MIT 18.901 Introduction to Topology, Fall 2004

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Collaborators:

All of this is my own work.

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0. Problem set 0

1. Determine which of the following statements are true for all sets A,B,C and D. If a double implication fails, determine whether one or the other of the possible implications holds. If an equality fails, determine whether the statement becomes true if the "equals" symbol is replaced by one or the other of the inclusion symbol $\subset$ or $\supset$.

   (a) $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cup C)$: no $\Rightarrow$

   (b) $A \subset B$ or $A \subset C \Leftrightarrow A \subset (B \cup C)$: yes

   (c) $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cap C)$: yes

   (d) $A \subset B$ or $A \subset C \Leftrightarrow A \subset (B \cap C)$: no $\Leftarrow$

   (e) $A - (A - B) = B$: no $\subset$

   (f) $A - (B - A) = A - B$: no $\supset$

   (g) $A \cap (B - C) = (A \cap B) - (A \cap C)$: yes

   (h) $A \cup (B - C) = (A \cup B) - (A \cup C)$: no $\supset$

   (i) $(A \cap B) \cup (A - B) = A$: no $\supset$

   (j) $A \subset C$ and $B \subset D \Rightarrow (A \times B) \subset (C \times D)$: yes

   (k) the converse of (j): no

   (l) the converse of (j) assuming that A and B are nonempty : yes

   (m) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$: yes

   (n) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$: yes

   (o) $A \times (B - C) = (A \times B) - (A \times C)$: yes

   (p) $(A - B) \times (C - D) = (A \times C - B \times C) - (A \times D)$: yes

(q) $(A \times B) - (C \times D) = (A - C) \times (B - D)$: no $\supset$

2. (a) If $g \circ f$ is injective what can you say about the surjectivity of g and f ?
   f is injective.

   (b) If $g \circ f$ is surjective what can you say about the surjectivity of g and f ?
   g is surjective.

1. Problem set 1

1. *Theorem.* If an ordered set A has the least upper bound property, then it has the greatest lower bound property.
   Proof. Suppose an ordered set A has the least upper bound property.(1)
   Let A_0 be an arbitrary nonempty subset of A that is bounded above. According to (1), A_0 has a least upper bound $b \in A$.
   Because b is a least upper bound, there is some subset A_1 of A such that: $A_1 \cap A_0 = \{b\}$ and $\forall (x, y) \in A_0 \times A_1$, $x \leq y$ and $b \leq y$.
   Thus, A_1 is bounded below with greatest upper bound b.
   Because this is true for an arbitrary subset A_0 of A with its associated set A_1 , it is true for all subsets of A, which shows that A has the greatest lower bound property. ■

2. Show that $\mathcal{P}(\mathbb{Z}_+)$ and $\mathbb{R}$ have the same cardinality.[*Hint* You may use the fact that every real number has a decimal expansion, which is unique if expansions that end in an infinite string of 9's or forbidden.]
   There is a bijective correspondence between $[0, 1)$ and $\mathbb{R}$ (ex: $(0, 1) \longrightarrow \mathbb{R} : x \longmapsto x/(1 - x^2)$). It is therefore equivalent to show the existence of a bijective correspondence between $\mathcal{P}(\mathbb{Z}_+)$ and $[0, 1)$.
   To do so, we will show the existence of an injection from $[0, 1)$ to $\mathcal{P}(\mathbb{Z}_+)$ and an injection from $\mathcal{P}(\mathbb{Z}_+)$ to $[0, 1)$. By the Canton-Bernstein theorem, this will prove the existence of the desired bijective correspondence.
   Injection from $[0, 1)$ to $\mathcal{P}(\mathbb{Z}_+)$:
   The function $f : [0, 1) \longrightarrow \mathcal{P}(\mathbb{Z}_+) : 0.a_1a_2 \cdots \longmapsto \{10 \cdot i + a_i \mid i \in \mathbb{Z}_+\} \subset \mathcal{P}(\mathbb{Z}_+)$ is the decimal representation of an real number in $[0, 1)$. Because the decimal representation is unique, this function is injective.
   Injection from $\mathcal{P}(\mathbb{Z}_+)$ to $[0, 1)$:
   The function $g : \mathcal{P}(\mathbb{Z}_+) \longrightarrow [0, 1) : \{a_1a_2, \ldots, a_jj\} \longmapsto 0.0a_100a_2 \ldots \underbrace{0 \ldots 0}_{i \text{ zeros}} a_i$ is injective and is built by replacing the operation of *addition* in f with *concatenation.*