# $p$-adic measures and Iwasawa cohomology

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**Abstract**

These are notes for a learning seminar on Euler systems. These notes basically lifted from other places. The main references are [Col99] and [Col04]. All typos are my own!

## 1 $p$-adic measures

By a $\mathbf{Q}_p$-Banach space we mean a complete normed $\mathbf{Q}_p$-vector space. In this talk all norms are assumed to satisfy the ultrametric inequality. For a compact totally disconnected Hausdorff space $X$ we let $\mathscr{C}^0(X, \mathbf{Q}_p)$ denote the space of continuous functions $X \to \mathbf{Q}_p$ equipped with the sup-norm.

**Definition 1.1.** *We define the space of $\mathbf{Q}_p$-valued $p$-adic measures as $\mathscr{D}_0(X, \mathbf{Q}_p) := \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(\mathscr{C}^0(X, \mathbf{Q}_p), \mathbf{Q}_p)$. Here $\underline{\mathrm{Hom}}_{\mathbf{Q}_p}$ denotes the internal Hom of $\mathbf{Q}_p$-Banach spaces (bounded linear maps equipped with the operator norm).*

The only totally obvious elements are the Dirac measures $\delta_x \in \mathscr{D}_0(X, \mathbf{Q}_p)$, given by evaluation for each $x \in X$. By definition there is a formal integration pairing

$$\int_X : \mathscr{C}^0(X, \mathbf{Q}_p) \times \mathscr{D}_0(X, \mathbf{Q}_p) \to \mathbf{Q}_p : (f, \mu) \mapsto \int_X f d\mu. \tag{1}$$

We note that $\mathscr{D}_0(X, \mathbf{Q}_p)$ is a module for $\mathscr{C}^0(X, \mathbf{Q}_p)$, where $f \cdot \mu$ is determined by $\int_X g(f \cdot \mu) = \int_X gf\mu$. We recall that "functions push back and measures push forward" so that $\mathscr{D}_0(-, \mathbf{Q}_p)$ is covariant in $X$. Further, for each $X, Y$ there is a natural in $X, Y$ map

$$\mathscr{D}_0(X, \mathbf{Q}_p) \times \mathscr{D}_0(Y, \mathbf{Q}_p) \to \mathscr{D}_0(X \times Y, \mathbf{Q}_p), \tag{2}$$

in order to define this map one has to use the isomorphism

$$\mathscr{C}^0(X, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} \mathscr{C}^0(Y, \mathbf{Q}_p) \xrightarrow{\sim} \mathscr{C}^0(X \times Y, \mathbf{Q}_p). \tag{3}$$

From this it follows formally that if $G$ is compact totally disconnected Hausdorff topological group then $\mathscr{D}_0(G, \mathbf{Q}_p)$ is a Hopf algebra object in $\mathbf{Q}_p$-Banach spaces. If $G$ is written multiplicatively then the product (convolution) of measures is given explicitly by the formula

$$\int_G f(x) d(\lambda \star \mu)(x) = \int_G \left( \int_G f(xy) \lambda(x) \right) d\mu(y). \tag{4}$$

If $\eta : G \to \mathbf{Z}_p^\times$ is a continuous character and $H \leqslant G$ is a finite index clopen subgroup, we get a $G$-equivariant homomorphism

$$\{\mu \in \mathscr{D}_0(G, \mathbf{Q}_p) : \|\mu\| \leqslant 1\} \to \mathbf{Z}_p[G/H] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(\eta) : \mu \mapsto \overline{\mu} \otimes \int_H \eta d\mu. \tag{5}$$

This will be used to define specialization maps later.

## 2 Amice transform

Now we specialise to the case when $G = \mathbf{Z}_p$. For an indeterminate $T$ and $x \in \mathbf{Z}_p$ we consider the power series

$$(1+T)^x = \sum_{n \geqslant 0} T^n \binom{x}{n}. \tag{6}$$

We are supposed to think of this as "continuous[1] in $x$ and analytic in $T$" so that if we integrate over $\mathbf{Z}_p$ we will get some kind of analytic function. More precisely we have the theorem of Amice and Mahler:

**Theorem 2.1.** *The map*

$$\mu \mapsto A_\mu(T) := \int_{\mathbf{Z}_p} (1+T)^x d\mu(x) := \sum_{n \geqslant 0} T^n \int_{\mathbf{Z}_p} \binom{x}{n} d\mu(x) \tag{7}$$

*is an isometry and gives an isomorphism of Hopf algebra objects with bounded functions on the rigid open unit disk:*

$$\mathscr{D}_0(\mathbf{Z}_p, \mathbf{Q}_p) \cong \left\{ \sum_{n \geqslant 0} a_n T^n : a_n \in \mathbf{Q}_p, \sup_n |a_n| < \infty \right\} = \mathbf{Z}_p[\![T]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p. \tag{8}$$

In particular $\mathbf{Z}_p[\![T]\!]$ is isomorphic to the unit ball in $\mathscr{D}_0(\mathbf{Z}_p, \mathbf{Q}_p)$. By pushforward functoriality $\mathscr{D}_0(\mathbf{Z}_p, \mathbf{Q}_p)$ carries an action of $\mathbf{Z}_p$. Under the Amice transform the action of $a \in \mathbf{Z}_p$ goes to the action

$$(a \cdot f)(T) := (1+T)^a f(T) \tag{9}$$

of $\mathbf{Z}_p$ on $\mathbf{Z}_p[\![T]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Multiplication of a measure by the continuous function $x = \mathrm{id} : \mathbf{Z}_p \to \mathbf{Q}_p$ goes to the operator $(1+T)\frac{d}{dT}$.

Let $\mathbf{Z}_p[\![\mathbf{Z}_p]\!]$ be the completed group ring. There is a $\mathbf{Z}_p$-equivariant isomorphism $\mathbf{Z}_p[\![T]\!] \xrightarrow{\sim} \mathbf{Z}_p[\![\mathbf{Z}_p]\!]$ determined by $T \mapsto \gamma - 1$ where $\gamma$ is the topological generator of $\mathbf{Z}_p$. This is not completely straightforward to prove: we direct the reader to [Was97, §7.1]. Using the $p-1$ branches of the $p$-adic logarithm we obtain the $p$-adic Mellin transform:

$$\mathscr{D}_0(\mathbf{Z}_p^\times, \mathbf{Q}_p) \xrightarrow{\sim} \mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p. \tag{10}$$

which is equivariant for the respective $\mathbf{Z}_p^\times$-actions.

Now let $(\varepsilon_n)_{n \geqslant 0}$ be a compatible system of $p$-power roots of unity in $\overline{\mathbf{Q}}_p$ with $\varepsilon_0 = 1$. Let $K/\mathbf{Q}_p$ be a finite extension of $\mathbf{Q}_p$. We do not assume that $\varepsilon_1 \in K$. Let $G_K := \mathrm{Gal}(\overline{K}/K)$, $K_n := K(\varepsilon_n)$ and $K_\infty := \bigcup_n K_n$. Put $\Gamma_K := \mathrm{Gal}(K_\infty/K)$ and $\Gamma_n := \mathrm{Gal}(K_\infty/K_n)$. Let $\chi : \Gamma_K \hookrightarrow \mathbf{Z}_p^\times$ be the cyclotomic character induced by $(\varepsilon_n)_{n \geqslant 0}$. By functoriality $\mathscr{D}_0(\Gamma_K, \mathbf{Q}_p)$ carries an action of $\Gamma_K$. By (10) we deduce a $G_K$-equivariant isomorphism

$$\mathscr{D}_0(\Gamma_K, \mathbf{Q}_p) \xrightarrow{\sim} \mathbf{Z}_p[\![\Gamma_K]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p. \tag{11}$$

## 3 Two definitions of Iwasawa cohomology

Let $T$ be a finite $\mathbf{Z}_p$-representation of $G_K$.

**Definition 3.1.** *(i) We define $H^i_{\mathrm{Iw}}(K, T) := \lim_n H^i(K_n, T)$, the transition maps here are the corestriction maps on Galois cohomology.*

*(ii) We define $H^i_{\mathrm{Iw}}(K, T) := H^i(K, \mathbf{Z}_p[\![\Gamma_K]\!] \otimes_{\mathbf{Z}_p} T)$.*

[1] Or locally-analytic.

**Example 3.2.** *Using the first definition of Iwasawa cohomology. By the Kummer map one has $H^1_{\mathrm{Iw}}(K, \mathbf{Z}_p(1)) = \lim_n K_n^\times$, the transition maps here are the norms.*

**Remark 3.3.** *Using the second definition of Iwasawa cohomology. Note that the actions of $\Gamma_K$ and $G_K$ on $\mathbf{Z}_p[\![\Gamma_K]\!] \otimes_{\mathbf{Z}_p} T$ commute (this is just because $\Gamma_K$ is abelian). Hence the $H^i_{\mathrm{Iw}}(K,T)$ defned as (ii) are $\mathbf{Z}_p[\![\Gamma_K]\!]$-modules.*

**Remark 3.4.** *Using the second definition of Iwasawa cohomology. We recall that $\mathbf{Z}_p[\![\Gamma_K]\!]$ can be regarded as the unit ball in $\mathscr{D}_0(\Gamma_K, \mathbf{Q}_p)$. Hence if $n \geqslant 0$ and $\eta : G_K \to \mathbf{Z}_p^\times$ is a continuous character, we get a $G_K$-equivariant homomorphism as in (5):*

$$\mathbf{Z}_p[\![\Gamma_K]\!] \to \mathbf{Z}_p[\mathrm{Gal}(K_n/K)] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(\eta) : \mu \mapsto \overline{\mu} \otimes \int_{\Gamma_n} \eta d\mu, \tag{12}$$

*which induces a* specialization homomorphism*:*

$$H^i_{\mathrm{Iw}}(K,T) \to H^i(K, \mathbf{Z}_p[\mathrm{Gal}(K_n/K)] \otimes_{\mathbf{Z}_p} T(\eta)) = H^i(K_n, T(\eta)). \tag{13}$$

*where we used Shapiro's Lemma. Hence we can think of Iwasawa cohomology as a gadget which simultaneously interpolates the Galois cohomology at all levels at $p$ and all twists at unramified characters with $p$-power conductor.*

**Lemma 3.5.** *The two definitions of $H^i_{\mathrm{Iw}}(K,T)$ are equivalent.*

*Proof.* By an application of Shapiro's lemma there is an isomorphism

$$\lim_n H^i(K_n, T) \cong \lim_n H^i(K, \mathbf{Z}_p[\mathrm{Gal}(K_n/K)] \otimes_{\mathbf{Z}_p} T). \tag{14}$$

where the transition maps on the right are induced by the maps $\mathbf{Z}_p[\mathrm{Gal}(K_{n+1}/K)] \to \mathbf{Z}_p[\mathrm{Gal}(K_n/K)]$. Now one wants to commute the limit with the $H^i(-)$. In order to do this, you will find that you have to use Mittag-Leffler in the following form: if $\{M_n\}_n$ is a tower of finite modules over the tower of rings $\{R_n\}_n = \{\mathbf{Z}_p[\mathrm{Gal}(K_n/K)]\}_n$ satisfying the sheaf condition $R_n \otimes_{R_{n+1}} M_{n+1} \xrightarrow{\sim} M_n$, then $R^1 \lim_n M_n = 0$. $\square$

Let $\eta : \Gamma_K \to \mathbf{Z}_p^\times$ be a continuous character. We recall that we can multiply a measure $\mu$ by $\eta$. It is easily seen that $g(\eta \cdot \mu) = \eta(\overline{g})^{-1}(\eta \cdot \mu)$ for $g \in G_K$. Hence there is an isomorphism of $\mathbf{Z}_p[G_K]$-modules $\mathbf{Z}_p[\![\Gamma_K]\!] \to \mathbf{Z}_p[\![\Gamma_K]\!] \otimes \mathbf{Z}_p(\eta)$ sending $\mu \mapsto (\eta \cdot \mu) \otimes e_\eta$. Hence we get a $\mathbf{Z}_p$-linear isomorphism $i_\eta$ as in the square:

$$\begin{tikzcd}
H^i_{\mathrm{Iw}}(K,T) \arrow[r, "\cong i_\eta"] \arrow[d, equal] & H^i_{\mathrm{Iw}}(K, T(\eta)) \arrow[d, equal] \\
H^i(K, \mathbf{Z}_p[\![\Gamma_K]\!] \otimes_{\mathbf{Z}_p} T) \arrow[r, "\cong"] & H^i(K, \mathbf{Z}_p[\![\Gamma_K]\!] \otimes_{\mathbf{Z}_p} T(\eta))
\end{tikzcd} \tag{15}$$

The isomorphism $i_\eta$ is not $\mathbf{Z}_p[\![\Gamma_K]\!]$-linear: the action gets twisted through $\eta$.

## 4 Coleman power series

Now we specialize to the case when $K = \mathbf{Q}_p$. We recall again that $H^1_{\mathrm{Iw}}(K, \mathbf{Z}_p(1)) = \lim_n K_n^\times \supseteq \lim_n \mathscr{O}_{K_n}^\times =: U_\infty$ by Kummer theory. Put $\pi_n := \varepsilon_n - 1$.

**Theorem 4.1.** *For every $u = (u_n)_{n \geqslant 1} \in U_\infty$ there is a unique power series $f_u(T) \in \mathbf{Z}_p[\![T]\!]^\times$ such that $f_u(\pi_n) = u_n$ for every $n \geqslant 1$.*

The *Coleman map* is the composite

$$U_\infty \xrightarrow{u \mapsto f_u} \mathbf{Z}_p[\![T]\!]^\times \xrightarrow{(1+T)\frac{d}{dT}\log} \mathbf{Z}_p[\![T]\!], \tag{16}$$

this will be used in later talks.

**Example 4.2.** *Let $a \in \mathbf{Z}_p^\times$. If $u_n = (\varepsilon_n^a - 1)/(\varepsilon_n - 1)$ is the system of cyclotomic units, then $f_u(T) = ((1+T)^a - 1)/T$.*

*Proof of Theorem 4.1. Uniqueness:* Follows from Weierstrass preparation (an analytic function on the open disk can only have finitely many zeros inside a closed disk of smaller radius).

*Existence:* We only give a sketch and direct the reader to [Col04, §7.3] for the details. Consider the ring of integers of the tilt: $\mathscr{O}^\flat_{\widehat{K}_\infty} := \lim_{x \mapsto x^p} \mathscr{O}_{\widehat{K}_\infty}/\pi_1$. This contains the element $\overline{\pi} := (\dots, \overline{\pi}_2, \overline{\pi}_1, 0)$ of norm $|\overline{\pi}|_\flat = p^{-p/(p-1)} < 1$. So $T \mapsto \overline{\pi}$ determines a map $\mathbf{F}_p[\![T]\!] \to \mathscr{O}^\flat_{\widehat{K}_\infty}$. As it turns out, the image of this map is the ring of integers in the *field of norms*: $E^+_{\mathbf{Q}_p} = \lim_{x \mapsto x^p} \mathscr{O}_{K_n}/\pi_1$. It is not hard to show that $E^+_{\mathbf{Q}_p}$ contains $\overline{u} = (\dots, \overline{u}_2, \overline{u}_1, \overline{u}_1^p)$. So certainly there exists $f \in \mathbf{Z}_p[\![T]\!]$ with $f(\overline{\pi}) = \overline{u}$, which gives the Coleman power series "approximately".

Now we use a fixed-point iteration to get the Coleman power series on the nose. To this end we introduce the operator $N : \mathbf{Z}_p[\![T]\!] \to \mathbf{Z}_p[\![T]\!]$ determined by $N(g)((1+T)^p - 1) = \prod_{\zeta^p = 1} g((1+T)\zeta - 1)$. It is not hard to show that $N(g)(\pi_n) = N_{K_{n+1}/K_n}(g(\pi_{n+1}))$ and $\pi_1 \mid (N(g) - g)$ for any $g$.

Taking our "approximate $f$" from before, we set $f_u := \lim_{k \to \infty} N^k(f)$. It can be shown that this converges when $f \in \mathbf{Z}_p[\![T]\!]^\times$, which holds when $|\overline{u}|_\flat = 1$, and fortunately one can easily reduce to this case. We have $N(f_u) = f_u$ by construction. Put $v_n = f_u(\pi_n)$, we know from the properties of $N$ above that $N_{K_{n+1}/K_n}(v_{n+1}) = v_n$ and $v_n = u_n (\operatorname{mod} \pi_1)$, we want to show that $v_n = u_n$.

It can be shown that the norm maps restrict to maps

$$1 + \pi_1^k \mathscr{O}_{K_{n+1}} \xrightarrow{N_{K_{n+1}/K_n}} 1 + \pi_1^{k+1} \mathscr{O}_{K_n}, \tag{17}$$

for every $n, k \geqslant 0$. Put $w_n = v_n/u_n$, then $w_n \in 1 + \pi_1 \mathscr{O}_{K_n}$ and $N_{K_{n+1}/K_n}(w_{n+1}) = w_n$. Hence by (17) and induction $w_n = N_{K_{n+k}/K_n}(w_{n+k}) \in 1 + \pi_1^{k+1} \mathscr{O}_{K_n}$ for every $k \geqslant 0$, so that $w_n = 1$. □

## References


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