Elliptic Optimal Control with Measure-Valued Controls

Predual reformulation, sparsity, and semismooth Newton numerics

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Roadmap (25 min)

- Motivation: localized actuation and sparsity
- PDE with measure right-hand side: very weak solutions
- Primal problem in $\mathcal{M}(\Omega)$: existence/uniqueness
- Fenchel duality $\Rightarrow$ predual (Hilbert space) with box constraints
- KKT + sparsity structure (active set)
- 1D gravity example: explicit Dirac solution + threshold
- Numerics: Moreau–Yosida + semismooth Newton / PDAS
- (Brief) alternative FEM route: nodal Dirac controls

Motivation: why measure-valued controls?

Localized actuation

Point sources/sinks, actuators, supports, injections are naturally *low-dimensional* objects: Dirac masses, sums of Diracs, line/surface measures.

Sparsity promotion

Replacing quadratic control cost by an L^1-type cost promotes sparsity. In PDE settings, the measure space $\mathcal{M}(\Omega)$ is the natural closure/relaxation.

Main challenge

The problem is nonsmooth *and* posed in a nonreflexive space $\mathcal{M}(\Omega)$.

Elliptic PDE with measure right-hand side

We consider

$$Ay = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega, \qquad u \in \mathcal{M}(\Omega).$$

Why not the usual H_0^1 formulation?

A general measure u is *not* in $H^{-1}(\Omega)$, so testing with $v \in H_0^1(\Omega)$ is too strong.

Very weak formulation (Dirichlet)

Choose a test space $V \hookrightarrow C_0(\Omega)$, e.g. $V = H^2(\Omega) \cap H_0^1(\Omega)$ (in $n = 2, 3$). Then $y \in L^1(\Omega)$ solves

$$\int_\Omega y\, A^* \varphi\, dx = \int_\Omega \varphi\, du \qquad \forall \varphi \in V.$$

Regularity and the control-to-state map

Elliptic smoothing for measure data

For $u \in \mathcal{M}(\Omega)$, the very weak solution satisfies

$$y \in W_0^{1,p}(\Omega) \quad \text{for all } 1 \le p < \frac{n}{n-1}, \qquad \|y\|_{W^{1,p}} \le C\,\|u\|_{\mathcal{M}(\Omega)}.$$

Control-to-state operator

Define

$$S : \mathcal{M}(\Omega) \to L^2(\Omega), \qquad Su = y(u),$$

using Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ (for suitable p, $n \in \{2, 3\}$).

Key takeaway

Even if u is singular (Dirac), the state y is a genuine function usable in an L^2 tracking term.

Primal sparse control problem in $\mathcal{M}(\Omega)$

Given desired state $z \in L^2(\Omega)$ and $\alpha > 0$:

$$\min_{u \in \mathcal{M}(\Omega)} J(u) := \frac{1}{2} \|Su - z\|^2_{L^2(\Omega)} + \alpha \|u\|_{\mathcal{M}(\Omega)}.$$

Interpretation

- $\frac{1}{2} \|Su - z\|^2$: match the target state
- $\alpha \|u\|_{\mathcal{M}}$: pay for total control mass $\Rightarrow$ sparse actuation

Two difficulties

- Nonsmooth term $\|u\|_{\mathcal{M}}$
- Nonreflexive space $\mathcal{M}(\Omega)$

Existence and uniqueness (direct method)

Compactness mechanism

- Minimizing sequence (u_n) bounded in $\mathcal{M}(\Omega)$
- Banach–Alaoglu: $u_n \overset{*}{\rightharpoonup} u^*$ in $\sigma(\mathcal{M}(\Omega), C_0(\Omega))$
- States $y_n = Su_n$ bounded in $W_0^{1,p}(\Omega)$
- Rellich: $y_n \to y^*$ strongly in $L^2(\Omega)$

Lower semicontinuity

$$\|u^*\|_{\mathcal{M}} \leq \liminf_{n\to\infty} \|u_n\|_{\mathcal{M}}, \qquad \|Su_n - z\|_{L^2}^2 \to \|Su^* - z\|_{L^2}^2 .$$

So u^* is optimal.

Uniqueness

Strict convexity of the L^2 tracking term (and injectivity of S) gives uniqueness of the minimizer.

Why a predual formulation?

Goal

Avoid discretizing measures directly and replace nonsmooth penalty by a simple constraint.

Core idea (Fenchel duality)

The conjugate of $\alpha \|\cdot\|_{\mathcal{M}}$ is the indicator of a box in $C_0(\Omega)$:

$$(\alpha \|\cdot\|_{\mathcal{M}})^*(\varphi) = \begin{cases} 0, & \|\varphi\|_\infty \le \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

Computational win

Measure norm $\Rightarrow$ *box constraint* on a smooth variable.

Predual problem (Hilbert space + box constraint)

In the measure-control setting, the dual variable satisfies

$$p \in H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow C_0(\Omega).$$

Predual formulation

$$\min_{p \in H^2(\Omega) \cap H^1_0(\Omega)} F(p) := \frac{1}{2} \|A^* p + z\|^2_{L^2(\Omega)} - \frac{1}{2} \|z\|^2_{L^2(\Omega)} \quad \text{s.t.} \quad \|p\|_\infty \leq \alpha.$$

Meaning

All nonsmoothness is now in a *simple* pointwise constraint.

First-order optimality: KKT system

Let $K = \{p \in H^2 \cap H_0^1 : \|p\|_\infty \le \alpha\}$.

Stationarity + normal cone

There exists a multiplier $\lambda^* \in N_K(p^*)$ such that

$$\nabla F(p^*) + \lambda^* = 0, \qquad \lambda^* \in N_K(p^*).$$

Identification of the primal control

The multiplier corresponds to the primal solution:

$$\lambda^* = -u^*.$$

So, once p^* is computed, the optimal measure control is recovered.

Sparsity structure from complementarity

Saturation $\Rightarrow$ support of the measure

From $p^* \in \alpha\,\partial\,\|u^*\|_{\mathcal{M}}$ one gets:

$$\|p^*\|_\infty \le \alpha, \qquad \langle u^*, p^* \rangle = \alpha\,\|u^*\|_{\mathcal{M}}\,.$$

Active set characterization

The measure cannot charge the inactive region:

$$|u^*|\big(\{x:\ |p^*(x)| < \alpha\}\big) = 0.$$

Hence u^* is supported where $|p^*| = \alpha$.

Sign information

$$u^* \ge 0 \text{ on } \{p^* = \alpha\}, \qquad u^* \le 0 \text{ on } \{p^* = -\alpha\}.$$

1D gravity example: tensioned string

Let $\Omega = (0,1)$ and consider a string under gravity + actuator force u:

$$-y''_{\text{phys}} = 1 + u \quad \text{in } \mathcal{D}'(0,1), \qquad y_{\text{phys}}(0) = y_{\text{phys}}(1) = 0.$$

Shift to fit the abstract model

Gravity-only state:

$$-y''_g = 1, \quad y_g(0) = y_g(1) = 0 \quad \Rightarrow \quad y_g(x) = \frac{x(1-x)}{2}.$$

Let $y := y_{\text{phys}} - y_g$, then

$$-y'' = u, \qquad z(x) = -y_g(x) = -\frac{x(1-x)}{2}.$$

Explicit optimal control: a single Dirac

In this example one can construct a symmetric solution where the constraint is active only at $x = \frac{1}{2}$.

Result: optimal control is a Dirac

$$u^* = \left(48\alpha - \frac{5}{8}\right) \delta_{1/2}.$$

Interpretation

The optimal actuator is concentrated exactly at the midpoint, where the dual variable saturates the box constraint.

Message

Sparse control is not just a slogan: it becomes a literal point actuator in 1D.

When is it optimal to apply *no control*?

The optimal control vanishes iff the unconstrained predual solution stays inside the box:

$$u^* \equiv 0 \quad \iff \quad \|p_0\|_\infty \leq \alpha,$$

where p_0 solves the unconstrained KKT equation.

Gravity example: sharp threshold

One computes

$$\|p_0\|_\infty = \frac{5}{384}.$$

Hence

$$u^* \equiv 0 \quad \iff \quad \alpha \geq \frac{5}{384}.$$

Economic meaning

Control appears only when its benefit exceeds its cost parameter α.

Numerics: Moreau–Yosida regularization of the box

Problem

The constraint $\|p\|_\infty \leq \alpha$ is nonsmooth (active-set structure).

Regularize the indicator of the box constraint (Moreau–Yosida), leading to a smooth approximation with parameter $\gamma > 0$.

Effect

- Allows Newton-type methods
- Retains active-set interpretation
- As $\gamma \to \infty$, solutions converge to the exact constrained problem

Semismooth Newton / primal-dual active set (PDAS)

High-level idea

Rewrite the optimality system as a semismooth equation and apply Newton with generalized derivatives.

Algorithmic structure

- Given (p^k, λ^k), define active set via saturation of $|p^k|$
- Solve a linearized PDE system on inactive set
- Update and repeat until active set stabilizes

Performance

Locally superlinear convergence (under standard assumptions).

Alternative numerical route: variational discretization (FEM sequel)

Different philosophy

Discretize only the *state* equation by FEM, keep $u \in \mathcal{M}(\Omega)$ continuous.

Key theorem (informal)

The discrete optimizer can be chosen as a *finite combination of Diracs at mesh nodes*:

$$u_h^* = \sum_{j=1}^{N(h)} \lambda_j \delta_{x_j}.$$

Comparison

- Predual approach: smooth variable + box constraints + Newton
- FEM sequel: state discretization induces nodal sparsity automatically

Conclusion

Main messages

- Measure controls model localized actuation and yield sparse optimal solutions
- Very weak solutions provide well-posed PDE state equations for $u \in \mathcal{M}(\Omega)$
- Fenchel duality gives a predual Hilbert-space formulation with $\|p\|_\infty \le \alpha$
- KKT structure explains sparsity: u^* lives on the active set $\{|p^*| = \alpha\}$
- 1D example: explicit Dirac optimal control $+$ sharp no-control threshold

Takeaway

Predual reformulation turns a difficult nonsmooth measure problem into a numerically friendly box-constrained PDE problem.

Thank you!