Plasma sheath in a tilted magnetic field: Closing of the diamagnetic currents; effect on plasma convection

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A plasma on open field lines that intersect endplates at a shallow angle, as in a tokamak with a toroidal limiter or poloidal divertor, is considered. A microscopic picture is developed, which shows the closure of the electron diamagnetic and electric drift currents approaching the wall by parallel (to **B**) electron current. A proper account of the radial variation of the ion "skin" current in the ion gyrosheath is made. The resulting sheath current-voltage characteristics are derived. Two applications are considered: the toroidal potential variations induced by a wavy limiter or divertor surface, which can lead to significant plasma convection, and the determination of the boundary conditions appropriate for macroscopic instabilities in such a configuration. © *1995 American Institute of Physics.*

I. INTRODUCTION

The problem of the plasma sheath in a magnetic field that intersects a conducting wall at a shallow angle has recently attracted considerable attention (see, for example, Refs. 1–5 and references therein). The reason for this is twofold: first, this problem is of importance for the physics of a tokamak scrape-off layer (SOL),[5–7] and, second, the problem contains a number of subtleties that lead to paradoxes and, sometimes, direct errors. In addition, there exist also other systems whose performance is strongly affected by the properties of the Debye sheath in a tilted magnetic field: magnetic mirrors; magnetized gas-discharge plasmas (in particular, the ones used for plasma processing applications); and space stations in positions where their surface is almost tangential to the Earth's magnetic field.

The geometry of the problem is illustrated by Fig. 1. The wall coincides with the $y=0$ plane, and the magnetic field has y and z components. In the geometry of a tokamak divertor plate, x would correspond to the radial direction, y to the poloidal direction, and z to the toroidal direction. We allow the plasma parameters in the bulk of the plasma to vary in the x (i.e., radial) direction. The natural radial variation of the electron temperature T_e and the close coupling of the plasma potential to T_e (through the requirement of near-ambipolarity of losses across the Debye sheath) generally leads to the existence of a radial electric field; accordingly, there exists an electric drift in the bulk of the (SOL) plasma directed (for small α's) almost perpendicularly to the wall. At $\alpha \ll 1$ the y component of this velocity is equal to

$$v_y = -\frac{cE_x}{B}\cos\alpha \approx -\frac{cE_x}{B} \equiv v_E. \qquad (1)$$

In the sheath itself, there exists, of course, a large y component of the electric field, which serves to repel the majority of electrons. The scale length Δ of the variation of plasma parameters in the x direction (the SOL thickness in a tokamak geometry) is assumed to be much larger than the ion gyroradius:

$$\Delta \gg \rho_i. \qquad (2)$$

In agreement with the real situation in a tokamak SOL, we consider the case when the intersection angle α, though much less than unity, is not excessively small, so that the uninhibited electron current normal to the wall is determined by the thermal streaming along the field lines and greatly exceeds the contributions from the ion thermal current and the electric drift current. In other words, we assume that inequalities

$$v_E/v_{te}, \quad (m_e/m_i)^{1/2} < \alpha \ll 1 \qquad (3)$$

hold, where $v_{te}=(2T_e/m_i)^{1/2}$. For a discussion of the opposite situation see Refs. 1 and 3.

In the case under consideration, in order to considerably reduce the electron current, a sheath potential should be positive and considerably exceed T_e/e (we define the sheath potential as the plasma potential a few ion gyroradii away from the wall; the wall potential is assumed to be zero). This means that almost all electrons get reflected from the sheath, and their distribution function is close to Maxwell–Boltzmann everywhere except the immediate vicinity of the wall.

In the present paper, we consider in some detail the problem of the closing of the plasma current near the wall. After a brief discussion of the sheath structure based on Refs. 1, 3, and 4 (Sec. II), we formulate the boundary conditions that relate the electron and ion currents to the wall to the sheath potential (Sec. III). Then, we apply these boundary conditions to the assessment of two particular problems: (1) possible effects caused by the presence of ripples on the surface of the limiter or the divertor plate (Sec. IV), and (2) formulation of the boundary conditions for the macroscopic,[8] electrostatic, or low-β instabilities (Sec. V).

In the first of the problems we find that even a ripple with a moderate amplitude can cause very vigorous plasma convection and strongly affect the SOL thickness. Therefore,

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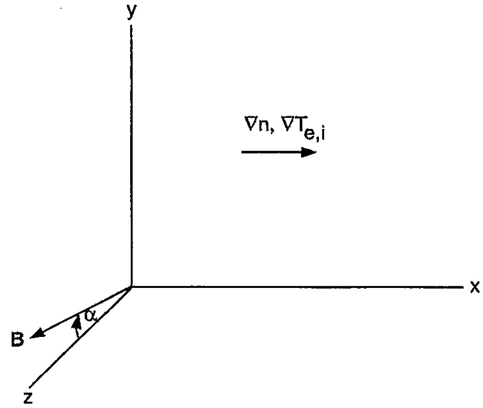

FIG. 1. Coordinate frame used in the analysis of the sheath structure. The wall coincides with the $y=0$ plane. Thick arrow shows the direction of the gradients of plasma parameters in the bulk of the plasma.

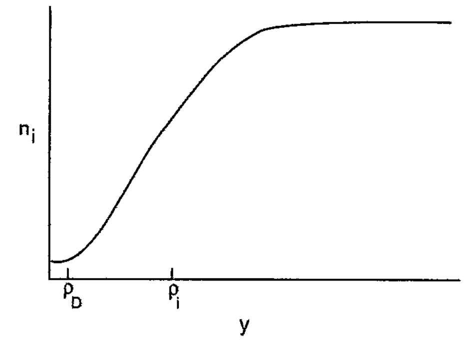

FIG. 2. Ion density profile for $\alpha \ll 1$, with y being a coordinate normal to the wall. At distances $y > \rho_D$ the electron density is equal to the ion density.

making the divertor plates or the limiter surface toroidally "wavy" may prove to be yet another tool for controlling the SOL structure.

In the second problem, that of the boundary conditions, we rederive the results that have been previously formulated on the basis of the phenomenological arguments in Ref. 7; we show how these boundary conditions should be modified to include the effects of ion diamagnetic drifts (Ref. 7 was addressing the situation when ion pressure was constant, and the ion diamagnetic drifts were absent).

In the Appendix, we formulate an equilibrium condition for the plasma confined between two perfectly reflected non-planar surfaces—a model problem that sheds some light on the physical nature of the effects considered in Sec. IV.

II. OVERALL PICTURE OF THE SHEATH STRUCTURE IN A TILTED MAGNETIC FIELD

In this section, we summarize those results from previous work[1,3,4] that are required for the purposes of our analysis. Here we discuss only the case when the angle α at which the field lines intersect the wall surface, is small. Also, we imply that the plasma flow to the wall is limited just by the free expansion velocity along the field lines, and not by the diffusion across the field lines (that might be the case in a plasma with strong ion–neutral collisions). As was shown in the aforementioned papers, the sheath then consists of two "subsheaths"—the ion subsheath with a thickness of the ion gyroradius ρ_i, and a much thinner electron subsheath (adjacent to the wall), whose thickness is of the order of Debye radius. The electron subsheath, in turn, may have a rather complex structure (which will be discussed in our forthcoming paper[9]), but this does not affect the content of our present paper.

The plasma density in the ion subsheath experiences a considerable drop from its value in the adjacent plasma. The reason for this is as follows: a "typical" ion with a pitch angle ~ 1, approaching the wall from the side of the plasma, gets closer to the wall during every gyroperiod only by a small amount $\sim \alpha\rho_i$. Therefore, near the wall, only ions with velocity vectors almost parallel to the walls are present. In other words, only a small fraction of the phase space is occupied by the ions near the wall, implying a small ion density. As is clear from the arguments just presented, this effect exists, even if the electron temperature is small so that there is no additional ion acceleration caused by the electric field.

The presence of the electric field (that repels electrons and is proportional to T_e) leads to the ion acceleration toward the walls [see Eq. (16) in Ref. 3]. Then, by virtue of the continuity equation, the ion density should drop even farther. For an electron temperature comparable with the ion temperature, the potential drop in the ion subsheath is of the order of plasma temperature divided by the unit charge; therefore, in this case (which is typical for the tokamak SOL), the normal component of the ion velocity at the wall side of the ion subsheath should become of the order of v_{ti}; accordingly, the ion density here becomes of the order of αn. The overall ion density distribution at distances from the wall greatly exceeding the Debye radius should qualitatively look, as shown in Fig. 2.

Since, for the reasons mentioned in the Introduction, the number of electrons penetrating through the potential barrier is very small, the electron distribution function inside the ion subsheath is close to Maxwellian. Their equilibrium along the magnetic field line is governed by the Boltzmann relationship:

$$\phi = \frac{T_e}{e} \ln n + C. \tag{4}$$

The plasma density in the ion subsheath drops by a large factor. Accordingly, the majority of electrons ($\sim 1-\alpha$, $\alpha \ll 1$) are reflected back into the plasma in the ion subsheath.

III. CLOSING OF THE PLASMA CURRENTS NEAR THE WALL

A. Closing of the electron current

The thickness of the ion subsheath is large compared to the electron gyroradius. Therefore, electron reflection can be described within the framework of the drift approximation. The equations of motion of the guiding center of a certain electron at $\alpha \ll 1$ read as

$$\dot{x}=-\frac{c}{B}\frac{\partial\phi}{\partial y}, \tag{5}$$

$$\dot{y}=\alpha v_\parallel+\frac{c}{B}\frac{\partial\phi}{\partial y}. \tag{6}$$

The coordinate system is that shown in Fig. 1. A convention regarding the sign of the parallel velocity is that it is considered positive if it has a positive projection on the y axis. The transverse energy of the electron is conserved because of the conservation of the magnetic moment; therefore, the energy conservation law acquires the form

$$v_\parallel^2-\frac{2e\phi}{m}=\text{const.} \tag{7}$$

In the course of reflection, an electron experiences a displacement in the x direction,

$$\delta x=-\frac{c}{B}\int_{-\infty}^{\infty}\frac{\partial\phi}{\partial y}dt=-\frac{c}{B}\int\frac{\partial\phi}{\partial y}\frac{dy}{v_y}, \tag{8}$$

where $\partial\phi/\partial y$ and v_y should be taken at the location of the electron guiding center. In the second integral, integration should be taken along the two branches of the electron trajectory (descending and ascending).

Because of the inequality (3), the last term in the equation (6) can be neglected. Then, by using the energy conservation law (7), one can easily obtain from (8) that

$$\delta x=-\frac{c}{\alpha B}\int\frac{\partial\phi}{\partial y}\frac{dy}{v_\parallel}=\frac{2mc}{\alpha eB}v_\parallel^{(\text{in})}, \tag{9}$$

where $v_\parallel^{(\text{in})}$ is the velocity of the incoming electron (above the sheath). For a thermal electron, the displacement δx is large compared to the electron gyroradius, but small compared to the SOL thickness Δ. The latter conclusion is based on the inequalities (2) and (3).

As after reflection an electron gets displaced in the x direction, and there exists an x component of the electric field in the bulk of the plasma, the parallel electron velocity changes, according to the energy integral (7),

$$v_\parallel^{(\text{out})}=-v_\parallel^{(\text{in})}+\delta v_\parallel,\quad \delta v_\parallel=eE_x\,\delta x/mv_\parallel^{(\text{in})}. \tag{10}$$

According to Eqs. (1) and (9),

$$\delta v_\parallel=2v_E/\alpha. \tag{11}$$

Relationships (9) and (11) allow us to find the conditions for the closing of the electron current near the wall. This can be done in the following way: Let $f(v_\parallel^{(\text{in})},v_\perp,x)$ be the guiding center distribution function of the incoming electrons on a field line that intersects the wall at some location x. The number of incoming electrons in some volume of the velocity space is

$$2\pi f(v_\parallel^{(\text{in})},v_\perp,x)v_\perp\,dv_\perp\,dv_\parallel. \tag{12}$$

The number of particles moving away from the wall *along the same field line* and having velocities in the volume element $2\pi v_\perp\,dv_\perp\,dv_\parallel^{(\text{out})}$ is, obviously,

$$2\pi f(-v_\parallel^{(\text{out})}+\delta v_\parallel,v_\perp,x-\delta x)v_\perp\,dv_\perp\,dv_\parallel^{(\text{out})}, \tag{13}$$

where f is the same function of its arguments, as in Eq. (12). Accordingly, the parallel current is

$$\begin{aligned}j_{\parallel e}&=-2\pi e\int_0^\infty v_\perp\,dv_\perp\Bigg(\int_0^\infty v_\parallel^{(\text{out})}\,dv_\parallel^{(\text{out})}f(-v_\parallel^{(\text{out})}\\&\quad+\delta v_\parallel,v_\perp,x-\delta x)+\int_{-\infty}^0 dv_\parallel^{(\text{in})}f(v_\parallel^{(\text{in})},v_\perp,x)\Bigg)\\&\cong-2\pi e\int_0^\infty v_\perp\,dv_\perp\int_{-\infty}^0 v_\parallel^{(\text{in})}\,dv_\parallel^{(\text{in})}\Bigg(\frac{\partial f}{\partial v_\parallel^{(\text{in})}}\,\delta v_\parallel\\&\quad+\frac{\partial f}{\partial x}\,\delta x\Bigg)\equiv\frac{env_E}{\alpha}-\frac{c}{\alpha B}\frac{\partial p_e}{\partial x}.\end{aligned} \tag{14}$$

This relationship shows on the microscopic level what happens with the electric drift and the diamagnetic current approaching the wall: they are closed by parallel current flowing from the wall.

It is instructive to consider in some more detail the current pattern in the simplest case when there is no electric field in the bulk of the plasma, so that only diamagnetic current flows across the field lines on the plasma side of the sheath. The normal component of this current is just $j_{ey}^{(D)}=-(c/B)dp_e/dx$. In the sheath itself, there appears a large x component of the diamagnetic current caused by the y component of the electron pressure gradient. When integrated over the sheath thickness, it produces what might be called a "surface diamagnetic current" $I_{ex}^{(D)}=cp_e/B$. Obviously, $j_{ey}^{(D)}+dI_{ex}^{(D)}/dx=0$, so that the normal component of the electron diamagnetic current approaching the wall gets automatically closed by the surface diamagnetic current. This is just a reflection of the fact that the divergence of the electron diamagnetic current, $\mathbf{j}_e^{(D)}=-c\mathbf{B}\times\nabla p_e/B^2$ is identically zero (as was correctly mentioned in this context in Ref. 4). However, as one can easily see, the x component of the electric drift current [which is inevitably present in the sheath because of the relationship (4)] exactly compensates the diamagnetic surface current, $I_{ex}^{(E)}=-I_{ex}^{(D)}$, so that the net electron surface current is just zero. The electric drift component of the surface current gets, in turn, closed through the current along the field lines. Therefore, for an observer interested in processes occurring in the bulk of the plasma, the net effect of the sheath processes can be described in simple words: "electron diamagnetic current approaching the wall gets closed by the parallel current flowing from the wall."

In the above, we were not considering the small group of electrons that overcome the potential barrier and reach the wall. Their presence, obviously, also gives a contribution to the parallel current. Assuming that the impinging electrons have a Maxwellian distribution, one can write the following final expression for the parallel electron current approaching the wall from the plasma:

$$j_{\parallel e}=en\left[\left(\frac{v_{te}}{2\pi^{1/2}}\right)\exp\left(-\frac{e\phi}{T_e}\right)+\frac{v_E}{\alpha}-\frac{c}{\alpha eBn}\frac{\partial p_e}{\partial x}\right]. \tag{15}$$

The normal component of the plasma current at the plasma side of the sheath is

$$j_{ne}=\alpha j_{\parallel e}-env_E+\frac{c}{B}\frac{\partial p_e}{\partial x}=\frac{\alpha env_{te}}{2\pi^{1/2}}\exp\left(-\frac{e\phi}{T_e}\right). \quad (16)$$

We see that electric and diamagnetic drifts do not contribute to the normal component of the electron current to the wall.

The simple form for the total normal current (16) may be understood as a consequence of the fact, shown above, that the x components of the electron diamagnetic and electric drift currents in the sheath sum to zero. We note that this situation is very different from that for the ions.

B. Plasma current on the plasma side of the sheath

The parallel component of the ion current at the plasma side of the sheath, for the perfectly absorbing wall, is

$$j_{\parallel i}=enu, \quad (17)$$

where u is the average parallel velocity of the ions [negative for the ions approaching the wall; see the comment after Eq. (6)]. Note that, in contrast to the electron parallel current, there are no contributions from electric drifts and diamagnetic currents. This is due to the fact that ions are totally absorbed by the wall. The normal component of the ion current (at $\alpha \ll 1$) is

$$j_{ni}=\alpha enu+env_E+\left(\frac{c}{B}\right)\frac{\partial p_i}{\partial x}. \quad (18)$$

When solving the problems of a plasma equilibrium and gross instabilities, one often needs the boundary conditions for the normal component of the current leaving the plasma. According to (16), the normal component of the current at the plasma side of the sheath is

$$j_n=en\left[\left(\frac{\alpha v_{te}}{2\pi^{1/2}}\right)\exp\left(-\frac{e\phi}{T_e}\right)+\alpha u+v_E\right]+\frac{c}{B}\frac{\partial p_i}{\partial x}. \quad (19)$$

Note the asymmetry in the contributions from the ion and electron diamagnetic current: the first *does* enter relationship (19) while the second *does not*. This asymmetry is caused by the different reflecting properties of the sheath with respect to the ions and the electrons: the former do not come back from the sheath, while the latter are almost totally reflected. For the plasma with approximately equal electron and ion temperatures of electrons and ions, the diamagnetic term is two to three times less than that produced by the electric drift.

Note also that the current density on the plasma side of the sheath differs from the expressions used in Refs. 4 and 5. The difference is caused by the proper account for the diamagnetic drifts carried out in the present paper. Note that this correction does not appreciably change the predictions of Ref. 5 regarding the current flowing in the SOL as, in this context, the correction plays a relatively insignificant role (it is, typically, two to three times less than the contribution of the electric drift).

C. Ion surface current and plasma current on the wall

Let us now consider the ion current in the sheath and at the wall. In what follows, relations (20)–(21) basically coincide with the ones published in a recent paper by Chankin and Stangeby.[4] We reproduce these relations here in order to make the further conclusions more clear. The y component of the exact momentum balance equation (valid for the arbitrary ion distribution function, even strongly deviating from the Maxwellian) reads as

$$0=-\frac{\partial \pi_{yy}}{\partial y}-\frac{\partial \pi_{yx}}{\partial x}+enE_y-\frac{j_{xi}B}{c}, \quad (20)$$

where π_{yy} and π_{yx} are the corresponding components of the full ion momentum flux tensor. Above the ion subsheath, π_{yy} is close to the transverse ion pressure (for small α). At the wall side of the ion subsheath, the momentum flux is much smaller, because of a lower ion density (see Sec. II of the present paper). The second term on the right-hand side of Eq. (20) is much smaller than the first one, as the scale length in the x direction (SOL thickness Δ) is much larger than the thickness of the ion subsheath ($\sim\rho_i$). With these observations made, and with the use of the relationship (4), one can integrate Eq. (20) across the sheath to obtain

$$I_{xi}=-\frac{c}{B}(p_{\perp i}+p_e), \quad (21)$$

where I_{xi} is the ion current flowing along the wall surface within the sheath in the x direction and $p_{\perp i}$ and p_e are taken just outside the sheath.

The part of the current caused by the electric field [the third term on the right-hand side of Eq. (20)] flows by virtue of a real displacement of the ions in the x direction (unlike the ion diamagnetic current). The magnitude of this displacement during an ion transit through the sheath is

$$\sim\frac{\rho_i}{\alpha u}\frac{cE_y}{B}\sim\frac{\rho_i}{\alpha}. \quad (22)$$

When α becomes too small, this displacement becomes formally larger than the scale length Δ of the variation of plasma parameters in the x direction. Therefore, the validity of the results derived in this section is subject to a constraint,

$$\alpha>\rho_i/\Delta. \quad (23)$$

One can note, parenthetically, that a constraint of the same type appears in the model problem of a perfectly reflecting wall (see the Appendix).

Since, according to Eq. (21), I_{ix} is changing along the surface, the normal component of the ion current density at the wall is different from that on the plasma side of the sheath:

$$j_n^{\text{wall}}=j_n+\frac{\partial I_{xi}}{\partial x}=en\left[\left(\frac{\alpha v_{te}}{2\pi^{1/2}}\right)\exp\left(-\frac{e\phi}{T_e}\right)+\alpha u+v_E\right]-\frac{c}{B}\frac{\partial p_e}{\partial x}. \quad (24)$$

Somewhat unexpectedly, the last term looks exactly like the electron diamagnetic current (of course, this is only a

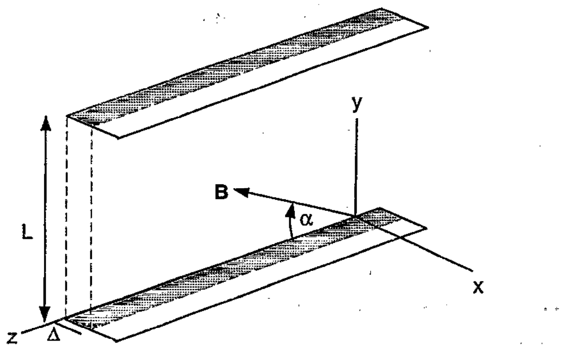

FIG. 3. "Rectified" geometry of the toroidal limiter (poloidal divertor). Shaded is the "wetted" part of the limiter surface.

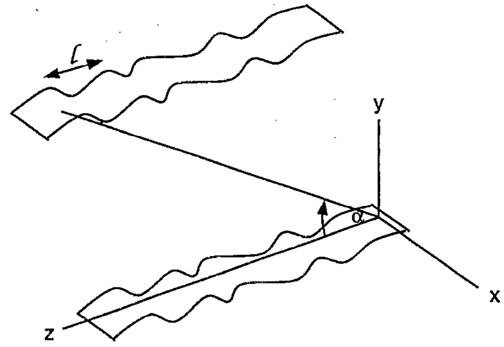

FIG. 4. "Wavy" limiter (divertor plate) in a rectified geometry.

formal likeness—all of the current I_{xi} is carried by the ions). Expression (24) is of particular interest in the case when the wall is made of a dielectric (or, as sometimes done in real experiments, is finely segmented, with no electrical contact between the segments). Then, the current to every particular piece of the wall must be zero, i.e., $j_n^{\text{wall}} = 0$.

IV. PLASMA CONVECTION CAUSED BY RIPPLES ON THE SURFACE OF THE TARGET PLATES

The results of the previous section allow us to predict an interesting phenomenon that should occur on the open field lines of a tokamak with a toroidal limiter or a poloidal divertor. We will consider this phenomenon in a so-called "rectified geometry"[10] (also see Refs. 5 and 7), in which the SOL gets unfolded in the poloidal direction, giving rise to a geometry shown in Fig. 3. As in the earlier sections, x corresponds to a radial coordinate, y to a poloidal coordinate, and z to a toroidal coordinate. The upper and lower plates in Fig. 3 correspond to two sides of a toroidal limiter or to two plates of a poloidal divertor. The magnetic field is considered as uniform. The plane $x=0$ corresponds to the last closed magnetic surface in a real geometry. We consider only the processes occurring on the open field lines, i.e., at $x>0$. For simplicity, we assume that the plates are normal to the poloidal component of the magnetic field.

Plasma diffusion from the tokamak interior and possible ionization of the neutral gas act as plasma sources, while the losses are caused by the plasma flow to the limiter (divertor plates). The problem of this stationary state was considered in some detail in Ref. 5. We assume that the sources possess toroidal symmetry (i.e., are independent of z). In this case a simple stationary state can be established, in which the plasma parameters are functions of radius (x) and the poloidal coordinate (y) only. Under quite realistic assumptions, the streamlines of the plasma flow (with possible electric drift included) lie within the surfaces $x=$const and the electrostatic potential does not depend on z.

As we show in this section, the situation changes dramatically when the limiters (or divertor plates) become wavy in the toroidal direction (as shown in Fig. 4), with the variation of the angle formed by the plates, with the $y=$const planes being of the order of or somewhat exceeding the angle α. We assume that the toroidal scale length of this variation l is small as compared to the poloidal circumference L of the device, so that the height of the nonuniformities is small, as compared with the distance L. In this latter case, the presence of nonuniformities does not considerably affect the longitudinal confinement time (which, for the perfectly absorbing plates is of the order of the ion transit time $L/\alpha v_{ti}$), and one might expect that the plasma state experiences only insignificant variations with respect to the case of the parallel plates. In reality, it turns out that the plasma flow in the SOL limited with "wavy" plates may change very considerably, with intense convection across the magnetic surfaces appearing.

To qualitatively show how this occurs, let us first assume that the plasma state remains the same as for parallel plates and see whether this state is compatible with the boundary conditions derived in the previous section. So we consider the plasma state in which the plasma density and electrostatic potential are independent of z. For simplicity, we assume that the electron temperature does not vary along the field lines, i.e., $T_e = T_e(x)$.

Because of the assumption of a Boltzmann distribution of electrons along the field lines, it follows that

$$\phi = \frac{T_e(x)}{e} \ln \frac{n(x,y)}{N(x)} + \psi(x), \tag{25}$$

where, by $N(x)$ and $\psi(x)$, we denote the plasma density and potential in the midplane (i.e., at $y=L/2$). According to our assumption they do not depend on z.

The transverse electron momentum balance equation reads as

$$0 = -\nabla_\perp p_e + en\, \nabla_\perp \phi + (1/c)\mathbf{j}_{\perp e} \times \mathbf{B}, \tag{26}$$

where $\mathbf{j}_{\perp e}$ is transverse electron current density. Solving this equation for $\mathbf{j}_{\perp e}$, we find

$$\mathbf{j}_{\perp e} = \frac{c}{B^2}(-\nabla_\perp p_e + en\, \nabla_\perp \phi) \times \mathbf{B}. \tag{27}$$

With this expression, it is easy to show that, for the aforementioned equilibrium (in which plasma parameters depend only on x and y),

$$\nabla \cdot \mathbf{j}_{\perp e}=\frac{cB_z}{B^2}\frac{\partial}{\partial y}\left\{n\left[\frac{\partial T_e}{\partial x}\left(\ln\frac{n}{N}-1\right)+e\frac{\partial\psi}{\partial x}\right]\right\}$$
$$=-\frac{cB_z}{B^2}\frac{\partial}{\partial y}\left(ne\frac{\partial\phi}{\partial x}-\frac{\partial p_e}{\partial x}\right)$$
$$=-e\frac{B_z}{B}\frac{\partial}{\partial y}[n(v_E+v_d)], \tag{28}$$

where

$$v_d \equiv -\frac{c}{eBn}\frac{\partial p_e}{\partial x} \tag{29}$$

is a quantity with the dimension of a velocity that is related in the obvious way to the electron diamagnetic current; v_E was introduced in relationship (1). Let us then use the electron continuity equation, which, for the stationary state, reads as

$$\frac{\partial j_{\parallel e}}{\partial s}+\nabla\cdot\mathbf{j}_{\perp e}=-eq_e, \tag{30}$$

where q_e is a volume source term (ionization and diffusion across the field lines). In what follows, to be more specific, we assume that the source terms for electrons and ions are locally equal:

$$q_e=q_i=q. \tag{31}$$

Integrating Eq. (30) along the field line, from just above the lower sheath to just below the upper sheath, taking into account Eq. (28), we obtain

$$j_{\parallel e2}-j_{\parallel e1}=-\int_{s_1}^{s_2}(eq+\nabla\cdot\mathbf{j}_{\perp e})ds=-eQ+(e/\alpha)$$
$$\times[n_2(v_{E2}+v_{d2})-n_1(v_{E1}+v_{d1})], \tag{32}$$

where we used equality (31) and the relationship $\alpha^{-1}\partial/\partial s=\partial/\partial y$. We have also made the approximation that, for small α, $B_z\approx B$. The subscripts "1" and "2" denote quantities at the already mentioned points on the field line (just above the lower sheath and just below the upper sheath, respectively); the quantity Q on the right-hand side of Eq. (32) is, obviously, equal to the number of ions produced in a flux tube of a unit cross section. Accordingly, this term can be estimated as

$$Q\equiv\int q\, ds\sim nu. \tag{33}$$

Using the expression (15) for the parallel component of the electron current at the sheath boundary, one finds that

$$j_{\parallel e1}=en_1\left[\frac{v_{te}}{2\pi^{1/2}}\exp\left(-\frac{e\phi_1}{T_e}\right)+\frac{v_{E1}}{\alpha_1}+\frac{v_{d1}}{\alpha_1}\right],$$
$$j_{\parallel e2}=-en_2\left[\frac{v_{te}}{2\pi^{1/2}}\exp\left(-\frac{e\phi_2}{T_e}\right)-\frac{v_{E2}}{\alpha_2}-\frac{v_{d2}}{\alpha_2}\right]. \tag{34}$$

Here we have taken into account the possible difference in the tilts of the upper and lower plates. In Eq. (34), we use the same convention regarding the sign of $j_\parallel$ as for $v_\parallel$ [see the comment following Eq. (6)]: $j_\parallel$ is positive if its projection on the y axis is positive. Substituting these expressions into Eq. (32), one obtains

$$\frac{v_{te}}{2\pi^{1/2}}\left[n_1\exp\left(-\frac{e\phi_1}{T_e}\right)+n_2\exp\left(-\frac{e\phi_2}{T_e}\right)\right]+n_1(v_{E1}+v_{d1})$$
$$\times\left(\frac{1}{\alpha_1}-\frac{1}{\alpha}\right)-n_2(v_{E2}+v_{d2})\left(\frac{1}{\alpha_2}-\frac{1}{\alpha}\right)=Q. \tag{35}$$

The values α_1 and α_2 should be taken on the same field line. The densities at the ends of the flux tube are related via the Boltzmann law:

$$n_2\exp\left(-\frac{e\phi_2}{T_e}\right)=n_1\exp\left(-\frac{e\phi_1}{T_e}\right). \tag{36}$$

Accordingly, we find the following expression for ϕ_1:

$$\phi_1=-\frac{T_e}{e}\ln\left\{\frac{\pi^{1/2}}{n_1 v_{te}}\left[Q-n_1(v_{E1}+v_{d1})\left(\frac{1}{\alpha_1}-\frac{1}{\alpha}\right)\right.\right.$$
$$\left.\left.+n_2(v_{E2}+v_{d2})\left(\frac{1}{\alpha_2}-\frac{1}{\alpha}\right)\right]\right\}. \tag{37}$$

As was noted in Ref. 5, under realistic conditions, the potential variation along the flux tube is relatively small as compared to the sheath potential. Therefore, one can consider ϕ_1 as the potential of the flux tube outside the sheath region. The source term Q, according to our assumption, does not depend on z. For $\alpha_1=\alpha_2=\alpha$, all other terms on the right-hand side of Eq. (37) depend only on x, and so does the flux-tube potential. However, if the angles α_1 and/or α_2 are varying in the toroidal direction, so does the flux-tube potential. This, in turn, means that plasma acquires a drift velocity in the x direction:

$$v_x\approx\frac{c}{B}\frac{\partial\phi}{\partial y}\approx\frac{c}{\alpha B}\frac{\partial\phi}{\partial z}. \tag{38}$$

Here we have used the fact that, for constant ϕ along the field line, $\partial/\partial y=\alpha^{-1}\partial/\partial z$. Velocities (38) are directed inward or outward, depending on the sign of the derivative.

Let us consider the case when electric and diamagnetic drifts give only a small contribution to the normal component of the plasma flow velocity on the walls. For comparable electron and ion temperatures this condition coincides with Eq. (23) and can be rewritten as

$$\alpha u \gg v_d. \tag{39}$$

Under this condition, the plasma should escape from both ends of the flux tube almost symmetrically, and we have $n_1\approx n_2$, $v_{E1}\approx v_{E2}$, etc.; in addition, the last three terms under the logarithm in Eq. (37) are small compared to the first one, and we obtain the following simple expression for the z-dependent part of the flux-tube potential:

$$\Delta\phi(x,z)\approx\frac{T_e}{e}\frac{n(v_E+v_d)}{Q}\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_1}\right), \tag{40}$$

where we have dropped the subscripts "1" and "2" on the plasma density and the drift velocity. Accordingly, taking account of Eqs. (33), (38), and (40), we obtain the following expression for the characteristic convective velocity:

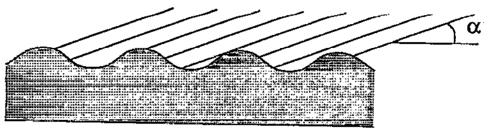

FIG. 5. Formation of the "shadows" on the limiter surface.

$$v_x \approx \frac{(v_E+v_d)cT_e}{\alpha ueB}\frac{\partial}{\partial z}\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_1}\right). \tag{41}$$

For illustrative purposes, we estimate the convective displacement for the case where the radial electric field is particularly small, so that the drift time of an ion through a ripple width is long compared to the ion parallel transit time; in this case, we have

$$\delta x_{\mathrm{conv}} \sim \frac{L}{\alpha u} v_x \sim \frac{Lv_d}{\alpha u}\frac{cT_e}{\alpha eB}\frac{\partial}{\partial z}\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_1}\right)$$
$$\sim \Delta\frac{2\pi L}{\alpha l}\left(\frac{v_d}{\alpha u}\right)^2, \tag{42}$$

where l is a characteristic wavelength of the ripples on the limiter surface. Clearly, for small enough l, this estimate exceeds Δ. However, as soon as δx_{conv} exceeds Δ, our derivations become invalid; we can only predict a considerable broadening of the SOL compared to the case with flat limiters. A self-consistent theory of the convective broadening is yet to be developed.

One can expect an especially strong convection in the case when the height of the ripples becomes such that "shadows" are formed on the limiter surface (Fig. 5). For a sinusoidally shaped limiter surface,

$$h = h_0 \sin(2\pi z/l), \tag{43}$$

this occurs when the amplitude h_0 of the ripples satisfies the condition

$$h_0 > \alpha l/2\pi. \tag{44}$$

Taking $l=60$ cm and $\alpha=6°$, we find that quite small ripples with $h_0>1$ cm already produce shadows.

Similarly to the situation described in the Appendix, the radial motion has a tendency to establish a plasma equilibrium in which the plasma parameters would become functions of the field line length between the two plates. In other words, for the z-dependent ripples, these motions tend to make equilibrium quantities functions of $z-y/\alpha$ only. This, of course, is incompatible with the distribution of the sources; hence, radial convection must appear.

The velocity described by Eq. (41) is an $\mathbf{E}\times\mathbf{B}$ velocity over the bulk of the SOL plasma, and, in the approximation of no variation of the potential along field lines, implies motion on constant-potential surfaces. These surfaces acquire a large radial extent when the shadow condition is approached. For electrons, a radial displacement per bounce [given by Eq. (9)] is produced during a round-trip transit through the ion sheath (resulting from the poloidal variation of the potential). These displacements are not constrained to lie on constant potential surfaces, and so could add up without bound.

In a poloidal divertor geometry, a considerable modification of the above considerations may be needed because of the role of the shear effect (caused by a rapid variation of the connection length L_{conn} with a radius in the vicinity of the separatrix). The shear effect gives rise to a stretching of the flux-tube cross section. This stretching is, obviously, insignificant if the imprint of the flux tube on the divertor plate has a length l in the z direction and width δ in the x direction, satisfying the condition

$$l > \delta\left(\frac{dL_{\mathrm{conn}}}{dx}\right). \tag{45}$$

If this condition is violated, some new effects, which are beyond the scope of the present paper, may appear.

To conclude this section: making a toroidal limiter or divertor plates deliberately wavy, should be an efficient means of controlling the SOL thickness. Development of a self-consistent theory of the convection thus induced goes beyond the scope of the present paper.

V. BOUNDARY CONDITIONS FOR MACROSCOPIC INSTABILITIES

When analyzing the behavior of the large-scale (with a scale considerably exceeding the ion gyroradius) electrostatic or low-β electromagnetic perturbations in a plasma bounded by material surfaces, one has to formulate the boundary conditions that would relate perturbations of plasma parameters near the wall to the perturbations of the normal component of the plasma current and the sheath potential. This boundary condition is required, for example, in studies of the flute perturbations[11] or a temperature gradient instability.[12] Of course, the description of the processes taking place near the walls, in terms of boundary conditions, is possible only for macroscopic perturbations, with scale lengths considerably exceeding the thickness of the ion subsheath (i.e., ρ_i).

For the case of normal intersection of the field lines with the walls, with the assumption that there is no perturbation of the electron temperature, this boundary condition was formulated in Ref. 11. Account of the electron temperature perturbation was introduced in Ref. 12. In Ref. 7, the generalization of the boundary conditions found in Ref. 12 to the case of the tilted magnetic field was made for the case where the ion pressure is initially uniform and therefore is not perturbed by the convective instabilities.

The results obtained in Sec. III allow us to derive this boundary condition for the general case (in particular, including the ion diamagnetic terms). To avoid overly lengthy equations, here we present only the results pertinent to small alphas. We base our derivation on the relationship (19). Taking the perturbation of this expression, we obtain

$$\tilde{j}_n=\frac{\tilde{n}}{n_0} j_{n0}+\alpha e n\left[\frac{v_{te}}{2\pi^{1/2}} \exp\left(-\frac{e\phi_0}{T_e}\right)\left(\left(\Lambda+\frac{1}{2}\right)\frac{\tilde{T}_e}{T_e} - \frac{e\tilde{\phi}}{T_e}\right)+\tilde{u}\right]+\frac{enc}{B}\frac{\partial\tilde{\phi}}{\partial x}-\frac{cn}{B}\left(\frac{1}{n}\frac{\partial \widetilde{p}_{\perp i}}{\partial x}\right), \tag{46}$$

where ~ and the subscript 0 denote perturbations and unperturbed quantities, respectively, and

$$\Lambda=-\ln\frac{2\pi^{1/2}}{v_{te}}\left(u-\frac{j_{n0}}{\alpha en}-\frac{enc}{\alpha B}\frac{\partial\phi_0}{\partial x}-\frac{c}{Bn_0}\frac{\partial p_{\perp i0}}{\partial x}\right). \tag{47}$$

This is the most general form of the boundary condition. For a practically important case when the unperturbed current is considerably smaller than the ion saturation current ($j_{n0} \ll \alpha enu_0$), one has

$$\Lambda \approx \ln\frac{v_{te}}{2\pi^{1/2}u} \approx 3-5. \tag{48}$$

For the special case considered in Ref. 7 (a uniform plasma density and ion temperature), one recovers Eq. (37) of Ref. 7. The general boundary condition (46) allows one to describe a much broader class of situations.

VI. CONCLUSION

We have considered the current-voltage characteristics of the plasma sheath in a strongly tilted magnetic field. A somewhat paradoxical result is that the normal component of the current is different on the plasma side of the sheath than on the wall. The difference is caused by the variation of the surface current flowing within the ion subsheath.

When one considers the boundary condition for the processes occurring in the bulk of the plasma, what matters is the current on the plasma side of the sheath. It is this boundary condition [Eq. (19)], which is most easily usable in SOL fluid codes (as part of an expression of current conservation on a flux tube) to determine the sheath potential. On the other hand, if the wall is made of an insulator, this imposes an obvious constraint that the current should vanish on the wall itself. This determines the plasma potential and the current on the plasma side of the sheath, as well.

Contributions of electron and ion electric drifts and diamagnetic currents enter the boundary conditions in a strongly asymmetric way: the ion contributions are significant, while electron contributions are vanishing. This asymmetry is caused by the fact that almost all the electrons get reflected from the sheath potential, while all the ions become absorbed by the wall. Recycling does not alter this conclusion as long as the mean-free path of the recycled neutrals is long compared to a gyroradius.

We presented two specific examples of the application of these boundary conditions. The first example pertains to the problem of the plasma equilibrium on the open field lines of the SOL in a tokamak with a toroidal limiter or a poloidal divertor. By considering the condition of current continuity for the electron component, we have shown that, if the surface of the limiter or the divertor plate is wavy in the toroidal direction, this can produce a vigorous plasma convection. The origin of this phenomenon is similar to the loss of the

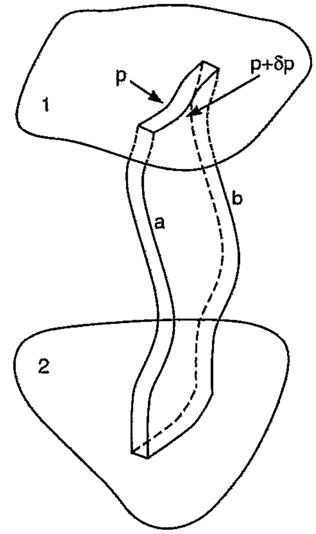

FIG. 6. Toward the derivation of the equilibrium condition for the perfectly reflecting end walls.

magnetohydrodynamic (MHD) equilibrium in a plasma in which the surfaces p=const do not coincide with the surfaces of constant specific flux tube volume. This example shows that one can strongly affect the SOL thickness by making the limiter surface slightly wavy, with a properly chosen amplitude. Further development of this research should give rise to a self-consistent convection theory in the case of the nonplanar limiting surfaces.

The second example is the formulation of the boundary conditions for electrostatic large-scale perturbations (e.g., of the flute type), on material surfaces for the case of a tilted magnetic field. We have derived this condition for the most general situation, where both electron and ion diamagnetic currents are present, as well as electric drift in the unperturbed electric field. This allows us to assess macroscopic instabilities of the SOL under a broad set of conditions.

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APPENDIX: EQUILIBRIUM OF PLASMA CONFINED BETWEEN REFLECTING SURFACES

In this appendix, we consider a model problem that sheds some light on the origin of the convection caused by the nonplanarity of the surfaces terminating the plasma (Sec. IV). Specifically, we consider a plasma confined along the field lines by perfectly reflecting end walls (Fig. 6). This system was mentioned in passing in Ref. 13, in conjunction with plasma stability in mirror devices. Its MHD stability was analyzed in Ref. 14 for the case of a uniform magnetic field and two planar mutually tilted end surfaces. Here we consider plasma equilibrium for the general case of a nonuniform magnetic field and end surfaces of the arbitrary shape. We assume that the confinement time is large enough so that collisions make the plasma isotropic, and, thereby, uniform along the field lines. A single-fluid description is sufficient for our purposes. Our derivation is similar to the one used in Ref. 15 for closed magnetic surfaces.

Let us consider two neighboring surfaces on which the plasma pressure is constant; let the pressure difference between the surfaces be δp. We apply the current continuity condition for the volume limited by two neighboring surfaces p=const, two side surfaces a and b formed by the field lines (Fig. 6), and two perfectly reflecting end surfaces 1 and 2. As is clear from the equilibrium equation,

$$-\nabla_{\perp} p+(1/c)(\mathbf{j}\times\mathbf{B})=0, \tag{A1}$$

the plasma current cannot have a component normal to the surfaces p=const. The plasma current through the end surfaces is also zero because they are perfectly reflecting. Therefore, the current continuity condition read as

$$I_a=I_b, \tag{A2}$$

where I_a and I_b are the currents through the side surfaces a and b. As these surfaces are formed by the field lines, only a perpendicular component of the plasma current contributes to I_a and I_b. The perpendicular component of the current density, according to (A1), is equal to

$$j_{\perp}=\frac{c\ \delta p}{\eta B}, \tag{A3}$$

where η is the (small) distance between the neighboring planes p=const. The surface element of the surface a is just $\eta\, ds$, where ds is the field line arc element. Therefore,

$$I_a=\int_1^2 j_{\perp}\eta\ ds=c\ \delta p\int_1^2 \frac{ds}{B}. \tag{A4}$$

Then, as δp is the same at both surfaces a and b, the condition (A2) is reduced just to

$$\left(\int_1^2 \frac{ds}{B}\right)_a=\left(\int_1^2 \frac{ds}{B}\right)_b. \tag{A5}$$

As the sections a and b were chosen arbitrarily, the condition (A5) just means that the integral,

$$U\equiv\int_1^2 \frac{ds}{B}, \tag{A6}$$

is constant over the surface of a constant pressure, or $p=p(U)$. If, initially, the p=const surfaces do not coincide with the surfaces U=const, then a polarization electric field immediately appears that sets plasma into convective motion that brings it to an equilibrium.

For a uniform magnetic field, the condition $p=p(U)$ is reduced just to the requirement that the pressure should be constant at the surfaces for which the distance L between the walls (counted along the field lines), is constant.

If the walls are slightly absorbing, one has to introduce the sources that would compensate (small) plasma losses. In order not to cause a considerable plasma convection, these sources should be distributed, so as to ensure the constancy of the pressure on the L=const surfaces.

Basically, the same condition emerges from our analysis in Sec. IV. What is different is that only electrons are well (for many bounces between the plates) confined and only for this plasma component is the condition $I_1=I_2=0$ approximately correct. Accordingly, we consider the current continuity for the electron component. The ions are a relatively "passive" element, as they do not stay in the system for a long time. In order to have an equilibrium free of a vigorous convection, electron sources should be distributed, so as to have the same strength on the surfaces L=const. For a toroidally symmetric device and flat limiter, this means that the sources should just be toroidally symmetric. For a "wavy" toroidal limiter (or divertor plates), according to (40), only the condition that $\alpha_1=\alpha_2=\alpha$ satisfies this requirement (the condition that $1/\alpha_1-1/\alpha_2=\text{const}\neq 0$ is incompatible with toroidal periodicity). Otherwise, for toroidally symmetric sources convection in the SOL appears.

One last comment: In the geometry of Fig. 3, at $\alpha\ll 1$, the ion experiences many bounces along the perfectly reflecting surface, before it eventually returns to the plasma. In the course of these bounces, it gets displaced in the x direction by a distance $\sim\rho_i/\alpha$. Therefore, again, in order to be justified in using the local description in terms of the x coordinate, one has to impose condition (22).

[1] R. Chodura, Phys. Fluids **25**, 1628 (1982).
[2] K.-U. Riemann, J. Phys. D Appl. Phys. **24**, 493 (1991).
[3] D. L. Holland, B. D. Fried, and G. J. Morales, Phys. Fluids B **5**, 1723 (1993).
[4] A. V. Chankin and P. C. Stangeby, Plasma Phys. Controlled Fusion **36**, 1485 (1994).
[5] R. H. Cohen and D. D. Ryutov, Comments Plasma Phys. Controlled Fusion **16**, 255 (1995).
[6] M. Tendler and V. Rozhansky, Comments Plasma Phys. Controlled Fusion **13**, 191 (1991).
[7] D. Farina, R. Pozzoli, and D. Ryutov, Plasma Phys. Controlled Fusion **35**, 1271 (1993).
[8] By "macroscopic," we mean instabilities with a scale length exceeding the ion gyroradius.
[9] R. H. Cohen and D. D. Ryutov, "Spreading particle trajectories near a perfectly reflecting surface in a tilted magnetic field," submitted for publication in Phys. Plasmas.
[10] P. C. Stangeby and G. M. McCracken, Nucl. Fusion **30**, 1225 (1990).
[11] W. B. Kunkel and J. U. Guillory, in *Proceedings of the 7th Conference on Phenomena in Ionized Gases*, Belgrade, 1965 (Gradjevinska Knjiga, Belgrade, Yugoslavia, 1966), Vol. 2, p. 702.
[12] H. L. Berk, D. D. Ryutov, and Yu. A. Tsidulko, Phys. Fluids B **3**, 1346 (1991).
[13] D. D. Ryutov, in *Physics of Mirrors, Reversed Field Pinches and Compact Tori*, Proceedings of the International School of Plasma Physics, Piero, Caldirola (Editrice Compositori, Bologna, 1988), Vol. 2, p. 791.
[14] D. Farina, R. Pozzoli, and D. Ryutov, Phys. Fluids, B **5**, 4055 (1993).
[15] V. D. Shafranov, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 2, p. 103.