Elliptic Optimal Control with Measure-Valued Controls

Predual reformulation, sparsity, and semismooth Newton numerics

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Roadmap

- Motivation: localized actuation and sparsity
- Setup PDE with measure data
- Optimization problem
- Optimality conditions and sparsity structure
- Numerics and regularization
- Experiments
- Conclusion

Motivation — why measure-valued controls?

Localized actuation in the real world

Many control mechanisms are *intrinsically concentrated* in space, e.g.

- **Thermal control:** laser heating on a tiny spot, or micro-heaters on a chip (point-like heat sources).
- **Semiconductor devices:** carrier injection/extraction at small contacts (point-electrodes).
- **Structural mechanics:** concentrated loads or actuators in beams/plates.

Sparsity promotion

Replacing a quadratic control cost by an L^1-type cost promotes sparse actuation. In PDE settings, the measure space $\mathcal{M}(\Omega)$ is the natural model.

Main challenge

The problem is nonsmooth *and* posed in the nonreflexive space $\mathcal{M}(\Omega)$.

Setup — Elliptic PDE with measure data

We consider

$$Ay = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega, \qquad u \in \mathcal{M}(\Omega).$$

Why not the usual H_0^1 formulation?

A general measure u is *not* in $H^{-1}(\Omega)$, so testing with $v \in H_0^1(\Omega)$ is not well-defined.

Very weak formulation (Dirichlet)

Choose a test space $V \hookrightarrow C_0(\Omega)$, e.g. $V = H^2(\Omega) \cap H_0^1(\Omega)$ (in $n = 2, 3$). Then there exist $y \in L^1(\Omega)$ that solves

$$\int_\Omega y\, A^* \varphi\, dx = \int_\Omega \varphi\, du \qquad \forall \varphi \in V.$$

Setup — Regularity and the solution map

Elliptic smoothing for measure data

For $u \in \mathcal{M}(\Omega)$, the very weak solution satisfies

$$y \in W_0^{1,p}(\Omega) \quad \text{for all } 1 \le p < \frac{n}{n-1}, \qquad \|y\|_{W^{1,p}} \le C\,\|u\|_{\mathcal{M}(\Omega)}.$$

Solution operator

Define

$$S : \mathcal{M}(\Omega) \to L^2(\Omega), \qquad Su = y(u),$$

using Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ (for suitable p, $n \in \{2,3\}$).

Key takeaway

Even if u is singular (Dirac), the state y is a genuine function usable in an L^2 tracking term.

Optimization problem — Primal sparse control problem in $\mathcal{M}(\Omega)$

Given desired state $z \in L^2(\Omega)$ and $\alpha > 0$:

$$\min_{u \in \mathcal{M}(\Omega)} J(u) := \frac{1}{2} \|Su - z\|^2_{L^2(\Omega)} + \alpha \|u\|_{\mathcal{M}(\Omega)} .$$

Interpretation

- $\frac{1}{2} \|Su - z\|^2$: match the target state
- $\alpha \|u\|_{\mathcal{M}}$: pay for total control mass $\Rightarrow$ sparse actuation

Two difficulties

- Nonsmooth term $\|u\|_{\mathcal{M}}$
- Nonreflexive space $\mathcal{M}(\Omega)$

Optimization problem — Existence and uniqueness (direct method)

Compactness mechanism

- Minimizing sequence (u_n) bounded in $\mathcal{M}(\Omega)$
- Banach–Alaoglu: $u_n \overset{*}{\rightharpoonup} u^*$ in $\sigma(\mathcal{M}(\Omega), C_0(\Omega))$
- States $y_n = Su_n$ bounded in $W_0^{1,p}(\Omega)$
- Rellich: $y_n \to y^*$ strongly in $L^2(\Omega)$

Lower semicontinuity

$$\|u^*\|_{\mathcal{M}} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{M}}\,, \qquad \|Su_n - z\|_{L^2}^2 \to \|Su^* - z\|_{L^2}^2\,.$$

So u^* is optimal.

Uniqueness

Strict convexity of the L^2 norm (and injectivity of S) gives uniqueness of the minimizer.

Optimization problem — Why a predual formulation?

Goal

Avoid discretizing measures directly and replace nonsmooth penalty by a simple constraint.

Core idea (Fenchel duality)

The conjugate of $\alpha \|\cdot\|_{\mathcal{M}}$ is the indicator of a box in $C_0(\Omega)$:

$$(\alpha \|\cdot\|_{\mathcal{M}})^*(\varphi) = \begin{cases} 0, & \|\varphi\|_\infty \leq \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

Computational win

Measure norm $\Rightarrow$ *box constraint* on a smooth variable.

Optimization problem — Deriving the predual (Step 1)

Step 1: Reduced primal formulation

Let $S : \mathcal{M}(\Omega) \to L^2(\Omega)$ denote the control-to-state map $Su = y(u)$. The primal problem reads

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2}\|Su - z\|^2_{L^2(\Omega)} + \alpha\|u\|_{\mathcal{M}(\Omega)}.$$

Optimization problem — Deriving the predual (Steps 2–5)

Step 2–4: Idea (Fenchel duality)

- **Step 2:** Dualize the quadratic tracking term

$$\frac{1}{2}\|Su - z\|_{L^2}^2 \; = \; \sup_{w \in L^2} \Big\{ \langle Su - z, w \rangle - \frac{1}{2}\|w\|_{L^2}^2 \Big\}.$$

- **Step 3:** Move S to the adjoint side: $\langle Su, w \rangle = \langle u, S^* w \rangle$, and set $p := S^* w \in C_0(\Omega)$.
- **Step 4:** Eliminate u using

$$\sup_{u \in \mathcal{M}(\Omega)} \{\langle u, p \rangle - \alpha \|u\|_{\mathcal{M}}\} = 0 \iff \|p\|_\infty \le \alpha.$$

Step 5: Predual problem

$$\min_{p \in H^2(\Omega) \cap H_0^1(\Omega)} \left[\frac{1}{2}\|A^* p + z\|_{L^2(\Omega)}^2 - \frac{1}{2}\|z\|_{L^2(\Omega)}^2 \right] \quad \text{s.t.} \quad \|p\|_\infty \le \alpha.$$

Optimization problem — Predual problem

In the measure-control setting, the dual variable satisfies

$$p \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow C_0(\Omega).$$

Predual formulation

$$\min_{p \in H^2(\Omega) \cap H_0^1(\Omega)} F(p) := \frac{1}{2} \|A^* p + z\|_{L^2(\Omega)}^2 - \frac{1}{2} \|z\|_{L^2(\Omega)}^2 \quad \text{s.t.} \quad \|p\|_\infty \leq \alpha.$$

Meaning

All nonsmoothness is now in a *simple* pointwise constraint.

Optimality conditions — KKT conditions for the predual problem

Step 1: Compute the derivative of F

$$\nabla F(p) = AA^* p + Az \in H_0^2(\Omega)^*.$$

Step 2: KKT condition via subdifferentials

Since F is convex and Fréchet differentiable and K is closed convex,

$$0 \in \partial(F + I_K)(p^*) = \nabla F(p^*) + \partial I_K(p^*). \iff \exists\, \lambda^* \in N_K(p^*)\ \nabla F(p^*) + \lambda^* = 0.$$

Step 3: KKT system (stationarity + variational inequality)

Find $(p^*, \lambda^*) \in H_0^2(\Omega) \times H_0^2(\Omega)^*$ such that

$$AA^* p^* + Az + \lambda^* = 0 \quad \text{in } H_0^2(\Omega)^*,$$

$$\langle \lambda^*,\, p - p^* \rangle_{H_0^{2*}, H_0^2} \le 0 \quad \forall\, p \in H_0^2(\Omega) \text{ with } \|p\|_{C_0} \le \alpha.$$

Optimality conditions — Primal identification and sparsity

Identify the primal solution u^*

From the saddle-point / KKT system one obtains

$$\lambda^* \in \partial I_K(p^*) = N_K(p^*) \qquad \Longleftrightarrow \qquad u^* := -\lambda^* \in \mathcal{M}(\Omega)$$

Sparsity / sign property

For every test function $\psi \in C_c(\Omega)$ with $\psi \geq 0$:

$$\langle u^*, \psi \rangle = 0 \quad \text{if } \operatorname{supp}(\psi) \subset \{|p^*| < \alpha\},$$

$$\langle u^*, \psi \rangle \geq 0 \quad \text{if } \operatorname{supp}(\psi) \subset \{p^* = \alpha\}, \qquad \langle u^*, \psi \rangle \leq 0 \quad \text{if } \operatorname{supp}(\psi) \subset \{p^* = -\alpha\}.$$

Numerics and regularization — Moreau–Yosida regularization of the box

Problem

The constraint $\|p\|_\infty \leq \alpha$ is nonsmooth (active-set structure).

Effect

Allows Newton-type methods

Regularized predual problem $(P^*_{M,c})$

For $c > 0$, let $p_c \in H^2_0(\Omega)$ be the unique minimizer of

$$\frac{1}{2}\|A^* p + z\|^2_{L^2(\Omega)} - \frac{1}{2}\|z\|^2_{L^2(\Omega)} + \frac{1}{2c}\|\max(0, c(p-\alpha))\|^2_{L^2} + \frac{1}{2c}\|\min(0, c(p+\alpha))\|^2_{L^2},$$

and define

$$\lambda_c := \max(0, c(p_c - \alpha)) + \min(0, c(p_c + \alpha)).$$

Then (p_c, λ_c) solves $AA^* p_c + Az + \lambda_c = 0$.

Theorem (Convergence as $c \to \infty$)

Let (p^*, λ^*) be the unique KKT solution of the unregularized box-constrained problem. Then, as $c \to \infty$,

$$p_c \to p^* \text{ strongly in } H_0^2(\Omega), \qquad \lambda_c \rightharpoonup \lambda^* \text{ weakly in } H_0^2(\Omega)^*.$$

Proof idea

1. (Key inequality) From the pointwise definition: $\langle \lambda_c, p_c \rangle_{L^2} \geq \frac{1}{c} \|\lambda_c\|_{L^2}^2$.
2. (Uniform bounds) Test $AA^* p_c + Az + \lambda_c = 0$ with p_c to bound $\|p_c\|_{H_0^2}$ and $\frac{1}{c}\|\lambda_c\|_{L^2}^2$.
3. (Subsequence limits) Extract $(p_c, \lambda_c) \rightharpoonup (\tilde{p}, \tilde{\lambda})$ in $H_0^2 \times H_0^{2*}$ (Banach–Alaoglu).
4. (Feasibility) The penalties $\max(0, p_c - \alpha)$, $\min(0, p_c + \alpha)$ vanish in $L^2 \Rightarrow |\tilde{p}| \leq \alpha$ a.e.
5. (Strong conv.) Use weak l.s.c. to get $\|A^* p_c\|_{L^2} \to \|A^* \tilde{p}\|_{L^2}$, hence $p_c \to \tilde{p}$ in H_0^2.
6. (Limit KKT) Limit in the variational inequality to show $(\tilde{p}, \tilde{\lambda})$ solves the unreg. KKT system.
7. (Uniqueness) Conclude $(\tilde{p}, \tilde{\lambda}) = (p^*, \lambda^*)$ and thus the whole family converges.

Goal

Solve the regularized optimality system $F(p) = 0$ for $p_c \in H_0^2(\Omega)$ via an active-set (piecewise linear) Newton iteration.

1. Choose an initial guess $p^0 \in H_0^2(\Omega)$ and set $k = 0$.
2. Repeat:
   (i) Define the active sets
   $$A_k^+ := \{x \in \Omega : \ p^k(x) > \alpha\}, \qquad A_k^- := \{x \in \Omega : \ p^k(x) < -\alpha\}, \qquad A_k := A_k^+ \cup A_k^- .$$
   (ii) Compute $p^{k+1} \in H_0^2(\Omega)$ by solving the linear equation (weak form)
   $$\langle A^* p^{k+1}, A^* v\rangle_{L^2} + c\langle p^{k+1}\chi_{A_k}, v\rangle_{L^2} = -\langle z, A^* v\rangle_{L^2} + c\alpha\langle \chi_{A_k^+} - \chi_{A_k^-}, v\rangle_{L^2} \quad \forall v \in H_0^2(\Omega).$$
   (iii) Set $k \leftarrow k + 1$.
3. Stop if $A_k^+ = A_{k-1}^+$ and $A_k^- = A_{k-1}^-$.

Experiments — Basic example

Experiments — Semiconductor example

Context. We consider a steady-state concentration y of a species in a bounded domain $\Omega \subset \mathbb{R}^2$ (with $y = 0$ on $\partial\Omega$). Species are *generated* in a localized subregion and can be *removed* by placing sinks. We want the sinks to be *spatially sparse*. This is a simplified description of a photodiode in steady-state operation

State equation (diffusion). With diffusion coefficient $D > 0$ and operator

$$A_D := -D\Delta, \qquad y|_{\partial\Omega} = 0,$$

the steady-state balance reads

$$A_D y \;=\; g - u \quad \text{in } \Omega,$$

where g is a prescribed generation term (here: constant on a small central square, 0 outside) and u is the sink control.

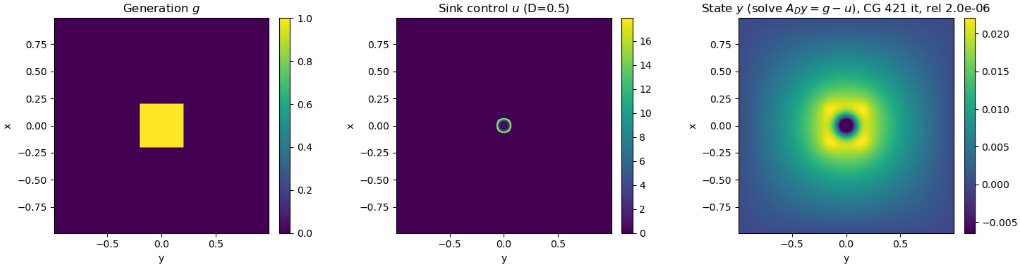
Optimal control problem (measure-sparse control). We set the desired target to $z \equiv 0$ (remove species everywhere) and penalize the sink by a *measure norm* to promote sparsity:

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2}\|y(u) - z\|_{L^2(\Omega)}^2 \;+\; \alpha\|u\|_{\mathcal{M}(\Omega)} \qquad \text{s.t.} \qquad A_D y = g - u,\; y|_{\partial\Omega} = 0.$$

Here $\alpha > 0$ controls the sparsity strength: larger α yields fewer/stronger sink locations.

Experiments — Semiconductor example

Numerical solution on $\Omega = [-1,1]^2$ *(Dirichlet BC),* $D = 0.5$*.*



The optimized sink u concentrates on a very thin ring surrounding the generation region. This structure is a direct consequence of the measure penalty, which promotes sparsity by localizing the control on sets of small measure. Despite its narrow support, the sink efficiently removes the species due to diffusion, resulting in a low concentration y throughout the domain.

Experiments — Semiconductor example

Alternative numerical route: variational discretization (FEM sequel)

Different philosophy

Discretize only the *state* equation by FEM, keep $u \in \mathcal{M}(\Omega)$ continuous.

Key theorem (informal)

The discrete optimizer can be chosen as a *finite combination of Diracs at mesh nodes*:

$$u_h^* = \sum_{j=1}^{N(h)} \lambda_j \delta_{x_j}.$$

Comparison

- Predual approach: smooth variable + box constraints + Newton
- FEM sequel: state discretization induces nodal sparsity automatically

Conclusion

Main messages

- Measure controls model localized actuation and yield sparse optimal solutions
- Very weak solutions provide well-posed PDE state equations for $u \in \mathcal{M}(\Omega)$
- Fenchel duality gives a predual Hilbert-space formulation with $\|p\|_\infty \leq \alpha$
- KKT structure explains sparsity: u^* lives on the active set $\{|p^*| = \alpha\}$

Takeaway

Predual reformulation turns a difficult nonsmooth measure problem into a numerically friendly box-constrained PDE problem.

Thank you!