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METHODOLOGICAL NOTES

On the Upper Bound in the Bohm Sheath Criterion

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Abstract—The question is discussed about the existence of an upper bound in the Bohm sheath criterion, according to which the Debye sheath at the interface between plasma and a negatively charged electrode is stable only if the ion flow velocity in plasma exceeds the ion sound velocity. It is stated that, with an exception of some artificial ionization models, the Bohm sheath criterion is satisfied as an equality at the lower bound and the ion flow velocity is equal to the speed of sound. In the one-dimensional theory, a supersonic flow appears in an unrealistic model of a localized ion source the size of which is less than the Debye length; however, supersonic flows seem to be possible in the two- and three-dimensional cases. In the available numerical codes used to simulate charged particle sources with a plasma emitter, the presence of the upper bound in the Bohm sheath criterion is not supposed; however, the correspondence with experimental data is usually achieved if the ion flow velocity in plasma is close to the ion sound velocity.

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1. INTRODUCTION

According to the well-known Bohm sheath criterion [1],

$$v_0 \geq c_s, \tag{1}$$

the ion flow velocity v_0 from plasma to a negatively biased absorbing collector must exceed the ion sound velocity $c_s = \sqrt{T_e/m_i}$.

But is there an upper bound?

In the recent work [2], called "New Formulation of the Bohm Sheath Criterion in Terms of Ion Sound Waves," its authors, A.E. Dubinov and L.A. Senilov, state that this bound exists and is defined by

$$c_s \leq v_0 \leq 1.58 c_s. \tag{2}$$

At the same time, there are many works in which the Bohm sheath criterion is stated in the form of the equality

$$v_0 = c_s. \tag{3}$$

As an example, we mention the surveys [3, 4] and references therein.

Let us try to elucidate which of three conditions (1)–(3) is valid.

Equality (3) appears in all known to us plasma models with a bulk ionization source the characteristic size of which L exceeds the Debye length λ_D [3, 5–10], $L \gg \lambda_D$. According to [11, 12], the generalized Bohm criterion [6], which replaces the classical criterion (1) if the cold ion approximation is put aside, is also satisfied marginally in the form of an equality. In the remarkable paper by J.E. Allen [12], where the physical meaning of condition (3) was discussed, it was shown that the generalized Bohm criterion in the form of an equality follows from the dispersion law for ion sound waves at $\omega = 0$. At a negative frequency $\omega = k(c_s - v_0)$ of a wave propagating backward against the ion flow, i.e., at $v_0 > c_s$, the perturbation is blown off forward by the flow. In this case, the current collector cannot have an effect on the processes in the ion source. If $v_0 < c_s$, then, as was shown in [13], a rarefaction wave forms [14], which propagates upstream the ion flow toward the source and interferes there with it. Thus, the fact that there is an upper bound on the ion flow velocity, as well as the interrelation between the Bohm criterion and ion sound dispersion, is certainly not new as was believed by the authors of [2].

Let us now turn to condition (2). Its derivation in [2] was unsupported by physical arguments. Dubinov and Senilov repeated the derivation of the Bohm criterion from hydrodynamic equations with cold collisionless ions. This derivation can be found in many textbooks and monographs (see, e.g., [8, 15–20]), but the authors of [2] analyzed the obtained solution not only in the region where the electric potential φ is less than the potential φ_p of the quasineutral part of plasma. Formally continuing the solution into the range $\varphi > \varphi_p$, they obtained the upper bound $v_0 < 1.58c_s$ on the ion flow velocity in the quasineutral region, because at $v_0 > 1.58c_s$ in the range $\varphi > \varphi_p$, "it is impossible to construct a continuous trajectory for a pseudoparticle starting from the hump top and going leftward" [2]. Although the authors of [2] note that the value of the upper bound depends on the plasma model, they do not discuss the physical meaning of this bound.

In a realistic formulation of the problem about a plasma emitter of positive ions [21], the ion source should be located deep in the plasma. If ions are produced with a low velocity, then an electric field accelerating ions to the velocity prescribed by the Bohm criterion in the form of equality (3) arises in the region of the source. Therefore, the range $\varphi > \varphi_p$ corresponds to the source region. This region is also called the ionization zone, or presheath. In the presheath, the continuity equation should be supplemented by a term responsible for ion generation. From heuristic considerations, it is usually assumed that a potential drop of $T_e/2e$ arises in the presheath, in which the ion flow velocity increases from zero to the speed of sound. A more rigorous theory [4] predicts a somewhat larger value of the potential drop if the ionization zone is much longer than the Debye length ($L \gg \lambda_{\mathrm{D}}$). According to this theory, the Bohm sheath criterion is satisfied in the form of equality (3).

The model (and not quite realistic) case of a source in the form of a delta function, when $L \ll \lambda_{\mathrm{D}}$, yields a solution that was studied by the authors of [2]. Earlier, this case was analyzed by R. Cohen and D. Ryutov [13], along with other, more realistic models with $L \gg \lambda_{\mathrm{D}}$.

2. LOCALIZED ION SOURCE

Cohen and Ryutov considered one-dimensional plasma placed between two absorbing plane walls (ion collectors). The localized ion source lied in the plane of symmetry of the problem $z = 0$, in the middle between the absorbing walls. It injected ions with a finite initial velocity v_{in} symmetrically to both sides. Due to the symmetry, the electric field $E = -\partial\varphi/\partial z$ in the plane $z = 0$ is zero, $E(0) = 0$. Moreover, it is sufficient to find the solution only in the half-space $z > 0$. As for the absorbing walls, it is supposed that they are under a large negative (relative to the plasma) potential and all electrons are reflected backward to the plasma in the unipolar sheath. It is also assumed that all ions pulled out by this potential from the plasma are absorbed by the collectors.

If the initial ion velocity v_{in} is equal to the speed of sound c_s, then the potential $\varphi_{\mathrm{in}} = \varphi(0)$ in the injection plane $z = 0$ is established at the level equal to the potential of the quasineutral plasma $\varphi_p = 0$; therefore, the ion velocity v_0 in the quasineutral region of the plasma (for which the Bohm criterion is just formulated) is also equal to c_s and the Bohm criterion is satisfied in the form of the equality $v_0 = c_s$. For $v_{\mathrm{in}} < c_s$, the potential φ_{in} at the injection point turns out to be larger than φ_p and there appears a Debye sheath near the source, where ions are accelerated to a supersonic velocity $v_0 > c_s$. In the limit $v_{\mathrm{in}} \to 0$, the ion velocity in the quasineutral part of plasma approaches the value $v_0 = 1.5852c_s$ which was interpreted in the work by Dubinov and Senilov [2] as the maximum velocity of the ion flow from the plasma emitter. However, we show that there exist solutions with a flow velocity $v_0 > 1.5852c_s$, although the boundary condition $E(0) = 0$ is not satisfied for them.

Let us consider a plasma layer with a given ion source

$$S_i(z, v_z) = S_0\delta(z)\left[\delta(v_z - v_{\mathrm{in}}) + \delta(v_z + v_{\mathrm{in}})\right] \tag{4}$$

between two parallel absorbing walls. We suppose that the function $S(z, v_z)$ is even in both arguments and sufficiently rapidly decreases beyond the region occupied by the source. In this case, the dependence on the coordinate z can be represented by the delta function, as is in formula (4). The initial velocity v_{in} is generally nonzero. As will be shown below, the potential $\varphi(z)$ has a maximum $\varphi_{\mathrm{in}} = \varphi(0)$ at the ion injection point; therefore, there are no ion stopping points, i.e., $m_i v_{\mathrm{in}}^2/2 + e\varphi_{\mathrm{in}} > e\varphi(z)$ for all $z > 0$. Therefore, the ion velocity

$$v_z = \sqrt{v_{\mathrm{in}}^2 + (2e/m_i)\left[\varphi_{\mathrm{in}} - \varphi(z)\right]} \tag{5}$$

does not vanish anywhere and the ion density

$$n_i = \frac{S_0}{v_z} \tag{6}$$

does not go into infinity. We assume that the electron density obeys the Boltzmann distribution

$$n_e = n_0 \exp(e\varphi/T_e). \tag{7}$$

Since the electric potential is defined to within a constant, this constant can be always chosen such that $n_i = n_e$ at $\varphi = 0$. Then,

$$n_0 = \frac{S_0}{\sqrt{v_{\mathrm{in}}^2 + 2e\varphi_{\mathrm{in}}/m_i}} \tag{8}$$

is the electron density in the quasineutral region the potential of which is set at zero, $\varphi_p = 0$.

We seek the solution to Poisson's equation

$$\frac{\partial^2\varphi}{\partial z^2} = 4\pi e(n_e - n_i), \tag{9}$$

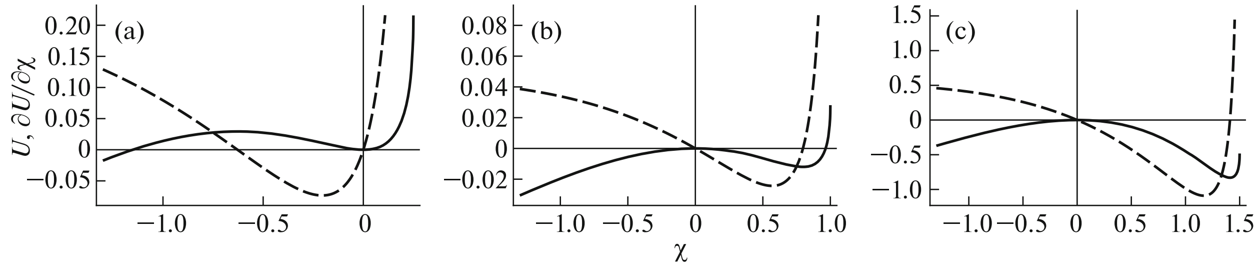

Fig. 1. Effective potential $U(\chi)$ (solid line) and its derivative $\partial U(\chi)/\partial\chi$ (dashed line) for u_0^2 = (a) 1/2, (b) 2, and (c) 3.

which describes the plasma ion emitter. It is implied that, beyond the plasma and near the collector, the potential varies according to the 3/2 power law [22, 23], as is in a diode with a space-charge-limited thermal emission. According to the terminology proposed in [24] (see also [4]), large negative values of the potential, $\varphi \lesssim -4T_e/e$, correspond to a unipolar sheath between which and the quasineutral plasma a Debye sheath is situated. Introducing the dimensionless variables

$$\chi = \frac{e\varphi}{T_e}, \quad \xi = \frac{z}{\lambda_D}, \quad u_0^2 = \frac{m_i v_{in}^2 + 2e\varphi_{in}}{T_e},$$

where $\lambda_D = \sqrt{T_e/4\pi e^2 n_0}$, we obtain from Poisson's equation (9) the equation

$$\frac{d^2\chi}{d\xi^2} = \exp(\chi) - \frac{u_0}{\sqrt{u_0^2 - 2\chi}} \equiv -\frac{\partial U}{\partial \chi} \qquad (10)$$

for the dimensionless potential χ.

Equation (10) admits a useful mechanical analogy with an imaginary pseudoparticle moving in the field of the effective potential

$$U(\chi) = 1 - \exp(\chi) - u_0\sqrt{u_0^2 - 2\chi} + u_0^2, \qquad (11)$$

where ξ has the meaning of time and χ is the coordinate of the pseudoparticle. The mechanical energy of such a pseudoparticle is an integral of motion,

$$W = \frac{1}{2}\left(\frac{d\chi}{d\xi}\right)^2 + U(\chi) = \text{const}. \qquad (12)$$

The effective potential $U(\chi)$ at $\chi = 0$ has either a minimum $U = 0$ if $u_0^2 < 1$ or a maximum $U = 0$ if $u_0^2 > 1$, as is shown in Fig. 1. Therefore, the point $\chi = 0$ is stationary for a pseudoparticle with the energy $W = 0$.

Let us consider the motion of a pseudoparticle with an energy $W \to 0+$ (i.e., $W \to 0$ at $W > 0$).

At $u_0^2 < 1$ (Fig. 1a), such a pseudoparticle is trapped in the vicinity of the minimum of the effective energy $\chi = 0$; therefore, there is no solution that would match the quasineutral region $\chi \approx 0$ to the unipolar sheath $\chi \lesssim -4$.

The situation is different if $u_0^2 > 1$, i.e., if Bohm criterion (1) is satisfied. In this case, the entire region $\chi < 0$ becomes available for a pseudoparticle with $W \to 0+$. It is easy to see that the "time" during which such a pseudoparticle reaches the absorbing wall (ion collector) somewhere in the region $\chi \to -\infty$ is very long because the pseudoparticle almost stops near the maximum of the effective potential (at $\chi = 0$) and stays there for a very long time before it begins to accelerate and continues to move toward the wall. In an actual system, a segment of the trajectory of such a pseudoparticle near $\chi = 0$ corresponds to a long quasineutral zone between the ion source and the Debye sheath. In this zone, the electric potential varies extremely slowly and the electric field is very close to zero.

In fact, the standard derivation of Bohm criterion (1) supposes that the pseudoparticle starts from a certain point $\chi \to 0-$ with a low velocity $d\chi/d\xi \to 0-$ (see, e.g., [4, 18, 20]). Below, we will continue this solution into the region $\chi > 0$ and see the formation of the second Debye sheath near the ion source, where the injected ions are preaccelerated.

It is significant that the range

$$1 \le u_0^2 \le 2.51286, \qquad (13)$$

which corresponds to criterion (2), contains the second zero of the function $U(\chi)$ at $\chi = \chi_{in} > 0$ (see Fig. 1b). The upper boundary of range (13) is determined from the equation $U(u_0^2/2) = 0$ (see below). The trajectory of a pseudoparticle that starts with a zero velocity $d\chi/d\xi = 0$ from a point close to $\chi = \chi_{in}$ satisfies the boundary condition $E(0) = 0$ in the ion source, which follows from the symmetry of the problem with respect to the plane $z = 0$. At the same time, for such a trajectory, we have $W \to 0$; therefore, the pseudoparticle "decelerates" near $\chi = 0$. This corresponds to the formation of a long quasineutral zone, which then goes over into the Debye sheath as is

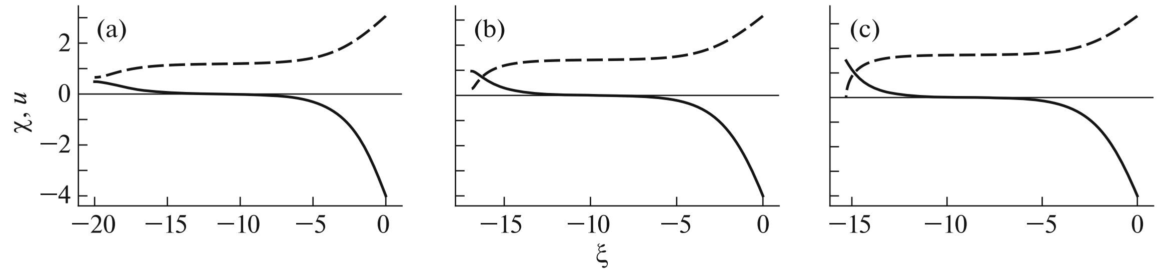

Fig. 2. Profile of the potential $\chi(\xi)$ (solid line) and local velocity u (dashed line) at $W = 0.5\times10^{-4}$ for u_0^2 = (a) 1.4, (b) 2.0, and (c) 3.0. The left peak in the profile of χ is the Debye preacceleration sheath formed near the localized ion source; the right well is the Debye sheath, which goes over into a unipolar sheath; and the plateau in the middle is the quasineutral zone, the length of which increases as $W \to 0$.

shown in Figs. 2a and 2b by the solid lines. The dimensionless local ion velocity

$$u = v_z/c_s = \sqrt{u_0^2 - 2\chi}$$

is shown in Fig. 2 by the dashed line. In Figs. 2a and 2b, its initial value $u_{\rm in} = \sqrt{u_0^2 - 2\chi_{\rm in}} = v_{\rm in}/c_s$ at the injection point is greater than zero. However, it tends to zero as $u_0^2 \to 2.51286$. In contrast, $\chi_{\rm in} \to 0$ and, consequently, $u_{\rm in} \to u_0 \to 1$ as $u_0^2 \to 1$. As is seen from Figs. 2a and 2b, injected ions are accelerated to a supersonic velocity over the time during which they move through the Debye sheath in the preliminary acceleration zone near the ion source, where $\chi > 0$. The acquired velocity remains nearly constant over the time during which ions move through the quasineutral plasma, where $\chi \approx 0$. Then, ions again begin to accelerate in the second Debye sheath, where $\chi < 0$.

For $u_0^2 > 2.51286$ (Fig. 1c), the second zero of the effective potential $\chi = \chi_{\rm in} > 0$ disappears, because the function $U(\chi)$ is complex at $\chi > u_0^2/2$. As a result, it becomes clear that the boundary condition $d\chi/d\xi = 0$, corresponding to the vanishing of the electric field $E \propto d\chi/d\xi$ in the plane of symmetry, cannot be satisfied.

If the condition $E(0) = 0$ is abandoned (by either abandoning the requirement of the problem symmetry or admitting that the electric field is discontinuous in the plane $z = 0$), then the point $U = 0$ is not distinguished in any way. Then, the source can be placed at any point in the domain of existence of the solution $\chi < u_0^2/2$, where $U \le 0$ (Figs. 2b, 2c). Here, the value of $\chi = u_0^2/2$ at the injection point corresponds to a zero initial ion velocity, $u_{\rm in} = 0$.

3. DISCUSSION

Although the Cohen–Ryutov model is quite artificial, it found practical application in the numerical codes used to calculate charged particle sources with a plasma emitter. For example, in the PBGUNS [25] and POISSON-2 [26, 27] codes, an imaginary source of ions is situated on the rear wall of the plasma chamber and ions are emitted from this wall with a prescribed velocity distribution. As far as we know, it is not supposed that the electric field vanishes on this wall; therefore, the corresponding distribution of the potential can be described by a segment of the solid line in Fig. 2. This segment starts from any point of the solid line to the left from the quasineutral zone if the initial ion velocity is less than the speed of sound. In the quasineutral zone, the flow is supersonic and the electric field is very close to zero. Therefore, the solution that starts from any point in the quasineutral zone describes a supersonic ion source in which the Debye preacceleration sheath does not appear. For such a source, one should set $\varphi_{\rm in} = 0$ in formula (8).

In actual ion sources with a plasma emitter (see, e.g., [28–33]), plasma is created in an arc or RF discharge excited in a magnetically insulated volume into which the gas is delivered. In the case of an RF discharge, models with an extended ionization zone, $L \gg \lambda_{\rm D}$, seem to be more adequate. All such models [3–12] predict that the Bohm sheath criterion is satisfied in the form of equality (2). In arc sources, the discharge is initiated by a strong electric field arising at the end of a point electrode. In such sources, the ion flow is significantly non-one-dimensional and the ion flow velocity probably can exceed the speed of sound [34, 35]. Simulations by the PBGUNS and POISSON-2 codes demonstrate good qualitative and quantitative agreement with results of experiments if the initial velocity of injected ions is set at a level close to the speed of sound [36].

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