Boundary Conditions for the Child–Langmuir Sheath Model

Mikhail S. Benilov

***Abstract*—A collision-free space-charge sheath formed by cold ions at a negative surface is considered. The method of matched asymptotic expansions is applied, small parameter being the ratio of the electron temperature to the sheath voltage. Two expansions are considered, one describing bulk of the sheath where the electron density is exponentially small as compared to the ion density, and another describing the outer section of the sheath in which the densities are comparable. Boundary conditions for equations describing the bulk are found such that the model have exponential accuracy. A physical meaning of these conditions is that the ions are accelerated in the outer section from the Bohm velocity to twice the Bohm velocity and that the voltage drop in the outer section equals $(3/2)(kT_e/e)$. The model predicts the electric field and ion velocity at the surface and the thickness of the ion layer to the accuracy of several percent for sheath voltages exceeding $3(kT_e/e)$.**

***Index Terms*—Plasma modeling, plasmas, sheath, space charge.**

I. Introduction

A CONVENIENT tool for describing space-charge sheaths at negatively biased surfaces is the model of electron-free sheath. This model is also referred to as the electron step-front model; in the case of a collisionless sheath formed by cold ions, the model is usually termed the Child–Langmuir sheath [1], [2].

A delicate point in this model is formulation of boundary conditions at the edge of the sheath. The boundary conditions used in the original Child–Langmuir model [1], [2] are those of zero ion velocity and zero electric field. These conditions imply that values of the ion velocity and of the electric field at the sheath edge are neglected in comparison with respective values in the bulk of the sheath, and are valid only in the first approximation. A question as to what are more accurate boundary conditions still has no satisfactory answer. Some authors set the ion velocity at the sheath edge equal to the Bohm velocity. It is well-known that this is an appropriate boundary condition for a sheath with nonzero electron density; however, it is unclear whether this condition is the most adequate one in the case of an electron-free sheath. In [3], a conclusion is drawn that the ions leave the quasineutrality region and enter the transition region with a value of the ion velocity (normalized by the Bohm velocity) $V \approx 0.806 < 1$ and leave the transition region entering the space-charge sheath region with the velocity $V \approx 2.12 > 1$. In [4], an assumption is suggested that the edge of an electron-free sheath is defined by the condition $n_e = 0.1 n_s$, where n_e is the local electron density, and n_s is the density of charged particle on the plasma side of the space-charge sheath. Another example of boundary conditions can be found in [5].

From a mathematical point of view, an electron-free sheath is an asymptotic model applicable in the limit of large values of the sheath voltage drop. An adequate tool for derivation of higher-approximation boundary conditions in such-type problems is the method of matched asymptotic expansions (e.g., [6]–[10]). This method has been widely employed in the theory of space-charge sheaths in low-pressure plasmas (e.g., [4], [11] and references therein), the small parameter being ε the ratio of the Debye length in the quasineutral plasma to the mean free path for collisions of ions and neutral particles. In order to describe asymptotically the model of electron-free sheath, one needs to consider another asymptotic parameter, $\chi = eU/kT_e$ (where U is the voltage drop in the sheath and T_e is the electron temperature), which is large.

In the framework of such an asymptotic approach, equations describing an electron-free sheath are of the exponential order of accuracy, since they are obtained by dropping the electron density term of the Poisson equation, which is exponentially small (with respect to χ). This, however, does not guarantee that the accuracy of the model of electron-free sheath also is exponential, since an algebraic-order (i.e., essentially higher) error can be introduced into the model by boundary conditions. An example of the Child–Langmuir model, for which the accuracy may be estimated to be $O(\chi^{-1/2})$, shows that the latter is indeed the case and that the error introduced in the model by the use of first-approximation boundary conditions is in fact rather high. Therefore, it is a derivation of higher-approximation boundary conditions which should be undertaken in order to improve accuracy of the model of electron-free sheath. In some cases, one can even derive such boundary conditions that the accuracy of the model be exponential.

Note that if an asymptotic model has the exponential accuracy, normally such a model is capable of giving accurate results even for values of the asymptotic parameter which are not strongly different from unity. Therefore, one can hope that if boundary conditions ensuring the exponential accuracy of a model of electron-free sheath have been found, such a model would predict results with a few percent accuracy already for surface potentials about, or even above, the floating potential.

The simplest model of electron-free sheath may be obtained by applying first the limit $\varepsilon \to 0$, and after that the limit $\chi \to \infty$. Obviously, this amounts to considering a collisionless sheath on a surface under a large negative potential. Such a model is considered in this work. Additionally, the assumption of cold ions is adopted. The model of a collisionless sheath has an exact solution in quadratures under this assumption, which allows one

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The author is with the Departamento de Física, Universidade da Madeira, 9000 Funchal, Portugal (e-mail: benilov@dragoeiro.uma.pt).

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to better illustrate a mathematical sense of asymptotic results and their accuracy. One can hope that asymptotic results which can be obtained for such a model, apart from being of interest by itself, will also provide a basis for applying a similar asymptotic treatment to more complex models, including those with account of collisions.

II. Model

Consider a collisionless sheath formed by cold ions at an absorbing surface under a negative potential. A system of governing equations includes the equation of continuity of ion flux, the equation of motion of an ion, the Boltzmann distribution for electrons, and the Poisson equation

$$n_i v_i = J_i, \qquad m_i v_i \frac{dv_i}{dy} = -e\frac{d\phi}{dy} \tag{1}$$

$$n_e = n_s \exp\frac{e\phi}{kT_e}, \qquad \varepsilon_0 \frac{d^2\phi}{dy^2} = -e(n_i - n_e). \tag{2}$$

where

y-axis	directed from the surface into the plasma;
n_i and n_e	number densities of ions and electrons;
v_i	velocity of the motion of ions in the direction to the surface;
J_i	density of the ion flux to the surface (since the ion flux is generated outside the sheath, J_i is considered as a given constant in the present analysis);
ϕ	electrostatic potential;
m_i	mass of ion;
n_s	constant governed by boundary conditions;
T_e	electron temperature (a given constant).

Choosing zero of potential at the "sheath edge" and applying the condition of quasineutrality and the Bohm criterion at the edge, one gets boundary conditions

$$v_i \to u_B, \qquad \phi \to 0, \qquad n_i \to \frac{J_i}{u_B}, \qquad n_e \to \frac{J_i}{u_B} \tag{3}$$

where $u_B = \sqrt{kT_e/m_i}$ is the Bohm velocity. It follows that n_s has the meaning of the charge-particle density at the sheath edge and equals J_i/u_B.

A boundary condition at the surface, $y = 0$, is obtained by specifying the local potential

$$\phi = -U \tag{4}$$

where U is the voltage drop in the sheath (a given positive quantity).

Introduce dimensionless variables

$$\eta = \frac{y}{h}, \qquad V = \frac{v_i}{u_B}, \qquad \Phi = \frac{e\phi}{kT_e} \tag{5}$$

where $h = (\varepsilon_0 kT_e/n_s e^2)^{1/2}$ is the Debye length estimated at the sheath edge. The equations assume the form

$$V\frac{dV}{d\eta} = -\frac{d\Phi}{d\eta}, \qquad \frac{d^2\Phi}{d\eta^2} = -\frac{1}{V} + e^{\Phi}. \tag{6}$$

Boundary conditions at the sheath edge, $\eta \to \infty$, and at the surface, $\eta = 0$, read, respectively

$$V = 1, \qquad \Phi = 0 \tag{7}$$

$$\Phi = -\chi \tag{8}$$

where $\chi = eU/kT_e$ is the dimensionless voltage drop in the sheath.

For purposes of the following analysis, it is convenient to introduce the dimensionless electric field, $E = d\Phi/d\eta$, and reformulate problem (6)–(8) in terms of unknown functions $W(V)$ and $\Phi(V)$, where $W = E^2$:

$$\frac{1}{2}\frac{dW}{dV} = 1 - Ve^{\Phi}, \qquad \frac{d\Phi}{dV} = -V \tag{9}$$

$$W(1) = 0, \qquad \Phi(1) = 0. \tag{10}$$

After this problem has been solved, one can determine the ion velocity at the surface, V_w, from equation $\Phi(V_w) = -\chi$ and to find a relation between V and η by means of the equation

$$\eta = \int_V^{V_w} \frac{V}{\sqrt{W}}\, dV. \tag{11}$$

III. Asymptotic Treatment

In this section, an asymptotic solution of the considered problem in the limit $\chi \to \infty$ is obtained by means of the method of matched asymptotic expansions (e.g., [6]–[10]). A physical meaning of the obtained asymptotic results is discussed in the next section.

Two asymptotic expansions should be considered in order to completely describe the solution. A straightforward (outer) asymptotic expansion of a solution to the problems (9) and (10) is applicable at V of the order unity and reads

$$W = W_1(V), \qquad \Phi = \Phi_1(V). \tag{12}$$

A problem governing functions W_1 and Φ_1 coincides with problems (9) and (10), the only difference being that W_1 and Φ_1 appear in place of W and Φ. A solution reads

$$W_1 = 2V + 2\exp\left(-\frac{V^2-1}{2}\right) - 4,$$

$$\Phi_1 = -\frac{V^2-1}{2}. \tag{13}$$

It should be emphasized that since this solution has been obtained by solving the exact problem without any simplifications, functions $W_1(V)$ and $\Phi_1(V)$ represent an exact solution to the problem considered. This is why straightforward asymptotic expansion (12) contains only one term.

Asymptotic behavior of functions $W_1(V)$ and $\Phi_1(V)$ at $V \to \infty$ is

$$W_1 = 2V - 4 + \text{t.s.t.}, \qquad \Phi_1 = -\frac{V^2}{2} + \frac{1}{2} \tag{14}$$

where t.s.t. stands for terms which are transcendentally (exponentially) small at $V \to \infty$.

The second (inner) asymptotic expansion reads

$$W = \sqrt{\gamma} W_2(V_2) + W_3(V_2) + \text{t.s.t.}$$
$$\Phi = \gamma \Phi_2(V_2) + \Phi_3(V_2) + \text{t.s.t.} \quad (15)$$

where $V_2 = V/\sqrt{\gamma}$ is a new independent variable varying in the range $0 < V_2 \leq 1$, $\gamma = V_w^2$ is a large parameter to be found as a part of a solution, and t.s.t. stands for terms which are exponentially small at $\chi \to \infty$. Substituting this expansion in (9) and expanding, one obtains the following equations governing functions $W_2(V_2)$, $W_3(V_2)$, $\Phi_2(V_2)$, and $\Phi_3(V_2)$

$$\frac{1}{2}\frac{dW_2}{dV_2} = 1, \quad \frac{d\Phi_2}{dV_2} = -V_2, \quad \frac{dW_3}{dV_2} = 0, \quad \frac{d\Phi_3}{dV_2} = 0. \quad (16)$$

Asymptotic matching of one-term asymptotic expansion (12) with two-term expansion (15) results in boundary conditions

$$V_2 \to 0:\ W_2 = 2V_2 + \cdots,\ \Phi_2 = -\frac{V_2^2}{2} + \cdots,$$
$$W_3 \to -4,\ \Phi_3 \to \tfrac{1}{2}. \quad (17)$$

A solution to problems (16) and (17) reads

$$W_2 = 2V_2, \qquad \Phi_2 = -\frac{V_2^2}{2}$$
$$W_3 = -4, \qquad \Phi_3 = \frac{1}{2}. \quad (18)$$

A relation between large parameters χ and γ can be found from boundary condition (8)

$$\gamma = 2\chi + 1 + \text{t.s.t.} \quad (19)$$

Now the asymptotic solution of the problem is complete. It is noteworthy that just two terms of the inner expansion are sufficient to obtain the exponential accuracy (with respect to χ). It is interesting to note for comparison that if a similar asymptotic treatment is applied to the problems (6)–(8), the inner expansion will involve all negative integer powers of large parameter $\sqrt{\chi}$. Therefore, the accuracy of such an approach will be $O(\chi^{-n/2})$, where n is the number of terms taken into account in the inner expansion, and one would need to take into account an infinite number of terms in order to obtain the exponential accuracy.

In the framework of the present analysis, the fact that the inner expansion contains only two algebraic (with respect to χ) terms is a consequence of the fact that the asymptotic behavior of the outer expansion, described by (12) and (14), contains just two algebraic (with respect to V) terms. The latter is due to the special choice of the independent and dependent variables. In other words, problem (6)–(8) was recast to unknown functions $W(V)$ and $\Phi(V)$ just in order to obtain the exponential accuracy without considering an infinite number of terms. Note that the fact that a proper choice of variables can in many cases yield the most satisfactory results is well known in perturbation methods; see, e.g., [6, Sec. 10.6].

IV. Results and Discussion

A. Asymptotic Structure of the Solution

Physical sense of the above asymptotic results is quite transparent. Expansion (12) describes the outer section of the space-charge sheath, where the electron and ion densities are comparable. Expansion (15) describes bulk of the space-charge sheath, where the electron density is exponentially small as compared to the ion density. These regions will be referred to as the ion-electron layer and the ion layer, respectively. Note that the latter term is similar to that employed in the theory of collision-dominated sheaths (e.g., [12], [13] and references therein); the former term is used here in place of "transitional region" [12], [13] in order not to confuse the region in question with a region separating a collisionless sheath from a presheath, which is usually termed the transition layer (e.g., [4]).

Thickness of the ion-electron layer may be estimated with the use of (11) and is of the order of h the Debye length evaluated at the sheath edge. Thickness of the ion layer is of the order of $h\chi^{3/4}$. For χ large enough, it substantially exceeds the thickness of the ion-electron layer. Note that quantity $h\chi^{3/4}$ may be represented as $(\varepsilon_0 eU/n_0 e^2)^{1/2}$ where $n_0 = n_s \chi^{-1/2}$ is a characteristic density of the ions inside the ion layer. Thus, the thickness of the ion layer is of the order of the local Debye length estimated in terms of characteristic energy eU.

Since the thickness of the ion-electron layer is much smaller than the thickness of the ion layer, the latter has a more or less distinct edge. On the other hand, no unique definition of this edge can be given to accuracy better than $O(\chi^{-3/4})$, which is the order of the ratio of the thickness of the ion-electron layer to the thickness of the ion layer.

Ion layer gives a dominating contribution to the total voltage drop in the sheath, the voltage in the ion-electron layer is of the order of $kT_e/e \ll U$.

The above-discussed asymptotic structure is schematically shown in Fig. 1.

B. Exponential-Accuracy Model of the Ion Layer

The above asymptotic analysis gives one a reason to believe that it is possible to develop a simple model of the ion layer having an exponential accuracy with respect to large parameter χ. This can be done as follows. A system of equations describing the ion layer may be obtained from (6) by neglecting the electron density term in the Poisson equation and may be written as

$$V\frac{dV}{d\eta} = -E, \qquad \frac{dE}{d\eta} = -\frac{1}{V}, \qquad \frac{d\Phi}{d\eta} = E. \quad (20)$$

A boundary condition at the surface is supplied by (8).

Problem (8), (20) may be easily solved in a general form (i.e., without specifying other boundary conditions, e.g., [14]). A convenient way to do it is as follows. Equations in (20) have first integrals

$$V - \frac{E^2}{2} = C_1, \qquad \Phi + \frac{V^2}{2} = C_2 \quad (21)$$

where C_1 and C_2 are arbitrary constants. Solving the first equation in (21) for V, substituting the result in the second equation

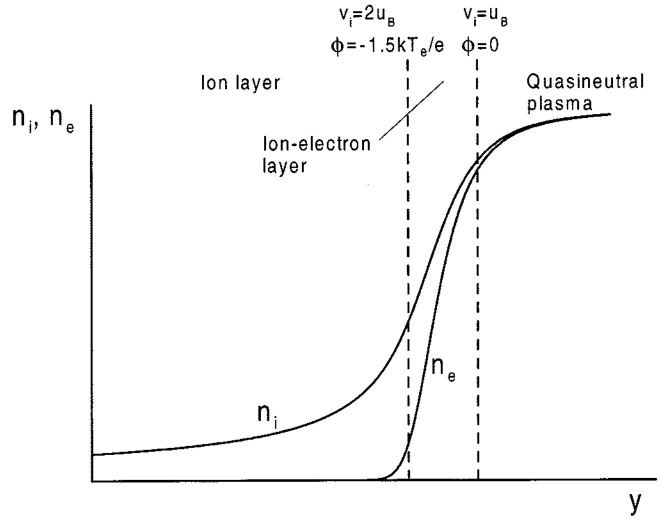

Fig. 1. Schematic of the asymptotic structure of the sheath.

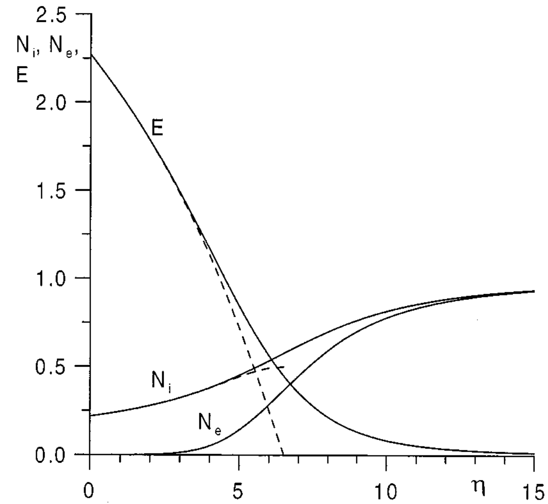

Fig. 2. Distributions of the ion and electron densities and of the electric field. $\chi = 10$. Solid lines: the exact solution and dashed lines: the exponential-accuracy model.

in (20), integrating the obtained equation for function $\eta(E)$, expressing E in terms of V, and making use of boundary condition (8), one can arrive at

$$\eta = \frac{2^{1/2}}{3}\left[\left(\sqrt{2\chi+2C_2}-C_1\right)^{3/2}-(V-C_1)^{3/2}\right] + 2^{1/2}C_1\left[\left(\sqrt{2\chi+2C_2}-C_1\right)^{1/2}-(V-C_1)^{1/2}\right]. \tag{22}$$

Equations (21) and (22) represent the desired general solution describing a general model of the ion layer: (22) describes function $V(\eta)$, while (21) relates E and Φ to V.

Constants C_1 and C_2 in the solutions (21) and (22) can be determined from asymptotic matching with a solution describing the ion-electron layer, which amounts to comparing (21) to (14) in which the exponentially small term has been dropped. A more transparent model, however, can be constructed in a somewhat different way. One can see from (21) and (22) that function $V(\eta)$ is decreasing and the solution cannot be extended beyond a point at which V decreases down to value $V = C_1$ or, in other words, at which $E = 0$. It is logical to consider this point as an edge of the ion layer. Thus, one arrives at the following definition of the edge of the ion layer: this is a point at which the electric field described by equations of the ion layer vanishes. It should be emphasized that this definition has been formulated without a reference to particular values of C_1 and C_2 and therefore applies to any model of the ion layer.

Note that since equations of the ion layer do not provide a good approximation in the vicinity of the edge (where the ion-electron layer is positioned), values of the electric field given by equations of the ion layer substantially diverge in this vicinity from exact values. In other words, what vanishes at the edge of the ion layer is not the (exact) electric field but rather the extrapolation of an approximate solution describing the field inside the layer. The exact value of the field at the edge of the ion layer is of course nonzero, although it is much smaller than the field in the bulk of the layer.

Thus, one of boundary conditions at the edge of the ion layer for equations describing distributions inside the layer is

$$E = 0. \tag{23}$$

Other boundary conditions at the edge of the ion layer may be obtained by using (14), which describes asymptotic behavior of a solution describing the ion-electron layer. Dropping the exponentially small term in this equation and setting $E = 0$, one finds

$$V = 2, \qquad \Phi = -\tfrac{3}{2}. \tag{24}$$

Note that boundary conditions (24) have quite a distinct physical sense: they imply that the ions are accelerated in the ion-electron layer from the Bohm velocity, $v_i = u_B$, on the plasma side of the ion-electron layer, to twice the Bohm velocity, $v_i = 2u_B$, at the boundary between the ion-electron layer and the ion layer; voltage drop in the ion-electron layer is $(3/2)(kT_e/e)$. This interpretation of boundary conditions (24) is illustrated by Fig. 2.

Using boundary conditions (23) and (24), one can determine constants C_1 and C_2: $C_1 = 2$, $C_2 = 1/2$. Now the solution describing the ion layer is complete. One can check that this solution conforms to the solution for the ion layer obtained in Section III. In accord to Section III, this solution has the exponential accuracy.

Thus, one arrives at the following model of the ion layer: the model comprises (20) and boundary conditions (8), (23), and (24). A solution to this model is given by (21) and (22) with $C_1 = 2$, $C_2 = 1/2$, and has the exponential accuracy. In the following, this model will be referred to as the exponential-accuracy model. It is identical to the Child–Langmuir model with exception of boundary conditions for V and Φ at the edge.

For purposes of comparison, we note that the respective boundary conditions in the framework of the Child–Langmuir model are

$$V = 0, \qquad \Phi = 0. \tag{25}$$

A solution to the Child–Langmuir model is given by (21) and (22) with $C_1 = C_2 = 0$.

Another model of the ion layer which can be used for comparison is the one in which boundary conditions for the ion velocity and potential at the edge are obtained by applying (7). In other words, in this model boundary, conditions at the edge of the ion-electron layer (which is the same as the edge of the space-charge sheath on the whole) are transferred to the edge of the ion layer. The model is described by (20), boundary condition (8) at the surface, and boundary conditions (23) and (7) at the edge. In the following, this model will be referred to as the ion layer-quasineutral plasma model. The respective solution is given by (21) and (22) with $C_1 = 1$ and $C_2 = 1/2$.

C. Comparison with an Exact Solution

Problems (6)–(8) have an exact solution in quadratures, which may be written as

$$\eta = 2^{-1/2} \int_{-\Phi}^{\chi} \left(\sqrt{1+2x} + e^{-x} - 2\right)^{-1/2} dx \tag{26}$$

$$V = \sqrt{1-2\Phi}, \qquad E = \sqrt{2}\left(\sqrt{1-2\Phi} + e^{\Phi} - 2\right)^{1/2}. \tag{27}$$

Neglecting in (26) and (27) terms of the order of $(-1/\Phi)^{1/2}$ and higher, one arrives at

$$\begin{aligned} V &= \frac{3^{2/3}}{2^{1/3}} (\Delta - \eta)^{2/3} \\ \Phi &= -\frac{3^{4/3}}{2^{5/3}} (\Delta - \eta)^{4/3} \\ E &= 6^{1/3} (\Delta - \eta)^{1/3} \end{aligned} \tag{28}$$

where

$$\Delta = \frac{2^{5/4}}{3} \chi^{3/4}. \tag{29}$$

One can check that this solution conforms to (21) and (22) with $C_1 = C_2 = 0$, i.e., it is the Child–Langmuir solution. It follows that accuracy of the Child–Langmuir model is of the order of $\chi^{-1/2}$.

Neglecting in (26) and (27) exponential terms, one arrives at equations which conform to (21) and (22) with $C_1 = 2$ and $C_2 = 1/2$, i.e., at the exponential-accuracy model. Comparing the exponential-accuracy model with the ion layer-quasi-neutral plasma model, one can see that asymptotic accuracy of the latter is $O(\chi^{-1/2})2$, i.e., the same as that of the Child–Langmuir model.

As an example, distributions of the dimensionless ion and electron densities, $N_i = 1/V$ and $N_e = e^{\Phi}$, and of the dimensionless electric field described by the exact solution (26) and (27) are shown in Fig. 2 for the case $\chi = 10$. Also shown are solutions for the ion density and for the electric field described by the exponential-accuracy model of the ion layer. One can see that the ion layer in the framework of the exponential-accuracy model occupies in this case the region $\eta \lesssim 6.50$. The agreement between the asymptotic and exact values inside the ion layer is good. One can see clearly that what vanishes at the edge of the ion layer is the extrapolation of a solution describing the electric field inside the layer, while the exact value of the electric field remains nonzero.

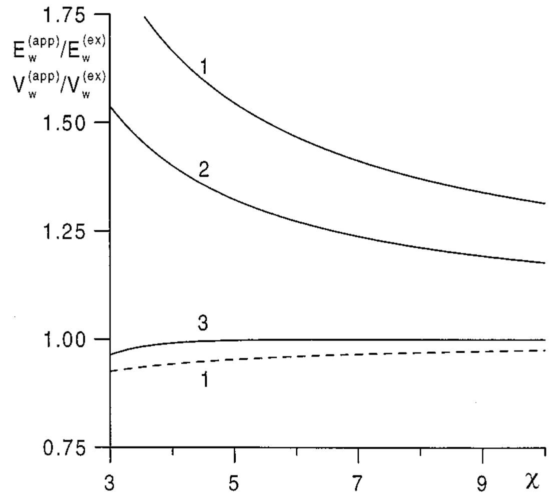

Fig. 3. Ratio of approximate values of the electric field at the surface (solid lines) and of the ion velocity at the surface (dashed line) given by different models to the exact value. 1: Child–Langmuir model. 2: Ion layer-quasi-neutral plasma model. 3: Exponential-accuracy model.

Of special interest for applications are values of the electric field and of the ion velocity at the surface as functions of the sheath voltage. For the Child–Langmuir model, it follows from (28) and (29)

$$E_w = 2^{3/4}\chi^{1/4}, \qquad V_w = 2^{1/2}\chi^{1/2}. \tag{30}$$

In the framework of the ion layer-quasineutral plasma model, one finds from (21) with $C_1 = 1$ and $C_2 = 1/2$

$$E_w = \sqrt{2}\left(\sqrt{2\chi+1} - 1\right)^{1/2}, \qquad V_w = \sqrt{2\chi+1}. \tag{31}$$

In the framework of the exponential-accuracy model, it follows from (21) and (22) with $C_1 = 2$ and $C_2 = 1/2$

$$E_w = \sqrt{2}\left(\sqrt{2\chi+1} - 2\right)^{1/2}, \qquad V_w = \sqrt{2\chi+1}. \tag{32}$$

Accuracy of these formulas is illustrated by Fig. 3, in which a ratio is shown of values given by these formulas to exact values [given by (27) with $\Phi = -\chi$]. As far as the electric field is concerned, the error of the Child–Langmuir model at $\chi = 5$, which is a typical value corresponding to a floating surface, is above 50%, and even at a rather high sheath voltage $\chi = 10$ the error exceeds 30%. Accuracy of the ion layer-quasi-neutral plasma model is higher, however the improvement is not drastic. On the contrary, the error of the value given by the exponential-accuracy model does not exceed 4% in the whole range of χ considered.

Values of the ion velocity at the surface given by the Child–Langmuir model are more accurate than values of the electric field given by the same model. A reason is that powers of χ in the expansion of function $V_w(\chi)$ at large χ vary with the step -1, rather than with the step $-1/2$, as in the expansion of function $E_w(\chi)$. As a consequence, the Child–Langmuir model predicts function $V_w(\chi)$ to the accuracy $O(\chi^{-1})$ and function $E_w(\chi)$ to the accuracy $O(\chi^{-1/2})$. Values of the ion velocity at the surface given by the exponential-accuracy and ion layer-quasi-neutral plasma models are exact and are not shown on the graph. A reason is that the second equation in (21) with $C_2 = 1/2$, describing function $V_w(\chi)$ in the framework of these models, represents an exact first integral of the problems (6)–(8) (the law of conservation of energy of the ions).

In order to compare approximate results for the thickness of the ion layer with an exact solution, one needs to identify the edge of the ion layer in the exact solution. Again, this cannot be done in a unique way. On the basis of the exponential-accuracy model, it is natural to adopt the following simple convention: the edge of the ion layer is a point at which $V = 2$ or, which is the same, $\Phi = -3/2$. In other words, we define the edge of the ion layer as a point at which the electron density equals to $2e^{-3/2} = 0.4463$ of the ion density. Making use of (26), one finds that the exact value of the above-defined thickness of the ion layer is

$$\Delta = 2^{-1/2} \int_{3/2}^{\chi} \left(\sqrt{1+2x} + e^{-x} - 2\right)^{-1/2} dx. \tag{33}$$

In the framework of the general model of the ion layer, the thickness of the ion layer is given by (22) with $V = C_1$. For the Child–Langmuir model, the result amounts to (29). The results for the ion layer-quasi-neutral plasma model and for the exponential-accuracy model are, respectively

$$\Delta = \frac{2^{1/2}}{3}\left(\sqrt{2\chi+1} - 1\right)^{3/2} + 2^{1/2}\left(\sqrt{2\chi+1} - 1\right)^{1/2} \tag{34}$$

$$\Delta = \frac{2^{1/2}}{3}\left(\sqrt{2\chi+1} - 2\right)^{3/2} + 2^{3/2}\left(\sqrt{2\chi+1} - 2\right)^{1/2}. \tag{35}$$

Note, however, that accuracy of (35) is algebraic rather than exponential, which can be seen as follows. Equation (35) may be obtained from (33) by dropping term e^{-x} in brackets on the right-hand side. This term is exponentially small in the bulk of the ion layer, however it is not exponentially small in the vicinity of the point $x = 3/2$. Thus, the exponential-accuracy model, while predicting quantities in the bulk of the ion layer and at the surface with exponential accuracy, predicts the thickness of the ion layer with algebraic accuracy.

It is easy, however, to improve the accuracy. Equation (33) may be rewritten as

$$\begin{aligned}\Delta =& 2^{-1/2} \int_{3/2}^{\chi} \left(\sqrt{1+2x} - 2\right)^{-1/2} dx \\ &+ \int_{3/2}^{\chi} 2^{-1/2} \left[\left(\sqrt{1+2x} + e^{-x} - 2\right)^{-1/2} \right. \\ &\left. - \left(\sqrt{1+2x} - 2\right)^{-1/2}\right] dx.\end{aligned} \tag{36}$$

The first term on the right-hand side is given by (35). The function under the second integral decreases exponentially at $x \to \infty$, therefore one can, to the exponential accuracy, set the upper limit in this integral equal to infinity. The obtained integral represents a constant, its numerical value being -0.9013. Thus, one arrives, to the exponential accuracy, at the following corrected formula for the thickness of the ion layer

$$\Delta = \Delta^{ee} - 0.9013 \tag{37}$$

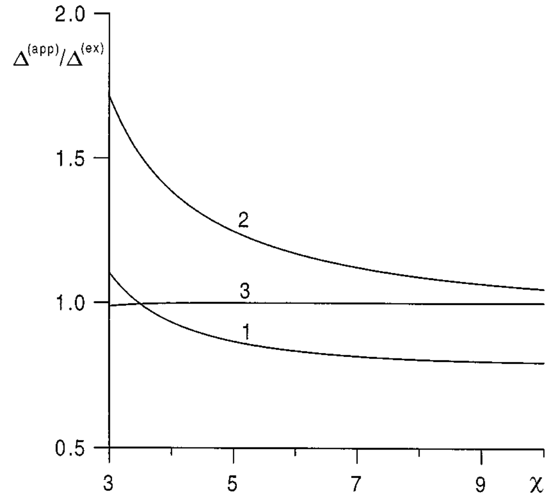

Fig. 4. Ratio of approximate values of the thickness of the ion layer given by different models to the exact value. 1: Child–Langmuir model. 2: Ion layer-quasi-neutral plasma model. 3: Exponential-accuracy model with correction.

where Δ^{ee} is a value of the thickness of the ion layer calculated in the framework of the exponential-accuracy model.

Comparison of results for the thickness of the ion layer obtained by means of the approximate and exact solutions is illustrated by Fig. 4, in which a ratio is shown of values given by (29), (34), and (37) to the exact value given by (33). One can see that the Child–Langmuir model provides a better accuracy than the ion layer-quasi-neutral plasma model in the range $\chi \lesssim 6$. At higher χ, a better accuracy is provided by the ion layer-quasi-neutral plasma model. However, the best accuracy in all the cases is provided by the exponential-accuracy model with correction, (37): the error is below 1.3% in the whole range of χ considered.

V. Concluding Remarks

At high values of the ratio χ of the sheath voltage to the electron temperature, the space-charge sheath comprises two sections: an ion-electron layer, which is an outer section of the sheath where the electron and ion densities are comparable, and an ion layer, in which the electron density is exponentially small as compared to the ion density. In a general case, an expansion describing the ion layer contains terms of algebraic orders in $1/\chi$, which originate from matching with an expansion describing the ion-electron layer. In other words, errors of algebraic orders in $1/\chi$ are introduced in models of the ion layer by boundary conditions at the edge of the layer. Modifying these conditions, one can improve accuracy of a model. In this work, a simple model with an exponentially small error is derived for the case of a collisionless sheath formed by cold ions. Equations describing the ion layer in this model are identical to those of the well-known Child–Langmuir model, however boundary conditions are different: they imply that the ions are accelerated in the ion-electron layer from the Bohm velocity to twice the Bohm velocity and that the voltage drop in the ion-electron layer equals $(3/2)(kT_e/e)$. The model predicts the electric field and ion velocity at the surface and the thickness of the ion layer to the accuracy of several percent for $\chi \geq 3$.

Comparing the above-described results with those of the previous authors cited in the Section I, one sees a resemblance between the physical picture found in this work and that of [3], provided that the terms "space-charge sheath region" and "transition region" of [3] are understood as referring to the ion layer and the ion-electron layer of the present work. Thus, the present treatment confirms the physics established in [3]. On the other hand, values 0.806 and 2.12 of the dimensionless ion velocity with which the ions enter the ion-electron layer and exit it, which have been obtained in [3] with the use of "gluing" of solutions in neighboring regions (i.e., matching them at one point), differ from values 1 and 2, arising from a regular asymptotic treatment.

Using the assumption [4] that the edge of the ion layer is defined by the condition $n_e = 0.1 n_s$, one finds that the local dimensionless potential and ion velocity equal $\Phi = -\ln 10 \approx -2.3$ and $V = \sqrt{1 + 2 \ln 10} \approx 2.4$, respectively. Again, these values differ from those derived in the present work, although the difference is not large.

The above results may be used also for analysis of numerical solutions obtained in the framework of more complex models. For example, in [5], numerical calculations were performed for collisionless or weakly collisional bounded plasmas. The sheath thickness was identified as a distance between the wall and a point at which the Bohm criterion is fulfilled, $V = 1$, and was found to be about four times larger than the value given by the Child–Langmuir model. In view of the present results, it should be noted that the thickness given by the Child–Langmuir model refers to the ion layer rather than to the sheath on the whole and should be compared with a numerical value defined as a distance between the wall and a point at which $V = 2$. For one of variants considered in [5] the latter distance can be taken from Fig. 3(d) [5] and is approximately 2.6, which is rather close to the respective Child–Langmuir value 2.5 cited in [5]. Note that this value is close also to that given by (37), which is 2.9 (for $\chi = 4.7$).

The above-mentioned boundary conditions, (24), have been derived from analysis of a solution in the ion-electron layer and are applicable also to problems in which collisions of ions and neutral particles are essential in the ion layer, provided that the collisions remain unessential in the ion-electron layer. The same applies to the formula for the thickness of the ion layer, (37); note that a dominating contribution to the second integral in (36) is given by a region in which $-\Phi = O(1)$, i.e., by the ion-electron layer. Consider, for example, the case when ions interact with neutral particles as rigid spheres and λ_i the mean free path for collisions of ions with neutral particles is comparable to the thickness of the ion layer. (Note that a solution describing the ion layer in this case in the framework of the fluid model with account of variable ion temperature and with the use of boundary conditions similar to (25) is given in [15].) In such a case, $\lambda_i = O(h\chi^{3/4})$ and the friction term of the momentum equation for the ion fluid is in the ion-electron layer of the order of $\chi^{-3/4}$ relative to the inertia and electric field terms. In such a situation, accuracy of (24) and (37) is likely to be $O(\chi^{-3/4})$, which, although no longer being exponential, is still higher than the accuracy of the first-approximation boundary conditions (25). One can derive also boundary conditions of enhanced accuracy, using an asymptotic treatment similar to the above.

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Mikhail S. Benilov received the Diploma and C.Sc. (Ph.D.) degrees in physics from the Moscow Institute for Physics and Technology in 1974 and 1978, respectively. In 1990, he received the degree of Doctor of Physical and Mathematical Sciences from the Institute for High Temperatures, USSR Academy of Sciences, Moscow. His thesis was concerned with the theory of electrostatic probes and electrodes in high-pressure flowing plasmas. He completed postgraduate courses at the Moscow Institute for Physics and Technology and at the Institute for Mechanics of Lomonosov Moscow State University in 1977.

He was with the Institute for High Temperatures, Russian Academy of Sciences, Moscow, where he led a group working in plasma and nonlinear physics, numerical modeling, and fluid dynamics. Since 1993, he has been Professor of Physics at the University of Madeira, Funchal, Portugal. In 1996–1998 and since June 2000 he has been serving as Chairman of the Department of Physics. His research interests include plasma physics (in particular, near-electrode phenomena and electrostatic probes), nonlinear physics, and fluid dynamics.

In 1990, Dr. Benilov was awarded the Alexander von Humboldt Research Fellowship and stayed for two years at the Ruhr-Universität Bochum, Germany, working on the theory of near-electrode phenomena.