

# Chapter 1

## Sets and functions

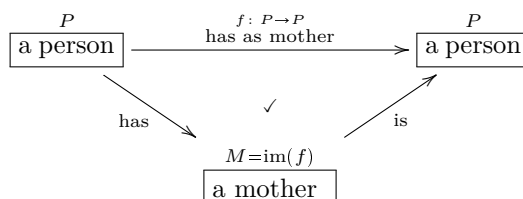
### 1.1 Ologs

#### 1.1.1 Facts

##### 1.1.1.1 Images

In this section we discuss a specific kind of fact, generated by any aspect. Recall that every function has an image, meaning the subset of elements in the codomain that are “hit” by the function. For example the function  $f(x) = 2 * x: \mathbb{Z} \rightarrow \mathbb{Z}$  has as image the set of all even numbers.

Similarly the set of mothers arises as is the image of the “has as mother” function, as shown below



*Exercise 1.1.1.2.* For each of the following types, write down a function for which it is the image, or say “not clearly an image type”

- ‘a book’
- ‘a material that has been fabricated by a process of type  $T$ ’
- ‘a bicycle owner’
- ‘a child’
- ‘a used book’
- ‘an inhabited residence’

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## 1.2 Products and coproducts

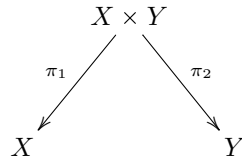
In this section we introduce two concepts that are likely to be familiar, although perhaps not by their category-theoretic names, product and coproduct. Each is an example of a large class of ideas that exist far beyond the realm of sets.

### 1.2.1 Products

**Definition 1.2.1.1.** Let  $X$  and  $Y$  be sets. The *product of  $X$  and  $Y$* , denoted  $X \times Y$ , is defined as the set of ordered pairs  $(x, y)$  where  $x \in X$  and  $y \in Y$ . Symbolically,

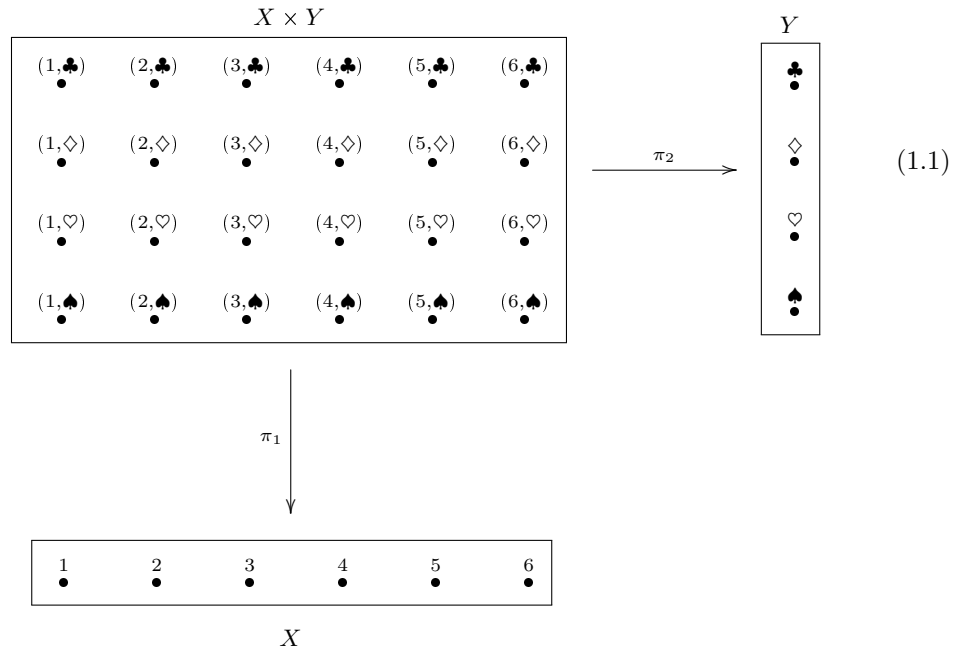
$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

There are two natural *projection functions*  $\pi_1: X \times Y \rightarrow X$  and  $\pi_2: X \times Y \rightarrow Y$ .



*Example 1.2.1.2.* [Grid of dots]

Let  $X = \{1, 2, 3, 4, 5, 6\}$  and  $Y = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ . Then we can draw  $X \times Y$  as a 6-by-4 grid of dots, and the projections as projections



*Application 1.2.1.3.* A traditional (Mendelian) way to predict the genotype of offspring based on the genotype of its parents is by the use of **Punnett squares**. If  $F$  is the set of possible genotypes for the female parent and  $M$  is the set of possible genotypes of the male parent, then  $F \times M$  is drawn as a square, called a Punnett square, in which every combination is drawn. ♦♦

*Exercise 1.2.1.4.* How many elements does the set  $\{a, b, c, d\} \times \{1, 2, 3\}$  have?  $\diamond$

*Application 1.2.1.5.* Suppose we are conducting experiments about the mechanical properties of materials, as in Application ???. For each material sample we will produce multiple data points in the set 'extension'  $\times$  'force'  $\cong \mathbb{R} \times \mathbb{R}$ .

$\diamond\diamond$

*Remark 1.2.1.6.* It is possible to take the product of more than two sets as well. For example, if  $A, B$ , and  $C$  are sets then  $A \times B \times C$  is the set of triples,

$$A \times B \times C := \{(a, b, c) \mid a \in A, b \in B, c \in C\}.$$

This kind of generality is useful in understanding multiple dimensions, e.g. what physicists mean by 10-dimensional space. It comes under the heading of *limits*, which we will see in Section ???.

*Example 1.2.1.7.* Let  $\mathbb{R}$  be the set of real numbers. By  $\mathbb{R}^2$  we mean  $\mathbb{R} \times \mathbb{R}$  (though see Exercise ???). Similarly, for any  $n \in \mathbb{N}$ , we define  $\mathbb{R}^n$  to be the product of  $n$  copies of  $\mathbb{R}$ .

According to [?], Aristotle seems to have conceived of space as something like  $S := \mathbb{R}^3$  and of time as something like  $T := \mathbb{R}$ . Spacetime, had he conceived of it, would probably have been  $S \times T \cong \mathbb{R}^4$ . He of course did not have access to this kind of abstraction, which was probably due to Descartes.

*Exercise 1.2.1.8.* Let  $\mathbb{Z}$  denote the set of integers, and let  $+: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  denote the addition function and  $\cdot: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  denote the multiplication function. Which of the following diagrams commute?

a.)

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} & \xrightarrow{(a,b,c) \mapsto (a \cdot b, a \cdot c)} & \mathbb{Z} \times \mathbb{Z} \\ (a,b,c) \mapsto (a+b,c) \downarrow & & \downarrow (x,y) \mapsto x+y \\ \mathbb{Z} \times \mathbb{Z} & \xrightarrow{(x,y) \mapsto xy} & \mathbb{Z} \end{array}$$

b.)

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{x \mapsto (x,0)} & \mathbb{Z} \times \mathbb{Z} \\ & \searrow \text{id}_{\mathbb{Z}} & \downarrow (a,b) \mapsto a \cdot b \\ & & \mathbb{Z} \end{array}$$

c.)

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{x \mapsto (x,1)} & \mathbb{Z} \times \mathbb{Z} \\ & \searrow \text{id}_{\mathbb{Z}} & \downarrow (a,b) \mapsto a \cdot b \\ & & \mathbb{Z} \end{array}$$

$\diamond$

### 1.2.1.9 Universal property for products

**Lemma 1.2.1.10** (Universal property for product). *Let  $X$  and  $Y$  be sets. For any set  $A$  and functions  $f: A \rightarrow X$  and  $g: A \rightarrow Y$ , there exists a unique function  $A \rightarrow X \times Y$*

such that the following diagram commutes<sup>1</sup>

$$\begin{array}{ccccc}
 & & X \times Y & & \\
 & \swarrow \pi_1 & \uparrow \text{---} & \searrow \pi_2 & \\
 X & & & & Y \\
 & \nwarrow \forall f & \downarrow \exists! & \nearrow \forall g & \\
 & & A & & 
 \end{array} \tag{1.2}$$

We might write the unique function as

$$f \times g: A \rightarrow X \times Y.$$

*Proof.* Suppose given  $f, g$  as above. To provide a function  $\ell: A \rightarrow X \times Y$  is equivalent to providing an element  $\ell(a) \in X \times Y$  for each  $a \in A$ . We need such a function for which  $\pi_1 \circ \ell = f$  and  $\pi_2 \circ \ell = g$ . An element of  $X \times Y$  is an ordered pair  $(x, y)$ , and we can use  $\ell(a) = (x, y)$  if and only if  $x = \pi_1(x, y) = f(a)$  and  $y = \pi_2(x, y) = g(a)$ . So it is necessary and sufficient to define

$$(f \times g)(a) = (f(a), g(a))$$

for all  $a \in A$ .

□

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<sup>1</sup>The symbol  $\forall$  is read “for all”; the symbol  $\exists$  is read “there exists”, and the symbol  $\exists!$  is read “there exists a unique”. So this diagram is intended to express the idea that for any functions  $f: A \rightarrow X$  and  $g: A \rightarrow Y$ , there exists a unique function  $A \rightarrow X \times Y$  for which the two triangles commute.