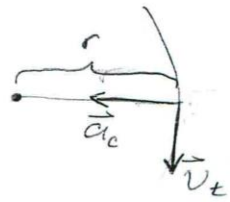
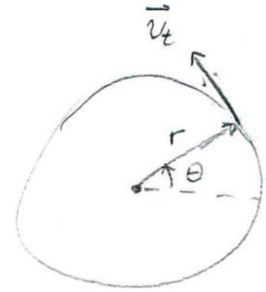
Part I

Changing Direction, Constant Speed (ChangeDirection)

$a_c = \dfrac{v_t^2}{r}$

$r = v_t^2 / a_c$





$$a(t) = \frac{d^2x}{dt^2}$$

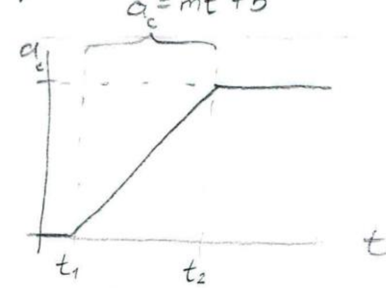
$$\frac{dx}{dt} = v(t) = \int a(t)\,dt = at + v_0$$

$$x(t) = \int v(t)\,dt = \int (at + v_0)\,dt = \frac{at^2}{2} + v_0 t + x_0$$

$\alpha = \dfrac{d^2\theta}{dt^2}$ ← angular acceleration

$\omega = \dfrac{d\theta}{dt} = \alpha t + \omega_0$ ← angular velocity

$\theta = \dfrac{\alpha t^2}{2} + \omega_0 t + \theta_0$ ← angle



circumference $= \pi(2r) = C$

period $= C / v_t = 2\pi r / v_t = T$

$$\omega = \frac{2\pi}{T} = \frac{2\pi v_t}{2\pi r} = \frac{v_t}{r}$$

$$\alpha = \frac{d\omega}{dt} = v_t \cdot \frac{d}{dt}\left(\frac{1}{r(t)}\right)$$

$$\alpha = v_t \cdot \frac{m}{v_t^2} = \frac{m}{v_t}, \quad t_1 \le t \le t_2$$

$$\omega = \frac{mt + b}{v_t} + \omega_0, \quad t_1 \le t \le t_2$$

$$\theta = \frac{m}{2v_t} t^2 + \omega_0 t + \theta_0$$

$r = v_t^2 / (mt + b), \quad t_1 \le t \le t_2$

$1/r = \dfrac{mt + b}{v_t^2}$, " "

$\dfrac{d}{dt}\left(\dfrac{1}{r}\right) = m / v_t^2$, " "

$\dfrac{v_t}{r} = \dfrac{mt + b}{v_t}$, " "

$$\Delta\theta = \theta(t_2) - \theta(t_1)$$

$$= \frac{m}{2v_t} t_2^2 + \omega_0 t_2 + \cancel{\theta_0} - \left(\frac{m}{2v_t} t_1^2 + \omega_0 t_1 + \cancel{\theta_0} \right)$$

$$= \frac{m}{2v_t} (t_2^2 - t_1^2) + \omega_0 (t_2 - t_1)$$

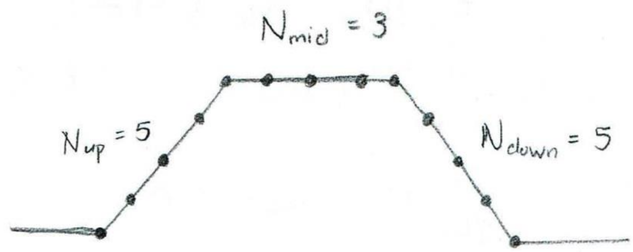
If $t_1 = 0$, $\Delta\theta = \frac{m}{2v_t} \Delta t^2 + \omega_0 \Delta t$

If $\Delta\theta$ is too great, solve for Δt, otherwise use this for ramping ac up and down.

Then, ~~$\Delta\theta_{desired} = 2\Delta\theta_{up\text{-}down}$ +~~ determine Δt for rest of turn knowing $\alpha = 0$, $\theta = \omega t + \theta_0$

$$\theta - \theta_0 = \omega t$$

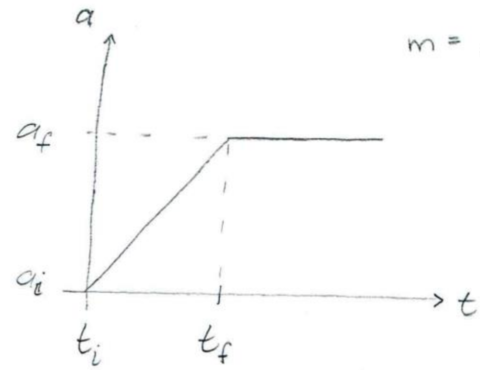
$$\Delta\theta = \omega \Delta t$$



Part II

Changing Speed, Constant Direction (ChangeSpeed)

$$a(t) = mt + b, \quad t_i \le t \le t_f$$



$$m = \frac{a_f - a_i}{t_f - t_i}$$

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int (mt + b)\,dt \\ &= \frac{mt^2}{2} + bt + v_0 \end{aligned}$$

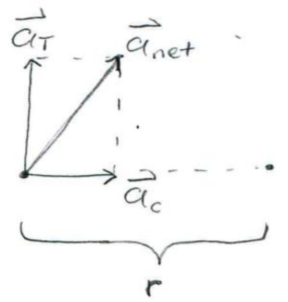
$$\begin{aligned} \Delta v &= v(t_f) - v(t_i) \\ &= \left(\frac{mt_f^2}{2} + bt_f + v_0\right) - \left(\frac{mt_i^2}{2} + bt_i + v_0\right) \\ &= \frac{m}{2}\left(t_f^2 - t_i^2\right) + b\left(t_f - t_i\right) \end{aligned}$$

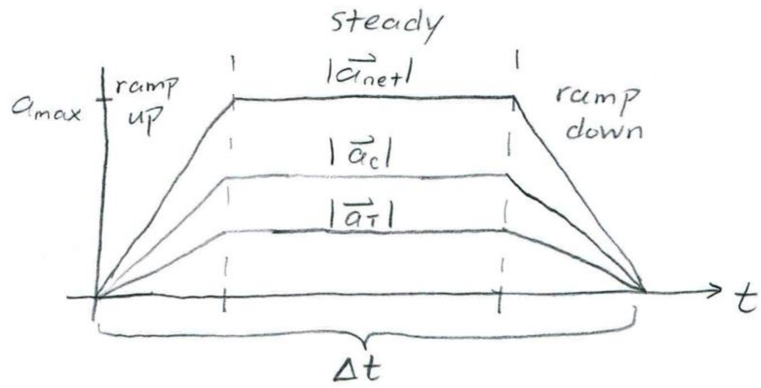
If $t_i = 0$, $\Delta v = \frac{m}{2} t_f^2 + bt_f$

$= \frac{m}{2}\Delta t^2 + b\Delta t$, because $\Delta t = t_f - t_i = t_f$

Part III

Changing Direction, Changing Speed (ChangeDirectionAndSpeed)





Constraints: $|a_{net}|^2 = |\vec{a}_c|^2 + |\vec{a}_T|^2$,

Speed must change by Δv_T (known) over time Δt (unknown),

angle must change by $\Delta\theta$ (known) over time Δt (unknown),

$k = |\vec{a}_c| / |\vec{a}_T|$ is constant but unknown.

$v_T(t) = \int a_T(t)\,dt$.

When $a_T(t)$ is constant (and $a_c(t)$ is constant),

$$v_T(t) = a_T \int dt = a_T t + v_{T0}$$

$$\omega = \frac{v_T}{r} \quad \text{and} \quad a_c = \frac{v_T^2}{r}, \quad \therefore\ r = \frac{v_T^2}{a_c}, \quad \omega = v_T \cdot \frac{a_c}{v_T^2} = \frac{a_c}{v_T},$$

$$\omega(t) = a_c / (a_T t + v_{T0})$$

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{a_c}{v_T}\right) = a_c \frac{d}{dt}\left(\frac{1}{v_T}\right) = a_c \frac{d}{dt}\left(v_T^{-1}\right) = a_c\left(-v_T^{-2}\frac{d}{dt}(v\right.$$

Now, $\frac{d}{dt}(v_T(t)) = a_T$, and

$$v_T^2 = (a_T t + v_{T0})^2 = a_T^2 t^2 + v_{T0}^2 + 2 a_T v_{T0} t.$$

$$\therefore\ \alpha(t) = -\frac{a_c a_T}{a_T^2 t^2 + 2 a_T v_{T0} t + v_{T0}^2}$$

$$\theta(t) = \int w(t)\,dt = \int \frac{a_c}{(a_T t + v_{T0})}\,dt = a_c \int \frac{1}{(a_T t + v_{T0})}\,dt$$

Let $w = a_T t + v_{T0}$, $dw = a_T\,dt$, $dt = dw/a_T$. Then,

$$a_c \int \frac{1}{w}\frac{dw}{a_T} = \frac{a_c}{a_T}\int \frac{dw}{w} = \frac{a_c}{a_T}\ln|w|, \text{ so}$$

$$\theta(t) = \frac{a_c}{a_T}\ln\left|a_T t + v_{T0}\right|.$$

Thus, for $a_T(t)$ and $a_c(t)$ constant,

$$\Delta v_T = v_T(t_2) - v_T(t_1) = a_T(t_2 - t_1) = a_T \Delta t,$$

$$\Delta\theta = \theta(t_2) - \theta(t_1) = \frac{a_c}{a_T}\left(\ln\left|a_T t_2 + v_{T0}\right| - \ln\left|a_T t_1 + v_{T0}\right|\right)$$

$$= \frac{a_c}{a_T}\ln\left(\frac{\left|a_T t_2 + v_{T0}\right|}{\left|a_T t_1 + v_{T0}\right|}\right).$$

When $a_T(t)$ is changing (and $a_c(t)$ is changing), slopes (jerk)

$$a_T(t) = m_T t + a_{T0}, \quad a_c(t) = m_c t + a_{c0}, \quad m_c/m_T = K,$$

$$v_T(t) = \int a_T(t)\,dt = \int (m_T t + a_{T0})\,dt = \frac{m_T}{2}t^2 + a_{T0}t + v_{T0}.$$

$$w(t) = \frac{a_c(t)}{v_T(t)} = \frac{m_c t + a_{c0}}{\frac{m_T}{2}t^2 + a_{T0}t + v_{T0}}.$$

$$\theta(t) = \int w(t)\,dt = \int \frac{m_c t + a_{c0}}{\frac{m_T}{2}t^2 + a_{T0}t + v_{T0}}\,dt$$

Use a table of integrals to solve.

$$\int \frac{mx+n}{ax^2+bx+c}\,dx = \begin{cases} \frac{m}{2a}\ln\left|ax^2+bx+c\right| + \frac{2an-bm}{a\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+C \\ \quad \text{for } 4ac-b^2>0, \\ \frac{m}{2a}\ln\left|ax^2+bx+c\right| - \frac{2an-bm}{a\sqrt{b^2-4ac}}\tanh^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right)+C \\ \quad \text{for } 4ac-b^2<0, \\ \frac{m}{2a}\ln\left|ax^2+bx+c\right| - \frac{2an-bm}{a(2ax+b)}+C \\ \quad \text{for } 4ac-b^2=0 \end{cases}$$

$m \to m_c$, $n \to a_{co}$, $a \to \frac{m_T}{2}$, $b \to a_{T0}$, $c \to v_{T0}$, $x \to t$,

$4ac-b^2 \longrightarrow 2m_T v_{T0} - a_{T0}^2$

$$\int \frac{m_c t + a_{co}}{\frac{m_T}{2}t^2 + a_{T0}t + v_{T0}}\,dt = \begin{cases} \frac{m_c}{m_T}\ln\left|\frac{m_T}{2}t^2 + a_{T0}t + v_{T0}\right| + \frac{m_T a_{co} - m_c a_{T0}}{\frac{m_T}{2}\sqrt{2m_T v_{T0} - a_{T0}^2}} \times \\ \quad \tan^{-1}\left(\frac{m_T t + a_{T0}}{\sqrt{2m_T v_{T0} - a_{T0}^2}}\right) + C \quad \text{for } 2m_T v_{T0} - a_{T0}^2 > 0, \\ \frac{m_c}{m_T}\ln\left|\frac{m_T}{2}t^2 + a_{T0}t + v_{T0}\right| - \frac{m_T a_{co} - m_c a_{T0}}{\frac{m_T}{2}\sqrt{a_{T0}^2 - 2m_T v_{T0}}} \times \\ \quad \tanh^{-1}\left(\frac{m_T t + a_{T0}}{\sqrt{a_{T0}^2 - 2m_T v_{T0}}}\right) + C \quad \text{for } 2m_T v_{T0} - a_{T0}^2 < 0, \\ \frac{m_c}{m_T}\ln\left|\frac{m_T}{2}t^2 + a_{T0}t + v_{T0}\right| - \frac{m_T a_{co} - m_c a_{T0}}{\frac{m_T}{2}(m_T t + a_{T0})} + C \\ \quad \text{for } 2m_T v_{T0} - a_{T0}^2 = 0 \end{cases}$$