Spreading particle trajectories near a perfectly reflecting surface in a tilted magnetic field

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A description is given for electron trajectories near a specularly reflecting surface in a magnetic field that intersects the surface at a shallow angle. A simple analytical theory of electron motion is presented. It is shown that an electron experiences a large number of reflections from the wall and makes a long path along the surface, before eventually returning to the plasma. The importance of this phenomenon for the sheath structure and electron cross-field transport is briefly discussed. © *1995 American Institute of Physics.*

I. INTRODUCTION

The problem we are considering in this paper arises in the analysis of sheath structure in a tilted magnetic field, when the intersection angle α of the field lines and the wall is small compared to unity.[1–7] The sheath in this case has a specific structure, consisting of two clearly distinguishable subsheaths—the ion, with a scale length of the order of the ion gyroradius ρ_i—and the electron, having a much smaller thickness. As the uninhibited electron current to the wall is much greater than the ion current, a potential barrier is formed that repels a majority of the impinging electrons. A portion of them is reflected within the ion subsheath, but a considerable fraction ($\sim\alpha$) comes closer to the wall and gets reflected from the electron subsheath near the wall. The thickness of the latter is roughly equal to electron Debye radius λ_{De} (see some more details below). We assume that, though the angle α is small, it is still larger than the square root of the electron-to-ion mass ratio $(m/M)^{1/2}$. For a tokamak divertor, a typical value of α is 0.1 while the square root of mass ratio is 1.5×10^{-2} (for deuterium plasma).

In the ion subsheath, the density drops to $\sim\alpha$ of the initial density, while, in order to maintain quasineutrality of losses to the walls, the electron density at the wall should be $(M/m)^{1/2}$ times less than the initial density. Therefore, the density in the electron subsheath should drop by another factor of $\alpha(M/m)^{1/2}$ with respect to the density on the wall side of the ion subsheath. The potential drop scales as the logarithm of the density ratio. This means that, in the aforementioned example (deuterium, $\alpha\sim0.1$), the potential drop within the ion subsheath is approximately $2T_e/e$, while the potential drop in the electron subsheath is approximately $1.5T_e$. For a Maxwellian distribution of electrons entering the sheath from the plasma side, the distribution of electrons approaching the electron subsheath remains Maxwellian (with a density $\sim\alpha$ of the initial density).

The qualitative consideration given above shows that the two-scale sheath structure ceases to exist at very small α's, $\alpha<(m/M)^{1/2}$—in agreement with the results of Refs. 2 and 4. We do not consider the case of very small α's in our paper.

Depending on the relationship between λ_{De} and the electron gyroradius ρ_e, one can have two very different types of electron motion in the electron subsheath. If condition $\lambda_{De}\gg\rho_e$ holds, electrons are magnetized and their motion can be described in the drift approximation. This problem has been considered in Ref. 7. In the present communication, we consider an opposite case,

$$\lambda_{De}\ll\rho_e. \tag{1}$$

We consider Cartesian geometry, with the magnetic field in the y-z plane at an angle α from the z axis and the wall in the x-z plane. We introduce the unit vector $\mathbf{n}$ normal to the surface and a unit vector $\mathbf{b}=\mathbf{B}/B$ along the magnetic field, both directed away from the surface. In order to facilitate comparison of the results of this paper with those of Refs. 5 and 7, we use the same orientation of the coordinate axes as in Refs. 5 and 7.

When condition (1) holds with a large margin, the reflecting potential barrier can be considered as a structureless reflecting surface. It turns out that in this (seemingly trivial) case the electron motion near the reflecting surface possesses some interesting features whose understanding is required for making correct conclusions regarding (1) the height of the potential barrier Φ, (2) the possible role of the surface imperfections, and (3) electron transport on open field lines.

It is clear, that only the electrons whose normal velocity at the wall is large enough

$$mv_y^2/2>e\Phi \tag{2}$$

penetrate the reflecting surface and are absorbed by the wall. The importance of a trajectory analysis can be illustrated by the following consideration.

Electrons approach the surface along the strongly tilted magnetic field lines. During every cyclotron period, electrons approach the wall only by a small distance $\sim\alpha\rho_e$ and, accordingly, hit the wall at a very shallow angle not exceeding $\alpha^{1/2}$ (see below). This, together with relationship (2), would suggest that $e\Phi$ should not exceed a few times αT_e, as otherwise there would be virtually no electrons reaching the wall. Hence, it would seem that the retarding potential should be very small. This conclusion is, however, incorrect: we will show that, in the course of the successive reflections, an electron inevitably passes through the state where it hits the wall almost perpendicularly to it.

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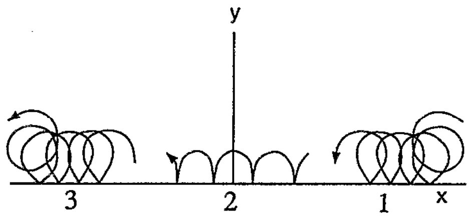

FIG. 1. Projection of the electron trajectory in the x-y plane. In phase 1 the electron experiences its first reflection from the wall and then bounces approximately $1/\alpha$ times, with a gradual increase of the collision angle from $\sim\alpha$ to ~ 1; at this phase $v_\parallel$ diminishes to $v_\parallel \ll v$. In phase 2 the electron experiences $\sim 1/\alpha$ reflections at an angle ~ 1, with a small value of $v_\parallel$. As $v_\parallel$ grows, electron enters phase 3 when it experiences $\sim 1/\alpha$ reflections, with a gradual decrease of the collision angle (caused by the increase of $v_\parallel$), and eventually leaves the surface. In order to not overload the picture, we have shown three phases separately (though they are part of the same trajectory).

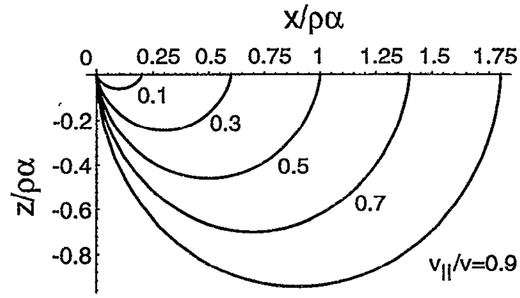

FIG. 2. Smoothed trajectories projected onto the x-z plane. The numbers by the curves mark the initial values of $v_\parallel/v$ electron pitch angles. The first collision with the wall occurs at the point (0,0) in all three cases.

II. TRAJECTORY ANALYSIS

We denote by $\mathbf{v}_1$ the electron velocity vector before a particular reflection and by $\mathbf{v}_2$ its velocity after reflection. As the normal component of velocity reverses its sign in the case of a specular reflection, while the tangential one remains unchanged, we obtain:

$$\mathbf{v}_2 = -\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_1) + [\mathbf{v}_1 - \mathbf{n}(\mathbf{n}\cdot\mathbf{v}_1)] = \mathbf{v}_1 - 2\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_1). \tag{3}$$

Projecting this expression on the unit vector $\mathbf{b}=\mathbf{B}/B$, we obtain

$$v_{2\parallel} - v_{1\parallel} = -2(\mathbf{n}\cdot\mathbf{b})(\mathbf{n}\cdot\mathbf{v}_1). \tag{4}$$

Note that Eqs. (3) and (4) are exact relations which do not exploit the smallness of α.

We recall that $\mathbf{b}$ is directed *away from* the surface, so that $(\mathbf{n}\cdot\mathbf{b}) = \sin\alpha > 0$. At the moment just before every reflection, the normal velocity component is directed towards the surface, i.e., $\mathbf{n}\cdot\mathbf{v}_1 < 0$. Accordingly, every collision is accompanied by a positive (directed away from the wall) change of the parallel velocity:

$$v_{2\parallel} - v_{1\parallel} > 0. \tag{5}$$

Initially, for the particle approaching the wall, $v_\parallel < 0$. For the particle leaving the wall, one must have $v_\parallel > 0$; hence $v_\parallel$ must change sign. As is clear from Eq. (4), the change of $v_\parallel$ at every reflection is small, $\Delta v_\parallel < \alpha v$. Thus, the change of $v_\parallel$ occurs in many small steps, and the particle, in order to reach a positive value of $v_\parallel$, must pass through a state in which $v_\parallel \ll v$ (phase 2 in Fig. 1). In this state the particle will experience approximately $1/\alpha$ reflections from the wall, at an angle close to 90°. Hence, in order to reflect a majority of the impinging electrons, the potential height $e\Phi$ must exceed T_e, just as in an unmagnetized plasma.

Another way to reach the same conclusion is to observe that, for the case of almost total reflection of the electrons from the electron subsheath, the electron distribution near the reflecting surface will be almost Maxwellian, and as is well known, a Maxwellian distribution (in fact, any isotropic distribution) is not affected at all by the presence of a magnetic field. So, one could conclude that the distribution function of electrons incident on the surface is just the same as in the unmagnetized plasma, and so should be Φ. The question that would remain unanswered in such an approach is whether there exist electron trajectories that are localized near the wall, not communicating with the bulk of the plasma. But our trajectory analysis shows that such trajectories are absent; the leak of the electrons to the wall will be compensated for by the influx of fresh electrons for all types of electron trajectories.

As a small fraction ϵ of electrons overcomes the potential barrier, the electron distribution function beyond the Debye sheath is slightly different from Maxwellian (there are no high-energy electrons moving away from the wall). Accordingly, there occurs a slight change of potential $\sim\epsilon T_e/e$ occurring at a scale of the order of ρ_e from the wall. But, because it is small, this potential variation does not affect our analysis in any important way.

To summarize the results of the qualitative trajectory analysis, we conclude that electron trajectory, in the projection on to the x-y plane, looks as shown in Fig. 1. First, electrons are being reflected from the surface at a shallow angle (phase 1 in Fig. 1). Then they bounce for some time, experiencing $\sim 1/\alpha$ reflections at angles ~ 1 (phase 2 in Fig. 1); during this phase the parallel velocity changes the sign. Then, the reflection angle decreases again, while the parallel velocity, being directed already away from the wall, gradually increases until eventually the electron leaves the wall. Because, between the first and the last collisions with the wall, the electron spends many gyroperiods bouncing near the surface, we call these very characteristic trajectories *spreading trajectories* (by analogy with spreading plants).

An interesting aspect of the electron trajectory is that, between the first and the last collisions with the wall, the electron traces out a long path along the wall. On each of three phases, the electron undergoes a displacement $\sim\rho_e/\alpha$ in the x direction. When observed in the projection on the x-y plane, the electron covers a quasicontinuous trajectory as shown in Fig. 2.

The smoothed electron trajectories can be described in a more quantitative way. To do that, we use the exact kinematic relationship (4) and two more relationships which (taken together) establish a link between two consecutive reflections. To find these additional two relationships, it is convenient to introduce an angle β which is measured in the

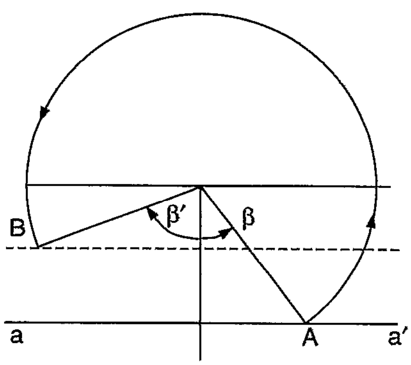

FIG. 3. Illustration of the definition of the angles β and β'. The situation shown in this figure corresponds to the particle in phase 1 of its bounces, when $v_\parallel$ is directed toward the wall.

plane perpendicular to the magnetic field and characterizes the position of an electron on its gyrocircle at its intersection point with the plane (Fig. 3). Because of the guiding center motion along the tilted field line, the horizontal line aa (which depicts the intersection of the reflecting plane and the gyrocircle plane) moves upward (with $v_\parallel>0$) or downward (with $v_\parallel<0$). The position of the electron on its gyrocircle just before the next impact with the wall can be characterized by the angle β', whose meaning is clear from Fig. 3. The relationship between β' and β reads:

$$v_{\parallel i}(\tan\alpha)(2\pi-\beta_i-\beta_i')=v_{\perp i}(\cos\beta_i-\cos\beta_i'). \tag{6}$$

The subscript i denotes the ith cycle of motion. When the electron gets reflected at the point B, new values of β, $v_\parallel$, and $v_\perp$, which apply for the next cycle of the gyromotion, are established ($\beta=\beta_{i+1}$, $v_\parallel=v_{\parallel i+1}$, $v_\perp=v_{\perp i+1}$).

The simplest way of finding β_{i+1} and $v_{\parallel i+1}$ is as follows: one can express the Cartesian components of the velocity v_x, v_y, and v_z just before the collision in terms of β_i', $v_{\parallel i}$, and $v_{\perp i}$, while expressing the same components after the collision in terms of β_{i+1}, $v_{\parallel i+1}$, and $v_{\perp i+1}$ and noting that v_x and v_z remain unchanged while v_y changes its sign. The three resultant equations allow us to express β_{i+1}, $v_{\parallel i+1}$, and $v_{\perp i+1}$ in terms of β_i', $v_{\parallel i}$, and $v_{\perp i}$. The results are

$$\tan\beta_{i+1}=\cos 2\alpha\,\tan\beta_i'+\frac{v_{\parallel i}}{v_{\perp i}}\frac{\sin 2\alpha}{\cos\beta_i'}, \tag{7}$$

$$v_{\parallel i+1}=v_{\perp i}\sin\beta_i'\sin 2\alpha+v_{\parallel i}\cos 2\alpha. \tag{8}$$

(At this stage we do not need $v_{\perp i+1}$, but it can be easily found from energy conservation once $v_{\parallel i+1}$ is known.)

Based on the relationship (6), one can find the maximum possible value of β' at which electrons can hit the wall for the first time. To do that, one should assume that, on the preceding cycle of the gyromotion, the electron has touched the wall tangentially, i.e., $\sin\beta_i=(v_{\parallel i}/v_{\perp i})\tan\alpha$. This gives the following equation for the maximum possible value of β' at first collision:

$$\left[2\pi-\beta'_{\max}-\arcsin\left(\frac{v_\parallel}{v_\perp}\tan\alpha\right)\right]\frac{v_\parallel}{v_\perp}=\left(1-\frac{v_\parallel^2}{v_\perp^2}\tan^2\alpha\right)^{1/2}-\cos\beta'_{\max}. \tag{9}$$

For small α's, the following approximate expression holds:

$$\beta'_{\max}\cong(4\pi\alpha v_\parallel/v_\perp)^{1/2}. \tag{10}$$

One should note that there is a large numerical factor entering expression (10). Therefore, validity of the following analysis for all the electrons hitting the wall requires that α be quite small, $\alpha<1/20$. Otherwise, the analysis pertains to the smaller group of electrons whose first collision with the wall occurs at small angles (such electrons are, obviously, present even if $\beta'_{\max}\sim 1$).

We will analyze in some detail the consequences of the recurrence relationships (4), (6), and (7) for the case $\alpha\ll 1$, when one can replace $\sin\alpha$ by α and $\cos\alpha$ by unity. In this case, clearly, the variation of both $v_\parallel$ and β between two successive cycles of the gyromotion is small (of the order of α). Using the appropriate expansions, one readily finds from the recurrence relationships that

$$\Delta\beta\equiv\beta_{i+1}-\beta_i\cong 2\alpha(v_\parallel/v_\perp)\{[(\pi-\beta)/\sin\beta]+\cos\beta\}. \tag{11}$$

The time within which this variation occurs is $\Delta t=2(\pi-\beta)/\omega$, where ω is the electron gyrofrequency. With the same accuracy, the change of $v_\parallel$ during one cycle of the gyromotion is [see Eq. (4)]:

$$\Delta v_\parallel=-2\alpha v_\perp\sin\beta. \tag{12}$$

Dividing $\Delta v_\parallel$ and $\Delta\beta$ by Δt, we obtain the differential equations for the smoothened particle motion:

$$\dot v_\parallel=\alpha\omega v_\perp\sin\beta/(\pi-\beta), \tag{13}$$

$$\dot\beta=\alpha\omega(v_\parallel/v_\perp)\{[(\pi-\beta)/\sin\beta]+\cos\beta\}/(\pi-\beta). \tag{14}$$

To form a closed set, these equations should be supplemented by the energy conservation law:

$$v_\parallel^2+v_\perp^2=v^2=\text{const}. \tag{15}$$

From Eqs. (13)–(15), we find one more integral of the system:

$$\frac{1}{2}\ln\frac{v^2-v_{\parallel 0}^2}{v^2-v_\parallel^2}=\int_0^\beta\frac{\sin^2\beta'\,d\beta'}{\pi-\beta'+\sin\beta'\cos\beta'}\equiv F(\beta). \tag{16}$$

Since we are assuming that the initial value of β is small [see the comment following Eq. (9)], we have replaced the lower integration limit by zero. A plot of the function $F(\beta)$ is presented in Fig. 4.

The equations for the position of the bouncing particle in the x-z plane read (with accuracy up to the leading terms in α):

$$\dot x=v_\perp\sin\beta/(\pi-\beta), \tag{17}$$

$$\dot z=v_\parallel. \tag{18}$$

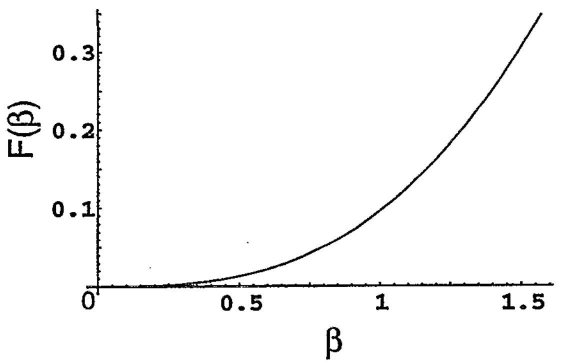

FIG. 4. The function $F(\beta)$.

The third integral of the system follows from Eqs. (13) and (17):

$$x - v_{\parallel}/\alpha\omega = \text{const.} \tag{19}$$

It shows that before the first and the last reflections, the particle is displaced in the x direction by a distance $2v_{\parallel 0}/\alpha\omega$, where $v_{\parallel 0}$ is the initial value of the parallel velocity.

Remarkably, this displacement is just the same as for the case $\lambda_{De} \gg \rho_e$ considered in Ref. 7. As was shown there, displacement in the x direction is responsible for the closing of the electron diamagnetic current on the plasma side of the sheath. Therefore, the conclusions of Ref. 7 on that issue can be also applied to that part of the electron diamagnetic current that reaches the electron subsheath.

Using relationships (15)–(19), one can calculate a smoothed electron trajectory in the x-z plane. The results of numerical integration for a set of initial values of $v_{\parallel}/v$ were presented in Fig. 2.

III. DISCUSSION

A model of the wall that perfectly reflects not only electrons but also ions is sometimes used for considering plasma equilibria in fusion devices (see, e.g., Ref. 7). In such a model the ions reflecting from the wall will have the same "spreading" trajectories as electrons. In particular, they will be displaced by a distance $\sim\rho_i/\alpha$ between the first and last collisions.

For a strictly flat surface and $\alpha \ll 1$ an electron leaves the surface with essentially the same values of $v_{\perp}$ and $v_{\parallel}$ as the ones it had during its initial collisions with the wall, and the trajectory has a symmetric character. However, if the surface becomes even slightly wavy, with the spatial scale of the nonuniformities of the order of ρ_e/α or less, then the electron gets reflected with different values of $v_{\perp}$ and $v_{\parallel}$ and gets displaced not only in the x but also in the z direction. To produce this effect, the variation of the tilt of the surface with respect to the $y=0$ plane can be quite small, of the order of α. Therefore, presence of even very slight "waviness" of the surface causes a scattering of electrons over the pitch angles and random displacements across the magnetic field lines. Whether this effect would cause a considerable anomalous electron transport, depends on the electron bounce frequency along the field lines.

If nonuniformities of the surface become "rough," with a spatial scale of the order of ρ_e or less, and variation of the tilt ~ 1, every collision of an electron with the wall causes change of the electron pitch angle ~ 1 and displacement of the guiding center across the field line $\sim\rho_e$. On average, the electron gets reflected from the wall after 1–2 collisions. The spatial diffusion in this case becomes slower than in the aforementioned case of "gentle" variations.

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