Plasma-Sheath Transition in the Magnetized Plasma-Wall Problem for Collisionless Ions

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Abstract—**A model for the collisionless plasma-wall problem under the action of an applied magnetic field is developed. The behavior of its solution is examined and found to be qualitatively consistent with experiment. The plasma and the sheath are then modeled separately to obtain the position of the quasi-neutral plasma boundary and the position of the edge of the electron-free sheath. It is shown that the plasma boundary can be specified as the point where the component of the ion velocity normal to the wall reaches the ion sound speed (Bohm criterion), and the sheath edge is specified as the point corresponding to Godyak's condition for the electric field. Studying the behavior near the plasma boundary and the sheath edge, the plasma solution and the solution of the space charge region are patched together to approximate the solution of the plasma-wall problem.**

Index Terms—**Magnetized plasma-wall problem, magnetized quasi-neutral plasma, magnetized sheath, patching, quasi-neutral plasma boundary, sheath edge.**

I. Introduction

THE BEHAVIOR of plasma and sheath characteristics under the action of an applied magnetic field is important in many applications including plasma probes and materials processing. In recent years, a great deal of interest has arisen in the possibility of using magnetized plasmas for hypersonic flow control, especially in mitigating the extreme heating that occurs in hypersonic flight. Given the high altitudes and concomitant low densities associated with flight at extreme speeds, sheaths are expected to play a prominent role in the interaction of the ionized gas flow with solid surfaces. It seems that the challenge in these applications is to understand and control the behavior of the ion flow toward an absorbing wall or electrode.

A number of articles can be found in the literature that study the kinetic model of magnetized plasmas [1]–[5]. Although the fluid model cannot provide all the information obtained by kinetic theory or particle simulation, it is believed to give a quite accurate description of the processes that occur in magnetized plasmas. In the present paper, we develop a hydrodynamic plasma-wall model and study the plasma and sheath characteristics for various magnitudes and directions of the magnetic field. Our model assumes direct ionization by electron-neutral collisions and low pressures so that ion-neutral collisions can be neglected. Our computations show that the magnetic field causes a quite strong ion drift normal to the magnetic and electric fields, especially in the sheath region near the wall. The fact that such behavior has been previously observed experimentally [6] motivates and partially validates our model.

It is important in many applications to find the relationship between the plasma and the sheath parameters. For that reason, the plasma and the sheath are modeled separately. There is a number of hydrodynamic models that deal with magnetized quasi-neutral plasmas. Probably the most prominent of them all is Chodura's [7] model for quasi-neutral collisionless plasma in an oblique magnetic field. According to Chodura, plasma particles pass through three regions on their way toward the wall. The first one is the quasi-neutral plasma region where the electric field is weak and the ions are aligned with the magnetic field. The second region (the Chodura layer) is the quasi-neutral magnetized presheath where the electric field becomes strong enough to deflect the ions from their motion along the magnetic field lines. The third region is a collisionless thin space charge sheath with strong electric field that directs the ions toward the wall. The ions enter the Chodura layer when their velocity component in the direction of the magnetic field reaches the ion sound speed (the Chodura criterion), and they exit the Chodura layer into the collisionless space charge sheath when the velocity component in the direction of the electric field (i.e., normal to the wall) reaches the ion-sound speed (the Bohm criterion). These findings by Chodura were further analyzed in [8] and [9] and justified by the inertia into the direction normal to both the electric and magnetic fields. However, Chodura's assumption that the ions are aligned with the magnetic field, as well as the neglect of ionizations and collisions in his model, yield an infinite magnetized presheath layer, which puts the applicability of this model into question. In fact, neither Chodura's theory, nor the alignment of the ions with the magnetic field, could be corroborated experimentally [6]. It seems that the Chodura layer is a consequence of the specific mathematical model [7] rather than a physical phenomenon. Indeed, it was shown in [10] and [11] that the Chodura layer can be eliminated if ion-neutral collisions are included into the model, while the Bohm criterion persists. In the present paper, Chodura's layer is eliminated by accounting for ionization. It is also shown that in this case, the assumption that the ions are aligned with the magnetic field can hold only near the plasma center for extremely strong magnetic fields. For such strong magnetic fields, however, the sheath is also magnetized.

Two approaches are found in the literature to study the non-magnetized plasma-wall problem. One approach is to use the method of matched asymptotic expansions [12] in order to approximate the solutions of the full plasma-wall problem. The

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asymptotic method was extended in [13] to include magnetized plasmas as well.

Although these asymptotic matching techniques are of theoretical value [14], their application to the plasma-wall problem is so algebraically involved that it is easier in practical applications to solve the full problem numerically.

Patching is the alternative approach to study the nonmagnetized plasma-wall problem [15]. In order to patch the plasma and sheath solutions, it is important to identify unambiguously the location of the plasma boundary (where quasi-neutrality is perturbed), as well as the location of the sheath edge. For nonmagnetized plasmas, the plasma boundary is specified by the Bohm criterion [16], namely as the point where the ions reach the ion-sound speed. According to Langmuir [17] and Bohm [16], the sheath is a thin region near the wall where no ionization takes place. As a result, the electron density in the sheath is negligible, and a strong electric field is caused by space charge [14]. The entrance to the sheath is called the sheath edge. It was shown in [14] that with no magnetic field present, the sheath edge is the point where the electric field is given by the Godyak [18] condition $|\mathbf{E}| = kT_e/\lambda_{D1}$, where k is the Boltzmann constant, T_e the electron temperature, and λ_{D1} the electron Debye radius at the plasma boundary. In this paper, we show that when a magnetic field is applied, Godyak's condition still holds, and we derive the velocity condition that specifies the sheath edge. In our model, the Bohm criterion holds for the ion-velocity component normal to the wall.

The plasma boundary and the sheath edge are located at different points in space, and there is a transition region between the quasi-neutral plasma and the sheath. In [15], for nonmagnetized plasmas, the transition region was modeled together with the sheath, and the solution of that space-charge model was patched with the plasma solution in order to approximate the full plasma-wall solution. The patching point in [15] was found analytically. In [19], an attempt was made to patch the solution of the collisional magnetized plasma model without ionization with the solution of the collisionless nonmagnetized sheath model. However, the range for potential patching points found numerically in [19] is quite large and it is not clear which specific patching point one has to choose in order to approximate the full solution. In fact, the patched solution is very sensitive to the choice of the patching point, and for this reason, the patching point cannot be chosen arbitrarily [15]. Furthermore, no comparison of the patched and full solutions were given in [19]. In this paper, we extend the results of [15] to include the magnetic field and find analytically, an adequate patching point for the plasma solution and the solution of the space charge region. We then show that the patched solution gives a good approximation of the full solution of the plasma-wall problem.

II. Plasma-Wall Problem

To model the planar bounded plasma-wall problem, we use the following hydrodynamic equations for ions and electrons. Continuity equations for the ion and electron flux

$$\operatorname{div}(n_i\mathbf{v}_i) = Zn_e \tag{1}$$
$$\operatorname{div}(n_e\mathbf{v}_e) = Zn_e. \tag{2}$$

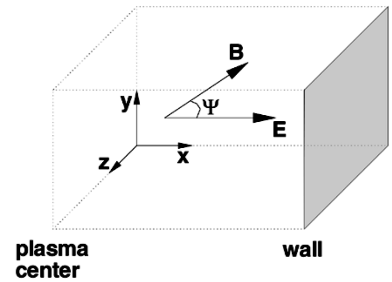


Fig. 1. Diagram of the magnetized plasma-wall problem: $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and Ψ is the angle between them.

Momentum equations for ions and electrons

$$\begin{aligned}\operatorname{div}(Mn_i\mathbf{v}_i\otimes\mathbf{v}_i + kT_in_i\mathbf{I}) + en_i(\nabla\varphi\\ - \mathbf{v}_i\times\mathbf{B}) + n_i\mathbf{F}_i = 0\end{aligned} \tag{3}$$
$$\begin{aligned}\operatorname{div}(mn_e\mathbf{v}_e\otimes\mathbf{v}_e + kT_en_e\mathbf{I}) - en_e(\nabla\varphi\\ - \mathbf{v}_e\times\mathbf{B}) + n_e\mathbf{F}_e = 0.\end{aligned} \tag{4}$$

The Poisson equation

$$\epsilon_0\nabla^2\varphi = -e(n_i - n_e). \tag{5}$$

In the equations, k is the Boltzmann constant, φ is the potential, m is the electron mass and M the ion mass, n_e is the electron density and n_i the ion density, T_e is the electron temperature and T_i is the ion temperature, $\mathbf{v}_e$ is the electron transport velocity and $\mathbf{v}_i$ the ion transport velocity, $\mathbf{F}_e$ is the electron frictional force and $\mathbf{F}_i$ the ion frictional force. Furthermore, Z is the frequency of ionization, ϵ_0 is the permittivity of free space, e is the electron charge, $\mathbf{B}$ is the magnetic field, and $\mathbf{I}$ is the identity matrix.

Equations (1)–(5) describe a weakly ionized plasma. In what follows, we assume cold ions (i.e., $T_i \approx 0$ is negligible) and constant electron temperature. We consider only the collisionless case ($\mathbf{F}_i = \mathbf{F}_e = 0$) with direct ionization (i.e., Z is constant). We assume the Boltzmann distribution for the electron density

$$n_e = n_0 \exp\left(\frac{e\varphi}{kT_e}\right) \tag{6}$$

where n_0 is the electron density at the plasma center. It is, therefore, sufficient to solve system (1)–(5) for $\mathbf{v}_i, n_i$, and φ.

In this paper, we consider the configuration illustrated in Fig. 1. For simplicity, we account only for variations in the plasma characteristics along the x-axis, normal to the wall. Thus, the electric field $\mathbf{E} = -\nabla\varphi$ is directed toward the wall. We are interested in the effect of a constant magnetic field $\mathbf{B}$ on the structure of the plasma parameters. Without loss of generality, we can choose the coordinate axes such that $\mathbf{B} = (B_x, B_y, 0)$, and the magnetic field forms an angle Ψ, with the electric field in the x-y-plane.

In this case, system (1)–(5) for cold ions and no ion-neutral collisions reduces to

$$(n_iv_x)' = Zn_e \tag{7}$$
$$\left(Mn_iv_x^2\right)' + en_i\varphi' + en_iv_zB_y = 0 \tag{8}$$
$$(Mn_iv_xv_y)' - en_iv_zB_x = 0 \tag{9}$$
$$(Mn_iv_xv_z)' - en_i(v_xB_y - v_yB_x) = 0 \tag{10}$$
$$\epsilon_0\varphi'' = -e(n_i - n_e) \tag{11}$$

where $\mathbf{v}_i = (v_x, v_y, v_z), n_e$ satisfies the Boltzmann equation (6), and the derivatives are with respect to x.

The plasma center is at $(x, y, z) = (0, 0, 0)$, and the wall is at $(x, y, z) = (R, 0, 0)$. A potential φ_w is imposed at the wall, which can be the floating potential, or a more negative potential created by an external electrical circuit. At the plasma center, we assume the following boundary conditions:

$$\begin{aligned} n_i(0) &= n_e(0) = n_0 \\ \varphi(0) &= 0 \\ \varphi'(0) &= 0 \\ \mathbf{v}_i(0) &= 0. \end{aligned} \tag{12}$$

We normalize system (7)–(11) by introducing dimensionless variables known in the literature as the plasma coordinates

$$\begin{aligned} \bar{x} &= x\frac{Z}{v_s} \\ \mathbf{b} &= \frac{\omega_c}{Z}(\cos\Psi, \sin\Psi, 0) = \frac{e}{MZ}\mathbf{B} \\ \eta(\bar{x}) &= -\frac{e}{kT_e}\varphi(x) \\ \mathbf{u}(\bar{x}) &= \frac{1}{v_s}\mathbf{v}_i(x) \\ y(\bar{x}) &= \frac{n_i(x)}{n_0} \\ y_e(\bar{x}) &= \frac{n_e(x)}{n_0} \end{aligned} \tag{13}$$

where

$$v_s = \left(\frac{kT_e}{M}\right)^{1/2}$$

is the ion sound speed, $\omega_c = e|\mathbf{B}|/M$ is the ion cyclotron frequency, $|\mathbf{B}|$ is the magnitude of the magnetic field, and $(\cos\Psi, \sin\Psi, 0)$ is a unit vector in the direction of the magnetic field.

For simplicity of exposition, we will use x instead of $\bar{x}$ for the normalized coordinate. Note that $x = 0$ corresponds to the plasma center, and $x_w = RZ/v_s$ corresponds to the position of the wall. In plasma coordinates, system (7)–(11) becomes

$$y_e = \exp(-\eta) \tag{14}$$
$$(yu_x)' = y_e \tag{15}$$
$$\left(yu_x^2\right)' - y\eta' + yu_z b_y = 0 \tag{16}$$
$$(yu_x u_y)' - yu_z b_x = 0 \tag{17}$$
$$(yu_x u_z)' - y(u_x b_y - u_y b_x) = 0 \tag{18}$$
$$\epsilon^2\eta'' = y - y_e \tag{19}$$

where $\epsilon = q_0 x_w, \mathbf{u} = (u_x, u_y, u_z), \mathbf{b} = (b_x, b_y, 0)$, and $q_0 = \lambda_{D0}/R$ is the nonneutrality parameter at the plasma center, $\lambda_{D0}^2 = \epsilon_0 kT_e/(n_0 e^2)$ is the electron Debye radius at the plasma center. The boundary condition (12) becomes

$$\begin{aligned} y(0) &= y_e(0) = 1 \\ \eta(0) &= 0 \\ \eta'(0) &= 0 \\ \mathbf{u}(0) &= 0. \end{aligned} \tag{20}$$

We use the notation

$$b_x = b\cos\Psi \quad \text{and} \quad b_y = b\sin\Psi \tag{21}$$

where $b = |\mathbf{b}|$ is the magnitude of the normalized magnetic field $\mathbf{b}$ and $\Psi \in [0, \pi/2]$ is the angle between the electric and magnetic fields.

Equations (14)–(19) can be rewritten in explicit form as follows:

$$\eta' = \psi \tag{22}$$
$$y' = \frac{2e^{-\eta}}{u_x} - \frac{\psi y}{u_x^2} + \frac{u_z y}{u_x^2}b_y \tag{23}$$
$$u_x' = \frac{\psi}{u_x} - \frac{e^{-\eta}}{y} - \frac{u_z}{u_x}b_y \tag{24}$$
$$u_y' = \frac{u_z}{u_x}b_x - \frac{e^{-\eta}u_y}{yu_x} \tag{25}$$
$$u_z' = b_y - \frac{u_y}{u_x}b_x - \frac{e^{-\eta}u_z}{yu_x} \tag{26}$$
$$\psi' = \frac{1}{\epsilon^2}(y - e^{-\eta}). \tag{27}$$

Observe that system (22)–(27) has a singularity at the plasma center $[u_x(0) = 0]$. In order to solve that system numerically, one needs to find a boundary condition for $x \neq 0$, but close to the center. Using Taylor expansions, we can prescribe the following boundary conditions near the center up to the leading order in ϵ:

$$\begin{aligned} u_x(h) &= h \\ u_y(h) &= \frac{b_x b_y}{4 + b_x^2}h \\ u_z(h) &= \frac{2b_y}{4 + b_x^2}h \\ \eta(h) &= \left(1 + \frac{b_y^2}{4 + b_x^2}\right)h^2 \\ \eta'(h) &= 2h\left(1 + \frac{b_y^2}{4 + b_x^2}\right) \\ y(h) &= y_e(h) = e^{-h^2} \end{aligned} \tag{28}$$

for some $h \approx 0$.

Fig. 2(a) shows the behavior of the ions in three-dimensional velocity space for different angles Ψ between the magnetic and electric fields ($\Psi = 0, \pi/6, \pi/3$, and $\pi/2$) and $b = 1$. Fig. 2(b) shows the ion velocity in the three-dimensional space for different magnitudes b of the magnetic field ($b = 1, 10, 50$, and 100) and $\Psi = \pi/3$. In these figures, and those that follow, we choose $\eta_w = 80$ and the nonneutrality parameter $q_1 = \lambda_{D1}/l = 0.01$ at the (quasi-neutral) plasma boundary $(l, 0, 0)$; λ_{D1} is the electron Debye radius at the plasma boundary.

When the magnetic field is normal to the wall (i.e., $\Psi = 0$), the ion velocity follows the x-axis, [i.e., $\mathbf{u} = (u_x, 0, 0)$]. In this case, there is no ion drift in the y-z-plane. This can also be seen by analyzing system (14)–(19). Indeed, if $\Psi = 0$, then $B_y = 0$, and the only solution that satisfies the boundary condition (20) is the one with $u_y = u_z = 0$. Thus, when the magnetic field is normal to the wall, the plasma-wall problem yields the same result as the plasma-wall problem without the magnetic field.

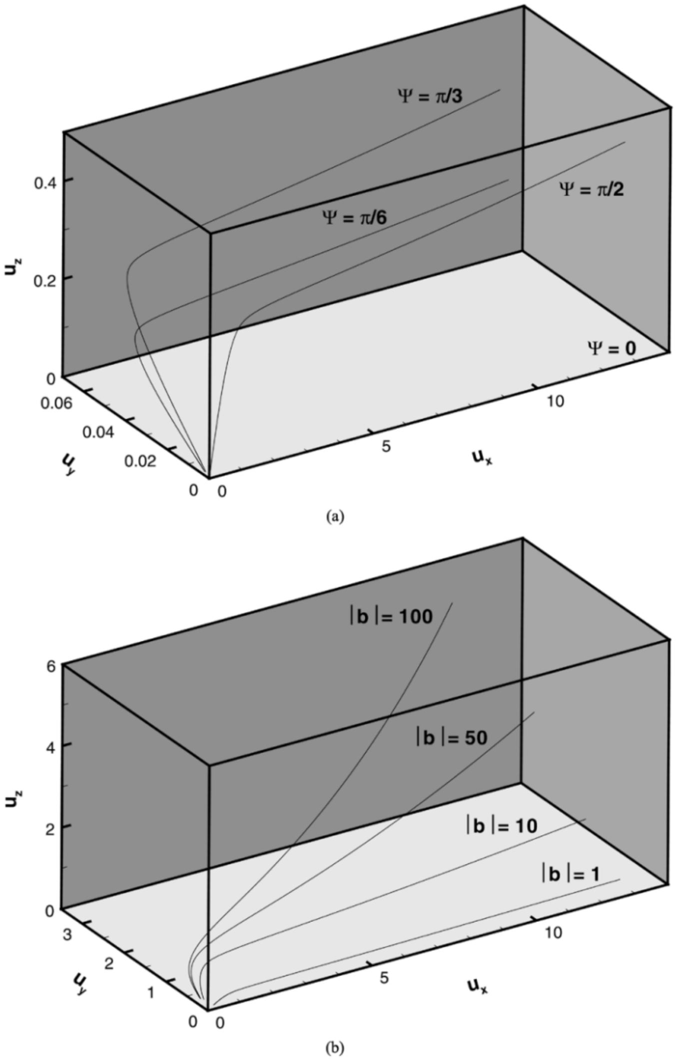


Fig. 2. Normalized ion velocity components (a) for different angles Ψ between the electric and magnetic fields with the magnitude of the normalized magnetic field $\mathbf{b} = 1$; (b) for $\Psi = \pi/3$ and different magnitudes of the normalized magnetic field. For each curve, the origin in the velocity space corresponds to the plasma center, whereas the endpoint with maximum velocity magnitude corresponds to the wall.

On the other hand, when the magnetic field is parallel to the wall (i.e., $\Psi = \pi/2$), the ion velocity is in the x-z-plane [i.e., $\mathbf{u} = (u_x, 0, u_z)$]. Indeed, if $\Psi = \pi/2$, then $B_x = 0$ and (17) yields $u_y = 0$. Note that in this case, the magnetic field is normal to the ion velocity (i.e., $\mathbf{v}_i \cdot \mathbf{B} = 0$), which means that there is no flow along the magnetic field lines. However, the drift into the z-direction (i.e., parallel to $\mathbf{E} \times \mathbf{B}$) is maximal. High magnitudes of the magnetic field will increase the drift along $\mathbf{E} \times \mathbf{B}$.

Under the action of the magnetic field, the behavior of the ion velocity is affected by both the drift along the magnetic field lines and the drift along $\mathbf{E} \times \mathbf{B}$, and those drifts depend on the magnitude of the magnetic field, as well as on the angle between the magnetic and electric fields. The drift velocity along the magnetic field lines is given by

$$u_{\|} = u_x \cos(\Psi) + u_y \sin(\Psi) \tag{29}$$

whereas the drift velocity along $\mathbf{E} \times \mathbf{B}$ in normalized coordinates is

$$u_{\perp} = u_z. \tag{30}$$

Fig. 3 shows the drift velocity $u_{\|}$ along the magnetic field lines and the drift velocity $u_{\perp}$ along $\mathbf{E} \times \mathbf{B}$ for $b = 1$. The drift velocity $u_{\|}$ is decreasing as the angle Ψ increases; it is maximal for $\Psi = 0$ and minimal ($u_{\|} = 0$) for $\Psi = \pi/2$. On the other hand, the drift velocity along $\mathbf{E} \times \mathbf{B}$ increases as the angle Ψ increases. It is maximal for $\Psi = \pi/2$ and minimal ($u_{\perp} = 0$) for $\Psi = 0$. A stronger magnetic field ($b > 1$) amplifies this behavior. Observe that near the wall, both drift velocities are higher than in the bulk plasma.

Recall that with no magnetic field present, the ion flow is aligned with the electric field and is normal to the wall. Under the action of the magnetic field, however, the drifts along the magnetic field lines and along $\mathbf{E} \times \mathbf{B}$ will change the angle at which the ion flow impacts the wall (see Fig. 2). One can estimate the deviation of the ion flow from the direction normal to the wall by computing the angle between the ion velocity and the electric field as follows:

$$\theta_E = \arccos\left(u_x / |\mathbf{u}|\right) \tag{31}$$

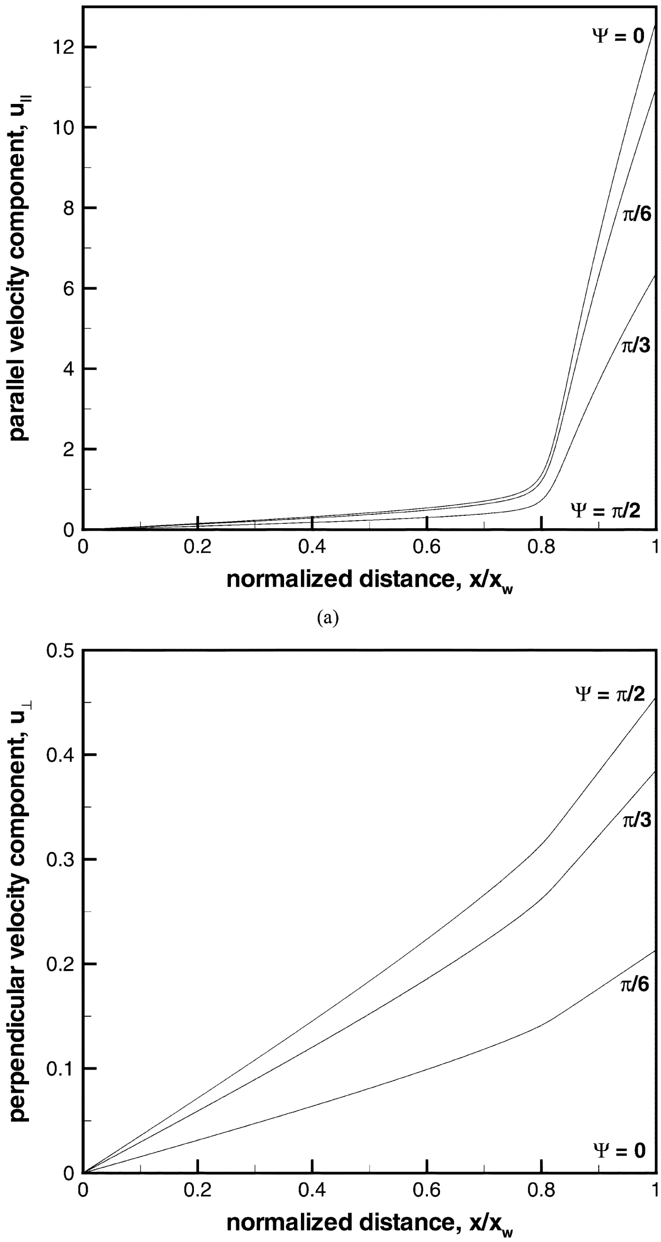
where $|\mathbf{u}| = (u_x^2 + u_y^2 + u_z^2)^{1/2}$ is the magnitude of $\mathbf{u}$.

Fig. 4 shows the angle θ_E for different values of Ψ and different magnitudes of the magnetic field. At the wall, the angle is largest for $\Psi = \pi/2$ and is decreasing with decreasing values of Ψ, reaching zero for $\Psi = 0$. A strong magnetic field ($b \geq 50$) forces the ions to follow the magnetic field lines throughout most of the bulk plasma and hinders alignment of their flow with the electric field in the sheath. For a given wall potential, the deviation from the normal direction is made more prominent by the magnitude of the magnetic field, rather then by the angle Ψ between the magnetic and the electric field.

The numerical results we have obtained so far provide an insight into some phenomena that occur when a magnetic field is applied to the plasma-wall problem. However, an important issue in applications is the sheath formation in the presence of a magnetic field. How are the bulk plasma and the sheath affected by the ion drift along the magnetic field lines and along $\mathbf{E} \times \mathbf{B}$? Where are the plasma boundary and the sheath edge? In the following sections, we will attempt to answer these questions.

III. Plasma Equations and Plasma Boundary

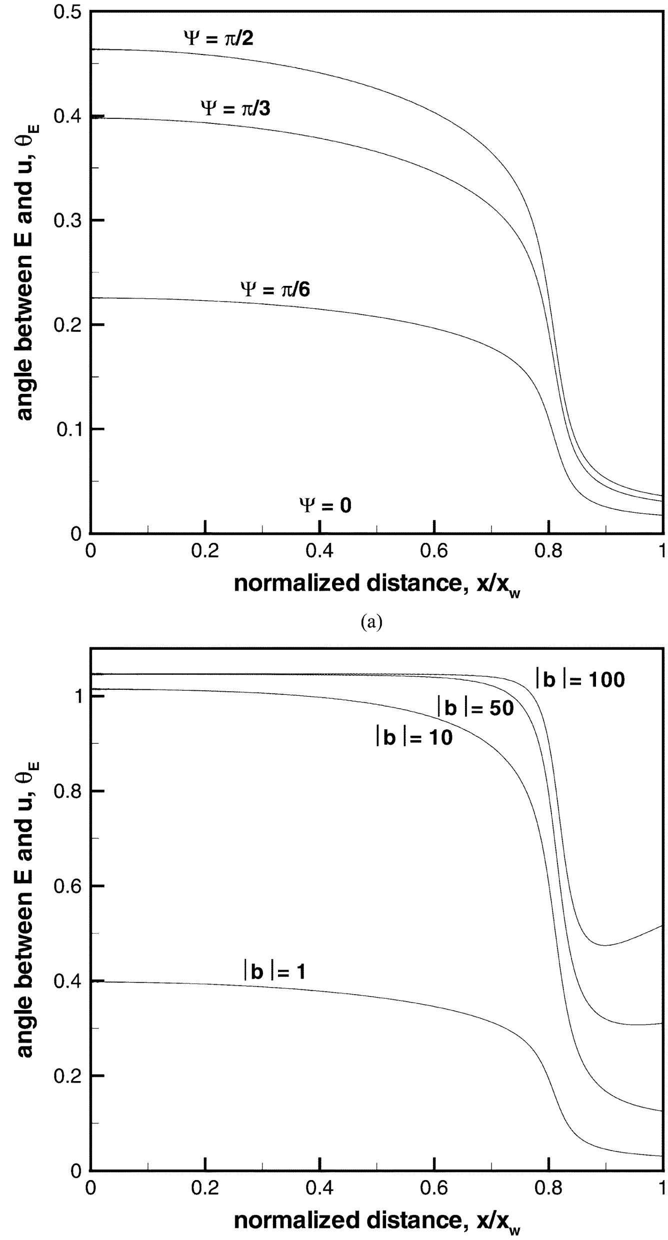
In the plasma-wall problem with no magnetic field, the plasma boundary is specified by the point where the plasma electric field reaches a singularity [20]. This is the point where the ion velocity reaches the ion sound speed and is known as the Bohm criterion [16]. Due to a persistent confusion over the definition of the sheath, it has been stated in the literature that the Bohm criterion specifies the position of the plasma-sheath boundary or the sheath edge (see the discussion in [21]). However, it was shown in [22] numerically and in [14] analytically that this is not the case. In fact, the Bohm criterion can be assumed to specify the point where the quasi-neutrality is violated



(a)

(b)

Fig. 3. Normalized drift velocities for different angles Ψ and $|\mathbf{b}| = 1$ (a) along the magnetic field lines; (b) along $\mathbf{E} \times \mathbf{B}$.



(a)

(b)

Fig. 4. Angle between the electric field and the ion velocity (a) for different angles Ψ and $|\mathbf{b}| = 1$; (b) for different magnitudes of the normalized magnetic field and $\Psi = \pi/3$.

(i.e., the plasma boundary), but it cannot specify the sheath edge. We will now investigate the appropriate formulation of the Bohm criterion at the plasma boundary when a magnetic field is present.

A normalized model for the plasma region can be obtained from (14)–(19) by assuming quasi-neutrality $(y = y_e)$, or asymptotically by letting $\epsilon \to 0$

$$y = \exp(-\eta) \tag{32}$$

$$(yu_x)' = y \tag{33}$$

$$\left(yu_x^2\right)' - y\eta' + yu_zb_y = 0 \tag{34}$$

$$(yu_xu_y)' - yu_zb_x = 0 \tag{35}$$

$$(yu_xu_z)' - y(u_xb_y - u_yb_x) = 0 \tag{36}$$

which is equivalent to the following system:

$$y = \exp(-\eta) \tag{37}$$

$$u_x' = \frac{1+u_x^2}{1-u_x^2} + \frac{u_xu_zb_y}{1-u_x^2} \tag{38}$$

$$\eta' = \frac{2u_x}{1-u_x^2} + \frac{u_zb_y}{1-u_x^2} \tag{39}$$

$$u_y' = -\frac{u_y}{u_x} + \frac{u_zb_x}{u_x} \tag{40}$$

$$u_z' = -\frac{u_z}{u_x} + \frac{u_xb_y - u_yb_x}{u_x}. \tag{41}$$

The boundary condition at the plasma center is given by (20). As for the full plasma-wall problem, we can use (28) as a boundary condition for solving the plasma equations (37)–(41) numerically, and avoid the singularity at $x = 0$, where $u_x = 0$. The plasma equations have a second singularity at the point where the ion velocity component normal to the wall reaches the ion sound speed $(u_x = 1)$. By analogy with the case with no magnetic field, we can define the plasma boundary as the point where the electric field reaches a singularity. We denote that point as $x = x_p$ and refer to it as the plasma boundary since the plasma solution cannot be extended beyond that point. We denote the values of the plasma characteristics at the plasma boundary by

$$\begin{aligned} u_x(x_p) &= u_{xp} = 1 \\ \eta(x_p) &= \eta_p \\ y_p &= \exp(-\eta_p) \\ u_y(x_p) &= u_{yp} \\ u_z(x_p) &= u_{zp}. \end{aligned} \tag{42}$$

Those values can be obtained numerically by solving the plasma equations (37)–(41) with the boundary condition (28).

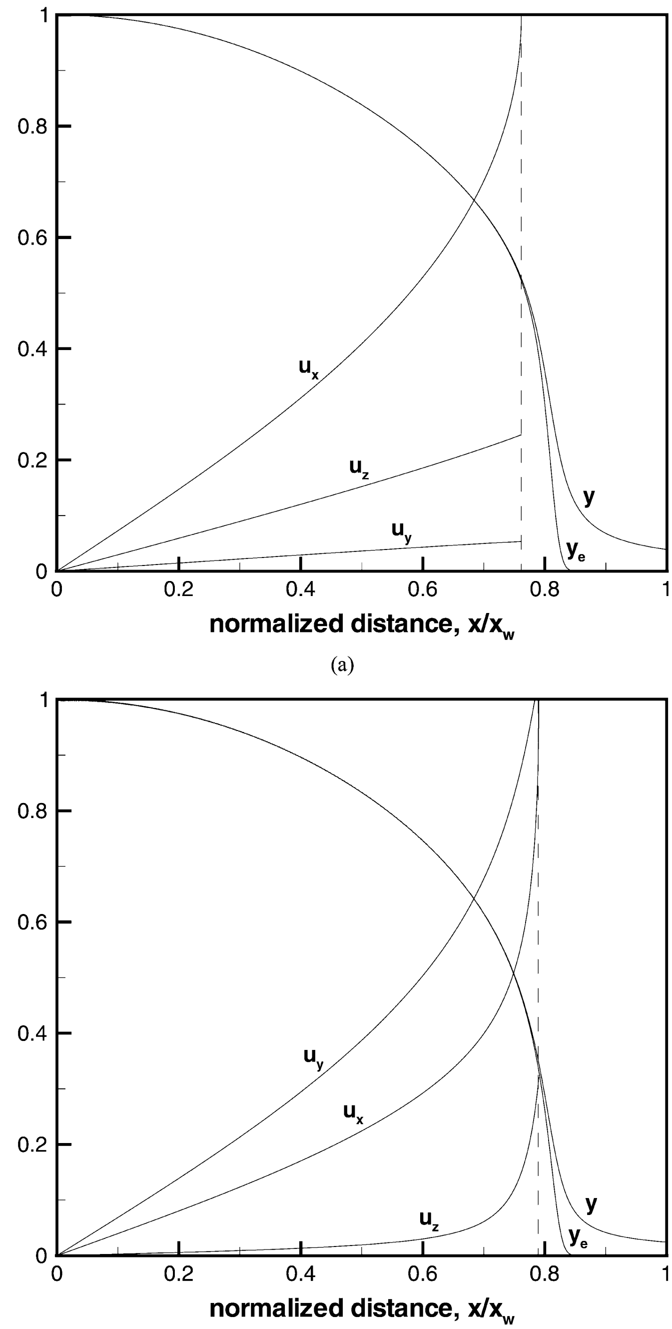
If $\Psi = \pi/2$, then $u_y = 0$ throughout the plasma region, whereas for $\Psi \neq \pi/2$, (34)–(35) yield

$$u_y = \frac{e^{\eta} - 1 - u_x^2}{\tan(\Psi) u_x} \tag{43}$$

and, in particular, at the plasma boundary

$$u_{yp} = \frac{e^{\eta_p} - 2}{\tan(\Psi)}. \tag{44}$$

The plasma equations suggest that only u_x can be used to justify the location of the plasma boundary. This is also supported by the numerical result shown in Fig. 5, where one can see the ion and electron density distributions predicted by the full model, as well as the components of the ion velocity predicted by the plasma equations, for $\Psi = \pi/3$ and various magnitudes of the magnetic field ($b = 1$ and $b = 100$). Note that for a moderate magnetic field $(b = 1), u_y < u_z < u_x$, whereas for a high magnetic field $(b = 100), u_z < u_x < u_y$. Neither Fig. 5 nor the plasma equations suggest any special values of u_y or u_z that specify the plasma boundary. However, $u_x = 1$ can be used consistently for any magnetic field to specify the point where the plasma-neutrality is violated, and one can take this to be the Bohm criterion.



(a)

(b)

Fig. 5. Plasma boundary according to the Bohm criterion for $\Psi = \pi/3$ and (a) for $|\mathbf{b}| = 1$; (b) $|\mathbf{b}| = 100$. The normalized components of the ion velocity are obtained from the plasma equations (37)–(41); the normalized number densities are obtained from the full plasma-wall model (22)–(27).

IV. Sheath Equations and Sheath Edge

According to Langmuir [17] and Bohm [16], the sheath is a thin region of a few Debye lengths where the electron density can be neglected. In the plasma-wall problem with no magnetic field, the sheath edge can be specified by Godyak's condition [18]

$$|\mathbf{E}| = \frac{kT_e}{\lambda_{D1}} \tag{45}$$

where $\mathbf{E}$ is the electric field and λ_{D1} is the electron Debye radius at the plasma boundary. This condition was obtained from the Boltzmann equation, neglecting the electron density in the sheath [19], [14]. Since the Boltzmann distribution for the electron density also holds in the model we are considering, Godyak's condition can be derived in the same way. Indeed, differentiating the Boltzmann equations in the x-direction at the sheath edge $(s, 0, 0)$, we find

$$\begin{aligned} \frac{e}{kT_e} n_e(s)\varphi'(s) = n_e'(s) &\approx \frac{n_e(s+\lambda_{D1}) - n_e(s)}{\lambda_{D1}} \\ &\approx -\frac{n_e(s)}{\lambda_{D1}} \end{aligned} \tag{46}$$

since $n_e(s+\lambda_{D1}) \approx 0$. This yields Godyak's condition, which can be interpreted as a balance between the electric field energy at the sheath edge $(s,0,0)$ and the electron kinetic energy at the plasma boundary $(l,0,0)$ in the direction of the electric field [14]

$$\frac{1}{2}\epsilon_0|\mathbf{E}|^2(s) = \frac{1}{2}n_e(l)kT_e. \tag{47}$$

Equation (46) suggests that there is a transition region between l and s which maintains a quasi-neutral plasma for $x < l$ and an ion sheath for $x > s$. In that region, the electron density drops from $n_e(l) \approx n_i(l)$ to $n_e(s) \approx 0$. In plasma coordinates, Godyak's condition (45) becomes

$$\eta'(x_s) = \frac{1}{q_1 x_p} = \frac{1}{\epsilon}y_p^{1/2} \tag{48}$$

where $q_1 = \lambda_{D1}/l$ is the nonneutrality parameter at the plasma boundary. Note that Godyak's condition does not depend explicitly on the magnitude or the direction of the magnetic field.

For $\Psi \neq \pi/2$, we can find the condition for the drift velocity along the magnetic field lines $u_{\|}$ at the sheath edge as follows. Consider once more the plasma-wall problem (14)–(19). Multiplying the Poisson equation (19) by η' and integrating using the boundary condition (20), we obtain

$$\frac{1}{2}\epsilon^2(\eta')^2 = yu_x^2 + \tan(\Psi)yu_xu_y + \exp(-\eta) - 1. \tag{49}$$

This equation for the electric field is valid in the region from the center to the wall. In the space charge region (transition region and sheath), we can assume a constant ion flux

$$yu_x = y_pu_{xp} = y_p \tag{50}$$

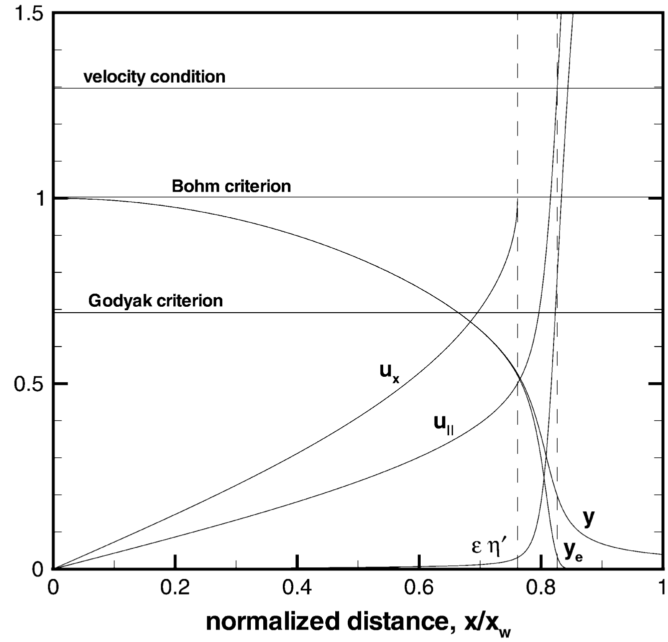
and, therefore, in the sheath region, the electric field of the plasma-wall problem can be approximated by

$$\frac{1}{2}\epsilon^2(\eta')^2 = y_pu_x + \tan(\Psi)y_pu_y + \exp(-\eta) - 1. \tag{51}$$

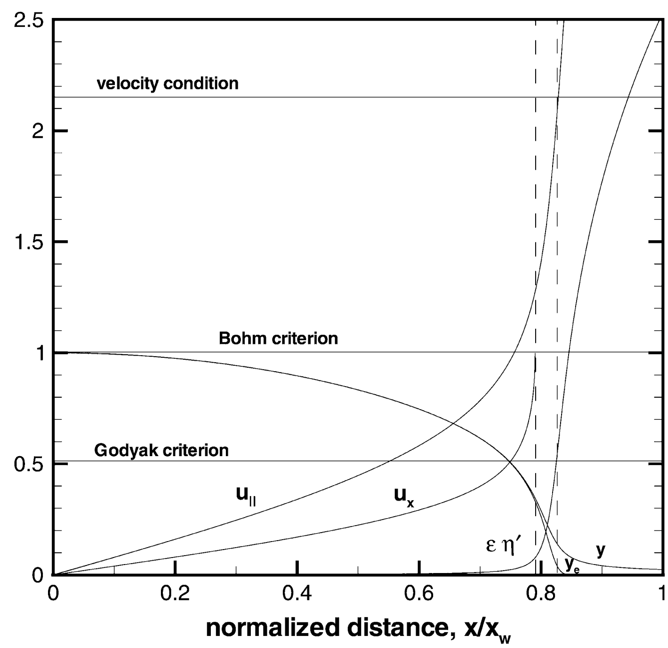
In particular, neglecting the electron density $\exp(-\eta)$, and using Godyak's condition (48), we obtain at the sheath edge

$$u_{\|} = u_x\cos(\Psi) + u_y\sin(\Psi) = \left(\frac{1}{2} + \frac{1}{y_p}\right)\cos(\Psi). \tag{52}$$

If $\Psi = 0$, then $y_p = 1/2$, and it follows that $u_x = 5/2$, which is exactly the value obtained in [14] and [15] for the case with no magnetic field. Fig. 6 shows that Godyak's condition for the electric field and condition (52) for the drift velocity along the magnetic field lines both specify the same sheath edge, and beyond that sheath edge, the electron density can be neglected. The width of the transition region between the plasma boundary and the sheath edge, where the electron density drop occurs, varies as the magnetic field increases. It seems that an increase in the magnitude of magnetic field causes an expansion of the quasi-neutral plasma into the transition region, while the sheath width remains practically unchanged.



(a)



(b)

Fig. 6. Plasma boundary according to the Bohm criterion, sheath edge according to the Godyak condition, and velocity condition (52) at the sheath edge for $\Psi = \pi/3$ and (a) $|\mathbf{b}| = 1$; (b) $|\mathbf{b}| = 100$. The normalized component of the ion velocity is obtained from the plasma equations (37)–(41); the normalized drift velocity, the normalized number densities and the electric field are obtained from the full plasma-wall model (22)–(27).

In the sheath region, the electron density $\exp(-\eta)$ is negligible, and the following sheath equations are obtained from the plasma-wall problem (14)–(19):

$$(yu_x)' = 0 \tag{53}$$
$$u_xu_x' - \eta' + u_zb_y = 0 \tag{54}$$
$$u_xu_y' - u_zb_x = 0 \tag{55}$$
$$u_xu_z' - u_xb_y + u_yb_x = 0 \tag{56}$$
$$\epsilon^2\eta'' = y. \tag{57}$$

Multiplying (57) by η' and integrating with the boundary condition at the sheath edge (48) and (52), we obtain the following equation for the electric field:

$$\frac{1}{2}\epsilon^2(\eta')^2 = yu_x(u_x + \tan(\Psi)u_y) + \frac{1}{2}y_p - yu_x\left(\frac{1}{2} + \frac{1}{y_p}\right) \tag{58}$$

where yu_x is constant. In order to solve the sheath model, we need to know the sheath characteristics at the sheath edge. Because the electron density is neglected in the sheath model, this boundary condition cannot be found directly from the plasma characteristics, i.e., (58) cannot be extended up to the plasma boundary. Indeed, for $y = y_p, u_x = u_{xp} = 1$, and $u_y = u_{yp}$ the right-hand side of (58) becomes negative. Therefore, one must account for the transition region between the quasi-neutral plasma and the sheath.

V. Accounting for the Transition Region

In the space charge region, which consists of the transition region and the sheath, assuming a constant ion flux norma l to the wall, system (14)–(19) reduces to

$$(yu_x)' = 0 \tag{59}$$
$$u_x u_x' - \eta' + u_z b_y = 0 \tag{60}$$
$$u_x u_y' - u_z b_x = 0 \tag{61}$$
$$u_x u_z' - u_x b_y + u_y b_x = 0 \tag{62}$$
$$\epsilon^2 \eta'' = y - e^{-\eta}. \tag{63}$$

In the sheath region, $\exp(-\eta)$ is negligible, and system (59)–(63) yields the sheath model (53)–(57). Unlike the sheath solution, the solution of (59)–(63) can be extended into the plasma region. Thus, multiplying the Poisson equation (63) by η' and integrating with arbitrary initial conditions $(y, u_x, u_y, u_z, \eta, \eta') = (y_1, u_{x1}, u_{y1}, u_{z1}, \eta_1, \eta_1')$ at some point $x = x_1$ in the plasma region, one obtains using (43) that

$$\begin{aligned}\frac{1}{2}\epsilon^2(\eta')^2 &= y_1u_{x1}(u_x + \tan(\Psi)u_y) + e^{-\eta} + \frac{1}{2}\epsilon^2(\eta_1')^2 \\ &\quad - y_1u_{x1}(u_{x1} + \tan(\Psi)u_{y1}) - e^{-\eta_1} \\ &= y_1u_{x1}(u_x + \tan(\Psi)u_y) + e^{-\eta} + \frac{1}{2}\epsilon^2(\eta_1')^2 - 1.\end{aligned} \tag{64}$$

Equation (64) should provide a connection between the plasma solution on one side and the sheath solution on the other. Prescribing the boundary condition (48) and (52) at the sheath edge and neglecting the electron density, we obtain the following patching condition:

$$\frac{1}{2}y_p = y_1u_{x1}\left(\frac{1}{2} + \frac{1}{y_p}\right) + \frac{1}{2}\epsilon^2(\eta_1')^2 - 1 \tag{65}$$

where y_1, u_{x1}, and η_1' satisfy the plasma equation. Observe that the derivation of the patching condition we have presented is valid only for $\Psi \neq \pi/2$. However, the patching condition (65) remains valid for $\Psi = \pi/2$ for reasons of continuity with respect to Ψ.

The values of the plasma characteristics at the patching point y_1, u_{x1}, η_1', and the position of the patching point x_1 can be obtained numerically by solving the plasma equation (37)–(41), but one can also obtain analytical approximations for them as follows. Set

$$u_x = 1 - \delta \tag{66}$$

where $\delta(x)$ is a function such that $\delta(x_p) = 0$. Substituting (66) into (38)–(39), we obtain up to the leading order in δ

$$-\delta' = \frac{1}{2\delta}(2 + u_z b_y) \tag{67}$$
$$\eta' = \frac{1}{2\delta}(2 + u_z b_y). \tag{68}$$

In particular

$$\eta' = -\delta' \tag{69}$$

hence

$$\eta = \eta_p - \delta \tag{70}$$

and quasi-neutrality (37) yields

$$y = y_p\left(1 + \delta + \frac{1}{2}\delta^2\right). \tag{71}$$

For sufficiently small values of δ, variations in u_z will be insignificant (see Fig. 5), and we can assume that $u_z = u_{zp}$ for all $x \approx x_p$. Solving (67) yields

$$x = x_p - \frac{1}{2 + u_{zp}b_y}\delta^2. \tag{72}$$

In order to find δ such that the patching condition (65) is satisfied, substitute (66), (68), and (71) into (65) and obtain

$$\begin{aligned}\delta &= \left(\frac{(2 + u_{zp}b_y)^2}{2(2 + y_p)}\right)^{1/4}\epsilon^{1/2} \\ &= \left(\frac{y_p(2 + u_{zp}b_y)^2}{2(2 + y_p)}\right)^{1/4}(x_p q_1)^{1/2}.\end{aligned} \tag{73}$$

Observe that with no magnetic field present, $b_y = 0, y_p = 1/2$, and we obtain the same patching point and the same values of the plasma characteristics at the patching point as in [15].

In Fig. 7, the plasma and the solution of the space charge region are patched at the patching point (72). As one can see, the patched solution approximates quite well the full solution in the region from the center up to the wall. At the wall, the relative error between the patched solution and the full solution in all sheath characteristics is about 1%–2% for $b = 10$, 1%–5% for $b = 50$, and 6%–14% for $b = 100$. The increase of the relative error at the wall for strong magnetic fields is due to the fact that we have used in our computations the analytical approximations of the plasma characteristics at the patching point where we assumed that $u_{z1} = u_{zp}$ rather than the patching condition (65). However, even for very high magnetic fields ($b = 100$), it is still within typical experimental accuracy.

For all $\Psi \in [0, \pi/2]$ and arbitrary magnitudes of the magnetic field, the sheath edge and the sheath characteristics at the sheath edge can be obtained from the corresponding values of the patched solution at the point where Godyak's condition (48) is satisfied.

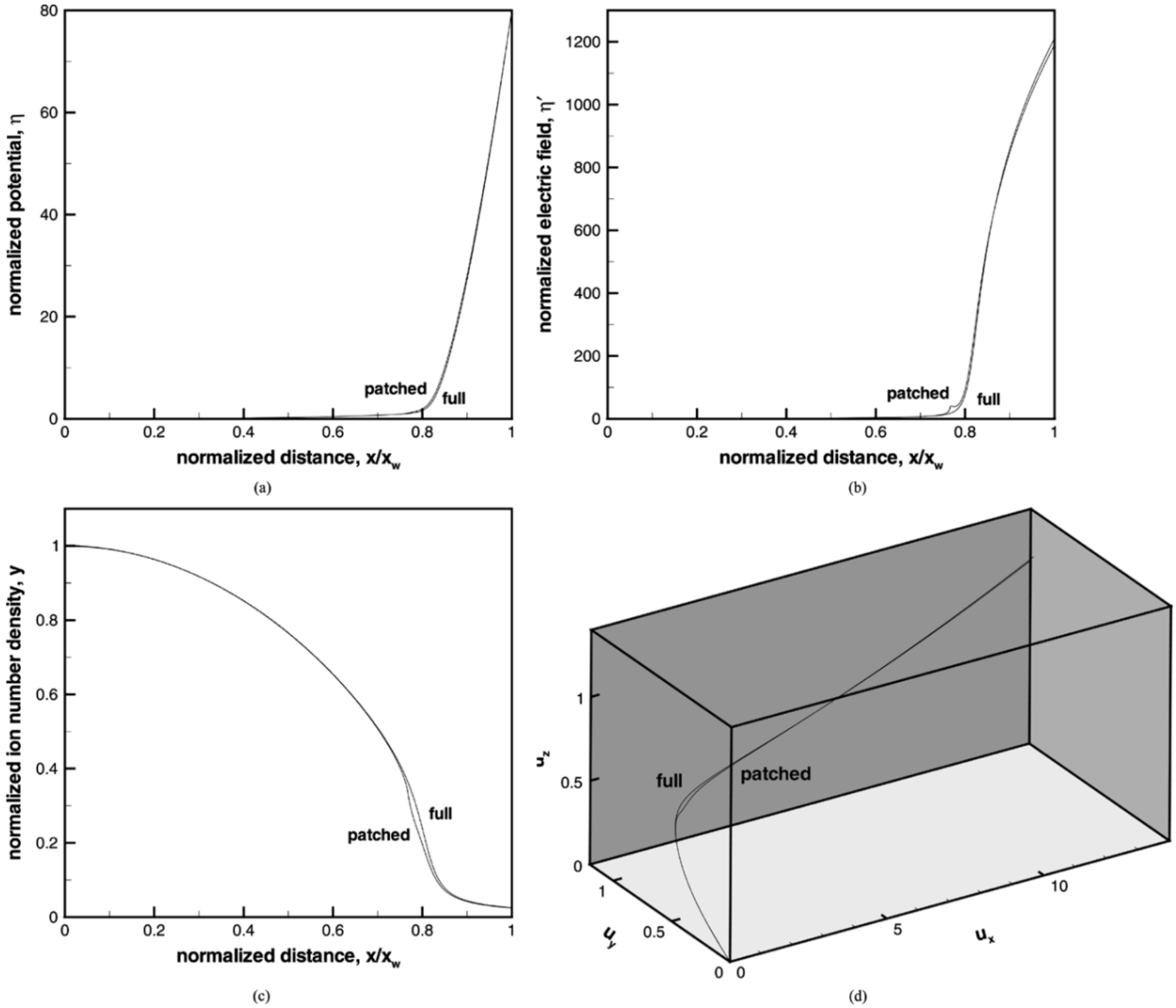


Fig. 7. Full and patched solutions of the plasma-wall problem for $\Psi = \pi/3$ and $|\mathbf{b}| = 10$: (a) normalized potential; (b) normalized electric field; (c) normalized ion density; (d) normalized ion velocity. The full solution is obtained by solving (22)–(27); the patched solution is obtained by patching the solution of the plasma equations (37)–(41) with the solution of the space charge (59)–(63) at the patching point (72).

VI. Conclusion

We have presented a model of the plasma-wall problem in the presence of an applied magnetic field. Analyzing this model, we have observed the behavior of the ion flow that has been reported in experiments [6], namely, a drift along the magnetic field lines and a drift normal to both the electric and magnetic fields. We have found that the drift velocity along the magnetic field lines decreases as the angle Ψ between the magnetic and electric fields increases. It is maximal for $\Psi = 0$ and zero for $\Psi = \pi/2$. In contrast, the drift velocity normal to the magnetic and electric fields is maximal for $\Psi = \pi/2$ and zero for $\Psi = 0$. Both drift velocities are amplified when the magnitude of the magnetic field is increased.

Under the action of a magnetic field, the ion flow deviates from the direction normal to the wall. At the wall, the angle between the ion flow and the electric field is largest for $\Psi = \pi/2$ and is decreasing with decreasing values of Ψ, reaching zero for $\Psi = 0$. A strong magnetic field ($b \geq 50$) forces the ions to follow the magnetic field lines throughout most of the bulk plasma and hinders their alignment with the electric field in the sheath. It seems that at the wall, one can decrease the deviation of the ion flow from the direction normal to the wall by increasing the wall potential, i.e., increasing the sheath width. For a given wall potential, the deviation from the normal direction is made more prominent by the magnitude of the magnetic field rather then by the angle Ψ between the magnetic and the electric fields.

We have derived the plasma and the sheath equations from our plasma-wall model and found the position of the quasi-neutral plasma boundary and the sheath edge. We have shown that the plasma boundary can be specified as the point where the component of the ion velocity normal to the wall reaches the ion sound speed (Bohm criterion). We have shown that the sheath edge is specified by Godyak's condition for the electric field $|\mathbf{E}| = kT_e/\lambda_{D1}$, and in the sheath, the electron density is negligible. For $\Psi \neq \pi/2$, we have also found the analytical representation of the ion drift velocity along the magnetic field lines at the sheath edge

$$u_{\parallel} = u_x \cos(\Psi) + u_y \sin(\Psi) = \left(\frac{1}{2} + \frac{1}{y_p}\right) \cos(\Psi).$$

Godyak's condition can be interpreted as a balance between the electric field energy at the sheath edge and the electron kinetic energy at the plasma boundary. This balance is maintained by the transition region between the plasma and the sheath where the electron density drop occurs. In order to account for the transition region, we have presented a model for the space charge region which includes the transition region and the sheath. We were able to patch the plasma solution with the solution of the space charge model and find analytical representations for the patching point and for the plasma characteristics at the patching point. The patched solution yields a good approximation of the full solution from the center up to the wall for realistic values of the magnetic field strength.

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