Locally analytic vectors of completed cohomology talk

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Abstract

This is the notes for a series of talks given at an informal "p-adic seminar" in Oxford in Trinity Term 2022.

Contents

1 Introduction to modular curves

1.1 Elliptic curves, complex tori, and lattices

Reference for this section is [KDSB73, Katz, Appendix 1.1]. Let $\Lambda \subseteq \mathbb{C}$ be a lattice. Then $\mathbb{C}/\Lambda$ is a complex torus. If

$$\wp_\Lambda(z) := \frac{1}{z^2} + {\sum_{\ell\in\Lambda}}' \left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2}\right), \tag{1}$$

is the Weierstrass $\wp$-function associated to Λ, then the map $\mathbb{C}/\Lambda \to \mathbb{P}^2_{\mathbb{C}}$ given by $z+\Lambda \mapsto [\wp_\Lambda(z) : \wp'_\Lambda(z) : 1] =: [x : y : 1]$, for $z \neq 0$, and $0 \mapsto [0:1:0]$, is holomorphic with holomorphic inverse, with image the curve E_Λ cut out (on $\mathbb{A}^2_{\mathbb{C}}$) by:

$$E_\Lambda : y^2 = 4x^3 - g_{2,\Lambda}x - g_{3,\Lambda}, \tag{2}$$

where $g_{2,\Lambda}$, $g_{3,\Lambda}$ are (rescaled) Eisenstein series. This sends the invariant differential dz to $d\wp(z)/\wp'(z) = dx/y$. In the other direction, if (E,ω) is an elliptic curve with invariant differential, then $\Lambda(E,\omega) = \left\{\int_\gamma \omega : \gamma \in H_1(E,\mathbb{Z})\right\}$ is a lattice in $\mathbb{C}$, called the *lattice of periods*. These operations are inverses, and note that $\Lambda(E,\lambda.\omega) = \lambda.\Lambda(E,\omega)$, for $\lambda\in\mathbb{C}$, so the bijection descends to isomorphism classes of complex tori (as Riemann surfaces). It also respects the addition structure [DS06, §1.4].

1.2 Modular forms

Reference for this section is [KDSB73, Katz, Appendix 1.1 & 1.2.] Usually, we think of a modular form (of full level $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ and weight k as either:

- A degree $-k$ homogeneous function $\mathbb{F}$ of isoclasses of complex elliptic curves with differential (E,ω): so $\mathbb{F}(E,\lambda\omega) = \lambda^{-k}\mathbb{F}(E,\omega)$,
- A degree $-k$ homogeneous function of all lattices $\Lambda \subseteq \mathbb{C}$: so $F(\lambda.\Lambda) = \lambda^{-k}F(\Lambda)$,
- An invariant (holomorphic) differential on $\mathbb{H}$ of degree $k/2$ for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{H}$.

The correspondence between these two notions is as follows: If $f(z)(dz)^{k/2}$ is an invariant differential, we get such a function F of lattices by setting $F(\Lambda) = \omega_2^{-k}f(\omega_1/\omega_2)$, where $\{\omega_1,\omega_2\}$ is a basis for Λ, with $\Im(\omega_1/\omega_2) > 0$. We obtain a function $\mathbb{F}$ of elliptic curves with differential by evaluating on the lattice of periods: $\mathbb{F}(E,\omega) := F(\Lambda(E,\omega))$.

It is common to isolate f from the differential, to arrive at the definition of a modular form of weight k as:

- A holomorphic function for $\tau\in\mathbb{H}$ satisfying the the transformation rule:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-k}f(\tau). \tag{3}$$

Since $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \in \Gamma(1)$, a modular form (of level 1) is 1-periodic and we can view it as a function of $q = e^{2\pi i\tau}$. This is equivalent to looking at its Fourier expansion:

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n(f)q^n \tag{4}$$

if this is finite-tailed, i.e. belongs to $\mathbb{C}((q))$ (resp. has no tail, i.e. belongs to $\mathbb{C}[[q]]$), then f is called meromorphic / holomorphic at infinity. If we unravel the definitions this is the same as asking for:

$$\mathbb{F}(E_\tau, dx/y) \in \mathbb{C}((q)) \text{ or } \mathbb{C}[[q]], \tag{5}$$

where E_τ is the family defined by:

$$E_\tau : y^2 = 4x^3 - \frac{1}{12}E_4(\tau)x - \frac{1}{216}E_6(\tau). \tag{6}$$

This family can be rewritten as a family defined over $\mathbb{Q}((q))$, known as the Tate elliptic curve Tate_q. The Eisenstein series are themselves modular forms, with Fourier expansions:

$$E_{2k} = \frac{1}{2\zeta(2k)} \sum_{(m,n)}{}' \frac{1}{(m+n\tau)^{2k}} = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{7}$$

and then the Tate family is defined by:

$$\mathrm{Tate}_q : y^2 = 4x^3 - \frac{1}{12}E_4(q)x - \frac{1}{216}E_6(q), \tag{8}$$

where $E_4(q), E_4(q) \in \mathbb{Q}((q))$ are the Eisenstein series now viewed as formal Laurent series in q. Therefore, we can obtain the q-expansions of modular forms by evaluation on the Tate family. This is how we will obtain q-expansions from the algebraic perspective.

1.3 The algebraic perspective.

From now on, all schemes are assumed to be at least over $\mathbb{Q}$.

Definition 1.1. *[Sai13, Definition 1.22] An elliptic curve over a $\mathbb{Q}$-scheme S is a proper smooth morphism $p : E \to S$, together with a choice of zero section $O : S \to E$, such that the fibers $E_{\overline{x}}$ of p over a geometric point $\bar{x} : \mathrm{Spec}(\overline{\mathbb{Q}}) \to S$ are isomorphic to an elliptic curve (= a connected algebraic curve of genus 1 over $\overline{\mathbb{Q}}$).*

Then, E/S carries the structure of an abelian group scheme over S, with O as its zero-section [KM85, Theorem 2.1.2]. We define $\underline{\omega}_{E/S} := p_*(\Omega^1_{E/S})$; it is a fact [KM85, §2.2.1] that this is a line bundle on S.

We follow [KDSB73, Katz, Appendix 1.2]. First, restrict to affine $S = \mathrm{Spec}(R)$. Imitating the previous, a modular form of weight k and level 1 is a "function" f sending pairs $(E/S, \omega) \to R$, where $\omega \in \underline{\omega}_{E/S}$ is a nowhere vanishing section, such that:

- $f(E/S, \omega)$ depends only on the S-isoclass of $(E/S, \omega)$.
- f is homogeneous of degree $-k$: $f(E, \lambda.\omega) = \lambda^{-k} f(E, \omega)$,
- f commutes with base change: if $g : R \to R'$ is any morphism, $S' = \mathrm{Spec}(R')$, then $f(E \times_S S'/S', (\mathrm{Spec}(g))^*\omega) = g(f(E/S, \omega))$.

The q−expansion of such a form is defined to be its value on the Tate elliptic curve (over $\mathrm{Spec}(\mathbb{Q}((q)))$. Given such a modular form, the element

$$f(E/S, \omega)\omega^{\otimes k} \in H^0(S, \underline{\omega}^{\otimes k}_{E/S}) \tag{9}$$

is a global section independent of the choice of ω. So finally, we can globalise the definition of a modular form of level 1 and weight k, meromorphic at ∞, to be a "function":

$$f : \left\{ \begin{matrix} \text{elliptic curves } E/S \\ \text{(over any base scheme } S) \end{matrix} \right\} \to H^0(S, \underline{\omega}_{E/S}^{\otimes k}), \tag{10}$$

such that $f(E/S)$ depends only on the isoclass of E/S over S, and:

- f commutes with base change: if $\varphi : S' \to S$ is any morphism of schemes, then $f(E \times_{S,\varphi} S'/S') = \varphi^* f(E/S)$.

1.4 Level structures.

Reference for this section is [DR73, §IV.3]. Fix $K \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be a congruence subgroup of level N. This means that there is a number N, called the *level*, such that:

$$K \supseteq \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})) =: \Gamma(N), \tag{11}$$

and moreover N is minimal with this property. Let $\overline{K}$ be the image of K in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Let N be the level of K, let $E[N] \subseteq E$ denote the sub-S-group scheme of N-torsion. A K-level structure on E is an equivalence class of isomorphisms of the form:

$$\iota : E[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2_S, \tag{12}$$

subject to $\iota \sim \iota'$ if $\iota = \overline{h} \circ \iota'$ for some $\overline{h} \in \overline{K}$. Denote the class by $[\iota]_K$. As in [DR73, Définition 3.2], we define the moduli functor:

$$\mathcal{M}_K(S) := \{\text{pairs } (E/S, [\iota]_K)\}/\sim, \tag{13}$$

where $(E/S, [\iota]_K) \sim (E'/S, [\iota']_K)$ if there is an isomorphism $\varphi : E \to E'$ over S with $\varphi^*[\iota]_K = [\iota']_K$.

A modular form of weight k and level K, meromorphic at ∞, is then a "function" f, which assigns to a class $[(E/S, [\iota]_K)]$ in $\mathcal{M}_K(S)$ (for any scheme S), an element of $H^0(S, \underline{\omega}_{E/S}^{\otimes k})$, compatible with base change.

The complex points of the Tate curve Tate_{q^N} are usually viewed as a complex torus (multiplicatively), as $\mathbb{C}^\times/q^{N\mathbb{Z}}$; then a trivialisation of its N-torsion is given by a maps of the form $(\mathbb{Z}/N\mathbb{Z})^2 \ni (i,j) \mapsto \zeta_N^i q^{mj}$. More generally, Tate_{q^N} admits level K-structures $[\iota]_K$ (not unique): the q-expansions of f are the values $f(\mathrm{Tate}(q^N)/\mathrm{Spec}(\mathbb{Q}((q))), [\iota]_K)$, as $[\iota]_K$ ranges [KDSB73, Katz, §1.2].

1.5 Modular curves and automorphic line bundles

If $N \geq 3$, then the moduli problem is representable by an affine $\mathbb{Q}$–scheme Y_K. See the remark under [DR73, Définition 3.2], combine with [KM85, Scholie 4.7.0] and use the rigidity of level N structures [KM85, Corollary 2.7.1]. This is the (open) modular curve of level K. By the general formalism of moduli problems, this implies the existence of a universal elliptic curve with level K structure, $(E_K/Y_K, [\tilde{\iota}]_K)$, such that every family $(E/S, [\iota]_K)$ is obtained uniquely as a base change of $(E_K/Y_K, [\tilde{\iota}]_K)$, i.e., for all S, there is a unique $\varphi : S \to Y_K$ such that

$$\begin{tikzcd} (E/S, [\iota]_K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (E_K/Y_K, [\tilde{\iota}]_K) \arrow[d] \\ S \arrow[r, "\exists! \varphi"] & Y_K \end{tikzcd} \tag{14}$$

is Cartesian. (The problem when the level is 2, for instance, is that $[-1]$ is still an automorphism of elliptic curves with level $\Gamma(2)$. Therefore the moduli problem is not rigid and so can't be representable). This means that we can redefine a modular form of weight k and level K, meromorphic at ∞, as a section $f \in H^0(Y_K, \underline{\omega}^{\otimes k}_{E_K/Y_K})$.

1.5.1 Modular forms holomorphic at ∞ and the compactified curve X_K.

Recall [Sai13, §2.1] that we can map:

$$\{\text{isoclasses of elliptic curves } E/S/\mathbb{Q}\} \to H^0(S, \mathcal{O}_S), \tag{15}$$

by sending an elliptic curve to its j-invariant j_E. Since the functor

$$H^0(S, \mathcal{O}_S) \cong \operatorname{Hom}(S, \mathbb{A}^1_{\mathbb{Q}}), \tag{16}$$

and on geometric points, an elliptic curve is uniquely determined up to isomorphism by its j-invariant, we tend to view $\mathbb{A}^1_{\mathbb{Q}}$ as a moduli space for isomorphism classes of elliptic curves, called the j-line. In any case, by Yoneda, we get a map $Y_K \to \mathbb{A}^1_{\mathbb{Q}}$, which extends to:

$$Y_K \to \mathbb{A}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}} = \text{“the projective } j\text{-line”}. \tag{17}$$

Then the compactification X_K is defined to be the normalisation [Aut, 29.53] of $Y_K \to \mathbb{P}^1_{\mathbb{Q}}$ [KDSB73, Katz, §1.4]. The upshot is that X_K is smooth and proper over $\mathbb{Q}$, and the boundary $X_K - Y_K$, (called the cusps), is a scheme finite étale over $\mathbb{Q}$, and there is an open immersion $Y_K \hookrightarrow X_K$ as an affine algebraic curve which is finite over $\mathbb{A}^1_{\mathbb{Q}}$ [KM85, Proposition 8.2.2].

$$\begin{tikzcd} Y_K \arrow[r, hook, "\text{open}"] \arrow[d, "j"] & X_K \arrow[d, "j"] \\ \mathbb{A}^1_{\mathbb{Q}} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{Q}} \end{tikzcd} \tag{18}$$

X_K represents a moduli problem of “generalised elliptic curves with level K structure”, and $X_K - Y_K$ can be identified with the isomorphism classes of the level-K structures on the Tate_{q^N}.

There is a line bundle $\underline{\omega}$ on X_K [KM85, §10.13], whose restriction to Y_K is $\underline{\omega}_{E_K/Y_K}$, and whose restriction to the cusps is only the $\mathbb{Q}[[q]]$-span of the canonical differential of the Tate elliptic curve. Therefore, sections $f \in H^0(X_K, \underline{\omega}^{\otimes k})$ correspond to modular forms of level K and weight k, holomorphic at ∞. This space is denoted $M_k(K, \mathbb{Q})$, the subspace $H^0(X_K, \underline{\omega}^{\otimes k}(-\infty))$ of forms vanishing at the cusps (cusp forms), is denoted $S_k(K, \mathbb{Q})$.

1.6 The locally symmetric spaces

For a lattice, e.g. $\Lambda \subseteq \mathbb{R}^2$, we define a level K-structure to be a trivialisation of the N-torsion (where N = the level of K) of $\mathbb{R}^2/\Lambda$, up to $\overline{K}$-isomorphism (just as with elliptic curves). There are bijections:

$$\begin{aligned} &(\text{elliptic curves over } \mathbb{C} \text{ with level } K \text{ structure})/\cong \\ \leftrightarrow&(\text{complex lattices with level } K \text{ structure})/\mathrm{GL}_1(\mathbb{C}) \\ \leftrightarrow&[(\text{lattices} \subseteq \mathbb{R}^2 \text{ with level } K \text{ structure}) \times (\text{complex structures on } \mathbb{R}^2)]/\mathrm{GL}_2(\mathbb{R}) \\ \leftrightarrow&[(\text{lattices} \subseteq \mathbb{Q}^2 \text{ with level } K \text{ structure}) \times (\text{complex structures on } \mathbb{R}^2)]/\mathrm{GL}_2(\mathbb{Q}) \\ \leftrightarrow&[(\text{lattices} \subseteq \mathbb{A}^2_f \text{ with level } K \text{ structure}) \times (\text{complex structures on } \mathbb{R}^2)]/\mathrm{GL}_2(\mathbb{Q}) \\ \leftrightarrow&\mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/K). \end{aligned} \tag{19}$$

The first identification in (19) was described in Section 1.1. For the second, recall that a complex structure on $\mathbb{R}^2$ is a homomorphism:

$$\psi : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}^2). \tag{20}$$

These carry a transitive $\mathrm{GL}_2(\mathbb{R})$-action by $M.\psi = M\psi M^{-1}$. You can check that if $\psi(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $\mathrm{Stab}(\psi) \cong \mathrm{GL}_1(\mathbb{C})$, so we identify complex structures on $\mathbb{R}^2$ with $\mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_1(\mathbb{C})$. This gives the second bijection in (19). For the third bijection in (19), we note that every $\mathrm{GL}_2(\mathbb{R})$-orbit is represented by a rational lattice. The fourth bijection in (19) comes from the correspondence:

$$\begin{aligned} \{\text{lattices } \Lambda_{\mathbb{Q}} \subseteq \mathbb{Q}^2\} &\leftrightarrow \{\text{lattices } \Lambda_{\mathbb{A}_f} \subseteq \mathbb{A}_f^2\} \\ \Lambda_{\mathbb{Q}} &\mapsto \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \\ \Lambda_{\mathbb{A}_f} \cap \mathbb{Q} &\leftarrow\!\shortmid \Lambda_{\mathbb{A}_f}. \end{aligned} \tag{21}$$

For the last bijection, we make two observations. Firstly, the lattice $\widehat{\mathbb{Z}}^2 \subseteq \mathbb{A}_f^2$ with the canonical level K structure from the class of:

$$\iota = \mathrm{id} : (\mathbb{A}_f^2/\widehat{\mathbb{Z}}^2)[N] = (\mathbb{Z}/N\mathbb{Z})^2 \to (\mathbb{Z}/N\mathbb{Z})^2, \tag{22}$$

is stabilised precisely by K, so since the action is transitive we can identify

$$\{\text{lattices} \subseteq \mathbb{A}_f^2 \text{ with level } K \text{ structure}\} = \mathrm{GL}_2(\mathbb{A}_f)/K. \tag{23}$$

Secondly, we note that:

$$\{\text{complex structures on } \mathbb{R}^2\} \cong \mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{H}^{\pm}, \tag{24}$$

because $\mathrm{GL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}^{\pm}$ by Möbius transformations, and the stabiliser of i is $\mathrm{GL}_1(\mathbb{C})$. So we get the identification on complex points (in fact on $\overline{\mathbb{Q}}$-points):

$$Y_K(\mathbb{C}) \leftrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/K) \leftrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A})/Z_\infty K_\infty K), \tag{25}$$

where $K_\infty = \mathrm{SO}_2(\mathbb{R})$, and $Z_\infty \subseteq \mathrm{GL}_2(\mathbb{R})$ is the diagonal torus. The latter description shows that the complex points have the structure of a locally symmetric space.

If $g \in \mathrm{GL}(\mathbb{A}_f)$ and $K', K \subseteq \mathrm{GL}(\mathbb{A}_f)$ are two congruence subgroups, such that $g^{-1}K'g \subseteq K$, then we get a well defined map:

$$\begin{aligned} \mathrm{GL}_2(\mathbb{A}_f)/K' &\to \mathrm{GL}_2(\mathbb{A}_f)/K \\ xK' &\mapsto xgK. \end{aligned} \tag{26}$$

By the formula (25), this induces a morphism

$$c_g : Y_{K'}(\mathbb{C}) \to Y_K(\mathbb{C}), \tag{27}$$

which is finite étale. More generally [DR73, §3.14], the map

$$(E/S, [\iota]_{K'}) \mapsto (E/S, [g \circ \iota]_K), \tag{28}$$

sends elliptic curves E/S with level K' structure to elliptic curves with level K structure, which yields a morphism of schemes $c_g : Y_{K'} \to Y_K$. This extends [DR73, Proposition 3.19] to the compactifications to give $c_g : X_{K'} \to X_K$.

1.7 Tame levels and completed cohomology

Consider the following general setup [Eme06b, §2.1]. Let G be a compact locally $\mathbb{Q}_p$-analytic group, with a decreasing neighbourhood basis of 1 by normal compact subgroups:

$$G = G_0 \supset G_1 \supset \cdots \supset G_r \supset \dots, \tag{29}$$

acting on a tower of right G-spaces with G-equivariant maps:

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_r \leftarrow \dots, \tag{30}$$

such that G_r acts trivially on X_r, and $X_r \to X_r'$ is Galois with Galois group G_r/G_r'. Let $\mathcal{V}_0$ be a local system of free finite rank $\mathbb{Z}_p$-modules on X_0 and $\mathcal{V}_r$ = the pullback to X_r. Then

$$\tilde{H}^n(\mathcal{V}) := \varprojlim_s \varinjlim_r H^n(X_r, \mathcal{V}_r/p^s) \tag{31}$$

is an admissible (in the sense of [Eme17, Proposition-Definition 6.2.3], or [ST02a, §3]), continuous $\mathbb{Q}_p$-Banach representation of G [Eme06b, Theorem 2.2.1]. (We can also do this with compact supports, and dually there is a completed homology).

We have to play this game in a more general setting to apply to the modular curves, but the gist is the same. Let K^p be a fixed compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$, and K_p an open compact subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, which we should view as being variable. Firstly, we have

$$H^i(Y_{K^pK_p}(\mathbb{C}), \underline{\mathbb{Z}/p^s}) \cong H^i_{\mathrm{Betti}}(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s), \tag{32}$$

where we endow $Y_{K^pK_p}(\mathbb{C})$ with the analytic topology, for the purposes of the Betti cohomology. [Here by $\underline{\mathbb{Z}/p^s}$, I mean the locally constant sheaf].

With this formalism, we consider the *completed cohomology of tame level* K^p:

$$\begin{aligned} \tilde{H}^i(K^p, \mathbb{Z}_p) &= \varprojlim_s \varinjlim_{K_p} H^i_{\mathrm{Betti}}(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s), \\ \tilde{H}^i(K^p, \mathcal{O}_{\mathbb{C}_p}) &= \tilde{H}^i(K^p, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}, \\ \tilde{H}^i(K^p, \mathbb{C}_p) &= \tilde{H}^i(K^p, \mathbb{Z}_p) \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p, \end{aligned} \tag{33}$$

It is an admissible $\mathbb{Q}_p$-Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Eme06b, Theorem 0.1], also the remark under (the proof of) [Eme06b, Theorem 2.2.16]). The action of $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ is as follows. For a compact open K_p, set $K_p' = gK_pg^{-1} \cap K_p$. Thus $g^{-1}K^pK_p'g \subset K^pK_p$, and as in (27), we get a finite étale map $Y_{K^pK_p'}(\mathbb{C}) \to Y_{K^pK_p}(\mathbb{C})$, and hence a pullback map on cohomology:

$$c_g^* : H^i_{\mathrm{Betti}}(Y_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s) \to H^i_{\mathrm{Betti}}(Y_{K^pK_p'}(\mathbb{C}), \mathbb{Z}/p^s), \tag{34}$$

thus via c_g^* we get an action on the directed system $\{H^i_{\mathrm{Betti}}(Y_{K^pK_p'}(\mathbb{C}), \mathbb{Z}/p^s)\}_{K_p \subseteq \mathrm{GL}_2(\mathbb{Q}_p)}$, and hence the direct limit

$$\varinjlim_{K_p} H^i_{\mathrm{Betti}}(Y_{K^pK_p'}(\mathbb{C}), \mathbb{Z}/p^s) \tag{35}$$

is endowed with a $\mathrm{GL}_2(\mathbb{Q}_p)$-action. This is compatible as s varies, leading to $\tilde{H}^i(K^p, \mathbb{Z}_p)$ being a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation.

The completed cohomology groups $\tilde{H}^i(K^p, \mathbb{Z}_p)$ are also a Galois representation, [Eme06a, §2.4]. By the comparison theorem for étale cohomology [AGV73, Exposé XI, Théorème

4.4], once an isomorphism of fields $\iota : \mathbb{C}_p \to \mathbb{C}$ (which exists for none other reason than that they are algebraically closed fields of the same cardinality), is fixed, we get a canonical isomorphism

$$H^i_{\text{ét}}(Y_{K^pK_p} \times_{\mathbb{Q}} \mathbb{C}_p, \mathbb{Z}/p^s) \cong H^i_{\text{Betti}}(Y_{K^pK_p'}(\mathbb{C}), \mathbb{Z}/p^s), \tag{36}$$

and $G_{\mathbb{Q}_p}$ acts on the left hand side: its action on the embeddings $\mathbb{Q} \hookrightarrow \mathbb{C}_p$ gives endomorphisms of $Y_{K^pK_p} \times_{\mathbb{Q}} \mathbb{C}_p$, the pullbacks of which induces an action on $\tilde{H}^i(K^p, \mathbb{Z}_p)$. It commutes with $\mathrm{GL}_2(\mathbb{Q}_p)$, so $\tilde{H}^i(K^p, \mathbb{Z}_p)$ becomes a $G_{\mathbb{Q}_p} \times \mathrm{GL}_2(\mathbb{Q}_p)$ representation. The locus where the action is differentiable, to an action of $\mathrm{Lie}(\mathrm{GL}_2(\mathbb{Q}_p)) = \mathfrak{gl}_2(\mathbb{Q}_p)$, is precisely the $\mathbb{Q}_p$-*locally analytic vectors*. Recall (see [ST02b, §3] or [Eme17, Definition 3.5.3]), that a representation V of a p-adic Lie group G is called locally analytic if the orbit map $\mathrm{ev}_v : g \mapsto gv \in \mathcal{C}^{la}(G, V)$; this is then differentiable to a map $d\mathrm{ev}_v \in \mathcal{C}^{la}(T(G), V)$ which restricts to a map $d_1\mathrm{ev}_v : T_1(G) = \mathrm{Lie}(G) \to V$, giving a Lie algebra representation. We denote these subspaces by:

$$\begin{aligned} \tilde{H}^i(K^p, \mathbb{Q}_p)^{\mathrm{la}} &\subseteq \tilde{H}^i(K^p, \mathbb{Q}_p), \\ \tilde{H}^i(K^p, \mathbb{C}_p)^{\mathrm{la}} &\subseteq \tilde{H}^i(K^p, \mathbb{C}_p), \end{aligned} \tag{37}$$

Then $\mathfrak{g} := \mathfrak{gl}_2(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ acts on the latter, restricting to an action of $\mathfrak{b} = \mathrm{Lie}(\mathrm{B}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. One of Lue Pan's main aims, is to compute a Hodge-Tate decomposition of $\tilde{H}^i(K^p, \mathbb{C}_p)^{\mathrm{la}}_{\mu_k}$, where μ_k is the character of $\mathfrak{b}$ sending $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ to kd.

1.8 The Hecke action on completed cohomology

The reference for this part is [Hid86, p.564-566]. Again, let $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ be an open compact, let $g \in \mathrm{GL}_2(\mathbb{A}_f)$, and set

$$K^g = gKg^{-1} \cap K, \quad K_g = g^{-1}Kg \cap K. \tag{38}$$

The group isomorphism $[g] : K_g \to K^g : x \mapsto gxg^{-1}$ induces an isomorphism $[g] : Y_{K_g} \to Y_{K^g}$. There are also natural maps $Y_{K_g} \to Y_K$, $Y_{K^g} \to Y_K$ induced by the inclusion of levels $K^g, K_g \subseteq K$: these are finite étale coverings, and hence we get a trace map on cohomology:

$$\mathrm{tr}_{Y_g/Y} : H^i(Y_{K_g}(\mathbb{C}), \mathbb{Z}/p^s) \to H^i(Y_K(\mathbb{C}), \mathbb{Z}/p^s), \tag{39}$$

and the pullback of $Y_{K^g} \to Y_K$ is called $\mathrm{res}_{Y^g/Y} : H^i(Y_K, \mathbb{Z}/p^s) \to H^i(Y_{K^g}, \mathbb{Z}/p^s)$. The composite $\mathrm{tr}_{Y_g/Y} \circ [g]^* \circ \mathrm{res}_{Y^g/Y}$ defines a endomorphism of $H^i(Y_K(\mathbb{C}), \mathbb{Z}/p^s)$ which depends only on the double coset KgK. This is called the Hecke operator and denoted by T_g or $[KgK]$. This induces an action of the Hecke algebra $\mathcal{H}(K\backslash\mathrm{GL}_2(\mathbb{A}_f)/K, \mathbb{Z}/p^s)$ of double cosets with coefficients in $\mathbb{Z}/p^s$. The multiplication in the Hecke algebra comes from identifying it with the algebra of compactly supported K-biinvariant functions on $\mathrm{GL}_2(\mathbb{A}_f)$ endowed with the convolution product. Let S be the finite set of primes ℓ where K_ℓ is not a hyperspecial[1] maximal compact subgroup of $\mathrm{GL}_2(\mathbb{Q}_\ell)$. These are called the ramified primes of K. We use the superscripts $\mathbb{A}_f^S$, K^S to denote the groups away from these primes. Then

$$\mathcal{H}^{\mathrm{sph}}(K, \mathbb{Z}/p^s) := \mathcal{H}(K^S\backslash\mathrm{GL}_2(\mathbb{A}_f^S)/K^S) \tag{40}$$

[1] K_ℓ is hyperspecial if $K_\ell \cong H(\mathbb{Z}_\ell)$ for some $H \leq \mathrm{GL}_2$ such that $H(\mathbb{Q}_\ell) = \mathrm{GL}_2(\mathbb{Q}_\ell)$ and $H_{\mathbb{F}_\ell}$ is connected reductive.

is called the spherical Hecke algebra. For $\ell \notin S$ denote $\mathcal{H}^{\mathrm{sph}}(K_\ell, \mathbb{Z}) := \mathcal{H}(K_\ell \backslash \mathrm{GL}_2(\mathbb{Q}_\ell)/K_\ell, \mathbb{Z})$, then, the Satake isomorphism [ST98, Chapter 4] (applied to GL_2), gives:

$$\mathcal{S} : \mathcal{H}^{\mathrm{sph}}(K_\ell, \mathbb{Z}) \otimes \mathbb{Z}[\ell^{\pm 1/2}] \xrightarrow{\sim} \mathbb{Z}[X_1^{\pm 1}, X_2^{\pm 1}]^{S_2} \otimes \mathbb{Z}[\ell^{\pm 1/2}], \tag{41}$$

in particular $\mathcal{H}^{\mathrm{sph}}(K_\ell, \mathbb{Z})$ injects into a commutative ring and so is commutative. Therefore the spherical Hecke algebra (40) is commutative.

Applying this to completed cohomology, we see that $\mathcal{H}^{\mathrm{sph}}(K^p K_p, \mathbb{Z}/p^s)$ acts on each $H^i(Y_{K^p K_p}(\mathbb{C}), \mathbb{Z}/p^s)$ and hence,

$$\varprojlim_s \varprojlim_{K_p} \mathcal{H}^{\mathrm{sph}}(K^p K_p, \mathbb{Z}/p^s) \text{ acts on } \tilde{H}^i(K^p, \mathbb{Z}_p), \tag{42}$$

and the same thing with $\mathbb{Z}_p$ replaced by $\mathbb{C}_p$, $\mathbb{Q}_p$ coefficients, etc. The left-hand side in (42) is called the big Hecke algebra. This commutes with the $\mathrm{GL}_2(\mathbb{Q}_p)$ and $G_{\mathbb{Q}_p}$-actions. This is how systems of Hecke eigenvalues arise in completed cohomology.

1.9 Why is completed cohomology important?

See Calegari-Emerton's survey article [CE12].

- As you can see from (25), the definition of completed cohomology generalises to arithmetic quotients of connected reductive groups G over $\mathbb{Q}$ - this is the full generality of Emerton's original definition [Eme06b, §2.2].

- It provides a candidate to extend (on the automorphic side) the (p−adic) Langlands correspondence, to allow the Galois side to be enlarged beyond representations which are just de Rham at p, and in general, with continous families of Hodge-Tate-Sen weights. See [Eme14, §2.1.6, §3].

- It can be used to give a construction of eigenvarieties. See [Eme06b, Theorem 0.7], also [Eme06b, §2.3].

- The Iwasawa dimensions of $\tilde{H}_i(K^p, \mathbb{Z}_p)$. If $G_0 \leq G$ is a small enough open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, then the completed *homology* groups $\tilde{H}_i(K^p, \mathbb{Z}_p)$ are finitely generated $\mathbb{Z}_p[[G_0]]$-modules. The Iwasawa dimensions of these modules are conjectured [CE12, Conjecture 1.5].

- The locally analytic vectors in completed cohomology are related to overconvergent modular forms, see [Pan22, Theorem 1.0.1, Theorem 1.0.2], also [Cam22, Theorem 1.1.7].

- It can be expressed as the sheaf cohomology of Scholze's infinite level modular curve, [Sch15, Theorem IV.2.1], also [Pan22, Theorem 4.4.6].

2 The Hodge-tate period map

2.1 The adic spaces

Fix a choice of p-adic complex numbers $\mathbb{C}_p$. Then $X_K \times_{\mathbb{Q}} \mathbb{C}_p$ is smooth and proper over $\mathbb{C}_p$. There is an adification[2] functor[3]:

$$\begin{matrix}\{\text{smooth proper schemes}/\mathbb{C}_p\} \\ \{\text{finite type affine schemes}/\mathbb{C}_p\}\end{matrix} \xrightarrow{(-)^{\mathrm{ad}}} \{\text{analytic adic spaces}/\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})\} : \tag{43}$$

firstly, you have a GAGA functor, given on affine schemes of finite type over $\mathbb{C}_p$ by $\mathrm{Spec}(\mathbb{C}_p[T_1,\dots,T_n]/I) \mapsto \bigcup_{i=0}^{\infty} \mathrm{Sp}(\mathbb{C}_p\langle p^{-i}T_1,\dots,p^{-i}T_n\rangle/I)$, and secondly an adification functor on analytic adic spaces, given on affinoids by $\mathrm{Sp}(A) \mapsto \mathrm{Spa}(A, A^\circ)$. This construction can be globalised, by gluing, they are functorial, and satisfy a universal property for morphisms of ringed spaces. Moreover, sheaves $\mathcal{F}$ on such schemes can be associated to sheaves $\mathcal{F}^{\mathrm{ad}}$ on the adification.

Denote by $\mathcal{X}_K := (X_K \times_{\mathbb{Q}} \mathbb{C}_p)^{\mathrm{ad}}$, $\mathcal{Y}_K := (X_K \times_{\mathbb{Q}} \mathbb{C}_p)^{\mathrm{ad}}$ the associated adic spaces to X_K, Y_K.

Theorem 2.1. *[Sch15, Theorem III.1.2] There is a unique perfectoid space $\mathcal{X}_{K^p}$ with:*

$$\mathcal{X}_{K^p} \sim \varprojlim_{K_p} \mathcal{X}_{K^pK_p}, \tag{44}$$

Here the $\sim$ means that $|\mathcal{X}_{K_p}| \xrightarrow{\sim} \varprojlim_{K_p} |\mathcal{X}_{K^pK_p}|$ on topological spaces, and on structure sheaves, that $\mathcal{X}_{K^p}$ has a cover by open affinoids $\mathrm{Spa}(A, A^+)$, such that $\varinjlim_{A_i} A_i \to A$ has dense image, where the limit is over all affinoid A_i such that the open immersion $\mathrm{Spa}(A, A^+) \hookrightarrow X_i$ factors through $\mathrm{Spa}(A_i, A_i^+)$. Similarly to Section 1.7, the inverse limit $\varprojlim_{K_p} \mathcal{X}_{K^pK_p}$ has a $\mathrm{GL}_2(\mathbb{Q}_p)$-action, which we transfer to $\mathcal{X}_{K^p}$. Scholze [Sch15, Theorem IV.2.1], [Pan22, Theorem 4.4.6], has shown that there is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$, $G_{\mathbb{Q}_p}$, and Hecke-equivariant isomorphism:

$$H^i(K^p, \mathbb{C}_p) \xrightarrow{\sim} H^i(\mathcal{X}_{K^p}, \mathcal{O}_{\mathcal{X}_{K^p}}). \tag{45}$$

Let $\mathscr{F}\ell = \mathbb{P}^{1,\mathrm{ad}}$ be the adic space associated to $\mathbb{P}^1_{\mathbb{C}_p}$. We will construct a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism $\pi_{HT} : \mathcal{X}_{K^p} \to \mathscr{F}\ell$, the Hodge-Tate period map. If we set $\mathcal{O}_{K^p} = \pi_{HT,*}\mathcal{O}_{\mathcal{X}_{K^p}}$, then it is a fact that:

$$H^i(\mathcal{X}_{K^p}, \mathcal{O}_{\mathcal{X}_{K^p}}) \cong H^i(\mathscr{F}\ell, \mathcal{O}_{K^p}). \tag{46}$$

Pan [Pan22, §4.2.6] defines a subsheaf $\mathcal{O}^{\mathrm{la}}_{K^p} \subseteq \mathcal{O}_{K^p}$ by:

$$\mathcal{O}^{\mathrm{la}}_{K^p}(U) = \mathcal{O}_{K^p}(U)^{K_p-\mathrm{la}}, \tag{47}$$

on quasi-compacts U, where $K_p \subseteq \mathrm{GL}_2(\mathbb{Q}_p)$ is an open compact stabilising U. Then Pan shows that:

Theorem 2.2. *[Pan22, Theorem 4.4.6] There is a $\mathrm{GL}_2(\mathbb{Q}_p)$ and Hecke-equivariant isomorphism:*

$$H^i(\mathscr{F}\ell, \mathcal{O}_{K^p})^{la} \cong H^i(\mathscr{F}\ell, \mathcal{O}^{la}_{K^p}). \tag{48}$$

The idea now is to study $\mathcal{O}^{\mathrm{la}}_{K^p}$ and π_{HT}. To define the latter properly, we will need p-adic Hodge theory for rigid analytic varieties [Sch13].

[2] For the definition of adic spaces see [Hub93].

[3] For the purposes of this functor, $\mathbb{C}_p$ may be replaced by any p-adic field.

2.2 The period sheaves

Let X be a scheme or adic space. Recall [Aut, 34.4] the étale site $X_{\text{ét}}$ of X is the site with underlying category $\mathsf{Sch}_{\text{ét}}/X$ (or $\mathsf{AdicSpaces}_{\text{ét}}/X$), and coverings given by jointly surjective families of étale morphisms $\{f_i : U_i \to V\}$ (over X).

As in [Sch13, Definition 3.9], let pro $-X_{\text{ét}}$ be the category of pro-objects of $X_{\text{ét}}$. Its objects are (small) cofiltered inverse limits of objects in $X_{\text{ét}}$. A morphism $\varprojlim_i U_i = U \to V = \varprojlim_i V_i$ in pro $-X_{\text{ét}}$ is called étale if there is a morphism $U_0 \to V_0$ making the following square Cartesian:

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ U_0 \arrow[r] & V_0 \end{tikzcd} \tag{49}$$

A map $\varprojlim_i U_i = U \to V$ is called pro-étale if it is a cofiltered inverse limit of étale morphisms $U_i \to V$ such that $U_j \to U_i$ is surjective finite étale for $j \gg i$. $X_{\text{proét}}$ is the site with underlying category given by the objects of pro $-X_{\text{ét}}$ that are pro-étale over X, and coverings given by jointly surjective (on underlying topological spaces) familes of pro-étale morphisms. The structure sheaf $\mathcal{O}_X$ on $X_{\text{proét}}$ is given on qcqs $U = \varprojlim_i U_i \in X_{\text{proét}}$ by $\mathcal{O}_X(U) = \varinjlim_i \mathcal{O}_{X_{\text{ét}}}(U_i)$.

Now let K be a characteristic 0 perfectoid field, let $K^+ \subseteq K$ be an open bounded valuation subring, and let X be a locally noetherian adic space over $\mathrm{Spa}(K, K^+)$. Call $U \in X_{\text{proét}}$ affinoid perfectoid if $U = \varprojlim_i \mathrm{Spa}(R_i, R_i^+)$ for affinoids $\mathrm{Spa}(R_i, R_i^+)$ such that (R, R^+) is an affinoid perfectoid (K, K^+) algebra, where $R^+ = (\varinjlim_i R_i^+)^\wedge_p$ and $R = R^+[1/p]$, and we write $\hat{U} = \mathrm{Spa}(R, R^+)$. One of the most important properties of $X_{\text{proét}}$ is that:

Theorem 2.3. *[Sch13, Corollary 4.7, Proposition 4.8] In this setup, the affinoid perfectoid U form a basis for $X_{proét}$.*

We now define [Sch13, §6] sheaves by giving them on such affinoid perfectoid U. Firstly, The completed structure sheaves $\widehat{\mathcal{O}}_X^+$ and $\widehat{\mathcal{O}}_X$: by

$$\widehat{\mathcal{O}}_X^+(U) = R^+ \quad \text{and} \quad \widehat{\mathcal{O}}_X(U) = R, \tag{50}$$

and the sheaves $\mathbb{A}_{\text{inf}}$ and $\mathbb{B}_{\text{inf}}$ by,

$$\mathbb{A}_{\text{inf}}(U) = W(R^{\flat +}) \quad \text{and} \quad \mathbb{B}_{\text{inf}}(U) = W(R^{\flat +})[1/p]. \tag{51}$$

Recall from the p-adic Hodge theory, that there is a surjective map $\theta : W(R^{\flat +}) \to R^+$, and $\ker\theta$ is principal generated by ξ[4]. As sheaves this is saying there are surjective maps $\theta : \mathbb{A}_{\text{inf}} \to \widehat{\mathcal{O}}_X^+$ and $\theta : \mathbb{B}_{\text{inf}} \to \widehat{\mathcal{O}}_X$. Define sheaves $\mathbb{B}_{\text{dR}}^+$ and $\mathbb{B}_{\text{dR}}$ by:

$$\mathbb{B}_{\text{dR}}^+ = \varprojlim_i \mathbb{B}_{\text{inf}}/(\ker\theta)^i \quad \text{and} \quad \mathbb{B}_{\text{dR}} = \mathbb{B}_{\text{dR}}[1/\xi]. \tag{52}$$

Lastly, we define the structural de Rham sheaves. Define $\mathcal{O}\mathbb{B}_{\text{inf}} = \mathcal{O}_X \otimes_{W(\kappa)} \mathbb{B}_{\text{inf}}$ and $\mathcal{O}\mathbb{B}_{\text{dR}}^+ = ((\mathcal{O}\mathbb{B}_{\text{inf}})^\wedge_p)^\wedge_{\ker\theta}$, i.e [Sch16, (3)].

$$\mathcal{O}\mathbb{B}_{\text{dR}}^+(U) = \varinjlim_i \varprojlim_j \left(R_i^+ \hat{\otimes}_{W(\kappa)} \mathbb{A}_{\text{inf}}(U)\right)[1/p]/(\ker\theta)^j, \tag{53}$$

[4]This is defined by the same formula as a perfectoid field: $\theta : \sum_i p^i[x_i] \mapsto \sum_i p^i x_i^\#$.

where κ is the residue field of K. Here the tensor product is p-adically completed, and the map $\theta : R_i^+\hat{\otimes}_{W(\kappa)}\mathbb{A}_{\mathrm{inf}}(U) \to R^+$ is the tensor product of the maps $R_i^+ \to R^+$ and $\mathbb{A}_{\mathrm{inf}}(U) \to R^+$. Also define $\mathcal{O}\mathbb{B}_{\mathrm{dR}} \coloneqq \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+[1/\xi]$. Therefore there are maps $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \to \widehat{\mathcal{O}}_X^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \widehat{\mathcal{O}}_X$. The structure sheaf is equipped with a connection[5] $\nabla : \mathcal{O}_X \to \Omega_X^1$ which we can extend $\mathbb{B}_{\mathrm{inf}}$-linearly, and then p–adically and $\ker\theta$-adically complete (and then invert ξ if you want), to get a $\mathbb{B}_{\mathrm{dR}}^+$-linear connection

$$\nabla : \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^1. \tag{54}$$

Then Scholze [Sch13] considers the following four categories:

1. $\mathbb{B}_{\mathrm{dR}}^+$-local systems $\mathbb{M}$ on $X_{\mathrm{proét}}$.
2. $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$-modules $\mathcal{M}$ with integrable connection $\nabla_{\mathcal{M}}$.
3. Filtered $\mathcal{O}_X$-modules $\mathcal{E}$ with filtration $\mathrm{Fil}^\bullet\mathcal{E}$ with integrable connection ∇ satisfying Griffiths transversality (this means that $\nabla\mathrm{Fil}^i\mathcal{E} \subseteq \mathrm{Fil}^{i-1}\mathcal{E}\otimes\Omega_X^1$).
4. Lisse $\mathbb{Z}_p$-sheaves $\mathbb{L}$ on $X_{\mathrm{proét}}$.

The first and second categories are equivalent [Sch13, Theorem 7.2] by:

$$\begin{aligned} \mathbb{M} &\mapsto (\mathbb{M}\otimes_{\mathbb{B}_{\mathrm{dR}}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}, \mathrm{id}\otimes\nabla) \\ \mathcal{M}^{\nabla_{\mathcal{M}}} &\hookleftarrow (\mathcal{M}, \nabla_{\mathcal{M}}). \end{aligned} \tag{55}$$

We say objects of the second and third categories are associated if

$$\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong \mathcal{M}\otimes_{\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+}\mathcal{O}\mathbb{B}_{\mathrm{dR}}, \tag{56}$$

compatibly with filtrations and connections, similarly objects of the first and third are called associated if:

$$\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong \mathbb{M}\otimes_{\mathbb{B}_{\mathrm{dR}}^+}\mathcal{O}\mathbb{B}_{\mathrm{dR}}, \tag{57}$$

compatibly with filtrations and connections. Any $\mathcal{E}$ belonging to the first category is associated with:

$$\mathbb{M} = \mathrm{Fil}^0(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{dR}})^{\nabla=0}. \tag{58}$$

This defines a fully faithful functor from the third to first categories [Sch13, Theorem 7.6]. A lisse $\mathbb{Z}_p$-sheaf $\mathbb{L}$ on $X_{\mathrm{proét}}$ is a locally finitely generated $\widehat{\mathbb{Z}}_p$-module, where $\widehat{\mathbb{Z}}_p$ is the inverse limit $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ of constant sheaves on the pro-étale site. We say it is associated to a $\mathbb{B}_{\mathrm{dR}}$-local system $\mathbb{M}$ if there is an isomorphism

$$\mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong \mathbb{M}\otimes_{\mathbb{B}_{\mathrm{dR}}^+}\mathcal{O}\mathbb{B}_{\mathrm{dR}}, \tag{59}$$

if moreover $\mathbb{M} = \mathbb{L}\otimes_{\widehat{\mathbb{Z}}_p}\mathbb{B}_{\mathrm{dR}}^+$ is in the image of (58) (i.e. admits an associated $\mathcal{E}$) we say that $\mathbb{L}$ is de Rham. This is precisely the situation in which we can pass between all four categories above.

[5] i.e., it is the first map in the de Rham complex.

2.3 Relative de Rham comparison theorem

If $f : X \to Y$ is a smooth proper map of such adic spaces, and $\mathcal{E}_X$ is a filtered $\mathcal{O}_X$-module with integrable connection (satisfying Griffiths transversality), then we can consider the relative de Rham complex of $\mathcal{O}_Y$-modules

$$\mathrm{DR}(\mathcal{E}_X) \coloneqq (0 \to \mathcal{E}_X \xrightarrow{\nabla_{X/Y}} \mathcal{E}_X \otimes \Omega^1_{X/Y} \to \dots). \tag{60}$$

The cohomology $H^i_{\mathrm{dR}}(\mathcal{E}_X/Y)$ of this complex are then $\mathcal{O}_Y$-modules. We can also consider the derived functors $R^i f_{*,\mathrm{pro\acute{e}t}}$ of pushforward on sheaves. If $\mathbb{L}$ is de Rham and associated to $\mathcal{E}_X$, then these are associated [Sch13, Theorem 8.8(ii)], i.e.,

$$R^i f_{*,\mathrm{pro\acute{e}t}} \mathbb{L} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(\mathcal{E}_X/Y) \otimes_{\mathcal{O}_Y} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}. \tag{61}$$

2.4 Variation of Hodge structures for complex analytic varieties

Let a X be a smooth variety over $\mathbb{C}$ (endowed with the analytic topology), or a complex manifold.

Definition 2.4. *[Sch73, §2] A (integral, pure of weight n) variation of Hodge structures (over X) is a $\mathbb{Z}$-local system $\underline{V}$ on X together with a decreasing filtration $\mathrm{Fil}^\bullet \mathcal{E}$ on $\mathcal{E} \coloneqq \underline{V} \otimes_{\mathbb{Z}} \mathcal{O}_X$ satifsying Griffiths transversality (i.e. $\nabla \mathrm{Fil}^i \mathcal{E} \subseteq \mathrm{Fil}^{i-1} \mathcal{E} \otimes \Omega^1_X$), which induces a pure Hodge structure of weight n on the fibers of $\underline{V}$.*

A Hodge structure on a finite rank free $\mathbb{Z}$-module V is a decomposition of $V_{\mathbb{C}} \coloneqq V \otimes_{\mathbb{Z}} \mathbb{C}$ into complex vector spaces $V^{i,j}$:

$$V_{\mathbb{C}} = \bigoplus_{i,j} V^{i,j}, \tag{62}$$

such that the complex conjugate $\overline{V^{i,j}} = V^{j,i}$. We write $d_{i,j} = \dim V^{i,j}$, and $(d_{i,j})_{i,j \in \mathbb{Z}}$ is called the Hodge weights of V. It is called pure of weight n if $\operatorname{rank} V = n$ and $i + j = n$ for all i, j, in which case we define a filtration by $\mathrm{Fil}^i V_{\mathbb{C}} = \bigoplus_{i' \leq i} V^{i', n-i'}$. We can recover $V^{i,j} = \mathrm{Fil}^i V_{\mathbb{C}} \cap \overline{\mathrm{Fil}^j V_{\mathbb{C}}}$, which is what "induces" means in Definition 2.4.

In the setting of Definition 2.4, let V be the fiber of $\underline{V}$, let $d_i = \operatorname{rank} \mathrm{Fil}^i \mathcal{E}$, and consider $\mathrm{Fl}^{\mathbf{d}}_{V_{\mathbb{C}}}$, the flag variety of decreasing filtrations of $\mathbb{C}$-subspaces V_i of $V_{\mathbb{C}}$ with $\dim V_i = d_i$. Its X-points (for $X/\mathbb{C}$) are given by:

$$\mathrm{Fl}^{\mathbf{d}}_{V_{\mathbb{C}}}(X) = \{(\mathcal{F}_i)_{i \in \mathbb{Z}} : V_{\mathbb{C}} \otimes \mathcal{O}_X \supseteq \dots \supseteq \mathcal{F}_i \supseteq \mathcal{F}_{i+1} \supseteq \dots \supseteq 0\}, \tag{63}$$

where each $\mathcal{F}_i$ is a vector bundle of rank d_i which is a locally direct summand. So if X is equipped with a variation of Hodge structures, $\mathrm{Fil}^\bullet \mathcal{E}$ determines a morphism $\pi_H : X \to \mathrm{Fl}^{\mathbf{d}}_{V_{\mathbb{C}}}$, the "period map".

Now let $f : X \to Y$ be a smooth proper morphism of varieties over $\mathbb{C}$. Then the relative de Rham cohomology $H^n_{\mathrm{dR}}(X/Y)$ is equipped with a decreasing filtration, called the Hodge filtration [Aut, §50.7], coming from the degeneration of the Hodge-de Rham spectral sequence [Del68, Théorème 5.5] $E_1^{i,j} = H^j(X, \Omega^i_{X/Y}) \Rightarrow H^n_{\mathrm{dR}}(X/Y)$ (here $n = i+j$). It also has the Gauss-Manin connection ∇ satisfying Griffiths transversality with respect to the Hodge filtration [Gri70, §2]. There is an isomorphism (coming from the compatibility of the Riemann-Hilbert correspondence in the derived category, with pushfowards, see for example [HTT08, Theorem 7.1.1]) of $\mathcal{O}_Y$-modules:

$$R^n f_* \underline{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_Y \cong H^n_{\mathrm{dR}}(X/Y), \tag{64}$$

and the filtration $\mathrm{Fil}^{\bullet}\mathcal{E}$ on $\mathcal{E} := H^n_{\mathrm{dR}}(X/Y)$ determines a Hodge structure on the fibers of $\underline{V} := R^n f_* \underline{\mathbb{Z}}$. In other words, $\underline{V}$ is a variation of Hodge structures on Y, and so we get a morphism $Y \to \mathrm{Fl}^{\mathbf{d}}_{V_{\mathbb{C}}}$.

For an elliptic curve $f : E \to S$ over any scheme $S/\mathbb{C}$, the relative de Rham cohomology $H^1_{\mathrm{dR}}(E/S)$ is a rank 2 vector bundle on S which sits in the exact sequence (the "Hodge-Tate filtration") [KDSB73, Katz, A1.2.1]:

$$0 \to \underline{\omega}_{E/S} \to H^1_{\mathrm{dR}}(E/S) \to \underline{\omega}^{-1}_{E/S} \to 0, \tag{65}$$

which determines a variation of Hodge structures $\underline{V} = R^1 f_* \underline{\mathbb{Z}}$ on S with $d_{-1,0} = d_{0,-1} = 1$. So we get a morphism $S \to \mathbb{P}(V) \cong \mathbb{P}^1_{\mathbb{C}}$. In particular, if $S = Y_K \otimes_{\mathbb{Q}} \mathbb{C}$ is the (open) modular curve of level K, and $E = E_K \times_{\mathbb{Q}} \mathbb{C}$ is the universal elliptic curve, then we get a map $Y_K \times_{\mathbb{Q}} \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$. Extending this, there is an exact sequence of vector bundles on $X_K \times_{\mathbb{Q}} \mathbb{C}$:

$$0 \to \underline{\omega}_K \to \mathrm{H}^1_{\mathrm{dR,log}} \to \underline{\omega}_K^{-1} \to 0 \tag{66}$$

and an extension $\underline{V_{\log}}$ of $\underline{V_{\log}}$ such that $\underline{V}_{\log} \otimes_{\mathbb{Z}} \mathcal{O}_{X_K} \cong \mathrm{H}^1_{\mathrm{dR,log}}$, and hence a variation of Hodge structures on $X_K \times_{\mathbb{Q}} \mathbb{C}$, which yields a period map $X_K \times_{\mathbb{Q}} \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$.

2.5 The Hodge-Tate period map

We follow [Pan22, §4.1.3]. Our aim will be to copy the period map from the previous section, for the perfectoid modular curve $\mathcal{X}_{K^p}$ over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. The substitute for variation of Hodge structures will be Scholze's p-adic Hodge theory for rigid analytic varieties [Sch13].

Let $f : \mathcal{E}_{K^pK_p} \to \mathcal{Y}_{K^pK_p}$ be the universal elliptic curve over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Let $\mathbb{L} = \widehat{\mathbb{Z}}_p$. Write $\hat{\underline{V}} = R^1 f_{*,\text{proét}} \widehat{\mathbb{Z}}_p$, then by (61) there is an isomorphism of sheaves on the proétale site:

$$\hat{\underline{V}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}} \cong H^1_{\mathrm{dR}}(\mathcal{E}_{K^pK_p}/\mathcal{Y}_{K^pK_p}) \otimes_{\mathcal{O}_{\mathcal{Y}_{K^pK_p}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR}}. \tag{67}$$

Using the theory of log adic spaces [DLLZ19b] [DLLZ19a], this isomorphism can be extended to $\mathcal{X} = \mathcal{X}_{K^pK_p}$, by equipping $\mathcal{X}_{K^pK_p}$ with the log structure defined by the divisor of its cusps. Similarly to (67), there is a comparison isomorphism

$$\hat{\underline{V}}_{\log} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \cong H^1_{\mathrm{dR,log}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}} \tag{68}$$

of sheaves on $\mathcal{X}_{\text{prokét}}$, the pro-Kummer étale site of log adic spaces which are pro-log-étale over $\mathcal{X}_{K^p}$, and $\mathbb{B}^+_{\mathrm{dR,log}}$ are log period sheaves, and as before $\hat{\underline{V}}_{\log}$ is a rank 2 $\widehat{\mathbb{Z}}_p$-local system on $\mathcal{X}_{\text{prokét}}$, and $H^1_{\mathrm{dR,log}}$ gets its filtration from the Hodge-Tate exact sequence

$$0 \to \underline{\omega}_{K^pK_p} \to H^1_{\mathrm{dR,log}} \to \underline{\omega}^{-1}_{K^pK_p} \to 0, \tag{69}$$

where $\underline{\omega}_{K^pK_p}$ is the automorphic line bundle[6]. Recall that $\mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}$ has the $\ker(\theta)$-adic filtration, so there is an inclusion

$$\mathrm{gr}^0 H^1_{\mathrm{dR,log}} \otimes_{\mathcal{O}_{\mathcal{X}}} \widehat{\mathcal{O}}_{\mathcal{X}} \hookrightarrow \mathrm{gr}^0(H^1_{\mathrm{dR,log}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR,log}}), \tag{70}$$

[6] We will also use the same notation for the sheaves on the pro-Kummer étale site

and the quotient can be identified with the rest of the degree 0 part, i.e. $\mathrm{gr}^1 H^1_{\mathrm{dR},\log} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}}(-1)$, because $\mathrm{gr}^\bullet H^1_{\mathrm{dR},\log}$ only lives in degrees $0,1$. So we get the filtration:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}^0 H_{\mathrm{dR},\log} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}} \arrow[r] \arrow[d, equal] & \mathrm{gr}^0 (H^1_{\mathrm{dR},\log} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}) \arrow[r] \arrow[d, "\sim"] & \mathrm{gr}^1 H_{\mathrm{dR},\log} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}}(-1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{\omega}^{-1}_{K^pK_p} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}} \arrow[r] & \underline{\hat{V}}_{\log} \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}} \arrow[r] & \underline{\omega}_{K^pK_p} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}}(-1) \arrow[r] & 0
\end{tikzcd} \tag{71}$$

where in the bottom line we took the 0^{th} graded part of the isomorphism (68) and used (69). This can be rewritten as:

$$0 \to \underline{\omega}^{-1}_{K^pK_p}(1) \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}} \to \underline{\hat{V}}_{\log}(1) \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_{\mathcal{X}} \to \underline{\omega}_{K^pK_p} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{\mathcal{X}} \to 0. \tag{72}$$

Now $\varprojlim_{K_p} \mathcal{X}_{K^pK_p} \sim \mathcal{X}_{K^p}$ is a cover of $\mathcal{X}_{K^pK_p}$ in the pro-Kummer étale site, and hence, restricting (72) to this cover and recalling Section 2.2, we get the exact sequence of sheaves over $\mathcal{X}_{K^p}$:

$$0 \to \underline{\omega}^{-1}_{K^p}(1) \to \underline{\hat{V}}_{\log}(1) \otimes \mathcal{O}_{\mathcal{X}_{K^p}} \to \underline{\omega}_{K^p} \to 0. \tag{73}$$

By choosing two sections of $\underline{\hat{V}}_{\log}(1)$, this is already enough to give a morphism to $\mathbb{P}^1_{\mathrm{ad}}$, but we can do better and make this canonical. The inverse limit $\varprojlim_{K_p} \mathcal{X}_{K^pK_p}$ can be calculated on the system of congruence subgroups $\Gamma(p^m)$, i.e $\varprojlim_m \mathcal{X}_{K^p\Gamma(p^m)}$. The maps are induced by the inclusions of level structures $\Gamma(p^{m+1}) \subseteq \Gamma(p^m)$. In particular, an S-point of this inverse limit gives rise to to an elliptic curve E_S/S together with a compatible system of trivialisations $\alpha_m : E_S[p^m] \to (\mathbb{Z}/p^m\mathbb{Z})^2_S$, that is to say, a trivialisation $\alpha : T_pE_S \to (\mathbb{Z}_p)^2_S$ of the Tate module over S. In particular the universal elliptic curve over $\mathcal{X}_{K^p}$ gives rise to a canonical trivialisation of the Tate module $\underline{\hat{V}}_{\log}(1)$ over $\mathcal{X}_{K^p}$, which we apply to (73):

$$0 \to \underline{\omega}^{-1}_{K^p}(1) \to (\mathbb{Q}_p^{\oplus 2})(1) \otimes \mathcal{O}_{\mathcal{X}_{K^p}} \to \underline{\omega}_{K^p} \to 0. \tag{74}$$

The images of that standard basis vectors e_1, e_2 in $\mathbb{Q}_p^{\oplus 2}$ give two sections that generate $\underline{\omega}_{K^p}$ and hence a morphism to $\mathscr{F}\ell = \mathbb{P}^{1,\mathrm{ad}}$. This is the Hodge-Tate period map π_{HT}. It is $\mathrm{GL}(\mathbb{Z}_p)$ equivariant because the action on $\mathcal{X}_{K^p}$ comes from composing with the level structure α. We can view it in the diagram:

$$\begin{tikzcd}
& \mathcal{X}_{K^p} \arrow[dl, "\pi_{K_p}"'] \arrow[dr, "\pi_{HT}"] & \\
\mathcal{X}_{K^pK_p} & & \mathscr{F}\ell
\end{tikzcd} \tag{75}$$

where π_{K_p} is the projection to finite level K^pK_p. Let $\omega_{\mathscr{F}\ell}$ be the tautological line bundle on $\mathscr{F}\ell$, and let $\underline{\omega}_{K^p} := \pi^*_{K_p}\underline{\omega}_{K^pK_p}$ be the pullback of the automorphic line bundle from any finite level. Then

Theorem 2.5. *[Pan22, Theorem 4.1.7][Sch15, Theorem III.3.] The Hodge-Tate period map π_{HT} is $\mathrm{GL}_2(\mathbb{Q}_p)$ and Hecke equivariant (for the trivial Hecke action on $\mathscr{F}\ell$). If $\mathscr{F}\ell \supset U_1 := \{[x_1 : x_2] : \|x_1\| \geq \|x_2\|\}$ (define U_2 similarly), and $\mathfrak{B}$ is the set of finite intersections of rational subsets of U_1, U_2, then every $U \in \mathfrak{B}$ has $V := \pi^{-1}_{HT}(U)$ affinoid perfectoid. There is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism of line bundles $\underline{\omega}_{K^p} \cong \pi^*_{HT}\omega_{\mathscr{F}\ell}$.*

The above construction of π_{HT} generalises straightforwardly to Siegel modular varieties. For the consruction of π_{HT} for Hodge type Shimura varieties see [CS17], for abelian type see [She17], for general Shimura varieties there is Hodge-Tate period map of diamonds constructed in [BP21] and [Cam22, §7].

3 Relative Sen theory

3.1 Classical Sen theory

Recalling the p−adic Hodge theory study group, the original (arithmetic) Sen theory is the following. $\overline{\mathbb{Q}}_p \supset K \supset \mathbb{Q}_p$ is a finite extension, K_∞/K is a ramified $\mathbb{Z}_p$-extension, $H := G_{K_\infty}$, $\Gamma := \mathrm{Gal}(K_\infty/K)$, with topological generator γ, $\Gamma_m := \Gamma^{p^m}$, $K_m := K_\infty^{\Gamma_m}$, and $\chi : \Gamma \xrightarrow{\sim} \mathbb{Z}_p$ is a choice of isomorphism. Here is a picture:

$$\begin{tikzcd} K \arrow[r, dash] \arrow[rrr, dash, bend left, "\Gamma_0=\Gamma"] & K_1 \arrow[r, dash] \arrow[rr, dash, bend right, "\Gamma_1"'] & K_2 \arrow[r, dotted, dash, "\Gamma_2"'] & K_\infty \arrow[r, dash, "H"] & \overline{K} \end{tikzcd}. \tag{76}$$

Each $\Gamma_m \cong$ an open subgroup of $\mathbb{Z}_p$ and so is a 1-dimensional p−adic Lie group. Let V be a f.d. $\mathbb{Q}_p$-Banach representation of G_K. Then for $m \gg 0$ one has an isomorphism of $\mathbb{C}_p$-semilinear G_K-representations:

$$\begin{gathered} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H,\Gamma_m-\mathrm{an}} \otimes_{K_m} \mathbb{C}_p \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_p, \text{ leading to} \\ (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H,\Gamma-\mathrm{la}} \otimes_{K_\infty} \mathbb{C}_p \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_p. \end{gathered} \tag{77}$$

The Γ-action on $V_\infty := (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H,\Gamma-\mathrm{la}}$ is differentiable, to an action of Lie(Γ), which turns out to be K_∞-linear. Explicitly, for $v \in V_\infty$, we can define

$$\theta_V(v) = \frac{1}{\log \chi(\gamma)} \left.\frac{d}{dt}\right|_{t=0} (\gamma^t v), \tag{78}$$

a canonical element in the image of $\mathrm{Lie}(\Gamma) \to \mathrm{End}_{K_\infty} V_\infty$, which commutes with the Γ-action. Extending scalars, we get the Sen operator $\theta_V \in \mathrm{End}_{\mathbb{C}_p} V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ which commutes with the action of G_K. One can decompose $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \bigoplus_\lambda (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)_\lambda$ into generalised eigenspaces for θ_V. In the case where $K = K(\mu_{p^\infty})$ and $\chi = \chi_{\mathrm{cyc}}$, V is Hodge-Tate if and only if θ_V acts semisimply with integer eigenvalues, in which case those are (minus) the Hodge-Tate weights. More generally, we call the Jordan form of θ_V the Hodge-Tate-Sen weights of V.

3.2 Theory of decompletions

The above theory has two features:

1. A "decompletion" to a subspace which is locally analytic for the action of a p-adic Lie group Γ appearing as a quotient of G_K.

2. Differentiation and analysis of the resulting Lie(Γ)-action.

The Tate-Sen formalism [BC08, BC16, Cam22] provides a recipe for the decompletion in quite general context. We follow [Cam22, §4]. Let Π be a profinite group, let (A, A^+)