The Bohm Plasma-Sheath Model and the Bohm Criterion Revisited

Natalia Sternberg and Valery Godyak, *Fellow, IEEE*

***Abstract*—The plasma–sheath model and the Bohm criterion introduced by Bohm are among the earliest attempts to separately model the plasma and the sheath and to find a way to join the plasma and the sheath solutions. Although there is hardly a paper on plasma–sheath modeling that does not quote the Bohm criterion, Bohm's paper and his results are widely misunderstood. The reason for this is that, in his paper, Bohm himself misinterpreted his result by concluding that the sheath edge coincides with the reference point of his plasma–sheath model. As a result, the criterion for the reference point obtained by Bohm to ensure monotonicity of his sheath solution (i.e., the Bohm criterion) was erroneously applied to the sheath edge and was used in literature as a criterion for sheath formation. In this paper, we show that the Bohm criterion when applied to the sheath edge contradicts Bohm's own definition of the sheath and cannot be obtained from Bohm's plasma–sheath model.**

***Index Terms*—Bohm criterion, plasma–sheath model.**

I. Introduction

It IS WELL known that a bounded plasma consists of bulk plasma and space-charge sheath. There is a transition region between the plasma and the sheath [1], [2]. The bulk plasma is characterized by quasi-neutrality, while the sheath is the region where no ionization occurs. Ionization is caused by electron collisions with gas atoms wherever electrons are present. Therefore, the lack of ionization in the sheath could be caused only by the absence of electrons. Indeed, it has been shown in [3], using asymptotic matching techniques, that the assumption of no ionization in the sheath implies that the electron density in the sheath is negligible. Thus, one can say that the sheath is an electron-free region, which is precisely Langmuir's [4, p. 970] view of the sheath, accepted by Bohm in [1, p. 79].

In literature, the plasma and the sheath are often modeled separately. No matter which model is used to describe the sheath, one has to address three separate issues: 1) what is the point of reference for the potential in the sheath model; 2) which point represents the sheath edge; and 3) which point can be used as a patching point in order to patch the plasma and the sheath solutions. In general, those are the three different points.

In [1], Bohm proposed a model, which he called the plasma–sheath model. That model can be derived from the standard fluid plasma-wall equations [5] by neglecting ionization but keeping the electron density. Bohm's goal was to find a solution that describes the sheath. He first specified a point of reference for the potential where the plasma is still quasi-neutral. Choosing a zero electric field at the reference point, he integrated the sheath model starting at that point. He then found that the solution he obtained is monotone only if the ion velocity at the reference point is greater than or equal to the ion sound speed. Bohm's result is purely mathematical and ensures existence of a monotone solution of his model. It is his misinterpretation of that mathematical result as a condition for the sheath formation that has been causing confusion in the literature.

Bohm justified his choice of a zero field as an initial condition for his solution by his claim that the electric field in the plasma is negligible. He was obviously trying to patch the plasma with the sheath solution, interpreting the reference point as the patching point. Thinking that his patching was successful, he further interpreted his solution as the sheath solution and the patching point as the sheath edge. Based on those interpretations, Bohm concluded that the sheath begins where the ions reach the ion sound speed. His conclusion is known in the literature as the Bohm criterion.

It is known today that Bohm's solution yields an infinite sheath [6]. It is known that Bohm's solution cannot be used for patching [2], [3], [7]. It is known that the Bohm criterion does not specify the sheath edge [3]. Nevertheless, the Bohm criterion has remained at the center of controversy about the position of the sheath edge. The present paper is an attempt to resolve this controversy by providing an insight into Bohm's model, his solution, and his conclusions that led to the formulation of the Bohm criterion.

In the present paper, using the geometric theory of differential equations [8], we give a mathematical analysis of the Bohm plasma–sheath model. Our analysis shows that Bohm's physical interpretation of his mathematical result, which led to the Bohm criterion, was based on his erroneous assumption that the reference point coincides with the patching point and the sheath edge. We show that Bohm's model and methods do not provide sufficient information about the position of the sheath edge. We show that a criterion for existence of a monotone solution of Bohm's plasma–sheath model is not sufficient for providing a criterion for the position of the sheath edge or for the sheath formation.

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N. Sternberg is with the Department of Mathematics and Computer Science, Clark University, Worcester, MA 01610 USA (e-mail: nsternberg@clarku.edu).

V. Godyak is with the OSRAM Sylvania, Beverly, MA 01915 USA (e-mail: valery.godyak@sylvania.com).

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II. Mathematical Analysis of the Bohm Sheath Model

The planar plasma-wall problem in the collisionless case with cold ions and no heat transfer can be described quite well by the following set of hydrodynamic equations [5]: the continuity equation of the ion flux

$$(nv)' = zn_{\mathrm{e}} \tag{1}$$

the momentum equation for ions

$$M(nv^2)' + en\varphi' = 0 \tag{2}$$

the Boltzmann distribution for electrons

$$kT_{\mathrm{e}}n_{\mathrm{e}}' = en_{\mathrm{e}}\varphi' \tag{3}$$

and the Poisson equation

$$\epsilon_0\varphi'' = -e(n - n_{\mathrm{e}}) \tag{4}$$

where n and n_{e} are the ion and electron densities, respectively, v is the ion velocity, φ is the potential, M is the ion mass, z is the frequency of ionization, e is the ion charge, T_{e} is the electron temperature, k is the Boltzmann constant, and ϵ_0 is the permittivity of free space. The derivatives are with respect to the space variable r.

The Bohm plasma–sheath model considered in [1] can be obtained from (1)–(4) by neglecting ionization and can be written as follows:

$$(nv)' = 0 \tag{5}$$

$$vv' + \frac{e}{M}\varphi' = 0 \tag{6}$$

$$\frac{n_e'}{n_e} = \frac{e}{kT_e}\varphi' \tag{7}$$

$$\varphi'' = -\frac{e}{\epsilon_0}(n - n_{\mathrm{e}}). \tag{8}$$

Bohm started the analysis of his plasma–sheath model by choosing a reference point in the quasi-neutral plasma (i.e., $n = n_{\mathrm{e}}$). At the reference point, he set the potential to φ_0, the ion velocity to v_0, and the ion and electron densities to n_0. His next step was to find a solution of (5)–(8) with the zero-field initial condition at the reference point, i.e., a solution through the point $(n_0, n_0, v_0, \varphi_0, 0)$. Observe that for all finite r, and for any choice of constants n_0, v_0, and φ_0

$$(n(r), n_{\mathrm{e}}(r), v(r), \varphi(r), E(r)) = (n_0, n_0, v_0, \varphi_0, 0) \tag{9}$$

is a constant solution (equilibrium solution) of Bohm's plasma–sheath model (5)–(8), where $E = -\varphi'$ is the electric field. According to the general theory of differential equations (namely, uniqueness of solutions with respect to initial data), there cannot be any other solution besides the equilibrium (9) which satisfies the initial condition prescribed by Bohm. There could, however, be solutions that approach the equilibrium (9) asymptotically as $|r| \to \infty$. One of those solutions was indeed found by Bohm [1]. In fact, his criterion of the existence of a "stable sheath" ($v_0 \geq v_s$) is precisely the condition of existence of a solution that approach the equilibrium (9) asymptotically as $r \to -\infty$. In Bohm's understanding, stable means asymptotically stable.

In order to explain Bohm's solution and his criterion, we now present a mathematical analysis of Bohm's plasma–sheath model. Observe first that integrating (5)–(7) with the initial condition (n_0, v_0, φ_0) yields

$$n = \frac{n_0 v_0}{v} \tag{10}$$

$$v = \sqrt{v_0^2 - \frac{2e}{M}(\varphi - \varphi_0)} \tag{11}$$

$$n_{\mathrm{e}} = n_0 \exp\left(\frac{e(\varphi - \varphi_0)}{kT_{\mathrm{e}}}\right). \tag{12}$$

Obviously, if $\varphi \to \varphi_0$, as $|r| \to \infty$, then $n \to n_0$, $n_{\mathrm{e}} \to n_0$, and $v \to v_0$. Thus, in order to find a solution that asymptotically approaches the equilibrium, it is sufficient to find a solution such that $E \to 0$ as $\varphi \to \varphi_0$. In order to find such solution, we use (10)–(12) and rewrite the Poisson equation (8) in the following form:

$$\varphi' = -E \tag{13}$$

$$E' = g(\varphi) \tag{14}$$

where

$$g(\varphi) = \frac{en_0}{\epsilon_0}\left[\frac{1}{\sqrt{1 - 2e(\varphi - \varphi_0)/(Mv_0^2)}} - \exp\left(\frac{e(\varphi - \varphi_0)}{kT_{\mathrm{e}}}\right)\right]. \tag{15}$$

System (13) and (14) is Hamiltonian [8] and has a first integral given by the total energy equation

$$H(\varphi, E) = \frac{1}{2}E^2 + G(\varphi) \tag{16}$$

where

$$G(\varphi) = -\frac{Mv_0^2 n_0}{\epsilon_0}\sqrt{1 - \frac{2e(\varphi - \varphi_0)}{Mv_0^2}} - \frac{n_0 kT_{\mathrm{e}}}{\epsilon_0}\exp\left(\frac{e(\varphi - \varphi_0)}{kT_{\mathrm{e}}}\right). \tag{17}$$

The orbits $(\varphi,\ E(\varphi))$ of (13) and (14) lie on the level curves $H =$ const. and are shown in Figs. 1–3, where the following normalizations were used:

$$y = \frac{n}{n_0} \quad u = \frac{v}{v_{\mathrm{s}}} \quad \eta = -\frac{e}{kT_{\mathrm{e}}}(\varphi - \varphi_0)$$
$$\psi = E\sqrt{\frac{\epsilon_0}{n_0 kT_{\mathrm{e}}}} \quad v_{\mathrm{s}} = \left(\frac{kT_{\mathrm{e}}}{M}\right)^{1/2} \tag{18}$$

with v_{s} being the ion sound speed. The relationship between the normalized potential and the normalized electric field in (18) shows that our normalization changed the scale of the

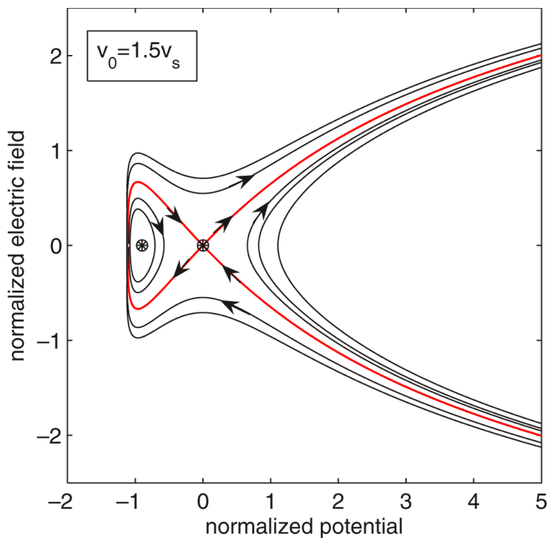

Fig. 1. Phase portrait of Bohm's plasma–sheath model for $v_0 = 1.5v_s$. The two equilibria are marked by dots: zero is a saddle, while the other one is a center. The red curves represent the stable (arrows point toward the equilibrium) and unstable (arrows point away from the equilibrium) manifolds.

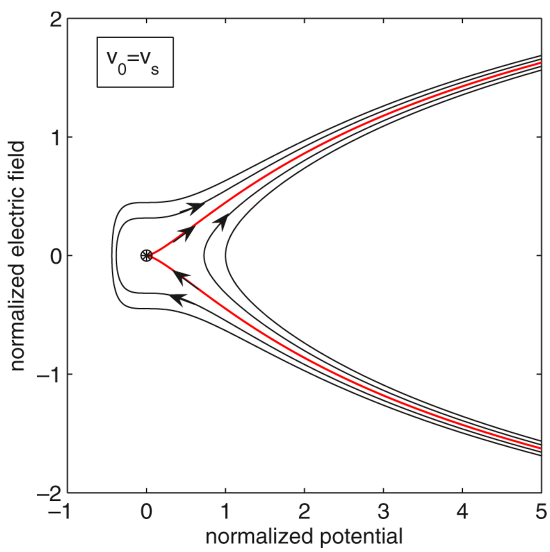

Fig. 2. Phase portrait of Bohm's plasma–sheath model for $v_0 = v_s$. The only zero equilibrium is marked by a dot; it is a saddle node. The red curves represent the stable (arrow points toward the equilibrium) and unstable (arrow points away from the equilibrium) manifolds. Bohm's solution is the unstable manifold.

problem. The relationship between the r and the sheath scale ξ is governed by the equation

$$dr = \lambda_{\mathrm{D0}} d\xi \tag{19}$$

where $\lambda_{\mathrm{D0}}^2 = \epsilon_0 k T_{\mathrm{e}} / (e^2 n_0)$ is the electron Debye radius at the equilibrium.

The level curves of H in normalized form are given by

$$\psi^2 = 2u_0^2 \sqrt{1 + \frac{2\eta}{u_0^2}} + 2e^{-\eta} + C \tag{20}$$

where C is an arbitrary constant, and u_0 (i.e., v_0) can be treated as a parameter of the problem. Note that $\eta \geq -u_0^2/2$.

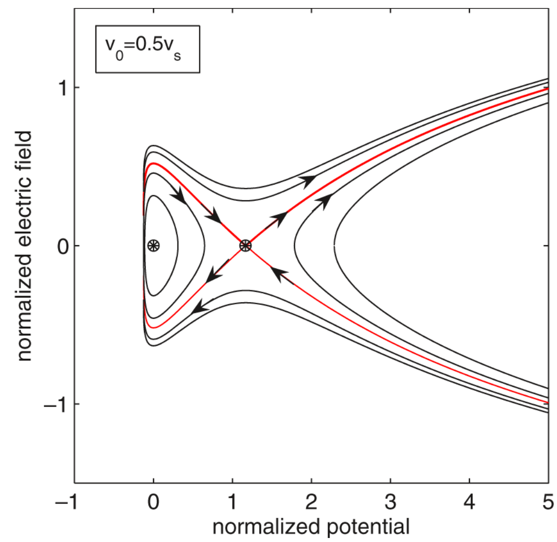

Fig. 3. Phase portrait of Bohm's plasma–sheath model for $v_0 = 0.5v_s$. The two equilibria are marked by dots: zero is a center, while the other one is a saddle. The red curves represent the stable and unstable manifolds.

Fig. 1 shows the phase portrait of the Bohm sheath model for $v_0 = 1.5v_s$, which is the plot of the normalized electric field ψ versus the normalized potential η. The arrows point in the direction in which r increases. One can see two constant orbits (equilibria) denoted by dots: $(\eta, \psi) = (0,\ 0)$ and $(\eta, \psi) = (-0.9,\ 0)$. The nonzero equilibrium is stable, but not asymptotically stable, which means that all orbits starting in a small neighborhood of that equilibrium remain in that neighborhood but do not approach the equilibrium. Moreover, those orbits are closed curves, and thus, they exhibit oscillatory behavior. An equilibrium with such properties is called a center [8]. The zero equilibrium is unstable in the sense that there are orbits that leave its neighborhood. No orbit can reach the equilibrium for a finite r since no two orbits can intersect. However, there are four special orbits in the neighborhood of zero shown in red. Two orbits approach zero asymptotically as $r \to \infty$. Those orbits are called stable manifolds. The two other orbits in red approach zero asymptotically as $r \to -\infty$. Those orbits are called unstable manifolds. The other orbits in the neighborhood of zero follow the flow of the stable and unstable manifolds. An equilibrium with such properties is called a saddle point [8].

Note that for $\eta < 0$, the stable and unstable manifolds merge, creating a loop which is called a soliton or a homoclinic orbit [8]. Such behavior is typical for Hamiltonian systems. For orbits originating inside the loop, the potential is oscillatory. For orbits originating outside the loop, the potential is piecewise monotone (decreasing when the electric field ψ is negative and increasing when the electric field is positive). The soliton separates the oscillatory solutions from the piecewise-monotone solutions. It is practically impossible to find the soliton or a point on the soliton by numerically solving the Bohm plasma–sheath model. Solving the plasma–sheath model, one could only find by trial and error points inside or outside the soliton [9]. For $\eta > 0$, the orbits eventually approach the unstable manifold as $r \to \infty$. Therefore, solving the plasma–sheath model with the initial conditions close to the equilibrium, one finds an approximation of the unstable manifold.

Although we have chosen a specific value for v_0, as long as $v_0 > v_{\rm s}$, the orbits of the Bohm sheath model will exhibit the same behavior as in Fig. 1. In particular, there will be one saddle point at $(\eta, \psi) = 0$, one center at $(\eta, \psi) = (\eta^*, 0)$ with $\eta^* < 0$, and one soliton.

Let us compare now Fig. 1 with Fig. 3 which shows the phase portrait of the Bohm sheath model for $v_0 = 0.5v_{\rm s}$. Although both figures look quite similar, the stability properties of the equilibria have changed. The zero has become a center, while the saddle point is now at $(\eta, \psi) = (\eta^*, 0)$ with $\eta^* > 0$. This behavior will persist as long as $v_0 < v_{\rm s}$.

The structural change that took place as the parameter v_0 moved from $v_0 > v_{\rm s}$ to $v_0 < v_{\rm s}$ suggests that the system underwent a bifurcation when $v_0 = v_{\rm s}$. Indeed, both equilibria move closer, as $v_0 \to v_{\rm s}$, and coalesce for $v_0 = v_{\rm s}$. The phase portrait of the Bohm sheath model for $v_0 = v_{\rm s}$ is shown in Fig. 2. There is only one equilibrium at zero, one stable manifold and one unstable manifold, both in red. Such a type of bifurcation is called a saddle-node bifurcation [8].

In summary, if $v_0 \geq v_{\rm s}$, the equilibrium (0, 0) (which is Bohm's reference point in our normalization) is a saddle, and there exist solutions that approach the equilibrium as $|r| \to \infty$ [see Figs. 1 and 2]. This is exactly Bohm's result. Bohm chose $v_0 = v_{\rm s}$ at the reference point and found the solution for $\psi > 0$ (i.e., $E > 0$) which approaches the equilibrium. In our normalization, that solution is obtained from (20), choosing $(u_0, \eta_0, \psi_0) = (1, 0, 0)$ as the boundary condition, and is given by

$$\psi = \left(2\sqrt{1+2\eta} + 2e^{-\eta} - 4\right)^{1/2}. \tag{21}$$

Fig. 2 shows that Bohm's solution (21) is precisely the unstable manifold of his plasma–sheath model for $v_0 = v_{\rm s}$.

The Bohm criterion $(v_0 \geq v_{\rm s})$ is the criterion of existence of the stable and unstable manifolds of the zero equilibrium of the model (5)–(8), which is a purely mathematical result. It is Bohm's interpretation of this result that puts it into a physical context. Bohm interpreted the reference point as the sheath edge and his solution as the sheath solution. His solution, however, is the unstable manifold and, therefore, cannot reach the reference point (equilibrium) for a finite r but asymptotically approaches it as $r \to -\infty$. Thus, Bohm's interpretation of his result makes the sheath infinite.

By choosing a specific boundary condition, Bohm found only one solution of the plasma–sheath model for $v_0 = v_{\rm s}$. There are however infinitely many choices for the parameter v_0 as well as for the initial conditions. Which choice is the best for approximating the plasma-wall problem? We will answer this question in the following section.

III. Relationship Between the Solutions of the Bohm Sheath Model and of the Plasma-Wall Problem

The symmetric plasma-wall problem is described by the unique solution of system (1)–(4) which satisfies the following boundary conditions at the center, $r = 0$:

$$n(0) = n_{\rm c} \quad v(0) = 0 \quad \varphi(0) = 0 \quad \varphi'(0) = 0 \tag{22}$$

and at the wall, $r = w$

$$n(w)v(w) = \gamma v_{\rm s} n_{\rm e}(w) \tag{23}$$

where $\gamma = [M/(2\pi m)]^{1/2}$. Equation (23) corresponds to the floating wall potential $\varphi(w) = \varphi_{\rm w}$. For the Bohm sheath model to be relevant, it should provide an approximation of that particular solution of the plasma-wall problem in the sheath region and at the wall. Furthermore, the solution of the Bohm sheath model that provides such an approximation should satisfy the boundary condition (23).

It is convenient to solve the plasma-wall problem using the following plasma coordinates referenced to the plasma center:

$$x = \frac{z}{v_{\rm s}} r \quad \overline{y}(x) = \frac{n(r)}{n_{\rm c}} \quad \overline{y}_{\rm e} = \frac{n_{\rm e}}{n_{\rm c}}$$
$$\overline{u}(x) = \frac{v(r)}{v_{\rm s}} \quad \overline{\eta}(x) = -\frac{e}{kT_{\rm e}}\varphi(r). \tag{24}$$

Comparing (24) with the normalization (18), (19), we find

$$\eta = \overline{\eta} + \frac{e}{kT_{\rm e}}\varphi_0 \quad dx = q_{\rm c} x_{\rm w} \sqrt{\frac{n_{\rm c}}{n_0}} d\xi \tag{25}$$

where $q_{\rm c} = \lambda_{\rm Dc}/w$ is the nonneutrality parameter at the center, and $x_{\rm w} = zw/v_{\rm s}$ is the normalized position of the wall. In particular, $q_{\rm c}x_{\rm w} = 4.036 \cdot 10^{-3}$, and $x_{\rm w} = 0.6286$ for argon (i.e., $\gamma = 108$) and $\lambda_{\rm Dl}/l = 0.01$, where l is the plasma half-length, and $\lambda_{\rm Dl}$ is the electron Debye radius at the plasma boundary, $r = l$, [3]. We have used those values to compute the solution of the plasma-wall problem and found that $\overline{\eta}(x_{\rm w}) \approx 5.35$ and $\overline{y}(x_{\rm w}) \approx 0.16$ at the wall.

The analysis of the Bohm sheath model given in the previous section did not depend on the specific choice of values for n_0 and φ_0 at the reference point. However, according to (25), in order to compare our solution of the plasma-wall problem with the solutions of the Bohm sheath model, one needs to know n_0 and φ_0. Those values are determined by the boundary condition at the wall and by the value of the parameter v_0. Indeed, using (10)–(12) together with (23), we find

$$\frac{e}{kT_{\rm e}}\varphi_0 = -\overline{\eta}(x_{\rm w}) + \ln\left(\frac{\gamma}{u_0}\right) \tag{26}$$

$$\frac{n_0}{n_{\rm c}} = \overline{y}(x_{\rm w})\sqrt{1 + \frac{2}{u_0^2}\ln\left(\frac{\gamma}{u_0}\right)} \tag{27}$$

where, as above, $u_0 = v_0/v_{\rm s}$.

Figs. 4–6 show the phase portrait of the Bohm plasma–sheath model for different parameter values v_0 together with the numerical solution of the plasma-wall problem (red curve). The curve with the arrow is the unstable manifold of the saddle point. As one can see, in the sheath region, there are good approximations of the plasma-wall problem by solutions of the plasma–sheath model for any v_0, but the unstable manifold is not always the best one. Within a given accuracy, there will

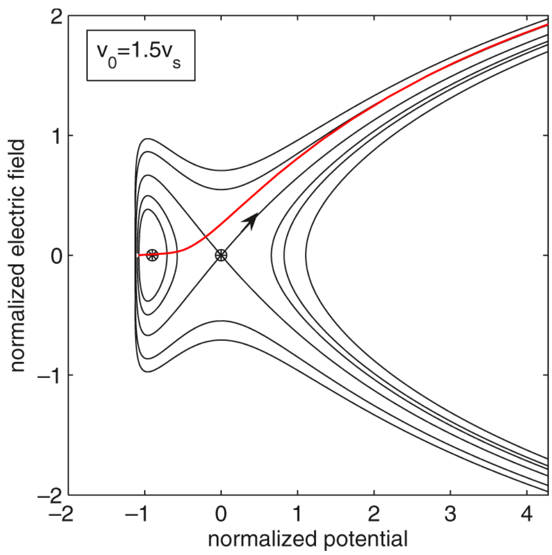

Fig. 4. Phase portrait of Bohm's plasma–sheath model for $v_0 = 1.5v_{\rm s}$ and the solution of the plasma-wall problem (red). The unstable manifold is marked with an arrow.

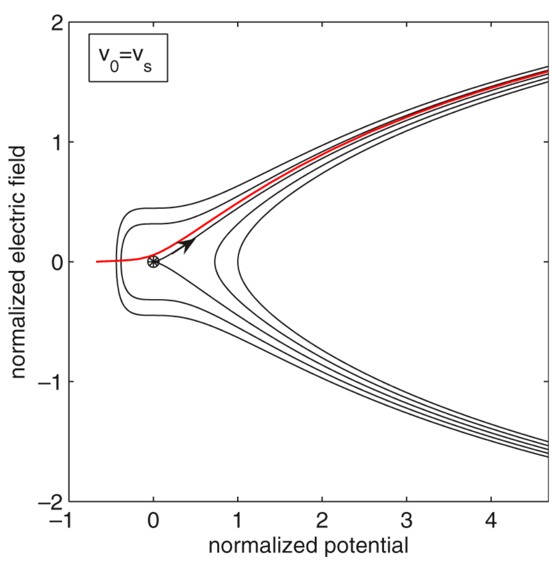

Fig. 5. Phase portrait of Bohm's plasma–sheath model for $v_0 = v_{\rm s}$ and the solution of the plasma-wall problem (red). Bohm's solution is marked with an arrow.

be infinitely many such approximations for any choice of the parameter value v_0. Thus, Bohm's idea that one can model the sheath only using the unstable manifold (i.e., a monotone solution that approaches the reference point) is not accurate, which brings his criterion into question.

The arbitrariness in the choice of v_0 brings ambiguity into the problem. Should there be a preference for v_0? How can one predict which solution of the Bohm plasma-sheath model will provide an adequate approximation of the plasma-wall problem? It is obvious by looking at Figs. 4 and 6 that there is no preferred value $v_0 > v_{\rm s}$ or $v_0 < v_{\rm s}$. Furthermore, one can neither predict which curve yields good approximations for a specific choice of $v_0 \neq v_{\rm s}$ nor can one find a point on such curve (i.e., an initial condition) without solving the plasma-wall problem first. Fig. 5, however, shows that for $v_0 = v_{\rm s}$,

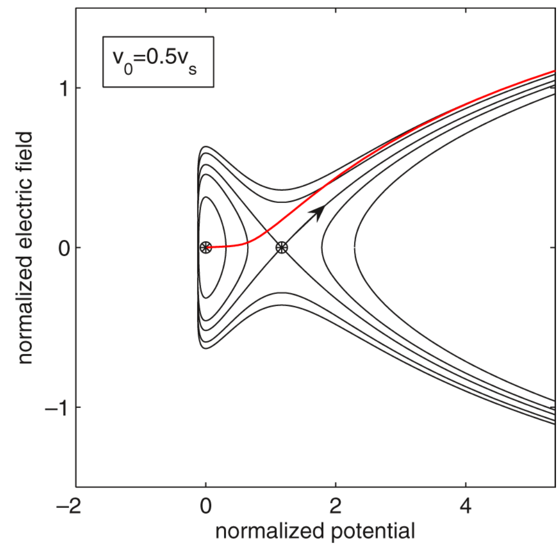

Fig. 6. Phase portrait of Bohm's plasma–sheath model for $v_0 = 0.5v_{\rm s}$ and the solution of the plasma-wall problem (red). The unstable manifold is marked with an arrow.

the unstable manifold provides a good approximation of the solution of the plasma-wall problem near the wall. Moreover, the unstable manifold is a special solution that approaches the equilibrium $(\varphi, E) \to 0$ as $r \to -\infty$. Using this as a boundary condition for the plasma–sheath model, one finds precisely Bohm's solution (21).

In summary, by choosing $v_0 = v_{\rm s}$, one can eliminate the ambiguity in the problem, which is probably the reason why Bohm made this choice. Furthermore, if $v_0 = v_{\rm s}$ is chosen as the reference point, Bohm's solution (21) can be used as the approximation of the plasma-wall problem in the sheath region.

IV. Patching Plasma and Sheath Solutions

In practice, solving the plasma-wall problem is not easy and may not give all the answers concerning plasma or sheath characteristics. For this reason, the plasma and sheath are often modeled separately, which should eliminate the need for solving the plasma-wall problem. When the plasma and sheath are modeled separately, the crucial question is how to patch the corresponding plasma and sheath solutions to approximate the solution of the plasma-wall model. Figs. 7–9 provide an insight into this problem. They show the plasma solution (blue curve), the solution of the plasma-wall model (red curve), and the phase portrait of the Bohm plasma–sheath model for different reference points v_0. The plasma model is obtained from the plasma-wall model (1)–(4) by assuming neutrality, i.e., by setting $n = n_{\rm e}$, [10], [11]. The plasma potential is

$$\frac{e}{kT_{\rm e}}\varphi = -\ln\left(1 + \frac{v^2}{v_{\rm s}^2}\right) \tag{28}$$

and the plasma electric field is

$$\frac{e}{kT_{\rm e}}E = -\frac{d\varphi}{dr} = \frac{z}{v_{\rm s}} \cdot \frac{2v/v_{\rm s}}{1 - v^2/v_{\rm s}^2}. \tag{29}$$

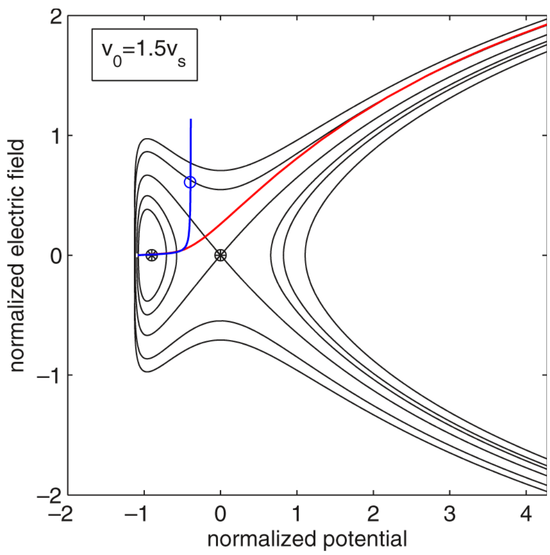

Fig. 7. Phase portrait of Bohm's plasma–sheath model for $v_0 = 1.5v_s$, the solution of the plasma-wall problem (red), and the plasma solution (blue). The possible patching point is marked by a blue dot.

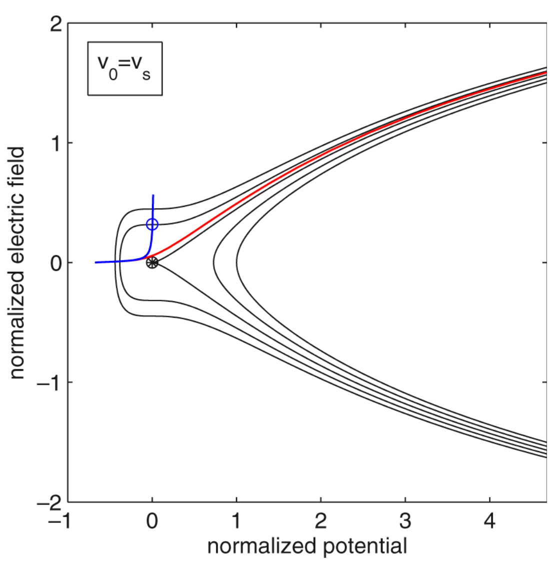

Fig. 8. Phase portrait of Bohm's plasma–sheath model for $v_0 = v_s$, the solution of the plasma-wall problem (red), and the plasma solution (blue). The possible patching point is marked by a blue dot.

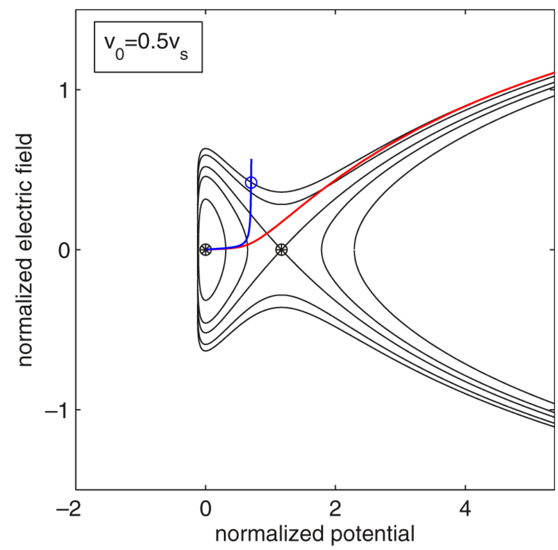

Fig. 9. Phase portrait of Bohm's plasma–sheath model for $v_0 = 0.5v_s$, the solution of the plasma-wall problem (red), and the plasma solution (blue). The possible patching point is marked by a blue dot.

As one can see in Figs. 7–9, the plasma solution approximates the plasma-wall problem in the quasi-neutral region before the gradient of the electric field becomes infinite. For $v_0 > v_s$ in Fig. 7, as well as for $v_0 < v_s$ in Fig. 9, the plasma solution intersects infinitely many sheath solutions. In particular, it intersects the sheath solution that approximates the plasma-wall solution at the point denoted by the blue circle. Therefore, one could use this point as a patching point. In that case, the plasma model could be solved up to the patching point, and the Bohm plasma–sheath model could be solved starting at the patching point. In other words, the patched curve follows the plasma solution up to the patching point and then switches to the sheath solution. As a result, one would obtain a good approximation of the plasma-wall problem in the quasi-neutral bulk plasma, in the sheath, and in the transition region. Note that, in general, the patching point is not the reference point.

As mentioned in the previous section, for $v_0 \neq v_s$, one does not know *a priori* which solution of the sheath model yields an adequate approximation of the plasma-wall problem. Because of this, the patching point can only be numerically found by trial and error, comparing the patched solution with the solution of the plasma-wall problem [9]. On the other hand, as mentioned in the previous section, for $v_0 = v_s$, the plasma-wall problem can be approximated by the unstable manifold of Bohm's model [i.e., by Bohm's solution (21)], thus eliminating the ambiguity in the choice of v_0 and in the choice of the approximation. However, the unstable manifold does not intersect the plasma solution, as shown in Fig. 8, and therefore, the two solutions can never be patched. Thus, one has to use another sheath solution which intersects the plasma solution and approximates the plasma-wall problem. Fig. 8 shows the patching point (blue circle) between the plasma solution and such a sheath solution. However, because of the proximity of the patching point to the equilibrium, the patching point needs to be chosen very carefully in order to achieve a good approximation of the plasma-wall problem. In [7], a technique for analytically finding an adequate patching point was developed, and it was found that the ion velocity at the patching point is $v = v_s[1 - 0.6(\lambda_{DI}/l)^{1/2}]$.

In summary, for any choice of v_0, one can find by trial and error a patching point that will provide an adequate approximation of the plasma-wall solution by the plasma and the sheath solutions. However, only by choosing $v_0 = v_s$ one can find analytically an adequate patching point without solving the plasma-wall problem.

V. Discussion

The Bohm criterion has been widely used in the literature and can be formulated as follows: the ions enter the sheath with the velocity equal to the ion sound speed. This statement

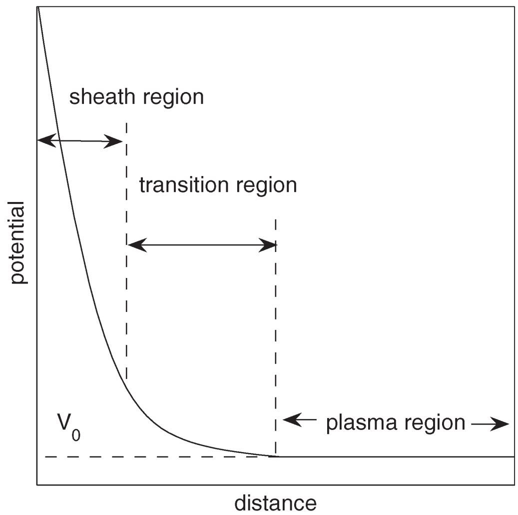

Fig. 10. Bohm's diagram of the plasma–sheath structure with his initial potential V_0 at the boundary between the plasma and the transition region.

originated from the very last sentence in Bohm's classical paper [1]: "Thus, it may be said roughly that a sheath begins when $V_0 = kT_e/2e$ [V is the potential], remembering that this means that beyond this point the potential begins to climb rapidly and the electron density to drop very rapidly." This sentence by Bohm, however, contradicts his solution [see (21)] that shows that $V \to V_0$, $dV/dx \to 0$, and $dn_e/dx \to 0$ as $x \to -\infty$. The potential begins to climb rapidly, and the electron density begins to drop rapidly only near the wall.

To explain his choice of V_0, Bohm included a diagram, which we reproduced in Fig. 10. In that diagram, Bohm marked the plasma, the sheath, and the transition region which separates the plasma from the sheath. According to his own diagram, the reference point V_0 is not the sheath edge but the boundary between the plasma and the transition region. Obviously, his diagram contradicts his conclusion.

To explain the three regions of the plasma-wall problem, Bohm [1] wrote: "Although there is no precise point at which the sheath begins, there is a transition region in which the plasma region, characterized by the stability of the neutral state of zero field, is in short distance replaced by the sheath region, characterized by negligible electron density." However, according to Bohm's solution, the transition region is not small but infinite. Furthermore, according to his diagram, the electric field in most of the transition region is quite small, and so is the gradient of the electron density. Therefore, the change in the electron density from $n_e = n$ for $V = V_0$ to $n_e < n$ inside the transition region occurs gradually. Thus, according to Bohm's diagram (see Fig. 10), the electron density in the transition region is not negligible, and therefore, V_0 cannot be the sheath edge.

By choosing V_0 as the sheath edge, did Bohm propose a step-front electron sheath model? If so, then beyond the point V_0, the electron density is negligible, and his equation for the electric field [which in our normalization is (21)] becomes

$$\psi = \left(2\sqrt{1+2\eta} - 4\right)^{1/2}. \tag{30}$$

However, at Bohm's sheath edge $\eta = 0$ (i.e., $V = V_0$), the electric field in (30) becomes imaginary. The reason for this contradiction is that the step-front formulation of the plasma–sheath model is inconsistent with the zero field in the plasma chosen by Bohm. Indeed, the ions can be accelerated only in a nonzero electric field.

Thus, Bohm's diagram and comment, although correct from the physics point of view, are inconsistent with his solution. What is the actual result of Bohm's paper [1], and what conclusion can be drawn based on that result?

In order to understand Bohm's results, one needs to understand how it was obtained. Bohm started by choosing a point of reference for the potential, where the plasma is still quasi-neutral, and assumed that the electric field in the quasi-neutral plasma can be set to zero. After setting a zero field at the reference point as an initial condition, he integrated his plasma–sheath model (5)–(8) and found a criterion ($V_0 \geq kT_e/2e$, or equivalently $v_0 \geq v_s$) that ensures monotonicity of the solution he obtained.

By choosing a point in the quasi-neutral plasma as the initial condition, Bohm was obviously attempting to patch the plasma with the solution of his plasma–sheath model (see his diagram in Fig. 10). Because of this, he mistook the reference point for the patching point and concluded that the reference point is the sheath edge. Thus, Bohm misinterpreted his own result, which caused its misinterpretation in the literature for over half a century.

Should have Bohm concluded that the ions exit the quasi-neutral plasma and enter the transition region with the ion sound speed? We know today that this statement is roughly true [2], [3], [10], [11]. The transition region obtained by asymptotic matching is centered at the point where the ions reach the ion sound speed, and it overlaps with the plasma region for lower ion velocities [2], [3]. This result, however, does not follow from Bohm's analysis. As we have shown in Section II, Bohm's solution is exactly the unstable manifold which approaches the reference point $(\eta_0, \psi_0) = (0, 0)$ only as $r \to -\infty$ [see Fig. 2]. Thus, if Bohm's solution were to represent the sheath with the sheath edge at $u_0 = 1$ (i.e., $v_0 = v_s$), then the sheath would be infinite. This same conclusion was reached in [6] using a different argument. Similarly, if Bohm's solution were to represent the transition region together with the sheath such that u_0 is the position of the boundary between the plasma and the transition region, then the plasma boundary would be infinitely removed from the wall.

Integrating the Poisson equation with the zero-field boundary condition [which is in our normalization $(\eta, \psi) = (0, 0)$], Bohm found that a monotone solution approaching (0, 0) [i.e., the unstable manifold of (0, 0)] exists only if $u_0 \geq 1$ (i.e., $v_0 \geq v_s$), while, when $u_0 < 1$ (i.e., $v_0 < v_s$), the potential in the neighborhood of zero becomes oscillatory. Bohm's result is shown in Figs. 1–3. Indeed, for $u_0 \geq 1$, the equilibrium (0, 0) is a saddle, and the unstable manifold exists, while for $u_0 < 1$, the equilibrium (0, 0) is a center, and the potential is oscillatory. By fixing the boundary condition before analyzing (5)–(8), Bohm missed all other solutions of the plasma–sheath model. Furthermore, he made conclusions about the sheath based on this one solution, without comparing it with the solution of the plasma-wall problem (1)–(4). He claimed that "a stable sheath is possible only when ions reach the sheath with the kinetic energy at least half the electron temperature" (in other words, when $v_0 \geq v_{\rm s}$), and "if the solutions are oscillatory, it is impossible for the voltage to increase indefinitely, as would be required for the sheath formation." However, the plasma-wall potential is never oscillatory. Thus, the oscillatory solution that Bohm mentioned cannot be the one representing the sheath.

Our results illustrated in Figs. 4–6 show that Bohm's statements about an oscillatory sheath are not accurate. We have found that for any choice of v_0 as the reference point, there are always solutions of the plasma–sheath model (5)–(8), which approximate the solution of the plasma-wall problem (1)–(4) in the sheath region. For $v_0 = v_{\rm s}$, one such approximation is indeed the unstable manifold, which is Bohm's solution given by (21) [see Fig. 5]. Hence, one could choose any v_0 as the reference point to set up the plasma–sheath model. Thus, contrary to Bohm's conclusion, which is widely accepted in the literature, the Bohm criterion is not a necessary condition for the formation or existence of a collisionless sheath. However, to avoid ambiguity in the choice of v_0, it is convenient to choose $v_0 = v_{\rm s}$.

Choosing $v_0 = v_{\rm s}$ as the reference point causes difficulties when one needs to patch the plasma with the sheath solution in order to approximate the plasma-wall problem from the center to the wall. Although for $v_0 = v_{\rm s}$ the unstable manifold approximates the solution of the plasma-wall problem in the sheath, it cannot be patched with the plasma solution [see Fig. 8]. There are, however, other solutions of the plasma–sheath model for $v_{\rm s} = v_0$ which can be used for patching. Those solutions approximate the plasma-wall problem and intersect the plasma solution [see Fig. 8]. Based on the proximity of those solutions to the unstable manifold (Bohm's solution), a technique was introduced in [7] to find one such solution and the corresponding patching point. Thus, although Bohm's solution is helpful in finding an approximation of the plasma-wall solution in the sheath, at the end, it is not the solution that can provide an approximation of the plasma-wall problem from the plasma center up to the wall. In general, the reference point chosen for the plasma–sheath model is not the patching point between the plasma and the sheath solution [see Figs. 7 and 8].

It remains to be clarified where the position of the sheath edge is according to Bohm's solution. Note that by choosing $v_0 = v_{\rm s}$, we have chosen the reference point at the plasma boundary, and therefore, the electron Debye radius at the reference point [see (19)] is in fact the electron Debye radius at the plasma boundary $r = l$, i.e., $\lambda_{\rm D0} = \lambda_{\rm Dl}$. Thus, according to (10), by setting $v_0 = v_{\rm s}$, we set $n/n_0 = n_{\rm e}/n_0 = 1$ at the plasma boundary. Furthermore, (10)–(12) [in the normalization (18)] are now reduced to

$$yu = 1 \tag{31}$$

$$u = \sqrt{1 + 2\eta} \tag{32}$$

$$y_{\rm e} = \exp(-\eta). \tag{33}$$

Equation (32) shows that beyond the reference point (i.e., beyond the plasma boundary), the ion velocity exceeds the ion sound speed. Hence, the ions enter the sheath with the velocity $u > 1$ (or, $v > v_{\rm s}$). This is Bohm's actual result.

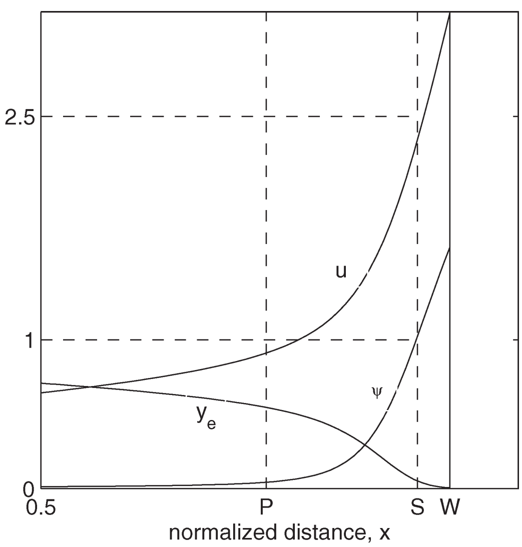

Fig. 11. Normalized ion velocity u, electron density $y_{\rm e}$, and electric field ψ of the plasma-wall problem for argon with $\lambda_{\rm Dl}/l = 0.01$. The plasma boundary is denoted by P, the sheath boundary by S, and the floating wall by W.

In order to find the value of the ion velocity at the sheath edge from Bohm's solution, recall that the electron density $e^{-\eta}$ is negligible in the sheath (see Bohm's definition of the sheath above as well as the reasoning in [3]). Therefore, the equation for the electric field (21) in the electron-free sheath can be reduced to (30), which, according to (31), is equivalent to

$$\psi^2 = 2u - 4. \tag{34}$$

According to [3] and [12], the electric field at the sheath edge is $E = kT_{\rm e}/(e\lambda_{\rm Dl})$ (i.e., $\psi = 1$ in our normalization). Substituting this value for ψ into (34) yields the value $u = 2.5$ (or $v = 2.5v_{\rm s}$) at the sheath edge [3]. Thus, the criterion for entering the sheath should be as follows: the ions enter the sheath with the velocity $v = 2.5v_{\rm s}$.

Note once more that one has to distinguish between the reference point of the plasma–sheath model and the sheath edge. Indeed, in the normalization (18), at the reference point, $(y, u, \eta, \psi) = (1, 1, 0, 0)$, while at the sheath edge, $(y, u, \eta, \psi) = (0.4, 2.5, 2.625, 1)$.

Fig. 11 shows the ion velocity $u = v/v_{\rm s}$, the electron density $y_{\rm e} = n/n_{\rm c}$, and the electric field $\psi = E\sqrt{\epsilon_0/(n_0 kT_{\rm e})}$ obtained by solving the plasma-wall problem (1)–(4) for argon with $\lambda_{\rm Dl}/l = 0.01$. In the figure, $W = 0.6286$ is the position of the floating wall [3], $P = 0.5708$ is the position of the plasma boundary [10], [11], and $S = 0.6184$ is the position of the sheath edge [3]. Observe that at the sheath edge, the electron density is indeed negligible. Recall that the ion velocity at the plasma boundary, predicted by the plasma model [10], [11], is $u(P) = 1$, and the ion velocity at the sheath edge, predicted by Bohm's sheath solution, is $u = 2.5$. According to Fig. 11, however, $u(P) < 1$ and $u(S) < 2.5$, although the values are quite close to the predicted ones. The reason for this discrepancy lies in the fact that the plasma and sheath solutions approximate well the plasma-wall problem away from the respective boundaries. On the other hand, as one can see, the prediction in [12] for the electric field $\psi(S) = 1$ is much more accurate. The reason for this is that the value for the electric field predicted in [12] was obtained from the Boltzmann electron distribution which is valid for the solution of the full plasma-wall problem (see [3] for a detailed discussion).

In light of all the advances made in the modeling of the plasma and sheath, we can now make the following statements. The ions exit the quasi-neutral plasma with a velocity slightly less than the ion sound speed (i.e., $v < v_{\mathrm{s}}$, but $v \approx v_{\mathrm{s}}$) [2], [3], [10], [11]. The ion velocity in the sheath is greater than ion sound speed (i.e., $v > v_{\mathrm{s}}$) [1]. The ion velocity at the sheath edge is $v \approx 2.5v_{\mathrm{s}}$ [3]. The electric field at the sheath edge is $E \approx kT_{\mathrm{e}}/(e\lambda_{\mathrm{Dl}})$ [3], [12]. Centered at the point where $v = v_{\mathrm{s}}$, there is a transition region where the ions are accelerated from a velocity less than the ion sound speed in the plasma to the velocity of about $2.5v_{\mathrm{s}}$ at the sheath edge [2], [3].

What does Bohm's solution describe? Does it describe a meaningful plasma-wall system? In general, Bohm's monotone solution describes a collisionless semi-infinite plasma-wall system without ionization, consisting of the sheath, the transition region, and the infinite quasi-neutral plasma. To be stable, such a system requires ion injection to equalize the ion sink to the wall. Bohm's zero-field boundary condition, together with the monotonicity requirement, yields that the ions must be injected with a speed greater than or equal to the ion sound speed.

There is no doubt that Bohm's pioneering paper [1] played a fundamental role in the modeling of the sheath. His model became a building block for other sheath models. His ideas and method became a foundation for much current research on plasma–sheath transition [13], [14]. Despite the fact that he did not find the position of the sheath edge, he found that in a collisionless sheath, the ion velocity exceeds the ion sound speed, which should be considered as the true Bohm criterion.

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Natalia Sternberg was born in Czernowitz, Ukraine, on April 7, 1954. She received the M.S. degree in mathematics from the University of Cologne, Cologne, Germany, in 1979, and the M.S. and Ph.D. degrees in applied mathematics from Brown University, Providence, RI, in 1982 and 1985, respectively.

From 1985 to 1987, she was a Visiting Assistant Professor with Northeastern University, Boston, MA, and Brown University. In 1987, she joined Clark University, Worcester, MA, where she is currently a Professor of mathematics and computer science.

Her research interest is in developing analytical and numerical methods to study the dynamical properties of solutions of differential equations that occur in physical and biological models. She is currently working on mathematical models in plasma science.

Valery Godyak (M'91–SM'95–F'98) was born in Chernowitz, Ukraine, on June 8, 1941. He received the M.S. degree in physics–electronics from St. Petersburg Technical University, St. Petersburg, U.S.S.R. (now Russia), in 1964 and the Ph.D. degree in plasma physics from Moscow State University (MSU), Moscow, Russia, in 1968.

In 1964, he became an Assistant Professor of physics with Ryazan Radio University, Ryazan, Russia. In 1968, he joined the Laboratory of Fusion Engineering, Institute of Electro-Physical Apparatus, St. Petersburg, where he conducted research and development on high-current relativistic electron accelerators, including field emission, electron optics, and diagnostics of relativistic electron beams. In 1972, he returned to the Physics Department, MSU, where he was a Team Leader, conducting basic research on RF discharges. In 1984, he immigrated to the U.S. and joined GTE Corporation (currently OSRAM Sylvania, Inc., Beverly, MA), where he is currently holding the position of a Corporate Scientist. His research interests are gas discharge physics, interaction of RF field with bounded plasma, electron kinetics and plasma electrodynamics, plasma diagnostics, light-source science and technology, plasma devices for material processing, and plasma system design.

Dr. Godyak is a winner of “OSRAM STAR,” “SIEMENS International Innovation Competition,” “James Clerk Maxwell Prize in Plasma Physics,” and “OSRAM SYLVANIA Innovation” awards for the development of RF lighting systems. He is a Fellow of the American Physical Society.