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Stability of the magnetized plasma-wall transition layer

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Under the condition $\lambda_D \ll \rho_i \ll \ell$ (where λ_D is the Debye length, ρ_i is the ion gyro-radius, and ℓ is the smallest relevant collision length) and for the case of the obliquity of the magnetic field to the wall, the magnetized plasma-wall transition layer can be split into the following sub-layers: the Debye sheath (DS), the magnetic pre-sheath (MPS), and the collisional pre-sheath (CPS). Thanks to the above-mentioned condition, it is possible to investigate these sublayers independently of each other. In this paper, the kinetic theory of the stability of the MPS and the CPS is presented (for the DS, a small parameter, relevant for the development of the linear theory, is not found). The ion gas was assumed to move with a constant velocity, and the presence of such an ion beam made it necessary to modify the form of the Bohm–Chodura criterion and the behavior of the electric potential in the CPS. The instability rate was found to be proportional to the square root of the time. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4931048]

I. INTRODUCTION

The past several decades have seen intensive experimental and theoretical investigations of the stability of the plasma-wall transition (PWT) layer without *a magnetic field*, as is clear from the large number of publications.[1–8] However, to the best of our knowledge, only a few of the published papers have taken into account the influence of *a magnetic field*.

In one of the earliest papers[9] on plasma-sheath stability, the instability of the current sheath in a magnetic field with respect to collision-free tearing is considered. The width of the sheath can be of the order of the ion Larmor radius or ever smaller. In such a very inhomogeneous situation, the electrostatic force acting on the particles owing to charge separation can no longer be neglected, as was the case earlier. It is clearly demonstrated that the current sheath is generally unstable against tearing only if the magnetic field goes through zero. A stability analysis of an inhomogeneous plane equilibrium is developed, finding the necessary and sufficient criteria for stability against tearing.

Reference 10 presents the theory of the universal drift modes' instability and shows that it can exist in a strongly inhomogeneous, collisionless, plasma sheath. A corresponding tractable methodology was developed to analyze the stability-related properties. Low-frequency drift waves are considered in a planar sheath of neutral plasma, with the thickness being of the order of the ion gyro-radius. The analysis is valid for arbitrary wave numbers and accounts for the shear in the equilibrium magnetic field. The conclusion is that the magnetic shear does not necessarily play a stabilizing role for these modes. This is in marked contrast to the results for a weakly inhomogeneous plasma medium.

In Ref. 11, the sheath and ballooning instabilities with high toroidal number modes in the scrape-off layer of the divertor tokamak are investigated. As an equilibrium state, a simple model of the X-point geometry and the parallel (as well as cross-field) equilibrium variation of temperature, density, and potential are chosen. The primary goal of this paper was to estimate (based on numerical solutions) the relative contributions of the curvature, the sheath, and the $E \times B$ shear to the stability. The stability analysis includes the influence of all these effects. The numerical solutions indicate two important modes: (i) an interchange-like MHD mode with the growth rate enhanced by the sheath's boundary conditions, and (ii) an $E \times B$ shear mode. The former is stabilized by the finite Larmor radius effects at high mode numbers. The growth rate of the latter increases with the mode number until the wavelength is of the order of the ion Langmuir radius. The results of Ref. 11 demonstrate a weak dependence of the growth rate on the scrape-off layer (SOL) density, the temperature, and the gradient scale length.

An analytical description of the magnetized plasma boundary layer, as a single unit, does not provide a satisfactory result because of the mathematical difficulties; therefore, the magnetized plasma-wall transition (MPWT) layer is usually split into three regions, i.e., the Debye sheath (DS), the magnetic pre-sheath (MPS), and the collision pre-sheath (CPS), with characteristic length scales λ_D (the electron Debye length), ρ_i (the ion gyro-radius), and ℓ (the smallest relevant collision length), respectively. Such a subdivision is convenient in the so-called "asymptotic three-scale (A3S) limit,"[12] when $\varepsilon_{Dm} = (\lambda_D/\rho_i) \to 0$ and $\varepsilon_{cm} = (\rho_i/\ell) \to 0$, which makes it possible to investigate these sub-layers separately. Such a simplification implies a detailed study of these three sub-layers, including their stability. The DS can be characterized as collisionless and non-neutral ($n_i \neq n_e$ with n_i and n_e the ion and electron number densities, respectively), the MPS as quasi-neutral $(n_i = n_e)$ and collisionless and the CPS as collisional and quasi-neutral.[13,14]

In the present paper, the stabilities of the CPS and the MPS sub-layers are investigated analytically in the kinetic

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approximation. For the description of the un-perturbed steady state, the results presented in Ref. 15 are used. In Sec. II, the main equations are given. It is assumed that the ion gas moves with a constant fluid velocity. The presence of such an ion beam made it necessary to change the form of the Bohm–Chodura condition and the shape of the electric potential in the CPS. These are realized in Sec. III. The stabilities of the CPS and the MPS are considered in Secs. IV and V, where the growing rates during the time of the instabilities are found. These rates are proportional to the square root of the time.

II. MAIN EQUATIONS

The plasma, placed in the semi-bounded region, is weakly ionized and composed of singly charged ions, electrons, and a cold neutral gas background, distributed uniformly. The z axis is directed perpendicular to the wall, and hence the plasma occupies the region $z \leq 0$ (see Fig. 1). Initially, the thermal motion of the ions is neglected, whereas the electrons follow the Boltzmann distribution $n_e = n_0 \exp\{e\Phi(t,z)/T_e\}$; here, $\Phi(t,z)$ is the electric potential. The electron temperature T_e is measured in energetic units. Under the condition $\lambda_D \ll \rho_i \ll \ell$, the method known as the motion of the guiding centers[16] can be used, as found in Ref. 15. As demonstrated in Ref. 15 (see the Appendix and item (iii) in Section IV there) in the CPS (and also in the MPS), under the conditions considered, the ions can only move in the positive z-axis direction. The ions are assumed to be produced by the electron-neutral impact ionization also accounted for by the source term that is proportional to the ionization frequency ν_i, and governed by the charge-exchange collisions with neutrals. For the ion velocity distribution function (VDF), we use the kinetic equation from Ref. 15 [see Eqs. (23)–(25)]

$$\frac{1}{c_s}\frac{\partial f_0}{\partial t} + v_\parallel \cdot \mathrm{Sin}\,\alpha \frac{\partial f_0}{\partial z} + \mathrm{Sin}\,\alpha \frac{\partial \varphi}{\partial z}\frac{\partial f_0}{\partial v_\parallel} + \frac{v_\parallel}{\lambda_{cx}} f_0$$
$$= \frac{1}{2\pi}\delta(v_\perp)\cdot\delta(v_\parallel)\left\{\frac{1}{\lambda_{cx}}\int_0^\infty v_\parallel \cdot f_0(t,z,v_\parallel)dv_\parallel + \frac{v_\parallel}{\lambda_i}\cdot n_e(t,z)\right\}, \tag{1}$$

where the following dimensionless quantities are introduced:

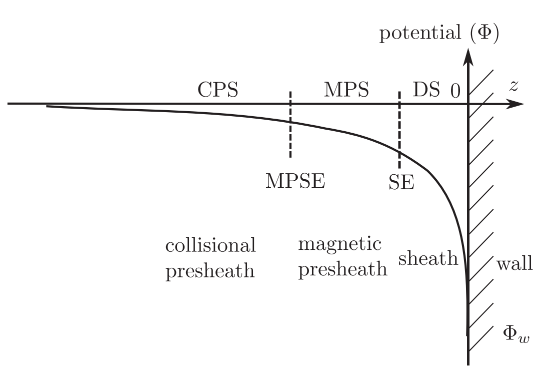

FIG. 1. MPWT geometry.

$$-e\Phi/T_e = \varphi\,, \quad c_s^3 f_0/n_0 \to f_0\,,$$
$$c_s/v_i = \lambda_i\,, \quad c_s = \sqrt{T_e/m_i}\,, \tag{2}$$

the normalized phase-space velocity and the densities are denoted with the same symbols as the non-normalized ones

$$\vec{v}/c_s \to \vec{v}\,, \quad n_{e,i}/n_0 \to n_{e,i}\,. \tag{3}$$

Here, n_0 is the particle density at $z \to -\infty$ (where $\varphi \to 0$). The first two terms on the right-hand side of Eq. (1) describe the charge-exchange collisions of the ions with the cold neutrals that have the frequency $v_{cx}(v_\parallel) = v_\parallel/\lambda_{cx}$, $v_\parallel \geq 0$ (where λ_{cx} is the ion mean-free path at their charge-exchange collisions with neutrals). The gas of neutrals is homogeneous in space (their number density $n_n = \mathrm{const}$) and follows the distribution function $f_n = n_n\delta(\vec{v})$. The Dirac δ-function appears in Eq. (1) just from the latter expression. In the MPWT theory, the model of the constant ion mean-free path, $\lambda_{cx} = \mathrm{const}$, during their charge-exchange collision with neutrals, is considered as a good approximation. For the analytical description of the stability, we extract from the VDF $f_0(t,z,\vec{v})$ and the electric potential $\varphi(t,z)$ their perturbed parts depending on the time

$$f_0(t,z,\vec{v}) = f_i(z,\vec{v}) + \delta f_i(t,z,\vec{v}), \tag{4}$$

$$\varphi(t,z) = \varphi(z) + \delta\varphi(t,z). \tag{5}$$

Obviously, $f_i(z,\vec{v})$ and $\varphi(z)$ describe the unperturbed state of the MPWT layer. Introducing the dimensionless coordinate, $x = (z/\ell)$ (where $\ell = \lambda_{cx}\cdot\sin\alpha$; α is the small inclination angle of the magnetic field to the wall) for the unperturbed ion VDF, we obtain

$$v_\parallel \frac{\partial f_i}{\partial x} + \frac{\partial \varphi}{\partial x}\frac{\partial f_i}{\partial v_\parallel} + v_\parallel \cdot f_i = \frac{1}{2\pi\cdot v_\perp}\delta(v_\perp)\cdot\delta(v_\parallel)C(x)\,, \tag{6}$$

$$C(x) = \int dv_\parallel \cdot v_\parallel \cdot f_i + \frac{\lambda_{cx}}{\lambda_i}\exp\{-\varphi\}\,, \tag{7}$$

$$\lambda_D \frac{\partial^2\varphi}{\partial z^2} = n_i - \exp(-\varphi)\,, \quad \lambda_D = \sqrt{T_e/4\pi e^2 n_0}\,. \tag{8}$$

For the ion density, the ion flux, and the ion temperature, we have

$$n_i(t,z) = \int dv_\parallel f_0(t,z,v_\parallel), \tag{9}$$

$$J_i(t,z) = n_i \cdot u_i = \int dv_\parallel \cdot v_\parallel f_0(t,z,v_\parallel), \tag{10}$$

$$n_i(t,z)\cdot T_i(t,z) = \int dv_\parallel (v_\parallel - u_i)^2 \cdot f_0(t,z,v_\parallel). \tag{11}$$

Equations (6)–(11) represent the system of partial integro-differential equations. Using the inequality $\lambda_D \ll \rho_i \ll \ell$ and splitting the MPWT layer into sub-layers, we can separate the CPS and the MPS from the DS, trying to describe, analytically, the stability of these two sub-layers separately. The stability properties of the DS now remain outside of the consideration, as a suitably small parameter that can be used for developing a stability theory for the DS is not known.

III. SHAPE OF THE POTENTIAL IN THE CPS

Using the results of Ref. 15, we can easily generalize the unperturbed solution of Eq. (6) for the case when ions move with the velocity v_0 and it satisfies the boundary condition $f_i(z,\vec{v}) \propto \delta(v_\parallel - v_0)$ at $\varphi = 0$. According to Eqs. (20)–(27) in Ref. 15, we find

$$f_i(z,\vec{v}) = \frac{1}{2\pi\cdot v_\perp}\delta(v_\perp)\cdot\Big\{\sqrt{2y_0}\cdot\delta(y-y_0-\varphi(x)) \times \exp\{-[x(\varphi)-x(0)]\} + C(\varphi-y)\cdot x'(\varphi-y) \times \exp\{-[x(\varphi)-x(\varphi-y)]\}\cdot \mathrm{H}(\varphi-y)\Big\}, \tag{12}$$

$$C(\varphi) = \int_0^\infty dy\cdot f_i(\varphi,y) + \frac{\lambda_{cx}}{\lambda_i}\cdot e^{-\varphi}, \tag{13}$$

where $x'(\varphi) = dx(\varphi)/d\varphi$, $y = v_\parallel^2/2$, $y_0 = v_0^2/2$, $\mathrm{H}(s)$ is the Heaviside step function, and $v_\parallel$ is the ion velocity along the magnetic field line. Due to the smallness of the electron Debye-length, $\lambda_D \ll \rho_i \ll \ell$, for the unperturbed states in the CPS and the MPS the quasi-neutrality condition is valid, $n_i = \exp(-\varphi)$, which means that the ion particle density is equal to the electron density. In terms of Eq. (12), the last equality takes the form of the equation for the electric potential φ

$$\exp\big[-\varphi + x(\varphi) - x(0)\big] = \int_0^\infty dy\frac{\sqrt{y_0}}{\sqrt{y}}\delta(y-y_0-\varphi) + \frac{1}{\sqrt{2}}\int_0^\varphi \frac{d\varphi'}{\sqrt{\varphi-\varphi'}}C(\varphi')\cdot x'(\varphi') \times \exp\big[x(\varphi') - x(0)\big]. \tag{14}$$

The potential profile resulting from Eq. (21) is shown in Fig. 2. Using Eq. (14), we can define, in analytical form, the electric potential close to the CPS edge, estimate φ_m—the potential at the CPS edge—and the Bohm–Chodura criterion in the presence of an ion beam with the velocity v_0.

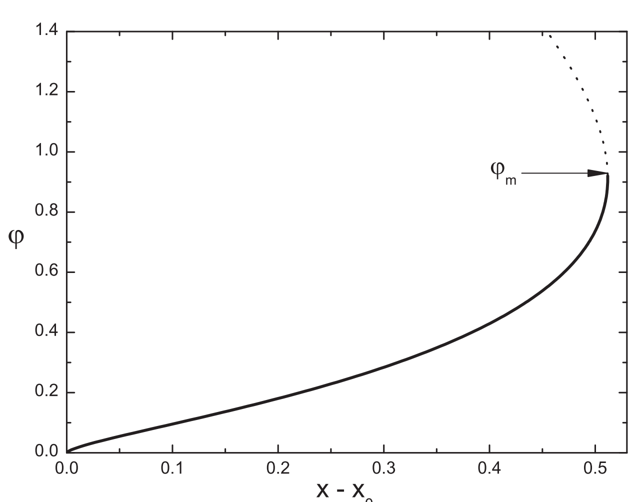

FIG. 2. The quasi-neutral potential profile $\varphi(x-x_0)$ for $y_0 = 0.1$ at $\lambda_{cx}/\lambda_i = 0.1$.

A. Behavior of the electric potential in the CPS

Multiplying Eq. (14) by $1/\sqrt{\varphi-\varphi'}$ and integrating over φ', we find

$$\int_0^\varphi d\varphi'\frac{e^{-\varphi'+x(\varphi')-x(0)}}{\sqrt{\varphi-\varphi'}} - 2\sqrt{y_0}\cdot\arcsin\sqrt{\frac{\varphi}{y_0+\varphi}} = \frac{\pi}{\sqrt{2}}\int_0^\varphi d\varphi'' C(\varphi'')\frac{\partial\cdot e^{x(\varphi'')-x(0)}}{\partial\varphi''}. \tag{15}$$

In obtaining the second term on the right-hand side of Eq. (15), we change the order of the integrations and use the relation

$$\int_{\varphi''}^{\varphi}\frac{d\varphi'}{\sqrt{(\varphi-\varphi')(\varphi'-\varphi'')}} = \pi\,. \tag{16}$$

According to Eqs. (10), (12), and (13), the ion flux in the unperturbed state is equal to

$$J_i(\varphi) = \sqrt{2y_0}\cdot\exp[-\{x(\varphi)-x(0)\}] + \int_0^\varphi d\varphi'\cdot C(\varphi') \times x'(\varphi')\cdot\exp[-\{x(\varphi)-x(\varphi')\}]\,, \tag{17}$$

where

$$C(\varphi) = J_i(\varphi) + \frac{\lambda_{cx}}{\lambda_i}\exp(-\varphi). \tag{18}$$

From Eq. (17), it follows that $\partial J_i(\varphi)/\partial\varphi = (\lambda_{cx}/\lambda_i)\exp(-\varphi)\cdot x'(\varphi)$, hence

$$J_i(\varphi) = \frac{\lambda_{cx}}{\lambda_i}\int_0^\varphi d\varphi'\cdot e^{-\varphi}x'(\varphi') + \sqrt{2y_0}, \tag{19}$$

$$C(\varphi) = \frac{\lambda_{cx}}{\lambda_i}\left\{\int_0^\varphi d\varphi'\cdot e^{-\varphi'}x'(\varphi') + e^{-\varphi}\right\} + \sqrt{2y_0}. \tag{20}$$

After straightforward calculations and grouping the coefficients in front of $x'(\varphi)$, we obtain

$$\Phi(\varphi)\frac{\partial x(\varphi)}{\partial\varphi} = \frac{\partial}{\partial\varphi}\overline{\Phi}(\varphi), \tag{21}$$

where

$$\Phi(\varphi) = \frac{\pi}{\sqrt{2}}\frac{\lambda_{cx}}{\lambda_i}\exp\big\{-\varphi + x(\varphi) - x(0)\big\} + \overline{\Phi}(\varphi), \tag{22}$$

and

$$\overline{\Phi}(\varphi) = 2\sqrt{y_0}\cdot\arcsin\sqrt{\frac{y_0}{y_0+\varphi}} + \int_0^\varphi\frac{d\varphi'}{\sqrt{\varphi-\varphi'}}\exp\big\{-\varphi' + x(\varphi') - x(0)\big\}. \tag{23}$$

From Eq. (21), it follows that at the point $x = x_m$ (or $\varphi = \varphi_m$), where $\partial\overline{\Phi}/\partial\varphi = 0$, or

$$-\frac{y_0}{\sqrt{\varphi}(y_0+\varphi)}+\frac{\partial}{\partial\varphi}\int_0^{\varphi}\frac{d\varphi'}{\sqrt{\varphi-\varphi'}}\times\exp\left\{-\varphi'+x(\varphi')-x(0)\right\}=0\,, \qquad (24)$$

the value $\partial x(\varphi)/\partial\varphi$ is equal to zero, which means that the electric field ($E\propto d\varphi/dx$) has a singularity, $E\to\infty$. This point is usually defined as an edge of the CPS. To find φ_m, we can present the expression (24) in a simpler form

$$\frac{\sqrt{\varphi_m}}{y_0+\varphi_m}=-\int_0^{\varphi_m}\frac{ds}{\sqrt{\varphi_m-s}}\frac{\partial}{\partial s}\exp\left\{-s+x(s)-x(0)\right\}. \qquad (25)$$

Fig. 3 shows the dependence of φ_m on the ion beam velocity y_0. Expanding the right-hand side of the relation (21) in powers of $\varphi_m-\varphi$ in the first non-vanishing approximation, we obtain

$$x_m-x=A\cdot(\varphi_m-\varphi)^2\,, \qquad (26)$$

where

$$A=-\frac{1}{2\Phi(\varphi_m)}\frac{\partial^2\overline{\Phi}}{\partial\varphi^2}\bigg|_{\varphi_m}\,. \qquad (27)$$

It should be mentioned that at the point $\varphi=\varphi_m$ according to Eq. (12) the ions with zero velocity along the magnetic field are absent, $f_i(\varphi_m,\vec{v}_\perp,v_\parallel=0)=0$.

B. Bohm–Chodura criterion

According to Eqs. (9)–(11), the expressions for the ion number density, the flux, and the ion temperature in the unperturbed state can be represented in the form

$$n_i(\varphi)=\left\{\sqrt{\frac{y_0}{y_0+\varphi}}+\int_0^{\varphi}\frac{dy}{\sqrt{2(\varphi-y)}}\overline{\overline{f}}_i(y)\right\}\times\exp\{-[x(\varphi)-x(0)]\}, \qquad (28)$$

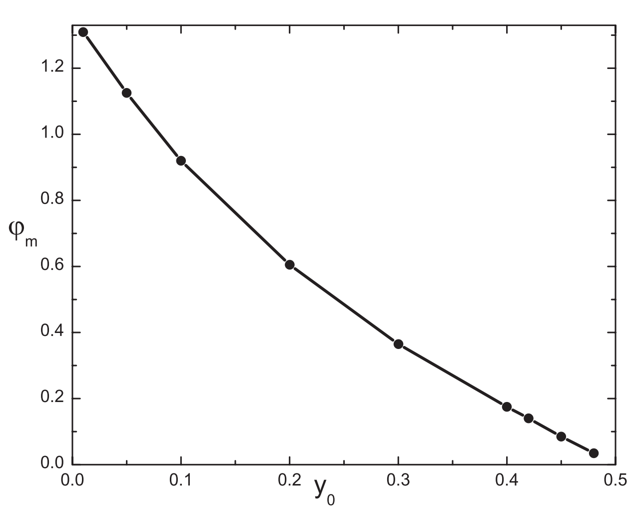

FIG. 3. Dependence of φ_m on the ion beam velocity y_0 at $\lambda_{cx}/\lambda_i=0.1$.

$$n_i(\varphi)\cdot u_i(\varphi)=\left\{\sqrt{2y_0}+\int_0^{\varphi}dy\cdot\overline{\overline{f}}_i(y)\right\}\times\exp\{-[x(\varphi)-x(0)]\}, \qquad (29)$$

$$n_i(\varphi)\cdot T_i(\varphi)=\left\{\sqrt{2y_0}\cdot\sqrt{2(y_0+\varphi)}+\int_0^{\varphi}dy\sqrt{2(\varphi-y)}\cdot\overline{\overline{f}}_i(y)\right\}\times\exp\{-[x(\varphi)-x(0)]\}-n_i(\varphi)\cdot u_i^2(\varphi), \qquad (30)$$

where

$$\overline{\overline{f}}_i(\varphi)=C(\varphi)\cdot x'(\varphi)\cdot\exp\{x(\varphi)-x(0)\}, \qquad (31)$$

[see the second term on the right-hand side of Eq. (12)]. After differentiating Eq. (30), we find

$$\left\{1+\frac{n_i}{T_i}\frac{\partial T_i/\partial\varphi}{\partial n_i/\partial\varphi}\right\}\frac{T_i}{n_i}\frac{\partial n_i}{\partial\varphi}=1+\frac{1}{n_i}\frac{\partial n_i}{\partial\varphi}\cdot u_i^2-2\frac{u_i}{n_i}\overline{\overline{f}}_i(\varphi)\times\exp\{-[x(\varphi)-x(0)]\}-x'(\varphi)\{T_i-u_i^2\}. \qquad (32)$$

Bearing in mind that at the CPS edge $x'(\varphi_m)=0$, and according to Eq. (31) $\overline{\overline{f}}_i(\varphi_m)=0$, due to the quasi-neutrality condition being fulfilled, $n_i(\varphi)=\exp(-\varphi)$, we obtain the Bohm–Chodura criterion in the marginal form

$$u_i^2(\varphi_m)=1+\gamma_i(\varphi_m)T_i(\varphi_m)\,, \qquad (33)$$

where

$$\gamma_i(\varphi_m)=1+\frac{n_i}{T_i}\frac{\partial T_i/\partial\varphi}{\partial n_i/\partial\varphi}\bigg|_{\varphi_m}, \qquad (34)$$

is the polytropic function of the ion gas at the CPS edge. According to the expression (29), it is possible to give Eq. (33) the following form:

$$\left\{\sqrt{2y_0}+\int_0^{\varphi_m}dy\cdot\overline{\overline{f}}_i(y)\right\}^2\cdot\exp\{-2[x(\varphi)-x(0)]+2\varphi_m\}=1+\gamma_i(\varphi_m)T_i(\varphi_m). \qquad (35)$$

Using Eq. (20), and after some manipulation, the Bohm–Chodura criterion (35) can be reduced to the form containing the ion beam velocity in a simple manner

$$\sqrt{2y_0}+\frac{\lambda_{cx}}{\lambda_i}\int_0^{\varphi_m}d\varphi'\cdot e^{-\varphi'}\cdot x'(\varphi')=\sqrt{1+\gamma_i(\varphi_m)T_i(\varphi_m)}\cdot e^{-\varphi_m}\,. \qquad (36)$$

As shown in Ref. 17, the coefficient can be $\gamma_i(\varphi_m)T_i(\varphi_m)<1$ (for instance in the Tonks–Langmuir model of the discharge).

IV. STABILITY OF THE CPS

We consider the perturbations of the ion VDF and the electric potential with the characteristic scale λ_D and introduce the dimensionless coordinate $\xi = z/\lambda_D \operatorname{Sin}\alpha$. We again assume that $\lambda_D \ll \ell$, then for the perturbations $\delta f_i(t,\xi,v_\parallel)$ and $\delta\varphi(t,z)$ from Eqs. (4), (5), and (8), we obtain

$$\frac{\partial \delta f_i}{\partial \tau} + v_\parallel \frac{\partial \delta f_i}{\partial \xi} + \frac{\partial \delta\varphi}{\partial \xi}\frac{\partial f_i(x,v_\parallel)}{\partial v_\parallel} = 0, \tag{37}$$

$$\frac{\partial^2 \delta\varphi}{\partial \xi^2} = \delta n_i - \delta n_e, \tag{38}$$

where $\tau = \omega_{pi}\cdot t$ $(\omega_{pi} = \sqrt{4\pi e^2 n_0/m_i}))$, δn_i and $\delta n_e = -\exp(-\varphi(x))\cdot\delta\varphi$ are the perturbed parts of the ion and electron number densities, respectively. The solution of Eq. (37) is

$$\delta f_i = -\frac{\partial f_i(\varphi,y)}{\partial v_\parallel}\frac{\partial}{\partial \xi}\int_{-\infty}^{\tau} d\tau' \delta\varphi\Big\{\xi - v_\parallel(\tau-\tau'),\tau'\Big\}. \tag{39}$$

In Eq. (39), the function $f_i(\varphi,y)$ is defined according to the expression (12). For the perturbation of the ion number density from Eq. (39), we obtain

$$\delta n_i = \int_0^\infty dv_\parallel \cdot \delta f_i = -\int_0^\infty dv_\parallel \frac{\partial f_i(\varphi,y)}{\partial y} \times \left\{\delta\varphi(\xi,t) - \int_{-\infty}^{\tau} d\tau' \frac{\partial}{\partial \tau'}\delta\varphi\{\xi - v_\parallel(\tau-\tau'),\tau'\}\right\}. \tag{40}$$

In Eq. (39), we brought the derivative by ξ under the integral. In the integrand of the second term of Eq. (40), the derivative by τ' is implied to take only over the last argument τ' of $\delta\varphi\{\xi - v_\parallel(\tau-\tau'),\tau'\}$. Bearing in mind the dependence of $f_i(\varphi,y)$ on φ [see Eq. (12)], we can write

$$\frac{\partial f_i}{\partial y} = -\frac{\partial f_i}{\partial \varphi} - x'(\varphi)\cdot f_i. \tag{41}$$

Substituting Eq. (41) into Eq. (40), it is clear that the first term from Eq. (41) cancels [due to the quasi-neutrality condition, $n_i = \exp(-\varphi)$] the term connected with the perturbation of the electron density. After taking the derivative with τ—time, the Poison's equation (38) acquires the form

$$\frac{\partial}{\partial \tau}\frac{\partial^2 \delta\varphi(\xi,\tau)}{\partial \xi^2} = -\frac{\partial n_i(\varphi)}{\partial \varphi}\frac{\partial}{\partial \tau}\delta\varphi(\xi,\tau) + \left\{\frac{\partial}{\partial \varphi} + x'(\varphi)\right\}\int_0^\infty dy \times f_i(\varphi,y)\cdot\frac{\partial}{\partial \xi}\int_{-\infty}^{\tau} d\tau' \frac{\partial}{\partial \tau'}\delta\varphi \times\Big\{\xi - v_\parallel(\tau-\tau'),\tau'\Big\}. \tag{42}$$

Further, in the second term on the right-hand side of Eq. (42), we can expand $\delta\varphi\{\xi - v_\parallel(\tau-\tau'),\tau'\}$ in a series of powers $v_\parallel(\tau-\tau')$, restricting ourselves to only the first two terms. Applying the condition

$$\frac{\partial}{\partial \tau}\delta\varphi \gg \sqrt{2y_0}\cdot \exp\{-[|x(0)| - |x(\varphi)|]\}\frac{\partial}{\partial \xi}\delta\varphi, \tag{43}$$

and using the un-equality, $\lambda_i > \lambda_{cx}$ [see Eq. (19)], we can give Eq. (42) the form

$$\frac{\partial}{\partial \tau}\frac{\partial^2 \delta\varphi}{\partial \xi^2} = e^{-\varphi}\frac{\partial \delta\varphi}{\partial \tau} + x'(\varphi)\cdot J_i(\varphi)\frac{\partial \delta\varphi}{\partial \xi}. \tag{44}$$

According to the Fourier expansion, we have

$$\delta\varphi(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\cdot\delta\varphi_k(\tau)\cdot\exp(ik\xi), \tag{45}$$

$$\delta\varphi_k(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\xi\cdot\delta\varphi(\xi,\tau)\cdot\exp(-ik\xi). \tag{46}$$

After finding the Fourier component from Eq. (44) $\delta\varphi_k(\tau)$ and using Eq. (45), we obtain

$$\delta\varphi(\xi,\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\xi'\cdot\delta\varphi(\xi+\xi',0)\int_0^\infty dk \times \cos\left\{k\xi' + \frac{kS(\varphi)}{k^2+e^{-\varphi}}\tau\right\}, \tag{47}$$

where $S(\varphi) = x'(\varphi)\cdot J_i(\varphi)$, and $\delta\varphi(\xi,0)$ is the initial perturbation of the electric potential. The integral over ξ' we can split into two integrals, reducing both to the form $\int_0^\infty d\xi'(\cdots)$ and introduce the new variables

$$k' = \frac{\sqrt{|\xi'|}}{\sqrt{S(\varphi)}\cdot\sqrt{\tau}}\,k\,. \tag{48}$$

At a later time, when

$$\frac{e^{-\varphi}}{S(\varphi)\cdot\tau} \ll 1\,, \tag{49}$$

the integration over k' gives[18]

$$\delta\varphi(\xi,\tau) \approx -\sqrt{\tau}\cdot\sqrt{S(\varphi)}\int_{\xi}^{\infty} d\xi' \frac{\delta\varphi(\xi',0)}{\sqrt{|\xi-\xi'|}} \times \Im_1\left\{2\sqrt{S(\varphi)}\cdot\sqrt{\tau}\cdot\sqrt{|\xi-\xi'|}\right\}, \tag{50}$$

where $\Im_1\{s\}$ is a Bessel function of the first kind. When introducing the variable (48), the inequality (49) allows us to consider the second term in the denominator of the argument of the Cosine in Eq. (47) as a small term. Obviously, according to Eq. (50), the perturbation of the potential is unstable and grows over time as $\sqrt{\tau}$, which can be easily shown, taking the initial distribution of the potential under the integral in Eq. (50) in the form $\delta\varphi(\xi',0) = a\delta(\xi'-\xi_0)$, where ξ_0 is the initial location, and a is the amplitude of the perturbation.

V. STABILITY OF THE MPS

For describing an un-perturbed state of the MPS, we use the ion VDF [given by Eq. (82) from Ref. 15] in the modified form

$$\bar{f}_i(\varphi,\vec{v}) = \bar{f}_0 + \alpha\cdot\bar{f}_1 = \frac{1}{\pi}\delta\left[v_z^2+v_y^2-2(\varphi-\varphi_m)\right]\cdot\bar{S}(v_x) - \alpha\left\{\frac{\partial\bar{f}_0}{\partial v_x}+v_x\frac{\partial\bar{f}_0}{\partial\varphi}\right\}\int_{\eta_m}^{\eta}d\eta' \times\frac{v_y-[\eta-\eta']}{\sqrt{v_z^2+v_y^2-2(\varphi-\varphi(\eta'))-\left(v_y-[\eta-\eta']\right)^2}} \times \mathrm{H}\left\{v_z^2+v_y^2-2(\varphi-\varphi(\eta'))-\left(v_y-[\eta-\eta']\right)^2\right\}\cdot\mathrm{H}(v_z)\cdot\mathrm{H}(-v_y)\,, \tag{51}$$

where

$$\bar{S}(\sqrt{2|y_x|}) = \delta\{y_x-\varphi_m-y_{0x}\}\cdot\exp\left[-\{x(\varphi_m)-x(0)\}\right] + C(\varphi_m-y_x)\cdot x'(\varphi_m-y_x) \times\exp\left[-\{x(\varphi_m)-x(\varphi_m-y_x)\}\right]\cdot\mathrm{H}(\varphi_m-y_x)\,. \tag{52}$$

Here, we introduce the dimensionless coordinate $\eta = z/\rho_i$, where φ_m is the potential at the CPS edge, $y_x = \frac{1}{2}v_x^2$ and $y_{0x} = \frac{1}{2}v_0^2$. We assume the characteristic scale-length for the perturbed ion VDF and the electric potential in the MPS to be again of the order of the electron Debye length. Under the condition $\lambda_D \ll \rho_i$ and neglecting the collision terms, the equation for the perturbed ion VDF, $\delta\bar{f}_i$, in the MPS acquires a similar form to Eq. (37) [only in Eq. (37), the expression for the unperturbed ion VDF $f_i(x, v_\parallel)$ must be replaced with $\bar{f}_i(\varphi,\vec{v})$ from Eq. (51)]

$$\frac{\partial\delta\bar{f}_i}{\partial\tau} + v_z\frac{\partial\delta\bar{f}_i}{\partial\xi} + \frac{\partial\delta\bar{\varphi}}{\partial\xi}\frac{\partial\bar{f}_i(\varphi,\vec{v})}{\partial v_z} = 0. \tag{53}$$

Here again, $\tau = \omega_{pi}t$ and $\xi = z/\lambda_D$. The solution of Eq. (53) we can represent in the form

$$\delta\bar{f}_i = -\frac{1}{v_z}\frac{\partial\bar{f}_i}{\partial v_z}\left[\delta\bar{\varphi}(\xi,t) - \frac{1}{v_z}\frac{\partial}{\partial\tau}\int_{-\infty}^{\xi}d\xi'\cdot\delta\bar{\varphi}\left\{\xi',\tau-\frac{\xi-\xi'}{v_z}\right\}\right]. \tag{54}$$

For the perturbation of the ion number density by means of Eq. (54), we find

$$\delta\bar{n}_i = \int d\vec{v}\cdot\delta\bar{f}_i = -\int_0^\infty dv_x\int_0^\infty dv_z\int_{-\infty}^0 dv_y\frac{1}{v_z}\frac{\partial\bar{f}_i}{\partial v_z} \times\left\{\delta\bar{\varphi}(\xi,t) - \frac{1}{v_z}\frac{\partial}{\partial\tau}\int_{-\infty}^{\xi}d\xi'\cdot\delta\bar{\varphi}\left(\xi',\tau-\frac{\xi-\xi'}{v_z}\right)\right\}. \tag{55}$$

On the right-hand side of Eq. (55), we can change the variable according to Eq. (51) as follows:

$$\frac{\partial\bar{f}_i}{v_z\partial v_z} = -\frac{\partial\bar{f}_i}{\partial\varphi}. \tag{56}$$

Then, the first term from the right-hand side of Eq. (55) cancels the electron-density perturbation term, as in the MPS the quasi-neutrality condition is also fulfilled, $n_i = \int d\vec{v}\cdot\bar{f}_i = \exp(-\varphi)$. Bearing in mind that in our case the inclination angle of the magnetic field to the wall is small, $\alpha \ll 1$, Poison's equation can be represented in the form

$$\frac{\partial^2\delta\bar{\varphi}}{\partial\xi^2} = -\frac{1}{\pi}N(\varphi)\frac{\partial}{\partial\varphi}\int_0^\infty dv_z\int_0^\infty dv_y \times\delta\left\{v_z^2+v_y^2-2(\varphi-\varphi_m)\right\} \times\frac{1}{v_z}\frac{\partial}{\partial\tau}\int_{-\infty}^{\xi}d\xi'\cdot\delta\varphi\left(\xi',\tau-\frac{\xi-\xi'}{v_z}\right), \tag{57}$$

where

$$N(\varphi_m) = \int_0^\infty dv_x\bar{S}(v_x). \tag{58}$$

After straightforward calculations Eq. (57) can be reduced to the form

$$\frac{\partial^2\delta\bar{\varphi}}{\partial\xi^2} = \frac{1}{2\pi}N(\varphi_m)\frac{\partial}{\partial\varphi}\cdot\sqrt{2(\varphi-\varphi_m)}\times\int_0^1 ds\frac{\partial}{\partial\xi}\int_{-\infty}^{\tau}d\tau'\cdot\delta\bar{\varphi} \times\left\{\xi-\sqrt{2(\varphi-\varphi_m)}\sqrt{1-s^2}(\tau-\tau'),\tau'\right\}. \tag{59}$$

Expanding the function $\delta\bar{\varphi}$ under the integral close the point ξ and assuming the fulfillment of the condition

$$\left[2(\varphi-\varphi_m)\right]^{-1/2}\frac{\partial\delta\bar{\varphi}}{\partial\tau} \gg \frac{\partial\delta\bar{\varphi}}{\partial\xi}, \tag{60}$$

we obtain the equation

$$\frac{\partial^2\delta\bar{\varphi}}{\partial\xi\partial\tau} = A\cdot\delta\bar{\varphi}\,,\quad A = \frac{1}{2\pi}N(\varphi_m)\frac{1}{\sqrt{2(\varphi-\varphi_m)}}. \tag{61}$$

By means of the transformation (46), we can construct from Eq. (61) an expression for the Fourier amplitude $\delta\bar{\varphi}_k(\tau)$, depending on the time, and using the inverse transformation (45) find the solution of Eq. (61)

$$\delta\bar{\varphi}(\xi,\tau) = -\sqrt{\tau}\cdot\sqrt{A}\int_{\xi}^{\infty}d\xi'\frac{\delta\bar{\varphi}(\xi',0)}{\sqrt{|\xi-\xi'|}}\Im_1\left\{2\sqrt{|\xi-\xi'|\cdot A\tau}\right\}, \tag{62}$$

where $\delta\bar{\varphi}(\xi,0)$ is the initial shape of the potential perturbation, and $\Im_1\{s\}$ is again a Bessel function of the first kind. Obviously, the potential perturbation grows in proportion to

$\sqrt{\tau}$. As for the case of the CPS, the instability theory developed here is valid for unperturbed potentials not far from the CPS edge, where $\varphi = \varphi_m$.

VI. SUMMARY

In this paper, we have used the results obtained in our previous paper.[15] These results serve to describe an unperturbed state of the MPWT layer. In the frame of the "asymptotic three-scale" limit the sub-layers—the CPS and the MPS can be investigated separately and their instability properties can be studied independently.

As in Ref. 15, the plasma is weakly ionized and both the charge exchange collisions of the ions with neutrals and the ions' creation during the neutrals' ionization by the electron impact are taken into account. The magnetic field intersects the wall at a small angle, $\alpha \ll 1$. It is assumed that the ion gas moves with a constant fluid velocity ("ion beam"). The corresponding modifications to the Bohm–Chodura criterion and the shape of the electric potential in the CPS are described. The dependence of the potential on the coordinate in analytical form in the vicinity of the CPS edge is also determined.

We can conclude that the instabilities of the CPS and the MPS can grow in proportion to the square root of the time, $\sqrt{t}$.

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