes I_p - \hat{L}_2 \otimes FK \otimes I_p$ and $R_6 = \hat{L}_2 \otimes FK_2 \otimes I_p$.

It follows from (37) that $\hat{\Xi}_F[k] = R_5^k \Xi_F[0] + \sum_{l=0}^{k-1} R_5^{k-l-1} R_6 \mathfrak{D}^{l_m-m}\hat{Z}[l]$. This together with Lemma 3 and the proof of Theorem 6 implies that

$$\left\|\hat{\Xi}_F[k]\right\|_2 \leq \sum_{l=0}^{k-1}\left\|R_5^{k-l-1}\right\|_2 \|R_6\|_2 \left\|\mathfrak{D}^{l_m-m}\hat{Z}[l]\right\|_2$$
$$+ \left\|R_5^k\right\|_2 \|\Xi_F[0]\|_2 \leq \Sigma_1[k] + \Sigma_2[k] \quad (38)$$

where $\Sigma_1[k] = M_4 \beta_4^k \|\Xi_F[0]\|_2$ and $\Sigma_2[k] = M_4 M_5 \|R_6\|_2 \sum_{l=0}^{k-1} \beta_4^{k-l-1}\beta_5^l$. It is calculated that $\lim_{k \to \infty} \Sigma_1[k] = 0$ and

$$\lim_{k \to \infty} \Sigma_2[k] = \begin{cases} M_4 M_5 \|R_6\|_2 \lim_{k \to \infty} \dfrac{\beta_5^k - \beta_4^k}{\beta_5 - \beta_4} = 0, & \text{if } \beta_4 \neq \beta_5 \\ M_4 M_5 \|R_6\|_2 \lim_{k \to \infty} k\beta_4^{k-1} = 0, & \text{if } \beta_4 = \beta_5. \end{cases}$$

Hence, $\lim_{k \to \infty} \|\hat{\Xi}_F[k]\|_2 = 0$. By Lemma 1, the containment problem is solved. ∎

D. Applications: Coordinated Control of Group of Mobile Robots

In order to demonstrate the practical value of the proposed algorithm, this section provides an application example: the coordinated control of a group of mobile robots [44].

Consider the scenario shown in Fig. 1, where a group of mobile robots are required to move across a partially unknown area through a narrow and safe tunnel. There are three master robots and three slave robots. As claimed in Section I, master robots are capable of self-navigation; and slave robots can measure the relative positions with neighbor robots.

Each robot is a differential drive mobile robot. The schematic diagram of the ith mobile robot is given in Fig. 4. The coordinate of the center point between two driving wheels is denoted by (x_i, y_i). The coordinate of the center of the ith
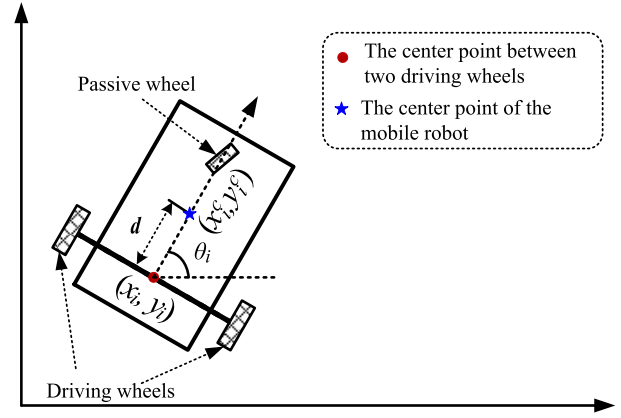

Fig. 4. Schematic of the ith nonholonomic mobile robot.

mobile robot is denoted by (x_i^c, y_i^c). The ith mobile robot's kinematics is described by

$$\begin{cases} \dot{x}_i(t) = v_i(t)\cos(\theta_i) \\ \dot{y}_i(t) = v_i(t)\sin(\theta_i) \\ \dot{\theta}_i(t) = \omega_i(t) \end{cases} \quad (39)$$

where θ_i is the orientation of the ith mobile robot with respect to the horizontal axis; v_i and ω_i are the linear velocity and the angular velocity of the ith robot, respectively.

Unfortunately, the proposed algorithm cannot be directly applied to this kind of mobile robots due to its nonlinear kinematics.

To deal with this challenge, we reformulate (39) at the point (x_i^c, y_i^c) by the feedback linearization [45]

$$\dot{\varphi}_i(t) = u_i(t)$$

where $\varphi_i(t) = (x_i^c(t), y_i^c(t))^T$, $u_i(t) = (u_i^x(t), u_i^y(t))$, $u_i^x(t) = v_i(t)\cos(\theta_i) - d\omega_i(t)\sin(\theta_i)$, and $u_i^y(t) = v_i(t)\sin(\theta_i) + d\omega_i(t)\cos(\theta_i)$. In the rest of this paper, the mobile robot is simply denoted by the coordinate of its center point [e.g., the ith robot is denoted by $\varphi_i(t)$].

Remark 4: According to the above analysis, the control inputs of the ith mobile robot are $v_i(t)$ and $w_i(t)$, which can be easily obtained from $u_i(t)$ by using the following transformation:

$$\begin{pmatrix} v_i(t) \\ \omega_i(t) \end{pmatrix} = \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\dfrac{1}{d}\sin(\theta_i) & \dfrac{1}{d}\cos(\theta_i) \end{pmatrix} u_i(t).$$

Due to the wide use of digital devices, it is assumed that only the sampled data at each sampling instant is available. Assume that the sampling period is $T = 1$. For any signal $s(t)$, the sampled data $s(kT)$ at the kth sampling instance is denoted by $s[k]$. By adopting the zero-order holder strategy, the ith robot's behavior is described by the following discrete-time difference equation:

$$\varphi_i[k+1] = \varphi_i[k] + u_i[k]. \quad (40)$$

Moreover, in this application, it is assumed that the robot's dynamics (40) is disturbed by the polynomial disturbance

TABLE II
COORDINATE VALUES OF ALL REFERENCE POINTS

Time (k)	Leader 1	2	3
0	(0, 25)	(20, −5)	(−10, −20)
30	(110, 8)	(130, −15)	(100, −35)
60	(200, 50)	(230, 35)	(210, 0)
90	(300, 120)	(335, 100)	(315, 70)
120	(405, 155)	(440, 130)	(410, 110)
150	(475, 150)	(510, 130)	(480, 110)

TABLE III
VALUES OF COEFFICIENTS $\mathbf{a}_j^i$ IN (41)

| Leader	1	2	3
a_0^i	$\begin{pmatrix} 0 \\ 25 \end{pmatrix}$	$\begin{pmatrix} 20.00 \\ -5.000 \end{pmatrix}$	$\begin{pmatrix} -10.00 \\ -20.00 \end{pmatrix}$
a_1^i	$\begin{pmatrix} 4.625 \\ -1.028 \end{pmatrix}$	$\begin{pmatrix} 4.100 \\ -1.392 \end{pmatrix}$	$\begin{pmatrix} 3.544 \\ -0.3833 \end{pmatrix}$
$a_2^i \times 10^2$	$\begin{pmatrix} -4.560 \\ -0.7963 \end{pmatrix}$	$\begin{pmatrix} -1.944 \\ 2.801 \end{pmatrix}$	$\begin{pmatrix} 0.7407 \\ -3.796 \end{pmatrix}$
$a_3^i \times 10^4$	$\begin{pmatrix} 5.092 \\ 10.77 \end{pmatrix}$	$\begin{pmatrix} 1.698 \\ 4.167 \end{pmatrix}$	$\begin{pmatrix} -1.389 \\ 15.05 \end{pmatrix}$
$a_4^i \times 10^6$	$\begin{pmatrix} 1.800 \\ -10.91 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -6.430 \end{pmatrix}$	$\begin{pmatrix} 1.029 \\ -1.337 \end{pmatrix}$
$a_5^i \times 10^8$	$\begin{pmatrix} 0 \\ 3.086 \end{pmatrix}$	$\begin{pmatrix} -0.3429 \\ 2.058 \end{pmatrix}$	$\begin{pmatrix} -0.3429 \\ 3.601 \end{pmatrix}$
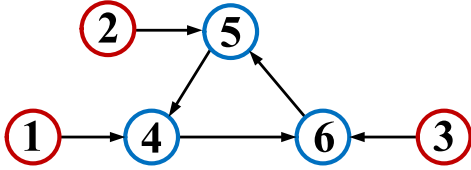
Fig. 5. Communication topology: nodes 1–3 denote master robots and nodes 4–6 represent slave robots.

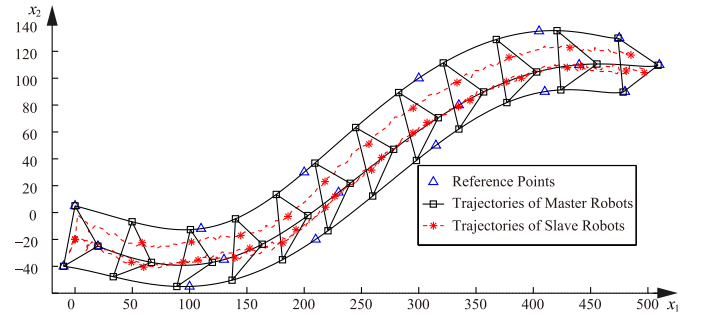

Fig. 6. Moving profiles of all robots.

input of the $(i − 3)$th slave robot ($i = 4, 5, 6$) is designed as

$$u_i[k] = \frac{1}{(1+d_i)} \sum_{l=0}^{5} \kappa_l \mathfrak{D}^{-l} \sum_{j=1}^{6} \alpha_{ij} \big(e_{ji}[k] + \rho_{ji}\eta_{ji}[k] \big) \quad (42)$$

where $\rho_{ji}\eta_{ji}[k]$ is the measurement noise; $\alpha_{ij} = 1$ if there is a directed edge from robot j to robot i, otherwise $\alpha_{ij} = 0$. By Theorem 4, the parameters $(k_6, k_5, \ldots, k_0)$ are set to be $(1.806, 0.4769, 0.0786, 0.0085, 5.660 \times 10^{-4}, 1.826 \times 10^{-5})$.

In order to verify the effectiveness of the algorithm (42), a simulation is carried out. The simulation results are shown in Fig. 6. All slave robots are convergent into the convex hull spanned by the master robots and move along with them in despite of the existence of the disturbance and the measurement noise.

$\delta_i[k] = \mathbf{1}_2 \otimes (1 + 0.2k − 0.01k^2 + 0.001k^3)$. The corresponding dynamics of the ith robot can therefore be written as

$$\varphi_i[k+1] = \varphi_i[k] + u_i[k] + \delta_i[k].$$

The task shown in Fig. 1 can be accomplished by adopting the control strategy introduced in Section I. For each master robot, we select six reference points inside the safe tunnel. The coordinates of these reference points are shown in Table II. By the polynomial interpolation, we can obtain the reference trajectory which goes through these points. The trajectory of the ith master robot is

$$\varphi_i[k] = \mathbf{a}_5^i k^5 + \mathbf{a}_4^i k^4 + \mathbf{a}_3^i k^3 + \mathbf{a}_2^i k^2 + \mathbf{a}_1^i k + \mathbf{a}_0^i. \quad (41)$$

The coefficients $\mathbf{a}_j^i$ in (41) are given in Table III. Because master robots are capable of self-navigation, they are assumed to be able to autonomously move along their reference trajectories.

The communication topology of the multirobot system is shown in Fig. 5. If there is a directed edge from node j to node i, then the ith robot can measure the relative position $e_{ji}[k]$ between itself and robot j. By Section V-B, the control

VI. CONCLUSION

A PI^n-type containment algorithm is proposed for solving the containment problem of MASs in both continuous-time domain and discrete-time domain. Leaders in the MAS are assumed to be the polynomial trajectories. Followers are described by the single-integrator dynamics and the high-order integral dynamics. It is proved that the proposed algorithm can solve the containment problem if for each follower, there is at least one leader which has a directed path to this follower. Compared to the previous results, this paper has the following contributions: 1) the containment problem is studied not only in the continuous-time domain but also in the discrete-time domain; 2) the proposed algorithm can solve the containment problem with dynamic leaders even if followers are described by the single-integrator dynamics; 3) there is no discontinuous sign function in the proposed algorithm; and 4) effects of both disturbance and measurement noise are taken into account. A potential application, the containment control of networked multiple mobile robots, is presented to demonstrate the practical value of the proposed algorithm.

APPENDIX

Lemma 4: For any sequence of numbers $\{x[k] | k = 0, 1, 2, \ldots\}$, the following formula holds:

$$\mathfrak{D}^n x[k] = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x[k+n-i], \quad n = 1, 2, \ldots. \quad (43)$$

Proof: If $n = 1$, the correctness of (43) can be easily verified. Assume when $n = m$, (43) is correct. It can be calculated that

$$\mathfrak{D}^{m+1}x[k] = \mathfrak{D}(\mathfrak{D}^m x[k]) = \mathfrak{D}\sum_{i=0}^{m}(-1)^i\binom{m}{i}x[k+m-i]$$

$$= \sum_{i=0}^{m}(-1)^i\binom{m}{i}(x[k+m-i+1] - x[k+m-i])$$

$$= \sum_{i=0}^{m}(-1)^i\binom{m}{i}x[k+m-i+1]$$

$$+ \sum_{i=1}^{m+1}(-1)^i\binom{m}{i-1}x[k+m-i+1]$$

$$= \sum_{i=1}^{m}(-1)^{i+1}\left(\binom{m}{i}+\binom{m}{i-1}\right)x[k+m-i+1]$$

$$+ x[k+m+1] + (-1)^{m+1}x[k]$$

$$= \sum_{i=1}^{m}(-1)^{i+1}\binom{m+1}{i}x[k+m-i+1]$$

$$+ x[k+m+1] + (-1)^{m+1}x[k]$$

$$= \sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}x[k+m+1-i].$$

By the mathematical induction, it is proved that (43) holds. ∎

Lemma 5: Define a random vector $v[k] = (\eta[k], \mathfrak{D}\eta[k], \ldots, \mathfrak{D}^n\eta[k])^T$, where $\eta[k]$ is the standard white noise. For $\forall m \in \{0, 1, 2\ldots, n\}$, there exists a finite positive constant G such that

$$\|\mathrm{E}(v[k]v^T[k+m])\|_F < G.$$

For $\forall m \geq n + 1$, the following equation holds:

$$\mathrm{E}(v[k]v^T[k+m]) = \mathbf{0}_{(n+1)\times(n+1)}.$$

Proof: The ith row and the jth column entry of $v[k]v^T[k+m]$ is $\mathfrak{D}^{i-1}\eta[k]\mathfrak{D}^{j-1}\eta[k+m]$. By Lemma 4, we know $\mathfrak{D}^{i-1}\eta[k]\mathfrak{D}^{j-1}\eta[k+m]$ is a linear combination of the following terms:

$$\eta[k+i-1-s]\eta[k+m+j-1-t]$$

where $s \in \{0, 1, \ldots, i-1\}$ and $t \in \{0, 1, \ldots, j-1\}$.

If $m \in \{0, 1, 2\ldots, n-1\}$, then $\forall s \in \{0, 1, \ldots, i-1\}$ and $\forall t \in \{0, 1, \ldots, j-1\}$

$$|\mathrm{E}(\eta[k+i-1-s]\eta[k+m+j-1-t])|_2$$
$$= \begin{cases} 0, & m+j-i+s-t \neq 0 \\ 1, & m+j-i+s-t = 0. \end{cases}$$

Hence, there must exist a finite positive constant G_1 such that

$$\left|\mathrm{E}\left(\mathfrak{D}^{i-1}\eta[k]\mathfrak{D}^{j-1}\eta[k+m]\right)\right|_2 < G_1$$

which results in that

$$\|\mathrm{E}(v[k]v^T[k+m])\|_F < \sqrt{(n+1)^2 G_1^2} = (n+1)G_1.$$

Let $G = (n+1)G_1 < \infty$. Then $\|\mathrm{E}(v[k]v^T[k+m])\|_2 < G$.

If $m \geq n+1$, then $k+m+j-1-t \geq k+n+1+j-1-(j-1) = k+n+1$ and $k+i-1-s \leq k+n+1-1-0 = k+n$. Hence, $\mathrm{E}(\eta[k+i-1-s]\eta[k+m+j-1-t]) = 0$, which leads to that $\mathrm{E}(\mathfrak{D}^{i-1}\eta[k]\mathfrak{D}^{j-1}\eta[k+m]) = 0$. Therefore, $\mathrm{E}(v[k]v^T[k+m]) = \mathbf{0}_{(n+1)\times(n+1)}$. ∎

Lemma 6: The standard white noise $\eta[k]$ has the following property:

$$\mathrm{E}\left(\mathfrak{D}^l\eta[k]\right) = 0, \quad l = 0, 1, 2, \ldots. \tag{44}$$

Proof: It is easy to see that $\mathrm{E}(\mathfrak{D}^0\eta[k]) = E(\eta[k]) = 0$. Assume that $\mathrm{E}(\mathfrak{D}^j\eta[k]) = 0$. Then it can be obtained that

$$\mathrm{E}\left(\mathfrak{D}^{j+1}\eta[k]\right) = \mathrm{E}\left(\mathfrak{D}^j\eta[k+1] - \mathfrak{D}^j\eta[k]\right)$$
$$= \mathrm{E}\left(\mathfrak{D}^j\eta[k+1]\right) - \mathrm{E}\left(\mathfrak{D}^j\eta[k]\right) = 0.$$

By the mathematic induction, it is proved that $\mathrm{E}(\mathfrak{D}^l\eta[k]) = 0, l = 0, 1, 2, \ldots$ ∎

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