p-adic Hodge tate talk

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September 27, 2022

Abstract

Let me know if there are mistakes and typos

Contents

1 Recap

Let K be a p-adic field. Recall the 1-d Tate module $\mathbb{Z}_p(1)$ with a choice of generator $t \in \mathbb{Z}_p(1)$. It is a G_K-module, with action given by:

$$g(t) = \chi(g)t, \tag{1}$$

and $\mathbb{Z}_p(i)$ ($i \in \mathbb{Z}$) is the free $\mathbb{Z}_p$-module with generator t^i where G_K acts by χ^i. Recall also, if $M \in G_K - \text{mod}$ we define its Tate twist $M(i) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$.

Set $\mathbb{C}_K \coloneqq \widehat{\overline{K}}$, which is a G_K-module since G_K can be identified with the group of isometric isomorphisms of $\mathbb{C}_K$. With this in mind, define the *Hodge-Tate period ring* B_{HT}:

$$B_{HT} = \bigoplus_{q \in \mathbb{Z}} \mathbb{C}_K(q) \simeq \mathbb{C}_K[t, t^{-1}]. \tag{2}$$

It is a graded ring, the multiplication comes from the maps $\mathbb{C}_K(q) \otimes \mathbb{C}_K(q') \to \mathbb{C}_K(q+q')$. The second isomorphism is the map from $c \otimes t^i \to ct^i$. You can see the G_K-action respects the grading. That $(B_{HT})^{G_K} \simeq K$ follows from:

Theorem 1.1 (Tate-Sen)**.** *For $i = 0, 1$ and any continuous character $\eta : G_K \to \mathbb{Z}_p^\times$, we have:*

$$H^i(G_K, \mathbb{C}_K(\eta)) \cong \begin{cases} 0 \text{ if } \eta(I_K) \text{ infinite,} \\ K \text{ if } \eta(I_K) \text{ finite.} \end{cases} \tag{3}$$

Of particular use is:

$$H^i(G_K, \mathbb{C}_K(n)) \cong \begin{cases} 0 & \text{if } n \neq 0 \\ K & \text{if } n = 0. \end{cases} \tag{4}$$

2 The equivalence of categories

We define the category:

$$\mathsf{Rep}_{\mathbb{C}_K}(G_K) = \left\{ \begin{array}{c} \text{f.d. } \mathbb{C}_K\text{-vector spaces } W \text{ equipped with} \\ \text{a continuous } \mathbb{C}_K\text{-semilinear } G_K\text{-action.} \end{array} \right\} \tag{5}$$

It is an abelian category endowed with tensors, direct sums, and duality satisfying all the usual properties. Semilinear means $g(cw) = g(c)g(w)$, for $c \in \mathbb{C}_K$ and $w \in W$. We define:

$$W\{q\} := W(q)^{G_K}, \tag{6}$$

this is a K-vector space. By left exactness of $(-)^{G_K}$ and the flat extension of scalars $K(-q) \otimes_K -$, we get an injection (K-linear, G_K-equivariant, where it's acting diagonally):

$$K(-q) \otimes_K W\{q\} \hookrightarrow K(-q) \otimes W(q) \simeq W, \tag{7}$$

the last isomorphism is from multiplication. Extending further to $\mathbb{C}_K$, we get maps $\mathbb{C}_K(-q) \otimes_K W\{q\} \hookrightarrow W$. Lastly, summing over all q, we get a map:

$$\xi_W : \bigoplus_q \mathbb{C}_K(-q) \otimes_K W\{q\} \to W. \tag{8}$$

The important lemma is:

Lemma 2.1 (Serre-Tate)**.** *ξ_W is injective.*

Therefore, $\sum_q \dim_K W\{q\} \le \dim_{\mathbb{C}_K} W$, and you see that equality here is the same as ξ_W being an isomorphism.

Definition 2.2. *$W \in \mathsf{Rep}_{\mathbb{C}_K}(G_K)$ is Hodge-Tate if ξ_W is an isomorphism. $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ is the full subcategory of Hodge-Tate objects.*

In which case, we define the Hodge-Tate weights $h_q = \dim_K W\{q\}$ for all q where this isn't 0.

[**Aside**: choosing a basis in each $W\{q\}$ gives a (non-canonical) isomorphism

$$W \cong \bigoplus_q \mathbb{C}_K(-q)^{h_q}. \tag{9}$$

The Tate-Sen theorem then shows that this can be taken as a definition of Hodge-Tate. By this description, it's easy to see that $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ is closed under tensors and direct sums. The dual has the negated weights.]

As usual, we are going to translate $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ into "semilinear algebraic data". For $W \in \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ consider

$$\underline{D}(W) = (B_{HT} \otimes W)^{G_K} = \bigoplus_q W\{q\}. \tag{10}$$

This defines a functor, and the description on the left, together with Lemma 2.1, shows us what the target category is:

$$\underline{D} : \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K) \to \mathsf{Gr}_{K,f} := \left\{ \begin{array}{c} \text{f.d. } \mathbb{Z}\text{-graded } K\text{-vector spaces } D, \\ \text{morphisms are grading preserving linear maps.} \end{array} \right\} \tag{11}$$

We can go back in the reverse direction. Let $D \in \mathsf{Gr}_{K,f}$. Then $B_{HT} \otimes_K D$ is a graded $\mathbb{C}_K$-vector space[1], and we set:

$$\underline{V}(D) = \mathrm{gr}^0(B_{HT} \otimes_K D) = \bigoplus_q \mathbb{C}_K(-q) \otimes_K D_q, \tag{12}$$

which gives an exact functor $\underline{V} : \mathsf{Gr}_{K,f} \to \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$. Now, we consider $\underline{V}(\underline{D}(-))$. Let $W \in \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$, and consider first the following composite, γ_W:

$$\begin{aligned} \gamma_W : B_{HT} \otimes_K \underline{D}(W) \hookrightarrow B_{HT} \otimes_K (B_{HT} \otimes_{\mathbb{C}_K} W) &\to B_{HT} \otimes_{\mathbb{C}_K} W \\ a \otimes b \otimes w &\mapsto ab \otimes w. \end{aligned} \tag{13}$$

It is G_K-equivariant, and grading preserving. Now consider this map in degree 0. It takes:

$$\underline{V}(\underline{D}(W)) = \mathrm{gr}^0(B_{HT} \otimes_K D) = \bigoplus_q \mathbb{C}_K(-q) \otimes W\{q\} \to W, \tag{14}$$

exactly as ξ_W. Therefore, by Lemma 2.1 it is an isomorphism. Even more: you can see that in degree n, γ_W is the $\mathbb{Z}_p(n)$-twist of ξ_W, and so γ_W is an isomorphism.

Next, consider $\underline{D}(\underline{V}(D))$. Since $\underline{V}(D)$ is Hodge-Tate, we get an isomorphism (G_K-equivariant, grading preserving):

$$\gamma_{\underline{V}(D)} : B_{HT} \otimes_K \underline{D}(\underline{V}(D)) \simeq B_{HT} \otimes_{\mathbb{C}_K} \underline{V}(D). \tag{15}$$

Now pass to G_K-invariants. We get the chain of equalities:

$$\begin{aligned} \underline{D}(\underline{V}(D)) &\simeq \bigoplus_r (\underline{V}(D) \otimes_{\mathbb{C}_K} \mathbb{C}_K(r))^{G_K} \\ &= \bigoplus_r (\bigoplus_q \mathbb{C}_K(r-q) \otimes_K D_q)^{G_K} \\ &= \bigoplus_r D_r = D, \end{aligned} \tag{16}$$

where we used the Tate-Sen theorem going from the second to the third line. Therefore:

Theorem 2.3. *The functors $\underline{D}$ and $\underline{V}$ are quasi-inverses, setting up an equivalence of categories:*

$$\underline{D} : \mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K) \rightleftarrows \mathsf{Gr}_{K,f} : \underline{V}. \tag{17}$$

2.1 The category $\mathsf{Rep}^{HT}_{\mathbb{Q}_p}(G_K)$.

Recall that $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$ is the category of continuous representations of G_K into f.d. $\mathbb{Q}_p$ vector spaces. (The formalism of admissible representations is directly applicable in this case but not directly for $\mathbb{C}_K$, because of the semilinearity).

Definition 2.4. *$V \in \mathsf{Rep}_{\mathbb{Q}_p}(G_K)$ is called Hodge-Tate if $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V \in \mathsf{Rep}_{\mathbb{C}_K}(G_K)$ is Hodge-Tate. The full subcategory is denoted $\mathsf{Rep}^{HT}_{\mathbb{Q}_p}(G_K)$.*

[1] Set $\mathrm{gr}^n(B_{HT} \otimes_K D) = \bigoplus_q \mathbb{C}_K(n-q) \otimes_K D_q$.

We define $D_{HT} : \mathsf{Rep}^{HT}_{\mathbb{Q}_p} \to \mathsf{Gr}_{K,f}$ by:

$$D_{HT}(V) = \underline{D}(\mathbb{C}_K \otimes_{\mathbb{Q}_p} V) = (B_{HT} \otimes_{\mathbb{Q}_p} V)^{G_K}. \tag{18}$$

The functor D_{HT} is faithful, but it is not full. To see this, by the Tate-Sen theorem (see talk 4), it follows that for any finitely ramified character η, $D_{HT}(\mathbb{Q}_p) \cong D_{HT}(\mathbb{Q}_p(\eta))$ but $\mathbb{Q}_p$, $\mathbb{Q}_p(\eta)$ admit no maps in $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$.

2.2 Properties of D and V

I would say to probably read Brinon and Conrad if you are interested in the details of these first two.

Proposition 2.5 (Exactness). *$\mathsf{Rep}_{\mathbb{C}_K}(G_K)$ (and so $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$) are stable under subobjects and quotients, and $\underline{D}$ (resp. D_{HT}) is exact on $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ (resp. $\mathsf{Rep}^{HT}_{\mathbb{Q}_p}(G_K)$).*

Proposition 2.6 (Compatibility with tensors and duals). *For W, $W' \in \mathsf{Rep}^{HT}_{\mathbb{C}_K}$, there are natural isomorphisms $\underline{D}(W \otimes W') \cong \underline{D}(W) \otimes \underline{D}(W')$ and a natural isomorphism $\underline{D}(W^\vee) \cong \underline{D}(W)^\vee$. Pretty much the same holds for D_{HT}.*

The above has all implicitly depended on the base field K. In the next proposition we make this explicit with the notation $\underline{D} = \underline{D}_K : \mathsf{Rep}_{\mathbb{C}_K}(G_K) \to \mathsf{Gr}_{K,f}$. Let K'/K be finite and $\widehat{K^{un}}$ be as usual, all contained in a fixed $\overline{K} \subset \mathbb{C}_K$.

Let $W \in \mathsf{Rep}_{\mathbb{C}_K}(G_K)$. Because $G_{K'} \subset G_K$, we get a natural map $K' \otimes_K \underline{D}_K(W) \to \underline{D}_{K'}(W)$ in $\mathsf{Gr}_{K',f}$, (and the same with $\widehat{K^{un}}$). Recall $G_{\widehat{K^{un}}} = I_K$. The below says "Hodge-Tate" is the same if we pass to a finite extension, or restrict to the inertia.

Proposition 2.7 (Scalar extension). *The just described maps in $\mathsf{Gr}_{K',f}$, $\mathsf{Gr}_{\widehat{K^{un}},f}$, are isomorphisms.*

As a warning note that $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ is not closed under extensions. An elementary proof is available in Brinon and Conrad but it might be best explained using Sen theory.

2.3 Why is it called p-adic Hodge theory?

Faltings proved:

Theorem 2.8. *If X is a smooth proper scheme over K, then the étale cohomology $H^i_{ét}(X_{\overline{K}}, \mathbb{Q}_p) \in \mathsf{Rep}^{HT}_{\mathbb{Q}_p}(G_K)$, and $D_{HT}(H^i_{ét}(X_{\overline{K}}, \mathbb{Q}_p)) \cong \bigoplus_q H^{i-q}(X, \Omega^q_{X/K})$, the Hodge cohomology.*

This is a $p-$adic analogue of the comparison between de Rham and singular cohomology for a smooth manifold (where the isomorphism comes from integration over cycles, Stokes's theorem).

3 Sen theory

The main idea of Sen theory is to differentiate the Galois action to get an operator called the Sen operator ϕ, and then seen how this controls the decomposition. It appears to be independent of the period ring formalism. We will see that being Hodge-Tate is the same as ϕ acting semisimply with integer eigenvalues.

3.1 Setup

We begin with the following simple result:

Proposition 3.1. *Let G be a top. group and let B be a top. ring with $G \curvearrowright B$ continuously. There is a natural bijection:*

$$H^1_{\text{cont}}(G, GL_d(B)) \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Isoclasses of free continuous } B\textit{-semilinear} \\ G\textit{-representations of rank } d. \end{array} \right\} \tag{19}$$

Proof. Let V be such a representation, and let $\alpha(g)$ be the matrix of g with respect to some basis. What I mean is that

$$g(e_i) = \sum_j a_{ij}(g) e_j, \text{ for some } a_{ij}(g) \in B, \tag{20}$$

and $\alpha(g) = (a_{ij}(g))$. Then $\alpha(gh) = \alpha(g)g(\alpha(h))$ (cocycle condition). If α' is the matrix wrt. some other basis, and P is the change of basis matrix, then $P\alpha'(g) = \alpha(g)g(P)$ (coboundary condition). Lastly, any cocycle defines a representation into B^d. □

First, notation. K_∞/K is a ramified $\mathbb{Z}_p$-extension living inside $\overline{K}$, $H := G_{K_\infty}$, $\Gamma := \mathrm{Gal}(K_\infty/K)$, with topological generator γ, $\chi : \Gamma \to \mathbb{Z}_p^\times$ is multiplicative character, $\Gamma_m := \Gamma^{p^m}$ with topological generator $\gamma_m := \gamma^{p^m}$, and $K_m := K_\infty^{\Gamma_m}$. The most important example of this would be a cyclotomic extension. Here is a picture:

$$\begin{tikzcd} K \arrow[r, dash] \arrow[rrr, dash, bend left, "\Gamma_0=\Gamma"] & K_1 \arrow[r, dash] \arrow[rr, dash, bend right, "\Gamma_1"'] & K_2 \arrow[r, dotted, dash] \arrow[r, dash, bend right, "\Gamma_2"'] & K_\infty \arrow[r, dash, "H"] & \overline{K} \end{tikzcd} . \tag{21}$$

Because of Proposition 3.1 we will start looking at various homology groups. Firstly, a "strong version" of Hilbert's theorem 90:

Proposition 3.2. $H^1_{\text{cont}}(H, GL_d(\mathbb{C}_K)) = 1$. *Therefore, by an "inflation-restriction" exact sequence (see Weibel 6.7.3), we get an iso:*

$$j : H^1_{\text{cont}}(\Gamma, GL_d(\widehat{K_\infty})) \simeq H^1_{\text{cont}}(G_K, GL_d(\mathbb{C}_K)) \tag{22}$$

We also have a "decompletion" result:

Proposition 3.3. *The natural map*

$$\iota : H^1_{\text{cont}}(\Gamma, GL_d(K_\infty)) \to H^1_{\text{cont}}(\Gamma, GL_d(\widehat{K_\infty})) \tag{23}$$

is an isomorphism, and any cocycle in $H^1_{\text{cont}}(\Gamma, GL_d(K_\infty))$ is cohomologous to a cocycle with values in $GL_d(K_r)$, if we take r large enough.

At the level of isoclasses of semilinear reps, this amounts to the following. Let W be a d–dimensional $\mathbb{C}_K$-semilinear rep of H, and set $\widehat{W}_\infty = W^H$. By Proposition 3.2, $W \simeq \widehat{W}_\infty \otimes_{\widehat{K_\infty}} \mathbb{C}_K$, and by Proposition 3.3, we can chase the isoclass $[\widehat{W}_\infty]$ back, to a representation W_r defined over some K_r such that:

$$W_r \otimes_{K_r} \widehat{K}_\infty \simeq \widehat{W}_\infty. \tag{24}$$

(Think of it like, the maps between the H^1_{cont}'s are extension of scalars, (when we view them as the isoclasses), and Galois descent is what undoes this). Now set

$$W_\infty = \{K\text{-finite vectors } w \in \widehat{W_\infty}\}, \tag{25}$$

where K-finite means that $\dim_K K\text{-span}(\Gamma w) < \infty$. This is a Γ-stable K_∞-vector space, containing W_r, and it is d–dimensional by a short argument using (24). Therefore, we have four d-dimensional semilinear reps over $K_r, K_\infty, \widehat{K_\infty}, \mathbb{C}_K$, respectively:

$$W_r \to W_\infty \to \widehat{W_\infty} \to W \tag{26}$$

where each is isomorphic to the next after extending scalars and inflating to the larger Galois group.

3.2 The Sen operator ϕ

Now, fix a K_r-basis $\{e_1, \dots, e_d\}$ of W_r. It will also be a K_∞-basis of W_∞ and a $\mathbb{C}_K$-basis of W. Let $\rho : \Gamma_r \to GL_d(K_r)$ be the matrix wrt this basis.

Definition 3.4. *The Sen operator ϕ associated to W is the linear endomorphism of W_r whose matrix wrt this basis is given by:*

$$\Phi = \log(\rho(\gamma_r))/\log(\chi(\gamma_r)), \tag{27}$$

and its linear extensions to W_∞, and W.

I am glossing over the fact that you can do these log's (because $\nu(\rho(\gamma_r) - 1) > c + d$, for $c, d \in \mathbb{Z}$ which come from Tate's normalised traces).

The main theorem is the following alternative characterisation:

Theorem 3.5. *The Sen operator ϕ is the unique K_∞-linear endomorphism of W_∞ with the following property.*

For all $w \in W_\infty$, there is an open subgroup Γ_ω of Γ such that:

$$\sigma(w) = \exp(\phi \log \chi(\sigma))w \tag{28}$$

for all $\sigma \in \Gamma$.

The expression $\exp(\phi \log \chi(\sigma))$ is a K_∞-linear endomorphism.

Proof. Write $w = \lambda_1 e_1 + \cdots + \lambda_d e_d$. There are $r_1, \dots, r_d$ such that $\lambda_i \in K_{r_i} = K_\infty^{\Gamma_{r_i}}$ (this is where we use the K-finiteness). Set $\Gamma_w = \Gamma_r \cap \Gamma_{r_1} \cap \cdots \cap \Gamma_{r_n}$. Every $\sigma \in \Gamma_w$ takes the form $\sigma = \gamma_r^a$ for some $a \in \mathbb{Z}_p$. Because $\rho(\gamma_r)$ takes values in K_r, we have that $\rho(\sigma) = \rho(\gamma_r)^a$. Now, as matrices with entries in K_r, we have:

$$\exp(\Phi \log \chi(\sigma)) = \exp(a \log \rho(\gamma_r)) = \rho(\gamma_r)^a = \rho(\sigma), \tag{29}$$

and because all the λ_i are fixed by σ, it follows that $\exp(\phi \log \chi(\sigma))w = \sigma(w)$, for all $\sigma \in \Gamma_w$. I am omitting uniqueness but it is not hard, maybe you can see it already. □

We may use the notation ϕ_W to denote dependence on W. In that case $\phi_{W_1 \oplus W_2} = \phi_{W_1} \oplus \phi_{W_2}$, $\phi_{W_1 \otimes W_2} = \phi_{W_1} \otimes 1 + 1 \otimes \phi_{W_2}$, and $\phi_{\mathrm{Hom}(W_1, W_2)} = (\phi_{W_1})^* - (\phi_{W_2})_*$. Now, it follows from the formula (28) that for $w \in W_\infty$:

$$\phi(w) = \frac{1}{\log \chi(\gamma)} \frac{d}{dx}\Big|_{x=0} (\gamma^x w) = \frac{1}{\log \chi(\gamma)} \lim_{n \to \infty} \frac{\gamma^{p^n}(w) - w}{p^n}. \tag{30}$$

It follows from this expression that ϕ is Γ-linear on W_∞ and G_K-linear on W. Also, if $w = t \in \mathbb{C}_K(q)$, then w is K-finite, and we calculate:

$$\phi(w) = \frac{1}{\log(\chi(\gamma))} \left.\frac{d}{dx}\right|_{x=0} (\chi(\gamma)^{qx} w) = qw. \tag{31}$$

So ϕ is just multiplication by q on $\mathbb{C}_K(q)$. Thus we see that ϕ acts semisimply with integer coefficients, if W is Hodge-Tate. We aim to prove the converse.

Theorem 3.6. $\ker\phi = W^{G_K} \otimes_K \mathbb{C}_K$.

Proof. The formula (30) shows that G-invariants belong to the kernel. Because ϕ is G_K-linear, $\ker\phi$ is G_K-stable. So consider $(\ker\phi)_\infty$ as before: we have $(\ker\phi)_\infty \otimes_{K^\infty} \mathbb{C}_K = \ker\phi$, and the Sen operator (which is 0), just comes from the one on $(\ker\phi)_\infty$ extended linearly. But by looking at formula (30) for one direction, and Theorem 3.5 for the other, we can see that $\phi(w) = 0$ is equivalent to Γw being finite, equivalently, the Γ-action is continous for the discrete topology on $(\ker\phi)_\infty$, equivalently, the Γ-action factors through an open subgroup Γ_r of Γ. Therefore, combining Hilbert's theorem 90 (that $H^1(\Gamma/\Gamma_r, GL_n(K_\infty)) = 1$) with Proposition 3.1 shows that $(\ker\phi)_\infty$ has a basis of G_K-invariants. $\square$

Now, using this, for $q \in \mathbb{Z}$ we can naturally identify $\ker(\phi + qI) = W(q)^{G_K} \otimes_K \mathbb{C}_K = W\{q\}$, whence it follows that:

Theorem 3.7. *W is Hodge-Tate iff ϕ acts semisimply with integer eigenvalues.*

By applying Theorem 3.6 to the representation $\mathrm{Hom}_{\mathbb{C}_K}(W_1, W_2)$ one can deduce:

Proposition 3.8. *$W_1, W_2 \in \mathsf{Rep}_{\mathbb{C}_K}(G_K)$ are isomorphic iff ϕ_{W_1} is similar to ϕ_{W_2}.*

Example 3.9. *Let ρ be the 2-d representation of G_K on $(\mathbb{C}_K)^2$ with matrix given by:*

$$\rho(\sigma) = \begin{pmatrix} 1 & \log\chi(\sigma) \\ 0 & 1 \end{pmatrix}. \tag{32}$$

It is an extension of $\mathbb{C}_K(0)$ by itself. Now, differentiation of $\rho(\sigma)^t$ as in (30), yields:

$$\phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{33}$$

which isn't semisimple, so ρ isn't Hodge-Tate and we see that $\mathsf{Rep}^{HT}_{\mathbb{C}_K}(G_K)$ isn't closed under extensions.

References