Elliptic Optimal Control with Measure-Valued Controls

Predual reformulation, sparsity, and semismooth Newton numerics

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Roadmap

- Motivation: localized actuation and sparsity
- Setup PDE with measure data
- Optimization problem
- Optimality conditions and sparsity structure
- Numerics and regularization
- Experiments
- Conclusion

Motivation — why measure-valued controls?

Localized actuation in the real world

Many control mechanisms are *intrinsically concentrated* in space, e.g.

- **Thermal control:** laser heating on a tiny spot, or micro-heaters on a chip (point-like heat sources).
- **Semiconductor devices:** carrier injection/extraction at small contacts (point-electrodes).
- **Structural mechanics:** concentrated loads or actuators in beams/plates.

Sparsity promotion

Replacing a quadratic control cost by an L^1-type cost promotes sparse actuation. In PDE settings, the measure space $\mathcal{M}(\Omega)$ is the natural model.

Main challenge

The problem is nonsmooth *and* posed in the nonreflexive space $\mathcal{M}(\Omega)$.

Setup — Elliptic PDE with measure data

We consider

$$Ay = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega, \qquad u \in \mathcal{M}(\Omega).$$

Why not the usual H_0^1 formulation?

A general measure u is *not* in $H^{-1}(\Omega)$, so testing with $v \in H_0^1(\Omega)$ is not well-defined.

Very weak formulation (Dirichlet)

Choose a test space $V \hookrightarrow C_0(\Omega)$, e.g. $V = H^2(\Omega) \cap H_0^1(\Omega)$ (in $n = 2, 3$). Then there exist $y \in L^1(\Omega)$ that solves

$$\int_\Omega y\, A^*\varphi \, dx = \int_\Omega \varphi \, du \qquad \forall \varphi \in V.$$

Setup — Regularity and the solution map

Elliptic smoothing for measure data

For $u \in \mathcal{M}(\Omega)$, the very weak solution satisfies

$$y \in W_0^{1,p}(\Omega) \quad \text{for all } 1 \le p < \frac{n}{n-1}, \qquad \|y\|_{W^{1,p}} \le C\,\|u\|_{\mathcal{M}(\Omega)}.$$

Solution operator

Define

$$S : \mathcal{M}(\Omega) \to L^2(\Omega), \qquad Su = y(u),$$

using Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ (for suitable p, $n \in \{2,3\}$).

Key takeaway

Even if u is singular (Dirac), the state y is a genuine function usable in an L^2 tracking term.

Optimization problem — Primal sparse control problem in $\mathcal{M}(\Omega)$

Given desired state $z \in L^2(\Omega)$ and $\alpha > 0$:

$$\min_{u \in \mathcal{M}(\Omega)} J(u) := \frac{1}{2} \|Su - z\|^2_{L^2(\Omega)} + \alpha \|u\|_{\mathcal{M}(\Omega)} .$$

Interpretation

- $\frac{1}{2}\|Su - z\|^2$: match the target state
- $\alpha \|u\|_{\mathcal{M}}$: pay for total control mass $\Rightarrow$ sparse actuation

Two difficulties

- Nonsmooth term $\|u\|_{\mathcal{M}}$
- Nonreflexive space $\mathcal{M}(\Omega)$

Optimization problem — Existence and uniqueness (direct method)

Compactness mechanism

- Minimizing sequence (u_n) bounded in $\mathcal{M}(\Omega)$
- Banach–Alaoglu: $u_n \overset{*}{\rightharpoonup} u^*$ in $\sigma(\mathcal{M}(\Omega), C_0(\Omega))$
- States $y_n = Su_n$ bounded in $W_0^{1,p}(\Omega)$
- Rellich: $y_n \to y^*$ strongly in $L^2(\Omega)$

Lower semicontinuity

$$\|u^*\|_{\mathcal{M}} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{M}}, \qquad \|Su_n - z\|_{L^2}^2 \to \|Su^* - z\|_{L^2}^2 .$$

So u^* is optimal.

Uniqueness

Strict convexity of the L^2 norm (and injectivity of S) gives uniqueness of the minimizer.

Optimization problem — Why a predual formulation?

Goal

Avoid discretizing measures directly and replace nonsmooth penalty by a simple constraint.

Core idea (Fenchel duality)

The conjugate of $\alpha \|\cdot\|_{\mathcal{M}}$ is the indicator of a box in $C_0(\Omega)$:

$$(\alpha \|\cdot\|_{\mathcal{M}})^*(\varphi) = \begin{cases} 0, & \|\varphi\|_\infty \leq \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

Computational win

Measure norm $\Rightarrow$ *box constraint* on a smooth variable.

Optimization problem — Deriving the predual (Step 1)

Step 1: Reduced primal formulation

Let $S : \mathcal{M}(\Omega) \to L^2(\Omega)$ denote the control-to-state map $Su = y(u)$. The primal problem reads

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2}\|Su - z\|_{L^2(\Omega)}^2 + \alpha \|u\|_{\mathcal{M}(\Omega)}.$$

Optimization problem — Deriving the predual (Steps 2–5)

Step 2–4: Idea (Fenchel duality)

- **Step 2:** Dualize the quadratic tracking term

$$\frac{1}{2}\|Su - z\|_{L^2}^2 \;=\; \sup_{w\in L^2}\left\{\langle Su - z, w\rangle - \frac{1}{2}\|w\|_{L^2}^2\right\}.$$

- **Step 3:** Move S to the adjoint side: $\langle Su, w\rangle = \langle u, S^* w\rangle$, and set $p := S^* w \in C_0(\Omega)$.
- **Step 4:** Eliminate u using

$$\sup_{u\in\mathcal{M}(\Omega)} \{\langle u, p\rangle - \alpha\|u\|_{\mathcal{M}}\} = 0 \iff \|p\|_\infty \le \alpha.$$

Step 5: Predual problem

$$\min_{p\in H^2(\Omega)\cap H_0^1(\Omega)} \left[\frac{1}{2}\|A^* p + z\|_{L^2(\Omega)}^2 - \frac{1}{2}\|z\|_{L^2(\Omega)}^2\right] \quad \text{s.t.} \quad \|p\|_\infty \le \alpha.$$

Optimization problem — Predual problem

In the measure-control setting, the dual variable satisfies

$$p \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow C_0(\Omega).$$

Predual formulation

$$\min_{p \in H^2(\Omega) \cap H_0^1(\Omega)} F(p) := \frac{1}{2} \|A^* p + z\|_{L^2(\Omega)}^2 - \frac{1}{2} \|z\|_{L^2(\Omega)}^2 \quad \text{s.t.} \quad \|p\|_\infty \leq \alpha.$$

Meaning

All nonsmoothness is now in a *simple* pointwise constraint.

Optimality conditions — KKT conditions for the predual problem

Step 1: Compute the derivative of F

$$\nabla F(p) = AA^* p + Az \in H_0^2(\Omega)^*.$$

Step 2: KKT condition via subdifferentials

Since F is convex and Fréchet differentiable and K is closed convex,

$$0 \in \partial(F + I_K)(p^*) = \nabla F(p^*) + \partial I_K(p^*). \iff \exists\, \lambda^* \in N_K(p^*)\ \nabla F(p^*) + \lambda^* = 0.$$

Step 3: KKT system (stationarity + variational inequality)

Find $(p^*, \lambda^*) \in H_0^2(\Omega) \times H_0^2(\Omega)^*$ such that

$$AA^* p^* + Az + \lambda^* = 0 \quad \text{in } H_0^2(\Omega)^*,$$

$$\langle \lambda^*,\, p - p^* \rangle_{H_0^{2*}, H_0^2} \le 0 \quad \forall\, p \in H_0^2(\Omega) \text{ with } \|p\|_{C_0} \le \alpha.$$

Optimality conditions — Primal identification and sparsity

Identify the primal solution u^*

From the saddle-point / KKT system one obtains

$$\lambda^* \in \partial I_K(p^*) = N_K(p^*) \qquad \Longleftrightarrow \qquad u^* := -\lambda^* \in \mathcal{M}(\Omega)$$

Sparsity / sign property

For every test function $\psi \in C_c(\Omega)$ with $\psi \geq 0$:

$$\langle u^*, \psi \rangle = 0 \quad \text{if } \operatorname{supp}(\psi) \subset \{|p^*| < \alpha\},$$

$$\langle u^*, \psi \rangle \geq 0 \quad \text{if } \operatorname{supp}(\psi) \subset \{p^* = \alpha\}, \qquad \langle u^*, \psi \rangle \leq 0 \quad \text{if } \operatorname{supp}(\psi) \subset \{p^* = -\alpha\}.$$

Numerics and regularization — Moreau–Yosida regularization of the box

Problem

The constraint $\|p\|_\infty \leq \alpha$ is nonsmooth (active-set structure).

Effect

Allows Newton-type methods

Regularized predual problem $(P^*_{M,c})$

For $c > 0$, let $p_c \in H^2_0(\Omega)$ be the unique minimizer of

$$\frac{1}{2}\|A^* p + z\|^2_{L^2(\Omega)} - \frac{1}{2}\|z\|^2_{L^2(\Omega)} + \frac{1}{2c}\|\max(0, c(p-\alpha))\|^2_{L^2} + \frac{1}{2c}\|\min(0, c(p+\alpha))\|^2_{L^2},$$

and define

$$\lambda_c := \max(0, c(p_c - \alpha)) + \min(0, c(p_c + \alpha)).$$

Then (p_c, λ_c) solves $AA^* p_c + Az + \lambda_c = 0$.

Numerics and regularization — Convergence of the Moreau–Yosida

Theorem (Convergence as $c \to \infty$)

Let (p^*, λ^*) be the unique KKT solution of the unregularized box-constrained problem. Then, as $c \to \infty$,

$$p_c \to p^* \text{ strongly in } H_0^2(\Omega), \qquad \lambda_c \rightharpoonup \lambda^* \text{ weakly in } H_0^2(\Omega)^*.$$

Proof idea

1. (Key inequality) From the pointwise definition: $\langle \lambda_c, p_c \rangle_{L^2} \geq \frac{1}{c} \|\lambda_c\|_{L^2}^2$.
2. (Uniform bounds) Test $AA^* p_c + Az + \lambda_c = 0$ with p_c to bound $\|p_c\|_{H_0^2}$ and $\frac{1}{c}\|\lambda_c\|_{L^2}^2$.
3. (Subsequence limits) Extract $(p_c, \lambda_c) \rightharpoonup (\tilde{p}, \tilde{\lambda})$ in $H_0^2 \times H_0^{2*}$ (Banach–Alaoglu).
4. (Feasibility) The penalties $\max(0, p_c - \alpha)$, $\min(0, p_c + \alpha)$ vanish in $L^2 \Rightarrow |\tilde{p}| \leq \alpha$ a.e.
5. (Strong conv.) Use weak l.s.c. to get $\|A^* p_c\|_{L^2} \to \|A^* \tilde{p}\|_{L^2}$, hence $p_c \to \tilde{p}$ in H_0^2.
6. (Limit KKT) Limit in the variational inequality to show $(\tilde{p}, \tilde{\lambda})$ solves the unreg. KKT system.
7. (Uniqueness) Conclude $(\tilde{p}, \tilde{\lambda}) = (p^*, \lambda^*)$ and thus the whole family converges.

Numerics and regularization — Algorithm: Semismooth Newton

Goal

Solve the regularized optimality system $F(p) = 0$ for $p_c \in H_0^2(\Omega)$ via an active-set (piecewise linear) Newton iteration.

1. Choose an initial guess $p^0 \in H_0^2(\Omega)$ and set $k = 0$.
2. Repeat:
   (i) Define the active sets
   $$A_k^+ := \{x \in \Omega : \ p^k(x) > \alpha\}, \qquad A_k^- := \{x \in \Omega : \ p^k(x) < -\alpha\}, \qquad A_k := A_k^+ \cup A_k^- .$$
   (ii) Compute $p^{k+1} \in H_0^2(\Omega)$ by solving the linear equation (weak form)
   $$\langle A^* p^{k+1}, A^* v\rangle_{L^2} + c\langle p^{k+1}\chi_{A_k}, v\rangle_{L^2} = -\langle z, A^* v\rangle_{L^2} + c\alpha\langle \chi_{A_k^+} - \chi_{A_k^-}, v\rangle_{L^2} \quad \forall v \in H_0^2(\Omega).$$
   (iii) Set $k \leftarrow k + 1$.
3. Stop if $A_k^+ = A_{k-1}^+$ and $A_k^- = A_{k-1}^-$.

Experiments — Basic example

Experiments — Semiconductor example

Experiments — Semiconductor example

Experiments — Semiconductor example

Alternative numerical route: variational discretization (FEM sequel)

Different philosophy

Discretize only the *state* equation by FEM, keep $u \in \mathcal{M}(\Omega)$ continuous.

Key theorem (informal)

The discrete optimizer can be chosen as a *finite combination of Diracs at mesh nodes*:

$$u_h^* = \sum_{j=1}^{N(h)} \lambda_j \delta_{x_j}.$$

Comparison

- Predual approach: smooth variable + box constraints + Newton
- FEM sequel: state discretization induces nodal sparsity automatically

Conclusion

Main messages

- Measure controls model localized actuation and yield sparse optimal solutions
- Very weak solutions provide well-posed PDE state equations for $u \in \mathcal{M}(\Omega)$
- Fenchel duality gives a predual Hilbert-space formulation with $\|p\|_\infty \le \alpha$
- KKT structure explains sparsity: u^* lives on the active set $\{|p^*| = \alpha\}$

Takeaway

Predual reformulation turns a difficult nonsmooth measure problem into a numerically friendly box-constrained PDE problem.

Thank you!