Automatica 48 (2012) 1415–1422

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Necessary and sufficient conditions for containment control of networked multi-agent systems[☆]

Huiyang Liu[a], Guangming Xie[a,b,1], Long Wang[a]

[a] The State Key Laboratory of Turbulence and Complex Systems, Center for Systems and Control, College of Engineering, Peking University, Beijing,100871, China

[b] School of Electrical and Electronics Engineering, East China Jiaotong University, Nanchang 330013, China

ARTICLE INFO

Article history:
Received 22 December 2010
Received in revised form
17 July 2011
Accepted 9 December 2011
Available online 5 June 2012

Keywords:
Containment control
Multi-agent systems
Sampled control
Directed network topology
Cooperative control

ABSTRACT

In this paper, containment control problems for networked multi-agent systems with multiple stationary or dynamic leaders are investigated. The topologies that characterize the interaction among the leaders and the followers are directed graphs. Necessary and sufficient criteria which guarantee the achievement of containment control are established for both continuous-time and sampled-data based protocols. When the leaders are stationary, the convergence for continuous-time protocol (sampled-data based protocol) is completely dependent on the topology structure (both the topology structure and the size of sampling period). When the leaders are dynamic, the convergence for continuous-time protocol (sampled-data based protocol) is completely dependent on the topology structure and the gain parameters (the topology structure, the gain parameters, and the size of sampling period). Moreover, the final states of all the followers are exclusively determined by the initial values of the leaders and the topology structure. In the stationary leaders case, all the followers will move into the convex hull spanned by the leaders, while in the dynamic leaders case, the followers will not only move into the convex hull but also move with the leaders with the same velocity. Finally, all the theoretical results are illustrated by numerical simulations.

© 2012 Elsevier Ltd. All rights reserved.

1. Introduction

As a class of engineered systems, cooperative control can be characterized as a collection of decision-making autonomous agents with processing capabilities, locally sensed information, and limited inter-agent communications, all seeking to achieve a collective objective. Areas of research that are related to cooperative control include 'distributed control systems', 'multi-agent systems', 'networked systems', as well as 'swarming'. In science, the researchers try to understand why certain types of collective behavior of swarms emerge and to design the local interaction rules for swarms so that the swarm exhibits a desired collective behavior. Investigations into fundamental aspects of swarm behavior have been widely reported, e.g., consensus (Jadbabaie, Lin, & Morse, 2003; Moreau, 2005; Olfati-Saber & Murray, 2004; Ren & Beard, 2005), formation control (Egerstedt & Hu, 2001; Lin, Francis, & Maggiore, 2005; Porfiri, Roberson, & Stilwell, 2007), rendezvous (Cortes, Martinez, & Bullo, 2006; Dimarogonas & Kyriakopoulos, 2007), flocking (Olfati-Saber, 2006; Tanner, Jadbabaie, & Pappas, 2007), coverage control (Gupta & Kumar, 1998; McNew, Klavins, & Egerstedt, 2007) and so on.

Recently, containment control strategies have been investigated a lot. The study of containment control is motivated by numerous natural phenomena and potential applications. For examples, the male silkworm moths will end up in the convex hull spanned by all the female silkworm moths by detecting pheromone released by females; a group of heterogenous agents moves from one target to another when only a portion of the agents is equipped with necessary sensors to detect the hazardous obstacles such that the agents who are not equipped will stay in a safety area formed by the equipped agents.

In the literature related to containment control problems, the interaction topologies are usually characterized by undirected graphs (Dimarogonas, Egerstedt, & Kyriakopoulos, 2006; Ji, Ferrari-Trecate, Egerstedt, & Buffa, 2008; Notarstefano, Egerstedt, & Haque, 2009, 2011). The interaction among different agents in physical systems may be directed due to heterogeneity and nonuniform communication/sensing powers. Therefore, some researchers have studied containment control problems where the interaction

☆ This work was supported by NSFC (Nos. 60774089, 10972003, 10926195, 10972002 and 60736022). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Giancarlo Ferrari-Trecate under the direction of Editor Ian R. Petersen.

E-mail addresses: huiyangliu@pku.edu.cn (H. Liu), xiegming@pku.edu.cn (G. Xie), longwang@pku.edu.cn (L. Wang).

[1] Tel.: +86 10 62765448; fax: +86 10 62765448.

0005-1098/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2012.05.010

topologies are characterized by directed graphs (Cao & Ren, 2009; Cao, Stuart, Ren, & Meng, 2010).

Inspired by the progress in the field, this paper tries to further investigate containment control problems with multiple stationary or dynamic leaders under directed topologies. The objective is containment of the followers in the convex hull of the leaders' positions. Some necessary and sufficient conditions will be established to guarantee the achievement of containment control for both continuous-time and sampled-data based protocols. When the leaders are stationary, the convergence for continuous-time protocol is completely dependent on the topology structure, and the convergence for sampled-data based protocol is not only dependent on the topology structure but also dependent on the size of the sampling period. When the leaders are dynamic, the convergence is dependent on the topology structure and gain parameters for the continuous-time protocol, and the topology structure, gain parameters and the size of the sampling period for the sampled-data based protocol. Moreover, we will show that the final states of all the followers are determined by the leaders' initial values and the topology structure. In the stationary leaders case, all the followers will move into the convex hull spanned by the leaders, while in the dynamic leaders case, the followers will not only move into the convex hull but also move with the leaders with the same velocity.

The outline of this paper is as follows: we next establish some of the basic notations that will be used in the paper. In Section 2, we list some basic definitions and results. We provide stability analysis of the underlying network model with multiple stationary leaders in Section 3 and dynamic leaders in Section 4, respectively. In Section 5, we give some simulation results. Then, we give a conclusion in Section 6.

Notations: let I_n be the $n \times n$ identity matrix and $\mathbf{1}_n = [1, \dots, 1]^T$ with dimension n. $\mathbb{i}$ is the imaginary unit. Given a complex number $\lambda \in \mathbb{C}$, $Re(\lambda)$, $Im(\lambda)$ and $|\lambda|$ are the real part, the imaginary part and the modulus of λ, respectively. $\Lambda(A)$ and $\rho(A)$ denote the eigenvalue set and the spectral radius of A, respectively. $\mathrm{diag}\{a_1, \dots, a_n\}$ and $\mathrm{triag}\{a_1, \dots, a_n\}$ represent the diagonal and upper triangular matrices, respectively.

2. Preliminaries

In this section, some basic concepts and results are introduced. For more details, please refer to Biggs (1974), Horn and Johnson (1987), Ogata (1995) and Rockafellar (1972).

Let $\mathscr{G} = (\mathscr{V}(\mathscr{G}), \mathscr{E}(\mathscr{G}), \mathscr{A}(\mathscr{G}))$ be a weighted directed graph with the set of vertices $\mathscr{V}(\mathscr{G}) = \{v_1, \dots, v_n\}$, the set of edges $\mathscr{E}(\mathscr{G}) \subseteq \mathscr{V}(\mathscr{G}) \times \mathscr{V}(\mathscr{G})$, and a weighted adjacency matrix $\mathscr{A}(\mathscr{G}) = [a_{ij}]$. An edge of $\mathscr{G}$ is denoted by $e_{ij} = (v_j, v_i)$, where v_j is called the parent vertex of v_i and v_i the child vertex of v_j. The adjacency elements associated with the edges are positive, i.e., $e_{ij} \in \mathscr{E}(\mathscr{G}) \Leftrightarrow a_{ij} > 0$. The set of *neighbors* of node v_i is denoted by $\mathscr{N}_i = \{v_j \in \mathscr{V}(\mathscr{G}) : (v_j, v_i) \in \mathscr{E}(\mathscr{G}), j \neq i\}$. A directed *path* in a directed graph $\mathscr{G}$ is a sequence $v_{i_1}, \dots, v_{i_k}$ of vertices such that for $s = 1, \dots, k-1$, $(v_{i_s}, v_{i_{s+1}}) \in \mathscr{E}(\mathscr{G})$, and a *weak path*, either $(v_{i_s}, v_{i_{s+1}})$ or $(v_{i_{s+1}}, v_{i_s}) \in \mathscr{E}(\mathscr{G})$. A directed graph $\mathscr{G}$ is *strongly connected* if between every pair of distinct vertices v_i, v_j in $\mathscr{G}$, there is a directed path that begins at v_i and ends at v_j, and is *weakly connected* if any two vertices can be jointed by a weak path. A *strong component* of a directed graph is an induced subgraph that is maximal, subject to being strongly connected. Since any subgraph consisting of only a vertex is strongly connected, it follows that each vertex lies in a strong component. A *directed tree* is a directed graph, where every node, except the root, has exactly one parent. A *directed forest* is a directed graph consisting of one or more directed trees, no two of which have a vertex in common. A *directed spanning tree* (*directed spanning forest*) is a directed tree (directed forest), which consists of all the nodes and some edges in $\mathscr{G}$.

The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ of $\mathscr{G}$ is defined as: $l_{ij} = \sum_{k=1,k\neq i}^{n} a_{ik}$ for $i = j$, and $l_{ij} = -a_{ij}$ for $i \neq j$, $i, j = 1, \dots, n$. The *in-degree* and *out-degree* of node v_i are, respectively, defined as: $\deg_{in}(v_i) = \sum_{v_j\in\mathscr{N}_i} a_{ij}$ and $\deg_{out}(v_i) = \sum_{v_j\in\mathscr{N}_i} a_{ji}$. The *degree matrix* is an $n \times n$ matrix defined as $D = [d_{ij}]$, where $d_{ij} = \deg_{in}(v_i)$ for $i = j$, otherwise, $d_{ij} = 0$. Then the Laplacian matrix of the graph $\mathscr{G}$ can be written as: $L = D - \mathscr{A}(\mathscr{G})$.

Lemma 1 (Taussky, 1949). *A complex matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is nonsingular if A is irreducible and $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ for all i with the inequality strict for at least one i.*

Lemma 2 (Horn & Johnson, 1987, *Geršgorin Disc Theorem*). *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then all the eigenvalues of A are located in the union of n discs $\bigcup_{i=1}^{n} \left\{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j=1,j\neq i}^{n} |a_{ij}|\right\}$.*

Give a polynomial: $p(s) = \varphi_n s^n + \dots + \varphi_1 s + \varphi_0$, where $\varphi_i \in \mathbb{C}, i = 0, 1, \dots, n$. Applying the bilinear transformation $s = \phi(\sigma) = \frac{\sigma+1}{\sigma-1}$ to $p(s)$, we get a new polynomial: $r(\sigma) = (\sigma-1)^n p\left(\frac{\sigma+1}{\sigma-1}\right) = \rho_0 + \rho_1\sigma + \dots + \rho_n\sigma^n$, where $\rho_i \in \mathbb{C}, i = 0, 1, \dots, n$. Then, the Schur stability of $p(s)$ is equivalent to the Hurwitz stability of $r(\sigma)$. Let $\rho_i = \alpha_i + \mathbb{i}\beta_i$, $\alpha_i, \beta_i \in \mathbb{R}$. Substituting $\sigma = \mathbb{i}\omega$ into $r(\sigma)$, we have $r(\mathbb{i}\omega) = m(\omega) + \mathbb{i}n(\omega)$, where $m(\omega) = \alpha_0 - \beta_1\omega - \alpha_2\omega^2 + \cdots$, and $n(\omega) = \beta_0 + \alpha_1\omega - \beta_2\omega^2 - \cdots$. In order to determine the Hurwitz stability of $r(\sigma)$, the following theorem is introduced.

Lemma 3 (Ogata, 1995, *Hermite–Biehler Theorem*). *The polynomial $r(\sigma)$ is Hurwitz stable if and only if the related pair $m(\omega), n(\omega)$ is interlaced and $m(0)n'(0) - m'(0)n(0) > 0$.*

Definition 1 (Rockafellar, 1972). A set $K \subset \mathbb{R}^m$ is said to be convex if $(1-\gamma)x + \gamma y \in K$ whenever $x \in K, y \in K$ and $0 < \gamma < 1$. The convex hull of a finite set of points $x_1, \dots, x_n \in \mathbb{R}^m$ is the minimal convex set containing all points $x_i, i = 1, \dots, n$, denoted by $\mathrm{co}\{x_1, \dots, x_n\}$. Particularly, $\mathrm{co}\{x_1, \dots, x_n\} = \left\{\sum_{i=1}^{n} \alpha_i x_i | \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\right\}$.

Definition 2 (Cao & Ren, 2009). For an n-agent system, an agent is called a *leader* if the agent has no neighbor, and a *follower* if the agent has at least one neighbor.

In this paper, we assume that there are $m, m < n$, leaders and $n - m$ followers. Denote the set of leaders as $\mathscr{R}$ and the set of followers as $\mathscr{F}$.

3. Containment control with stationary leaders

Consider a group of single-integrator agents given by

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \dots, n. \tag{1}$$

where $x_i(t), u_i(t) \in \mathbb{R}^N$ are the state and the control input of the ith agent, respectively.

3.1. Convergence analysis for continuous-time protocol

The following protocol is proposed in Cao and Ren (2009):

$$\begin{aligned} u_i(t) &= \sum_{j\in\mathscr{F}\cup\mathscr{R}} a_{ij}(t)(x_j(t) - x_i(t)), \quad i \in \mathscr{F} \\ u_i(t) &= 0, \quad i \in \mathscr{R}. \end{aligned} \tag{2}$$

Then the agent dynamics can be summarized as: $\dot{x}(t) = -(L \otimes I_N)x(t)$, where L is the Laplacian matrix. According to the definitions of leader and follower, L can be partitioned as:

$$\begin{bmatrix} L_{\mathscr{F}\mathscr{F}} & L_{\mathscr{F}\mathscr{R}} \\ \mathbf{0}_{m\times(n-m)} & \mathbf{0}_{m\times m} \end{bmatrix}, \tag{3}$$

where $L_{\mathscr{F}\mathscr{F}} \in \mathbb{R}^{(n-m)\times(n-m)}$ and $L_{\mathscr{F}\mathscr{R}} \in \mathbb{R}^{(n-m)\times m}$.

Lemma 4. *$L_{\mathscr{F}\mathscr{F}}$ is invertible if and only if the directed graph $\mathscr{G}$ has a directed spanning forest.*

Proof. Sufficiency. Given a directed graph $\mathscr{G}$. Let $\mathscr{G}_{\mathscr{F}} = (\mathscr{F}(\mathscr{G}_{\mathscr{F}}), \mathscr{E}(\mathscr{G}_{\mathscr{F}}), \mathscr{A}(\mathscr{G}_{\mathscr{F}}))$ be the induced subgraph corresponding to the follower set. The sufficiency is proved through the following two steps.

Step 1: $\mathscr{G}_{\mathscr{F}}$ is weakly connected. According to the results in Xiao and Wang (2006), the Laplacian matrix of the followers' subgraph has the following form:

$$L(\mathscr{G}_{\mathscr{F}}) = \begin{bmatrix} L_{11} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L_{22} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & L_{mm} & \mathbf{0} \\ L_{m+1,1} & L_{m+1,2} & \cdots & L_{m+1,m} & L_{m+1,m+1} \end{bmatrix},$$

where $L_{ii}, i = 1, \ldots, m$ is the Laplacian matrix of the ith strong component $\mathscr{G}_i$, and all the eigenvalues of $L_{m+1,m+1}$ have positive parts.

From the partition of L, it follows that $L_{\mathscr{F}\mathscr{F}} = L(\mathscr{G}_{\mathscr{F}}) + \Delta_{\mathscr{G}_{\mathscr{F}}}$, where $\Delta_{\mathscr{G}_{\mathscr{F}}} = \mathrm{diag}\{\Delta_1, \ldots, \Delta_m, \Delta_{m+1}\}$ is a diagonal matrix with nonnegative entries. According to the definition of the directed spanning forest, we know that $\Delta_i \neq 0, i = 1, \ldots, m$. Otherwise, we assume $\Delta_i = 0$ for at least one $i = 1, \ldots, m$, then for the agents in strong component $\mathscr{G}_i$, there does not exist a directed path from any of the leaders to them. This is in contradiction with the definition of the directed spanning forest. From Lemma 1, we can obtain that $L_{ii} + \Delta_i, i = 1, \ldots, m$ are nonsingular. Noticing that all the eigenvalues of $L_{m+1,m+1}$ have positive real parts and Δ_{m+1} is a diagonal matrix with nonnegative entries, $L_{m+1,m+1} + \Delta_{m+1}$ is also nonsingular. Therefore, $L_{\mathscr{F}\mathscr{F}}$ is invertible.

Step 2: $\mathscr{G}_{\mathscr{F}}$ is disconnected. It contains some weakly connected components. Suppose the number of the weakly connected components is p. By performing a permutation on $L_{\mathscr{F}\mathscr{F}}$, we can rewrite it as:

$$L_{\mathscr{F}\mathscr{F}} = \begin{bmatrix} L(\mathscr{G}_{\mathscr{F}_1}) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & L(\mathscr{G}_{\mathscr{F}_2}) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & L(\mathscr{G}_{\mathscr{F}_p}) \end{bmatrix} + \begin{bmatrix} \Delta_{\mathscr{G}_{\mathscr{F}_1}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \Delta_{\mathscr{G}_{\mathscr{F}_2}} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \Delta_{\mathscr{G}_{\mathscr{F}_p}} \end{bmatrix}$$

with $L(\mathscr{G}_{\mathscr{F}_i}), i = 1, \ldots, p$ being the Laplacian of the ith weakly connected component. It is easy to see that the block diagonal structure comes from the fact that the nodes from the subset $\mathscr{G}_{\mathscr{F}_i}$ are disconnected with the nodes belonging to $\mathscr{G}_{\mathscr{F}_j}, j \neq i$, thus the agents in $\mathscr{G}_{\mathscr{F}_i}$ and $\mathscr{G}_{\mathscr{F}_j}$ do not interact each other. $\Delta_{\mathscr{G}_{\mathscr{F}_i}} \neq 0$ for $i = 1, \ldots, p$. Otherwise, $\Delta_{\mathscr{G}_{\mathscr{F}_i}} = 0$ means that the weakly connected component $\mathscr{G}_{\mathscr{F}_i}$ is not connected with the leader set. It contradicts the fact that $\mathscr{G}$ has a directed spanning forest.

For each weakly connected component, according to the analysis in Step 1, we know that $L_{\mathscr{G}_{\mathscr{F}_i}} + \Delta_{\mathscr{G}_{\mathscr{F}_i}}, i = 1, \ldots, p$ is nonsingular. Thus, $L_{\mathscr{F}\mathscr{F}}$ is invertible.

Necessity. If a directed graph $\mathscr{G}$ does not have a directed spanning forest, there exists at least one follower in the group who does not receive any information from the leader set. Suppose that the strong component which contains the follower is $\mathscr{G}_{\mathscr{F}_q}$, then the corresponding $\Delta_{\mathscr{F}_q} = 0$. This leads to the singularity of $L(\mathscr{G}_{\mathscr{F}_q}) + \Delta_{\mathscr{F}_q}$, and furthermore the singularity of $L_{\mathscr{F}\mathscr{F}}$. □

Remark 1. From the definitions of united directed spanning tree given in Cao and Ren (2009) and directed spanning forest, we know that the following two conditions are equivalent.

(1) A directed graph $\mathscr{G}$ has a united directed spanning tree.
(2) A directed graph $\mathscr{G}$ has a directed spanning forest.

However, a directed spanning forest is a definition given by almost all the monographs on algebraic graph theory. Therefore, in this paper, we use a directed spanning forest to characterize the interaction topologies.

The next theorem shows a necessary and sufficient condition that can guarantee the achievement of containment control.

Theorem 1. *Suppose that the directed network topology is fixed. Using (2) for (1), all followers will converge to the stationary convex hull spanned by the stationary leaders for arbitrary initial conditions if and only if the directed graph $\mathscr{G}$ contains a directed spanning forest. Furthermore, the final positions of the followers are given by $(-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_N)x_{\mathscr{R}}(0)$, where $x_{\mathscr{R}}(0)$ is the initial values of the leader set.*

Proof. Sufficiency. From Lemmas 2 and 4, we can obtain that all the eigenvalues of $L_{\mathscr{F}\mathscr{F}}$ have positive real parts. Then $\lim_{t\to\infty} e^{-L_{\mathscr{F}\mathscr{F}}t} = \mathbf{0}_{(n-m)\times(n-m)}$. Therefore,

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} e^{-\left(\begin{bmatrix} L_{\mathscr{F}\mathscr{F}} & L_{\mathscr{F}\mathscr{R}} \\ \mathbf{0}_{m\times(n-m)} & \mathbf{0}_{m\times m} \end{bmatrix}\otimes I_N\right)t} \begin{bmatrix} x_{\mathscr{F}}(0) \\ x_{\mathscr{R}}(0) \end{bmatrix}$$
$$= \begin{bmatrix} (-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_N)x_{\mathscr{R}}(0) \\ x_{\mathscr{R}}(0) \end{bmatrix},$$

where $x_{\mathscr{F}}(0)$ is the initial values of the follower set.

It is noted that $L_{\mathscr{F}\mathscr{F}} = L(\mathscr{G}_{\mathscr{F}}) + \Delta_{\mathscr{G}_{\mathscr{F}}} = (D(\mathscr{G}_{\mathscr{F}}) + \Delta_{\mathscr{G}_{\mathscr{F}}}) - \mathscr{A}(\mathscr{G}_{\mathscr{F}})$ and $D(\mathscr{G}_{\mathscr{F}}) + \Delta_{\mathscr{G}_{\mathscr{F}}}$ is a diagonal matrix. Define $\eta = \max_{i=1,\ldots,n-m}(d_{ii} + (\Delta_{\mathscr{G}_{\mathscr{F}}})_{ii})$. Rewrite $L_{\mathscr{F}\mathscr{F}}$ as $L_{\mathscr{F}\mathscr{F}} = \eta I_{n-m} - M$, where $M \in \mathbb{R}^{(n-m)\times(n-m)}$ is a nonnegative matrix. We can see that the entries of M and $\mathscr{A}(\mathscr{G}_{\mathscr{F}})$ are equal except the diagonal entries. Noticing that all the eigenvalues of $L_{\mathscr{F}\mathscr{F}}$ have positive real parts, we have $\rho(M) < \eta$. Letting $T = \frac{1}{\eta}M$, then T is a nonnegative matrix. It follows that $L_{\mathscr{F}\mathscr{F}}^{-1} = \frac{1}{\eta}(I_{n-m} - T)^{-1}$. Let us consider the infinite series $(I_{n-m} - T)^{-1} = \sum_{k=1}^{+\infty} T^k$. Since $L_{\mathscr{F}\mathscr{F}}$ is invertible, then the above series is convergent. We can make a conclusion that $L_{\mathscr{F}\mathscr{F}}^{-1} \geq 0$. Since all the entries of $L_{\mathscr{F}\mathscr{R}}$ are nonpositive, then all the entries of $-L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}$ are nonnegative. Therefore, all the entries of $-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_N$ are nonnegative.

From the property of the Laplacian matrix L that $\left(\begin{bmatrix} L_{\mathscr{F}\mathscr{F}} & L_{\mathscr{F}\mathscr{R}} \\ \mathbf{0}_{m\times(n-m)} & \mathbf{0}_{m\times m} \end{bmatrix}\mathbf{1}_n\right) \otimes \mathbf{1}_N = \mathbf{0}_{n\times 1} \otimes \mathbf{1}_N$, it follows $(L_{\mathscr{F}\mathscr{F}}\mathbf{1}_{n-m}) \otimes \mathbf{1}_N + (L_{\mathscr{F}\mathscr{R}}\mathbf{1}_m) \otimes \mathbf{1}_N = \mathbf{0}_{(n-m)\times 1} \otimes \mathbf{1}_N$, then $(-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_N)(\mathbf{1}_m \otimes \mathbf{1}_N) = \mathbf{1}_{n-m} \otimes \mathbf{1}_N$. From Definition 1, we know that all the followers converge to the convex hull spanned by the leaders.

Necessity. When the directed graph $\mathscr{G}$ does not have a directed spanning forest, there exists at least one follower such that it does not belong to any one of the directed trees. Then the follower will not receive any information from the leaders. It follows that the position of this follower is independent of the positions of the leaders $\forall t \geq 0$. This results in that containment may not be achieved. □

3.2. Convergence analysis for sampled-data based protocol

Here a sampled-data based protocol is induced from (2) by using period sampling technology and zero-order hold circuit. Let $h > 0$ be the sampling period and $k = 0, 1, \ldots$ the sampling instants, the obtained protocol is given as:

$$\begin{aligned} &u_i(t) = \sum_{j\in\mathscr{F}\cup\mathscr{R}} a_{ij}(x_j(kh) - x_i(kh)), \quad i \in \mathscr{F}, \\ &u_i(t) = 0, \quad i \in \mathscr{R}, \\ &\quad \text{if } t \in [kh, kh + h). \end{aligned} \tag{4}$$

By using the protocol (4) for (1), the network dynamics is summarized as: $x(kh + h) = ((I - hL) \otimes I_N)x(kh)$. By partition (3), the network dynamics can be written as:

$$x(kh + h) = (\Sigma \otimes I_N)x(kh), \quad k = 0, 1, 2, \ldots, \tag{5}$$

where $\Sigma = \begin{bmatrix} I_{n-m} - hL_{\mathscr{F}\mathscr{F}} & -hL_{\mathscr{F}\mathscr{R}} \\ \mathbf{0}_{m\times(n-m)} & I_m \end{bmatrix}$.

From the convergence analysis for the continuous-time protocol, we have the following corollary immediately.

Corollary 1. *Suppose that the directed network topology is fixed. Using (4) for (1), all followers will converge to the stationary convex hull formed by the stationary leaders for arbitrary initial conditions if and only if the directed graph $\mathscr{G}$ has a directed spanning forest and the sampled period satisfies $h < \min_{\lambda\in\Lambda(L_{\mathscr{F}\mathscr{F}})} \frac{2Re(\lambda)}{|\lambda|^2}$. Furthermore, the final positions of the followers are given by $(-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_N)x_{\mathscr{R}}(0)$.*

Proof. The verification is similar to that of Theorem 1 except for the Schur stability analysis instead. □

4. Containment control with dynamic leaders

Consider a group of double-integrator agents given by

$$\dot{x}_i(t) = v_i(t), \qquad \dot{v}_i(t) = u_i(t), \quad i = 1, \ldots, n, \tag{6}$$

where $x_i(t) \in \mathbb{R}^N$ and $v_i(t) \in \mathbb{R}^N$ are the state and velocity of the ith agent, and $u_i(t) \in \mathbb{R}^N$ is the control input.

4.1. Convergence analysis for continuous-time protocol

We propose the following protocol:

$$\begin{aligned} u_i(t) &= k_1 \sum_{j\in\mathscr{F}\cup\mathscr{R}} a_{ij}(x_j - x_i) \\ &\quad + k_2 \sum_{j\in\mathscr{F}\cup\mathscr{R}} a_{ij}(v_j - v_i), \quad i \in \mathscr{F} \\ u_i(t) &= 0, \quad i \in \mathscr{R}, \end{aligned} \tag{7}$$

where $k_1, k_2 > 0$ are gain parameters to be designed.

Denote $\xi(t) = [\xi_1^T(t), \ldots, \xi_n^T(t)]^T$, where $\xi_i(t) = [x_i^T(t), v_i^T(t)]^T$. System (6) with protocol (7) can be summarized: $\dot{\xi}(t) = ((I_n \otimes E - L \otimes F) \otimes I_N)\xi(t)$, where $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$and $F = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$. Dueto partition (3), the dynamics can be rewritten as:

$$\dot{\xi}(t) = (\Phi \otimes I_N)\xi(t) \tag{8}$$

where

$$\Phi = \begin{bmatrix} L_1 & L_2 \\ \mathbf{0}_{2m\times(2n-2m)} & I_m \otimes E \end{bmatrix}, \quad \begin{aligned} L_1 &= I_{n-m} \otimes E - L_{\mathscr{F}\mathscr{F}} \otimes F, \\ L_2 &= -L_{\mathscr{F}\mathscr{R}} \otimes F. \end{aligned}$$

Lemma 5. *System (8) can achieve containment control asymptotically if and only if Φ has exactly $2m$ zero eigenvalues and all the other eigenvalues have negative real parts. Specifically, the final dynamics of leaders is given as: $\xi_{\mathscr{R}}(t) = \left(\left(I_m \otimes \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) \otimes I_N\right) \xi_{\mathscr{R}}(0)$, and the final dynamics offollowers is given as: $((-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2) \otimes I_N)\xi_{\mathscr{R}}(t)$.*

Proof. Sufficiency. From the expression of Φ, we know that all the eigenvalues of Φ can be obtained by solving the equation: $\det(sI_{2n-2m} - L_1) \cdot s^{2m} = 0$. It is easy to see that 0 is an eigenvalue of Φ with multiplicity $2m$. All the other eigenvalues of Φ are the same as that of L_1. Suppose all the eigenvalues of L_1 have negative real parts, we have $\lim_{t\to+\infty} e^{L_1 t} = \mathbf{0}_{(2n-2m)\times(2n-2m)}$.

Now, let us consider the solution of Eq. (8):

$$\begin{aligned} \xi(t) &= e^{(\Phi\otimes I_N)t}\xi(0) \\ &= \left(\begin{bmatrix} e^{L_1 t} & \Xi(t) \\ \mathbf{0}_{2m\times(2n-2m)} & I_m \otimes \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{bmatrix} \otimes I_N\right) \begin{bmatrix} x_{\mathscr{F}}(0) \\ x_{\mathscr{R}}(0) \end{bmatrix}, \end{aligned}$$

where $\Xi(t) = L_1^{-1}(e^{L_1 t} - I_{n-m} \otimes I_2)L_2 + L_1^{-2}(e^{L_1 t} - I_{n-m} \otimes I_2 - L_1 t)L_2(I_m \otimes E)$.

From the facts that $\lim_{t\to+\infty} e^{L_1 t} = \mathbf{0}_{(2n-2m)\times(2n-2m)}$ and $L_{\mathscr{F}\mathscr{F}} \otimes F = I_{n-m} \otimes E - L_1$, we have $\lim_{t\to+\infty} \Xi(t) = (-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2)\left(I_m \otimes \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right)$. Therefore,

$$\begin{aligned} \lim_{t\to+\infty} \xi(t) &= \begin{bmatrix} \left((-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2)\left(I_m \otimes \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right)\right) \otimes I_N \\ \left(I_m \otimes \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) \otimes I_N \end{bmatrix} \xi_{\mathscr{R}}(0) \\ &= \begin{bmatrix} ((-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2) \otimes I_N)\xi_{\mathscr{R}}(t) \\ \xi_{\mathscr{R}}(t) \end{bmatrix}. \end{aligned}$$

Similarly, from the proof of Theorem 1, we can obtain that $(-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2) \otimes I_N$ is nonnegative and $(-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}} \otimes I_2) \otimes I_N)((\mathbf{1}_m \otimes \mathbf{1}_2) \otimes \mathbf{1}_N) = (\mathbf{1}_{n-m} \otimes \mathbf{1}_2) \otimes \mathbf{1}_N$. Therefore, system (8) achieves containment control asymptotically.

Necessity. Suppose that the sufficient condition that Φ has exactly $2m$ zero eigenvalues and all the other eigenvalues have negative real parts does not hold. Noticing that Φ has at least $2m$ zero eigenvalues, the fact that the sufficient condition does not hold implies that Φ has either more than $2m$ zero eigenvalues or it has $2m$ zero eigenvalues but has at least another eigenvalue that has a positive real part. In either case, it can be verified that $\lim_{t\to\infty} e^{\Phi t}$ has a rank larger than $2m$. Note that containment control is reached asymptotically if and only if the states and velocities of all the followers are determined by the leaders' states and velocities and the number of leaders is m. As a result, the rank of $\lim_{t\to\infty} e^{\Phi t}$ cannot exceed $2m$. This results in a contradiction. □

Now we will give the main result of containment control with multiple dynamic leaders under continuous-time protocol.

Theorem 2. *Suppose that the directed network topology is fixed. Using (7) for (6), all followers will always converge to the convex hull spanned by the dynamic leaders as $t \to +\infty$ for any initial conditions $x_i(0), v_i(0), i \in \mathscr{R} \cup \mathscr{F}$, if and only if the directed graph $\mathscr{G}$ has a directed spanning forest and the gain parameters satisfy $\frac{k_2^2}{k_1} > \max_{\lambda\in\Lambda(L_{\mathscr{F}\mathscr{F}})} \frac{Im^2(\lambda)}{Re(\lambda)|\lambda|^2}$.*

Proof. According to Lemma 5, we only need to show that all the eigenvalues of L_1 have negative real parts. Let $\lambda_1, \ldots, \lambda_{n-m}$ denote the eigenvalues of $L_{\mathscr{F}\mathscr{F}}$. We can find an invertible matrix W such that $W^{-1}L_{\mathscr{F}\mathscr{F}}W = \text{triag}\{\lambda_1, \ldots, \lambda_{n-m}\}$. Then we have

$$\begin{aligned} (W^{-1} \otimes I_2)L_1(W \otimes I_2) &= I_{n-m} \otimes E - \text{triag}\{\lambda_1, \ldots, \lambda_{n-m}\} \otimes F \\ &= \text{triag}\left\{\begin{bmatrix} 0 & 1 \\ -k_1\lambda_1 & -k_2\lambda_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -k_1\lambda_{n-m} & -k_2\lambda_{n-m} \end{bmatrix}\right\}. \end{aligned}$$

The eigenvalues of L_1 can be obtained by solving the equation $\prod_{i=1}^{n-m} a_i(s) = 0$, where $a_i(s) = \det \begin{bmatrix} s & -1 \\ k_1\lambda_i & s + k_2\lambda_i \end{bmatrix}$, $i = 1, \ldots, n-m$.Noticing that all $a_i(s), i = 1, \ldots, n-m$ have the same form, we can analyze them uniformly. Let $\lambda \in \Lambda(L_{\mathscr{F}\mathscr{F}})$ represent the eigenvalues $\lambda_1, \ldots, \lambda_{n-m}$. We only need to determine Hurwitz stability of the polynomial: $a(s) = s^2 + k_2\lambda s + k_1\lambda$. Let $\lambda = Re(\lambda) + \mathbb{i}Im(\lambda)$, where $Re(\lambda) > 0$. Substituting $s = \mathbb{i}\omega$ into $a(s)$, we get $a(\mathbb{i}\omega) = -\omega^2 + k_2(Re(\lambda) + \mathbb{i}Im(\lambda))\mathbb{i}\omega + k_1(Re(\lambda) + \mathbb{i}Im(\lambda))$. Then $m(\omega) = -\omega^2 - k_2Im(\lambda)\omega + k_1Re(\lambda)$ and $n(\omega) = k_2Re(\lambda)\omega + k_1Im(\lambda)$.

From Lemma 3, the polynomial $a(s)$ is Hurwitz stable if and only if the following conditions hold:

(a) the polynomial $m(\omega)$ has two distinct roots $m_1 < m_2$;
(b) the interlaced condition holds, i.e., $m_1 < n_1 < m_2$, where n_1 is the unique root of the polynomial $n(\omega)$;
(c) $m(0)n'(0) - m'(0)n(0) > 0$.

Considering condition (a), equation $m(\omega) = 0$ has two distinct roots if and only if its discriminant is larger than zero. That is, $\Delta_m = k_2^2Im^2(\lambda) + 4k_1Re(\lambda) > 0$. It holds obviously.

By using the radical formula, we obtain two roots of $m(\omega) = 0$,

$$m_1 = \frac{-k_2Im(\lambda) - \sqrt{k_2^2Im^2(\lambda) + 4k_1Re(\lambda)}}{2} \quad \text{and}$$
$$m_2 = \frac{-k_2Im(\lambda) + \sqrt{k_2^2Im^2(\lambda) + 4k_1Re(\lambda)}}{2}.$$

The root of $n(\omega) = 0$ is $n_1 = -\frac{k_1Im(\lambda)}{k_2Re(\lambda)}$. According to condition (b), we have $\frac{-k_2Im(\lambda) - \sqrt{k_2^2Im^2(\lambda)+4k_1Re(\lambda)}}{2} < -\frac{k_1Im(\lambda)}{k_2Re(\lambda)} < \frac{-k_2Im(\lambda) + \sqrt{k_2^2Im^2(\lambda)+4k_1Re(\lambda)}}{2}$. It is equivalent to $\frac{k_2^2}{k_1} > \frac{Im^2(\lambda)}{Re(\lambda)|\lambda|^2}$.

From the expressions of $m(\omega)$ and $n(\omega)$, we have $m(0) = k_1Re(\lambda)$, $n(0) = k_1Im(\lambda)$, $m'(0) = -k_2Im(\lambda)$, and $n'(0) = k_2Re(\lambda)$. According to condition (c), we have $m(0)n'(0) - m'(0)n(0) = Re^2(\lambda)k_1k_2 + k_1k_2Im^2(\lambda) = k_1k_2|\lambda|^2 > 0$. □

4.2. Convergence analysis for sampled-data based protocol

We assume that each agent can only receive the information of states and velocities from its neighbors at sampling times. A sampled-data based protocol is induced from (7) by using periodic sampling technology and a zero-order hold circuit:

$$\begin{aligned} u_i(t) &= k_1 \sum_{j\in\mathscr{F}\cup\mathscr{R}} a_{ij}(x_j(kh) - x_i(kh)) \\ &\quad + k_2 \sum_{j\in\mathscr{F}\cup\mathscr{R}} a_{ij}(v_j(kh) - v_i(kh)), \quad i \in \mathscr{F} \\ u_i(t) &= 0, \quad i \in \mathscr{R}, \\ &\text{if } t \in [kh, kh+h),\ k = 0, 1, 2, \ldots. \end{aligned} \tag{9}$$

Discretizing Eq. (6) with sampling period h, we obtain:

$$\begin{cases} x_i(kh+h) = x_i(kh) + hv_i(kh) + \dfrac{1}{2}h^2u_i(kh), \\ v_i(kh+h) = v_i(kh) + hu_i(kh), \quad k = 0, 1, 2, \ldots, \end{cases} \tag{10}$$

where $x_i(kh), v_i(kh)$ represent the state and velocity of the ith agent at the kth sampling time.

Denote $\xi(kh) = [\xi_1^T(kh), \ldots, \xi_n^T(kh)]^T$, where $\xi_i(kh) = [x_i^T(kh), v_i^T(kh)]^T$. The network dynamics is summarized: $\xi(kh+h) = ((I_n \otimes G - L \otimes H) \otimes I_N)\xi(kh)$, where $G = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} \frac{1}{2}k_1h^2 & \frac{1}{2}k_2h^2 \\ k_1h & k_2h \end{bmatrix}$.According to partition (3), the system dynamics can be written as:

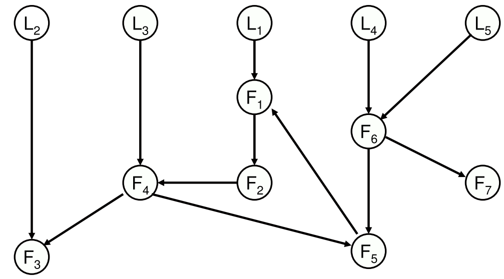

Fig. 1. A directed graph which has a directed spanning forest for a group of agents with five leaders and seven followers. $L_i, i = 1, \ldots, 5$ denote the leaders and $F_i, i = 1, \ldots, 7$ denote the followers.

$$\xi(kh+h) = (\Gamma \otimes I_N)\xi(kh), \quad k = 0, 1, 2, \ldots, \tag{11}$$

where

$$\Gamma = \begin{bmatrix} L_3 & L_4 \\ \mathbf{0}_{2m\times(2n-2m)} & I_m \otimes G \end{bmatrix}, \quad \begin{aligned} L_3 &= I_{n-m} \otimes G - L_{\mathscr{F}\mathscr{F}} \otimes H, \\ L_4 &= -L_{\mathscr{F}\mathscr{R}} \otimes H. \end{aligned}$$

Lemma 6. *System (11) can achieve containment control asymptotically if and only if Γ has eigenvalue 1 with multiplicity exactly $2m$ and all the other eigenvalues lie in the unit circle. Furthermore, the final dynamics of leaders is given as $\xi_{\mathscr{R}}(t) = \left(\left(I_m \otimes \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) \otimes I_N\right)\xi_{\mathscr{R}}(0)$ and the final dynamics of followers is given as $((-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2) \otimes I_N)\xi_{\mathscr{R}}(t)$.*

Proof. From the expression of Γ, we can obtain all the eigenvalues of Γ by solving the equation $\det(sI_{2n-2m} - L_3) \cdot (s-1)^{2m} = 0$. It is obvious 1 is an eigenvalue of Γ with multiplicity $2m$. All the other eigenvalues of Γ are the same as that of L_3. Suppose that all the eigenvalues of L_3 lie in the unit circle, then we have $\lim_{k\to\infty} L_3^k = \mathbf{0}_{(2n-2m)\times(2n-2m)}$.

Consider the solution of Eq. (11):

$$\begin{aligned} \xi(kh+h) &= (\Gamma^{k+1} \otimes I_N)\xi(0) \\ &= \left(\begin{bmatrix} L_3^{k+1} & S_k \\ \mathbf{0}_{2m\times(2n-2m)} & I_m \otimes G^{k+1} \end{bmatrix} \otimes I_N\right) \begin{bmatrix} \xi_{\mathscr{F}}(0) \\ \xi_{\mathscr{R}}(0) \end{bmatrix}, \end{aligned}$$

where $S_k = L_3^kL_4 + L_3^{k-1}L_4(I_m \otimes G) + \cdots + L_3L_4(I_m \otimes G^{k-1}) + L_4(I_m \otimes G^k)$. From the fact that $L_{\mathscr{F}\mathscr{F}} \otimes H = I_{n-m} \otimes G - L_3$, by some calculations, we have $S_k = L_3^{k+1}((L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2) - (L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes G^{k+1}$. Therefore,

$$\begin{aligned} &\lim_{t\to+\infty} \xi(kh+h) \\ &= \begin{bmatrix} (-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2)\left(I_m \otimes \begin{bmatrix} 1 & (k+1)h \\ 0 & 1 \end{bmatrix} \otimes I_N\right)\xi_{\mathscr{R}}(0) \\ \left(I_m \otimes \begin{bmatrix} 1 & (k+1)h \\ 0 & 1 \end{bmatrix} \otimes I_N\right)\xi_{\mathscr{R}}(0) \end{bmatrix}. \end{aligned}$$

For any $t \geq 0$, there exists a nonnegative integer k such that $t \in [kh, kh+h)$. Obviously, $t \to +\infty$ is equivalent to $k \to +\infty$. From periodic sampling technology and zero-order hold circuit theory, $\lim_{t\to+\infty}(x_i(t) - x_i(0) - tv_i(0)) = \lim_{k\to+\infty}(x_i(k) - x_i(0) - khv_i(0)) = 0, i \in \mathscr{R}$ and $\lim_{t\to+\infty} v_i(t) = \lim_{k\to+\infty} v_i(k), i \in \mathscr{R}$. Therefore, $\xi_{\mathscr{R}}(t) \to \left(I_m \otimes \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right)\xi_{\mathscr{R}}(0)$. It follows that $\xi_{\mathscr{F}}(t) \to ((-(L_{\mathscr{F}\mathscr{F}}^{-1}L_{\mathscr{F}\mathscr{R}}) \otimes I_2) \otimes I_N)\xi_{\mathscr{R}}(t)$.

Necessity. Suppose that the sufficient condition that Γ has exactly $2m$ 1 eigenvalues and all the other eigenvalues lie in the unit circle does not hold. Noticing that Γ has at least $2m$ 1

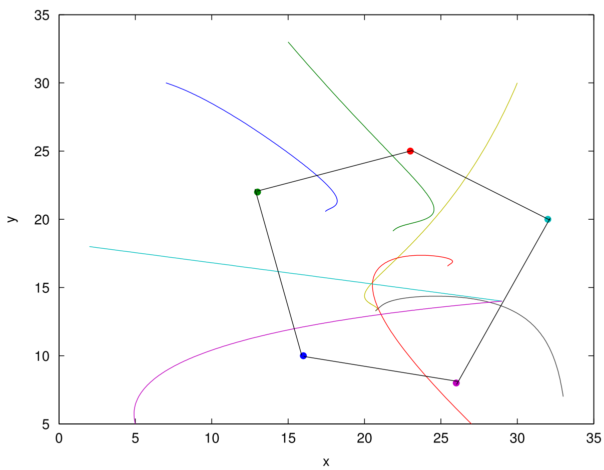

Fig. 2. Trajectories of all the agents under continuous-time protocol.

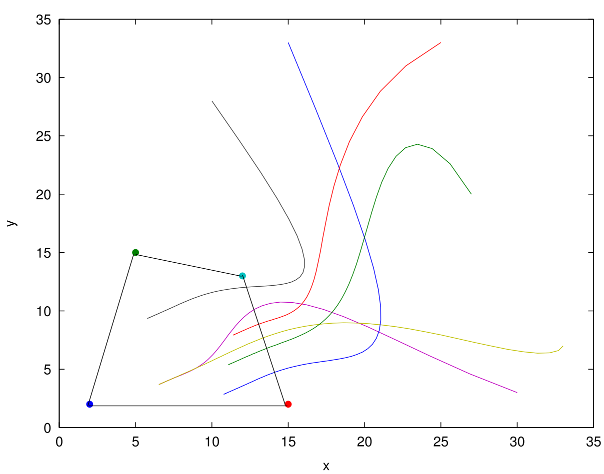

Fig. 3. Trajectories of all the agents under sampled-data based protocol.

eigenvalues, the fact that the sufficient condition does not hold implies that Γ has either more than $2m$ 1 eigenvalues or it has $2m$ 1 eigenvalues but has at least another eigenvalue that lies outside of the unit circle. In either case, it can be verified that $\lim_{k\to\infty}\Gamma^k$ has a rank larger than $2m$. Note that containment control is reached asymptotically if and only if the states and velocities of all the followers are determined by the leaders' states and velocities. As a result, the rank of $\lim_{k\to\infty}\Gamma^k$ cannot exceed $2m$. This results in a contradiction. □

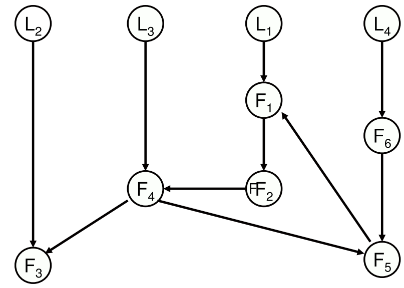

Fig. 5. A directed graph which has a directed spanning forest for a group of agents with four leaders and six followers. L_i, $i = 1, \ldots, 4$ denote the leaders and F_i, $i = 1, \ldots, 6$ denote the followers.

Theorem 3. *Suppose that the directed network topology is fixed. Using (9) for (6), all followers will converge to the dynamic convex hull spanned by the leaders as $k \to +\infty$ for any initial conditions $x_i(0), v_i(0), i \in \mathscr{R} \cup \mathscr{F}$, if and only if the directed graph $\mathscr{G}$ has a directed spanning forest and k_1, k_2, and the sampled period h are chosen from the set $S = S_\lambda \cap S_h$, where $S_\lambda = \cap_{\lambda\in\Lambda(L_{\mathscr{F}\mathscr{F}})}\{(h, k_1, k_2) : 8k_1 Im^2(\lambda) < (2Re(\lambda) - k_2|\lambda|^2 h)(2k_2 - k_1 h)^2|\lambda|^2\}$ and $S_h = \left\{(h, k_1, k_2) : h < \frac{2k_2}{k_1}\right\}$.*

Proof. According to Lemma 6, we only need to verify that all the eigenvalues of L_3 lie in the unit circle. By using the similar analysis, we can obtain $b(s) = s^2 + \left(\frac{1}{2}k_1\lambda h^2 + k_2\lambda h - 2\right)s + \frac{1}{2}k_1\lambda h^2 - k_2\lambda h + 1$ instead of the polynomial $a(s)$ in the proof of Theorem 2. However, for discrete-time systems, we should analyze Schur stability instead of Hurwitz stability of continuous-time systems. By applying the bilinear transformation $s = \frac{\sigma+1}{\sigma-1}$, we get $r(\sigma) = k_1\lambda h^2\sigma^2 + (2k_2 h\lambda - k_1\lambda h^2)\sigma + 4 - 2k_2\lambda h$. We rewrite it as $r_1(\sigma) = \sigma^2 + \left(\frac{2k_2}{k_1 h} - 1\right)\sigma + \frac{4\bar{\lambda}}{k_1|\lambda|^2h^2} - \frac{2k_2}{k_1 h}$. Then the polynomial $b(s)$ is Schur stable if and only if the polynomial $r_1(\sigma)$ is Hurwitz stable. The Hurwitz stability analysis of $r_1(\sigma)$ is similar to that of $a(s)$ in the proof of Theorem 2. Then we can obtain that all the eigenvalues of L_3 lie in the unit circle if and only if k_1, k_2 and h are chosen from the set S. □

5. Simulations

In this section, we present several simulation results to validate the previous theoretical results.

Example 1. When the interaction pattern is chosen as in Fig. 1. It can be noted that $\mathscr{G}$ has a directed spanning forest. The simulation result using (2) for (1) is shown in Fig. 2. We can see that all

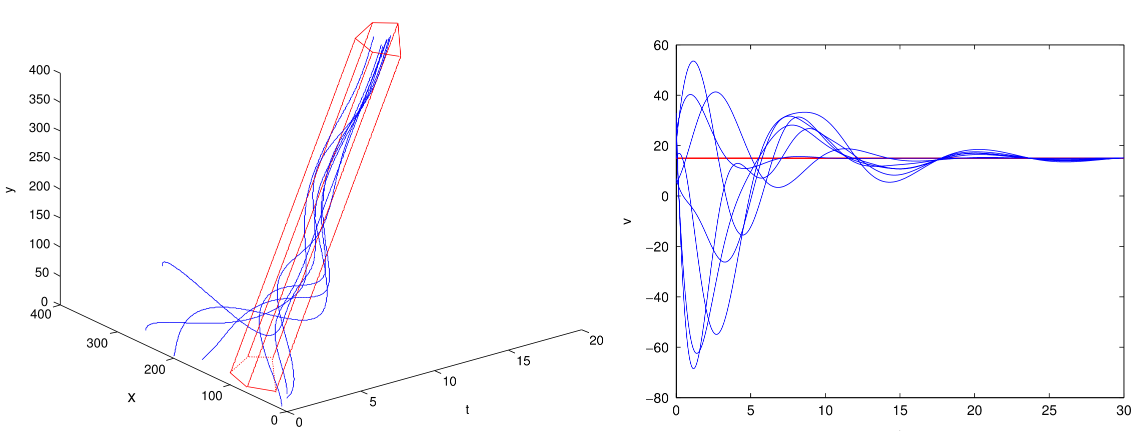

Fig. 4. State and velocity trajectories of all the agents under continuous-time protocol.

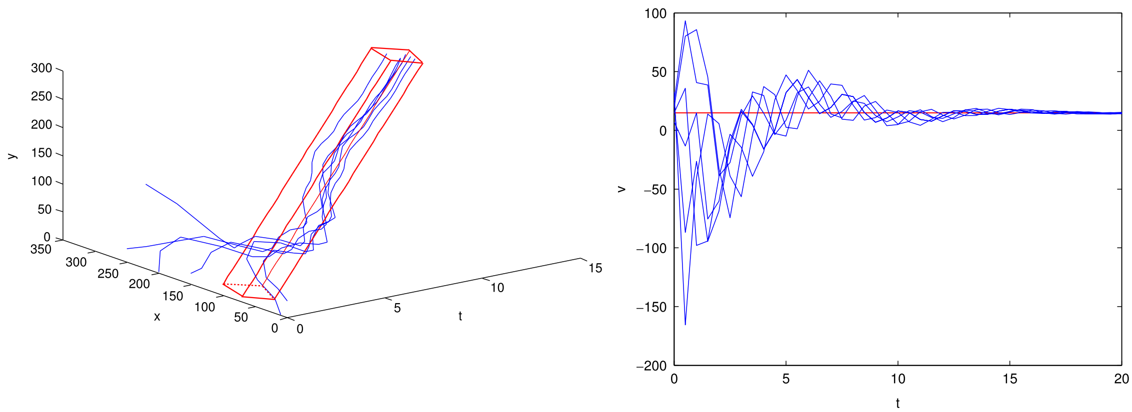

Fig. 6. State and velocity trajectories of all the agents under sampled-data based protocol.

followers ultimately converge to the convex hull formed by the stationary leaders and the final positions of the followers are constant.

Example 2. Still consider Fig. 1. From Theorem 2, we can obtain $\frac{k_2^2}{k_1} > 0.1172$. Now, we choose $k_1 = 0.5$ and $k_2 = 0.5$. Fig. 4 shows the state and velocity trajectories of all the agents. It is obvious that all followers ultimately converge to the convex hull formed by the dynamic leaders and the final positions of the followers are also dynamic. The velocities of all the followers will reach an agreement with the leaders and traveling in the convex hull spanned by the leaders with the same velocity.

Example 3. Suppose the interaction pattern is chosen as in Fig. 5. It can be noted that $\mathscr{G}$ has a directed spanning forest. By calculation, the sampled period should satisfy $h < 0.7418$. We choose $h = 0.1$. The simulation result using (4) for (1) is shown in Fig. 3. We can see that all followers ultimately converge to the convex hull formed by the stationary leaders and the final positions of the followers are constant.

Example 4. Still consider Fig. 5. From Theorem 3, we can obtain $(0.8852 - k_2h)(2k_2 - k_1h)^2 1.5661 > k_1, k_2h < 0.7418$ and $k_1h < 2k_2$. We choose $h = 0.5$, $k_1 = 1$ and $k_2 = 1.2$. Fig. 6 shows the state and velocity trajectories of all the agents. The conclusion is the same as the continuous-time case.

6. Conclusion

This paper has studied containment control of continuous-time multi-agent systems with multiple stationary or dynamic leaders under continuous-time and sampled-data based protocols. A directed graph has been used to represent information exchange among agents. We have established several necessary and sufficient conditions to guarantee the achievement of containment control such that all the followers move into the convex hull spanned by the stationary and dynamic leaders. We have also provided some simulations to illustrate the results. Our future work will concern containment control under switching topology.

References

Biggs, N. (1974). *Algebraic graph theory*. Cambridge, UK: Cambridge Univ. Press.

Cao, Y., & Ren, W. 2009. Containment control with multiple stationary or dynamic leaders under a directed interaction graph. In *Joint 48th IEEE conference on decision and control and 28th Chinese control conference*, pp. 3014–3019.

Cao, Y., Stuart, D., Ren, W., & Meng, Z. 2010. Distributed containment control for double-integrator dynamics: algorithms and experiments. In *2010 American control conference*, pp. 3830–3835.

Cortes, J., Martinez, S., & Bullo, F. (2006). Robust Rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Transactions on Automatic Control*, *51*(8), 1289–1298.

Dimarogonas, D.V., Egerstedt, M., & Kyriakopoulos, K.J. 2006. A leader-based containment control strategy for multiple unicycles. In *Proceedings of the 45th IEEE conference on decision and control* pp. 5968–5973.

Dimarogonas, D. V., & Kyriakopoulos, K. J. (2007). On the rendezvous problem for multiple nonholonomic agents. *IEEE Transactions on Automatic Control*, *52*(5), 916–922.

Egerstedt, M., & Hu, X. (2001). Formation constrained multi-agent control. *IEEE Transactions on Robotics and Automation*, *17*(6), 947–951.

Gupta, P., & Kumar, P.R. 1998. Critical power for asymptotic connectivity in wireless networks, *Stochastic analysis, control, optimization and applications*.

Horn, R. A., & Johnson, C. R. (1987). *Matrix analysis*. Cambridge, UK: Cambridge Univ. Press.

Jadbabaie, A., Lin, J., & Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, *48*(6), 988–1001.

Ji, M., Ferrari-Trecate, G., Egerstedt, M., & Buffa, A. (2008). Containment control in mobile networks. *IEEE Transactions on Automatic Control*, *53*(8), 1972–1975.

Lin, Z., Francis, B., & Maggiore, M. (2005). Necessary and sufficient graphical conditions for formation control of unicycles. *IEEE Transactions on Automatic Control*, *50*(1), 121–127.

McNew, J. M., Klavins, E., & Egerstedt, M. (2007). Solving coverage problems with embedded graph grammars. In *Hybrid systems: computation and control* (pp. 413–427). Springer-Verlag.

Moreau, L. (2005). Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, *50*(2), 169–182.

Notarstefano, G., Egerstedt, M., & Haque, M. 2009. Rendezvous with multiple, intermittent leaders. In *Joint 48th IEEE conference on decision and control and 28th Chinese control conference*, pp. 2733–2738.

Notarstefano, G., Egerstedt, M., & Haque, M. (2011). Containment in leader-follower networks with switching communication topologies. *Automatica*, *47*, 1035–1040.

Ogata, K. (1995). *Discrete-time control systems* (2nd ed.). Englewood Cliffs, NJ: Prentice-Hall.

Olfati-Saber, R. (2006). Flocking for multi-agent dynamic systems: algorithms and theory. *IEEE Transactions on Automatic Control*, *51*(3), 401–420.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Porfiri, M., Roberson, D. G., & Stilwell, D. J. (2007). Tracking and formation control of multiple autonomous agents: a two-level consensus approach. *Automatica*, *43*, 1318–1328.

Ren, W., & Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing integration topologies. *IEEE Transactions on Automatic Control*, *50*(5), 655–661.

Rockafellar, R. T. (1972). *Convex analysis*. New Jersey: Princeton University Press.

Tanner, H. G., Jadbabaie, A., & Pappas, G. J. (2007). Flocking in fixed and switching networks. *IEEE Transactions on Automatic Control*, *52*(5), 863–868.

Taussky, O. (1949). A recurring theorem on determinants. *American Mathematical Monthly*, *10*, 672–676.

Xiao, F., & Wang, L. 2006. Consensus behavior of agents in networked systems under general communication topologies. In *Proceedings of the 2006 IEEE international symposium on intelligent control*. pp. 862–867.

Huiyang Liu received the B.S. and M.S. degrees in Applied Mathematics from Zhengzhou University in 2004 and 2007, respectively. She is currently working toward the Ph.D. degree in complex systems at Peking University. Her current research interests are in the fields of coordination of multi-agent systems, consensus problems, containment control and swarm dynamics. Email: huiyangliu@pku.edu.cn.

Guangming Xie received his B.S. degrees in Applied Mathematics and Computer Science and Technology, his M. E. degree in Control Theory and Control Engineering, and his Ph.D. degree in Control Theory and Control Engineering from Tsinghua University, Beijing, China in 1996, 1998, and 2001, respectively. Then he worked as a postdoctoral research fellow in the Center for Systems and Control, Department of Mechanics and Engineering Science, Peking University, Beijing, China from July 2001 to June 2003. In July 2003, he joined the Center as a lecturer. Now he is an Associate Professor of Dynamics and Control. He is also a Guest Professor of East China Jiaotong University. He is an Editorial Advisory Board member of the International Journal of Advanced Robotic Systems and an Editorial Board Member of the Journal of Information and Systems Science. His research interests include hybrid and switched systems, networked control systems, and multi-agent systems, multi-robot systems, and bio-mimetic robotics. Email: xiegming@pku.edu.cn.

Long Wang was born in Xian, China on Feb. 13, 1964. He received his Bachelor's, Master's, and Doctorate degrees in Dynamics and Control from Tsinghua University and Peking University in 1986, 1989, and 1992, respectively. He has held research positions at the University of Toronto, the University of Alberta, Canada, and the German Aerospace Center, Munich, Germany. He is currently a Cheung Kong Chair Professor of Dynamics and Control, and the Director of the Center for Systems and Control of Peking University. He is also a Guest Professor at Wuhan University and Beihang University. He serves as the Editor-in-Chief of the International Journal of Information and Systems Science, an Editor of the Journal of Mechanical Science and Technology, an executive member of the Chinese Society of Artificial Intelligence and Chinese Society of Systems Simulation, a panel member of the Division of Information Science, National Natural Science Foundation of China, and a member of the IFAC (International Federation of Automatic Control) Technical Committee on Networked Systems. He is on the editorial boards of Science in China, Progress in Natural Science, Journal of Intelligent Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Control and Decision, Information and Control, Journal of Applied Mathematics and Computation, Journal of Intelligent and Robotic Systems, and International Journal of Computer Systems. His research interests are in the fields of complex networked systems, information dynamics, collective intelligence, and bio-mimetic robotics. Email: longwang@pku.edu.cn.