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Distributed fixed-time orientation synchronization with application to formation control

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***Abstract*— This paper studies the formation tracking control problem in the agents' local co-ordinate frame without any special assumption, such as positive definiteness, on the initial orientation matrix. When follower agents do not have knowledge of their absolute orientation with respect to a global reference frame, formation control is difficult and special assumptions on initial orientation matrix are generally imposed for orientation synchronization. To address this, a distributed fixed time orientation synchronization law is first presented, using only relative orientation measurements, which aligns the co-ordinate frames of agents almost globally in finite time and locally in fixed time. This law is then used in cascade with an acceleration command for formation tracking control in a leader-follower set-up. Global asymptotic convergence of formation tracking error is proved for displacement based formation. For bearing-only formation control, semi-global uniformly asymptotic stability is established. Simulations illustrate the applicability of the results.**

I. INTRODUCTION

Formation control of multi-agent systems is an active area of research because of its applicability to various domains. Formations can be used for environment monitoring, search and rescue operations, under-water marine exploration, coordinated satellite and aircraft control, etc. Depending on the sensing capabilities of agents, formation control schemes can be classified as position based, displacement based, distance based, or bearing based [1], [2]. Generally, for formation control, agents should have a common sense of orientation with respect to a global frame of reference [3]–[5], or have synchronized body-fixed orientation [1], [6], [7]. In the absence of absolute attitude information with respect to a global frame of reference, the formation control problem becomes extremely challenging. To tackle this issue, the approach adopted is to either implement formation control while simultaneously achieving orientation synchronization using relative orientation measurements [8]–[13], or to estimate orientations up-to a common frame of reference and use these estimates to achieve a desired formation [14]–[16].

However, existing orientation synchronization approaches use strict assumptions on initial orientation of agents. For example, the results in [8]–[10], [17], [18] use assumptions such as the angle corresponding to initial orientation matrix in axis-angle representation to belong to the interval $(-\pi/2, \pi/2)$, which corresponds to positive definiteness of the initial orientation matrix. In [11]–[13], the authors assume the difference between the maximum and minimum of all initial orientation angles of the agents to be less than π, restricting initial orientation angles to an interval of length less than π. In [19], although consensus control of rigid bodies has been discussed, leader-follower formation tracking was not addressed. A recent paper, [20], studied the finite time orientation consensus of rigid bodies using Morse-Lyapunov (M-L) functions. However, the applicability of this consensus algorithm to a leader-follower formation tracking control has not been explored. Motivated by this, the current paper proposes a fixed-time orientation consensus algorithm. Using our algorithm, attitude consensus can be achieved in a fixed time locally and in finite time almost globally. Fixed time convergence is appealing because the upper bound on the convergence time is independent of the initial states [21]. Also, we consider formation tracking control for agents modeled as double integrators, when the agents do not have their global orientation information to begin with. The control law for each agent uses only relative displacement and relative orientation measurements with respect to neighbors. We prove global asymptotic convergence of the error to zero for this case.

Bearing based formation control has garnered a lot of attention recently and is currently an active area of research [6], [8], [22]. With the advances in image processing techniques, it is easy to obtain accurate bearing measurements using passive sensors like camera, from pixel co-ordinates. The aim of bearing-only control is to achieve formation control using only bearing measurements, hence it is ideal when bearing-only sensors like camera [2] are available. In [8], [23] bearing-only formation stabilization was reported. But, it is important to investigate moving target formation for practical applications. In [22] bearing-only formation tracking control of multi-agent systems was studied. However, the control laws were in a global frame of reference. Though in [9] bearing-only formation control in agents' local frame of reference was studied, assumption of positive definiteness on the initial orientation matrix of the agents was imposed. We provide a rigorous proof of semi- global uniformly asymptotic convergence for formation tracking control of constant velocity leaders in the context of bearing-only formation. Additionally, we do not use any strict assumption, such as positive-definiteness of initial orientation matrix, for attitude/orientation synchronization. Hence, desired formations can be achieved for almost all initial orientation matrices. Furthermore, the control laws presented here do not require any communication between agents and use relative measurements with respect to neighbors only.

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II. PRELIMINARIES

A. Graph Theory

For n agents in $\mathbb{R}^3$ the communication topology is an undirected graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the node set and $\mathcal{E}$ is the edge set. The set containing neighbors of agent i is given by $\mathcal{N}_i := \{j|(i,j) \in \mathcal{E}\}$ and $|\mathcal{N}_i| = n_i$. The symmetric adjacency and laplacian matrices of $\mathcal{G}$ are $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$, respectively. $H \in \mathbb{R}^{m\times n}$ denotes the incidence matrix of the oriented graph. These definitions and other details can be found in [24].

B. Attitude kinematics

To represent the attitudes of rigid bodies, we use rotation matrices in real special orthogonal group, $SO(3) = \{Q \in \mathbb{R}^{3\times 3}|Q^TQ = I_3, det(Q) = 1\}$, where $I_3 \in \mathbb{R}^{3\times 3}$ is the identity matrix. $SO(3)$ representation of attitudes are preferred over other representations [25]. $SO(3)$ has a structure of a Lie group and the associated Lie algebra is represented by $\mathfrak{so}(3)$. For $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ and $X \in \mathfrak{so}(3)$, the operations $vec(\cdot)$ and $(\cdot)^\times$ are given by $x^\times = X = [0, -x_3, x_2; x_3, 0, -x_1; -x_2, x_1, 0]$ and $vec(X) = [x_1, x_2, x_3]^T$. The superscript i is used to represent the vector quantities in $\sum_i$. For example, angular velocity measured by agent i in its own frame is given by w_i^i. The rotational kinematics of the i^{th} agent is

$$\dot{Q}_i = Q_i(w_i^i)^\times. \tag{1}$$

Attitude is represented by the principal rotation vector $\Theta \in \mathbb{R}^3$. The set containing all principle vectors is a closed ball of radius π. Each vector inside the ball uniquely represents one axis of rotation and its magnitude represents the angle of rotation around that axis. For the points on surface, axis of rotation is the line joining the antipodal points and angle of rotation is π. Axis-angle and $SO(3)$ representations are related by the exponential map $Q = e^{\Theta^\times} \in SO(3)$ [17].

C. Agent dynamics

Agents are modeled as double integrators. The position and velocity in global ($\sum_w$) and body-fixed local frame of reference ($\sum_i$) of the agents are represented by $p_i, v_i \in \mathbb{R}^3$ and $p_i^i, v_i^i \in \mathbb{R}^3$, respectively. Agent dynamics is given by

$$\dot{p_i}^i = v_i^i; \quad \dot{v_i}^i = u_i^i, \tag{2}$$

where u_i^i is the control input to be designed. The relative displacement and bearing measurement obtained by i^{th} agent from a neighbor $j \in \mathcal{N}_i$ is $p_{ij}^i := p_j^i - p_i^i$ and $g_{ij}^i := \frac{p_{ij}^i}{\|p_{ij}^i\|}$. The measured quantities in $\sum_w$ and $\sum_i$ are related by Q_i, e.g. $g_{ij} = Q_i g_{ij}^i$, where g_{ij} is the measured bearing in $\sum_w$. The positions of all nodes in $\sum_w$ is the vector $p = [p_1^T, \ldots, p_n^T]^T$.

III. MAIN RESULTS

A. Fixed-time orientation synchronization

In this section a fixed-time attitude synchronization law, w_i^i, is designed using relative orientation measurements $Q_{ij} := Q_i^TQ_j$. Let Q_c be the synchronized orientation. The communication graph $\mathcal{G}$ is assumed to be fixed and connected. Let , $A_{ij} \in \mathbb{R}^3$ be a diagonal matrix with distinct positive diagonal entries satisfying $A_{ij}(1,1) > A_{ij}(2,2) > A_{ij}(3,3) \geq 1$ and $\frac{1}{p} \in (0, \frac{1}{2})$ be a positive real number. The orientation synchronization law for agent i, having attitude kinematics as in (1), is given by:

$$w_i^i = \frac{S_i}{\left(S_i^TS_i\right)^{(1/p)}} + \left(S_i^TS_i\right)^{(1/p)} S_i, \tag{3}$$

where $S_i = \sum_{j\in\mathcal{N}_i} vec(Q_i^TQ_jA_{ij} - A_{ij}Q_j^TQ_i)$.

Theorem 1: For agents whose attitude kinematics is given by (1), the orientation synchronization law (3) guarantees that the synchronization error $\|Q_i - Q_j\|, \forall (i,j) \in \mathcal{E}$ converges to zero in finite time almost globally and in fixed time locally.

Proof: Consider the Lyapunov candidate $V = \sum_{i=1}^n \sum_{j\in\mathcal{N}_i} \langle A_{ij}, (I_3 - Q_j^TQ_i)\rangle$, where $\langle A, B\rangle = trace(A^TB)$. It can be shown V is positive definite and has a set of non-degenerate critical points $E_c = \{Q_{ij}|Q_{ij} = I_3, \mathrm{diag}(-1,1,-1), \mathrm{diag}(1,-1,-1), \mathrm{diag}(-1,-1,1)\}$ on $SO(3)$. The critical point I_3 is a stable equilibrium which is also a minima. Out of other critical points in E_c, one is a maxima and others are saddle points having stable and unstable manifolds around them [25]. Differentiating V and using the fact that for a real square matrix B and vector w we have $trace(Bw^\times) = w^Tvec(B^T - B)$, and as each connected pair (i,j) is counted twice in $\dot{V}$, it is symmetric with respect to relative attitude Q_{ij}, we get

$$\begin{aligned}\dot{V} &= -\sum_{i=1}^n \sum_{j\in\mathcal{N}_i} trace(A_{ij}(\dot{Q}_j^TQ_i + Q_j^T\dot{Q}_i)) \\ &= -2\sum_{i=1}^n (w_i^i)^T \sum_{j\in\mathcal{N}_i} vec(Q_i^TQ_jA_{ij} - A_{ij}Q_j^TQ_i) \\ &= -2\sum_{i=1}^n (w_i^i)^T S_i. \end{aligned} \tag{4}$$

Substituting w_i^i from (3) in (4) we get

$$\dot{V} = -2\sum_{i=1}^n (S_i^TS_i)^{1-\frac{1}{p}} - 2\sum_{i=1}^n (S_i^TS_i)^{1+\frac{1}{p}}. \tag{5}$$

Now, $\dot{V}$ is negative semi-definite and hence asymptotic convergence to the largest invariant set E_c is established. By [26, Lemma 2] $\exists$ a neighborhood $\mathcal{S} \subset SO(3)$ given by $\mathcal{S} = \{Q \in SO(3)|Q(i,i) \geq 0, Q(i,j)Q(j,i) \leq 0\}$ about the identity (i.e $Q_j^TQ_i \approx I_3$) so that $2\sum_{i=1}^n S_i^TS_i \geq V$. holds inside $\mathcal{S}$. Using [27, Lemma 3, Lemma 4] and equation (5), we get

$$\begin{aligned}\dot{V} &\leq -\left(2\sum_{i=1}^n S_i^TS_i\right)^{1-\frac{1}{p}} - \frac{1}{(2n)^{\frac{1}{p}}}\left(2\sum_{i=1}^n S_i^TS_i\right)^{1+\frac{1}{p}} \\ \implies \dot{V} &\leq -(\alpha V^{1-\frac{1}{p}} + \beta V^{1+\frac{1}{p}}) \end{aligned} \tag{6}$$

where $\alpha = 1$ and $\beta = \frac{1}{(2n)^{1/p}}$. Hence, from equations (5), (6) and [28, Lemma 1] we conclude that for all initial states, except those on the critical points other than I_3, or on the stable manifolds around the other critical points (sets

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of measure zero), system will converge to the set $\mathcal{S}$ in a finite time and then, once inside $\mathcal{S}$, further converge to I_3 in a fixed time. Thus, fixed-time convergence is guaranteed when starting from inside $\mathcal{S}$ (local) while almost global convergence is guaranteed in finite time. ■

Remark 1: The proposed control law is continuous unlike discontinuous control laws generally used in the literature [29], [30], which lead to chattering. The *almost* global convergence is not due to the control law used. Rather, it is a property of continuous control laws on $SO(3)$ that no such law can provide global convergence results due to the existence of nowhere dense measure zero sets that are the complement of the domain of attraction [31].

B. Displacement based formation tracking control

This subsection presents a formation control law when agents can only sense their relative displacement, relative velocity, and relative orientation with respect to their neighbors. Followers do not need their orientation information with respect to any global frame of reference. It is assumed that the leader, moving at a constant velocity, v_c, with respect to $\sum_w$, is controlled independently. The goal is to achieve and maintain a formation by the follower agents. Let p^*_{ij} be the specified desired formation in $\sum_c$, where $\sum_c$ is the consensus frame of reference. The formation control problem can then be thought to achieve a desired formation specified by $p^{'}_{ij} = Q_c p^*_{ij}$ in the global co-ordinate frame $\sum_w$. We define $\bar{L} = \mathcal{L} \otimes I_3$ and $p^{'} = [(p_1)^{'}, \ldots, (p^{'}_n)^T] \in \mathbb{R}^{3n}$ where $p^{'}_i = Q_c p^*_i$. $\bar{L}$ can also be represented as the partitioned matrix $\bar{L} = \begin{bmatrix} \bar{L}_{ll} & \bar{L}_{lf} \\ \bar{L}_{fl} & \bar{L}_{ff} \end{bmatrix}$. We assume one leader, i.e. $n_l = 1$ and hence $\bar{L}_{ll} \in \mathbb{R}^{3\times3}, \bar{L}_{ff} \in \mathbb{R}^{3(n-1)\times3(n-1)}$, and $\bar{L}_{lf}, \bar{L}_{fl} \in \mathbb{R}^{3\times3(n-1)}$. We propose the following control law for the follower agents ($i \in \mathcal{V}_f$ where $\mathcal{V}_f$ is the set of all nodes that represent followers) :

$$u^i_i = \sum_{j\in\mathcal{N}} [k_p(p^i_{ij} - \frac{1}{2}(Q^T_i Q_j + I_3)p^*_{ij}) + k_v v^i_{ij}], \quad (7)$$

where $k_p, k_v > 0$ are the control gains.

Theorem 2: For agents modeled as in (2), the proposed control law (7) with the orientation synchronization law (3) in cascade will guarantee that the formation tracking error will go to zero globally asymptotically.

Proof: In global frame of reference ($\sum_w$) the control law (7) can be written as: $\dot{v}_i = \sum_{j\in\mathcal{N}_i} [k_p(p_{ij} - Q_c p^*_{ij}) + k_v v_{ij}] + g_i(t), \quad i \in \mathcal{V}_f$, where $g_i(t) := \sum_{j\in\mathcal{N}_i} k_p(Q_c - \frac{1}{2}(Q_j + Q_i))p^*_{ij}$, which, in compact form is

$$\begin{aligned} &\dot{v}_f = -k_p[\bar{L}_{fl} \quad \bar{L}_{ff}]p + k_p[\bar{L}_{fl} \quad \bar{L}_{ff}]p^{'} \\ &\quad - k_v[\bar{L}_{fl} \quad \bar{L}_{ff}]v + g \\ \iff &\dot{v}_f = -k_p \left(\bar{L}_{fl}p_1 + \bar{L}_{ff}p_f - [\bar{L}_{fl} \quad \bar{L}_{ff}]p^{'}\right) \\ &\quad - k_v \left(\bar{L}_{fl}v_1 + \bar{L}_{ff}v_f\right) + g, \end{aligned} \quad (8)$$

where $g = [g^T_2, \ldots, g^T_n]^T$.

Since $\mathcal{G}$ contains a spanning tree and the true position and orientation is known to the root node, i.e. , the leader 1, from [15, Theorem 3], it follows that the network is localizable. The desired positions of the followers can be given as

$$\hat{p}_f = -\bar{L}^{-1}_{ff}\bar{L}_{fl}p_1 + \bar{L}^{-1}_{ff}[\bar{L}_{fl} \quad \bar{L}_{ff}]p^{'}. \quad (9)$$

The tracking error is $\delta_{p_f} = p_f(t) - \hat{p}_f(t) = p_f(t) + \bar{L}^{-1}_{ff}\bar{L}_{fl}p_1 - \bar{L}^{-1}_{ff}[\bar{L}_{fl} \quad \bar{L}_{ff}]p^{'}$ and velocity error is $\delta_{v_f} = \dot{\delta}_{p_f} = v_f(t) + \bar{L}^{-1}_{ff}\bar{L}_{fl}v_c$. Hence, from (9) it follows that $\dot{v}_f(t) = -k_p\bar{L}_{ff}\delta p_f - k_v\bar{L}_{ff}\delta v_f + g$ and $\dot{\delta}_{v_f} = \dot{v}_f(t) + \bar{L}^{-1}_{ff}\bar{L}_{fl}\dot{v}_c = \dot{v}_f(t)$. The error dynamics is thus

$$\begin{bmatrix} \dot{\delta}_{p_f} \\ \dot{\delta}_{v_f} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -k_p\bar{L}_{ff} & -k_v\bar{L}_{ff} \end{bmatrix} \begin{bmatrix} \delta_{p_f} \\ \delta_{v_f} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ g \end{bmatrix}. \quad (10)$$

Due to the existence of a spanning tree, $\bar{L}_{ff}$ is invertible. Let the eigenvalues of $\bar{L}_{ff}$ be $\lambda_i > 0, i = 1, \ldots, 3(n-1)$. It can be shown that eigenvalues of the state matrix of the system in (10) are $\lambda = (-k_v\lambda_i \pm \sqrt{k_v^2\lambda_i^2 - 4k_p\lambda_i})/2$ which have negative real parts. Now, $g_i(t) = \sum_{j\in\mathcal{N}_i} k_p(0.5(E_i+E_j))p^*_{ij}$ with $E_i = Q_i - Q_c, \implies ||g_i(t)|| \le 0.5k_p(n_i||E_i|| + \sum_{j\in\mathcal{N}_i} ||E_j||)$. From Theorem 1 it is known that E_i will go to zero in finite time $\forall i$, hence it follows that the input $g(t)$ will be bounded and will go to zero in finite time. As the forced system (10) is linear in the states and input, it is globally Lipschitz. With the globally exponentially stable nominal system, the forced system is input to state stable (I.S.S) [32, Lemma 4.6]. Hence, the system is globally asymptotically stable. ■

C. Bearing-only formation tracking control

In this subsection, bearing-only formation tracking control is considered. The control law used is

$$u^i_i = k_p \sum_{j\in\mathcal{N}_i} (g^i_{ij} - \frac{1}{2}(I_3 + Q^T_i Q_j)g^*_{ij}) + k_v \sum_{j\in\mathcal{N}_i} \dot{g}^i_{ij}. \quad (11)$$

The stability analysis for the above control law (11) in cascade with (3) is carried out. We do not assume that the initial orientation matrix is positive definite, as in [9], so the bearing-only formation control law (11) in cascade with (3) works for more general initial configurations. We only assume that the graph $\mathcal{G}$ is fixed and connected, and the bearing constraints $g^{'}_{ij} = Q_c g^*_{ij}$ in $\sum_w$ with the constant velocity leaders v_c ensure unique target formation [33]. Substituting the control law (11) in (2), the closed loop dynamics in global frame of reference $\sum_w$ is $\dot{p}_i = v_i, \ i \in \mathcal{V}_f; \dot{v}_i = \dot{Q}_i v_i + Q_i u^i_i = k_p \sum_{j\in\mathcal{N}_i}(g_{ij} - Q_c g^*_{ij}) + k_v \sum_{j\in\mathcal{N}_i} \dot{g}_{ij} + h(t)$, where $h_i(t) = Q_i(w^i_i)^\times v_i - k_v \sum_{j\in\mathcal{N}_i} \dot{Q}_i g^i_{ij} + k_p \sum_{j\in\mathcal{N}_i}(Q_c - 0.5(Q_i+Q_j))g^*_{ij}$. Let, $g^{'}_i = Q_c g^*_i, g^{'} = [(g^{'}_1)^T, \ldots, (g^{'}_m)^T]^T$ and $g = [g^T_1, \ldots, g^T_m]^T$. The closed loop system above can be written in compact form as

$$\dot{p} = v; \dot{v} = M(k_p(g - g^{'}) + k_v\dot{g}) + h, \quad (12)$$

where $M = -\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{3n_f} \end{bmatrix} \bar{H}^T$ and $h = [0_{nl}, h^T_1, \ldots, h^T_{n_f}]$ with $\bar{H} = H \otimes I_3$. Let, $\delta_p := p - p^{'}, \delta_v := v - v^{'}, \delta_g := g - g^{'}$ where $p^{'}, v^{'} \in \mathbb{R}^{3n}$ be desired velocity and position

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in $\sum_c$ and $\mathcal{X} := [\delta_p, \delta_v, \delta_g]$. Let, $\tilde{E} = [E_1, \dots, E_n]$ with $E_i = Q_i - Q_c$. The error system is expressed by

$$\dot{\mathcal{X}} = [\dot{\delta}_p^T, \dot{\delta}_v^T, \dot{\delta}_g^T]^T \implies \dot{\mathcal{X}} = F(\mathcal{X}, \dot{\delta}_g(t)) + G(\mathcal{X}, \tilde{E}) \quad (13)$$

where, $F(\mathcal{X}, \dot{\delta}_g(t)) = \begin{bmatrix} 0_{3n} & I_{3n} & 0_{3n\times 3n} \\ 0_{3n} & 0_{3n} & k_p M \\ 0_{3m} & 0_{3m} & 0_{3m\times 3m} \end{bmatrix} \mathcal{X} + \begin{bmatrix} 0_{3n} \\ k_v M \dot{\delta}_g(t) \\ \dot{\delta}_g(t) \end{bmatrix}$ is the nominal system and $G(\mathcal{X}, \tilde{E}) = [0_{3n}^T, h^T, 0_{3m}^T]^T \in \mathbb{R}^{6n+3m}$ is the perturbed one.

Theorem 3: The perturbed system (13) is semi-globally uniformly asymptotically stable.

Proof: We will use [5, Lemma 2.1] for the proof and verify the conditions therein to be satisfied. Consider the Lyapunov candidate $V(\mathcal{X}, t) = k_p e^T (g - g') + \frac{1}{2}\delta_v^T \delta_v$. By [22, Lemma 2], V is positive definite and using [32, Lemma 4.3] first sub-condition of [C1] in [5, Lemma 2.1] is satisfied. By [22, Theorem 3] the nominal system in equation (13) is globally uniformly asymptotically stable, and hence the second sub-condition of [C1] holds.

Now, $\frac{\partial V}{\partial \mathcal{X}} = [k_p \delta_g^T \bar{H}^T, \delta_v^T, k_p(\delta_p^T \bar{H}^T + (e')^T)]$ where $e' = \bar{H}p'$ and hence $\left\|\frac{\partial V}{\partial \mathcal{X}}\right\| \leq \sqrt{k_p^2 \|\delta_g\|^2 \left\|\bar{H}\right\|^2 + \|\delta_v\|^2 + k_p^2(\|\delta_p\|^2 \left\|\bar{H}\right\|^2 + \|e'\|^2)}$. Here, $\|\delta_g\| \leq ||g|| + \left\|g'\right\| \leq \sqrt{\sum_{k=1}^m ||g_k||^2} + \sqrt{\sum_{k=1}^m ||g_k'||^2} \leq 2\sqrt{m} \quad (\because ||g_k|| \leq 1 \quad \forall k = 1, \dots m)$ and hence $\left\|\frac{\partial V}{\partial \mathcal{X}}\right\| \leq \sqrt{\|\delta_v\|^2 + k_1 \|\delta_p\|^2 + k_2^2}$, where $k_1 = k_p^2 \left\|\bar{H}\right\|^2$ and $k_2 = 4mk_p^2 \left\|\bar{H}\right\|^2 + k_p^2 \left\|e'\right\|^2$. Let $k_3 = \max_{||\mathcal{X}|| \leq \zeta} (||\delta_v||^2 + k_1 ||\delta_p||^2)$. Then $\left\|\frac{\partial V}{\partial \mathcal{X}}\right\| \leq c_2$, where $c_2 = \sqrt{k_3 + k_2^2}$ and hence fourth sub-condition of [C1] is also satisfied. We assume the initial position and velocity is inside an arbitrary large set which is given a priori. Hence, $\mathcal{X}(t_0) \in E := \{\mathcal{X} : ||\delta_p|| \leq B_p, ||\delta_v|| \leq B_v\}$. So, $\mathcal{X} \in E \implies ||\mathcal{X}|| = \sqrt{||\delta_p|| + ||\delta_v|| + ||\delta_g||} \leq \sqrt{B_p^2 + B_v^2 + 4m} = k_4$. Now when $0 < \zeta \leq ||\mathcal{X}|| \leq k_4 \implies ||\delta_v||^2 \geq \zeta^2 - (\delta_p^2 + 4m) \geq \zeta^2 - (B_p^2 + 4m) := k_5$. Let $\zeta > \sqrt{B_p^2 + 4m} \implies k_5 > 0$.

Next, $c_1 V \geq c_1 \frac{||\delta_v||^2}{2} > \frac{c_1 k_5}{2}$, and $\left\|\frac{\partial V}{\partial \mathcal{X}}\right\| ||\mathcal{X}|| \leq \sqrt{||\delta_v||^2 + k_1||\delta_p||^2 + k_2^2}\sqrt{||\delta_p||^2 + ||\delta_v||^2 + ||\delta_g||^2} \leq k_6||\mathcal{X}||^2 \leq k_6 k_4^2$, where $k_6 = \max(1, \sqrt{k_1}, k_2, \sqrt{4m})$. Hence, $\left\|\frac{\partial V}{\partial \mathcal{X}}\right\| ||\mathcal{X}|| - c_1 V \leq k_6 k_4^2 - \frac{c_1 k_5}{2} := k_7$. Taking, $c_1 > \frac{2k_6 k_4^2}{k_5}$, k_7 will be less than zero. Thus third sub-condition of [C1] is satisfied. Further, $\tilde{E}$ will always be bounded and will go to zero in finite time (from Theorem 1). Also, integral of norm of any bounded function over a finite interval is finite. Hence, the condition in [C2] of [5, Lemma 2.1] will be satisfied. Now, $||G(\mathcal{X}, \tilde{E})|| = ||h||$, where $h_i = (E_i + Q_c)(w_i^i)^\times v_i - k_v \sum_{j\in\mathcal{N}_i} E_i g_{ij}^i - k_p \sum_{j\in\mathcal{N}_i}(E_i + E_j) g_{ij}'$. Let, $\alpha_i = (S_i^T S_i)^{\frac{1}{p}}$. Then, as $w_i^i = \frac{S_i}{\alpha_i} + \alpha_i S_i \implies ||(w_i^i)^\times|| \leq (\alpha_i + \frac{1}{\alpha_i})||(S_i)^\times||$. Now, $S_i = \sum_{j\in\mathcal{N}_i} vec((E_i^T + Q_c^T)(E_j + Q_c)A_{ij} - A_{ij}(E_j^T + Q_c^T)(E_i + Q_c))$

TABLE I

FIRST SET OF INITIAL ORIENTATION CONFIGURATION

Agent (i)	Axis of Rotation ($\Theta_i(t_0)$)	Corresponding Angle (degree)
1	$[0.5, -0.15, -0.68]^T$	49.1176
2	$[0, 0, 0]^T$	0
3	$[-0.15, -0.07, -0.17]^T$	13.5949
4	$[-0.57, -0.05, 1.51]^T$	92.5198
5	$[-0.37, -0.35, -0.17]^T$	30.7642
6	$[0, 0, -2.15]^T$	123.1859
7	$[0, 0, 0]^T$	0
8	$[-0.59, -2, 1.09]^T$	134.8120

$= \sum_{j\in\mathcal{N}_i} vec((E_i^T E_j A_{ij} - A_{ij} E_j^T E_i) + (Q_c^T E_j A_{ij} - A_{ij} E_j^T Q_c) + (E_i^T Q_c A_{ij} - A_{ij} Q_c^T E_i))$. Hence, $||(S_i)^\times|| \leq 2n_i ||A_{ij}|| ||\tilde{E}|| (||\tilde{E}|| + 2) \implies ||(w_i^i)^\times|| \leq ||\tilde{E}|| \beta_i(||\tilde{E}||)$, where β_i is a function of $||\tilde{E}||$ given by $\beta_i(||\tilde{E}||) = 2n_i(\alpha_i + \frac{1}{\alpha_i})||A_{ij}||(||\tilde{E}|| + 2)$. Also, $||v_i|| \leq ||v_c|| + ||\delta_{v_i}|| \leq ||v_c|| + ||\mathcal{X}||$, $||g_{ij}^i|| \leq 1$, $||g_{ij}'|| \leq 1$ and $||\dot{E}_i|| = ||\dot{Q}_i|| \leq ||Q_i|| ||(w_i^i)^\times|| \leq ||(w_i^i)^\times||$. Hence, $||h_i|| \leq ||\tilde{E}|| \left(\beta_{1i}(||\tilde{E}||)||\mathcal{X}|| + \beta_{2i}(||\tilde{E}||)\right)$, where $\beta_{1i}(||\tilde{E}||) = (||\tilde{E}|| + 1)\beta_i(||\tilde{E}||)$ and $\beta_{2i}(||\tilde{E}||) = \beta_{1i}(||\tilde{E}||)||v_c|| + (k_v(||\tilde{E}|| + 1)^2 \beta_i(||\tilde{E})||) + 2k_p n_i)$. From this it follows that $||h|| \leq \sum_{i=1}^{n_f} ||h_i|| \leq ||\tilde{E}||(\Theta_1(||\tilde{E}||)||\mathcal{X}|| + \Theta_2||\tilde{E}||)$, where $\Theta_1(||\tilde{E}||) = \sum_{i=1}^{n_f} \beta_{1i}$ and $\Theta_2(||\tilde{E}||) = \sum_{i=1}^{n_f} \beta_{2i}$.

Hence, condition [C3] of [5, Lemma 2.1] is satisfied. ■

IV. ILLUSTRATIVE EXAMPLES

In this section, numerical examples and simulations with 8 agents are considered to validate our analytical results.

A. Attitude synchronization

The initial orientation configuration data in axis-angle representation is specified in TABLE I. The diagonal entries of the diagonal matrix A_{ij} are $[1.5, 1.3, 1]^T \ \forall (i, j) \in \mathcal{E}$ and $\frac{1}{p} = 0.3$. The Lie-group variational integrator proposed in [34] is used for the numerical calculations. Fig. 1 shows the orientation error norms for initial configuration from TABLE I. It is clear that norms of orientation errors (from identity matrix) converge to a constant value for all agents. The goal is not to obtain convergence of error to zero, but to a common value. Here, all attitudes achieve consensus in finite time with the control law (3). From Fig 1, the convergence time obtained is less than 0.4 seconds for the control law proposed in here and the convergence time is about 1.4 s for the control law proposed in [20] indicating slower convergence than our proposed control.

B. Displacement based tracking control

Next, the control law (7) in cascade with (3), is employed for displacement based formation tracking control. The initial positions of the 8 agents are $[0, 0, 0]^T, [-3, 4, 1]^T, [2, 1, 3]^T, [1, 2, 0]^T, [-2, 5, 1]^T, [2, 1, -1]^T, [0, 2, 4]^T, [3, 0, 1]^T$, respectively. A cube in $\sum_c$, specified by the displacement constraints $p_{14}^* = p_{23}^* = p_{56}^* = p_{87}^* = [0, -3, 0]^T, p_{43}^* = p_{12}^* = p_{58}^* = p_{67}^* = [3, 0, 0]^T, p_{64}^* = p_{73}^* = p_{51}^* = p_{82}^* = [0, 0, 3]^T$

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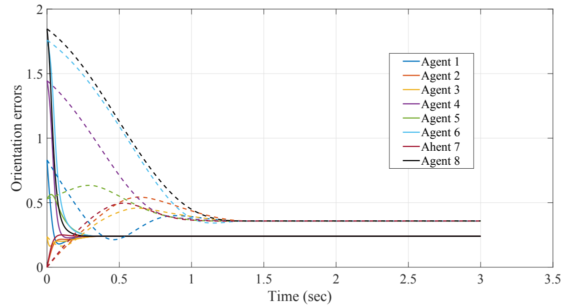

Fig. 1. Orientation errors $||Q_i - I_3|| \quad (i \in \mathcal{V})$ for the law proposed here with solid lines and for the law proposed in [20] with dashed lines

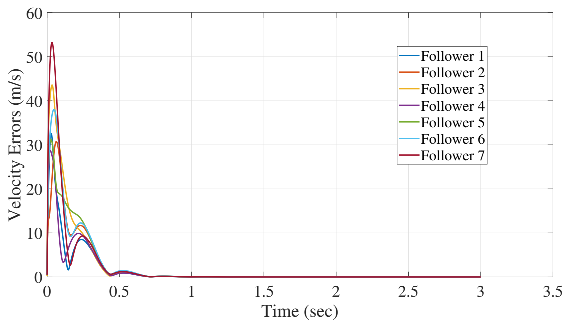

Fig. 2. Velocity errors in displacement based control $||v_i - v_c|| \quad (i \in \mathcal{V}_f)$

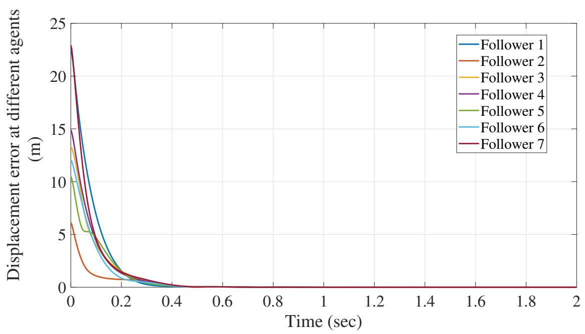

Fig. 3. Total Displacement error $||\sum_{j \in \mathcal{N}_i} p_{ij}^i - p_{ij}^*|| \quad (i \in \mathcal{V}_f)$

and $p_{84}^* = \frac{1}{\sqrt{3}}[-3,-3,3]^T$ is the target formation. There is one leader, $n_l = 1$, and $n_f = 7$. The leader's constant velocity is $v_c = [0.3, 0.3., 0.3]$ m/s in global frame $\sum_w$. The initial orientations are as in TABLE I. The initial velocities of all followers are zero. The design parameters $k_p = 100$ and $k_v = 200$. Figs. 2 and 3 show velocity errors and the displacement errors going to zero indicating that target formation is achieved.

C. Bearing only tracking control

In this section, the control law (11) in cascade with (3) for the formation control is employed. A cube, specified by the bearing constraints $g_{1,4}^* = g_{2,3}^* = g_{5,6}^* = g_{8,7}^* = [0,-1,0]^T, g_{4,3}^* = g_{1,2}^* = g_{5,8}^* = g_{6,7}^* = [1,0,0]^T, g_{6,4}^* = g_{7,3}^* = g_{5,1}^* = g_{8,2}^* = [0,0,1]^T$ and $g_{8,4}^* = \frac{1}{\sqrt{3}}[-1,-1,1]^T$ in agents' consensus co-ordinate frame $\sum_c$, is the target formation in $\sum_c$, with $n_l = 2$ and $n_f = 6$. The velocities of both the leaders are $v_c = [0.3, 0.3., 0.3]$m/s in global frame $\sum_w$. The initial positions and velocities used for simulation are given in TABLE II. The design parameters are $k_p = 10$ and $k_v = 5$. Figs. 4 and 5 show velocity errors and the total bearing error going to zero. Fig. 6 shows the trajectories of agents in global reference frame ($\sum_w$). The initial positions are marked by circles (o) and final positions are marked by stars ($*$). Final formation is as desired in $\sum_w$. The initial orientations of agents are as in TABLE I, which suggests that the initial angles corresponding to the rotation matrices can be greater than $\frac{\pi}{2}$.

TABLE II

INITIAL POSITIONS AND VELOCITIES FOR BEARING-ONLY CONTROL

Agent (i)	$p_i(t_0)$ (m)	$v_i(t_0)$ (m/s)
1	$[0,3,3]^T$	$[0.3,0.3,0.3]^T$
2	$[3,3,3]^T$	$[0.3,0.3,0.3]^T$
3	$[4,0,4]^T$	$[0.12,0.14,0]^T$
4	$[0,0,5]^T$	$[0.1,0.2,0.1]^T$
5	$[1,3,-1]^T$	$[0.1,0.2,0.12]^T$
6	$[2,1,-1]^T$	$[0.14,0.13,-0.1]^T$
7	$[4,-2,3]^T$	$[0.2,-0.1,0]^T$
8	$[2,2,3]^T$	$[-0.12,0,0.2]^T$

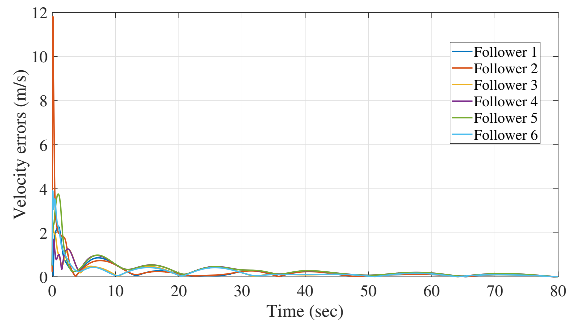

Fig. 4. The Velocity errors in bearing-only control $||v_i - v_c|| \quad (i \in \mathcal{V}_f)$

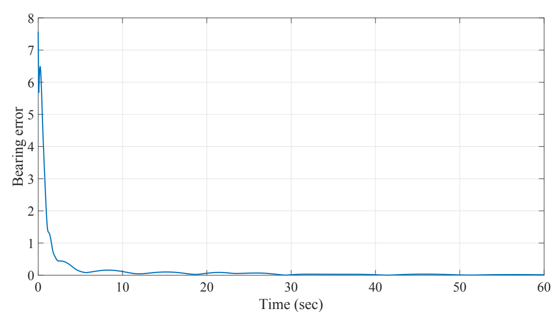

Fig. 5. The Bearing error $\sum_{k \in \mathcal{E}} ||g_k - g_k^{'}||$

V. CONCLUSIONS

In this paper formation control in agents' local co-ordinate system was facilitated by a newly proposed orientation synchronization protocol which resulted in faster, fixed-time convergence of orientation errors. Double integrator model was considered for agents. Achieving time varying formation

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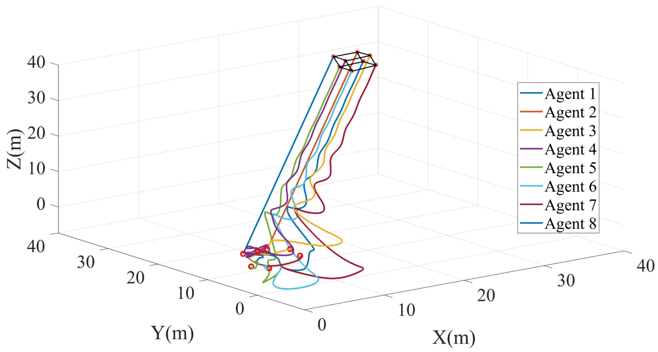

Fig. 6. Trajectory of Agents in Bearing only control

in agents' local co-ordinate system is an interesting topic for future exploration. Studying bearing-only formation control for agents modeled by more general dynamical systems is another potential research direction.

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