MIT 18-S-S996: Category theory for scientists, Spring 2013
Homework and personal notes

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Collaborators:

All of this is my own work.

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1 Introduction

2 The category of sets

2.1 Sets and function

2.1.1 Sets

Exercise 1

The set of all sets of $A = \{1, 2, 3\}$, also called the power set of A, is

$$\mathcal{P}(A) = \{\varnothing, A, 1, 2, 3, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Exercise 2

(a) Because each element of the set of photo-receptive cells (PR) connects to exactly one element of the set of retino-ganglial cells(RG) and no elements of PR connect to two elements of RG, the only valid function would be from $PR \to RG$.

(b) Each neuron forms at least one synapse with other neurons(otherwise it dies). Thus, the set of function-like connections between brain parts is a subset of the set of all biologically possible connections between brain parts.
It therefore seems plausible that the connection patterns that exists between other areas of the brain are function-like.

Exercise 3

As depicted in (2.2), the elements of the codomain of f that receive at least one arrows are y_1, y_2, y_4. Therefore $im(f) = \{y_1, y_2, y_4\}$.

Exercise 4

(a) $A = \{1, 2, 3, 4, 5\}$ and $B = \{x, y\}$.
Since each element in A has $|B|$ choices, by the product rule, the total number of functions from A to B is

$$|\mathrm{Hom}_{\mathrm{Set}}(A, B)| = \underbrace{|B| \times |B| \times \cdots \times |B|}_{|A| \text{ times}} = 2^5 = 32.$$

(b) Similarly, since each element of B has $|A|$ choices, the total number of functions from B to A is

$$|\mathrm{Hom}_{\mathrm{Set}}(B, A)| = \underbrace{|A| \times |A| \times \cdots \times |A|}_{|B| \text{ times}} = 5^2 = 25.$$

Exercise 5

(a) One set A such that for all sets X there's exactly one element in

$$\mathrm{Hom}_{\mathrm{Set}}(X, A) \text{ is } \; A = \varnothing$$

. This is because all sets contain the empty set and the empty set can only be mapped onto itself.

(b) Similarly, one set B such that for all sets X there's exactly one element in $\mathrm{Hom}_{\mathrm{Set}}(B, X)$ is $B = \varnothing$.

Exercise 6

Let X be a set with cardinal n.

(a) The first element of X can be mapped to n elements of X, the second to $(n-1)$, and so on and the k-th to $(n-(k-1))$, thus there are

$$n \times (n-1) \times \cdots \times 1 = n!$$

isomorphisms from X to itself.

(b) By convention, $0! = 1$ so the formula above holds when $n = 0$ and $X = \varnothing$.

Exercise 7

There is no one-to-one correspondence between "types" of the elements of the sets A and B so there's no obvious choice for which element of B will be associated to each element of the codomain of f. For this reason isn't one particular "canonical funcion" $A \to \{1, 2, 3, 4, 5\}$

corresponding to f.

Exercise 8

Suppose we have found a set A such that for any set X, there's an isomorphism of sets

$$X \cong \mathrm{Hom}_{\mathrm{Set}}(A, X)$$

then, the two sets must have the same cardinal:

$$|X| = |\mathrm{Hom}_{\mathrm{Set}}(A, X)|$$

Because $\mathrm{Hom}_{\mathrm{Set}}(A, X)$ is the set of all functions from A to X, its cardinal must be $|X|^{|A|}$, hence:

$$|X| = |X|^{|A|}$$

From here we can split into 2 cases, namely $|X| < 1$ and $|X| > 1$.
If $|X| > 1$, applying the logarithm to both sides gives:

$$\log|X| = |A|\log|X|$$
$$|A| = 1$$

therefore, only sets A with only one element can be picked.
If $|X| < 1$, then any set A can be picked.
Because the desired property must hold for all sets X, the only valid choices for A are among sets containing only one element.

Exercise 9

(a) If A = {a,b,c,d} and $f : \underline{10} \to A = \{a, b, c, c, b, a, d, d, a, b\}$, then $f(4) = c$.

(b) $s : \underline{7} \to \mathbb{N}$ given by $s(i) = i^2$ can be written as a sequence

$$s = (1, 4, 9, 16, 25, 36, 49).$$

Exercise 10

(a) $|\{5,6,7\}| = 3$;

(b) $|\mathbb{N}| = \infty$;

(c) $|\{n \in \mathbb{N} \mid n \leqslant 5\}| = 6$.

2.2 Commutative diagrams

2.3 Ologs

Exercise 1

Olog that captures the parent-child relationship:

$$\begin{tikzcd}
 & \boxed{\text{no children}} \\
\boxed{\text{a person}} \arrow[ur, "\text{can have}"] \arrow[dr, "\text{can have}"'] & \\
 & \boxed{\text{a child}}
\end{tikzcd}$$

Exercise 2

Olog for human nuclear biological families:

$$
\begin{tikzcd}
\boxed{\text{a man}} \arrow[dr, "\text{is}"] \arrow[drrr, "\text{is}"] \arrow[drr, phantom, "\checkmark"] & & & \\
& \boxed{\text{a parent}} \arrow[r, "\text{has a}"] & \boxed{\text{a child}} \arrow[r, "\text{is}"] & \boxed{\text{a person}} \\
\boxed{\text{a woman}} \arrow[ur, "\text{is}"] \arrow[urrr, "\text{is}"] \arrow[urr, phantom, "\checkmark"] & & &
\end{tikzcd}
$$

Exercise 3

Given x, an operational land-line phone, consider the following. We know that x is an operational land-line phone, which is assigned to a phone number, which has an area code, which corresponds to a region
that we'll call $P(x)$.
We also know that x is an operational land-line phone, which is a physical phone, which is currently located in a region
that we'll call $Q(x)$.
Fact: whenever x is an operational land-line phone, we will have $P(x) = Q(x)$.

Exercise 4

If the box "an operational land-line phone" was replaced with the box "an operational mobile phone", the region in which it is currently located would not not always match the region corresponding to the mobile phone's phone number's area code and diagram would not commute.

Exercise 5

(a) “a book”: “not clearly an image type”

(b) “a material that has been fabricated by a process of type T.”

f : material $\to$ material : “has been fabricated by a process of type T”

(c) “a bicycle owner”

$$f : \text{owner} \to \text{owner} : \text{“owns a bicycle”}$$

(d) “a child” : not clearly an image type.

(e) “a used book”

$$f : \text{book} \to \text{book} : \text{“is used”}$$

(f) “an inhabited residence”

$$f : \text{residence} \to \text{residence} : \text{“is inhabited”}$$

2.4 Products and coproducts

Exercise 1

$\{a, b, c, d\} \times \{1, 2, 3\}$ has $|\{a, b, c, d\}| \times |\{1, 2, 3\}| = 4 \times 3 = 12$ items.

Exercise 2

1. The following diagram commutes:

$$\begin{tikzcd}
\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \arrow[r, "(a,b,c)\mapsto(a\cdot b,a\cdot c)"] \arrow[d, "(a,b,c)\mapsto(a+b,c)"'] \arrow[dr, phantom, "\checkmark"] & \mathbb{Z} \times \mathbb{Z} \arrow[d, "(x,y)\mapsto x+y"] \\
\mathbb{Z} \times \mathbb{Z} \arrow[r, "(x,y)\mapsto xy"'] & \mathbb{Z}
\end{tikzcd}$$

2. The following diagram doesn't commute:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "x\mapsto(x,0)"] \arrow[dr, "\mathrm{id}_{\mathbb{Z}}"'] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "(a,b)\mapsto a\cdot b"] \\
& \mathbb{Z}
\end{tikzcd}$$

3. The following does:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "x\mapsto(x,1)"] \arrow[dr, "\mathrm{id}_{\mathbb{Z}}"'] \arrow[dr, phantom, "\checkmark", near start, shift left=3ex] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "(a,b)\mapsto a\cdot b"] \\
& \mathbb{Z}
\end{tikzcd}$$

Exercise 3

$$\begin{aligned}
&|\mathrm{Hom}_{\mathrm{Set}}(A,X)| \times |\mathrm{Hom}_{\mathrm{Set}}(A,Y)| \\
&\qquad = |X|^{|A|} \times |Y|^{|A|} \\
&\qquad = (|X| \times |Y|)^{|A|} \\
&\qquad = |\mathrm{Hom}_{\mathrm{Set}}(A, X\times Y)|
\end{aligned}$$

Exercise 4

(a) By the universal product rule, for any function $f : X \times Y \mapsto X$ and a function $g : X \times Y \mapsto Y$ there's a unique function $f \times g : X \mapsto Y \times Y \times X$.

If $\pi_1 : X \times Y \mapsto X$ and $\pi_2 : X \times Y \mapsto Y$, we can define the swap function $s : X \times Y \to Y \times X$ as $s = \pi_2 \times \pi_1$, so that the diagram below commutes:

$$
\begin{tikzcd}
 & & X \times Y \arrow[dll, "\pi_1"'] \arrow[drr, "\pi_2"] \arrow[dd, dashed, "s=\pi_2\times\pi_1"] & & \\
X & \checkmark & & \checkmark & Y \\
 & & Y \times X \arrow[ull, "\pi_2"] \arrow[urr, "\pi_1"'] & &
\end{tikzcd}
$$

(b) To show that s is a isomorphism, we must find a function $t : Y \times X \mapsto X \times Y$ such that:
$t \circ s = \mathrm{id}_{X\times Y}$ and
$s \circ t = \mathrm{id}_{Y\times X}$
The obvious choice is $t : Y \times X \mapsto X \times Y$ defined as $t = \pi_2 \times \pi_1$. It is obvious because for any projection p, $p \circ p = \mathrm{id}$.

Exercise 5

Because a phone is either a cellphone or a land-line phone and not both at the same time, we can think of $\ulcorner aphone \urcorner$ as the coproduct of $\ulcorner acellphone \urcorner$ and $\ulcorner alandline - phone \urcorner$.

Exercise 6

$$\mathrm{Hom}_{\mathrm{Set}}(X, A) \sqcup \mathrm{Hom}_{\mathrm{Set}}(Y, A) = \mathrm{Hom}_{\mathrm{Set}}(X \sqcup Y, A)$$

2.5 Finite limits in Set

Exercise 1

A photon has both the properties of a wave and the properties of a particle, therefore we can label a photon that has the properties of a wave "W" and one that has the properties of a particle "P" and let $C = W \sqcup P$ be the set of photons that have the properties of a wave labeled as "W" and those that have the properties of the properties of a particle labeled as "P".

Exercise 2

$$
\begin{tikzcd}
x_1 \arrow[r] & z_1 & y_1 \arrow[l] \\
x_2 \arrow[r] & z_2 & y_2 \arrow[l] \\
x_3 \arrow[ur] & z_3 & y_3 \arrow[dl] \\
 & z_4 & y_4 \arrow[uul]
\end{tikzcd}
$$

The pullback of the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ presented is the subset $\{z_1\}$ of Z. For any other subset of Z, either one of the following is true:

- either f^{-1} or g^{-1} is not a function(ex: $f^{-1}(z_2) = x_2$ and $f^{-1}(z_2) = x_3$).
- some element of the chosen subset of Z is not in the domain or codomain of either f,g,f^{-1} or g^{-1}.

Exercise 3

We're given the diagram of functions below:

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

1. Suppose $Y = \varnothing$, because any function whose domain is the empty set has as codomain the emptyset, the fiber product $X \times_Z Y$ must be the empty set.

2. Suppose Y is any set but Z has exactly one element, then $X \times_Z Y$ is a set containing at most one element.

Exercise 4

1. A ⌜person whose favorite color is blue⌝ is a ⌜person⌝, not a ⌜color⌝.

2. The labels seem inappropriate because a ⌜dog whose owner is a woman⌝ is not a ⌜person⌝.

3. The labels seems inappropriate because the nature of the element of the set ⌜a good fit⌝ aren't fully specified.

Exercise 5

$$
\begin{tikzcd}
& D \times_F I\text{: a pair } (d, f) \arrow[ddl, bend right, "\text{yields as } d"'] \arrow[ddr, bend left, "\text{yields as} f"] & \\
& \overset{P}{\text{a person}} \arrow[u, dashed, "\text{collection of a DNA sampleand a fingerprint}"] \arrow[dl, "\text{has}"'] \arrow[dr, "\text{has}"] & \\
\overset{D}{\text{DNA material}} \arrow[dr, "\text{defines}"] & & \overset{F}{\text{a fingerprint}} \arrow[dl, "\text{defines}"'] \\
& \overset{I}{\text{an identity}} &
\end{tikzcd}
$$