

Barr–Beck–Lurie in families

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1 Barr–Beck–Lurie in families

In this section we present a generalization of the result of [GHK22, Proposition 4.4.5] which is adapted to our setting.

Proposition 1.1. *Given a diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{U} & \mathcal{D} \\ & \searrow p & \swarrow r \\ & \mathcal{B} & \end{array} \quad (1)$$

in \mathbf{Cat}_∞ such that:

- (i) p and r are coCartesian fibrations and U preserves coCartesian edges;
- (ii) U has a left adjoint $F : \mathcal{D} \rightarrow \mathcal{C}$ such that $pF \simeq r$;
- (iii) The adjunction $F \dashv U$ restricts in each fiber to an adjunction $F_b \dashv U_b$. For all $b \in \mathcal{B}$, the functor U_b is conservative, and \mathcal{C}_b admits colimits of U_b -split simplicial objects, which U_b preserves.
- (iv) For any edge $e : b \rightarrow b'$ in \mathcal{B} , the coCartesian covariant transport $e_! : \mathcal{C}_b \rightarrow \mathcal{C}_{b'}$ preserves geometric realizations of U_b -split simplicial objects.

Then, the adjunction $F \dashv U$ is monadic.

Remark 1.2. *In view of the Barr–Beck–Lurie theorem, condition (iii) in Proposition prop1.1 is equivalent to:*

- (iii)' The adjunction $F \dashv U$ restricts in each fiber to a monadic adjunction $F_b \dashv U_b$.

Proof of Proposition prop1.1. We verify the conditions of the Barr–Beck–Lurie theorem [Lur17, Theorem 4.7.3.5].

First we show that U is conservative. We can argue in exactly the same way as [GHK22, Proposition 4.4.5]. Suppose that $f : c \rightarrow c'$ is a morphism in \mathcal{C} such that Uf is an equivalence in \mathcal{D} . Then $e := qUf \simeq pf$ is an equivalence in \mathcal{B} . One can factor f as $c \xrightarrow{\varphi} e_!c \xrightarrow{f'} c'$ where φ is a coCartesian lift of e and f' is a morphism in the fiber $\mathcal{C}_{b'}$ above $b' := p(c')$. Since φ is coCartesian lift of an equivalence, it is an equivalence. Because of the fiberwise monadicity assumption (iii), f' is an equivalence. Therefore f is an equivalence and U is conservative.

Now we will show that \mathcal{C} admits and U preserves colimits of U -split simplicial objects. Let $q : \Delta^{\text{op}} \rightarrow \mathcal{C}$ be a U -split simplicial object, so that Uq extends to a diagram $\widetilde{Uq} : \Delta_{-\infty}^{\text{op}} \rightarrow \mathcal{D}$. Let $f : \Delta_{-\infty}^{\text{op}} \rightarrow \mathcal{B}$ be the underlying diagram in \mathcal{B} . There is a morphism

$$\Delta^1 \times \Delta_{-\infty}^{\text{op}} \rightarrow \Delta_{-\infty}^{\text{op}} \quad (2)$$

which is the identity on $\{0\} \times \Delta_{-\infty}^{\text{op}}$ and carries $\{1\} \times \Delta_{-\infty}^{\text{op}}$ to $[-1] \in \Delta_{-\infty}^{\text{op}}$. It sends each horizontal morphism $\{0\} \times [n] \rightarrow \{1\} \times [n]$ to the unique morphism $[n] \rightarrow [-1]$. Consider the composite

$$P : \Delta^1 \times \Delta_{-\infty}^{\text{op}} \rightarrow \Delta_{-\infty}^{\text{op}} \xrightarrow{f} \mathcal{B}. \quad (3)$$

Now we will take a coCartesian lifts, using the exponentiation for coCartesian fibrations [Lur18, Tag 01VG].

- ★ Let Q be a coCartesian lift of $P|_{\Delta^1 \times \Delta_{-\infty}^{\text{op}}}$ to \mathcal{C} . Then Q is a natural transformation between q and a morphism $q' : \Delta^{\text{op}} \rightarrow \mathcal{C}_b$, where b is the image under f of $[-1] \in \Delta_{-\infty}^{\text{op}}$.
- ★ Let \widetilde{UQ} be a coCartesian lift of P to \mathcal{D} . Then \widetilde{UQ} is a natural transformation between \widetilde{Uq} and a morphism $\widetilde{Uq}' : \Delta_{-\infty}^{\text{op}} \rightarrow \mathcal{C}_b$.

These natural transformations Q and \widetilde{UQ} are uniquely characterised by the property that their components are coCartesian edges [Lur18, Tag 01VG]. Because of the assumption (i) that U preserves coCartesian edges, this unicity implies that $UQ \simeq \widetilde{UQ}|_{\Delta^1 \times \Delta_{-\infty}^{\text{op}}}$. In particular $Uq' : \Delta^{\text{op}} \rightarrow \mathcal{C}_b$ extends to the split simplicial object $\widetilde{Uq}' : \Delta_{-\infty}^{\text{op}} \rightarrow \mathcal{C}_b$. By the fiberwise monadicity assumption (iii), this implies that q' extends to a colimit diagram $\bar{q}' : (\Delta^{\text{op}})^{\triangleright} \rightarrow \mathcal{C}_b$ such that \widetilde{Uq}' is also a colimit diagram. By assumption (iv) and [Lur09, Proposition 4.3.1.10] it then follows that \bar{q}' and $U\bar{q}'$, when regarded as diagrams in \mathcal{C} and \mathcal{D} respectively, are p -colimit diagrams. Now we can argue as in [Lur09, Corollary 4.3.1.11]. We have a commutative diagram

$$\begin{array}{ccc} (\Delta^1 \times \Delta^{\text{op}}) \amalg_{\{1\} \times \Delta_{-\infty}^{\text{op}}} (\{1\} \times (\Delta^{\text{op}})^{\triangleright}) & \xrightarrow{(Q, \bar{q}')} & \mathcal{C} \\ \downarrow & \nearrow s & \downarrow p \\ (\Delta^1 \times \Delta^{\text{op}})^{\triangleright} & \xrightarrow{(f|_{(\Delta^{\text{op}})^{\triangleright}}) \circ \pi} & \mathcal{B} \end{array} \quad (4)$$

in which $\pi : (\Delta^1 \times \Delta^{\text{op}})^{\triangleright} \rightarrow (\Delta^{\text{op}})^{\triangleright} = \Delta_+^{\text{op}} \subseteq \Delta_{-\infty}^{\text{op}}$ denotes the morphism which is the identity on $\{0\} \times \Delta^{\text{op}}$ and which carries $(\{1\} \times \Delta^{\text{op}})^{\triangleright}$ to the cone point. Because the left map is an inner fibration there exists a lift s as indicated by the dashed arrow. Consider now the map $\Delta^1 \times (\Delta^{\text{op}})^{\triangleright} \rightarrow (\Delta^1 \times \Delta^{\text{op}})^{\triangleright}$ which is the identity on $\Delta^1 \times \Delta^{\text{op}}$ and carries the other vertices of $\Delta^1 \times (\Delta^{\text{op}})^{\triangleright}$ to the cone point. Let \bar{Q} denote the composition

$$\Delta^1 \times (\Delta^{\text{op}})^{\triangleright} \rightarrow (\Delta^1 \times \Delta^{\text{op}})^{\triangleright} \xrightarrow{s} \mathcal{C} \quad (5)$$

and define $\bar{q} := \bar{Q}|_{\{0\} \times (\Delta^{\text{op}})^{\triangleright}}$. Then \bar{Q} is a natural transformation from \bar{q} to \bar{q}' which is componentwise coCartesian. Then [Lur09, Proposition 4.3.1.9] implies that \bar{q} is a p -colimit diagram which fits into the diagram

$$\begin{array}{ccc} \Delta^{\text{op}} & \xrightarrow{q} & \mathcal{C} \\ \downarrow & \nearrow \bar{q} & \downarrow p \\ (\Delta^{\text{op}})^{\triangleright} & \xrightarrow{f|_{(\Delta^{\text{op}})^{\triangleright}}} & \mathcal{B} \end{array} \quad (6)$$

By assumption (i), $U\overline{Q}$ is a natural transformation from $U\overline{q}$ to $U\overline{q}'$ which is componentwise coCartesian. Hence [Lur09, Proposition 4.3.1.9] implies that $U\overline{q}$ is a p -colimit diagram. The underlying diagram $f|_{(\Delta^{\text{op}})^{\triangleright}}$ of \overline{q} in \mathcal{B} extends to the split simplicial diagram f and hence admits a colimit in \mathcal{B} . Hence [Lur09, Proposition 4.3.1.5(2)] implies that \overline{q} and $U\overline{q}$ are colimit diagrams in \mathcal{C} and \mathcal{D} respectively. Hence \mathcal{C} admits and U preserves geometric realizations of U -split simplicial objects. \square

References

- [GHK22] David Gepner, Rune Haugseng, and Joachim Kock. ∞ -operads as analytic monads. *Int. Math. Res. Not. IMRN*, (16):12516–12624, 2022.
- [Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
- [Lur17] Jacob Lurie. Higher Algebra. Available at <https://www.math.ias.edu/~lurie/papers/HA.pdf>, 2017.
- [Lur18] Jacob Lurie. Kerodon. <https://kerodon.net>, 2018.