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The influence of collisions on the plasma sheath transition

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In the asymptotic limit $\lambda_D/\lambda \to 0$ (where λ_D is the electron Debye length and λ represents the ion mean free path) the boundary layer of a collision dominated plasma is split up in a collision free planar sheath (scale λ_D) and a quasi-neutral presheath (scale λ). The ''sheath edge'' separating the presheath and the sheath is defined by the Bohm criterion. To clear up controversial statements on the validity of the plasma sheath concept and on the role of the Bohm criterion for finite values of λ_D/λ, a simple fluid model of the ions is used to account for collisions and for space charges in the boundary layer. It is shown that the asymptotic solution on an ''intermediate scale'' (scale length $\lambda^{1/5}\lambda_D^{4/5}$) is suitable to match the presheath and sheath solutions smoothly and to construct a convenient approximation for small but finite λ_D/λ. The intermediate scale, again, is closely related to the Bohm criterion. Various attempts to derive a ''generalized'' Bohm criterion accounting for collisions are inconsistent. © *1997 American Institute of Physics.* [S1070-664X(97)01811-9]

I. INTRODUCTION

In the numerical plasma simulation it is frequently possible without serious difficulties to account for space charges in the entire plasma volume. Such analyses, however, are not always favorable to reveal general aspects and similarity relations. Moreover, there are special situations, where the numerical expenditure is essentially increased by a self-consistent solution of Poisson's equation in the whole plasma volume. Consequently, it may be necessary or at least convenient to use the concept of a quasi-neutral plasma (presheath) and a collision free planar sheath. This concept is strictly valid only in the asymptotic limit $\lambda_D/L \to 0$, where λ_D is the typical electron Debye length of the boundary layer, and L represents the smallest competing length (e.g. the ion mean free path λ). The ''sheath edge'' separating the two model zones is governed by the Bohm criterion.[1,2] A detailed discussion of the plasma-sheath concept and the Bohm criterion may be found in Refs. 2 and 3.

For practical applications it is important to judge the validity of the asymptotic concept for finite values $\varepsilon = \lambda_D/L$. This is in particular true for the boundary layer of collision dominated plasmas ($L=\lambda$) where we have a relatively small charged particle density (and, correspondingly, a relatively large Debye length) in the boundary layer. Consequently, the basic assumptions of the two scale analysis are frequently not or only poorly fulfilled in collision dominated plasmas, and there is no reason to be surprised if a numerical analysis yields ion velocities too small to fulfill the Bohm criterion.[4–6]

For this reason different attempts have been made to ''generalize'' the Bohm criterion accounting for collisions.[7–10] These attempts, however, are based on an arbitrary definition of the presheath and sheath regions that is not consistent with the two scale concept. In particular, it must be emphasized that the solutions of Poisson's equation depend rather delicately on the plasma boundary conditions, and that it is not allowed to start at an *arbitrarily* defined ''sheath edge'' with *arbitrarily* chosen or postulated initial conditions.[2,11] In a Comment[12] to Ref. 9, the errors introduced by this arbitrariness were explicitly exhibited.

The problem of a consistent definition of the sheath edge for finite $\varepsilon = \lambda_D/\lambda$ is closely related to a general problem originating from the ''sheath edge singularity'': In the asymptotic theory the sheath edge is *defined*[2] by a singularity of the electric field terminating the quasi-neutral presheath. Due to this singularity, it is not possible—and this holds for arbitrarily small values of ε—to match the presheath and sheath solutions smoothly. This consequence was occasionally regarded as a lack[13] or even a contradiction[8,9] of the two scale analysis. Actually it shows that the environment of the sheath edge must be considered on an ''intermediate scale'' accounting for space charge *and* for collisions (or other relevant presheath processes). Corresponding consistent investigations for special problems were presented by Lam[14] and Su[15] (probe theory), by Franklin and Ockendon[16] (Tonks–Langmuir model of the collision-less plasma column) and by Riemann[17] (kinetic theory of the collisional presheath). Unfortunately these intermediate scale analyses were not observed or forgotten in later publications dealing with a matching of plasma and sheath.

In this paper we want to expand in more detail on the ideas and arguments sketched in the Comment (Ref. 12). Our aim is to clear up persistent misunderstandings on the plasma-sheath concept and to discuss its correct application as well as its validity limits for finite values of $\varepsilon = \lambda_D/\lambda$. To this end we start from a very simple fluid model to describe the ions of the boundary layer of a collision dominated plasma. In a subsequent paper we shall extend the analysis to more general and/or more sophisticated models of the plasma-sheath transition.

The model assumptions and basic equations are presented in Section II. In Section III we start from numerical results for various values of λ_D/λ to discuss the usual two scale analysis and the matching difficulty. In Section IV we investigate the plasma-sheath transition on the intermediate scale. This intermediate scale provides the basis for a consistent matching and a convenient approximation presented in Section V. The results and conclusions are summarized in Section VI.

II. MODEL AND BASIC EQUATIONS

To discuss the basic features of the plasma-sheath transition, we use a very simple model of the boundary layer: We consider a one-dimensional weakly ionized plasma in front of a plane absorbing wall. The wall position is given by $z=z_w$, and the plasma is characterized by $z<z_w$. The electric potential Φ is governed by Poisson's equation

$$n_i-n_e=-\frac{\varepsilon_0}{e}\frac{d^2\Phi}{dz^2}. \tag{1}$$

The wall is sufficiently negative and the electrons are assumed to be in Boltzmann equilibrium

$$n_e=n_{\mathrm{ch}}\exp\frac{e\Phi}{kT_e} \tag{2}$$

with a characteristic charged particle density n_{ch} and the electron temperature T_e. To calculate the ion density n_i, we start from the cold fluid equations

$$n_iu_i=j_i, \tag{3}$$

$$m_iu_i\frac{du_i}{dz}=-e\frac{d\Phi}{dz}-\nu_c(u_i)m_iu_i. \tag{4}$$

Here m_i and u_i designate the mass and the (average) velocity of the ions. For the sake of clearness and transparency, we restrict ourselves in this paper to the boundary layer of a *collision dominated* plasma and neglect ionization. Consequently, the ion current density j_i in the continuity equation (3) is constant. In the momentum balance (4) we use a collision frequency $\nu_c(u_i)$ to account for ion friction. (To be specific, let assume charge exchange with cold neutrals.) To discuss some special aspects related to the collision model we shall refer in the evaluation to the special cases

(a) constant collision frequency, $\nu_c(u_i)=$ const and/or
(b) constant mean free path, $\nu_c(u_i)=u_i/\lambda$.

The system of equations (1)–(4) for the unknown quantities n_i, u_i, n_e, and Φ may be simplified if we use the characteristic quantities

$$c_s=\sqrt{\frac{kT_e}{m_i}},\quad \lambda_D=\sqrt{\frac{n_{\mathrm{ch}}e^2}{\varepsilon_0kT_e}},\quad \text{and}\quad \lambda=\frac{c_s}{\nu_c(c_s)} \tag{5}$$

(ion sound velocity, Debye length, and ion mean free path) and introduce the dimensionless variables

$$s=\frac{z}{\ell},\quad \varphi=-\frac{e\Phi}{kT_e}, \tag{6}$$

$$n_\pm=\frac{n_{i,e}}{n_{\mathrm{ch}}},\quad u=\frac{u_i}{c_s},\quad \nu(u)=\frac{\nu_c(u_i)}{\nu_c(c_s)}. \tag{7}$$

Further we are free to choose

$$n_{\mathrm{ch}}=\frac{j_i}{c_s} \tag{8}$$

by adjusting the zero point of the potential. We then have

$$n_+=1/u\quad \text{and}\quad n_-=e^{-\varphi}, \tag{9}$$

and arrive at the basic equation system

$$u\frac{du}{ds}-\frac{d\varphi}{ds}=-\frac{\ell}{\lambda}u\nu(u), \tag{10}$$

$$\frac{1}{u}-e^{-\varphi}=\left(\frac{\lambda_D}{\ell}\right)^2\frac{d^2\varphi}{ds^2} \tag{11}$$

[with $\nu(u)=O(1)$ and $\nu(1)=1$] for the two unknown quantities u and φ. Suitable initial conditions for φ, φ', and u in the plasma region ($s\to-\infty$) may be obtained from the quasineutral approximation (see Section III A). (The wall potential may be simply adjusted by a parallel shift of the solution because the equations are homogeneous and we are free to define the position $s=0$.)

III. TWO SCALE ANALYSIS

So far we did *not* specify the characteristic length ℓ. Of course it is self-suggesting to choose one of the two following possibilities:

(o) ''Outer'' or ''presheath'' scale $\ell_o=\lambda$:

$$s_o=x=z/\lambda. \tag{12}$$

(i) ''Inner'' or ''sheath'' scale $\ell_i=\lambda_D$:

$$s_i=\xi=z/\lambda_D. \tag{13}$$

For a fixed finite ratio $\varepsilon=\lambda_D/\lambda\ll1$ these two scales are naturally equivalent. In the limit $\varepsilon\to0$, however, we must clearly distinguish the two scales. This can be seen from Fig. 1 where we have plotted the results of a numerical integration of Eqs. (10) and (11) for the special case $\nu_c=$ const ($\nu=1$).

Figure 1(a) shows the potential profiles for various finite values of ε on the *presheath* scale $x=z/\lambda_D$ (the *outer* scale in the terminology of asymptotic two scale analysis). All profiles show a logarithmic shape for $x\to-\infty$ indicating that a small residual field in the boundary layer is necessary to overcome ion friction for $j_i=$ const. (To describe the undisturbed plasma, ionization and/or non-planar geometry must be accounted for.[18]) For small values of ε we can distinguish the steep potential gradient of the ''sheath'' ($x>0$) and the weak gradient of the ''presheath'' ($x<0$). In the limit $\varepsilon\to0$ the sheath solution degenerates to a vertical line, and the presheath solution runs into a field singularity at $\varphi=0$ or $u=1$. We have chosen this ''sheath edge singularity'' of the asymptotic ($\varepsilon=0$) solution to define the position $x=0$ (see below). For larger values of ε (say $\varepsilon>0.1$) it appears questionable to distinguish a presheath and a sheath region.

The presentation of the same results on the *sheath* scale $\xi=z/\lambda_D$ is shown in Fig. 1(b). Here, the curves are plotted with a parallel shift ξ_w corresponding to a wall potential $\varphi=3$. For small values of ε we observe a gradual fading away of the strong sheath field. In the limit $\varepsilon\to0$, the potential tends asymptotically to the limiting value $\varphi=0$ of the sheath edge, indicating that the presheath is infinitely remote on the sheath scale for $\varepsilon=0$. For larger values of ε, again, distinct sheath and presheath regions cannot be recognized.

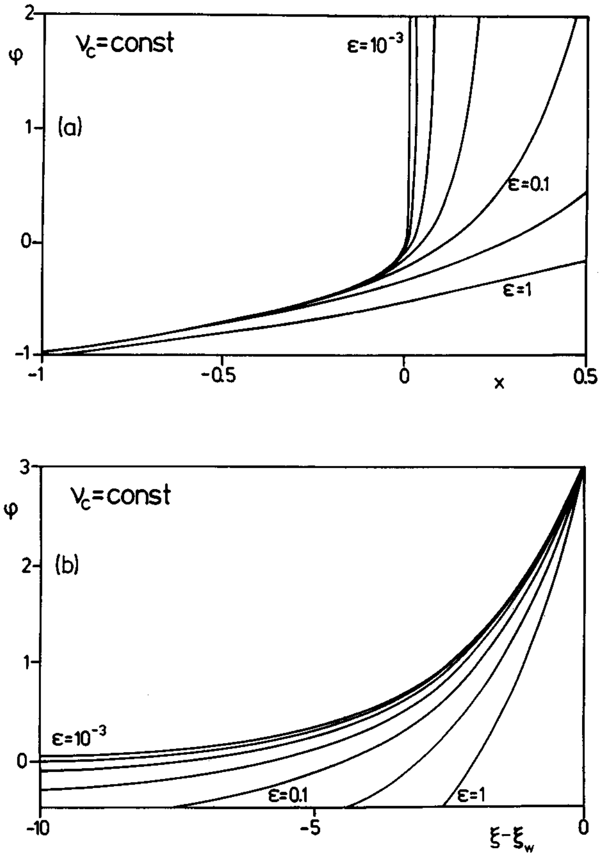

FIG. 1. Profiles of the potential $\varphi=-e\Phi/kT_e$ for $\nu_c=$ const. $\varepsilon=\lambda_D/\lambda$, $\varepsilon^2=10^{-6}$, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 0.1, 1. (a) Presheath scale $x=z/\lambda$. (b) Sheath scale $\xi=z/\lambda_D$. (The curves refer to a wall potential $\varphi=3$.)

This conclusion can be confirmed by an inspection of the space charge plotted in Fig. 2 as a function of the ion velocity $u=u_i/c_s$. Only in the limiting case $\varepsilon=0$ we find a precisely defined space charge (sheath) region $u>1$ and a neutral (presheath) region $u<1$. The common ''boundary'' of these regions (sheath edge) is defined by the Bohm criterion $u=1$ $(u_i=c_s)$. For all finite (and in particular for larger) values of ε a subdivision of the boundary layer in a sheath and a presheath appears artificial and somewhat arbitrary.

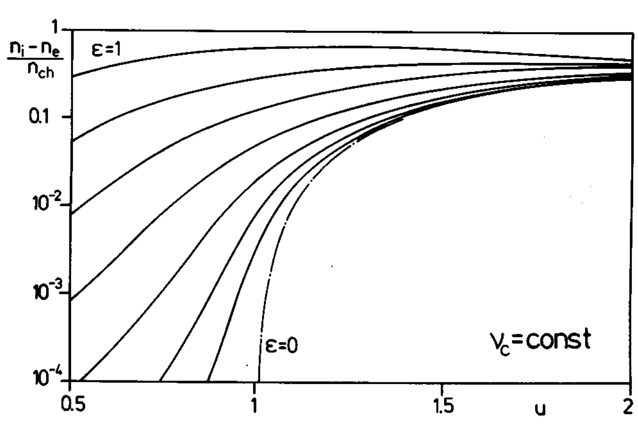

FIG. 2. The space charge (n_+-n_-) as a function of the ion velocity $u=u_i/c_s$ for $\nu_c=$ const and $\varepsilon^2=0$, 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 0.1, and 1.

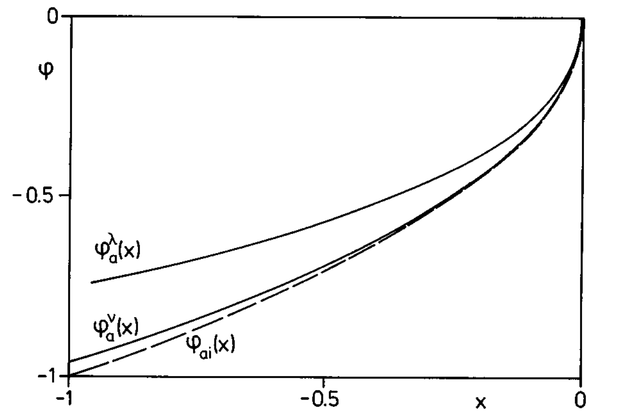

FIG. 3. The outer (presheath) solutions $\varphi_o^\nu(x)$ for $\nu_c=$ const [cf. Eq. (16)] and $\varphi_o^\lambda(x)$ for $\lambda=$ const [cf. Eq. (17)]. Dashed line: The inner expansion $\varphi_{oi}(x)$ of the outer solutions [cf. Eq. (18)]. $(x=z/\lambda,\ \varphi=-e\Phi/kT_e)$.

The question of a reasonable definition of a sheath and presheath region will be answered in the next section. Before we do that we shall first provide the asymptotic theory ($\varepsilon=0$) as a basis for the further discussion.

A. Asymptotic presheath theory

We use the space coordinate $x=z/\lambda$ of the outer scale (presheath, $\ell=\lambda$); see Eq. (12). In the asymptotic limit $\varepsilon=\lambda_D/\lambda=0$ Eq. (11) turns to the quasi-neutrality relation

$$\varphi=\ln u, \tag{14}$$

and we obtain from Eq. (10)

$$\frac{1-u^2}{u^2\nu(u)}du=dx. \tag{15}$$

This equation describes the build up of the flow velocity $u<1$ in the presheath. In the limit $u\to1$ (ion sound barrier, Bohm criterion) it breaks down with a singularity representing the sheath edge.[2,19,20] We have used this sheath edge singularity to define the origin $x=0$ of our space coordinate.

For some special cases the integration can be performed explicitly. In particular we obtain (a) for constant collision frequency ν_c $(\nu=1)$

$$x=2-2\cosh\varphi_o=-\varphi_o^2-\tfrac{1}{12}\varphi_o^4-\ \ldots \tag{16}$$

and (b) for constant mean free path λ $(\nu=u)$

$$x=\tfrac{1}{2}[1-e^{-2\varphi_o}-2\varphi_o]=-\varphi_o^2+\tfrac{2}{3}\varphi_o^3-\ \ldots\ . \tag{17}$$

Here we have used an index o to mark the approximation of the *outer* scale. In both special (and all other) cases we obtain, approaching the sheath edge, $x\to-\varphi_o^2$ $(x\to0)$. The limiting parabola

$$\varphi_{oi}(x)=-\sqrt{-x} \tag{18}$$

is called the ''*inner expansion*'' of the outer solution $\varphi_o(x)$. It is plotted in Fig. 3 (dashed line) together with the outer solutions $\varphi_o(x)$ for the cases $\nu_c=$ const and $\lambda=$ const. The case $\nu_c=$ const is distinguished by a very close agreement of φ_o and φ_{oi} [which can also be recognized directly from Eq.

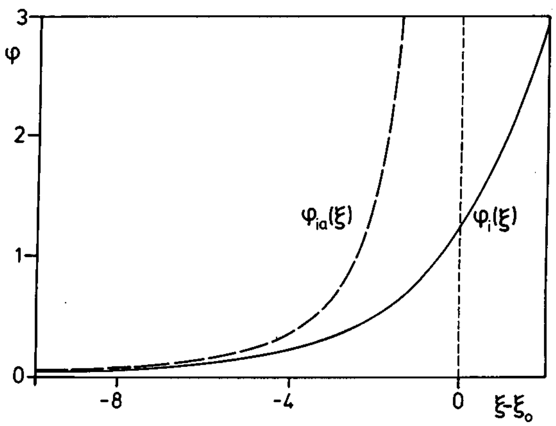

FIG. 4. The inner (sheath) solution $\varphi_i(\xi)$ and its outer expansion $\varphi_{io}(\xi)$ [cf. Eq. (22)]. ($\xi = z/\lambda_D$, $\varphi = -e\Phi/kT_e$).

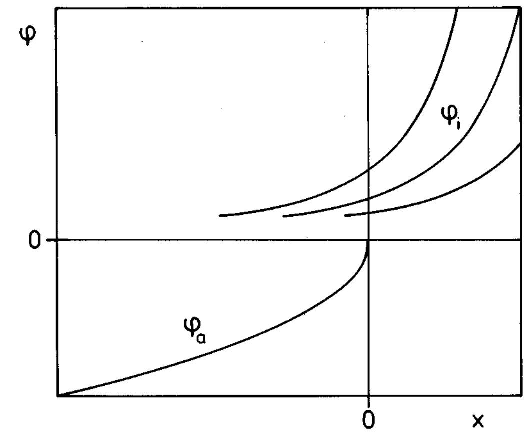

FIG. 5. The matching problem of the inner (sheath) solution φ_i and the outer (presheath) solution φ_o for fixed $\varepsilon = \lambda_D/\lambda$ (schematically). Note the ambiguity in φ_i due to the unknown constant ξ_0 [cf. Eq. (22)].

(16)]. For this reason the case $\nu_c =$ const is somewhat untypical and we consider the collision model $\lambda =$ const for comparison.

B. Asymptotic sheath theory

The sheath (inner scale, $\ell = \lambda_D$) is representeded by the space coordinate $\xi = z/\lambda_D = x/\varepsilon$ [cf. Eq. (13)]. In the asymptotic limit $\varepsilon = \lambda_D/\lambda = 0$ now space charge is essential but collisional friction can be neglected. Consequently Eq. (10) turns to the energy conservation

$$u^2 = 1 + 2\varphi, \tag{19}$$

where we have chosen the integration constant in accordance with the definition $u = 1$ (Bohm criterion) of the sheath edge in the preceding subsection. Substituting u from Eq. (19) and using the ''sheath edge boundary condition''

$$\varphi_i \to 0 \quad \text{for } \varphi \to 0 \quad (\xi \to -\infty), \tag{20}$$

Eq. (11) may be integrated in the form[2]

$$\frac{1}{2}\left(\frac{d\varphi_i}{d\xi}\right)^2 = \sqrt{1+2\varphi_i} + e^{-\varphi_i} - 2\,. \tag{21}$$

The second integration must be performed numerically. To discuss the matching problem, however, we can make an expansion to find the solution near the sheath edge ($\varphi_i \to 0$, $\xi \to -\infty$) and arrive at the ''*outer expansion*'' of the inner solution

$$\varphi_{io}(\xi) = \frac{6}{(\xi - \xi_0)^2} \tag{22}$$

with an arbitrary integration constant ξ_0 accounting for the homogeneity of the problem. The inner solution $\varphi_i(\xi)$ and its outer expansion $\varphi_{io}(\xi)$ are shown in Fig. 4. Two facts should be observed: First, different collision models need not to be distinguished, the sheath solution is ''universal.'' This is a consequence of the universal boundary condition at the sheath edge (Bohm criterion). Secondly, there is *no* satisfactory agreement of the solution $\varphi_i(\xi)$ and its expansion $\varphi_{io}(\xi)$ for potential values exceeding 0.1 (or at most 0.2).

C. The matching problem

A smooth matching of the inner and outer solutions requires a common range of validity. This is expressed by the *matching condition*[21]

$$\varphi_{oi}(x) = \varphi_{io}(\varepsilon^{-1}x) \quad (\varepsilon \to 0). \tag{23}$$

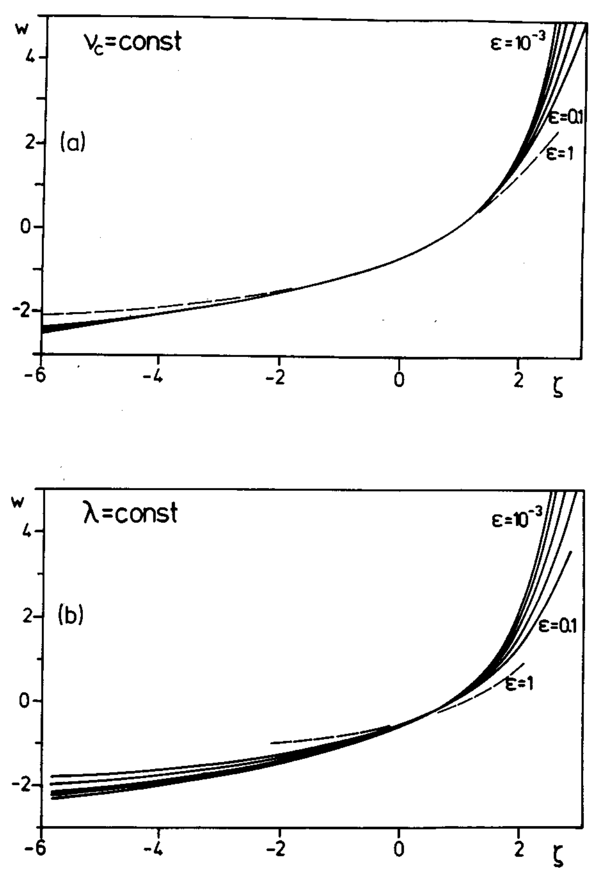

FIG. 6. Profiles of the potential $w = -\varepsilon^{-2/5} e\Phi/kT_e$ on the intermediate scale $\zeta = z/(\lambda_D^{4/5}\lambda^{1/5})$ for $\varepsilon^2 = 10^{-6}$, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}; broken line: $\varepsilon = 1$ ($\varepsilon = \lambda_D/\lambda$). (a) $\nu_c =$ const, (b) $\lambda =$ const.

An inspection of Eqs. (18) and (22) shows that the matching condition is *not* fulfilled: Plotting the presheath and sheath solutions for a fixed value of ε into a common diagram (Fig. 5), the presheath solution φ_o runs with an infinite field strength into the sheath edge ($\varphi=0$), whereas the sheath solution tends to the sheath edge with zero field. Moreover, the integration constant ξ_0 remains undefined because the Debye length cannot be resolved on the presheath scale.

This "mismatching" is no contradiction pointing to an inconsistency of the theory,[8,9] but it shows that a smooth matching of the potential profile requires an additional analysis on an *intermediate* scale[2,14–17,22] accounting for collisions *and* for space charge. For many applications this problem may be disregarded, because the sheath extension can be neglected, and the ion velocity is hardly influenced by the small transition region. The intermediate scale gains growing *practical* importance with increasing $\varepsilon=\lambda_D/\lambda$, in *principle*, however, it must be considered for arbitrarily small ε.

IV. THE INTERMEDIATE SCALE

In the transition region we use a new scale length $\ell_m=\delta\lambda$ and write

$$\zeta=\frac{z}{\ell_m}=\frac{x}{\delta}. \tag{24}$$

(The scale factor δ is not *a priori* evident, but we must naturally expect $\varepsilon\ll\delta\ll 1$.) The corresponding basic equation system [cf. Eqs. (10), (11)] reads

$$u\frac{du}{d\zeta}-\frac{d\varphi}{d\zeta}=-\delta u\,\nu(u), \tag{25}$$

$$\frac{1}{u}-e^{-\varphi}=\frac{\varepsilon^2}{\delta^2}\,\frac{d^2\varphi}{d\xi^2}. \tag{26}$$

To compare the orders of magnitude of the different terms, we remember that we have $\varphi=O(x^{1/2})$ [cf. Eq. (18)] and $u\approx 1+\varphi$ [cf. Eq. (14)] when we enter the transition region $x=O(\delta)$ from the presheath. Consequently, we use new dependent variables $w=O(1)$ and $v=O(1)$ defined by

$$\varphi=\delta^{1/2}w(\zeta)\quad\text{and}\quad u=1+\delta^{1/2}v(\zeta). \tag{27}$$

Presuming now that the left hand sides of Eqs. (25), (26) have the same orders of magnitude, we can compare the friction term $O(\delta)$ in Eq. (25) and the space charge term $O(\delta^{1/2}\varepsilon^2/\delta^2)$ in Eq. (26) and conclude that the appropriate intermediate scale to describe the transition region is given by

$$\delta=\varepsilon^{4/5}\quad\text{or}\quad \ell_m=\lambda^{1/5}\lambda_D^{4/5}. \tag{28}$$

To demonstrate the similarity rule expressed by this scale we re-plot the potential profiles of Fig. 1 ($\nu_c=$const) and the corresponding profiles for $\lambda=$const. The impressive result is shown in Fig. 6: The figures exhibit, indeed, a transition region distinguished by more or less common profiles near the sheath edge for all $\varepsilon\ll 1$. In particular in the special case $\nu_c=$const, the profiles show a close agreement up to values $\varepsilon\sim 1$ (dashed line) exceeding by far the validity range of the asymptotic theory. Evidently, there is a limiting curve $\varepsilon=0$ neither identical with the outer solution φ_o nor with the inner solution φ_i. To find this limiting ($\varepsilon=0$) profile, we expand in the lowest powers of δ. Observing $\nu(1)=1$, we obtain from Eqs. (25), (26)

$$w'-v'=\delta^{1/2}(1+vv')+O(\delta), \tag{29}$$

$$w-v=\delta^{1/2}(w''+\tfrac{1}{2}w^2-v^2)+O(\delta), \tag{30}$$

where the prime $'$ denotes a derivative with respect to ζ. Integrating now Eq. (29) and comparing with Eq. (30), we obtain in the lowest order in δ

$$v=w,\qquad\text{and} \tag{31}$$

$$\frac{d^2w}{d\zeta^2}=w^2+\zeta. \tag{32}$$

Here, the integration constant is chosen in accord with the relation $u\to 1+\varphi$ for $\varphi\to 0$ [see Eqs. (14) and (19)]. Eq. (32) is a *universal* differential equation for the potential variation ($\delta\to 0$) on the intermediate scale *not* depending on the special collision model. A physical interpretation is given in Appendix A. Apart from that it is clear that Eq. (32) represents Poisson's equation. For $\zeta\to-\infty$ we approach the presheath region and expect quasi–neutrality. The corresponding approximation is given by $\zeta=-w_l^2$ or

$$w_l(\zeta)=-\sqrt{-\zeta}. \tag{33}$$

Obviously the "*left expansion*" w_l of the intermediate solution agrees with the inner expansion φ_{oi} [cf. Eq. (18)] of the outer solution:

$$\varepsilon^{2/5}w_l(\varepsilon^{-4/5}x)=\varphi_{oi}(x). \tag{34}$$

Approaching, on the other hand, the sheath, the term ζ representing collisions (see Appendix A) may be neglected and we obtain from Eq. (32) $w_r''=w_r^2$. Multiplying by w_r' and integrating twice with the sheath edge boundary condition $w_r'=0$ for $w=0$ [see Eq. (20)] we obtain $\zeta=\zeta_0-(6/w_r)^{1/2}$ or

$$w_r(\zeta)=\frac{6}{(\zeta-\zeta_0)^2}. \tag{35}$$

Comparing with Eq. (22), we see that the "*right expansion*" w_r of the intermediate solution is identical with the outer expansion φ_{io} of the inner solution:

$$\varepsilon^{2/5}w_r(\varepsilon^{1/5}\xi)=\varphi_{io}(\xi). \tag{36}$$

Initial conditions for a numerical integration of Eq. (32) can be found from the approximation (33); details are given in Appendix B. The solution runs into a singularity at $\zeta=\zeta_0=3.4417$ [cf. Eq. (35)]. Consequently, the integration constant $\xi_0=\varepsilon^{-1/5}\zeta_0$—which was undefined on the sheath scale—is now determined by the intermediate scale analysis.

The intermediate solution $w(\zeta)$ and the expansions $w_l(\zeta)$ and $w_r(\zeta)$ are shown in Fig. 7. The plot suggests very convincingly, that $w(\zeta)$ represents the "missing link" which is required for a smooth matching of the presheath and sheath solutions. We shall show that it also provides a convenient basis to construct an approximation for small but finite λ_D/λ.

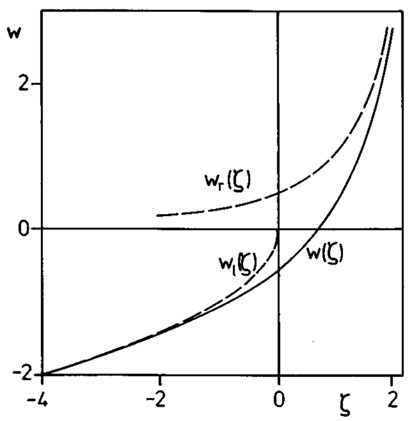

FIG. 7. Asymptotic intermediate solution $w(\zeta)$ [see Eq. (32)] and its expansions $w_l(\zeta)$ [representing the presheath solution, see Eq. (33)] and $w_r(\zeta)$ [representing the sheath solution, see [Eq. (35)]. ($\zeta = z/(\lambda_D^{4/5}\lambda^{1/5})$, $w = -\varepsilon^{-2/5} e\Phi/kT_e$, $\varepsilon = \lambda_D/\lambda$.)

V. CONSISTENT MATCHING AND APPROXIMATIONS FOR FINITE λ_D/λ

A. Matched asymptotic expression for $\lambda_D/\lambda \to 0$

The conjecture suggested by Fig. 7 is, indeed, confirmed analytically: Eq. (34) represents the matching condition of the outer (presheath) solution and the intermediate solution; correspondingly, the matching condition for the intermediate solution and the inner (sheath) solution is given by Eq. (36). According to the rules of asymptotic two scale analysis,[21,22] we can consequently construct a uniformly valid ($\varepsilon = \lambda_D/\lambda \to 0$) asymptotic expression

$$\varphi(x) = \varphi_o(x) - \varphi_{oi}(x) + \varepsilon^{2/5} w(\varepsilon^{-4/5}x) - \varphi_{io}\left(\frac{x}{\varepsilon}\right) + \varphi_i\left(\frac{x}{\varepsilon}\right) \tag{37}$$

with

$$\varphi_{oi}(x) = -\sqrt{-x}, \quad \varphi_{io}(\xi) = \frac{6}{(\xi - \xi_0)^2},$$

and

$$\xi_0 = 3.41167\varepsilon^{-1/5}.$$

Here, we have explicitly used the transformations $\zeta = \varepsilon^{-4/5}x$ and $\xi = x/\varepsilon$. The functions $\varphi_o(x)$ and $\varphi_{oi}(x)$, which are defined only for $x \leqslant 0$, are assumed to be zero for $x > 0$. By construction, in Eq. (37) $\varphi_o(x)$ is the only term contributing in the intrinsic presheath region, and $\varphi_i(x/\varepsilon)$ is the only term contributing in the intrinsic sheath region. In the transition region the solution is essentially given by $\varepsilon^{2/5}w(\varepsilon^{-4/5}x)$ because the terms $\varphi_o(x) - \varphi_{oi}(x)$ and/or $\varphi_i(x/\varepsilon) - \varphi_{io}(x/\varepsilon)$ cancel out due to the matching conditions.

The matched asymptotic expression (37) depends on ε via the arguments of the different functions (φ_o, φ_i, and w) but is, strictly speaking, valid only in the limit $\varepsilon \to 0$. Comparing Eq. (37) with numerical results, one finds that it is actually limited to very small values $\varepsilon \leqslant O(10^{-4})$. The reason can be seen from Figs. 4 and 7: A good representation of the solution by the matched asymptotic expression requires that there is an interval where both φ_i and $\varepsilon^{2/5}w(\zeta)$ are well represented by $\varphi_{io}(\xi) = \varepsilon^{2/5} w_r(\zeta)$. From Fig. 4 we consequently conclude $\varphi < 0.1$, and from Fig. 7 we read $w > 3$ or $\varphi > 3\varepsilon^{2/5}$. Combining these conditions yields $\varepsilon < 2 \cdot 10^{-4}$. This strict limitation comes from the fact that the asymptotic representation of the *sheath* is an order of magnitude worse than the asymptotic representation of the *presheath*: Whereas space charges give only corrections of the order ε^2 on the presheath scale, the influence of collisions in the sheath is $\sim\varepsilon$. Due to this asymmetry, the intermediate scale is separated only by a scale factor $\varepsilon^{1/5}$ from the sheath scale. (Accordingly, the validity of the zero order asymptotic theory is subject to the condition $\varepsilon^{2/5} \ll 1$.)

B. An approximation procedure for finite λ_D/λ

For $\varepsilon > 2 \cdot 10^{-4}$, in principle, a higher order asymptotic theory was necessary. This, however, results in a useless expenditure, because the basic equations on the different scales then become more complicated than the original equations. On the other hand, we come to a reasonable procedure if we make use of the fact that the breakdown of the zero order asymptotic solution depends primarily on the influence of collisions in the sheath. Since a numerical integration is anyhow required to solve Poisson's equation, there is hardly any additional effort necessary, to account for collisions calculating the sheath solution.

Consequently we can start from a matched asymptotic expression for the *outer* (presheath) and *intermediate* scale (transition region) only, and use the transition region to merge into a numerical sheath solution accounting for collisions. The decisive advantage of this procedure is that it saves us the numerical integration of Poisson's equation in the entire plasma volume and that it yields a *consistent* set of *initial conditions* in the transition region to solve the sheath equations. In particular it is convenient, that a *universal* function $w(\zeta)$ is sufficient to modify the quasi-neutral presheath approximation for this purpose. The proposed procedure consists in the following steps:

(1) Start from the *quasi-neutral* solution $\varphi_o(x)$.
(2) Choose a suitable "sheath limit" $x_s \geqslant 0$ (see below) and improve $\varphi_o(x)$ for $x \leqslant x_s$ by the matched asymptotic expression

$$\varphi(x) = \begin{cases} \varphi_o(x) + \sqrt{-x} + \varepsilon^{2/5} w(\varepsilon^{-4/5}x) & (x \leqslant 0) \\ \varepsilon^{2/5} w(\varepsilon^{-4/5}x) & (x \geqslant 0). \end{cases} \tag{38}$$

(3) Use the starting values [see Eq. (38)]

$$\varphi_s = \varphi(x_s) = \varepsilon^{2/5} w(\varepsilon^{-4/5}x_s), \quad \varphi_s' = \varphi'(x_s),$$

and

$$u_s = 1 + \varphi_s \tag{39}$$

for a numerical integration of the "sheath region" $x > x_s$.

Note that the starting values for the sheath integration are *universal* and do *not* depend on the the special presheath solution φ_0. The limit $x_s \geqslant 0$ therefore plays the role of a

TABLE I. Initial conditions for an integration of the sheath region.

x_s	φ_s	φ_s'	u_s	ρ_s/n_{ch}
0	$-0.5496\varepsilon^{2/5}$	$0.6244\varepsilon^{-2/5}$	$1-0.55\varepsilon^{2/5}$	$0.3020\varepsilon^{4/5}$
$0.7152\varepsilon^{4/5}$	0	$0.9602\varepsilon^{-2/5}$	1	$0.7152\varepsilon^{4/5}$
$\varepsilon^{4/5}$	$0.3067\varepsilon^{2/5}$	$1.2126\varepsilon^{-2/5}$	$1+0.31\varepsilon^{2/5}$	$1.094\varepsilon^{4/5}$

''generalized sheath edge'' for finite $\varepsilon=\lambda_D/\lambda \ll 1$. With respect to this interpretation we emphasize the following four points:

(i) There is no unambiguous *definition* of a ''sheath edge'' x_s for finite values of ε. On the other hand, the choice of x_s is *not* arbitrary: x_s must be chosen within the validity range of the intermediate scale, say within the interval $[0,\varepsilon^{4/5}]$ corresponding to $\zeta_s \in [0,1]$. In the limit $\varepsilon \to 0$ this results in $x_s - x_w \to 0$ and $\xi_s - \xi_w \to -\infty$: The sheath edge moves to the wall on the presheath scale and to $-\infty$ on the sheath scale.

(ii) The potential φ_s and the field strength φ_s' at the sheath boundary x_s cannot be chosen arbitrarily, but the corresponding initial conditions are uniquely defined by Eq. (39). In particular, the field strength (in its dimensional form) at the sheath edge is *not* given by[7,8] $E_0 = kT_e/e\lambda_D$ but by

$$E_s \sim \frac{kT_e}{e\lambda_D^{2/5}\lambda^{3/5}} = \varepsilon^{3/5}E_0. \tag{40}$$

(iii) Similarly, the *space charge* at the sheat boundary x_s cannot be chosen arbitrarily, but has a finite value

$$\rho_s \sim \varepsilon^{4/5} n_{ch} \tag{41}$$

(see Table I) increasing with ε. In this connection it is important to bear in mind that ρ is *not* neglected for $x < x_s$ but described in the approximation of the intermediate scale. The sheath limit is not the point where first space charges *occur* (for finite ε they occur everywhere!) but the point where space charge effects become *dominant.*

(iv) Corresponding to the freedom to choose x_s, there is some *ambiguity* in the ''limiting ion velocity'' $u_s = 1+\varphi_s = 1 \pm O(\varepsilon^{2/5})$ at the sheath boundary. It should be observed, however, that there is no definite shift in u_s, and that the special choice $x_s = 0.7152\varepsilon^{4/5}, \varphi_s = 0$ (see Table I) confirms the *universal* validity of the *original* Bohm criterion $u_s = 1$ ($u_i = c_s$) for (small but) finite ε. This must be clearly distinguished from the general effect of collisional friction to lower the ion flux to the wall.

In Table I we have listed three examples for a consistent choice of the initial conditions ($\varepsilon \ll 1$) together with the corresponding space charge ρ_s. The resulting approximations for the second example [$\varphi_s = 0$ (dashed line), $u_s = 1$] are compared with the exact numerical solutions in Fig. 8. We find an excellent agreement for $\varepsilon = \lambda_D/\lambda \leq = 0.01$ and a satisfactory agreement for $\varepsilon = O(0.1)$.

In the special case $\nu_c =$ const [Fig. 8(a)] the approximation appears reasonable up to values $\varepsilon \sim 1$, and may even be improved by choosing the initial conditions at smaller values of x_s. This, however, is *no* indication for a general rule but reflects the very close agreement of φ_o and φ_{oi} in the special case $\nu_c =$ const discussed in Sec. III A (see Fig. 3). The

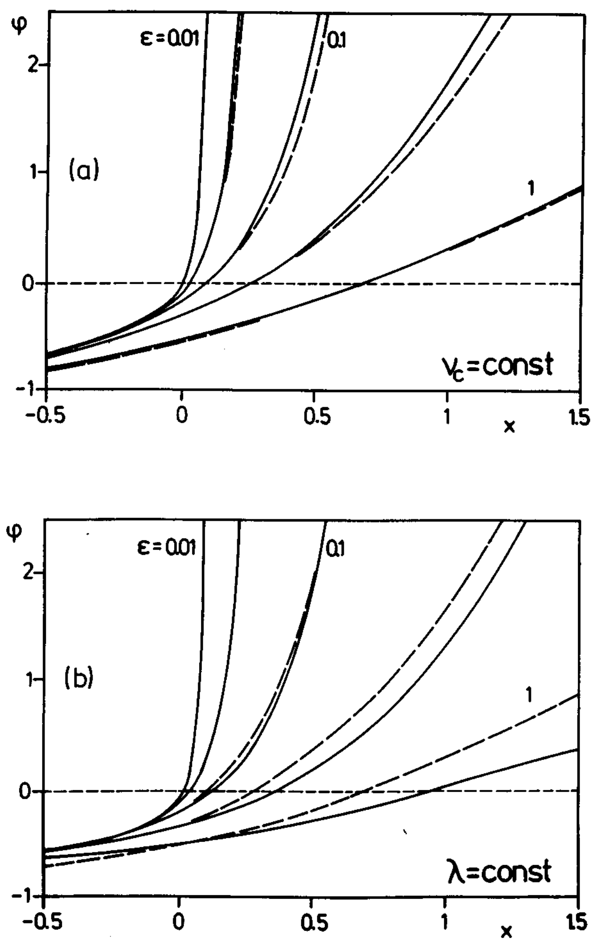

FIG. 8. Comparison of the exact potential profiles (solid curves) and the approximation of Sec. V B (dashed curves) with the initial condition $\varphi_s = 0$ (indicated by a broken line). $x = z/\lambda$, $\varphi = -e\Phi/kT_e$. $\varepsilon^2 = (\lambda_D/\lambda)^2 = 10^{-4}$, 10^{-3}, 10^{-2}, 0.1, and 1. (a) $\nu_c =$ const, (b) $\lambda =$ const.

general characteristics of the transition region and the resulting approximation are more likely represented by the case $\lambda =$ const shown in Fig. 8(b).

VI. SUMMARY AND FINAL DISCUSSION

The well-known and widely used subdivision of the boundary layer of a collision dominated plasma into a quasineutral presheath (typical extension λ) and into a collision free sheath (typical extension λ_D) is strictly valid only in the asymptotic case $\varepsilon = \lambda_D/\lambda \to 0$. Ambiguities in the application and interpretation for finite values of ε are closely related to the sheath edge singularity preventing a smooth matching of the presheath and sheath solutions (Sec. III). Various heuristic attempts[7–10] to overcome this difficulty by a modification of the Bohm criterion and/or by artificial assumptions of special sheath boundary conditions are inconsistent and lead to unreasonable results.[12]

To exhibit the problem and to demonstrate its consistent solution, we have chosen a simple fluid model of the boundary layer (Sec. II; more sophisticated models will be investigated in a later paper). A smooth matching of the presheath and sheath solutions must be performed on an intermediate

TABLE II. Numerical values of the function $w(\zeta)$.

ζ	w	ζ	w	ζ	w	ζ	w
−10	−3.1635	−3	−1.7442	0.5	−0.1917	2.4	5.6442
−8	−2.8303	−2	−1.4380	1	+0.3067	2.6	8.9540
−6	−2.4529	−1	−1.0614	1.5	1.0986	2.8	15.941
−5	−2.2408	−0.5	−0.8295	2	2.6515	3	35.356
−4	−2.0072	0	−0.5496	2.2	3.7994	3.2	133.90

scale[14–17] accounting as well for collisions as for space charge (Sec. IV). The characteristic length for this intermediate scale is given by $\ell_m=\lambda^{1/5}\lambda_D^{4/5}$. The asymptotic analysis results in a universal function $w(z/\ell_m)$ representing the potential profile on the intermediate scale. This function [defined by Eq. (32) and listed in Table II] allows to construct a uniformly valid asymptotic expression avoiding the singularity [Sec. V A, Eq. (37)]. Its validity is restricted, however to small values $\varepsilon\leqslant 2\cdot 10^{-4}$. A convenient practical procedure to construct approximations for larger values $\varepsilon<1$ can be obtained by using the intermediate scale to formulate consistent boundary conditions for a numerical integration of the sheath equation (Sec. V B).

In the asymptotic limit $\varepsilon=\lambda_D/\lambda\rightarrow 0$ the sheath edge (the onset of space charge) is uniquely related to the marginal validity $u_s=1$ of the Bohm criterion. This is in accord with an interpretation of the Bohm criterion in terms of a presheath sound barrier[2,3,19,20] and of a uni-directed signal flux through the collision-less sheath region in front of an absorbing wall.[2,3,23,24] For finite ε, the space charge is built up gradually, and there is no unique *a priori* definition of a sheath edge. A consistent generalization of the two scale concept for small but finite ε can be achieved by choosing the ''sheath edge'' x_s within the validity limits of the intermediate scale (see Sec. V B). The corresponding space charge $\rho_s\sim n_{\rm ch}\lambda_D^{4/5}/\lambda^{4/5}$ and electric field strength $E_s\sim kT_e/e\lambda_D^{2/5}\lambda^{3/5}$ at the sheath edge confirm the asymptotic $(\lambda_D/\lambda\rightarrow 0)$ limits of a vanishing space charge ρ_s, of a vanishing *sheath* field $eE_s\lambda_D/kT_e\rightarrow 0$, and of an infinite *presheath* field $eE_s\lambda/kT_e\rightarrow\infty$ at the sheath edge.

An *exact* definition of a critical velocity separating plasma and sheath ($u_s=1$, Bohm criterion), again, can be given only in the limiting case $\varepsilon=0$. According to the uncertainty of the exact sheath edge position x_s, the limiting ion velocity u_s is defined for small but finite ε only within certain limits, say, within the interval

$$1-0.5\varepsilon^{2/5}<u_s<1+0.3\varepsilon^{2/5}$$

(see Table I). Consequently, with growing ε, the ''sheath criterion'' becomes more and more fuzzy but shows no definite shift in u_s. (Naturally, strong collisional friction tends to reduce *all* ion velocities. This, however, must not be confused with a shift of a ''*critical*'' velocity indicating that space charge effects become dominant.) Correspondingly, there is no reason and no basis to formulate a new modified Bohm criterion accounting for collisions in the sheath. In particular, there is no scale separation for $\lambda_D\sim\lambda$ and, consequently, there is no critical ion velocity separating the scales. For collisional conditions with $\varepsilon>1$, finally, the ''continuum theory'' (see, e.g., Refs. 25 and 26) applies. In this continuum theory the Debey region (extending over many mean free paths) and the plasma region (governed by a third length scale $L\gg\lambda_D$) can be matched smoothly, and there is *no* sheath condition.

APPENDIX A: PHYSICAL INTERPRETATION OF THE INTERMEDIATE SCALE

To interpret the basic equation (32) of the intermediate scale, we return to the more familiar presheath variables and obtain

$$\varepsilon^2\frac{d^2\varphi}{dx^2}=\varphi^2+x. \tag{A1}$$

Obviously, the right hand side must represent the space charge $\varrho=n_+-n_-$. The first contribution can be understood from the *collision free* acceleration of the ions [see Eqs. (11) and (19)]:

$$\varrho_{\rm cf}=(1+2\varphi)^{-1/2}-e^{-\varphi}=\varphi^2+O(\varphi^3). \tag{A2}$$

This contribution is proportional to φ^2 and not to φ because the Bohm criterion is fulfilled *marginally*.[2]

The additional influence of collisions is caused by an enhancement of the ion density due to friction:

$$\left(\frac{dn_+}{dx}\right)_{\rm c}=\frac{d}{dx}\left(\frac{1}{u}\right)_{\rm c}=-\frac{1}{u^2}\left(\frac{du}{dx}\right)_{\rm c}=\frac{\nu(u)}{u^2}=1\quad(u=1). \tag{A3}$$

Integration yields the second (collisional) contribution $\varrho_{\rm c}=x$. Following the arguments backwards, an inspection of Eq. (A1) shows, that the intermediate scale is *necessarily* defined by the transformation (24), (27), and (28).

APPENDIX B: NUMERICAL INTEGRATION OF THE INTERMEDIATE SOLUTION

To obtain starting values w_1 and w_1' for a numerical integration of Eq. (32), we make use of the quasi-neutral approximation (33) and improve the result by one iteration step. This yields

$$w_1^2=\tfrac{1}{4}|\zeta_1|^{-3/2}-\zeta_1\quad\text{and}\quad w_1'=\tfrac{3}{8}|\zeta_1|^{-5/2}-12w_1. \tag{B1}$$

Choosing $\zeta_1\leqslant -10$, the integration may be performed without difficulties by a Runge–Kutta procedure. The position $\zeta_0=3.4117$ is most easily found using a variable step-width and comparing with Eq. (35). For comparison some results are listed in Table II.

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