Barr–Beck–Lurie in families

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1 Barr–Beck–Lurie in families

In this section we present a generalization of the result of [GHK22, Proposition 4.4.5] which is adapted to our setting.

Proposition 1.1. *Given a diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "U"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[dl, "r"] \\ & \mathcal{B} & \end{tikzcd}$$

in Cat_∞ *such that:*

(i) p *and* r *are coCartesian fibrations and* U *preserves coCartesian edges;*

(ii) U *has a left adjoint* $F : \mathcal{D} \to \mathcal{C}$ *such that* $pF \simeq r$*;*

(iii) The adjunction $F \dashv U$ *restricts in each fiber to an adjunction* $F_b \dashv U_b$*. For all* $b \in \mathcal{B}$*, the functor* U_b *is conservative, and* $\mathcal{C}_b$ *admits colimits of* U_b*-split simplicial objects, which* U_b *preserves.*

(iv) For any edge $e : b \to b'$ *in* $\mathcal{B}$*, the coCartesian covariant transport* $e_! : \mathcal{C}_b \to \mathcal{C}_{b'}$ *preserves geometric realizations of* U_b*-split simplicial objects.*

Then, the adjunction $F \dashv U$ *is monadic.*

Remark 1.2. *In view of the Barr–Beck–Lurie theorem, condition (iii) in Proposition 1.1 is equivalent to:*

(iii)′ The adjunction $F \dashv U$ *restricts in each fiber to a monadic adjunction* $F_b \dashv U_b$*.*

Proof of Proposition 1.1. We verify the conditions of the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5].

First we show that U is conservative. We can argue in exactly the same way as [GHK22, Proposition 4.4.5]. Suppose that $f : c \to c'$ is a morphism in $\mathcal{C}$ such that Uf is an equivalence in $\mathcal{D}$. Then $e := qUf \simeq pf$ is an equivalence in $\mathcal{B}$. One can factor f as $c \xrightarrow{\varphi} e_!c \xrightarrow{f'} c'$ where φ is a coCartesian lift of e and f' is a morphism in the fiber $\mathcal{C}_{b'}$ above $b' := p(c')$. Since φ is coCartesian lift of an equivalence, it is an equivalence. Because of the fiberwise monadicity assumption (iii), f' is an equivalence. Therefore f is an equivalence and U is conservative.

Now we will show that $\mathcal{C}$ admits and U preserves colimits of U-split simplicial objects. Let $q : \Delta^{\mathsf{op}} \to \mathcal{C}$ be a U-split simplicial object, so that Uq extends to a diagram $\widetilde{Uq} : \Delta^{\mathsf{op}}_{-\infty} \to \mathcal{D}$. Let $f : \Delta^{\mathsf{op}}_{-\infty} \to \mathcal{B}$ be the underlying diagram in $\mathcal{B}$. There is a morphism

$$\Delta^1 \times \Delta^{\mathsf{op}}_{-\infty} \to \Delta^{\mathsf{op}}_{-\infty} \tag{1}$$

which is the identity on $\{0\} \times \Delta^{\mathsf{op}}_{-\infty}$ and carries $\{1\} \times \Delta^{\mathsf{op}}_{-\infty}$ to $[-1] \in \Delta^{\mathsf{op}}_{-\infty}$. It sends each horizontal morphism $\{0\} \times [n] \to \{1\} \times [n]$ to the unique morphism $[n] \to [-1]$. Consider the composite

$$P : \Delta^1 \times \Delta^{\mathsf{op}}_{-\infty} \to \Delta^{\mathsf{op}}_{-\infty} \xrightarrow{f} \mathcal{B}. \tag{2}$$

Now we will take a coCartesian lifts, using the exponentiation for coCartesian fibrations [Lur18, Tag 01VG].

- ⋆ Let Q be a coCartesian lift of $P|_{\Delta^1 \times \Delta^{\mathsf{op}}}$ to $\mathcal{C}$. Then Q is a natural transformation between q and a morphism $q' : \Delta^{\mathsf{op}} \to \mathcal{C}_b$, where b is the image under f of $[-1] \in \Delta^{\mathsf{op}}_{-\infty}$.
- ⋆ Let $\widetilde{UQ}$ be a coCartesian lift of P to $\mathcal{D}$. Then $\widetilde{UQ}$ is a natural transformation between $\widetilde{Uq}$ and a morphism $\widetilde{Uq'} : \Delta^{\mathsf{op}}_{-\infty} \to \mathcal{C}_b$.

These natural transformations Q and $\widetilde{UQ}$ are uniquely characterised by the property that their components are coCartesian edges [Lur18, Tag 01VG]. Because of the assumption (i) that U preserves coCartesian edges, this unicity implies that $UQ \simeq \widetilde{UQ}\Big|_{\Delta^1 \times \Delta^{\mathsf{op}}}$. In particular $Uq' : \Delta^{\mathsf{op}} \to \mathcal{C}_b$ extends to the split simplicial object $\widetilde{Uq'} : \Delta^{\mathsf{op}}_{-\infty} \to \mathcal{C}_b$. By the fiberwise monadicity assumption (iii), this implies that q' extends to a colimit diagram $\overline{q}' : (\Delta^{\mathsf{op}})^{\triangleright} \to \mathcal{C}_b$ such that $U\overline{q}'$ is also a colimit diagram. By assumption (iv) and [Lur09, Proposition 4.3.1.10] it then follows that $\overline{q}'$ and $U\overline{q}'$, when regarded as diagrams in $\mathcal{C}$ and $\mathcal{D}$ respectively, are p-colimit diagrams. Now we can argue as in [Lur09, Corollary 4.3.1.11]. We have a commutative diagram

$$\begin{tikzcd}
(\Delta^1 \times \Delta^{\mathsf{op}}) \coprod_{\{1\} \times \Delta^{\mathsf{op}}} (\{1\} \times (\Delta^{\mathsf{op}})^{\triangleright}) \arrow[r, "{(Q, \overline{q}')}"] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\
(\Delta^1 \times \Delta^{\mathsf{op}})^{\triangleright} \arrow[r, "(f|_{(\Delta^{\mathsf{op}})^{\triangleright}}) \circ \pi"'] \arrow[ur, dashed, "s"] & \mathcal{B}
\end{tikzcd}$$

in which $\pi : (\Delta^1 \times \Delta^{\mathsf{op}})^{\triangleright} \to (\Delta^{\mathsf{op}})^{\triangleright} = \Delta^{\mathsf{op}}_{+} \subseteq \Delta^{\mathsf{op}}_{-\infty}$ denotes the morphism which is the identity on $\{0\} \times \Delta^{\mathsf{op}}$ and which carries $(\{1\} \times \Delta^{\mathsf{op}})^{\triangleright}$ to the cone point. Because the left map is an inner fibration there exists a lift s as indicated by the dashed arrow. Consider now the map $\Delta^1 \times (\Delta^{\mathsf{op}})^{\triangleright} \to (\Delta^1 \times \Delta^{\mathsf{op}})^{\triangleright}$ which is the identity on $\Delta^1 \times \Delta^{\mathsf{op}}$ and carries the other vertices of $\Delta^1 \times (\Delta^{\mathsf{op}})^{\triangleright}$ to the cone point. Let $\overline{Q}$ denote the composition

$$\Delta^1 \times (\Delta^{\mathsf{op}})^{\triangleright} \to (\Delta^1 \times \Delta^{\mathsf{op}})^{\triangleright} \xrightarrow{s} \mathcal{C} \tag{3}$$

and define $\overline{q} := \overline{Q}\big|_{\{0\} \times (\Delta^{\mathsf{op}})^{\triangleright}}$. Then $\overline{Q}$ is a natural transformation from $\overline{q}$ to $\overline{q}'$ which is componentwise coCartesian. Then [Lur09, Proposition 4.3.1.9] implies that $\overline{q}$ is a p-colimit diagram which fits into the diagram

$$\begin{tikzcd}
\Delta^{\mathsf{op}} \arrow[r, "q"] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\
(\Delta^{\mathsf{op}})^{\triangleright} \arrow[r, "f|_{(\Delta^{\mathsf{op}})^{\triangleright}}"'] \arrow[ur, "\overline{q}"] & \mathcal{B}
\end{tikzcd}$$

By assumption (i), $U\overline{Q}$ is a natural transformation from $U\overline{q}$ to $U\overline{q}'$ which is componentwise coCartesian. Hence [Lur09, Proposition 4.3.1.9] implies that $U\overline{q}$ is a p-colimit diagram. The underlying diagram $f|_{(\Delta^{\mathrm{op}})^{\triangleright}}$ of $\overline{q}$ in $\mathcal{B}$ extends to the split simplicial diagram f and hence admits a colimit in $\mathcal{B}$. Hence [Lur09, Proposition 4.3.1.5(2)] implies that $\overline{q}$ and $U\overline{q}$ are colimit diagrams in $\mathcal{C}$ and $\mathcal{D}$ respectively. Hence $\mathcal{C}$ admits and U preserves geometric realizations of U-split simplicial objects. $\square$

References

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