Elliptic Optimal Control with Measure-Valued Controls

Predual reformulation, sparsity, and semismooth Newton numerics

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Roadmap

- Motivation: localized actuation and sparsity
- Setup PDE with measure data
- Optimization problem
- Optimality conditions and sparsity structure
- Numerics and regularization
- Experiments
- Alternative route (optional)
- Conclusion

Motivation: why measure-valued controls?

Desired properties

- **Localized actuation:** heating spots, micro-heaters, point electrodes, concentrated loads.
- **Natural model:** point sources $\approx$ Dirac measures $\Rightarrow u \in \mathcal{M}(\Omega)$.
- **Sparsity:** L^1-type penalties promote concentrated/active-set solutions.

Main challenge: nonsmooth cost in a nonreflexive space.

Setup — Elliptic PDE with measure data

We consider

$$Ay = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega, \qquad u \in \mathcal{M}(\Omega).$$

Why not the usual H_0^1 formulation?
A general measure u is *not* in $H^{-1}(\Omega)$, so testing with $v \in H_0^1(\Omega)$ is not well-defined.

Very weak formulation (Dirichlet)
Choose a test space $V \hookrightarrow C_0(\Omega)$, e.g. $V = H^2(\Omega) \cap H_0^1(\Omega)$ (in $n = 2, 3$). Then there exist $y \in L^1(\Omega)$ that solves

$$\int_\Omega y\, A^* \varphi \, dx = \int_\Omega \varphi \, du \qquad \forall \varphi \in V.$$

Setup — Regularity and the solution map

Elliptic smoothing for measure data
For $u \in \mathcal{M}(\Omega)$, the very weak solution satisfies

$$y \in W_0^{1,p}(\Omega) \quad \text{for all } 1 \leq p < \frac{n}{n-1}, \qquad \|y\|_{W^{1,p}} \leq C\,\|u\|_{\mathcal{M}(\Omega)}.$$

Solution operator
Define

$$S : \mathcal{M}(\Omega) \to L^2(\Omega), \qquad Su = y(u),$$

using Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ (for suitable p, $n \in \{2,3\}$).

Key takeaway: Even if u is singular (Dirac), the state y is a genuine function usable in an L^2 tracking term.

Optimization problem — Primal sparse control problem in $\mathcal{M}(\Omega)$

Given desired state $z \in L^2(\Omega)$ and $\alpha > 0$:

$$\min_{u \in \mathcal{M}(\Omega)} J(u) := \frac{1}{2} \|Su - z\|^2_{L^2(\Omega)} + \alpha \|u\|_{\mathcal{M}(\Omega)} .$$

Interpretation

- $\frac{1}{2} \|Su - z\|^2$: match the target state
- $\alpha \|u\|_{\mathcal{M}}$: pay for total control mass $\Rightarrow$ sparse actuation

Two difficulties

- Nonsmooth term $\|u\|_{\mathcal{M}}$
- Nonreflexive space $\mathcal{M}(\Omega)$

Existence and uniqueness (direct method)

Primal problem: $\min\limits_{u \in \mathcal{M}(\Omega)} \dfrac{1}{2}\|Su - z\|^2_{L^2(\Omega)} + \alpha \|u\|_{\mathcal{M}(\Omega)}.$

Existence. Let (u_n) be a minimizing sequence. Then

$$\|u_n\|_{\mathcal{M}} \leq C \quad \Rightarrow \quad u_n \overset{*}{\rightharpoonup} u^* \text{ in } \mathcal{M}(\Omega) \quad \text{(Banach–Alaoglu).}$$

Moreover, $y_n := Su_n$ is bounded in $W_0^{1,p}(\Omega)$, hence

$$Su_n \to Su^* \text{ in } L^2(\Omega) \quad \text{(Rellich).}$$

Lower semicontinuity of $\|\cdot\|_{\mathcal{M}}$ gives $\|u^*\|_{\mathcal{M}} \leq \liminf_n \|u_n\|_{\mathcal{M}}$, so u^* is optimal.

Uniqueness. The term $\frac{1}{2}\|Su - z\|^2_{L^2}$ is strictly convex in Su. If S is injective, then u^* is unique.

Optimization problem — Why a predual formulation?

Goal

Avoid discretizing measures directly and replace nonsmooth penalty by a simple constraint.

Core idea (Fenchel duality)

The conjugate of $\alpha \|\cdot\|_{\mathcal{M}}$ is the indicator of a box in $C_0(\Omega)$:

$$(\alpha \|\cdot\|_{\mathcal{M}})^*(\varphi) = \begin{cases} 0, & \|\varphi\|_\infty \leq \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

Computational win

Measure norm $\Rightarrow$ *box constraint* on a smooth variable.

Predual reformulation (Fenchel duality)

Let $S : \mathcal{M}(\Omega) \to L^2(\Omega)$ denote the control-to-state map $Su = y(u)$. The primal problem reads

$$\inf_{u \in \mathcal{M}(\Omega)} \frac{1}{2}\|Su - z\|^2_{L^2(\Omega)} + \alpha \|u\|_{\mathcal{M}(\Omega)}.$$

- Tracking term dualization: $\frac{1}{2}\|Su - z\|^2_{L^2} = \sup_{w \in L^2} \Big\{ \langle Su - z, w \rangle - \frac{1}{2}\|w\|^2_{L^2} \Big\}$.
- Adjoint move: $\langle Su, w \rangle = \langle u, S^* w \rangle$, set $p := S^* w \in C_0(\Omega)$.
- Eliminating u: $\sup_{u \in \mathcal{M}(\Omega)} \{ \langle u, p \rangle - \alpha \|u\|_{\mathcal{M}} \} = 0 \iff \|p\|_\infty \le \alpha$.

Predual: $\inf_{p \in H^2 \cap H^1_0} \Big[\frac{1}{2}\|A^* p + z\|^2_{L^2} - \frac{1}{2}\|z\|^2_{L^2} \Big]$ s.t. $\|p\|_\infty \le \alpha$.

Optimization problem — Predual problem

In the measure-control setting, the dual variable satisfies

$$p \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow C_0(\Omega).$$

Predual formulation

$$\min_{p \in H^2(\Omega) \cap H_0^1(\Omega)} F(p) := \frac{1}{2} \|A^* p + z\|_{L^2(\Omega)}^2 - \frac{1}{2} \|z\|_{L^2(\Omega)}^2 \quad \text{s.t.} \quad \|p\|_\infty \le \alpha.$$

Meaning

All nonsmoothness is now in a *simple* pointwise constraint.

KKT conditions for the predual problem

Predual: $\min_{p \in H^2 \cap H_0^1} F(p)$ s.t. $\|p\|_\infty \le \alpha$, where

$$F(p) = \tfrac{1}{2}\|A^* p + z\|_{L^2}^2 - \tfrac{1}{2}\|z\|_{L^2}^2.$$

Gradient: $\nabla F(p) = AA^* p + Az \in (H_0^2(\Omega))^*$.

Let $K := \{p \in C_0(\Omega) : \|p\|_\infty \le \alpha\}$. Optimality is

$$0 \in \partial(F + I_K)(p^*) = \nabla F(p^*) + \partial I_K(p^*). \iff \exists\, \lambda^* \in N_K(p^*) \;\text{ s.t. }\; \nabla F(p^*) + \lambda^* = 0.$$

KKT system: find (p^*, λ^*) with

$$AA^* p^* + Az + \lambda^* = 0, \qquad \lambda^* \in N_K(p^*).$$

Equivalently to $\lambda^* \in N_K(p^*)$ (variational inequality):

$$\langle \lambda^*, p - p^* \rangle \le 0 \qquad \forall\, p \in K.$$

Optimality conditions — Primal identification and sparsity

From KKT: $\lambda^* \in N_K(p^*) \subset \mathcal{M}(\Omega)$. Define the primal optimizer by

$$u^* := -\lambda^* \in \mathcal{M}(\Omega).$$

Support / sparsity (inactive set): for $\psi \in C_c(\Omega)$, $\psi \geq 0$,

$$\text{supp}(\psi) \subset \{|p^*| < \alpha\} \quad \Longrightarrow \quad \langle u^*, \psi \rangle = 0.$$

Sign on the active set:

$$\text{supp}(\psi) \subset \{p^* = \alpha\} \Rightarrow \langle u^*, \psi \rangle \geq 0, \qquad \text{supp}(\psi) \subset \{p^* = -\alpha\} \Rightarrow \langle u^*, \psi \rangle \leq 0.$$

Interpretation: u^* concentrates on $\{|p^*| = \alpha\}$ (active set).

Numerics and regularization — Moreau–Yosida regularization of the box

Problem: The constraint $\|p\|_\infty \leq \alpha$ is nonsmooth (active-set structure).

Regularized predual problem $(P^*_{M,c})$**:**
For $c > 0$, let $p_c \in H^2_0(\Omega)$ be the unique minimizer of

$$\frac{1}{2}\|A^*p + z\|^2_{L^2(\Omega)} - \frac{1}{2}\|z\|^2_{L^2(\Omega)} + \frac{1}{2c}\|\max(0, c(p-\alpha))\|^2_{L^2} + \frac{1}{2c}\|\min(0, c(p+\alpha))\|^2_{L^2},$$

and define

$$\lambda_c := \max(0, c(p_c - \alpha)) + \min(0, c(p_c + \alpha)).$$

Then (p_c, λ_c) solves $AA^*p_c + Az + \lambda_c = 0$.

Convergence of Moreau–Yosida regularization

Theorem. Let (p^*, λ^*) solve the unregularized KKT system. Then as $c \to \infty$,

$$p_c \to p^* \text{ in } H_0^2(\Omega), \qquad \lambda_c \rightharpoonup \lambda^* \text{ in } (H_0^2(\Omega))^*.$$

Proof idea .

1. **Key inequality.** From the pointwise definition of λ_c one obtains $\langle \lambda_c, p_c \rangle_{L^2(\Omega)} \geq \frac{1}{c} \|\lambda_c\|^2_{L^2(\Omega)}$.
2. **Uniform bounds.** Test $AA^* p_c + Az + \lambda_c = 0$ with p_c to get bounds on $\|p_c\|_{H_0^2(\Omega)}$ and $\frac{1}{c}\|\lambda_c\|^2_{L^2(\Omega)}$.
3. **Compactness.** Extract a subsequence such that

$$p_c \rightharpoonup \tilde{p} \text{ in } H_0^2(\Omega), \qquad \lambda_c \rightharpoonup \tilde{\lambda} \text{ in } H_0^2(\Omega)^*.$$

4. **Feasibility in the limit.** The penalties $\max(0, p_c - \alpha)$ and $\min(0, p_c + \alpha)$ vanish in $L^2(\Omega)$ as $c \to \infty$, hence $|\tilde{p}| \leq \alpha$ a.e.
5. **Limit KKT + uniqueness.** Pass to the limit in $AA^* p_c + Az + \lambda_c = 0$ implies that $(\tilde{p}, \tilde{\lambda})$ solves the unregularized KKT system.

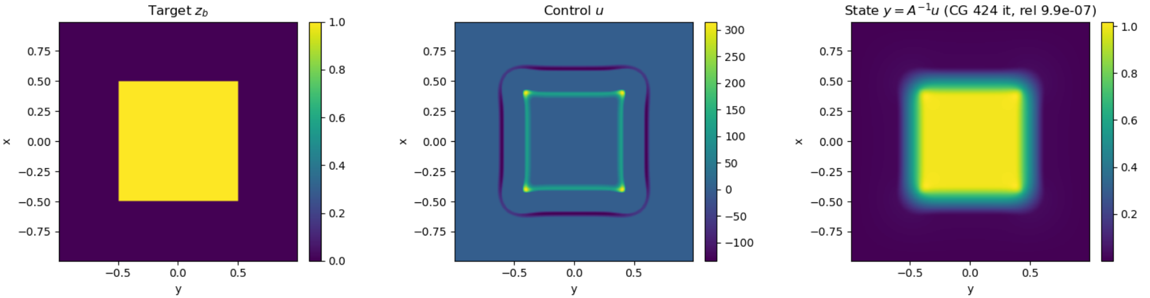
Numerics and regularization — Algorithm: Semismooth Newton

Goal

Solve the regularized optimality system $F(p) = 0$ for $p_c \in H_0^2(\Omega)$ via an active-set (piecewise linear) Newton iteration.

1. Choose an initial guess $p^0 \in H_0^2(\Omega)$ and set $k = 0$.
2. Repeat:
   (i) Define the active sets
   $$A_k^+ := \{x \in \Omega : \; p^k(x) > \alpha\}, \qquad A_k^- := \{x \in \Omega : \; p^k(x) < -\alpha\}, \qquad A_k := A_k^+ \cup A_k^-.$$
   (ii) Compute $p^{k+1} \in H_0^2(\Omega)$ by solving the linear equation (weak form)
   $$\langle A^* p^{k+1}, A^* v\rangle_{L^2} + c\langle p^{k+1}\chi_{A_k}, v\rangle_{L^2} = -\langle z, A^* v\rangle_{L^2} + c\alpha\langle \chi_{A_k^+} - \chi_{A_k^-}, v\rangle_{L^2} \quad \forall v \in H_0^2(\Omega).$$
   (iii) Set $k \leftarrow k + 1$.
3. Stop if $A_k^+ = A_{k-1}^+$ and $A_k^- = A_{k-1}^-$.

Experiments — Basic example



Qualitative validation. To sanity-check our implementation, we compare the computed control/state patterns against the benchmark examples for the square target z_b in Clason–Kunisch (2009).

- the optimal control concentrates on a thin ring near the interface of the target set,
- the resulting state reproduces the plateau inside the target with a smooth transition layer near the boundary,
- corner effects and sign changes align with the qualitative patterns reported in the reference.

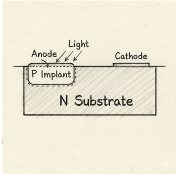
Experiments — Semiconductor example

Design trade-off. When light is absorbed in the semiconductor it generates electron–hole pairs. These carriers must be collected at the **p–n junction** (anode implant) and extracted at the contacts.

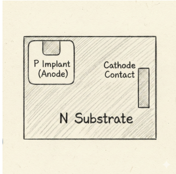
A central design objective is to balance:

- **Carrier collection efficiency:** the collecting region should be placed/extended such that carriers reach it before recombination or boundary loss.
- **Capacitance minimization:** the junction capacitance scales approximately with the **junction area**, $C_j \propto A_{\text{junction}}$, and a larger capacitance reduces bandwidth and increases noise.

Goal: find an optimal junction geometry that achieves *high collection* while keeping the *junction area small*.



Cross-section



Top view (layout)

Experiments — Semiconductor example

Context. We consider a steady-state concentration y of a species in a bounded domain $\Omega \subset \mathbb{R}^2$ (with $y = 0$ on $\partial\Omega$). Species are *generated* in a localized subregion and can be *removed* by placing sinks. We want the sinks to be *spatially sparse*. This is a simplified description of a photodiode in steady-state operation

State equation (diffusion). With diffusion coefficient $D > 0$ and operator

$$A_D := -D\Delta, \qquad y|_{\partial\Omega} = 0,$$

the steady-state balance reads

$$A_D y \; = \; g - u \quad \text{in } \Omega,$$

where g is a prescribed generation term (here: constant on a small central square, 0 outside) and u is the sink control.

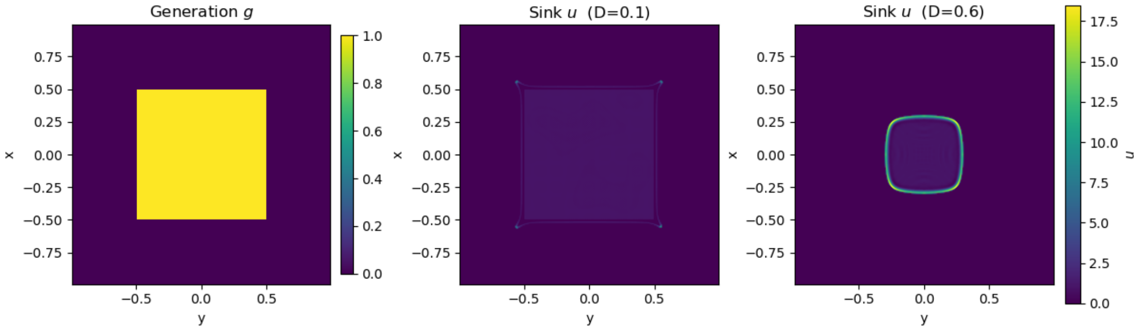
Optimal control problem (measure-sparse control). We set the desired target to $z \equiv 0$ (remove species everywhere) and penalize the sink by a *measure norm* to promote sparsity:

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2}\|y(u) - z\|^2_{L^2(\Omega)} \; + \; \alpha\|u\|_{\mathcal{M}(\Omega)} \qquad \text{s.t.} \qquad A_D y = g - u, \; y|_{\partial\Omega} = 0.$$

Here $\alpha > 0$ controls the sparsity strength: larger α yields fewer/stronger sink locations.

Experiments — Semiconductor example

Numerical solution on $\Omega = [-1,1]^2$ *(Dirichlet BC)*



For low diffusitivity the optimized sink u concentrates on pointlike structures surrounding the generation region. In contrast, in higher diffusivity environments, a ring like structure appears similar like a net for collecting the carriers. This structure is a direct consequence of the measure penalty, which promotes sparsity by localizing the control on sets of small measure.

Alternative numerical route: variational discretization (SparseFEM)

Different "philosophy" Discretize only the *state* equation by FEM, keep $u \in \mathcal{M}(\Omega)$ continuous.

Key theorem (informal) The discrete optimizer can be chosen as a *finite combination of Diracs at mesh nodes*:

$$u_h^* = \sum_{j=1}^{N(h)} \lambda_j \delta_{x_j}.$$

Comparison

- Predual approach: smooth variable + box constraints + Newton
- FEM sequel: state discretization induces nodal sparsity automatically

Conclusion

Main messages

- Measure controls model localized actuation and yield sparse optimal solutions
- Very weak solutions provide well-posed PDE state equations for $u \in \mathcal{M}(\Omega)$
- Fenchel duality gives a predual Hilbert-space formulation with $\|p\|_\infty \leq \alpha$
- KKT structure explains sparsity: u^* lives on the active set $\{|p^*| = \alpha\}$

Takeaway

Predual reformulation turns a difficult nonsmooth measure problem into a numerically friendly box-constrained PDE problem.

Thank you!