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Existence and stability of the plasma sheath resonance near the ion plasma frequency

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Abstract. – We present a theory of plasma sheath resonance in the neighborhood of the ion plasma frequency. Properly implementing the boundary conditions on the moving sheath edge leads to the possibility of reactive current compensation in the plasma sheath. We derive conditions which certain plasma parameters must satisfy in order for this resonance to exist and to be stable. The theory explains experimental results which have appeared to disagree with each other as well as with certain theoretical assertions.

Experimental results on the resonance properties of plasma sheaths in the neighborhood of the ion plasma frequency, published over the last few decades, appear to be contradictory. In certain experiments these resonances did not appear at all [1], in others they did appear but were very weak [2], and sometimes they appeared to be unstable and resulted in tens of kW of rf power in the external circuits [3]. Moreover, these resonances and their instabilities appear to be inconsistent with a theoretical assertion that the capacitive reactance of the plasma sheath does not vanish [4, 5], as well as with another assertion regarding sheath stability [6].

In this paper we show that variations of the sheath thickness, combined with plasma inhomogeneities near the sheath edge, may cause a resonance due to the vanishing of the sheath reactance. If at the same time the sheath resistance becomes negative, the resonance will become unstable and strong rf current oscillations will appear in the external circuit. We show that ion transit-time effects are able and sufficient to cause this negative resistance.

We will consider a planar collisionless plasma sheath in front of an absorbing wall. This wall is negatively biased so that the potential fall U across the sheath is much higher than the floating potential $U_{\mathrm{F}} \approx T_{\mathrm{e}} \ln(M/m)^{1/2}$. Here T_{e} is the electron temperature in potential units, and M and m are the ion and electron masses. Under these conditions the plasma electrons penetrate inside the sheath only to a very short distance. The frequencies ω considered here, which are around the ion plasma frequency ω_{i}, are much smaller than the electron plasma frequency ω_{e}. Then a step-like electron density profile can follow the potential changes instantaneously and the sheath properties can be characterized by the ion dynamics only. The motion of the ion along its trajectory, in the self-consistent field of the ionic space charge, may be described by the Llewellyn equation [7]

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = \frac{e}{M}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{4\pi e}{M} J(t)\,, \tag{1}$$

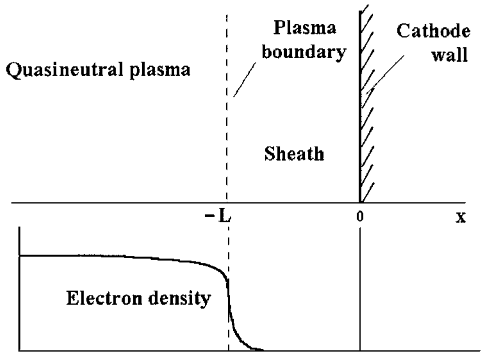

Fig. 1. – The plasma with electron-free sheath near the cathode wall (top) and the spatial distribution of electron density (bottom). The electron density falls sharply at the plasma sheath boundary.

where $\mathrm{d}/\mathrm{d}t$ is the time derivative along the trajectory, E is the electric field in the sheath, $J(t)$ is the density of the total (displacement and convective) current through the sheath and x is the position of the ion in the sheath. This equation describes the sheath dynamics subject to appropriate boundary conditions corresponding to the initial conditions for each ion crossing the boundary. We choose the frame of reference in such a way that its origin coincides with the wall position and the plasma is located on the left side (fig. 1). We mark the position of the plasma sheath boundary as $x = -L$. On the left side of this boundary the electron density is equal to the ion density.

The first initial condition, namely the instantaneous position of the sheath edge $x(t) = -L(t)$, obviously should be chosen in such a way that the potential fall between the sheath-plasma boundary and the cathode surface at $x = 0$ must be equal to the voltage $U(t)$ across the sheath. An important point here is that the sheath edge has been supposed to be movable. Turning to the second initial condition, we will utilize the Bohm criterion (BC). This criterion says that in the steady-state case the cold ions enter the sheath with the ion sound velocity $c_\mathrm{s} \geq (T_\mathrm{e}/M)^{1/2}$ due to the acceleration in the quasineutral plasma where the potential fall does not exceed T_e [8]. When the sheath thickness changes the sheath edge also moves. In this case, as was shown both analytically and numerically [9], the ions enter the sheath at least with the same velocity c_s in the frame of reference of the plasma boundary. Thus in the laboratory frame the velocity of the ions c_i at the sheath-plasma boundary may be written as $c_\mathrm{i} \geq c_\mathrm{s} + v_\mathrm{L}$, where $v_\mathrm{L}(t)$ is the velocity of the boundary. This condition, called the dynamic Bohm criterion (DBC), does not change BC itself but merely provides a way to use it in nonstationary cases. As usual we will apply BC in its marginal (equality) form [8]. The third initial condition, namely the value of $\mathrm{d}^2x/\mathrm{d}t^2$ at the plasma boundary, can be obtained assuming zero electric field at this boundary: $E[-L(t)] = 0$.

Assuming that the wall is negatively biased by a dc voltage U_0 and a small ac voltage $\widetilde{U}\mathrm{e}^{-i\omega t}$ ($\widetilde{U} \ll U_0$), we have to find the relationship between the ac current-density amplitude $\widetilde{J}$ and the ac voltage amplitude $\widetilde{U}$. In linear approximation this relationship appears as $\widetilde{U} = Z(\omega)\widetilde{J}$, where $Z(\omega)$ is the impedance of the cathode sheath per unit area. Our object is to calculate this impedance. Note that the function $Z(\omega)$ should describe the sheath properties as an element in an electrical circuit.

The total current density J through the sheath consists of the displacement current density J_d and the convective current density J_c. We have $J = enc_\mathrm{i} + (1/4\pi)\partial E/\partial t$; $J_\mathrm{c} = enc_\mathrm{i}$; $J_\mathrm{d} = (1/4\pi)\partial E/\partial t$. Because the total current density is the same in any cross-section of the sheath, we will calculate it in the most convenient place. The sheath-plasma boundary is such a place, because it is possible to determine the total current density through it, using the initial conditions mentioned above. In order to determine the value of $\partial E/\partial t$ in the sheath close to the boundary, let us assume that at the moment t_0 an ion has crossed the boundary located at $x_0 = -L(t_0)$. Also we will consider a short time period Δt so that within the period $(t_0; t_0+\Delta t)$ the values of c_i and v_L remain constant. So at the moment $t_0+\Delta t$ the ion position is $x_0 + c_\mathrm{i}\Delta t$ and the electric field there, according to eq. (1), should be $E = 4\pi J\Delta t$. Here we used the fact that according to the third initial condition, $E = 0$ at t_0. During this Δt the plasma boundary has moved to the point $x_0 + v_\mathrm{L}\Delta t$. The distance between the ion and the boundary is now $(c_\mathrm{i} - v_\mathrm{L})\Delta t$. This means that at a certain time later, namely $(c_\mathrm{i} - v_\mathrm{L})\Delta t/v_\mathrm{L}$, the boundary will pass the point $x_0 + c_\mathrm{i}\Delta t$. Keeping in mind that at the boundary $E = 0$, one may write $\partial E/\partial t = -4\pi J v_\mathrm{L}/(c_\mathrm{i} - v_\mathrm{L})$ which yields $J = enc_\mathrm{i} - env_\mathrm{L}$ and $J_\mathrm{d} = -env_\mathrm{L}$.

The time-dependent variation of the convective current density enc_i can in general be written as $\delta J_\mathrm{c} = ec_\mathrm{s}\delta n + en_0\delta c_\mathrm{i}$, where n_0 and δn are the unperturbed quasi-neutral plasma density and its variation near the sheath-plasma boundary and δc_i is the variation of the ion initial velocity at the sheath edge. Here the unperturbed ion velocity is c_s. Thus, neglecting the density variation in the displacement current term, the variation of the total current density may be written as

$$\delta J = en_0\delta c_\mathrm{i} + ec_\mathrm{s}\delta n - en_0 v_\mathrm{L}\,. \tag{2}$$

In the case of the static BC $\delta c_\mathrm{i} = 0$ and eq. (2) yields $\delta J = ec_\mathrm{s}\delta n - en_0 v_\mathrm{L}$. Thus even in the case of $\delta n = 0$, *i.e.* a uniform steady-state plasma, δJ does not vanish [4]. In the present DBC case $\delta c_\mathrm{i} = v_\mathrm{L}$ and correspondingly the variation of the total current density $\delta J = ec_\mathrm{s}\delta n$ is determined by the density variation of the quasineutral plasma only. This time-dependent plasma density variation at the sheath-plasma boundary may be caused by the steady-state density inhomogeneity near this boundary: indeed the variation of the sheath thickness leads to density variation at the sheath edge. Another reason for the plasma density variation might be the ionization dependence on the discharge voltage. However, this cannot play a significant role because ionization and de-ionization processes are very slow compared with the ion plasma frequency [10]. On the other hand, the steady-state plasma density is usually influenced strongly by the presence of the sheath which leads to the density inhomogeneity near the sheath edge.

To calculate the sheath impedance, following the method described in refs. [4, 7], eq. (1) was integrated along the ions trajectories in a self-consistent space-charge field using the initial conditions mentioned above. Consequently, $Z(\omega)$ can be written as

$$Z(\theta) = -\frac{4\pi c_\mathrm{s}\tau^2}{\theta^2}\left\{\frac{F_\mathrm{R}(\theta) - i\alpha\theta \mathrm{e}^{-i\theta} + 2\alpha\left(\mathrm{e}^{-i\theta} - 1\right) + i\beta(\theta_\mathrm{i}/\theta)\left(\mathrm{e}^{-i\theta} - 1\right)}{(1-\alpha) - i\beta(\theta_\mathrm{i}/\theta)}\right\}. \tag{3}$$

Here τ is the ion transit time, $\theta = \omega\tau$ is the ion transit angle, and $\theta_\mathrm{i} = \omega_\mathrm{i}\tau$. The parameter α is introduced in such a way that the case $\alpha = 0$ corresponds to the initial condition determined by the steady-state BC and the case $\alpha = 1$ corresponds to the initial condition determined by DBC. The parameter $\beta = |\partial n_0/\partial x| r_\mathrm{d}/n_0$, where r_d is the Debye length, describes the plasma inhomogeneity ($\beta = 0$ corresponds to uniform plasma). The function $F_\mathrm{R}(\theta)$ coincides with the one calculated by Rosa [4]:

$$F_\mathrm{R}(\theta) = i\theta + \theta_\mathrm{i}^2\left[\frac{i\theta}{6} + \frac{i}{\theta}\left(1 + \mathrm{e}^{-i\theta}\right) + \frac{2}{\theta^2}\left(\mathrm{e}^{-i\theta} - 1\right)\right].$$

If $\alpha = \beta = 0$ then the behavior of the impedance $Z(\theta)$ is determined by $F_\mathrm{R}(\theta)$ and its imaginary part $\mathrm{Im}Z(\theta)$ never vanishes for any value of θ, which means the absence of

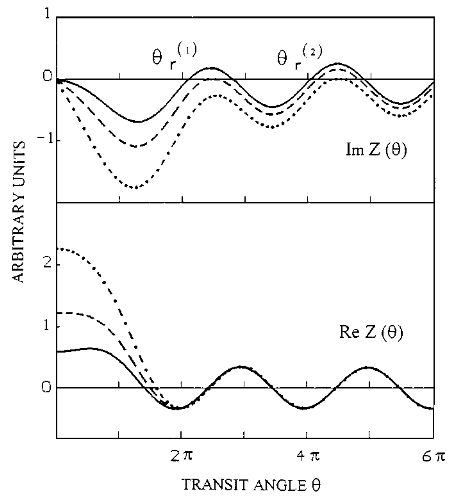

Fig. 2. – The dependence of $\mathrm{Im}Z(\theta)$ (top) and $\mathrm{Re}Z(\theta)$ (bottom) for different values of R_n/ℓ. Full, dashed and dot-dashed lines correspond to $R_n/\ell = 1.2$, 1.0, and 0.8, respectively. The first resonance $\theta_\mathrm{r}^{(1)}$ exists if $\ell \lesssim R_n$ and disappears when $\ell > R_n$. $\theta_\mathrm{r}^{(1)}$ and $\theta_\mathrm{r}^{(2)}$ are the first and second unstable resonances.

resonances. Some authors have interpreted such non-resonance properties of the cathode sheath as meaning that "the sheath impedance is essentially capacitive", because "inductance is lacking in the equivalent circuit" [5]. Such a "capacitor-like" behavior of the sheath appears because the steady-state BC was used as an initial condition. When instead of the static condition one uses the dynamic one for the $Z(\theta)$ calculations, an "inductance" appears in the equivalent circuit and resonances arise. Numerical calculation of the impedance with the conditions $\alpha = 1$ (DBC use) and $\beta \neq 0$ (non-uniform plasma) shows that $\mathrm{Im}Z(\theta)$ can reach zero which indicates the appearance of resonance. Analysis of eq. (3) shows that this resonance appears only in cases where $\beta\theta_\mathrm{i}^3 \leq 29$. Keeping in mind the assumption $U \gg U_\mathrm{F}$, and using the Child-Langmuir law, one can obtain $\theta_\mathrm{i} \approx 3(U/T_\mathrm{e})^{1/4}$ [11]. Defining the scale of the inhomogeneity of the quasineutral plasma as $R_n = r_\mathrm{d}/\beta$, this condition for the appearance of resonance may be written as

$$R_n \gtrsim \ell \approx r_\mathrm{d}(U/T_\mathrm{e})^{3/4}\,, \tag{4}$$

with a clear meaning: the value of R_n near the sheath edge should be larger than the thickness ℓ of the plasma sheath (the formula for ℓ on the right is also derived from the Child-Langmuir law). The plasma inhomogeneity is determined mainly by the spatial distribution of ionization and the device geometry but not by the plasma density itself, the discharge current or the sheath voltage fall. In contrast, the sheath thickness does depend directly on the voltage fall across the sheath and, as a consequence, by varying the voltage one is able to satisfy the condition (4).

It is seen in fig. 2 (top) that the smallest transit angle θ satisfying $\mathrm{Im}Z(\theta) = 0$ (the first resonance) should be between 2π and 3π. The value of the ratio R_n/ℓ defines the resonance frequencies ω_r where $\mathrm{Im}Z(\theta_\mathrm{r}) = 0$. In the marginal case $R_n = \ell$ one has $\theta_\mathrm{r} \approx 2.4\pi$ (fig. 2 top). The ion transit time might be estimated as $\tau \approx 3/\omega_\mathrm{i}(U/T_\mathrm{e})^{1/4}$ and correspondingly the resonance frequency $f_\mathrm{r} = \omega_\mathrm{r}/2\pi$ as evaluated from $\omega_\mathrm{r}\tau = \theta_\mathrm{r}$ is

$$f_\mathrm{r} \approx 2.5(U/T_\mathrm{e})^{-1/4} f_\mathrm{i}\,, \tag{5}$$

where $f_\mathrm{i} = \omega_\mathrm{i}/2\pi$. Typically $f_\mathrm{r} \approx 0.5 f_\mathrm{i}$–$0.8 f_\mathrm{i}$. The real part of the impedance $\mathrm{Re}Z(\theta)$ (fig. 2 bottom) becomes negative here, which implies the instability of this resonance in the absence of losses.

Concerning the experiments related with this kind of resonance one should note that the sheath thickness *increases* with the voltage across the sheath only when the stationary bulk plasma density does not depend on that voltage (*e.g.*, Langmuir probes). On the contrary, the sheath thickness might *decrease* with the growth of the voltage across the sheath when the stationary plasma density increases with that voltage (*e.g.*, cathode fall in a gas discharge). Correspondingly the experiments should be divided into the two following groups.

The first group consists of experiments done with a biased body (probes) located in an externally prepared plasma, mainly hot-cathode discharge plasma [1, 2]. In these experiments the stationary bulk plasma density and the probe bias U were independent. Therefore, as is seen from eq. (4), in order to get the resonance the probe bias should be below a certain value. Unfortunately the plasma inhomogeneity near the probes was not measured or controlled. This may clarify why some authors observed the resonance [2] while others did not [1]. Also one should not expect unstable or very sharp resonances because the low probe voltage contradicts the assumption $U \gg U_\mathrm{F}$. This leads to a significant number of plasma electrons penetrating inside the sheath up to the probe surface. This electron current "shunts" the sheath and weakens the resonance. Indeed these authors used very sensitive methods such as a balance method or sensitive receivers in order to observe the resonance. In case when the condition (4) is not satisfied the resonance does not exist, though near a transit angle θ of about 2π the sheath capacitance is minimal (fig. 2 top). This was clearly observed experimentally [5].

The second situation appears in experiments where the body located in the plasma acts as a cathode. In such a case, as we have already mentioned, the stationary bulk plasma density and the cathode fall voltage are strictly dependent. For example, in unstable hollow-cathode discharges [3, 10, 12] the plasma density *vs.* the cathode fall voltage could be approximated as $n_0 \sim U^\gamma$ with $\gamma \gtrsim 4$, although T_e has been kept constant. Therefore $\ell \sim U^{-5/4}$, *i.e.* in order to reach the threshold, one should increase the voltage. Fortunately in a number of these experiments the plasma inhomogeneity was measured. It was found that when the hollow cathode diameter $2R_0$ was comparable with its length, the plasma density near the cathode wall was approximately one order of magnitude less than in the cathode cylinder axis. This is correct for all the discharge currents and corresponding densities investigated. Fitting the plasma profile with a parabola, it is easy to obtain that near the wall $R_n/R_0 \approx 0.055$. Since this ratio does not depend on the cathode material, the cathode radius or the sort of gas, one may compare R_n with the sheath thickness ℓ when the cathode fall U reaches the threshold value.

TABLE I. – *Comparison of the near-threshold values of the scale of inhomogeneity R_n, the sheath thickness ℓ and the measured and calculated frequencies for different cathode materials, cathode sizes and gases.*

Material & gas	Length (cm)	$2R_0$ (cm)	ℓ (cm)	R_n (cm)	f_r(exp) (MHz)	f_r(cal) (MHz)
Fe-He	10	8.5	0.17	0.26	25	32
Fe-He	5	3.5	0.074	0.11	28	34
Fe-Xe	20	13.5	0.13	0.26	14	11
Cu-He	20	26	0.35	0.76	18	17
Cu-He	10	13.5	0.2	0.38	26	29

For this comparison we have chosen the data for hollow cathodes with diameter/length ratio within the range 0.7–1.5 [3]. These results are presented in table I together with the measured and calculated frequencies near the threshold. It is seen that even such crude estimates give us very reasonable results: The calculated sheath thicknesses are always somewhat smaller than the corresponding values of R_n, *i.e.* the theoretical threshold (eq. (4)) is exceeded by a factor of 1.5 to 2 only. This is understood because we did not take into account any collisional losses [10] nor any longitudinal plasma inhomogeneities [12]. The frequency discrepancies are within 20%. Even for very short or very long cylinders with extreme R_n/R_0 values the discrepancy between R_n and ℓ does not exceed a factor of 3.

We have to note that the present theory predicts not just one but a series of unstable resonances with increasing frequencies (see fig. 2): $\theta_{\mathrm{r}}^{(n)} \simeq n\theta_{\mathrm{r}}^{(1)}$. On the other hand, it is easy to show that the ion velocity spreading determined by ion temperature of $\sim$1 eV should "kill" all the resonances when $n \geq 4$. The experimentally observed instability, as a rule, was identified with the first resonance, very seldom with the second one and never with the third or any higher resonances.

Concerning the above-quoted "lack of inductance" in the equivalent circuit of the cathode sheath, we have to note that DBC cancels this "lacking". Indeed, when the voltage across the sheath increases, the sheath edge moves away from the cathode and this reduces the initial ion velocity (in the laboratory frame). In other words, the sheath edge movement tries to compensate the current changes and this is an "inductor-like" behavior.

To conclude, a theory of the low-frequency plasma sheath resonance has been presented. We have found the plasma parameters responsible for the creation and stability of this kind of resonance, namely the plasma inhomogeneity and the plasma sheath thickness. In the light of this theory the apparent contradictions among published experimental results have been eliminated. The theory allows us to predict when such an instability in plasma-containing devices will occur. Moreover, if desired, one can select the appropriate plasma parameters so as to produce intense rf oscillations or to prevent them.

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