The limits of the Bohm criterion in collisional plasmas

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(Received 27 January 2015; accepted 9 May 2015; published online 27 May 2015)

The sheath formation within a low-pressure collisional plasma is analysed by means of a two-fluid model. The Bohm criterion takes into account the effects of the electric field and the inertia of the ions. Numerical results yield that these effects contribute to the space charge formation, only, if the collisionality is lower than a relatively small threshold. It follows that a lower and an upper limit of the drift speed of the ions exist where the effects treated by Bohm can form a sheath. This interval becomes narrower as the collisionality increases and vanishes at the mentioned threshold. Above the threshold, the sheath is mainly created by collisions and the ionisation. Under these conditions, the sheath formation cannot be described by means of Bohm like criteria. In a few references, a so-called upper limit of the Bohm criterion is stated for collisional plasmas where the momentum equation of the ions is taken into account, only. However, the present paper shows that this limit results in an unrealistically steep increase of the space charge density towards the wall, and, therefore, it yields no useful limit of the Bohm velocity. *© 2015 AIP Publishing LLC*.
[http://dx.doi.org/10.1063/1.4921656]

I. INTRODUCTION

Tonks and Langmuir[1] treated the sheath edge between a quasi-neutral plasma and the sheath in front of a non-conducting wall. They took into consideration the positive column under free-fall conditions. The model of the quasi-neutral plasma ends at a point where the electric field would tend to infinity. It means the sheath begins there where the space charge has to be taken into account. However, this definition of the position of the sheath edge remains a little arbitrarily as shown in Ref. 1.

Dealing with a collisionless sheath between a quasi-neutral plasma and a negatively charged isolating wall, Bohm[2] has given an ingeniously simple criterion to define the sheath as the region where the ions had attained at least a drift speed υ_i large enough to form a space charge zone. The lower limit of this drift speed results in the well known Bohm velocity υ_B (see Sec. II). It means the sheath is formed in the region where $\upsilon_i > \upsilon_B$—in accord with Bohm's intention. The sheath edge is distinctly determined as the position where $\upsilon_i = \upsilon_B$ holds. The velocity distribution of the ions is taken into consideration at the sheath edge and within the sheath by Harrison and Thompson.[3] In this case, a similar but more general expression results for the sheath edge than that one given by Bohm.

Furthermore, it is well known that the basic equations of the much employed one-fluid model of a quasi-neutral plasma shows a singularity as the ion drift velocity attains the Bohm speed[4–6] where collisions between the ions and the neutral gas can be included.[4] In this case, the condition $\upsilon_i = \upsilon_B$ determines the position where the quasi-neutral description of the plasma breaks down. The Bohm criterion is obtained with an equality sign since the sheath is not included.

Forrest and Franklin[7] employed a two-fluid model and took into account the space charge density throughout the plasma and the sheath and collisionless and collisional conditions. The space charge density is small within the plasma region, but not zero. These results yield $\upsilon_i > \upsilon_B$ at the wall if the collisionality is weak and $\upsilon_i < \upsilon_B$ in the strongly collisional case. Consequently, the quasi-neutral approximation of this fluid model does not lead to precise results at the wall. In addition, the sheath begins gradually in collisional plasmas.

The usefulness of the Bohm criterion diminishes substantially as the influence of the collisions increases (e.g., Refs. 7, 8, 14, and 16). Several so called collisionally modified Bohm criteria (e.g., Refs. 14 and 16–24) were taken into consideration to obtain a relevant definition of the sheath edge under these conditions. However, under substantially collisional conditions, the sheath cannot be treated as infinitesimally small compared with the mean free path of the ions and other appropriate dimensions of the plasma. Therefore, in the strict sense of the word a so called collisionally modified Bohm criterion is no Bohm criterion (see also Ref. 25). In particular, this conclusion is valid if the effects taken into account by Bohm do not dominate the sheath formation. The statement by Tonks and Langmuir[1] defining the sheath edge is a non Bohm like criterion as well (see chap. 3). The present paper will analyse the contribution of several effects to the sheath formation within the plasma, especially the limits of the efficiency of these effects.

The Bohm criterion takes into account the various effects of the electric field on the ions and the electrons and the inertia of the ions, only. However, in low pressure discharges, the generation of ions and electrons and the collisions of the ions and the electrons with neutral particles have to be taken into consideration. The differences between these effects on the density distributions of the ions and the electrons contribute to the formation of a space charge density as well. The efforts to include these effects in the definition of the sheath edge have led to a controversial discussion and a lot of papers about this matter during several decades. We refer to a few textbooks[8,9] and several review papers,[10–16] mainly, where more details and references can be found.

The aim of the present paper is to determine the contribution of the called effects to the space charge density and the electric field within plasma, separately. We want to find out the range of parameters where the effects of the field and the ion inertia dominate the formation of the boundary sheath, really. A simple and clear physical model is used. Additional effects can be easily taken into account since the basic equations are well known.

To this end, in Sec. II the basic equations, the boundary conditions, and a few analytical results are specified. Numerical results are given in Sec. III. The assertions in the literature about the existence of an upper limit of the Bohm velocity are critically analysed in Sec. IV. Conclusions are drawn in Sec. V. A few further details are mentioned in the Appendix.

II. MODEL AND BASIC EQUATIONS

A one-dimensional two-fluid model for a compressible media is taken into consideration (e.g., Refs. 7–10, 12, and 14–16). The plasma is assumed to consist of electrons and singly charged positive ions in an immobile neutral gas. A planar geometry with insulating walls is treated. Steady-state conditions and homogeneity parallel to the walls are assumed to prevail. The electrons and ions are generated by electron collisions within the plasma. They recombine at the wall. The walls are charged negatively with respect to the plasma. The inertia of the ions and elastic collisions between ions and atoms and the charge exchange are taken into account. The inertia of the electrons is neglected. The neutral gas density and the electron temperature T_e are taken as constant. The ion temperature is assumed to be zero. The space charge is taken into account. Electrons arrive at the wall due to their thermal energy. The ions are accelerated by the electric field towards the wall. They are retarded by collisions with the neutral gas and their mass.

The basic equations for the ions are the equation of continuity

$$(\mathrm{d}/\mathrm{d}x)n_i\upsilon_i = n_e\nu_{ni}, \tag{1}$$

and the equation of momentum transfer

$$n_i m_i \upsilon_i(\mathrm{d}\upsilon_i/\mathrm{d}x) = n_i eE - m_i\upsilon_i(n_e\nu_{ni} + n_i\nu_{ic}). \tag{2}$$

Here, x denotes the distance from the mid-plane, n_i and n_e are the number densities of the ions and the electrons, respectively, υ_i is the drift velocity of the ions, ν_{ni} is the ionisation frequency, ν_{ic} is the collision frequency between ions and neutral atoms, m_i is the ion mass, and e is the positive elementary charge, and in the following, k is the Boltzmann constant and ε_0 is the permittivity of the vacuum. The equation of the momentum transfer of the electrons can be reduced to

$$kT_e(\mathrm{d}/\mathrm{d}x)n_e = -n_e eE. \tag{3}$$

The electric field is determined by the Poisson equation

$$\varepsilon_o(\mathrm{d}/\mathrm{d}x)E = e(n_i - n_e). \tag{4}$$

From Eq. (1) and the corresponding equation of the electrons, we obtain

$$n_e\upsilon_e = n_i\upsilon_i, \tag{5}$$

and from Eqs. (1) and (2) follows:

$$\upsilon_i(\mathrm{d}n_i/\mathrm{d}x) = -n_i eE/m_i\upsilon_i + 2n_e\nu_{ni} + n_i\nu_{ic}. \tag{6}$$

At the mid-plane, i.e., $x = 0$, the boundary conditions read

$$\mathrm{d}n_i/\mathrm{d}x = \mathrm{d}n_e/\mathrm{d}x = 0, \quad \upsilon_i = \upsilon_e = 0, \quad E = 0. \tag{7}$$

Subsequently, $n_i(0)$ and $n_e(0)$ are written n_{i0} and n_{e0}, where n_{e0} has to be given. The relation n_{i0}/n_{e0} can be derived from Eqs. (1)–(7) (see, for instance, Refs. 2 and 6). At the wall, i.e., $x = L$, the boundary condition[7,8,16]

$$\upsilon_e(L) = \upsilon_{ew} \quad \text{with} \quad \upsilon_{ew} = (kT_e/2\pi m_e)^{1/2} \tag{8}$$

determines the floating potential. Equation (8) is a convenient approximation, in spite of the fact that it is something incorrect. A detailed description of the conditions adjacent to a wall and relevant references can be found, for instance, by Franklin[8] and also in Ref. 7.

The complete set of the basic Eqs. (1)–(4) with the relevant boundary conditions (7) and (8) can be solved numerically, only. However, a few features can be detected already analytically. The region in which the sheath is formed can be described by $\mathrm{d}E/\mathrm{d}x > 0$ and by means of $(\mathrm{d}/\mathrm{d}x)(n_i - n_e) > 0$ as well. Perhaps, the second condition is a little more relevant than the first one, since E tends to infinity as υ_i tends to υ_B when a quasi-neutral plasma is taken into account (e.g., Refs. 7–9).

Solving Eqs. (1)–(4) by power series in x one obtains (e.g., Refs. 7, 8, and 16)

$$(n_{i0}/n_{e0} - 1) = (2\,n_{e0}/n_{i0} + \nu_{ic}/\nu_{in})\,{\nu_{in}}^2/{\omega_{pi}}^2 > 0, \tag{9}$$

at the mid-plane where $\omega_{pi} = (e^2 n_{e0}/\varepsilon_0 m_i)^{1/2}$ is the ion plasma frequency.[9] Due to (9) Eq. (4) yields $\varepsilon_0(dE/dx)_{x=0} > 0$ and further $E = (e/\varepsilon_0)(n_{i0} - n_{e0})\cdot x$ if $0 < x/L \ll 1$. It is known that $n_i/n_e - 1$ is very small within a weakly collisional plasma out of the sheath.

Equations (3), (6), and (4) yield

$$\frac{\varepsilon_0}{e}\frac{\mathrm{d}^2E}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}(n_i - n_e) = \frac{n_i eE}{m_i{\upsilon_i}^2}\left(\frac{{\upsilon_i}^2}{{\upsilon_c}^2} - 1\right) + \frac{2n_e\nu_{ni} + n_i\nu_{ic}}{\upsilon_i}, \tag{10}$$

where $\upsilon_c = (n_i/n_e)^{1/2}\upsilon_B$ is a critical speed[15,16] with the Bohm velocity $\upsilon_B = (kT_e/m_i)^{1/2}$. The velocity υ_c was interpreted as the modified Bohm velocity[15,16] or as the local ion sound speed[19] with the non-neutrality taken into account. One can expect that the sheath is formed there where the space charge density $\rho = e(n_i - n_e)$ increases steeply from small positive values in the plasma core to large values in front of the wall. Both of the terms on the RHS of Eq. (10) should be positive therewith a steep gradient of ρ can exist.

However, the first term is positive only if

$$\upsilon_i > \upsilon_c. \tag{11}$$

Dealing with an almost quasi-neutral and collisionless plasma core, the inequality (11) is reduced to the Bohm criterion at the sheath edge. Bohm had taken into consideration the first term on the RHS of (10), only. In this case, Eqs. (1)–(4) and (10) can be integrated analytically (e.g., Refs. 2, 8, 9, and 12) (see the Appendix). The second term on the RHS of the Eq. (10) is everywhere positive between the centre and the wall. It occurs when the ionisation or/and collisions between ions and neutral particles are taken into account. Under conditions in which the first term is negative across the section of the plasma a sheath can be originated as well if the absolute amount of this term is substantially smaller than the amount of the second term. The derivative (10) of n_i–n_e, further n_i–n_e itself and E will be calculated numerically as functions of x in Sec. III.

Due to (7) and (10) the space charge density ρ has a minimum at $x=0$ where (9) and

$$(eE/m_i\upsilon_i)_{x=0} = 2\nu_{ni}(n_{e0}/n_{i0}) + \nu_{ic} \tag{12}$$

hold. In addition, at the mid-plane the curvature of ρ is positive as derived analytically in Ref. 16.

In Sec. III, it will be shown that the condition (11) is not necessary to form a positive sheath in front of the wall if the second term on the RHS of (10) is large sufficiently. In this case, a sheath can occur even if the first term on the RHS of (10) is negative everywhere between the centre and the wall.

III. NUMERICAL RESULTS

We introduce the Debye length $\lambda_D = (\varepsilon_0 kT_e/n_e e^2)^{1/2}$, the lengths $\lambda_{ni} = \upsilon_B/\nu_{ni}$, $\lambda_{ic} = \upsilon_B/\nu_{ic}$, the parameters $a_0 = (\lambda_D/\lambda_{ni})^2$, $K = \lambda_{ni}/\lambda_{ic}$, and the normalised variables $\xi = x/\lambda_{ni}$, $N_i = n_i/n_{e0}$, $N_e = n_e/n_{e0}$, $V_i = \upsilon_i/\upsilon_B$, $V_e = \upsilon_e/\upsilon_B$, and $F = E/E_0$, where $E_0 = kT_e/e\lambda_{ni}$. The normalised version of the basic equations and the boundary conditions can easily be derived from Eqs. (1)–(7). For instance, instead of Eqs. (4) and (10), we obtain

$$\mathrm{d}F/\mathrm{d}\xi = (N_i - N_e)/a_0, \tag{13}$$

$$(\mathrm{d}/\mathrm{d}\xi)(N_i - N_e) = B + C, \tag{14}$$

$$\text{where } B = (FN_i/V_i^2)(V_i^2/V_c^2 - 1), \tag{15}$$

$$C = (2N_e + KN_i)/V_i. \tag{16}$$

Here, $V_c = (N_i/N_e)^{1/2}$ can be considered as the normalised critical speed. B describes the contributions of the electric field and the ion inertia to the formation of a positively charged boundary sheath as treated by Bohm. C gives the effects of the collisions and the ionisation processes to the generation of a positive space charge density. The complete set of the normalised equations and several numerical results can be found in Refs. 7, 8, and 16. In the present paper, B and C are investigated in the main. By the way, the abbreviations B and C are to put in mind of Bohm and collisions, respectively.

Setting $\xi = 0$ then B tends to $-\infty$ and C to $+\infty$. The sum $B + C$ results in zero[16] at the mid-plane. It is known that there $0 < N_i(0) - N_e(0) \ll 1$ follows, where $N_e(0) = 1$ (e.g., Refs. 7, 8, and 16). The numerical results shown in Figures 1–5 are obtained for an argon plasma where $m_i/m_e = 7.25 \times 10^4$ and $V_{ew} = \upsilon_{ew}/\upsilon_B = (m_i/m_e 2\pi)^{1/2} = 108.5$. The sign of B is determined by the relation V_i/V_c. This relation as function of x/L is plotted in Ref. 16. Figures 1 and 2 depict B and C as functions of x/L and V_i with a_0 and K as the parameters. $B \geq 0$ holds in an interval between the two zero points $V_i = V_{i1}$ and $V_i = V_{i2}$ of B. Owing to Eq. (15), the zero points follow from $V_i^2 = V_c^2$ if the collisionality K is small. The results are almost $V_{i1} = 1.009$ at the sheath edge and approximately $V_{i2} = 2.543$ in front of the wall if $K = 0$. The interval becomes narrower and the magnitude of B decreases as K increases. Figures 1(b) and 2(b) allow to compare the contributions of both the terms B and C to the formation of the sheath. The contribution of the first term dominates in a weakly collisional plasma, only.

It is interesting to see that B and accordingly $V_i/V_c - 1$ are positive, only, if K is smaller than a threshold K_{thr}. Therefore, in the domain $B \leq 0$ the formation of the space charge density is distinctly dominated by C, which means by

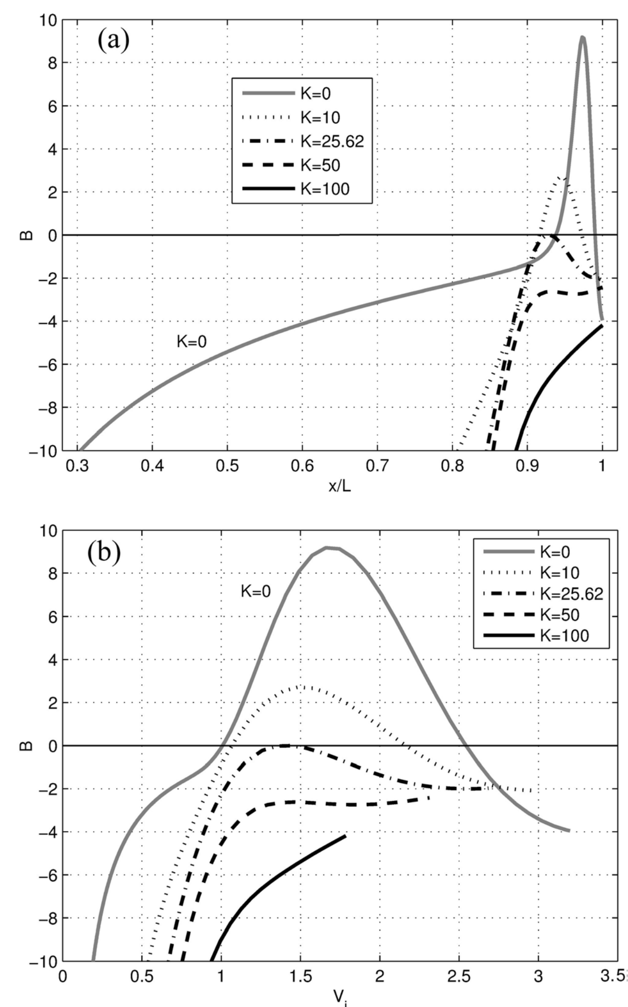

FIG. 1. The dimensionless effect B of the electric field (Bohm effect) as functions of x/L (a) and the dimensionless ion drift speed V_i (b) for $a_0 = 10^{-5}$ with K as the parameter.

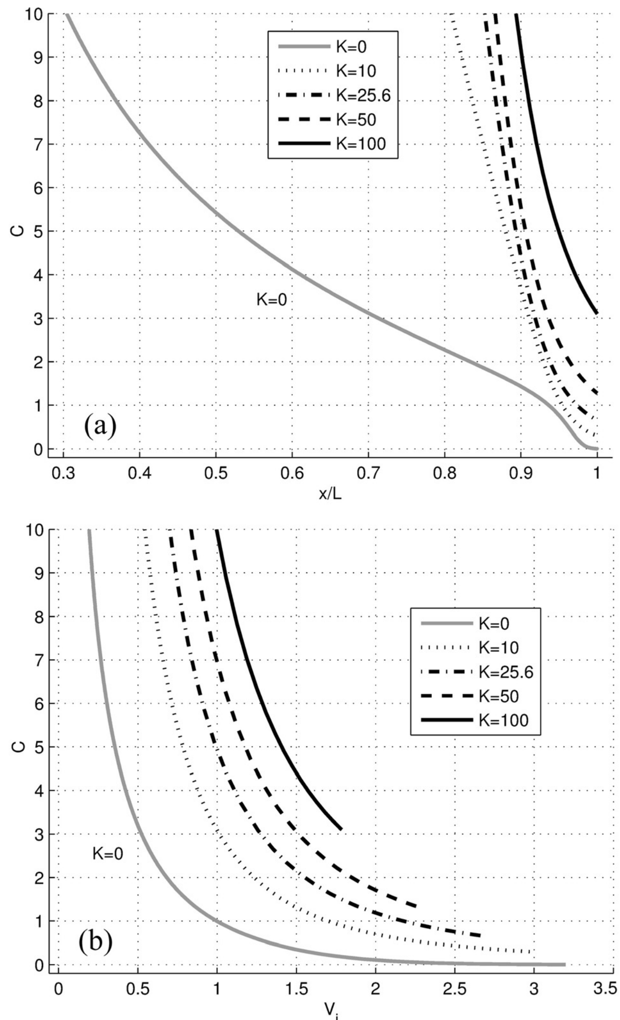

FIG. 2. The dimensionless effect C of the ionisation and the collisions as functions of x/L (a) and the dimensionless ion drift speed V_i (b) for $a_0 = 10^{-5}$ with K as the parameter.

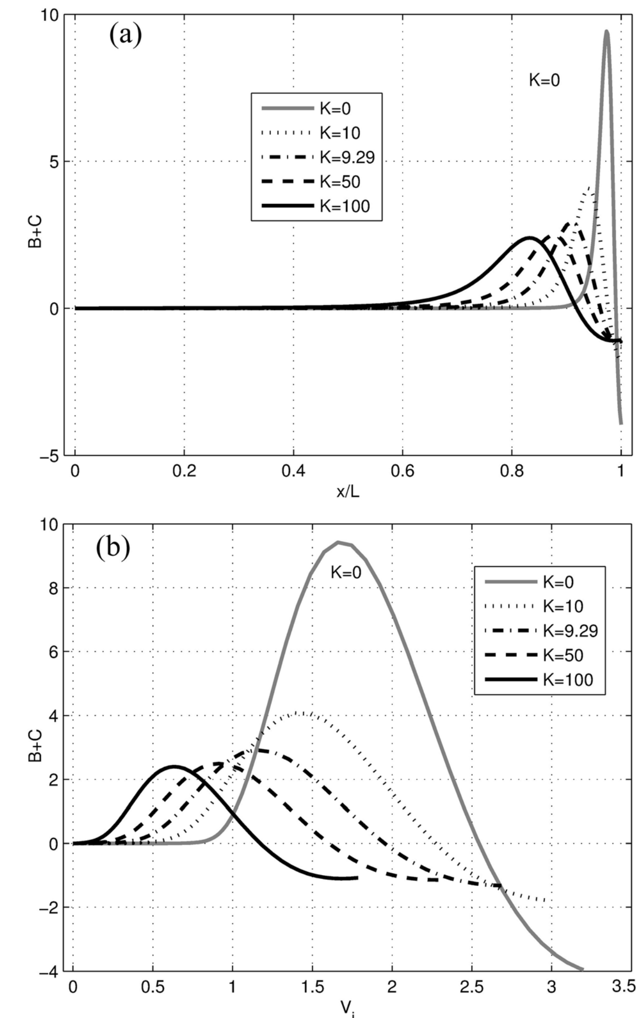

FIG. 3. The dimensionless increase $B + C$ of the space charge density as functions of x/L (a) and the dimensionless ion drift speed V_i (b) for $a_0 = 10^{-5}$ with K as the parameter.

the generation of ions and electrons and collisions between ions and atoms. No Bohm criterion exists if $B \leq 0$ holds throughout the plasma.

Furthermore, in the domain $0 \leq K \leq K_{thr}$, the critical velocity V_c actually increases as K increases (Figure 1). The extension of the space charge region to smaller V_i values with rising K is caused by the collisions, alone (Figure 4).

The basic equations and $\mathrm{d}B/\mathrm{d}\xi = 0$ show that the maximum of B depends on K and via F or E also on a_0. It is independent of m_i/m_e. Setting $K = K_{thr}$ the maximum of B amounts $B = 0$ where $V_c/V_i = 1.0$ and $V_i = V_{i1} = V_{i2}$ follow. Using $a_0 = 10^{-4}$, 10^{-5}, and 10^{-6} the results are $K_{thr} = 9.295$, 25.62, and 66.88 and $V_i = 1.419$, 1.408, and 1.404, respectively. Due to ${V_i}^2 = {V_c}^2 = N_i/N_e$, all these examples lead to almost $N_e/N_i = 0.50$.

As a degree for the collisionality is used $\lambda_D/\lambda_{ic} = K\sqrt{a_0}$ as well (e.g., Refs. 14 and 16). Taking into account $a_0 = 10^{-4}$, 10^{-5}, and 10^{-6} and the appropriate values of K_{thr}, we obtain for the collisionality at the threshold $\lambda_D/\lambda_{ic} = 0.093$, 0.081, 0.067. These results imply that above these relatively small values of the collisionality, no Bohm like criterion can occur. For this reason, it is understandable that other statements[14–16,23,24] were used to define a reasonable beginning of the sheath, at least approximately.

Figure 3 plots $B + C$ as functions of x/L and V_i with K as the parameter and $a_0 = 10^{-5}$. The derivative (14) of N_i–N_e,

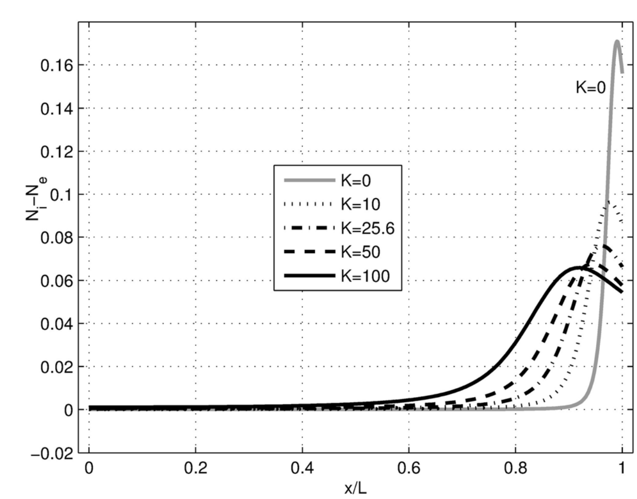

FIG. 4. The dimensionless space charge density $N_i - N_e$ as function of x/L for $a_0 = 10^{-5}$ with K as the parameter.

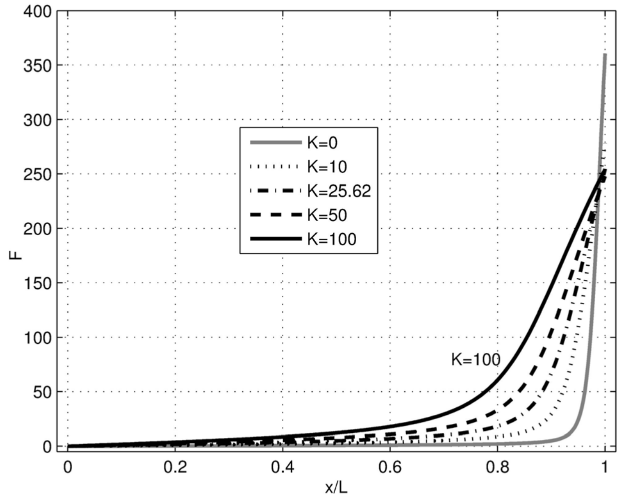

FIG. 5. The dimensionless electric field strength F as function of x/L for $a_0 = 10^{-5}$ with K as the parameter.

that means $B + C$, has a minimum at $x = 0$, passes through a maximum within the sheath and becomes negative in front of the wall. Figures 4 and 5 depict $N_i - N_e$ and F as functions of x/L with K as parameter. The space charge density, normalised as $N_i - N_e$, increases up to a maximum close to the wall as x grows and is positive everywhere between the centre and the wall. According to that the electric field increases monotonically from the centre to the wall. Comparing Figures 3(a) and 4(a) with Figure 5, it can be seen that $N_i - N_e$ and F start to increase steeply towards the wall in an interval in which $(\mathrm{d}/\mathrm{d}x)(N_i - N_e)$ grows steeply equally.

Further results of N_i, N_e, $N_i - N_e$, V_i, and F or the electric potential can be found, for instance, in Refs. 7, 8, 15, and 16.

It can be seen from Eq. (13) that $a_0 \to 0$ leads to the quasi-neutral approximation. It is known[8,12] that instead of λ_{ni} also λ_{ic} or L can be used to define a_0, the other parameters, and the normalised variables.

Moreover, Tonks and Langmuir (Ref. 1, p. 902) define the sheath edge at a point where the LHS of Poisson's equation (4) becomes equal to a certain fractional part of either of the terms on the RHS. For example, this statement can be written as $\delta = n_i/n_e - 1$ with a suitable factor δ. Taking into consideration, collisionless and weakly collisional plasmas $\delta \ll 1$ is relevant, only. In this case, $a_0 = 10^{-5}$, $K = 0$, and $V_i = 1$ lead to $\delta = 0.017$. A given set of a_0, K, and δ yields the corresponding results of V_i and x/L at the sheath edge. A few other non-Bohm like sheath criteria are treated numerically in Ref. 16.

IV. ADDITIONAL CONDITIONS WITHIN THE PLASMA

We introduce $\Omega_1 = \nu_{ni}(n_e/n_i) + \nu_{ic}$ and $\Omega_2 = 2\nu_{ni}(n_e/n_i) + \nu_{ic}$ as characteristic frequencies. In a plasma investigated here $\mathrm{d}\upsilon_i/\mathrm{d}x > 0$, $\mathrm{d}n_i/\mathrm{d}x < 0$, and $\mathrm{d}E/\mathrm{d}x > 0$ hold. From Eqs. (2), (6), and (4), one obtains

$$eE/m_i\upsilon_i > \Omega_1, \tag{17}$$

$$eE/m_i\upsilon_i > \Omega_2, \tag{18}$$

and $n_i > n_e$. Both the inequalities (17) and (18) were taken into consideration throughout the plasma in the papers.[15,16]

Using the limiting value with the equal sign, the relation (17) yields the well known ion drift speed $\upsilon_i = eE/m_i\Omega_1$ through a plasma if the ion motion is in collisional equilibrium with the field (e.g., Ref. 25). Due to $\mathrm{d}\upsilon_i/\mathrm{d}x = 0$, this velocity is constant. Further, $\mathrm{d}n_i/\mathrm{d}x = n_e\nu_{ni}/\upsilon_i$ follows. Using the equal sign (18) yields a constant n_i throughout the plasma, and $\mathrm{d}\upsilon_i/\mathrm{d}x = \nu_{ni}n_e/n_i$. Alone n_e would decrease with rising x in both the cases.

Taking into account the limiting cases, then both the relations (17) and (18) lead to an unrealistically steep increase of the space charge density as x grows.[16] Therefore, these limiting values yield no useful upper limits of the Bohm velocity in contrast to conclusions given in the literature.[20–22] This hint is valid for collisionless and collisional plasmas. In addition, the inequalities (17) and (18) are self-evident under the conditions of the treated plasmas.

Furthermore, (17) and (18) lead to lower limits of E/υ_i, and not to separated values of E and υ_i. A solution of the complete set of the relevant equations is necessary to give a correct relation between E and υ_i. The Bohm criterion valid in the collisionless case does not depend on the electric field, explicitly, and hence it can yield a definite velocity alone.

However, there are plasmas where a region with a negative space charge density can exist between a positive plasma core and the boundary sheath. Such plasmas can occur if the generation of charged particles is localised on a narrow region at the centre or the ionisation profile decreases steeply in a small zone.[26,28] These effects can be amplified by a recombination in the volume.[8,27] It means conditions can occur where (17), (18), and $n_i > n_e$ do not hold throughout the whole plasma.

V. CONCLUSIONS

The Bohm criterion takes into account the effects of the electric field and the ion inertia on the formation of the positive space charge sheath in front of a negatively charged wall, only. In a real plasma collisions between ions and neutral particles and the generation of ions and electrons contribute to the sheath formation as well. The analysis of a simple much employed two-fluid model yields that the effects treated by Bohm contribute to the sheath formation, only, if the collisionality $K = \lambda_{ni}/\lambda_{ic}$ or $\lambda_D/\lambda_{ic} = K\sqrt{a_0}$ is lower than a relatively small threshold. The sheath formation is mainly determined by collisions and the ionisation above the threshold. These effects cause the extension of the sheath towards the centre and to ion drift velocities smaller than the Bohm speed as K increases.

The effects taken into account by Bohm create a positive space charge in an interval where $\upsilon_i \geq \upsilon_B \times (n_i/n_e)^{1/2}$. Consequently, a lower and an upper limit of the drift speed of the ions exist where the effects treated by Bohm can form a sheath. Outside this interval the effect of the electric field decreases the increase of the space charge density from the centre to the wall. The interval becomes narrower as K increases and vanishes if K is greater than the mentioned threshold. Therefore, it is not relevant to denote a sheath criterion as a modified Bohm criterion if the influence of the collisions is larger than the mentioned threshold. It is understandable that other statements were taken into consideration to describe the onset of the sheath conveniently.

During the last years from time to time an upper limit of the Bohm velocity is taken into consideration for collisional plasmas. This value follows from the momentum equation of the ions. However, it can be easily seen that this limit results in an unrealistically steep increase of the space charge density towards the wall. Therefore, this limit yields no useful upper limit of the Bohm speed in collisional plasmas.

ACKNOWLEDGMENTS

The authors thank the referee for useful comments.

APPENDIX: REMARKS TO THE SHEATH EDGE IN A COLLISIONLESS PLASMA, TO THE QUASI-NEUTRAL PLASMA MODEL AND THE RELATIONS TO A GENERAL PRESSURE BALANCE

It is known[2,8,9] that the Eq. (10) can be integrated analytically together with Eqs. (1)–(6) if $\nu_{in} = \nu_{ic} = 0$. Using these conditions, Bohm[2] had treated the sheath edge adjacent to a quasi-neutral plasma core. The sheath edge is assumed to be on $x = x_s = 0$ with $n_i(0) = n_e(0) = n_s$, $\upsilon_i(0) = \upsilon_s$, $E(0) = 0$, and $\Phi(0) = 0$, where Φ is the electrical potential. As usual, there are $\mathrm{d}E/\mathrm{d}x = -\phi$ and $\Phi(x) \leq 0$ if $x \geq 0$. Equations (3), (2), and (6) or (5) yield $n(x)/n_s = \exp(e\phi/kT_e)$, $(\upsilon_i/\upsilon_s)^2 = 1 - 2e\phi/m_i\upsilon_s^2$, and $n_i/n_s = (1 - 2e\phi/m_i\upsilon_s^2)^{-1/2}$. Therewith Eq. (10) is integrated once. Expanding the results as power series in Φ at the sheath edge leads to $(\varepsilon_0/en_s)(\mathrm{d}E/\mathrm{d}x) = (e\phi/m_i\upsilon_s^2)(1 - \upsilon_s^2/\upsilon_B^2)$, where $\mathrm{d}E/\mathrm{d}x > 0$ if $\upsilon_s > \upsilon_B$ (e.g., Ref. 8). Following Bohm and integrating the term $E\mathrm{d}E/\mathrm{d}x$ yields $\varepsilon_0 E^2 = (n_s e^2\phi^2/m_i\upsilon_s^2)(\upsilon_s^2/\upsilon_B^2 - 1)$ where E^2 is real if $\upsilon_s > \upsilon_B$. In all the terms $(\mathrm{d}/\mathrm{d}x)(n_i - n_e)$, $\mathrm{d}E/\mathrm{d}x$, and E^2 the factors determining the sign are the same. In this case, the sheath edge can be clearly defined by $\upsilon_i = \upsilon_B$. The sheath is formed in the region where $\upsilon_i > \upsilon_B$. The space charge and the electric field are positive throughout the sheath. Corresponding with Bohm's intent the sheath edge describes the beginning of the sheath at the end of the quasi-neutral plasma core.

The inconsistencies as $E = 0$ at $x \leq 0$ existing in the model used by Bohm can be eliminated joining a presheath (e.g., Refs. 8, 9, 12, 13, and 29).

Moreover, it is known[8,9] that in the quasi-neutral model of the plasma, i.e., $n_i = n_e = n$ and $\upsilon_i = \upsilon_e = \upsilon$, the singularities of the basic equations can be easily derived from the Eqs. (1)–(4). Adding (2) and (3), the electric field E is eliminated and by means of (1) follow: $\mathrm{d}n/\mathrm{d}x \to -\infty$ and $\mathrm{d}\upsilon/\mathrm{d}x \to \infty$ if $\upsilon_i \to \upsilon_B$. Using (6) and (3), setting $(\mathrm{d}/\mathrm{d}x)(n_i - n_e) = 0$ and restricting to $\upsilon_i < \upsilon_B$ yields[4,8,16]

$$(eE/m_i\upsilon_i) = (2\nu_{ni} + \nu_{ic})/(1 - \upsilon_i^2/\upsilon_B^2). \tag{A1}$$

A pressure balance at the beginning of the sheath is given in Refs. 12 and 18. It is shown in Ref. 29 that a general pressure balance throughout the plasma and the sheath can be easily derived. By means of (1), Eq. (2) can be written as

$$(d/dx)n_i m_i \upsilon_i^2 = n_i eE - m_i\upsilon_i n_i \nu_{ic}. \tag{A2}$$

We add the momentum equations of the ions (A2), the electrons (3), and the neutral atoms and eliminate the space charge density ρ by means of the Poisson equation (4). Integrating this sum from the mid-plane outwardly leads to

$$n_i m_i \upsilon_i^2 + n_e kT_e + n_n m_n \upsilon_n^2 - \varepsilon_0 E^2/2 = n_{e0} kT_e, \tag{A3}$$

where n_{e0} is the electron density on the mid-plane. The index n denotes the neutral gas. The pressures of the ion gas and the neutral gas have to be added to (A3) if the temperatures of both the gases do not vanish. Under steady-state conditions, $n_i\upsilon_i + n_n\upsilon_n = 0$ holds. More general results with respect to (A3) as given here can be found in Ref. 30.

In[24] $\varepsilon_0 E^2/2 > \delta n_e kT_e$ with a factor $\delta < 1$ was proposed as a sheath criterion, different from Bohm's criterion (see also Refs. 15, 16, and 23). This statement to define the sheath edge is similar to that one used by Tonks and Langmuir.[1] Furthermore, assuming $\upsilon_n = 0$ and taking into account Bohm's sheath model then near the beginning of the sheath the pressure balance (A3) yields the same result of E^2 as given above in this chapter.

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