Asymptotically correct collisional presheaths

Geoffrey L. Main
School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332

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Few exact solutions for collisional presheaths exist because of the difficulty of simultaneously satisfying both the collisional Boltzmann equation and the Poisson equation. The exact solutions that do exist are for very specialized collision terms such as constant cross-section charge exchange with cold neutrals. The present paper presents an asymptotic method which is applicable to a variety of collision terms and is applied in particular to constant collision frequency charge exchange with noncold neutrals. Constant collision frequency and constant cross-section collision with cold neutral results are also presented. The first-order terms for the presheath potential rise and ion distribution functions are calculated and it is shown that second- and higher-order terms can be calculated using a multiexponential expansion for presheath potential rise. The first-order cold neutral constant cross-section results correspond well to the exact solution. The calculated presheath potential rises are of the order expected from the Bohm criterion, and in some of the specialized cold neutral cases, exactly $kT_e/2$. The presheath potential rise is reduced by a neutral plasma potential gradient which accelerates ions toward the presheath. In all cases the collisional presheath is asymptotically matched to both the neutral plasma and the collisionless sheath.

I. INTRODUCTION

The majority of plasma–surface interaction work matches a neutral plasma to a collisionless sheath without detailed consideration of a collisional presheath. However, the collisional presheath structure is of great interest. Sheath theory, beginning with Bohm,[1] tends to assume that the plasma ion distribution is cold so that a minimum presheath potential rise may be calculated, which makes the collisionless sheath self-consistent. Harrison and Thompson[2] generalize the Bohm criterion to noncold ion distributions; however, the result is sensitive to the density of the low energy tail of the ion distribution, which in turn is strongly affected by the collisional presheath. And, a second difficulty in the absence of a collisional presheath is that the collisionless sheath and the surface beyond it may return no ions or a nonthermal distribution of ions which the collisional presheath must match to the neutral plasma region.

Some exact solutions exist for presheaths; notable is the work of Ecker and Kanne[3] and Riemann,[4] who derive exact solutions for collision terms based on charge exchange with cold neutrals and Emmert *et al.*,[5] who derive an exact collisionless solution in which there is an ionization source. In the present paper an asymptotically correct collisional presheath theory is developed which can be applied to a less restrictive range of collision terms. Potential in the presheath is expanded as a multiexponential series and the distribution functions are expanded in terms of presheath potential rise. First-order approximations are calculated for both constant collision frequency and constant cross-section charge exchange collisions.

II. FIRST-ORDER ASYMPTOTIC POTENTIAL FORMULATION

In this section it is assumed that the potential in the collisional presheath is of the form

$$U = U_0 + \Delta U = \alpha x + e^{\beta x}, \tag{1}$$

where $U_0 = \alpha x$ is the assumed linear potential in the neutral plasma and $\Delta U = e^{\beta x}$ is the additional potential rise in the collisional presheath, as shown in Fig. 1. In this paper the convention used is that $U = q\phi$, where q is the electron charge and ϕ is potential in electron volts so that U has units of energy. In addition, potential is defined in the reverse of the usual sign convention so that increasing potential repels electrons. With these conventions, the Boltzmann equation can be written as

$$\frac{dU}{dx}\left(v\frac{\partial f}{\partial U} \pm \frac{1}{m}\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial t}\right)_c. \tag{2}$$

In Eq. (2) and those following, the $\pm$ denotes the sign of the charged species in question; the upper sign refers to positively charged ions and the lower sign to electrons. The Boltzmann equation is expressed in terms of ΔU, which will be the expansion variable in the presheath:

$$v\beta\Delta U\frac{\partial f}{\partial \Delta U} \pm \frac{1}{m}(\beta\Delta U + \alpha)\frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial t}\right)_c. \tag{3}$$

The distribution function is then expanded as

$$f = f_0(v) + \Delta U f_1(v) + \Delta^2 f_2(v) + \cdots, \tag{4}$$

so that the derivatives are

$$\frac{\partial f}{\partial \Delta U} = f_1(v) + 2\Delta U f_2(v) + 3\Delta U^2 f_3(v) + \cdots \tag{5}$$

and

$$\frac{\partial f}{\partial v} = \frac{\partial f_0}{\partial v}(v) + \Delta U\frac{\partial f_1}{\partial v}(v) + \Delta U^2\frac{\partial f_2}{\partial v}(v) + \cdots. \tag{6}$$

Substitution of (5) and (6) into the Boltzmann equation (3) yields the terms

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$$1: \quad \pm \frac{\alpha}{m} \frac{\partial f_0}{\partial v}(v) = \left[\left(\frac{\partial f}{\partial t}\right)_c\right]_1, \tag{7a}$$

$$\Delta U: \quad v\beta f_1(v) \pm \frac{\beta}{m} \frac{\partial f_0}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_1}{\partial v}(v) = \left[\left(\frac{\partial f}{\partial t}\right)_c\right]_{\Delta U}, \tag{7b}$$

$$\Delta U^2: \quad 2v\beta f_2(v) \pm \frac{\beta}{m} \frac{\partial f_1}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_2}{\partial v}(v) = \left[\left(\frac{\partial f}{\partial t}\right)_c\right]_{\Delta U^2}, \tag{7c}$$

$$\vdots$$

$$\Delta U^n: \quad nv\beta f_n(v) \pm \frac{\beta}{m} \frac{\partial f_{n-1}}{\partial v}(v) \pm \frac{\alpha}{m} \frac{\partial f_n}{\partial v}(v) = \left[\left(\frac{\partial f}{\partial t}\right)_c\right]_{\Delta U^n}. \tag{7d}$$

The quantity β, representing the presheath potential rise, is determined from the Poisson equation

$$\frac{d^2U}{dx^2} = 4\pi q^2\left(\int_{-\infty}^{\infty} f_i(v,\Delta U)dv - \int_{-\infty}^{\infty} f_e(v,\Delta U)dv\right), \tag{8}$$

where q is the electron charge. It is assumed that the ions are singly ionized for simplicity. The Poisson equation (8) is expanded as

$$\beta^2\Delta U = 4\pi q^2\left[\left(\int_{-\infty}^{\infty} f_{i1}(v)dv - \int_{-\infty}^{\infty} f_{e1}(v)dv\right)\Delta U + \left(\int_{-\infty}^{\infty} f_{i2}(v)dv - \int_{-\infty}^{\infty} f_{e2}(v)dv\right)\Delta U^2 + \cdots\right], \tag{9}$$

where charge neutrality at $\Delta U = 0$ has eliminated the terms containing f_{i0} and f_{e0}:

$$n_0 = \int_{-\infty}^{\infty} f_{i0}(v)dv = \int_{-\infty}^{\infty} f_{e0}(v)dv.$$

The quantity n_0 is the neutral plasma density of the asymptotic presheath, not of the neutral plasma.

III. FIRST-ORDER SOLUTION WITH A CONSTANT COLLISION FREQUENCY CHARGE EXCHANGE COLLISION TERM

The constant collision frequency charge exchange collision term is modeled as

$$\left(\frac{\partial f_i}{\partial t}\right)_c = \frac{1}{\tau n_n}\left(f_n(v)\int_{-\infty}^{\infty} f_i(u)du - f_i(v)\int_{-\infty}^{\infty} f_n(u)du\right), \tag{10}$$

where $f_n(v)$ is the neutral distribution and τ is the collision time. Previous work has assumed cold neutrals and results in an integral equation which is solvable only for constant collision cross section.[4]

A. Zero plasma potential gradient ($\alpha = 0$)

In this case Eqs. (7) become

$$\begin{aligned}
1:&\quad 0 = \frac{1}{\tau n_n}\left(f_n(v)\int_{-\infty}^{\infty} f_{i0}(u)du - f_{i0}(v)\int_{-\infty}^{\infty} f_n(u)du\right),\\
\Delta U:&\quad v\beta f_{i1}(v) + \frac{\beta}{m}\frac{\partial f_{i0}}{\partial v}(v) = \frac{1}{\tau n_n}\left(f_n(v)\int_{-\infty}^{\infty} f_{i1}(u)du - f_{i1}(v)\int_{-\infty}^{\infty} f_n(u)du\right),\\
&\quad\vdots\\
\Delta U^n:&\quad nv\beta f_{in}(v) + \frac{\beta}{m}\frac{\partial f_{i(n-1)}}{\partial v}(v) = \frac{1}{\tau n_n}\left(f_n(v)\int_{-\infty}^{\infty} f_{in}(u)du - f_{in}(v)\int_{-\infty}^{\infty} f_n(u)du\right).
\end{aligned} \tag{11}$$

Under the assumption that the neutral distribution is Maxwellian $f_n(v) = n_n\sqrt{m/2\pi kT}\,\exp(-mv^2/2kT)$, the solution to (11) is

$$\begin{aligned}
f_{i0} &= Cf_n(v),\\
f_{i1}(v) &= (1/kT)f_{i0}(v),\\
&\ \ \vdots\\
f_{in}(v) &= (1/nkT)f_{i(n-1)}(v).
\end{aligned} \tag{12}$$

Thus

$$f_i(v,\Delta U) = C\,e^{(\Delta U/kT)}f_n(v), \tag{13}$$

which is the expected result. In this case the mean ion velocity is zero throughout the collisional presheath since charge exchange collisions conserve ions and the mean ion velocity in the neutral plasma is zero. Thus, if $\alpha = 0$, constant collision frequency charge exchange collisions do not shift the ion distribution upward in velocity. This presheath can be matched to a collisionless sheath only if the collisionless sheath returns all the ions entering it from the collisional presheath.

With electron density assumed to follow

$$n_e(\Delta U) = n_0 e^{(-\Delta U/kT_e)},$$

the Poisson equation (9) yields, to first order,

$$\beta^2 = 4\pi q^2 n_0 (1/kT + 1/kT_e),$$

which is the length scale of the Debye length. Thus for $\alpha = 0$ the collisional presheath is not distinct from the collisionless sheath since there is no separate collisional presheath length scale.

B. Nonzero plasma potential gradient ($\alpha \neq 0$)

Under this condition there is a net flux of ions from the plasma into the sheath, which allows the construction of a collisional presheath that accelerates the ions and depopulates the ion distribution of returning ions. Thus the collisional presheath may be correctly matched to the collisionless sheath which returns no ions. In this case (7a) and (7b) can be written as

$$\frac{\alpha}{m}\frac{\partial f_0}{\partial v}(v) = \frac{1}{\tau n_n}[f_n(v)n_0 - n_n f_0(v)], \tag{14a}$$

$$v\beta f_1(v) + \frac{\beta}{m}\frac{\partial f_0}{\partial v} + \frac{\alpha}{m}\frac{\partial f_1}{\partial v}(v) = \frac{1}{\tau n_n}[f_n(v)n_1 - n_n f_1(v)]. \tag{14b}$$

The solution to Eqs. (14) are

$$f_0(v) = n_0 \exp\left(\frac{-mv}{\alpha\tau}\right)\frac{m}{\alpha\tau}\sqrt{\frac{m}{2\pi kT}}\int_{-\infty}^{v}\exp\left(-\frac{mu^2}{2kT} + \frac{mu}{\alpha\tau}\right)du \tag{15}$$

and

$$f_1(v) = \exp\left(-\frac{\beta mv^2}{2\alpha} - \frac{mv}{\alpha\tau}\right)\left[\int_0^v \exp\left(\frac{\beta mu^2}{2\alpha} + \frac{mu}{\alpha\tau}\right)\left[\frac{n_1 m}{\alpha\tau}\sqrt{\frac{m}{\alpha\pi kT}}\exp\left(\frac{mu^2}{2kT}\right) - \frac{\beta}{\alpha}\frac{\partial f_0}{\partial v}(u)\right]du + C\right], \tag{16}$$

where

$$n_0 = \int_{-\infty}^{\infty} f_0(v)dv \tag{17}$$

and

$$n_1 = \int_{-\infty}^{\infty} f_1(v)dv. \tag{18}$$

The constant of integration in (15) has been set so that f_0 goes to zero at $-\infty$; f_0 goes to zero at ∞ regardless of the constant of integration. Equation (17) is immediately satisfied by (15). The constant of integration C in (16) must be set so that (18), which represents self-consistency, is satisfied. It can be seen from (16) that f_1 goes to zero at $-\infty$ and ∞ regardless of the constant C. From (18), then

$$\begin{aligned} C = n_1\bigg[1 - \int_{-\infty}^{\infty}\exp\left(-\frac{\beta mv^2}{2\alpha} - \frac{mv}{\alpha\tau}\right)\int_0^v \exp\left(\frac{\beta mu^2}{2\alpha} + \frac{mu}{\alpha\tau}\right)\frac{m}{\alpha\tau}\sqrt{\frac{m}{2\pi kT}}\exp\left(-\frac{mu^2}{2kT}\right)du\,dv\bigg] \\ \times\left[\sqrt{\frac{2\pi\alpha}{m\beta}}\exp\left(\frac{m}{2\alpha\beta\tau^2}\right)\right]^{-1} + \left[\int_{-\infty}^{\infty}\exp\left(-\frac{\beta mv^2}{2\alpha} - \frac{mv}{\alpha\tau}\right)\int_0^v \frac{\beta}{\alpha}\exp\left(\frac{\beta mu^2}{2\alpha} + \frac{mu}{\alpha\tau}\right)\frac{\partial f_0}{\partial v}(u)du\,dv\right] \\ \times\left[\sqrt{\frac{2\pi\alpha}{m\beta}}\exp\left(\frac{m}{2\alpha\beta\tau^2}\right)\right]^{-1}. \end{aligned} \tag{19}$$

The exponential sheath rise β is determined from the Poisson equation under the simplifying assumption that

$$n_e = \int_{-\infty}^{\infty} f_e(v)dv = n_0 \exp\left(-\frac{\Delta U}{kT_e}\right). \tag{20}$$

One might expect that the approximation should be $n_e = n_0 \exp(-U/kT_e)$; however, this cannot be true in the asymptotic presheath because n_e must approach n_0 as U approaches negative infinity. With (20) the Poisson equation (9) to first order becomes

$$\beta^2 = 4\pi q^2 (n_1 + n_0/kT_e). \tag{21}$$

Since the ion density is only calculated to first order, the same will be done for the electron density in (20).

To obtain a particular solution it is assumed here that the collisionless sheath to which the collisional presheath is joined at $\Delta U = \Delta U^*$ returns no ions. In particular,

$$\int_{-\infty}^{0} f(v,\Delta U^*)dv = 0, \tag{22}$$

or

$$\int_{-\infty}^{0} f_0(v)dv + \Delta U^* \int_{-\infty}^{0} f_1(v)dv = 0 \tag{23}$$

and

$$\int_{-\infty}^{0} vf(v,\Delta U^*)dv = 0, \tag{24}$$

or

$$\int_{-\infty}^{0} vf_0(v)dv + \Delta U^* \int_{-\infty}^{0} vf_1(v)dv = 0. \tag{25}$$

Because the approximation is only first order, it is not possible to impose the condition that $f(v)$ is uniformly zero for returning ions. Equations (23) and (25) represent zero returning ion density and zero returning ion flux. When higher-order terms are included, the conditions of zero returning ion momentum flux, zero returning ion energy flux, etc., can be applied in succession. Equations (21), (23), and (25) are solved for n_1, β, and ΔU^*, with all other quantities assumed constant. Equation (21) immediately satisfies the Bohm criterion at $\Delta U = \Delta U^*$ for the first-order approximation

$$n_1 + n_0/kT_e > 0. \tag{26}$$

The Poisson equation (21) can be written as

$$\beta^2 \lambda_{\mathrm{D}}^2 = 1 + kT_e(n_1/n_0), \tag{27}$$

where

$$\lambda_{\mathrm{D}} = \sqrt{kT_e/4\pi q^2 n_0} \tag{28}$$

is the Debye length. It is expected that the length scale of the presheath should be of the order $\beta = 1/\lambda_i$, where λ_i is the ion mean free path. In the circumstance that the Debye length is small compared to the ion mean free path, the product $\beta^2\lambda_{\mathrm{D}}^2$ is small and

$$n_1 = -n_0/kT_e. \tag{29}$$

The neutral plasma region is matched to the collisional presheath also at $\Delta U = \Delta U^*$, as shown in Fig. 1, to produce a three-scale uniform asymptotic solution. In particular, assuming constant collision frequencies, the momentum equations become

$$\left(kT_i - \frac{m_i \Gamma_i^2}{n^2}\right)\frac{dn}{dx} = n\frac{dU}{dx} - \frac{m_i \Gamma_i}{\tau} \tag{30}$$

and

$$\left(kT_e - \frac{m_e \Gamma_e^2}{n^2}\right)\frac{dn}{dx} = -n\frac{dU}{dx} - \frac{m_e \Gamma_e}{\tau_e}, \tag{31}$$

where

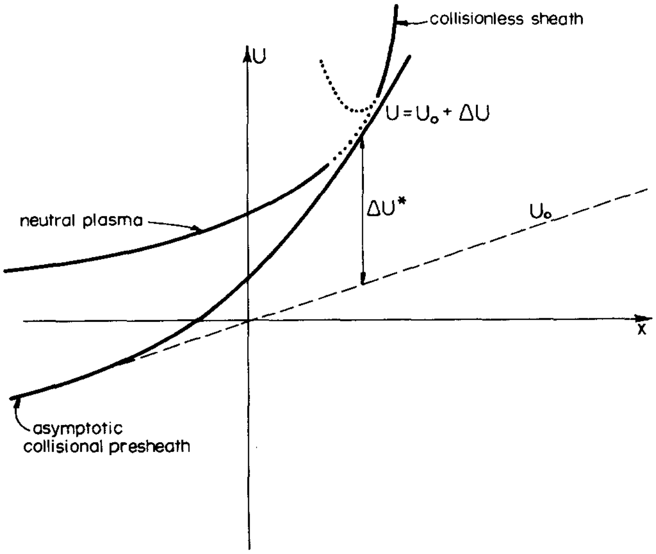

FIG. 1. Asymptotically correct potential in the collisional presheath.

$$n = n_0(1 - \Delta U^*/kT_e), \tag{32}$$

$$\frac{dn}{dx} = -n_0\beta\frac{\Delta U^*}{kT_e}, \tag{33}$$

and

$$\frac{dU}{dx} = \alpha + \beta\Delta U^*. \tag{34}$$

The quantity n is the plasma density at the matching point ΔU^* and Γ_i and Γ_e are, respectively, the ion and electron net fluxes.

Nondimensionalization results in

$$A = (\alpha\tau/m)\sqrt{m/2kT}, \tag{35}$$

$$B = \beta kT/\alpha, \tag{36}$$

$$R_e = T_e/T, \tag{37}$$

$$\omega = \sqrt{m/2kT}\; v, \tag{38}$$

where A and R_e are the parameters and B is a function of A and R_e. The quantity A represents the nondimensional asymptotic presheath potential gradient, B represents the nondimensional exponential presheath rise, R_e is the electron to neutral temperature ratio, and ω is the nondimensional velocity. The distribution functions can then be written as

$$F_0(\omega,A) = \frac{f_0(v)}{n_0\sqrt{m/2kT}} = \frac{\exp(-\omega/A)}{\sqrt{\pi}\,A}\int_{-\infty}^{\omega}\exp\Big(-\xi^2 + \frac{\xi}{A}\Big)d\xi \tag{39}$$

and

$$\begin{aligned} F_1(\omega,A,B) &= \frac{f_1(v)}{(n_0/kT_e)\sqrt{m/2kT}} \\ &= \frac{\exp(-B\omega^2 - \omega/A)}{\sqrt{\pi}}\bigg(\int_0^{\omega}\exp\Big(B\xi^2 + \frac{\xi}{A}\Big)\bigg\{-\frac{1}{A}\exp(-\xi^2) \\ &\quad - R_eB\bigg[\frac{1}{A}\exp(-\xi^2) - \frac{1}{A^2}\exp\Big(-\frac{\xi}{A}\Big)\int_{-\infty}^{\xi}\exp\Big(-\eta^2 + \frac{\eta}{A}\Big)d\eta\bigg]\bigg\}d\xi + C\bigg), \end{aligned} \tag{40}$$

where

$$\begin{aligned} C = &-\bigg[1 - \frac{1}{\sqrt{\pi}A}\int_{-\infty}^{\infty}\exp\Big(-B\omega^2 - \frac{\omega}{A}\Big)\int_0^{\omega}\exp\Big(B\xi^2 + \frac{\xi}{A} - \xi^2\Big)d\xi\, d\omega\bigg]\bigg[\sqrt{\frac{1}{B}}\exp\Big(\frac{1}{4BA^2}\Big)\bigg]^{-1} \\ &+ \frac{R_eB}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\Big(-B\omega^2 - \frac{\omega}{A}\Big)\int_0^{\omega}\exp\Big(B\xi^2 + \frac{\xi}{A}\Big) \\ &\times\bigg[\frac{1}{A}\exp(-\xi^2) - \frac{1}{A^2}\exp\Big(-\frac{\xi}{A}\Big)\int_{-\infty}^{\xi}\exp\Big(-\eta^2 + \frac{\eta}{A}\Big)d\eta\bigg]d\xi\, d\omega\bigg[\sqrt{\frac{1}{B}}\exp\Big(\frac{1}{4BA^2}\Big)\bigg]^{-1}. \end{aligned} \tag{41}$$

Thus (23) and (25) become

$$\int_{-\infty}^{0} F_0(\omega,A)d\omega + \frac{\Delta U^*}{kT_e}\int_{-\infty}^{0} F_1(\omega,A,B)d\omega = 0 \tag{42}$$

and

$$\int_{-\infty}^{0} \omega F_0(\omega,A)d\omega + \frac{\Delta U^*}{kT_e}\int_{-\infty}^{0} \omega F_1(\omega,A,B)\, d\omega = 0. \tag{43}$$

Figure 2 presents the presheath potential rise $\Delta U^*/kT_e$ and the nondimensional exponential rise B as a function of the nondimensional asymptotic presheath potential gradient A for a range of electron to neutral temperature ratios R_e. As would be intuitively expected, the presheath potential rise decreases with increasing A. Figure 3 presents the ion distribution functions at the neutral plasma–collisional presheath interface $F_0(\omega)$, the first-order correction to the distribution function $F_1(\omega)$, and the resulting distribution function at the collisional presheath–collisionless sheath interface $F_0(\omega) + \Delta U^*F_1(\omega)$. Although the resulting distribution is not uniformly zero for $\omega < 0$, its net returning density and flux are zero by (42) and (43). It is expected that higher-order corrections to the distribution function and potential with the corresponding application of higher-order moment conditions of zero returning momentum, energy, etc., will converge the returning distribution function toward a uniform zero.

In the limit of cold neutrals, the constant collision frequency charge exchange solution is considerably simplified. Equations (14a) and (14b) become

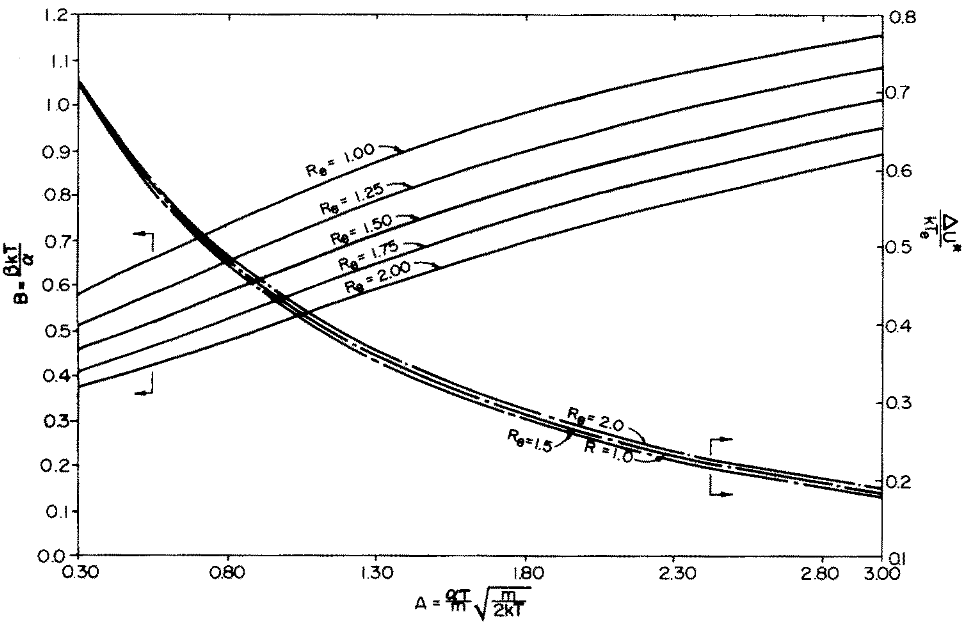

FIG. 2. Constant collision frequency presheath potential rise.

$$\frac{\alpha}{m}\frac{\partial f_0}{\partial v}(v)=\frac{1}{\tau n_n}[n_n\delta_n(v)n_0-n_nf_0(v)] \tag{44}$$

and

$$v\beta f_1(v)+\frac{\beta}{m}\frac{\partial f_0}{\partial v}+\frac{\alpha}{m}\frac{\partial f_1}{\partial v}(v)=\frac{1}{\tau n_n}[n_n\delta_n(v)n_1-n_nf_1(v)]. \tag{45}$$

The solutions to (44) and (45) are

$$f_0(v)=\begin{cases}n_0(m/\alpha\tau)\exp(-mv/\alpha\tau), & v>0,\\ 0, & v<0,\end{cases} \tag{46}$$

and

$$f_1(v)=\begin{cases}\exp\left(-\frac{\beta mv^2}{2\alpha}-\frac{mv}{\alpha\tau}\right)\left[C^+ +\frac{\beta}{\alpha}n_0\left(\frac{m}{\alpha\tau}\right)^2\int_0^v\exp\left(\frac{\beta mu^2}{2\alpha}\right)du\right], & v>0,\\ \exp\left(-\frac{\beta mv^2}{2\alpha}-\frac{mv}{\alpha\tau}\right)(C^-), & v<0,\end{cases} \tag{47}$$

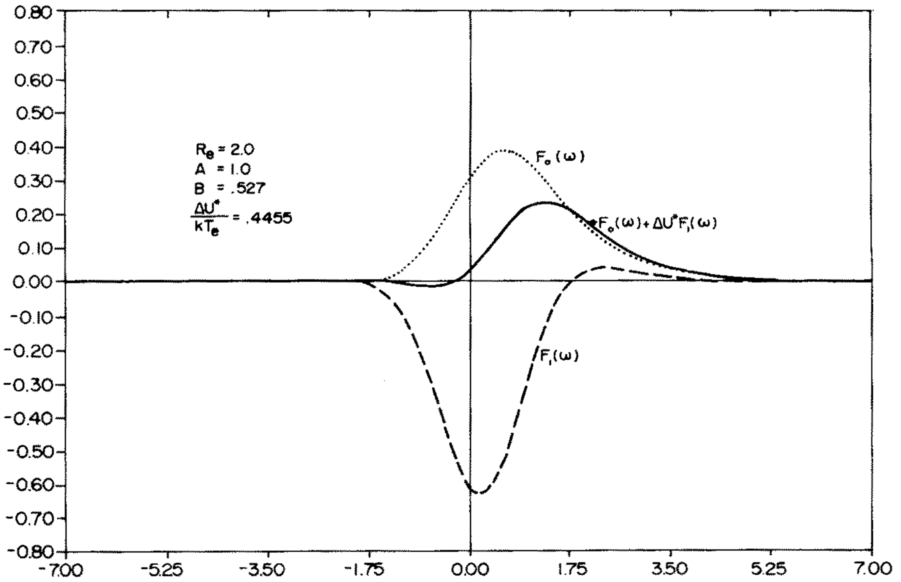

FIG. 3. Constant collision frequency ion distributions in the neutral plasma and at the presheath–sheath boundary.

such that

$$C^+ - C^- = (m/\alpha\tau)[n_1 - (\beta/\alpha)n_0]. \quad (48)$$

Equation (46) immediately satisfies $n_0 = \int_{-\infty}^{\infty} f_0(v)dv$. No returning ions implies that

$$C^- = 0 \quad (49)$$

and

$$C^+ = (m/\alpha\tau)[n_1 - (\beta/\alpha)n_0] \quad (50)$$

since f_0 on $v < 0$ is already zero. The final condition is then that $n_1 = \int_0^\infty f_1(v)dv$, or

$$n_1 = \int_0^\infty \exp\left(-\frac{\beta m v^2}{2\alpha} - \frac{mv}{\alpha\tau}\right)\left[\frac{n_1 m}{\alpha\tau} - \frac{\beta n_0}{\alpha}\frac{m}{\alpha\tau} + \frac{\beta n_0}{\alpha}\left(\frac{m}{\alpha\tau}\right)^2 \int_0^v \exp\left(\frac{\beta m u^2}{2\alpha}\right)du\right]dv. \quad (51)$$

The application of $n_1 = -n_0/kT_e$ yields

$$\frac{\beta k T_e}{\alpha} = \left\{1 - \int_0^\infty \exp\left(-\frac{\beta\alpha\tau^2}{2m}\xi^2 - \xi\right)d\xi\right\}\left\{\int_0^\infty \exp\left(-\frac{\beta\alpha\tau^2}{2m}\xi^2 - \xi\right)\left[1 - \int_0^\xi \exp\left(\frac{\beta\alpha\tau^2}{2m}\eta^2\right)dn\right]d\xi\right\}^{-1}. \quad (52)$$

In this case, ΔU^* is defined by

$$f_0(0^+) + \Delta U^* f_1(0^+) = 0, \quad (53)$$

which yields

$$\Delta U^*/kT_e = 1/(\beta k T_e/\alpha + 1), \quad (54)$$

as expected. In the limit of $\beta\alpha\tau^2/2m \to 0$ we have

$$\beta k T_e/\alpha = 1 \quad (55)$$

and

$$\Delta U^* = kT_e/2, \quad (56)$$

which corresponds to the Bohm criterion. Figure 4 presents the variation of $B = \beta k T_e/\alpha$, with $\beta\alpha\tau^2/2m$ for the cold neutral case. A particular β for the parameters can be conveniently found by drawing a line from the origin, with slope $2mkT_e/\tau^2\alpha^2$, so that the intersection is the solution. Figure 5 presents an example cold neutral ion distribution. Examination of the ion distribution function at $v = 0$ shows that the slope is discontinuous. This is because the neutral source is a delta function at $v = 0$. It appears that the Bohm criterion cannot be satisfied at ΔU^* because the integral $\int_0^\infty [f(v)/v^2]dv$ is singular; however, the use of this integral in the Bohm criterion assumes that the ions accelerated are not replaced. In this case the ions accelerated from $v = 0$ are replaced by ions from the cold neutral distribution which, of course, is a delta function at $v = 0$.

IV. FIRST-ORDER SOLUTION WITH A QUASICONSTANT CROSS-SECTION COLLISION TERM

First-order asymptotic solutions can also be developed for a quasiconstant cross-section collision term

$$\left(\frac{\partial f}{\partial t}\right)_c = \sigma\left(\int_{-\infty}^{\infty} f_n(v)f(u)|v-u|du - \int_{-\infty}^{\infty} f(v)f_n(u)|v-u|du\right). \quad (57)$$

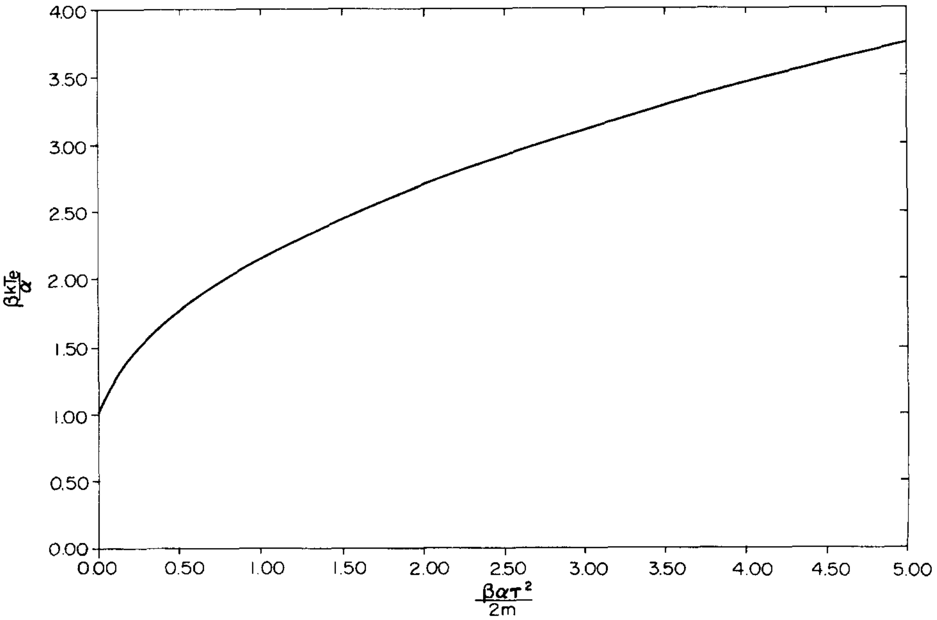

FIG. 4. Constant collision frequency presheath rise with cold neutrals.

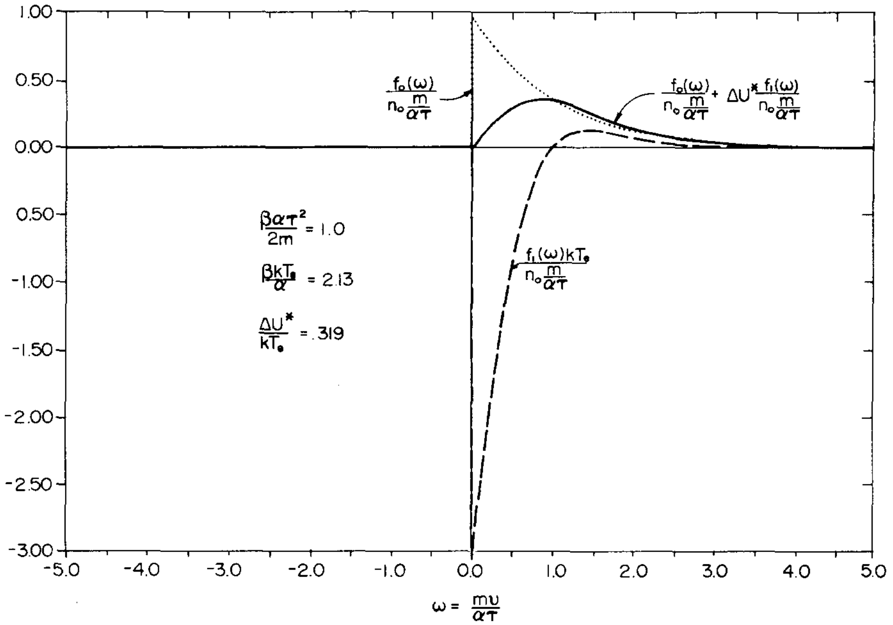

FIG. 5. Constant collision frequency ion distribution with cold neutrals.

This collision term is not really constant cross section because it is a one-dimensional representation which does not take into account average velocities in the other two dimensions. However, this collision term corresponds to that commonly called constant cross section. The application of this term leads to a set of integro-differential equations which can be at least approximately solved, and in the cold neutral case it leads to readily soluble first-order differential equations. The cold neutral case presented here corresponds to that which can be solved exactly (Riemann[4]). Unfortunately, though, the exact solution method is not extensible to noncold neutrals. The cold neutral collision term is

$$\left(\frac{\partial f}{\partial t}\right)_c = \sigma n_n \delta(v) \int_{-\infty}^{\infty} f(u)|u|du - \sigma f(v) n_n |v| \tag{58}$$

and the zero-order Boltzmann equation term (7a) becomes

$$\frac{\alpha}{m}\frac{\partial f_0}{\partial v}(v) = \sigma n_n \delta(v) \int_{-\infty}^{\infty} f_0(u)|u|du - \sigma n_n |v| f_0(v), \tag{59}$$

for which the solution is

$$f_0(v) = \begin{cases} n_0\sqrt{\dfrac{2}{\pi}}\sqrt{\dfrac{\sigma m n_n}{\alpha}} \exp\left(-\dfrac{\sigma m n_n}{2\alpha}v^2\right), & v>0, \\ 0, & v<0. \end{cases} \tag{60}$$

The first-order Boltzmann term is

$$v\beta f_1(v) + \frac{\beta}{m}\frac{\partial f_0}{\partial v}(v) + \frac{\alpha}{m}\frac{\partial f_1}{\partial v}(v) = \sigma n_n \delta(v) \int_{-\infty}^{\infty} f_1(u)|u|du - \sigma n_n |v| f_1(v), \tag{61}$$

for which the solution is

$$f_1(v) = \begin{cases} \exp\left[-\dfrac{1}{2}\left(\dfrac{\beta m}{\alpha} + \dfrac{\sigma m n_n}{\alpha}\right)v^2\right]\left\{\dfrac{n_0}{m}\sqrt{\dfrac{2}{\pi}}\left(\sqrt{\dfrac{\sigma m n_n}{\alpha}}\right)^3\left[\exp\left(\dfrac{\beta m v^2}{2\alpha}\right) - 1\right] + C^+\right\}, & v>0, \\ \exp\left[-\dfrac{1}{2}\left(\dfrac{\beta m}{\alpha} - \dfrac{\sigma m n_n}{\alpha}\right)v^2\right](C^-), & v<0. \end{cases} \tag{62}$$

The jump condition at $v = 0$ must be satisfied in (61):

$$C^+ - C^- = \frac{\delta m n_n}{\alpha}\int_{-\infty}^{\infty} f_1(u)|u|du - \frac{\beta}{\alpha}n_0\sqrt{\frac{2}{\pi}}\sqrt{\frac{\sigma m n_n}{\alpha}}. \tag{63}$$

No returning ions, $C^- = 0$, and the application of (63) to (62) yields

$$C^+ = -n_0(\beta/\alpha)\sqrt{2/\pi}\sqrt{\sigma m n_n/\alpha}. \tag{64}$$

The collisional presheath–collisionless sheath boundary ΔU^* is again

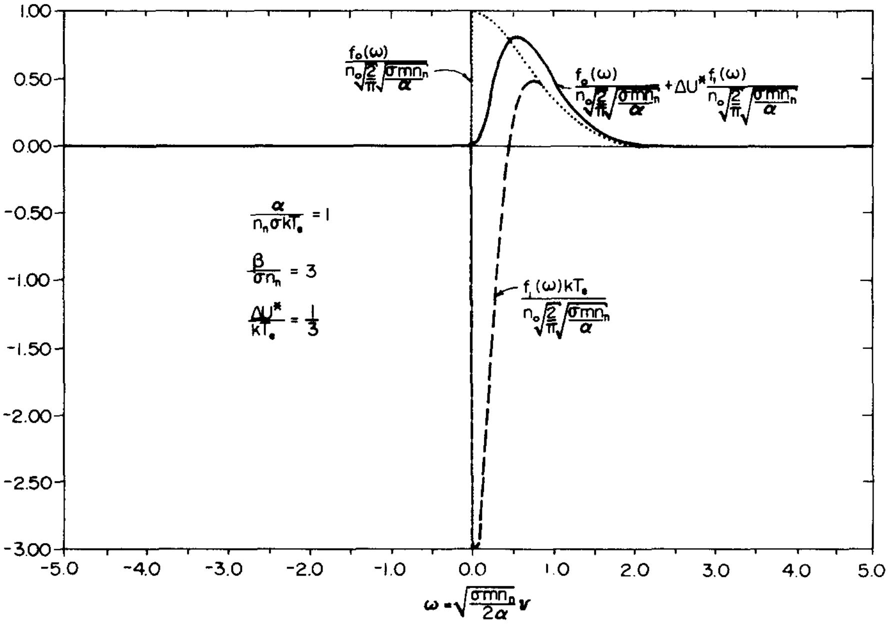

FIG. 6. Constant cross section ion distribution with cold neutrals.

$$0 = f(0^+) = f_0(0^+) + \Delta U^* f_1(0^+), \tag{65}$$

which yields

$$\Delta U^*/kT_e = \alpha/\beta kT_e. \tag{66}$$

Equation (62) is integrated to

$$n_1 = \int_{-\infty}^{\infty} f_1(v)dv = \frac{n_0 n_n \sigma}{\alpha}\left(1 - \sqrt{1 + \frac{\beta}{\sigma n_n}}\right) \tag{67}$$

and applied to the Poisson equation (8) to produce

$$\left(\frac{\beta}{\sigma n_n}\right)^2 = \left(\frac{4\pi q^2 n_0}{kT_e}\right)^2 \left(\frac{1}{\sigma n_n}\right)^2 \left[\frac{n_n \sigma k T_e}{\alpha}\left(1 - \sqrt{1 + \frac{\beta}{\sigma n_n}}\right) + 1\right]. \tag{68}$$

Under the assumption that the Debye length is short compared to the ion mean free path,

$$(4\pi q^2 n_0/kT_e)^2 (1/\sigma n_n)^2 \gg 1,$$

Eq. (68) results in

$$\beta/\sigma n_n = \alpha/n_n \sigma k T_e (2 + \alpha/n_n \sigma k T_e) \tag{69}$$

and

$$\Delta U^*/kT_e = 1/(2 + \alpha/n_n \sigma k T_e). \tag{70}$$

The Bohm criterion is satisfied at ΔU^* to the first order by virtue of (68). And interestingly, the presheath potential rise for $\alpha = 0$ is exactly that required by the cold ion Bohm criterion. Figure 6 presents the results for cold neutrals with $\alpha/n_n \sigma k T_e = 1$. From the ion distribution at ΔU^*, the mean ion velocity into the sheath can be determined to be $\bar{v} = 1.06\sqrt{kT_e/m_i}$, while the exact solution of Riemann gives $\bar{v} = 1.27\sqrt{kT_e/m_i}$; thus the first-order asymptotic result appears close.

V. CONCLUSIONS

It has been shown that approximate collisional presheath solutions can be obtained for a variety of collision terms. In particular the constant collision frequency case has been solved approximately, whereas previous attempts at exact solutions have found this case intractable. In addition, it has been shown that higher-order corrections can be made a regular and tractable fashion. Also the return of ions from the collisionless sheath can be treated.

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APPENDIX: MULTIEXPONENTIAL FORMULATION

In the previous sections we have calculated only the first-order terms in the ion distribution and presheath potential rise. Also, we have implicitly made the same first-order approximation for electrons:

$$n_e = n_0(1 - \Delta U/kT_e). \tag{A1}$$

A complete multiexponential expansion can also be constructed that correctly calculates the second- and higher-order terms. Potential in the presheath is

$$U = U_0 + \Delta U + a_2\Delta U^2 + a_3\Delta U^3 + \cdots, \tag{A2}$$

where $U_0 = \alpha x$ and $\Delta U = \exp(\beta x)$. Thus

$$\frac{dU}{dx} = \alpha + \beta\Delta U + 2\beta a_2\Delta U^2 + 3\beta a\Delta U^3 + \cdots \tag{A3}$$

and

$$\frac{d(\Delta U)}{dU} = \frac{\beta\Delta U}{a + \beta\Delta U + 2\beta a_2\Delta U^2 + 3\beta a_3\Delta U^3 + \cdots}, \tag{A4}$$

which transforms the Boltzmann equation

$$\frac{dU}{dx}\left(v\frac{\partial f}{\partial\Delta U}(v)\frac{\partial\Delta U}{\partial U} \pm \frac{1}{m}\frac{\partial f}{\partial v}(v)\right) = \left(\frac{\partial f}{\partial t}\right)_c \tag{A5}$$

into

$$v\beta\Delta U\frac{\partial f}{\partial\Delta U}(v) \pm \frac{1}{m}(\alpha + \beta\Delta U + 2\beta a_2\Delta U^2 + \cdots)\frac{\partial f}{\partial v}(v) = \left(\frac{\partial f}{\partial t}\right)_c, \tag{A6}$$

or

$$1{:}\quad \pm\frac{\alpha}{m}\frac{\partial f_0}{\partial v}(v) = \left[\left(\frac{\partial f}{\partial t}\right)_c\right]_1, \tag{A7a}$$

$$\Delta U{:}\quad v\beta f_1(v) \pm \frac{\beta}{m}\frac{\partial f_0}{\partial v}(v) \pm \frac{\alpha}{m}\frac{\partial f_1}{\partial v}(v) = \left[\left(\frac{\partial f}{\partial t}\right)_c\right]_{\Delta U}, \tag{A7b}$$

$$\Delta U^2{:}\quad 2v\beta f_2(v) \pm \frac{2\beta a_2}{m}\frac{\partial f_0}{\partial v}(v) \pm \frac{\beta}{m}\frac{\partial f_1}{\partial v}(v) \pm \frac{\alpha}{m}\frac{\partial f_2}{\partial v}(v) = \left[\left(\frac{\partial f}{\partial t}\right)_c\right]_{\Delta U^2}, \tag{A7c}$$

$$\vdots$$

$$\Delta U^n{:}\quad nv\beta f_n(v) \pm \frac{n\beta a_n}{m}\frac{\partial f_0}{\partial v}(v) \pm \frac{(n-1)\beta a_{n-1}}{m}\frac{\partial f_1}{\partial v}(v) \pm \cdots \pm \frac{\beta}{m}\frac{\partial f_{n-1}}{\partial v}(v) \pm \frac{\alpha}{m}\frac{\partial f_n}{\partial v}(v) = \left[\left(\frac{\partial f}{\partial t}\right)_c\right]_{\Delta U^n}. \tag{A7d}$$

The Poisson equation (8) becomes

$$\begin{aligned}\beta^2\Delta U &+ (2\beta)^2 a_2\Delta U^2 + (3\beta)^2 a_3\Delta U^3 + \cdots \\ &= 4\pi q^2\Bigg[\Delta U\left(\int_{-\infty}^{\infty} f_{i1}(v)dv - \int_{-\infty}^{\infty} f_{e1}(v)dv\right) \\ &\quad + \Delta U^2\left(\int_{-\infty}^{\infty} f_{i2}(v)dv - \int_{-\infty}^{\infty} f_{e2}(v)dv\right) + \cdots\Bigg].\end{aligned} \tag{A8}$$

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