MIT 18-S-S996: Category theory for scientists, Spring 2013
Homework and personal notes

Armand Sumo – armandsumo@gmail.com

August 5, 2021

Collaborators:

All of this is my own work.

---

1 Introduction

2 The category of sets

2.1 Sets and function

2.1.1 Sets

Exercise 1

The set of all sets of $A = \{1, 2, 3\}$, also called the power set of A, is

$$\mathcal{P}(A) = \{\varnothing, A, 1, 2, 3, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

2.1.2 Functions

Exercise 1

a.) Because each element of the set of photo-receptive cells (PR) connects to exactly one element of the set of retino-ganglial cells(RG) and no elements of PR connect to two elements of RG, the only valid function would be from $PR \to RG$. Each neuron forms at least one synapse with other neurons(otherwise it dies). Thus, the set of function-like connections between brain parts is a subset of the set of all biologically possible connections between brain parts. It therefore seems plausible that the connection patterns that exists between other areas of the brain are function-like.

Exercise 2

As depicted in (2.2), the elements of the codomain of f that receive at least one arrows are y_1, y_2, y_4. Therefore $im(f) = \{y_1, y_2, y_4\}$.

Exercise 3

a.) $A = \{1, 2, 3, 4, 5\}$ and $B = \{x, y\}$.
Since each element in A has $|B|$ choices, by the product rule, the total number of functions from A to B is

$$|\mathrm{Hom}_{\mathrm{Set}}(A, B)| = \underbrace{|B| \times |B| \times \cdots \times |B|}_{|A| \text{ times}} = 2^5 = 32.$$

Similarly, since each element of B has $|A|$ choices, the total number of functions from B to A is

$$|\mathrm{Hom}_{\mathrm{Set}}(B, A)| = \underbrace{|A| \times |A| \times \cdots \times |A|}_{|B| \text{ times}} = 5^2 = 25.$$

Exercise 4

a.) One set A such that for all sets X there's exactly one element in

$$\mathrm{Hom}_{\mathrm{Set}}(X, A) \text{ is } \ A = \varnothing$$

. This is because all sets contain the empty set and the empty set can only be mapped onto itself. Similarly, one set B such that for all sets X there's exactly one element in $\mathrm{Hom}_{\mathrm{Set}}(B, X)$ is $B = \varnothing$.

Exercise 5

Let X be a set with cardinal n.

a.) The first element of X can be mapped to n elements of X, the second to $(n-1)$, and so on and the k-th to $(n-(k-1))$, thus there are

$$n \times (n-1) \times \cdots \times 1 = n!$$

isomorphisms from X to itself. By convention, $0! = 1$ so the formula above holds when $n = 0$ and $X = \varnothing$.

Exercise 6

There is no one-to-one correspondence between "types" of the elements of the sets A and B so there's no obvious choice for which element of B will be associated to each element of the codomain of f. For this reason isn't one particular "canonical funcion" $A \to \{1, 2, 3, 4, 5\}$
corresponding to f.

Exercise 7

Suppose we have found a set A such that for any set X, there's an isomorphism of sets

$$X \cong \mathrm{Hom}_{\mathrm{Set}}(A, X)$$

then, the two sets must have the same cardinal:

$$|X| = |\mathrm{Hom}_{\mathrm{Set}}(A, X)|$$

Because $\mathrm{Hom}_{\mathrm{Set}}(A, X)$ is the set of all functions from A to X, its cardinal must be $|X|^{|A|}$, hence:

$$|X| = |X|^{|A|}$$

From here we can split into 2 cases, namely $|X| < 1$ and $|X| > 1$.
If $|X| > 1$, applying the logarithm to both sides gives:

$$\begin{aligned} \log|X| &= |A|\log|X| \\ |A| &= 1 \end{aligned}$$

therefore, only sets A with only one element can be picked.
If $|X| < 1$, then any set A can be picked.
Because the desired property must hold for all sets X, the only valid choices for A are among sets containing only one element.

Exercise 8

a.) If A = {a,b,c,d} and $f : \underline{10} \to A = \{a, b, c, c, b, a, d, d, a, b\}$, then $f(4) = c$. $s : \underline{7} \to \mathbb{N}$ given by $s(i) = i^2$ can be written as a sequence

$$s = (1, 4, 9, 16, 25, 36, 49).$$

Exercise 9

a.) $|\{5, 6, 7\}| = 3$ $|\mathbb{N}| = \infty$ $|\{n \in \mathbb{N} \mid n \leqslant 5\}| = 6$

2.2 Commutative diagrams

2.3 Ologs

2.3.1 Types

2.3.2 Aspects

Exercise 1

Valid Olog that captures the parent-child relationship:

$$
\begin{tikzcd}
 & \boxed{\text{no children}} \\
\boxed{\text{a person}} \arrow[ur, "\text{may have}"] \arrow[dr, "\text{may have}"'] & \\
 & \boxed{\text{a child}}
\end{tikzcd}
$$