Locally analytic vectors of completed cohomology talk

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Abstract

This is the notes for a series of talks given at an informal "p-adic seminar" in Oxford in Trinity Term 2022.

Contents

1 Introduction to modular curves

1.1 Elliptic curves, complex tori, and lattices

Reference for this section is [KDSB73, Katz, Appendix 1.1]. Let $\Lambda \subseteq \mathbb{C}$ be a lattice. Then $\mathbb{C}/\Lambda$ is a complex torus. If

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\ell\in\Lambda}{}' \left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2} \right), \tag{1}$$

is the Weierstrass $\wp$-function associated to Λ, then the map $\mathbb{C}/\Lambda \to \mathbb{P}^2_{\mathbb{C}}$ given by $z + \Lambda \mapsto [\wp_\Lambda(z) : \wp'_\Lambda(z) : 1] =: [x : y : 1]$, for $z \neq 0$, and $0 \mapsto [0 : 1 : 0]$, is holomorphic with holomorphic inverse, with image the curve E_Λ cut out (on $\mathbb{A}^2_{\mathbb{C}}$) by:

$$E_\Lambda : y^2 = 4x^3 - g_{2,\Lambda}x - g_{3,\Lambda}, \tag{2}$$

where $g_{2,\Lambda}$, $g_{3,\Lambda}$ are (rescaled) Eisenstein series. This sends the invariant differential dz to $d\wp(z)/\wp'(z) = dx/y$. In the other direction, if (E, ω) is an elliptic curve with invariant differential, then $\Lambda(E,\omega) = \left\{ \int_\gamma \omega : \gamma \in H_1(E,\mathbb{Z}) \right\}$ is a lattice in $\mathbb{C}$, called the *lattice of periods*. These operations are inverses, and note that $\Lambda(E, \lambda.\omega) = \lambda.\Lambda(E,\omega)$, for $\lambda \in \mathbb{C}$, so the bijection descends to isomorphism classes of complex tori (as Riemann surfaces). It also respects the addition structure [DS06, §1.4].

1.2 Modular forms

Reference for this section is [KDSB73, Katz, Appendix 1.1 & 1.2.] Usually, we think of a modular form (of full level $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ and weight k as either:

- A degree $-k$ homogeneous function $\mathbb{F}$ of isoclasses of complex elliptic curves with differential (E,ω): so $\mathbb{F}(E, \lambda\omega) = \lambda^{-k}\mathbb{F}(E,\omega)$,
- A degree $-k$ homogeneous function of all lattices $\Lambda \subseteq \mathbb{C}$: so $F(\lambda.\Lambda) = \lambda^{-k}F(\Lambda)$,
- An invariant (holomorphic) differential on $\mathbb{H}$ of degree $k/2$ for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{H}$.

The correspondence between these two notions is as follows: If $f(z)(dz)^{k/2}$ is an invariant differential, we get such a function F of lattices by setting $F(\Lambda) = \omega_2^{-k} f(\omega_1/\omega_2)$, where $\{\omega_1, \omega_2\}$ is a basis for Λ, with $\Im(\omega_1/\omega_2) > 0$. We obtain a function $\mathbb{F}$ of elliptic curves with differential by evaluating on the lattice of periods: $\mathbb{F}(E,\omega) := F(\Lambda(E,\omega))$.

It is common to isolate f from the differential, to arrive at the definition of a modular form of weight k as:

- A holomorphic function for $\tau \in \mathbb{H}$ satisfying the the transformation rule:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-k} f(\tau). \tag{3}$$

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$, a modular form (of level 1) is 1-periodic and we can view it as a function of $q = e^{2\pi i\tau}$. This is equivalent to looking at its Fourier expansion:

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n(f) q^n \tag{4}$$

if this is finite-tailed, i.e. belongs to $\mathbb{C}((q))$ (resp. has no tail, i.e. belongs to $\mathbb{C}[[q]]$), then f is called meromorphic / holomorphic at infinity. If we unravel the definitions this is the same as asking for:

$$\mathbb{F}(E_\tau, dx/y) \in \mathbb{C}((q)) \text{ or } \mathbb{C}[[q]], \tag{5}$$

where E_τ is the family defined by:

$$E_\tau : y^2 = 4x^3 - \frac{1}{12}E_4(\tau)x - \frac{1}{216}E_6(\tau). \tag{6}$$

This family can be rewritten as a family defined over $\mathbb{Q}((q))$, known as the Tate elliptic curve Tate_q. The Eisenstein series are themselves modular forms, with Fourier expansions:

$$E_{2k} = \frac{1}{2\zeta(2k)} \sum_{(m,n)}{}' \frac{1}{(m+n\tau)^{2k}} = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{7}$$

and then the Tate family is defined by:

$$\mathrm{Tate}_q : y^2 = 4x^3 - \frac{1}{12}E_4(q)x - \frac{1}{216}E_6(q), \tag{8}$$

where $E_4(q), E_4(q) \in \mathbb{Q}((q))$ are the Eisenstein series now viewed as formal Laurent series in q. Therefore, we can obtain the q-expansions of modular forms by evaluation on the Tate family. This is how we will obtain q-expansions from the algebraic perspective.

1.3 The algebraic perspective.

From now on, all schemes are assumed to be at least over $\mathbb{Q}$.

Definition 1.1. *[Sai13, Definition 1.22] An elliptic curve over a $\mathbb{Q}$-scheme S is a proper smooth morphism $p : E \to S$, together with a choice of zero section $O : S \to E$, such that the fibers $E_{\overline{x}}$ of p over a geometric point $\bar{x} : \mathrm{Spec}(\overline{\mathbb{Q}}) \to S$ are isomorphic to an elliptic curve (= a connected algebraic curve of genus 1 over $\overline{\mathbb{Q}}$).*

Then, E/S carries the structure of an abelian group scheme over S, with O as its zero-section [KM85, Theorem 2.1.2]. We define $\underline{\omega}_{E/S} := p_*(\Omega^1_{E/S})$; it is a fact [KM85, §2.2.1] that this is a line bundle on S.

We follow [KDSB73, Katz, Appendix 1.2]. First, restrict to affine $S = \mathrm{Spec}(R)$. Imitating the previous, a modular form of weight k and level 1 is a "function" f sending pairs $(E/S, \omega) \to R$, where $\omega \in \underline{\omega}_{E/S}$ is a nowhere vanishing section, such that:

- $f(E/S, \omega)$ depends only on the S-isoclass of $(E/S, \omega)$.
- f is homogeneous of degree $-k$: $f(E, \lambda.\omega) = \lambda^{-k} f(E, \omega)$,
- f commutes with base change: if $g : R \to R'$ is any morphism, $S' = \mathrm{Spec}(R')$, then $f(E \times_S S'/S', (\mathrm{Spec}(g))^* \omega) = g(f(E/S, \omega))$.

The q–expansion of such a form is defined to be its value on the Tate elliptic curve (over $\mathrm{Spec}(\mathbb{Q}((q)))$. Given such a modular form, the element

$$f(E/S, \omega)\omega^{\otimes k} \in H^0(S, \underline{\omega}^{\otimes k}_{E/S}) \tag{9}$$

is a global section independent of the choice of ω. So finally, we can globalise the definition of a modular form of level 1 and weight k, meromorphic at ∞, to be a "function":

$$f : \begin{Bmatrix} \text{elliptic curves } E/S \\ \text{(over any base scheme } S) \end{Bmatrix} \to H^0(S, \underline{\omega}_{E/S}^{\otimes k}), \tag{10}$$

such that $f(E/S)$ depends only on the isoclass of E/S over S, and:

- f commutes with base change: if $\varphi : S' \to S$ is any morphism of schemes, then $f(E \times_{S,\varphi} S'/S') = \varphi^* f(E/S)$.

1.4 Level structures.

Reference for this section is [DR73, §IV.3]. Fix $K \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be a congruence subgroup of level N. This means that there is a number N, called the *level*, such that:

$$K \supseteq \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})) =: \Gamma(N), \tag{11}$$

and moreover N is minimal with this property. Let $\overline{K}$ be the image of K in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Let N be the level of K, let $E[N] \subseteq E$ denote the sub-S-group scheme of N-torsion. A K-level structure on E is an equivalence class of isomorphisms of the form:

$$\iota : E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2_S, \tag{12}$$

subject to $\iota \sim \iota'$ if $\iota = \overline{h} \circ \iota'$ for some $\overline{h} \in \overline{K}$. Denote the class by $[\iota]_K$. As in [DR73, Définition 3.2], we define the moduli functor:

$$\mathcal{M}_K(S) := \{\text{pairs } (E/S, [\iota]_K)\}/\sim, \tag{13}$$

where $(E/S, [\iota]_K) \sim (E'/S, [\iota']_K)$ if there is an isomorphism $\varphi : E \to E'$ over S with $\varphi^*[\iota]_K = [\iota']_K$.

A modular form of weight k and level K, meromorphic at ∞, is then a "function" f, which assigns to a class $[(E/S, [\iota]_K)]$ in $\mathcal{M}_K(S)$ (for any scheme S), an element of $H^0(S, \underline{\omega}_{E/S}^{\otimes k})$, compatible with base change.

The complex points of the Tate curve Tate_{q^N} are usually viewed as a complex torus (multiplicatively), as $\mathbb{C}^\times/q^{N\mathbb{Z}}$; then a trivialisation of its N-torsion is given by a maps of the form $(\mathbb{Z}/N\mathbb{Z})^2 \ni (i,j) \mapsto \zeta_N^i q^{mj}$. More generally, Tate_{q^N} admits level K-structures $[\iota]_K$ (not unique): the q-expansions of f are the values $f(\mathrm{Tate}(q^N)/\mathrm{Spec}(\mathbb{Q}((q))), [\iota]_K)$, as $[\iota]_K$ ranges [KDSB73, Katz, §1.2].

1.5 Modular curves and automorphic line bundles

If $N \geq 3$, then the moduli problem is representable by an affine $\mathbb{Q}$–scheme Y_K. See the remark under [DR73, Définition 3.2], combine with [KM85, Scholie 4.7.0] and use the rigidity of level N structures [KM85, Corollary 2.7.1]. This is the (open) modular curve of level K. By the general formalism of moduli problems, this implies the existence of a universal elliptic curve with level K structure, $(E_K/Y_K, [\tilde{\iota}]_K)$, such that every family $(E/S, [\iota]_K)$ is obtained uniquely as a base change of $(E_K/Y_K, [\tilde{\iota}]_K)$, i.e., for all S, there is a unique $\varphi : S \to Y_K$ such that

$$\begin{tikzcd} (E/S, [\iota]_K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (E_K/Y_K, [\tilde{\iota}]_K) \arrow[d] \\ S \arrow[r, "\exists! \varphi"] & Y_K \end{tikzcd} \tag{14}$$

is Cartesian. (The problem when the level is 2, for instance, is that $[-1]$ is still an automorphism of elliptic curves with level $\Gamma(2)$. Therefore the moduli problem is not rigid and so can't be representable). This means that we can redefine a modular form of weight k and level K, meromorphic at ∞, as a section $f \in H^0(Y_K, \underline{\omega}^{\otimes k}_{E_K/Y_K})$.

1.5.1 Modular forms holomorphic at ∞ and the compactified curve X_K.

Recall [Sai13, §2.1] that we can map:

$$\{\text{isoclasses of elliptic curves } E/S/\mathbb{Q}\} \to H^0(S, \mathcal{O}_S), \tag{15}$$

by sending an elliptic curve to its j-invariant j_E. Since the functor

$$H^0(S, \mathcal{O}_S) \cong \operatorname{Hom}(S, \mathbb{A}^1_{\mathbb{Q}}), \tag{16}$$

and on geometric points, an elliptic curve is uniquely determined up to isomorphism by its j-invariant, we tend to view $\mathbb{A}^1_{\mathbb{Q}}$ as a moduli space for isomorphism classes of elliptic curves, called the j-line. In any case, by Yoneda, we get a map $Y_K \to \mathbb{A}^1_{\mathbb{Q}}$, which extends to:

$$Y_K \to \mathbb{A}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}} = \text{"the projective } j\text{-line"}. \tag{17}$$

Then the compactification X_K is defined to be the normalisation [Aut, 29.53] of $Y_K \to \mathbb{P}^1_{\mathbb{Q}}$ [KDSB73, Katz, §1.4]. The upshot is that X_K is smooth and proper over $\mathbb{Q}$, and the boundary $X_K - Y_K$, (called the cusps), is a scheme finite étale over $\mathbb{Q}$, and there is an open immersion $Y_K \hookrightarrow X_K$ as an affine algebraic curve which is finite over $\mathbb{A}^1_{\mathbb{Q}}$ [KM85, Proposition 8.2.2].

$$\begin{tikzcd} Y_K \arrow[r, hook, "\text{open}"] \arrow[d, "j"] & X_K \arrow[d, "j"] \\ \mathbb{A}^1_{\mathbb{Q}} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{Q}} \end{tikzcd} \tag{18}$$

X_K represents a moduli problem of "generalised elliptic curves with level K structure", and $X_K - Y_K$ can be identified with the isomorphism classes of the level-K structures on the Tate_{q^N}.

There is a line bundle $\underline{\omega}$ on X_K [KM85, §10.13], whose restriction to Y_K is $\underline{\omega}_{E_K/Y_K}$, and whose restriction to the cusps is only the $\mathbb{Q}[[q]]$-span of the canonical differential of the Tate elliptic curve. Therefore, sections $f \in H^0(X_K, \underline{\omega}^{\otimes k})$ correspond to modular forms of level K and weight k, holomorphic at ∞. This space is denoted $M_k(K, \mathbb{Q})$, the subspace $H^0(X_K, \underline{\omega}^{\otimes k}(-\infty))$ of forms vanishing at the cusps (cusp forms), is denoted $S_k(K, \mathbb{Q})$.

1.6 The locally symmetric spaces

For a lattice, e.g. $\Lambda \subseteq \mathbb{R}^2$, we define a level K-structure to be a trivialisation of the N-torsion (where N = the level of K) of $\mathbb{R}^2/\Lambda$, up to $\overline{K}$-isomorphism (just as with elliptic curves). There are bijections:

$$\begin{aligned} &(\text{elliptic curves over } \mathbb{C} \text{ with level } K \text{ structure})/\cong \\ \leftrightarrow&(\text{complex lattices with level } K \text{ structure})/\mathrm{GL}_1(\mathbb{C}) \\ \leftrightarrow&[(\text{lattices} \subseteq \mathbb{R}^2 \text{ with level } K \text{ structure}) \times (\text{complex structures on } \mathbb{R}^2)]/\mathrm{GL}_2(\mathbb{R}) \\ \leftrightarrow&[(\text{lattices} \subseteq \mathbb{Q}^2 \text{ with level } K \text{ structure}) \times (\text{complex structures on } \mathbb{R}^2)]/\mathrm{GL}_2(\mathbb{Q}) \\ \leftrightarrow&[(\text{lattices} \subseteq \mathbb{A}_f^2 \text{ with level } K \text{ structure}) \times (\text{complex structures on } \mathbb{R}^2)]/\mathrm{GL}_2(\mathbb{Q}) \\ \leftrightarrow&\mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/K). \end{aligned} \tag{19}$$

The first identification in (19) was described in Section 1.1. For the second, recall that a complex structure on $\mathbb{R}^2$ is a homomorphism:

$$\psi : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}^2). \tag{20}$$

These carry a transitive $\mathrm{GL}_2(\mathbb{R})$-action by $M.\psi = M\psi M^{-1}$. You can check that if $\psi(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $\mathrm{Stab}(\psi) \cong \mathrm{GL}_1(\mathbb{C})$, so we identify complex structures on $\mathbb{R}^2$ with $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_1(\mathbb{C})$. This gives the second bijection in (19). For the third bijection in (19), we note that every $\mathrm{GL}_2(\mathbb{R})$-orbit is represented by a rational lattice. The fourth bijection in (19) comes from the correspondence:

$$\begin{aligned} \{\text{lattices } \Lambda_{\mathbb{Q}} \subseteq \mathbb{Q}^2\} &\leftrightarrow \{\text{lattices } \Lambda_{\mathbb{A}_f} \subseteq \mathbb{A}_f^2\} \\ \Lambda_{\mathbb{Q}} &\mapsto \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \\ \Lambda_{\mathbb{A}_f} \cap \mathbb{Q} &\leftarrow\!\shortmid \Lambda_{\mathbb{A}_f}. \end{aligned} \tag{21}$$

For the last bijection, we make two observations. Firstly, the lattice $\widehat{\mathbb{Z}}^2 \subseteq \mathbb{A}_f^2$ with the canonical level K structure from the class of:

$$\iota = \mathrm{id} : (\mathbb{A}_f^2/\widehat{\mathbb{Z}}^2)[N] = (\mathbb{Z}/N\mathbb{Z})^2 \to (\mathbb{Z}/N\mathbb{Z})^2, \tag{22}$$

is stabilised precisely by K, so since the action is transitive we can identify

$$\{\text{lattices} \subseteq \mathbb{A}_f^2 \text{ with level } K \text{ structure}\} = \mathrm{GL}_2(\mathbb{A}_f)/K. \tag{23}$$

Secondly, we note that:

$$\{\text{complex structures on } \mathbb{R}^2\} \cong \mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{H}^{\pm}, \tag{24}$$

because $\mathrm{GL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}^{\pm}$ by Möbius transformations, and the stabiliser of i is $\mathrm{GL}_1(\mathbb{C})$. So we get the identification on complex points (in fact on $\overline{\mathbb{Q}}$-points):

$$Y_K(\mathbb{C}) \leftrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/K) \leftrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A})/Z_\infty K_\infty K), \tag{25}$$

where $K_\infty = \mathrm{SO}_2(\mathbb{R})$, and $Z_\infty \subseteq \mathrm{GL}_2(\mathbb{R})$ is the diagonal torus. The latter description shows that the complex points have the structure of a locally symmetric space.

If $g \in \mathrm{GL}(\mathbb{A}_f)$ and $K', K \subseteq \mathrm{GL}(\mathbb{A}_f)$ are two congruence subgroups, such that $g^{-1}K'g \subseteq K$, then we get a well defined map:

$$\begin{aligned} \mathrm{GL}_2(\mathbb{A}_f)/K' &\to \mathrm{GL}_2(\mathbb{A}_f)/K \\ xK' &\mapsto xgK. \end{aligned} \tag{26}$$

By the formula (25), this induces a morphism

$$c_g : Y_{K'}(\mathbb{C}) \to Y_K(\mathbb{C}), \tag{27}$$

which is finite étale. More generally [DR73, §3.14], the map

$$(E/S, [\iota]_{K'}) \mapsto (E/S, [g \circ \iota]_K), \tag{28}$$

sends elliptic curves E/S with level K' structure to elliptic curves with level K structure, which yields a morphism of schemes $c_g : Y_{K'} \to Y_K$. This extends [DR73, Proposition 3.19] to the compactifications to give $c_g : X_{K'} \to X_K$.

1.7 Tame levels and completed cohomology

Consider the following general setup [Eme06b, §2.1]. Let G be a compact locally $\mathbb{Q}_p$-analytic group, with a decreasing neighbourhood basis of 1 by normal compact subgroups:

$$G = G_0 \supset G_1 \supset \cdots \supset G_r \supset \ldots, \tag{29}$$

acting on a tower of right G-spaces with G-equivariant maps:

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_r \leftarrow \ldots, \tag{30}$$

such that G_r acts trivially on X_r, and $X_r \to X_r'$ is Galois with Galois group G_r/G_r'. Let $\mathcal{V}_0$ be a local system of free finite rank $\mathbb{Z}_p$-modules on X_0 and $\mathcal{V}_r$ = the pullback to X_r. Then

$$\tilde{H}^n(\mathcal{V}) \coloneqq \varprojlim_s \varinjlim_r H^n(X_r, \mathcal{V}_r/p^s) \tag{31}$$

is an admissible (in the sense of [Eme17, Proposition-Definition 6.2.3], or [ST02a, §3]), continuous $\mathbb{Q}_p$-Banach representation of G [Eme06b, Theorem 2.2.1]. (We can also do this with compact supports, and dually there is a completed homology).

We have to play this game in a more general setting to apply to the modular curves, but the gist is the same. Let K^p be a fixed compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$, and K_p an open compact subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, which we should view as being variable. Firstly, we have

$$H^i(Y_{K^pK_p}(\mathbb{C}), \underline{\mathbb{Z}/p^s}) \cong H^i_{\mathrm{Betti}}(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s), \tag{32}$$

where we endow $Y_{K^pK_p}(\mathbb{C})$ with the analytic topology, for the purposes of the Betti cohomology. [Here by $\underline{\mathbb{Z}/p^s}$, I mean the locally constant sheaf].

With this formalism, we consider the *completed cohomology of tame level* K^p:

$$\begin{aligned} \tilde{H}^i(K^p, \mathbb{Z}_p) &= \varprojlim_s \varinjlim_{K_p} H^i_{\mathrm{Betti}}(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s), \\ \tilde{H}^i(K^p, \mathcal{O}_{\mathbb{C}_p}) &= \tilde{H}^i(K^p, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}, \\ \tilde{H}^i(K^p, \mathbb{C}_p) &= \tilde{H}^i(K^p, \mathbb{Z}_p) \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p, \end{aligned} \tag{33}$$

It is an admissible $\mathbb{Q}_p$-Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Eme06b, Theorem 0.1], also the remark under (the proof of) [Eme06b, Theorem 2.2.16]). The action of $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ is as follows. For a compact open K_p, set $K_p' = gK_pg^{-1} \cap K_p$. Thus $g^{-1}K^pK_p'g \subset K^pK_p$, and as in (27), we get a finite étale map $Y_{K^pK_p'}(\mathbb{C}) \to Y_{K^pK_p}(\mathbb{C})$, and hence a pullback map on cohomology:

$$c_g^* : H^i_{\mathrm{Betti}}(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s) \to H^i_{\mathrm{Betti}}(Y_{K^pK_p'}(\mathbb{C}), \mathbb{Z}/p^s), \tag{34}$$

thus via c_g^* we get an action on the directed system $\{H^i_{\mathrm{Betti}}(Y_{K^pK_p'}(\mathbb{C}), \mathbb{Z}/p^s)\}_{K_p \subseteq \mathrm{GL}_2(\mathbb{Q}_p)}$, and hence the direct limit

$$\varinjlim_{K_p} H^i_{\mathrm{Betti}}(Y_{K^pK_p'}(\mathbb{C}), \mathbb{Z}/p^s) \tag{35}$$

is endowed with a $\mathrm{GL}_2(\mathbb{Q}_p)$-action. This is compatible as s varies, leading to $\tilde{H}^i(K^p, \mathbb{Z}_p)$ being a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation.

The completed cohomology groups $\tilde{H}^i(K^p, \mathbb{Z}_p)$ are also a Galois representation, [Eme06a, §2.4]. By the comparison theorem for étale cohomology [AGV73, Exposé XI, Théorème

4.4], once an isomorphism of fields $\iota : \mathbb{C}_p \to \mathbb{C}$ (which exists for none other reason than that they are algebraically closed fields of the same cardinality), is fixed, we get a canonical isomorphism

$$H^i_{\text{ét}}(Y_{K^pK_p} \times_{\mathbb{Q}} \mathbb{C}_p, \mathbb{Z}/p^s) \cong H^i_{\text{Betti}}(Y_{K^pK'_p}(\mathbb{C}), \mathbb{Z}/p^s), \tag{36}$$

and $G_{\mathbb{Q}_p}$ acts on the left hand side: its action on the embeddings $\mathbb{Q} \hookrightarrow \mathbb{C}_p$ gives endomorphisms of $Y_{K^pK_p} \times_{\mathbb{Q}} \mathbb{C}_p$, the pullbacks of which induces an action on $\tilde{H}^i(K^p, \mathbb{Z}_p)$. It commutes with $\mathrm{GL}_2(\mathbb{Q}_p)$, so $\tilde{H}^i(K^p, \mathbb{Z}_p)$ becomes a $G_{\mathbb{Q}_p} \times \mathrm{GL}_2(\mathbb{Q}_p)$ representation. The locus where the action is differentiable, to an action of $\mathrm{Lie}(\mathrm{GL}_2(\mathbb{Q}_p)) = \mathfrak{gl}_2(\mathbb{Q}_p)$, is precisely the $\mathbb{Q}_p$-*locally analytic vectors*. Recall (see [ST02b, §3] or [Eme17, Definition 3.5.3]), that a representation V of a p-adic Lie group G is called locally analytic if the orbit map $\mathrm{ev}_v : g \mapsto gv \in \mathcal{C}^{la}(G, V)$; this is then differentiable to a map $d\mathrm{ev}_v \in \mathcal{C}^{la}(T(G), V)$ which restricts to a map $d_1\mathrm{ev}_v : T_1(G) = \mathrm{Lie}(G) \to V$, giving a Lie algebra representation. We denote these subspaces by:

$$\begin{aligned} \tilde{H}^i(K^p, \mathbb{Q}_p)^{\mathrm{la}} &\subseteq \tilde{H}^i(K^p, \mathbb{Q}_p), \\ \tilde{H}^i(K^p, \mathbb{C}_p)^{\mathrm{la}} &\subseteq \tilde{H}^i(K^p, \mathbb{C}_p), \end{aligned} \tag{37}$$

Then $\mathfrak{g} \coloneqq \mathfrak{gl}_2(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ acts on the latter, restricting to an action of $\mathfrak{b} = \mathrm{Lie}(\mathrm{B}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. One of Lue Pan's main aims, is to compute a Hodge-Tate decomposition of $\tilde{H}^i(K^p, \mathbb{C}_p)^{\mathrm{la}}_{\mu_k}$, where μ_k is the character of $\mathfrak{b}$ sending $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ to kd.

1.8 The Hecke action on completed cohomology

The reference for this part is [Hid86, p.564-566]. Again, let $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ be an open compact, let $g \in \mathrm{GL}_2(\mathbb{A}_f)$, and set

$$K^g = gKg^{-1} \cap K, \quad K_g = g^{-1}Kg \cap K. \tag{38}$$

The group isomorphism $[g] : K_g \to K^g : x \mapsto gxg^{-1}$ induces an isomorphism $[g] : Y_{K_g} \to Y_{K^g}$. There are also natural maps $Y_{K_g} \to Y_K$, $Y_{K^g} \to Y_K$ induced by the inclusion of levels $K^g, K_g \subseteq K$: these are finite étale coverings, and hence we get a trace map on cohomology:

$$\mathrm{tr}_{Y_g/Y} : H^i(Y_{K_g}(\mathbb{C}), \mathbb{Z}/p^s) \to H^i(Y_K(\mathbb{C}), \mathbb{Z}/p^s), \tag{39}$$

and the pullback of $Y_{K^g} \to Y_K$ is called $\mathrm{res}_{Y^g/Y} : H^i(Y_K, \mathbb{Z}/p^s) \to H^i(Y_{K^g}, \mathbb{Z}/p^s)$. The composite $\mathrm{tr}_{Y_g/Y} \circ [g]^* \circ \mathrm{res}_{Y^g/Y}$ defines a endomorphism of $H^i(Y_K(\mathbb{C}), \mathbb{Z}/p^s)$ which depends only on the double coset KgK. This is called the Hecke operator and denoted by T_g or $[KgK]$. This induces an action of the Hecke algebra $\mathcal{H}(K\backslash\mathrm{GL}_2(\mathbb{A}_f)/K, \mathbb{Z}/p^s)$ of double cosets with coefficients in $\mathbb{Z}/p^s$. The multiplication in the Hecke algebra comes from identifying it with the algebra of compactly supported K-biinvariant functions on $\mathrm{GL}_2(\mathbb{A}_f)$ endowed with the convolution product. Let S be the finite set of primes ℓ where K_ℓ is not a hyperspecial[1] maximal compact subgroup of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. These are called the ramified primes of K. We use the superscripts $\mathbb{A}_f^S$, K^S to denote the groups away from these primes. Then

$$\mathcal{H}^{\mathrm{sph}}(K, \mathbb{Z}/p^s) \coloneqq \mathcal{H}(K^S\backslash\mathrm{GL}_2(\mathbb{A}_f^S)/K^S) \tag{40}$$

[1] K_ℓ is hyperspecial if $K_\ell \cong H(\mathbb{Z}_\ell)$ for some $H \leq \mathrm{GL}_2$ such that $H(\mathbb{Q}_\ell) = \mathrm{GL}_2(\mathbb{Q}_\ell)$ and $H_{\mathbb{F}_\ell}$ is connected reductive.

is called the spherical Hecke algebra. For $\ell \notin S$ denote $\mathcal{H}^{\mathrm{sph}}(K_\ell, \mathbb{Z}) := \mathcal{H}(K_\ell \backslash \mathrm{GL}_2(\mathbb{Q}_\ell)/K_\ell, \mathbb{Z})$, then, the Satake isomorphism [ST98, Chapter 4] (applied to GL_2), gives:

$$\mathcal{S} : \mathcal{H}^{\mathrm{sph}}(K_\ell, \mathbb{Z}) \otimes \mathbb{Z}[\ell^{\pm 1/2}] \xrightarrow{\sim} \mathbb{Z}[X_1^{\pm 1}, X_2^{\pm 1}]^{S_2} \otimes \mathbb{Z}[\ell^{\pm 1/2}], \tag{41}$$

in particular $\mathcal{H}^{\mathrm{sph}}(K_\ell, \mathbb{Z})$ injects into a commutative ring and so is commutative. Therefore the spherical Hecke algebra (40) is commutative.

Applying this to completed cohomology, we see that $\mathcal{H}^{\mathrm{sph}}(K^pK_p, \mathbb{Z}/p^s)$ acts on each $H^i(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s)$ and hence,

$$\varprojlim_s \varprojlim_{K_p} \mathcal{H}^{\mathrm{sph}}(K^pK_p, \mathbb{Z}/p^s) \text{ acts on } \tilde{H}^i(K^p, \mathbb{Z}_p), \tag{42}$$

and the same thing with $\mathbb{Z}_p$ replaced by $\mathbb{C}_p$, $\mathbb{Q}_p$ coefficients, etc. The left-hand side in (42) is called the big Hecke algebra. This commutes with the $\mathrm{GL}_2(\mathbb{Q}_p)$ and $G_{\mathbb{Q}_p}$-actions. This is how systems of Hecke eigenvalues arise in completed cohomology.

1.9 Why is completed cohomology important?

See Calegari-Emerton's survey article [CE12].

- As you can see from (25), the definition of completed cohomology generalises to arithmetic quotients of connected reductive groups G over $\mathbb{Q}$ - this is the full generality of Emerton's original definition [Eme06b, §2.2].
- It provides a candidate to extend (on the automorphic side) the (p–adic) Langlands correspondence, to allow the Galois side to be enlarged beyond representations which are just de Rham at p, and in general, with continous families of Hodge-Tate-Sen weights. See [Eme14, §2.1.6, §3].
- It can be used to give a construction of eigenvarieties. See [Eme06b, Theorem 0.7], also [Eme06b, §2.3].
- The Iwasawa dimensions of $\tilde{H}_i(K^p, \mathbb{Z}_p)$. If $G_0 \leq G$ is a small enough open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, then the completed *homology* groups $\tilde{H}_i(K^p, \mathbb{Z}_p)$ are finitely generated $\mathbb{Z}_p[[G_0]]$-modules. The Iwasawa dimensions of these modules are conjectured [CE12, Conjecture 1.5].
- The locally analytic vectors in completed cohomology are related to overconvergent modular forms, see [Pan22, Theorem 1.0.1, Theorem 1.0.2], also [Cam22, Theorem 1.1.7].
- It can be expressed as the sheaf cohomology of Scholze's infinite level modular curve, [Sch15, Theorem IV.2.1], also [Pan22, Theorem 4.4.6].

2 The Hodge-tate period map

2.1 The adic spaces

Fix a choice of p-adic complex numbers $\mathbb{C}_p$. Then $X_K \times_{\mathbb{Q}} \mathbb{C}_p$ is smooth and proper over $\mathbb{C}_p$. There is an adification[2] functor[3]:

$$\begin{matrix}\{\text{smooth proper schemes}/\mathbb{C}_p\} \\ \{\text{finite type affine schemes}/\mathbb{C}_p\}\end{matrix} \xrightarrow{(-)^{\mathrm{ad}}} \{\text{analytic adic spaces}/\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})\} : \tag{43}$$

firstly, you have a GAGA functor, given on affine schemes of finite type over $\mathbb{C}_p$ by $\mathrm{Spec}(\mathbb{C}_p[T_1, \ldots, T_n]/I) \mapsto \bigcup_{i=0}^{\infty} \mathrm{Sp}(\mathbb{C}_p\langle p^{-i}T_1, \ldots, p^{-i}T_n\rangle/I)$, and secondly an adification functor on analytic adic spaces, given on affinoids by $\mathrm{Sp}(A) \mapsto \mathrm{Spa}(A, A^{\circ})$. This construction can be globalised, by gluing, they are functorial, and satisfy a universal property for morphisms of ringed spaces. Moreover, sheaves $\mathcal{F}$ on such schemes can be associated to sheaves $\mathcal{F}^{\mathrm{ad}}$ on the adification.

Denote by $\mathcal{X}_K := (X_K \times_{\mathbb{Q}} \mathbb{C}_p)^{\mathrm{ad}}$, $\mathcal{Y}_K := (X_K \times_{\mathbb{Q}} \mathbb{C}_p)^{\mathrm{ad}}$ the associated adic spaces to X_K, Y_K.

Theorem 2.1. *[Sch15, Theorem III.1.2] There is a unique perfectoid space $\mathcal{X}_{K^p}$ with:*

$$\mathcal{X}_{K^p} \sim \varprojlim_{K_p} \mathcal{X}_{K^pK_p}, \tag{44}$$

Here the $\sim$ means that $|\mathcal{X}_{K_p}| \xrightarrow{\sim} \varprojlim_{K_p} |\mathcal{X}_{K^pK_p}|$ on topological spaces, and on structure sheaves, that $\mathcal{X}_{K^p}$ has a cover by open affinoids $\mathrm{Spa}(A, A^+)$, such that $\varinjlim_{A_i} A_i \to A$ has dense image, where the limit is over all affinoid A_i such that the open immersion $\mathrm{Spa}(A, A^+) \hookrightarrow X_i$ factors through $\mathrm{Spa}(A_i, A_i^+)$. Similarly to Section 1.7, the inverse limit $\varprojlim_{K_p} \mathcal{X}_{K^pK_p}$ has a $\mathrm{GL}_2(\mathbb{Q}_p)$-action, which we transfer to $\mathcal{X}_{K^p}$. Scholze [Sch15, Theorem IV.2.1], [Pan22, Theorem 4.4.6], has shown that there is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$, $G_{\mathbb{Q}_p}$, and Hecke-equivariant isomorphism:

$$H^i(K^p, \mathbb{C}_p) \xrightarrow{\sim} H^i(\mathcal{X}_{K^p}, \mathcal{O}_{\mathcal{X}_{K^p}}). \tag{45}$$

Let $\mathscr{F}\ell = \mathbb{P}^{1,\mathrm{ad}}$ be the adic space associated to $\mathbb{P}^1_{\mathbb{C}_p}$. We will construct a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism $\pi_{HT} : \mathcal{X}_{K^p} \to \mathscr{F}\ell$, the Hodge-Tate period map. If we set $\mathcal{O}_{K^p} = \pi_{HT,*}\mathcal{O}_{\mathcal{X}_{K^p}}$, then it is a fact that:

$$H^i(\mathcal{X}_{K^p}, \mathcal{O}_{\mathcal{X}_{K^p}}) \cong H^i(\mathscr{F}\ell, \mathcal{O}_{K^p}). \tag{46}$$

Pan [Pan22, §4.2.6] defines a subsheaf $\mathcal{O}^{\mathrm{la}}_{K^p} \subseteq \mathcal{O}_{K^p}$ by:

$$\mathcal{O}^{\mathrm{la}}_{K^p}(U) = \mathcal{O}_{K^p}(U)^{K_p\text{-}\mathrm{la}}, \tag{47}$$

on quasi-compacts U, where $K_p \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$ is an open compact stabilising U. Then Pan shows that:

Theorem 2.2. *[Pan22, Theorem 4.4.6] There is a $\mathrm{GL}_2(\mathbb{Q}_p)$ and Hecke-equivariant isomorphism:*

$$H^i(\mathscr{F}\ell, \mathcal{O}_{K^p})^{la} \cong H^i(\mathscr{F}\ell, \mathcal{O}^{la}_{K^p}). \tag{48}$$

The idea now is to study $\mathcal{O}^{\mathrm{la}}_{K^p}$ and π_{HT}. To define the latter properly, we will need p-adic Hodge theory for rigid analytic varieties [Sch13].

[2] For the definition of adic spaces see [Hub93].

[3] For the purposes of this functor, $\mathbb{C}_p$ may be replaced by any p-adic field.

2.2 The period sheaves

Let X be a scheme or adic space. Recall [Aut, 34.4] the étale site $X_{\text{ét}}$ of X is the site with underlying category $\mathsf{Sch}_{\text{ét}}/X$ (or $\mathsf{AdicSpaces}_{\text{ét}}/X$), and coverings given by jointly surjective families of étale morphisms $\{f_i : U_i \to V\}$ (over X).

As in [Sch13, Definition 3.9], let $\text{pro}-X_{\text{ét}}$ be the category of pro-objects of $X_{\text{ét}}$. Its objects are (small) cofiltered inverse limits of objects in $X_{\text{ét}}$. A morphism $\varprojlim_i U_i = U \to V = \varprojlim_i V_i$ in $\text{pro}-X_{\text{ét}}$ is called étale if there is a morphism $U_0 \to V_0$ making the following square Cartesian:

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ U_0 \arrow[r] & V_0 \end{tikzcd} \tag{49}$$

A map $\varprojlim_i U_i = U \to V$ is called pro-étale if it is a cofiltered inverse limit of étale morphisms $U_i \to V$ such that $U_j \to U_i$ is surjective finite étale for $j \gg i$. $X_{\text{proét}}$ is the site with underlying category given by the objects of $\text{pro}-X_{\text{ét}}$ that are pro-étale over X, and coverings given by jointly surjective (on underlying topological spaces) familes of pro-étale morphisms. The structure sheaf $\mathcal{O}_X$ on $X_{\text{proét}}$ is given on qcqs $U = \varprojlim_i U_i \in X_{\text{proét}}$ by $\mathcal{O}_X(U) = \varinjlim_i \mathcal{O}_{X_{\text{ét}}}(U_i)$.

Now let K be a characteristic 0 perfectoid field, let $K^+ \subseteq K$ be an open bounded valuation subring, and let X be a locally noetherian adic space over $\text{Spa}(K, K^+)$. Call $U \in X_{\text{proét}}$ affinoid perfectoid if $U = \varprojlim_i \text{Spa}(R_i, R_i^+)$ for affinoids $\text{Spa}(R_i, R_i^+)$ such that (R, R^+) is an affinoid perfectoid (K, K^+) algebra, where $R^+ = (\varinjlim_i R_i^+)^\wedge_p$ and $R = R^+[1/p]$, and we write $\hat{U} = \text{Spa}(R, R^+)$. One of the most important properties of $X_{\text{proét}}$ is that:

Theorem 2.3. *[Sch13, Corollary 4.7, Proposition 4.8] In this setup, the affinoid perfectoid U form a basis for $X_{proét}$.*

We now define [Sch13, §6] sheaves by giving them on such affinoid perfectoid U. Firstly, The completed structure sheaves $\widehat{\mathcal{O}}_X^+$ and $\widehat{\mathcal{O}}_X$: by

$$\widehat{\mathcal{O}}_X^+(U) = R^+ \quad \text{and} \quad \widehat{\mathcal{O}}_X(U) = R, \tag{50}$$

and the sheaves $\mathbb{A}_{\text{inf}}$ and $\mathbb{B}_{\text{inf}}$ by,

$$\mathbb{A}_{\text{inf}}(U) = W(R^{\flat +}) \quad \text{and} \quad \mathbb{B}_{\text{inf}}(U) = W(R^{\flat +})[1/p]. \tag{51}$$

Recall from the p-adic Hodge theory, that there is a surjective map $\theta : W(R^{\flat +}) \to R^+$, and $\ker\theta$ is principal generated by ξ[4]. As sheaves this is saying there are surjective maps $\theta : \mathbb{A}_{\text{inf}} \to \widehat{\mathcal{O}}_X^+$ and $\theta : \mathbb{B}_{\text{inf}} \to \widehat{\mathcal{O}}_X$. Define sheaves $\mathbb{B}_{\text{dR}}^+$ and $\mathbb{B}_{\text{dR}}$ by:

$$\mathbb{B}_{\text{dR}}^+ = \varprojlim_i \mathbb{B}_{\text{inf}}/(\ker\theta)^i \quad \text{and} \quad \mathbb{B}_{\text{dR}} = \mathbb{B}_{\text{dR}}[1/\xi]. \tag{52}$$

Lastly, we define the structural de Rham sheaves. Define $\mathcal{O}\mathbb{B}_{\text{inf}} = \mathcal{O}_X \otimes_{W(\kappa)} \mathbb{B}_{\text{inf}}$ and $\mathcal{O}\mathbb{B}_{\text{dR}}^+ = ((\mathcal{O}\mathbb{B}_{\text{inf}})^\wedge_p)^\wedge_{\ker\theta}$, i.e [Sch16, (3)].

$$\mathcal{O}\mathbb{B}_{\text{dR}}^+(U) = \varinjlim_i \varprojlim_j \left(R_i^+ \hat{\otimes}_{W(\kappa)} \mathbb{A}_{\text{inf}}(U)\right)[1/p]/(\ker\theta)^j, \tag{53}$$

[4] This is defined by the same formula as a perfectoid field: $\theta : \sum_i p^i [x_i] \mapsto \sum_i p^i x_i^\#$.

where κ is the residue field of K. Here the tensor product is p-adically completed, and the map $\theta : R_i^+ \hat{\otimes}_{W(\kappa)} \mathbb{A}_{\inf}(U) \to R^+$ is the tensor product of the maps $R_i^+ \to R^+$ and $\mathbb{A}_{\inf}(U) \to R^+$. Also define $\mathcal{O}\mathbb{B}_{\mathrm{dR}} := \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+[1/\xi]$. Therefore there are maps $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \to \widehat{\mathcal{O}}_X^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \widehat{\mathcal{O}}_X$. The structure sheaf is equipped with a connection[5] $\nabla : \mathcal{O}_X \to \Omega_X^1$ which we can extend $\mathbb{B}_{\inf}$-linearly, and then p–adically and $\ker\theta$-adically complete (and then invert ξ if you want), to get a $\mathbb{B}_{\mathrm{dR}}^+$-linear connection

$$\nabla : \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^1. \tag{54}$$

Then Scholze [Sch13] considers the following four categories:

1. $\mathbb{B}_{\mathrm{dR}}^+$-local systems $\mathbb{M}$ on $X_{\text{proét}}$.
2. $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$-modules $\mathcal{M}$ with integrable connection $\nabla_{\mathcal{M}}$.
3. Filtered $\mathcal{O}_X$-modules $\mathcal{E}$ with filtration $\mathrm{Fil}^\bullet\mathcal{E}$ with integrable connection ∇ satisfying Griffifths transversality (this means that $\nabla \mathrm{Fil}^i\mathcal{E} \subseteq \mathrm{Fil}^{i-1}\mathcal{E} \otimes \Omega_X^1$).
4. Lisse $\mathbb{Z}_p$-sheaves $\mathbb{L}$ on $X_{\text{proét}}$.

The first and second categories are equivalent [Sch13, Theorem 7.2] by:

$$\begin{aligned} \mathbb{M} &\mapsto (\mathbb{M} \otimes_{\mathbb{B}_{\mathrm{dR}}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}, \mathrm{id} \otimes \nabla) \\ \mathcal{M}^{\nabla_{\mathcal{M}}} &\leftarrow\!\shortmid (\mathcal{M}, \nabla_{\mathcal{M}}). \end{aligned} \tag{55}$$

We say objects of the second and third categories are associated if

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong \mathcal{M} \otimes_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}, \tag{56}$$

compatibly with filtrations and connections, similarly objects of the first and third are called associated if:

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong \mathbb{M} \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}, \tag{57}$$

compatibly with filtrations and connections. Any $\mathcal{E}$ belonging to the first category is associated with:

$$\mathbb{M} = \mathrm{Fil}^0(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}})^{\nabla=0}. \tag{58}$$

This defines a fully faithful functor from the third to first categories [Sch13, Theorem 7.6]. A lisse $\mathbb{Z}_p$-sheaf $\mathbb{L}$ on $X_{\text{proét}}$ is a locally finitely generated $\widehat{\mathbb{Z}}_p$-module, where $\widehat{\mathbb{Z}}_p$ is the inverse limit $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ of constant sheaves on the pro-étale site. We say it is associated to a $\mathbb{B}_{\mathrm{dR}}$-local system $\mathbb{M}$ if there is an isomorphism

$$\mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong \mathbb{M} \otimes_{\mathbb{B}_{\mathrm{dR}}^+} \mathcal{O}\mathbb{B}_{\mathrm{dR}}, \tag{59}$$

if moreover $\mathbb{M} = \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{dR}}^+$ is in the image of (58) (i.e. admits an associated $\mathcal{E}$) we say that $\mathbb{L}$ is de Rham. This is precisely the situation in which we can pass between all four categories above.

[5] i.e., it is the first map in the de Rham complex.

2.3 Relative de Rham comparison theorem

If $f : X \to Y$ is a smooth proper map of such adic spaces, and $\mathcal{E}_X$ is a filtered $\mathcal{O}_X$-module with integrable connection (satisfying Griffiths transversality), then we can consider the relative de Rham complex of $\mathcal{O}_Y$-modules

$$\mathrm{DR}(\mathcal{E}_X) := (0 \to \mathcal{E}_X \xrightarrow{\nabla_{X/Y}} \mathcal{E}_X \otimes \Omega^1_{X/Y} \to \dots). \tag{60}$$

The cohomology $H^i_{\mathrm{dR}}(\mathcal{E}_X/Y)$ of this complex are then $\mathcal{O}_Y$-modules. We can also consider the derived functors $R^i f_{*,\mathrm{pro\acute{e}t}}$ of pushforward on sheaves. If $\mathbb{L}$ is de Rham and associated to $\mathcal{E}_X$, then these are associated [Sch13, Theorem 8.8(ii)], i.e.,

$$R^i f_{*,\mathrm{pro\acute{e}t}} \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(\mathcal{E}_X/Y) \otimes_{\mathcal{O}_Y} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}. \tag{61}$$

2.4 Variation of Hodge structures for complex analytic varieties

Let a X be a smooth variety over $\mathbb{C}$ (endowed with the analytic topology), or a complex manifold.

Definition 2.4. *[Sch73, §2] A (integral, pure of weight n) variation of Hodge structures (over X) is a $\mathbb{Z}$-local system $\underline{V}$ on X together with a decreasing filtration $\mathrm{Fil}^\bullet \mathcal{E}$ on $\mathcal{E} := \underline{V} \otimes_{\mathbb{Z}} \mathcal{O}_X$ satifsying Griffiths transversality (i.e. $\nabla \mathrm{Fil}^i \mathcal{E} \subseteq \mathrm{Fil}^{i-1}\mathcal{E} \otimes \Omega^1_X$), which induces a pure Hodge structure of weight n on the fibers of $\underline{V}$.*

A Hodge structure on a finite rank free $\mathbb{Z}$-module V is a decomposition of $V_{\mathbb{C}} := V \otimes_{\mathbb{Z}} \mathbb{C}$ into complex vector spaces $V^{i,j}$:

$$V_{\mathbb{C}} = \bigoplus_{i,j} V^{i,j}, \tag{62}$$

such that the complex conjugate $\overline{V^{i,j}} = V^{j,i}$. We write $d_{i,j} = \dim V^{i,j}$, and $(d_{i,j})_{i,j\in\mathbb{Z}}$ is called the Hodge weights of V. It is called pure of weight n if $\operatorname{rank} V = n$ and $i + j = n$ for all i, j, in which case we define a filtration by $\mathrm{Fil}^i V_{\mathbb{C}} = \bigoplus_{i' \le i} V^{i',n-i'}$. We can recover $V^{i,j} = \mathrm{Fil}^i V_{\mathbb{C}} \cap \overline{\mathrm{Fil}^j V_{\mathbb{C}}}$, which is what "induces" means in Definition 2.4.

In the setting of Definition 2.4, let V be the fiber of $\underline{V}$, let $d_i = \operatorname{rank} \mathrm{Fil}^i \mathcal{E}$, and consider $\mathrm{Fl}^{\mathbf{d}}_{V_{\mathbb{C}}}$, the flag variety of decreasing filtrations of $\mathbb{C}$-subspaces V_i of $V_{\mathbb{C}}$ with $\dim V_i = d_i$. Its X-points (for $X/\mathbb{C}$) are given by:

$$\mathrm{Fl}^{\mathbf{d}}_{V_{\mathbb{C}}}(X) = \{(\mathcal{F}_i)_{i\in\mathbb{Z}} : V_{\mathbb{C}} \otimes \mathcal{O}_X \supseteq \dots \supseteq \mathcal{F}_i \supseteq \mathcal{F}_{i+1} \supseteq \dots \supseteq 0\}, \tag{63}$$

where each $\mathcal{F}_i$ is a vector bundle of rank d_i which is a locally direct summand. So if X is equipped with a variation of Hodge structures, $\mathrm{Fil}^\bullet \mathcal{E}$ determines a morphism $\pi_H : X \to \mathrm{Fl}^{\mathbf{d}}_{V_{\mathbb{C}}}$, the "period map".

Now let $f : X \to Y$ be a smooth proper morphism of varieties over $\mathbb{C}$. Then the relative de Rham cohomology $H^n_{\mathrm{dR}}(X/Y)$ is equipped with a decreasing filtration, called the Hodge filtration [Aut, §50.7], coming from the degeneration of the Hodge-de Rham spectral sequence [Del68, Théorème 5.5] $E_1^{i,j} = H^j(X, \Omega^i_{X/Y}) \Rightarrow H^n_{\mathrm{dR}}(X/Y)$ (here $n = i+j$). It also has the Gauss-Manin connection ∇ satisfying Griffiths transversality with respect to the Hodge filtration [Gri70, §2]. There is an isomorphism (coming from the compatibility of the Riemann-Hilbert correspondence in the derived category, with pushfowards, see for example [HTT08, Theorem 7.1.1]) of $\mathcal{O}_Y$-modules:

$$R^n f_* \underline{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_Y \cong H^n_{\mathrm{dR}}(X/Y), \tag{64}$$

and the filtration $\mathrm{Fil}^\bullet \mathcal{E}$ on $\mathcal{E} := H^n_{\mathrm{dR}}(X/Y)$ determines a Hodge structure on the fibers of $\underline{V} := R^n f_* \underline{\mathbb{Z}}$. In other words, $\underline{V}$ is a variation of Hodge structures on Y, and so we get a morphism $Y \to \mathrm{Fl}^{\mathbf{d}}_{V_{\mathbb{C}}}$.

For an elliptic curve $f : E \to S$ over any scheme $S/\mathbb{C}$, the relative de Rham cohomology $H^1_{\mathrm{dR}}(E/S)$ is a rank 2 vector bundle on S which sits in the exact sequence (the "Hodge-Tate filtration") [KDSB73, Katz, A1.2.1]:

$$0 \to \underline{\omega}_{E/S} \to H^1_{\mathrm{dR}}(E/S) \to \underline{\omega}^{-1}_{E/S} \to 0, \tag{65}$$

which determines a variation of Hodge structures $\underline{V} = R^1 f_* \underline{\mathbb{Z}}$ on S with $d_{-1,0} = d_{0,-1} = 1$. So we get a morphism $S \to \mathbb{P}(V) \cong \mathbb{P}^1_{\mathbb{C}}$. In particular, if $S = Y_K \otimes_{\mathbb{Q}} \mathbb{C}$ is the (open) modular curve of level K, and $E = E_K \times_{\mathbb{Q}} \mathbb{C}$ is the universal elliptic curve, then we get a map $Y_K \times_{\mathbb{Q}} \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$. Extending this, there is an exact sequence of vector bundles on $X_K \times_{\mathbb{Q}} \mathbb{C}$:

$$0 \to \underline{\omega}_K \to \mathrm{H}^1_{\mathrm{dR,log}} \to \underline{\omega}_K^{-1} \to 0 \tag{66}$$

and an extension $\underline{V_{\log}}$ of $\underline{V_{\log}}$ such that $\underline{V}_{\log} \otimes_{\mathbb{Z}} \mathcal{O}_{X_K} \cong \mathrm{H}^1_{\mathrm{dR,log}}$, and hence a variation of Hodge structures on $X_K \times_{\mathbb{Q}} \mathbb{C}$, which yields a period map $X_K \times_{\mathbb{Q}} \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$.

2.5 The Hodge-Tate period map

We follow [Pan22, §4.1.3]. Our aim will be to copy the period map from the previous section, for the perfectoid modular curve $\mathcal{X}_{K^p}$ over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. The substitute for variation of Hodge structures will be Scholze's p-adic Hodge theory for rigid analytic varieties [Sch13].

Let $f : \mathcal{E}_{K^p K_p} \to \mathcal{Y}_{K^p K_p}$ be the universal elliptic curve over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Let $\mathbb{L} = \widehat{\mathbb{Z}}_p$. Write $\underline{\hat{V}} = R^1 f_{*,\text{proét}} \widehat{\mathbb{Z}}_p$, then by (61) there is an isomorphism of sheaves on the proétale site:

$$\underline{\hat{V}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \cong H^1_{\mathrm{dR}}(\mathcal{E}_{K^p K_p} / \mathcal{Y}_{K^p K_p}) \otimes_{\mathcal{O}_{\mathcal{Y}_{K^p K_p}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}. \tag{67}$$

Using the theory of log adic spaces [DLLZ19b] [DLLZ19a], this isomorphism can be extended to $\mathcal{X} = \mathcal{X}_{K^p K_p}$, by equipping $\mathcal{X}_{K^p K_p}$ with the log structure defined by the divisor of its cusps. Similarly to (67), there is a comparison isomorphism

$$\underline{\hat{V}}_{\log} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \cong H^1_{\mathrm{dR,log}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \tag{68}$$

of sheaves on $\mathcal{X}_{\text{prokét}}$, the pro-Kummer étale site of log adic spaces which are pro-log-étale over $\mathcal{X}_{K^p}$, and $\mathbb{B}^+_{\mathrm{dR,log}}$ are log period sheaves, and as before $\underline{\hat{V}}_{\log}$ is a rank 2 $\widehat{\mathbb{Z}}_p$-local system on $\mathcal{X}_{\text{prokét}}$, and $H^1_{\mathrm{dR,log}}$ gets its filtration from the Hodge-Tate exact sequence

$$0 \to \underline{\omega}_{K^p K_p} \to H^1_{\mathrm{dR,log}} \to \underline{\omega}^{-1}_{K^p K_p} \to 0, \tag{69}$$

where $\underline{\omega}_{K^p K_p}$ is the automorphic line bundle[6]. Recall that $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$ has the $\ker(\theta)$-adic filtration, so there is an inclusion

$$\mathrm{gr}^0 H^1_{\mathrm{dR,log}} \otimes_{\mathcal{O}_{\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}} \hookrightarrow \mathrm{gr}^0(H^1_{\mathrm{dR,log}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}), \tag{70}$$

[6] We will also use the same notation for the sheaves on the pro-Kummer étale site

and the quotient can be identified with the rest of the degree 0 part, i.e. $\mathrm{gr}^1 H^1_{\mathrm{dR,log}} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}}(-1)$, because $\mathrm{gr}^\bullet H^1_{\mathrm{dR,log}}$ only lives in degrees $0,1$. So we get the filtration:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{gr}^0 H_{\mathrm{dR,log}} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}} \arrow[r] \arrow[d, equal] & \mathrm{gr}^0(H^1_{\mathrm{dR,log}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}) \arrow[r] \arrow[d, "\sim"] & \mathrm{gr}^1 H_{\mathrm{dR,log}} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}}(-1) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \underline{\omega}^{-1}_{K^pK_p} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}} \arrow[r] & \underline{\hat{V}}_{\mathrm{log}} \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}} \arrow[r] & \underline{\omega}_{K^pK_p} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}}(-1) \arrow[r] & 0 \end{tikzcd} \tag{71}$$

where in the bottom line we took the 0^{th} graded part of the isomorphism (68) and used (69). This can be rewritten as:

$$0 \to \underline{\omega}^{-1}_{K^pK_p}(1) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}} \to \underline{\hat{V}}_{\mathrm{log}}(1) \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}} \to \underline{\omega}_{K^pK_p} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}} \to 0. \tag{72}$$

Now $\varprojlim_{K_p} \mathcal{X}_{K^pK_p} \sim \mathcal{X}_{K^p}$ is a cover of $\mathcal{X}_{K^pK_p}$ in the pro-Kummer étale site, and hence, restricting (72) to this cover and recalling Section 2.2, we get the exact sequence of sheaves over $\mathcal{X}_{K^p}$:

$$0 \to \underline{\omega}^{-1}_{K^p}(1) \to \underline{\hat{V}}_{\mathrm{log}}(1) \otimes \mathcal{O}_{\mathcal{X}_{K^p}} \to \underline{\omega}_{K^p} \to 0. \tag{73}$$

By choosing two sections of $\underline{\hat{V}}_{\mathrm{log}}(1)$, this is already enough to give a morphism to $\mathbb{P}^1_{\mathrm{ad}}$, but we can do better and make this canonical. The inverse limit $\varprojlim_{K_p} \mathcal{X}_{K^pK_p}$ can be calculated on the system of congruence subgroups $\Gamma(p^m)$, i.e $\varprojlim_m \mathcal{X}_{K^p\Gamma(p^m)}$. The maps are induced by the inclusions of level structures $\Gamma(p^{m+1}) \subseteq \Gamma(p^m)$. In particular, an S-point of this inverse limit gives rise to to an elliptic curve E_S/S together with a compatible system of trivialisations $\alpha_m : E_S[p^m] \to (\mathbb{Z}/p^m\mathbb{Z})^2_S$, that is to say, a trivialisation $\alpha : T_pE_S \to (\mathbb{Z}_p)^2_S$ of the Tate module over S. In particular the universal elliptic curve over $\mathcal{X}_{K^p}$ gives rise to a canonical trivialisation of the Tate module $\underline{\hat{V}}_{\mathrm{log}}(1)$ over $\mathcal{X}_{K^p}$, which we apply to (73):

$$0 \to \underline{\omega}^{-1}_{K^p}(1) \to (\mathbb{Q}_p^{\oplus 2})(1) \otimes \mathcal{O}_{\mathcal{X}_{K^p}} \to \underline{\omega}_{K^p} \to 0. \tag{74}$$

The images of that standard basis vectors e_1, e_2 in $\mathbb{Q}_p^{\oplus 2}$ give two sections that generate $\underline{\omega}_{K^p}$ and hence a morphism to $\mathscr{F}\ell = \mathbb{P}^{1,\mathrm{ad}}$. This is the Hodge-Tate period map π_{HT}. It is $\mathrm{GL}(\mathbb{Z}_p)$ equivariant because the action on $\mathcal{X}_{K^p}$ comes from composing with the level structure α. We can view it in the diagram:

$$\begin{tikzcd} & \mathcal{X}_{K^p} \arrow[dl, "\pi_{K_p}"'] \arrow[dr, "\pi_{HT}"] & \\ \mathcal{X}_{K^pK_p} & & \mathscr{F}\ell \end{tikzcd} \tag{75}$$

where π_{K_p} is the projection to finite level K^pK_p. Let $\omega_{\mathscr{F}\ell}$ be the tautological line bundle on $\mathscr{F}\ell$, and let $\underline{\omega}_{K^p} := \pi^*_{K_p}\underline{\omega}_{K^pK_p}$ be the pullback of the automorphic line bundle from any finite level. Then

Theorem 2.5. *[Pan22, Theorem 4.1.7][Sch15, Theorem III.3.] The Hodge-Tate period map π_{HT} is $\mathrm{GL}_2(\mathbb{Q}_p)$ and Hecke equivariant (for the trivial Hecke action on $\mathscr{F}\ell$). If $\mathscr{F}\ell \supset U_1 := \{[x_1 : x_2] : \|x_1\| \geq \|x_2\|\}$ (define U_2 similarly), and $\mathfrak{B}$ is the set of finite intersections of rational subsets of U_1, U_2, then every $U \in \mathfrak{B}$ has $V := \pi^{-1}_{HT}(U)$ affinoid perfectoid. There is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism of line bundles $\underline{\omega}_{K^p} \cong \pi^*_{HT}\omega_{\mathscr{F}\ell}$.*

The above construction of π_{HT} generalises straightforwardly to Siegel modular varieties. For the consruction of π_{HT} for Hodge type Shimura varieties see [CS17], for abelian type see [She17], for general Shimura varieties there is Hodge-Tate period map of diamonds constructed in [BP21] and [Cam22, §7].

3 Relative Sen theory

3.1 Classical Sen theory

Recalling the p–adic Hodge theory study group, the original (arithmetic) Sen theory is the following. $\overline{\mathbb{Q}}_p \supset K \supset \mathbb{Q}_p$ is a finite extension, K_∞/K is a ramified $\mathbb{Z}_p$-extension, $H \coloneqq G_{K_\infty}$, $\Gamma \coloneqq \mathrm{Gal}(K_\infty/K)$, with topological generator γ, $\Gamma_m \coloneqq \Gamma^{p^m}$, $K_m \coloneqq K_\infty^{\Gamma_m}$, and $\chi : \Gamma \xrightarrow{\sim} \mathbb{Z}_p$ is a choice of isomorphism. Here is a picture:

$$\begin{tikzcd} K \arrow[r, dash] \arrow[rrr, dash, bend left, "\Gamma_0=\Gamma"] & K_1 \arrow[r, dash] \arrow[rr, dash, bend right, "\Gamma_1"'] & K_2 \arrow[r, dotted, dash] \arrow[r, dash, bend right, "\Gamma_2"'] & K_\infty \arrow[r, dash, "H"] & \overline{K} \end{tikzcd} . \tag{76}$$

Each $\Gamma_m \cong$ an open subgroup of $\mathbb{Z}_p$ and so is a 1-dimensional p–adic Lie group. Let V be a f.d. $\mathbb{Q}_p$-Banach representation of G_K. Then for $m \gg 0$ one has an isomorphism of $\mathbb{C}_p$-semilinear G_K-representations:

$$\begin{gathered} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H,\Gamma_m-\mathrm{an}} \otimes_{K_m} \mathbb{C}_p \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_p, \text{ leading to} \\ (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H,\Gamma-\mathrm{la}} \otimes_{K_\infty} \mathbb{C}_p \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_p. \end{gathered} \tag{77}$$

The Γ-action on $V_\infty \coloneqq (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H,\Gamma-\mathrm{la}}$ is differentiable, to an action of $\mathrm{Lie}(\Gamma)$, which turns out to be K_∞-linear. Explicitly, for $v \in V_\infty$, we can define

$$\theta_V(v) = \frac{1}{\log \chi(\gamma)} \left.\frac{d}{dt}\right|_{t=0} (\gamma^t v), \tag{78}$$

a canonical element in the image of $\mathrm{Lie}(\Gamma) \to \mathrm{End}_{K_\infty} V_\infty$, which commutes with the Γ-action. Extending scalars, we get the Sen operator $\theta_V \in \mathrm{End}_{\mathbb{C}_p} V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ which commutes with the action of G_K. One can decompose $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \bigoplus_\lambda (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)_\lambda$ into generalised eigenspaces for θ_V. In the case where $K = K(\mu_{p^\infty})$ and $\chi = \chi_{\mathrm{cyc}}$, V is Hodge-Tate if and only if θ_V acts semisimply with integer eigenvalues, in which case those are (minus) the Hodge-Tate weights. More generally, we call the Jordan form of θ_V the Hodge-Tate-Sen weights of V.

3.2 Theory of decompletions

The above theory has two features:

1. A "decompletion" to a subspace which is locally analytic for the action of a p-adic Lie group Γ appearing as a quotient of G_K.

2. Differentiation and analysis of the resulting $\mathrm{Lie}(\Gamma)$-action.

The Tate-Sen formalism [BC08, BC16, Cam22] provides a recipe for the decompletion in quite general context. We follow [Cam22, §4]. Let Π be a profinite group, let (A, A^+)

be a uniform affinoid $\mathbb{Q}_p$-algebra with $\Pi \curvearrowright (A, A^+)$ continuously[7], and let $\chi : \Pi \twoheadrightarrow \mathbb{Z}_p^d$ with $H = \ker \chi$. Let $\{\varpi^\epsilon\}_{\epsilon \in I} \subseteq A^H$ be a system of topologically nilpotent units such that $|\varpi^{\epsilon+\delta}|_x = |\varpi^\epsilon|_x |\varpi^\delta|_x$ for every $x \in \mathrm{Spa}(A, A^+)$, and such that:

- The norm $\|\cdot\|$ on (A, A^+) making A^+ the unit ball and $|\varpi^\epsilon A^+|$ the ball of radius $p^{-\epsilon}$ is nonarchimidean and submultiplicative,
- Π acts by isometries on $(A, \|\cdot\|)$.

If the data (A, Π, χ) satisfies the Colmez-Sen-Tate axioms (CST0)-(CST2) as formulated in [Cam22, §4.1], it is called a Sen theory. The additional axioms (CST1*), (CST2*) give rise to a "strongly decomposable Sen theory".

By [Cam22, Theorem 2.2.6], a continuous $\mathbb{Q}_p$–Banach representation of a compact p–adic Lie group G is locally analytic if and only if there is a lattice $V^0 \subseteq V$, stable under G such that $G \curvearrowright V^0/p$ factors through a finite quotient. This motivates:

Definition 3.1. *[Cam22, Defintion 4.2.5] A continuous A-semilinear Banach representation $\rho : \Pi \times V \to V$ is called locally analytic if there is an open subgroup $\Pi' \subseteq \Pi$ stabilising a lattice $V^0 \subseteq V$ such that Π' acts trivially on V^0/ϖ^ϵ, for some $\epsilon > 0$.*

Example 3.2. *[Cam22, Example 4.2.8], if $\Pi \twoheadrightarrow G$, a compact p–adic Lie group, and W is a (not necessarily finite-dimensional) locally analytic $\mathbb{Q}_p$-Banach representation of G, then $W \widehat{\otimes}_{\mathbb{Q}_p} A$ is a locally analytic representation of Π.*

Definition 3.1 generalises to LB, Fréchet and LF representations. If V is as in Definition 3.1, and $H' \subseteq H$ is open, define the Sen functor

$$S_{H'}(V) := (V^{H'})^{\Gamma_{H'}-\mathrm{la}}, \tag{79}$$

where $\Gamma_{H'} = N_{H'}/H'$, note this is a p-adic Lie group. Axiom (CST1) says (in particular), that for any open subgroup, $H' \subseteq H$, we have closed subalgebras $\{A_{H',n}\}_{n \geq n(H')}$, on which $\Gamma_{H'}$ acts locally analytically, and $A_{H',n}$-linear projections, ("Tate's normalised traces"), $R_{H',n} : A^{H'} \to A_{H',n}$ which are compatible and converge to the identity, $\lim_{n\to\infty} R_{H',n}(x) = x$.

Theorem 3.3. *[Cam22, Theorem 4.3.3] If (A, Π, χ) is a Sen theory, V is a locally analytic A-Banach representation of Π as in Definition 3.1, $\Pi' \subset \Pi$ is an open normal subgroup[8] and $H' = \Pi' \cap H$, then:*

- *V contains a unique[9] $A_{H',n}$-submodule $S_{H',n}(V)$, fixed by H' and stable by Π, which is a locally analytic $\Gamma_{H'}$-representation, and there is an isomorphism of A-semilinear representations*
$$A \widehat{\otimes}_{A_{H',n}} S_{H',n}(V) = V. \tag{80}$$
- *If (A, Π, χ) is strongly decomposable then $S_{H'}(V) = \varinjlim_n S_{H',n}(V)$, so that*
$$A \widehat{\otimes}_{A^{H'}} S_{H'}(V) = V, \tag{81}$$
as A-semilinear representations.

[7] We also assume $A^+/p^s = \varinjlim_{H' \subset_o H} A^{+,H'}$.

[8] Subject to technical hypothesis

[9] There are additional technical constraints that I don't understand well.

If we assume that the Tate's normalised traces in (CST1) are constructed in the same way as the Ax-Sen-Tate theorem, we can differentiate the action $\Pi'/H' \curvearrowright S_{H'}(V)$ and get a $A^{H'}$-linear action $\mathrm{Lie}(\Pi'/H') \times S_{H'}(V) \to S_{H'}(V)$, equivalently, a map $S_{H'}(V) \to S_{H'}(V) \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\Pi'/H')^\vee$, which we extend A–linearly to get

$$\mathrm{Sen}(V) : V \to V \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\Pi'/H')^\vee. \tag{82}$$

If $V = W \widehat{\otimes}_{\mathbb{Q}_p} A$ as in Example 3.2, then there is a "universal" Sen operator, i.e the action can always be seen to factor through

For example, if $A = \mathbb{C}_p$, $\Pi = G_K$, and $V = W \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$, where W is a locally analytic G and $\chi : G_K \to \mathbb{Z}_p^d$, in the situation of Theorem 3.3. Then $\Gamma_{H'}$ is a subquotient of G_K which is a d-dimensional compact p-adic Lie group. Say $H' = G_{K_\infty}$, and $\Gamma_{H'} = \mathrm{Gal}(K_\infty/L)$ for some L/K finite. This turns out to be a strongly decomposable Sen theory, and we get an isomorphism

$$\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} W \cong \mathbb{C}_p \widehat{\otimes}_{K_\infty} (\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} W)^{H', \Gamma_{H'}-\mathrm{la}}. \tag{83}$$

Another example is the following. If $A = \mathbf{A}^\dagger$ as in [BC08, §4.2], $\Pi = G_K$, $\chi = \chi_{\mathrm{cyc}}$, $H_K = \ker \chi_{\mathrm{cyc}}$. The resulting Sen theory allows us to decomplete Galois representations to overconvergent (ϕ, Γ)-modules. Let $\mathbf{B}_K^\dagger := \mathbf{A}_K^\dagger[1/p]$, $\mathbf{B}^\dagger = \mathbf{A}^\dagger[1/p]$ and let V be d-dimensional $\mathbb{Q}_p$-representation of G_K. Set $D^\dagger(V) = (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_K}$. Then

Proposition 3.4. *[BC08, Proposition 4.2.6] $D^\dagger(V)$ is a rank d (ϕ, Γ)-module over $\mathbf{B}_K^\dagger$ and there is a natural isomorphism of G_K-modules*

$$D^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}^\dagger \cong V \otimes_{\mathbb{Q}_p} \mathbf{B}^\dagger. \tag{84}$$

The most important examples come from geometry. Let G be a p-adic Lie group and let V be a locally analytic $\mathbb{Q}_p$-Banach representation of G. Let $X/\mathrm{Spa}(K, K^+)$ be a (log fs) smooth adic space, and let $\tilde{X} \to X$ be a (log) G-Galois proétale covering of X. For example one can take $X = \mathcal{X}_{K^p K_p}$, and $\tilde{X} = \mathcal{X}_{K^p}$, the finite level and perfectoid modular curves, and then $G = K_p \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$. As a simpler example take $X = \mathrm{Spa}(\mathbb{C}_p\langle T^{\pm 1}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T^{\pm 1}\rangle)$, $\tilde{X} = \mathrm{Spa}(\mathbb{C}_p\langle T^{\pm 1/p^\infty}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T^{\pm 1/p^\infty}\rangle)$, then $G = \mathbb{Z}_p$. $k \in G = \mathbb{Z}_p$ acts by $k \cdot T^{\frac{i}{p^m}} = \zeta_{p^m}^{ik} T^{\frac{i}{p^m}}$, for $i \in \mathbb{Z}$. In this case one can show that the locally analytic vectors are $\cup_{m \geq 0} \mathbb{C}_p\langle T^{\pm 1/p^m}\rangle$, on which the Lie algebra action is trivial.

Let $\underline{V}$ be the local system on $X_{\text{prokét}}$ associated to V, then our aim will be to explain:

Theorem 3.5. *[Cam22, Theorem 5.0.3] Associated to V there is a morphism*

$$\theta_{\tilde{X}}(V) : \underline{V} \widehat{\otimes}_{\widehat{\mathbb{Q}_p}} \widehat{\mathcal{O}}_X \to \underline{V} \widehat{\otimes}_{\widehat{\mathbb{Q}_p}} \widehat{\mathcal{O}}(-1) \otimes_{\mathcal{O}} \Omega_X^1(\log). \tag{85}$$

such that $\theta_{\tilde{X}}(V) \wedge \theta_{\tilde{X}}(V) = 0$. It is functorial in V and compatible with pullbacks, and factors through the universal Sen operator

$$\theta_{\tilde{X}} : \widehat{\mathcal{O}}_X \to \underline{\mathrm{Lie}(G)} \widehat{\otimes}_{\widehat{\mathbb{Q}_p}} \widehat{\mathcal{O}}(-1) \otimes_{\mathcal{O}} \Omega_X^1(\log) \tag{86}$$

3.3 Why is $\tilde{H}^i(K^p, \mathbb{C}_p)^{\mathrm{la}}$ a $\widetilde{\mathcal{D}}$-module?

Recall the sheaf $\mathcal{O}_{K^p} := \pi_{HT,*}\mathcal{O}_{\mathcal{X}_{K^p}}$ and the subsheaf $\mathcal{O}_{K^p}^{\mathrm{la}}$ of locally analytic vectors for the $\mathrm{GL}_2(\mathbb{Q}_p)$ action. The image of $\mathcal{O}_{\mathscr{F}\ell} \to \pi_{HT,*}\mathcal{O}_{\mathcal{X}_{K^p}}$ is contained in $\mathcal{O}_{K^p}^{\mathrm{la}}$. In otherwise we have inclusions

$$\mathcal{O}_{\mathscr{F}\ell} \subseteq \mathcal{O}_{K^p}^{\mathrm{la}} \subseteq \mathcal{O}_{K_p}. \tag{87}$$

We differentiate $\mathrm{GL}_2(\mathbb{Q}_p) \curvearrowright \mathcal{O}^{\mathrm{la}}_{K^p}$ and then extend scalars to get a $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C}_p)$-action on $\mathcal{O}^{\mathrm{la}}_{K^p}$. Recall points $x \in \mathrm{Fl}_{/\mathbb{C}_p}$ are associated to Borel subalgebras $\mathfrak{n}_x \subseteq \mathfrak{b}_x \subseteq \mathfrak{g}$. As in [Bei83], define:

$$\begin{aligned}
\mathfrak{g}^0 &:= \mathcal{O}_{\mathrm{Fl}} \otimes \mathfrak{g}, \\
\mathfrak{b}^0 &:= \{f \in \mathfrak{g}^0 : \forall x \in \mathrm{Fl}, f(x) \in \mathfrak{b}_x\}, \\
\mathfrak{n}^0 &:= \{f \in \mathfrak{g}^0 : \forall x \in \mathrm{Fl}, f(x) \in \mathfrak{n}_x\}, \\
U^0 &:= \mathcal{O}_{\mathrm{Fl}} \otimes U(\mathfrak{g}), \\
\tilde{\mathcal{D}} &:= U^0/U^0 \cdot \mathfrak{n}^0,
\end{aligned} \tag{88}$$

the $\mathcal{O}_{\mathrm{Fl}}$-module structure on, for example, $U^0 := \mathcal{O}_{\mathrm{Fl}} \otimes U(\mathfrak{g})$, comes from the rule $[x, f] = \alpha(x)f$ for $x \in \mathfrak{g} \subseteq U(\mathfrak{g})$, $f \in \mathcal{O}_{\mathrm{Fl}}$, where $\alpha : \mathfrak{g} \to T_{\mathrm{Fl}}$ is the usual map of Lie algebras, and we view the tangent space T_{Fl} as a subsheaf of the sheaf of differential operators (which is how it acts on $\mathcal{O}_{\mathrm{Fl}}$). We use the same notation for the sheaves on the adic space $\mathscr{F}\ell$. The $\mathfrak{g}^0$-action on $\mathcal{O}^{\mathrm{la}}_{K^p}$ is the $\mathfrak{g}$-action extended $\mathcal{O}_{\mathscr{F}\ell}$-linearly, i.e. it is $(f \otimes x) \cdot s = f.(x \cdot s)$ for $f \in \mathcal{O}_{\mathscr{F}\ell}, x \in \mathfrak{g}, s \in \mathcal{O}_{\mathscr{F}\ell}$.

Theorem 3.6. *[Pan22, Theorem 4.2.7] The action restricted to $\mathfrak{n}^0$ is trivial, i.e. $\mathcal{O}^{la}_{K^p}$ is a $\tilde{\mathcal{D}}$-module.*

The key ingredient is the following theorem:

Theorem 3.7. *There is a natural isomorphism $\pi^*_{HT}\mathfrak{n}^0 \otimes_{\mathcal{O}_{\mathcal{X}_{K^p}}} \Omega^1_{\mathcal{X}_{K^p}} \otimes \mathcal{O}_{\mathcal{X}_{K^p}}(-1) \cong \theta_{\mathcal{X}_{K_p}}$, where $\theta_{\mathcal{X}_{K^p}}$ is the universal Sen operator as in* (86)*, viewed as a sub-bundle of $\mathfrak{g}\widehat{\otimes}_{\overline{\mathbb{Q}_p}}\widehat{\mathcal{O}}(-1)\otimes_{\mathcal{O}} \Omega^1_{\mathcal{X}}(\log)$.*

For Theorem 3.6 we want to show the action map $\mathfrak{n}^0 \otimes_{\mathcal{O}_{\mathscr{F}\ell}} \mathcal{O}^{\mathrm{la}}_{K^p} \to \mathcal{O}^{\mathrm{la}}_{K^p}$ is 0. Since π_{HT} is surjective and $\mathcal{O}_{\mathcal{X}_{K^p}} \to \Omega^1_{\mathcal{X}_{K^p}}$ is flat, this is the same as (by doing pullback then tensoring) showing that

$$\left(\pi^*_{HT}\mathfrak{n}^0 \otimes_{\mathcal{O}_{\mathcal{X}_{K^p}}} \Omega^1_{\mathcal{X}_{K^p}} \otimes \mathcal{O}_{\mathcal{X}_{K^p}}(-1)\right) \otimes \pi^*_{HT}\mathcal{O}^{\mathrm{la}}_{K^p} \to \Omega^1_{\mathcal{X}_{K^p}} \otimes \mathcal{O}_{\mathcal{X}_{K^p}}(-1) \otimes \pi^*_{HT}\mathcal{O}^{\mathrm{la}}_{K^p} \tag{89}$$

is 0, equivalently by Theorem 3.7, that the action of the Sen operator

$$\theta_{\mathcal{X}_{K^p}} \otimes_{\mathcal{O}_{\mathcal{X}_{K^p}}} \pi^*_{HT}\mathcal{O}^{\mathrm{la}}_{K^p} \to \pi^*_{HT}\mathcal{O}^{\mathrm{la}}_{K^p}, \tag{90}$$

is 0. It suffices to check this locally, and by affineness properties of π_{HT}, we may assume that we can cover $\mathcal{X}_{K^p}$ by perfectoids $V = \mathrm{Spa}(B, B^+)$ and $\mathscr{F}\ell$ by affinoids such that $\pi_{HT}(V) = U$. By definition of $\mathcal{O}^{\mathrm{la}}_{K^p}$, this amounts to showing $\theta_{\mathcal{X}_{K^p}}$ acts by 0 on $B^{K_p-\mathrm{la}}$ (where $K_p \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$ is chosen small enough to stabilise U). Let us explain concretely how the Sen operator arises in this situation. There is a ring extension $B_\infty \to B$ with Galois group $\Gamma = \mathbb{Z}_p(1)$. Let $\chi : G \times \Gamma \to \mathbb{Z}_p$, then $(B_\infty, K_p \times \Gamma, \chi)$ forms a (strongly decomposable) Sen theory. Recall

$$B^{K_p-\mathrm{la}} = (\mathcal{C}^{\mathrm{la}}(K_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B)^{K_p} = (\mathcal{C}^{\mathrm{la}}(K_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_\infty)^{K_p \times \Gamma}, \tag{91}$$

and, by definition, the Sen operator on the $K_p \times \Gamma$-representation $\mathcal{C}^{\mathrm{la}}(K_p, \mathbb{Q}_p) \otimes B_\infty$ arises from the Lie(Γ)-action on $(\mathcal{C}^{\mathrm{la}}(K_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_\infty)^{K_p, \Gamma-\mathrm{la}}$. It follows that $B^{K_p-\mathrm{la}}$ is annihilated by the Sen operator. Thus, we have proven Theorem 3.6 modulo Theorem 3.7.

3.4 p-adic Simpson correspondence

A consequence of (86) is that it is enough to calculate $\theta_{\mathcal{X}_{K^p}}$ for a faithful representation of $\mathfrak{g} = \mathrm{Lie}(K_p)$, so we choose V to be the standard representation $\mathbb{Q}_p^{\oplus 2}$ of $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$. It turns out that the Sen operator associated to the local system $\underline{V}$ on $\mathcal{X}_{K^p}$ is nothing but the Higgs bundle on the other side of the p-adic Simpson correspondence:

Theorem 3.8. *Let $X/\mathrm{Spa}(K, K^+)$ be a (log fs) smooth adic space. Then there is a natural functor*

$$\mathcal{H} : \begin{Bmatrix} \mathbb{Q}_p\text{-local systems} \\ \mathbb{L} \text{ on } X_{\acute{e}t} \end{Bmatrix} \to \begin{Bmatrix} \mathrm{Gal}(\mathbb{C}_p/K)\text{-equivariant Higgs bundles on } X_{C,an} \\ \theta : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_{\mathcal{X}}(-1) \end{Bmatrix} \tag{92}$$

then one has an equality in the derived category $R\nu_*(\mathbb{L} \otimes \widehat{\mathcal{O}}_X) \cong R\Gamma(\theta, \mathcal{H}(\mathbb{L}))$, where the latter is the Dolbeault complex. (explain what a Higgs bundle is). Explicitly, the correspondence is given by taking the 0th graded piece of

$$\mathbb{L} \mapsto R\mu_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}) \tag{93}$$

where μ is the projection from the pro(k)ét to the analytic site. Recall that for $\mathbb{L} = \underline{V}_{\log}$, we also had the de Rham comparison

$$\underline{V} \otimes \mathcal{O}\mathbb{B}^+_{dR} \cong H^1_{dR,log} \otimes \mathcal{O}\mathbb{B}^+_{dR} \tag{94}$$

compatible with filtrations and connections. In particular Griffiths transversality implies

$$\nabla(\mathrm{Fil}^i H^1_{dR,log} \otimes \mathrm{Fil}^j \mathcal{O}\mathbb{B}^+_{dR}) \subseteq (\mathrm{Fil}^{i-1} H^1_{dR,log} \otimes \mathrm{Fil}^j \mathcal{O}\mathbb{B}^+_{dR} + \mathrm{Fil}^i H^1_{dR,log} \otimes \mathrm{Fil}^{j-1} \mathcal{O}\mathbb{B}^+_{dR}) \otimes \Omega^1_X \tag{95}$$

where Ω^1 lives in degree 1. This has the consequence that $\theta = \mathrm{gr}^0(\nabla)$ preserves the Hodge-Tate filtration on $\underline{V}_{\log} \otimes \mathcal{O}_{\mathcal{X}}$. Combined with the fact that it is nilpotent, and that π_{HT} is defined by the Hodge-Tate filtration, this implies that $\theta_{\mathcal{X}_{K^p}} \subseteq \pi^*_{HT}\mathfrak{n}^0 \otimes \Omega^1_X(-1)$.

4 Hodge-Tate decomposition of completed cohomology

The Hodge-Tate decomposition of completed cohomology comes from the interplay of three actions: the Lie algebra action of $\mathfrak{gl}_2(\mathbb{C}_p)$, the "horizontal" Cartan action, and the action of $G_{\mathbb{Q}_p}$.

(The horizontal Cartan). By Theorem 3.6, $\mathfrak{n}^0$ acts trivially on $\mathcal{O}^{\mathrm{la}}_{K^p}$ and hence we get an action of $\mathfrak{b}^0/\mathfrak{n}^0$ on it. Then $\mathfrak{b}^0/\mathfrak{n}^0$ is a trivial $\mathfrak{g}$-equivariant vector bundle on $\mathscr{F}\ell$ with $\Gamma(\mathscr{F}\ell, \mathfrak{b}^0/\mathfrak{n}^0) \cong \mathfrak{h} =: \mathfrak{b}/\mathfrak{n}$. The resulting map $\mathfrak{h} \to \mathfrak{b}^0/\mathfrak{n}^0$ induces a "horizontal" action of the Cartan:

$$\theta_{\mathfrak{h}} : \mathfrak{h} \curvearrowright \mathcal{O}^{\mathrm{la}}_{K^p} \tag{96}$$

(The Lie algebra). This action commutes with the $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C}_p)$-action on $\mathcal{O}^{\mathrm{la}}_{K^p}$ (induced by differentiating $\mathrm{GL}_2(\mathbb{Q}_p) \curvearrowright \mathcal{O}^{\mathrm{la}}_{K^p}$). If $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{C}_p)$ and $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ is the diagonal Cartan, then $-\theta_{\mathfrak{h}}$ induces an action of $S(\mathfrak{h}_0)$ which extends the action of $Z(U(\mathfrak{g}_0)) = S(\mathfrak{h}_0)^{W,\bullet}$.

(The Galois action). On the other hand, the completed cohomology $\tilde{H}^i(K^p, \mathbb{C}_p)^{\mathrm{la}} = \tilde{H}^i(\mathscr{F}\ell, \mathcal{O}^{\mathrm{la}}_{K^p})$ is a $\mathbb{C}_p$-semilinear $G_{\mathbb{Q}_p}$-representation. As in the classical (arithmetic) Sen theory, it is endowed with a $\mathbb{C}_p$-linear endomorphism θ_{Sen}, the Sen operator, whose eigenvalues are, by definition, (-1) times the Hodge-Tate weights. They are related by:

Theorem 4.1. $\theta_{Sen} = \theta_{\mathfrak{h}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *as endomorphisms of $H^i(K^p, \mathbb{C}_p)^{la}$.*

We can consider eigenspaces for the $\mathfrak{b} \subseteq \mathfrak{g}$ action. Let $\mu : \mathfrak{b} \to \mathbb{C}_p$ be a character, with $\mu\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = k_1 a + k_2 d$. Since the horizontal action of $S(\mathfrak{h}_0)$ extends that of $Z(U(\mathfrak{g}_0)) = S(\mathfrak{h}_0)^{W,\bullet}$, it follows that the μ-isotypic part $\tilde{H}^1(K^p, \mathbb{C}_p)^{\mathrm{la}}_{\mu}$ decomposes into eigenspaces for the horizontal Cartan (whenever $\mu\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \neq -1$) action as:

$$\begin{aligned} \tilde{H}^1(K^p, \mathbb{C}_p)^{\mathrm{la}}_{\mu} &= \left(\tilde{H}^1(K^p, \mathbb{C}_p)^{\mathrm{la}}_{\mu}\right)^{(k_2, k_1)} \oplus \left(\tilde{H}^1(K^p, \mathbb{C}_p)^{\mathrm{la}}_{\mu}\right)^{(k_1+1, k_2-1)}, \\ &= M_{\mu,1} \oplus M_{\mu,w}, \end{aligned} \tag{97}$$

i.e. the possible weights for the $\theta_{\mathfrak{h}}$-action are the dotted W-orbit of μ. (Explain this in the talk). Combining with Theorem 4.1, it follows that (97) is a Hodge-Tate decomposition of μ-isotypic part of completed cohomology into weight $-k_1$ and $1-k_2$ parts. This decomposition can be refined further. Let $\chi : G_{\mathbb{Q}} \to \mathbb{Z}_p^{\times}$ be the p-adic cyclotomic character, and let ε_p be the character $\mathbb{Q}_p^{\times} \to \mathbb{Z}_p^{\times} : x \mapsto x|x|$. Let $B \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$ be the (upper-triangular) Borel and let $e_1^i e_2^j t^k$ be the character of $B \times G_{\mathbb{Q}_p}$ given by

$$e_1^i e_2^j t^k \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, g\right) = a^i d^j \varepsilon_p(ad)^k \chi_p(g)^{i+j+k}. \tag{98}$$

Set $-\mu(h) = k \neq 1$. Let

$$N_{k,1} := M_{\mu,1} \otimes e_1^k t^{k_2}, \quad N_{k,w} := M_{\mu,w} \otimes e_1 e_2^{1-k} t^{-1-k_1}, \tag{99}$$

be the twists of $M_{\mu,1}$, $M_{\mu,w}$ regarded as $B \times G_{\mathbb{Q}_p}$-representations. These spaces are closely related to overconvergent modular forms. Let $G = Bw_0B \sqcup B$ be the Bruhat decomposition of $G = \mathrm{GL}_2(\mathbb{Q}_p)$, where w_0 is the nontrivial element of the Weyl group. Then $G/B = Bw_0B/B \sqcup \infty$, where $\infty = B/B$. By abuse we also write ∞ as a point of $\mathscr{F}\ell$. For a character χ of $\mathfrak{h}$ we write $\mathcal{O}^{\mathrm{la},\chi}_{K^p}$ for the χ-isotypic part for the horizontal Cartan action $\theta_{\mathfrak{h}}$. Then Lue Pan defines overconvergent modular forms of weight χ and tame level K^p as $M^{\dagger}_{\chi}(K^p) = \mathcal{O}^{\mathrm{la},\chi}_{K^p,\infty}$, the fiber at ∞. It is clearly a $B \times G_{\mathbb{Q}_p} \times \mathbb{T}_{K^p}$-representation. If $M^{\dagger}_k(K^p\Gamma(p^n))$ are the overconvergent modular forms as defined in Katz-Mazur chapter 13, then $M^{\dagger}_k(K^p) := \varinjlim_n M^{\dagger}_k(K^p\Gamma(p^n))$ is related to $M^{\dagger}_{\chi}(K^p)$ as follows. If $\chi = (k_1, k_2)$ with $k_1 - k_2 = k$, then $M^{\dagger}_{\chi}(K^p) \otimes e_2^{-k} t^{n_1} \cong M^{\dagger}_{\chi}(K^p)$ as $B \times G_{\mathbb{Q}_p} \times \mathbb{T}_{K^p}$ representations. An alternative definition of $M^{\dagger}_{\chi}(K^p)$ is as follows. Let $\omega^{k,\mathrm{sm}}_{K^p} \subseteq \pi_{HT,*}\omega^{\otimes k}_{K^p}$ be the smooth sections for the $\mathrm{GL}_2(\mathbb{Q}_p)$-action (say what this means). Then $M^{\dagger}_k(K^p) = \omega^{\mathrm{sm}}_{K^p,\infty}$. Classical automorphic forms of tame level K^p and weight k are defined as global sections:

$$M_k(K^p) := H^0(\mathscr{F}\ell, \omega^{\mathrm{sm}}_{K^p}) = \varinjlim_{K^p} H^0(\mathcal{X}_{K^pK_p}, \omega^{\otimes k}_{K^pK_p}), \tag{100}$$

they are a $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p} \times \mathbb{T}_{K^p}$ representation also. We can now state

Theorem 4.2. *[Pan22, Theorem 5.4.2] $N_{k,w} \cong M^{\dagger}_{2-k}(K^p)$ for all $k \neq 2$. If $k = 2$ then there is an exact sequence*

$$0 \to M^{\dagger}_0(K^p)/M_0(K^p) \to N_{2,w} \to M_0(K^p) \to 0 \tag{101}$$

If $k \leq -1$ there is an exact sequence

$$0 \to \varinjlim_{K_p} H^1(\mathcal{X}_{K^pK_p}, \omega^{\otimes k}) \to N_{k,1} \to M^{\dagger}_k(K^p) \to 0. \tag{102}$$

If $k = 0$ there is an exact sequence

$$0 \to \varinjlim_{K^p} H^1(\mathcal{X}_{K^pK_p}, \mathcal{O}_{\mathcal{X}_{K^pK_p}}) \to N_{0,1} \to M_0^\dagger(K^p)/M_0(K^p) \to 0, \tag{103}$$

and if $k \geq 2$ there is an isomorphism $N_{k,1} \cong M_k^\dagger(K^p)/M_k(K^p)$.

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