ANN KIDDER
HW #6

11.3

48) $F(x,y,z) = x+2y+3z-\sin(xyz)$

$$\frac{dF}{dx} = 1 - yz\cos(xyz)$$

$$\frac{dF}{dy} = 2 - xz\cos(xyz)$$

$$\frac{dF}{dz} = 3 - xy\cos(xyz)$$

$$\boxed{\begin{aligned} \frac{dz}{dx} &= -\frac{dF/dx}{dF/dz} = -\left(\frac{1-yz\cos(xyz)}{3-xy\cos(xyz)}\right) \\ \frac{dz}{dy} &= -\frac{dF/dy}{dF/dz} = -\left(\frac{2-xz\cos(xyz)}{3-xy\cos(xyz)}\right) \end{aligned}}$$

54) $v_x = \frac{y(x-y)-xy}{(x-y)^2} = \frac{xy-y^2-xy}{(x-y)^2} = \boxed{\frac{-y^2}{(x-y)^2}}$

$v_y = \frac{x(x-y)-xy(-1)}{(x-y)^2} = \frac{x^2-xy+xy}{(x-y)^2} = \boxed{\frac{x^2}{(x-y)^2}}$

$v_{xx} = \frac{y^2(2)\cancel{(x-y)}}{(x-y)^{\cancel{4}3}} = \boxed{\frac{2y^2}{(x-y)^3}}$

$v_{xy} = \frac{-2y(x-y)^{\cancel{2}}+y^2(2)\cancel{(x-y)}(-1)}{(x-y)^{\cancel{4}3}} = \frac{-2xy+2y^2-2y^2}{(x-y)^3} = \boxed{\frac{-2xy}{(x-y)^3}}$

$v_{yx} = \frac{2x(x-y)^2-x^2(2)\cancel{(x-y)}}{(x-y)^{\cancel{4}3}} = \frac{2x^2-2xy-2x^2}{(x-y)^3} = \boxed{\frac{-2xy}{(x-y)^3}}$

$v_{yy} = \frac{+x^2(2)\cancel{(x-y)}\cancel{(-1)}}{(x-y)^{\cancel{4}3}} = \boxed{\frac{2x^2}{(x-y)^3}}$

68) a) $\boxed{f_x \text{ is decreasing}}$ because as you walk to the right of P (increasing x values) f(x,y) decreases

b) $\boxed{f_y \text{ is increasing}}$ because as you walk 'up' (increasing y values) f(x,y) increases.

c) $\boxed{f_{xx} \text{ is decreasing}}$ because the rate ~~that~~ the values are decreasing slows down as you walk at to the right.

d) $\boxed{f_{xy} \text{ is increasing.}}$ As we move up in the y direction, the curves are further apart in the x direction.

11.5

22) $u=\sqrt{r^2+s^2}$ $\quad r=y+x\cos t$, $\quad s=x+y\sin t.$

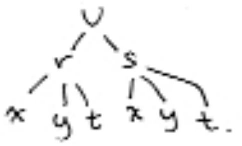
$$\frac{du}{dr}=\frac{1}{\cancel{2}}(r^2+s^2)^{-\frac{1}{2}}(\cancel{2}r)=\frac{r}{\sqrt{r^2+s^2}}$$

$$\frac{du}{ds}=\frac{1}{\cancel{2}}(r^2+s^2)^{-\frac{1}{2}}(\cancel{2}s)=\frac{s}{\sqrt{r^2+s^2}}$$

$$\frac{dr}{dy}=1 \qquad \frac{ds}{dx}=1$$

$$\frac{dr}{dx}=\cos t \qquad \frac{ds}{dy}=\sin t$$

$$\frac{dr}{dt}=-x\sin t. \qquad \frac{ds}{dt}=y\cos t.$$



$$\frac{du}{dy}=\left(\frac{du}{dr}\right)\left(\frac{dr}{dy}\right)+\left(\frac{du}{ds}\right)\left(\frac{ds}{dy}\right)=\left(\frac{r}{\sqrt{r^2+s^2}}\right)(1)+\left(\frac{s}{\sqrt{r^2+s^2}}\right)(\sin t)=\frac{r}{\sqrt{r^2+s^2}}+\frac{s(\sin t)}{\sqrt{r^2+s^2}}=\frac{r+s(\sin t)}{\sqrt{r^2+s^2}}$$

$$\frac{du}{dx}=\left(\frac{du}{dr}\right)\left(\frac{dr}{dx}\right)+\left(\frac{du}{ds}\right)\left(\frac{ds}{dx}\right)=\left(\frac{r}{\sqrt{r^2+s^2}}\right)(\cos t)+\left(\frac{s}{\sqrt{r^2+s^2}}\right)(1)=\frac{r\cos t+s}{\sqrt{r^2+s^2}}$$

$$\frac{du}{dt}=\left(\frac{du}{dr}\right)\left(\frac{dr}{dt}\right)+\left(\frac{du}{ds}\right)\left(\frac{ds}{dt}\right)=\left(\frac{r}{\sqrt{r^2+s^2}}\right)(-x\sin t)+\left(\frac{s}{\sqrt{r^2+s^2}}\right)(y\cos t)=\frac{sy\cos t-xr\sin t}{\sqrt{r^2+s^2}}$$

at $x=1, y=2, t=0$ $\quad \dfrac{du}{dx}=\dfrac{3\cos(0)+1}{\sqrt{9+1}}=\dfrac{4}{\sqrt{10}}$

$r=2+\cos(0)=3$

$s=1+2\cdot\sin(0)=1.$ $\quad \dfrac{du}{dy}=\dfrac{3+\sin(0)}{\sqrt{10}}=\dfrac{3}{\sqrt{10}}$

$$\frac{du}{dt}=\frac{2(\cos(0))-3\sin(0)}{\sqrt{10}}=\frac{2}{\sqrt{10}}$$

$$\boxed{\frac{du}{dx}=\frac{4}{\sqrt{10}} \quad \frac{du}{dy}=\frac{3}{\sqrt{10}} \quad \frac{du}{dt}=\frac{2}{\sqrt{10}}}$$

26) $F(x,y) = 1 + ye^{x^2} - y^5 - x^2y^3$

$F_x = 2xye^{x^2} - 2xy^3$

$F_y = e^{x^2} - 5y^4 - 3x^2y^2$

$$\frac{dy}{dx} = -\left(\frac{2xye^{x^2} - 2xy^3}{e^{x^2} - 5y^4 - 3x^2y^2}\right) = -\frac{2xy(e^{x^2} - y^2)}{e^{x^2} - 5y^4 - 3x^2y^2}$$

32) $F(x,y,z) = \ln(x+z) - yz$

$F_x = \frac{1}{x+z}$

$F_y = -z$

$F_z = \frac{1}{x+z} - y$

$$\frac{dz}{dx} = \frac{-\frac{1}{x+z}}{\frac{1}{x+z} - y} = -\frac{\frac{1}{x+z}}{\frac{1-y(x+z)}{x+z}} = -\left(\frac{1}{\cancel{x+z}}\right)\left(\frac{\cancel{x+z}}{1-y(x+z)}\right) = \boxed{-\frac{1}{1-y(x+z)}}$$

$$\frac{dz}{dy} = -\frac{(-z)}{\frac{1-y(x+z)}{x+z}} = \left(\frac{z}{1}\right)\left(\frac{x+z}{1-y(x+z)}\right) = \boxed{\frac{xz+z^2}{1-y(x+z)}}$$

43) $\frac{dz}{dr} = \left(\frac{dz}{dx}\right)\left(\frac{dx}{dr}\right) + \left(\frac{dz}{dy}\right)\left(\frac{dy}{dr}\right)$

$\frac{dz}{d\theta} = \left(\frac{dz}{dx}\right)\left(\frac{dx}{d\theta}\right) + \left(\frac{dz}{dy}\right)\left(\frac{dy}{d\theta}\right)$

$\frac{dx}{dr} = \cos\theta \qquad \frac{dx}{d\theta} = -r\sin\theta$

$\frac{dy}{dr} = \sin\theta \qquad \frac{dy}{d\theta} = r\cos\theta$.

$$\left(\frac{dz}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dz}{d\theta}\right)^2 = \left(\frac{dz}{dx}\right)^2\cos^2\theta + 2\left(\frac{dz}{dx}\right)\left(\frac{dz}{dy}\right)\cos\theta\sin\theta + \left(\frac{dz}{dy}\right)^2\sin^2\theta + \frac{1}{\cancel{r^2}}\left[\left(\frac{dz}{dx}\right)^2 \cancel{r^2}\sin^2\theta - 2\left(\frac{dz}{dx}\right)\left(\frac{dz}{dy}\right)\cancel{r^2}\sin\theta\cos\theta + \left(\frac{dz}{dy}\right)^2\cancel{r^2}\cos^2\theta\right]$$

$$= \left(\frac{dz}{dx}\right)^2\underbrace{\left[\cos^2\theta + \sin^2\theta\right]}_{=1} + \cancel{2\left(\frac{dz}{dx}\right)\left(\frac{dz}{dy}\right)\cos\theta\sin\theta} - \cancel{2\left(\frac{dz}{dx}\right)\left(\frac{dz}{dy}\right)\cos\theta\sin\theta} + \left(\frac{dz}{dy}\right)^2\underbrace{\left[\sin^2\theta + \cos^2\theta\right]}_{=1}$$

$$= \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2.$$

QED.

53) $\frac{dz}{dx} = \frac{-F_x}{F_z}$

$\frac{dx}{dy} = \frac{-F_y}{F_x}$ therefore $\boxed{\left(\frac{dz}{dx}\right)\left(\frac{dx}{dy}\right)\left(\frac{dy}{dz}\right) = \left(-\frac{\cancel{F_x}}{\cancel{F_z}}\right)\left(+\frac{\cancel{F_y}}{\cancel{F_x}}\right)\left(+\frac{\cancel{F_z}}{\cancel{F_y}}\right) = -1.}$

$\frac{dy}{dz} = -\frac{F_z}{F_y}$

11.6

44a) From problem 32 we know:

$$\frac{dz}{dx} = -\frac{1}{1+y(x+z)} \quad \text{at } (0,0,1) \ \frac{dz}{dx} = -1$$

$$\frac{dz}{dy} = \frac{xz+z^2}{1-y(x+z)} \quad \text{at } (0,0,1) \ \frac{dz}{dy} = 1$$

Therefore, $z-1 = -1(x-0)+1(y-0)$

$$\boxed{z = 1-x+y}$$

b) The normal line is given by $\frac{x-x_0}{F_x} = \frac{y-y_0}{F_y} = \frac{z-z_0}{F_z}$ where (x_0, y_0, z_0) is the point of tangency.

Therefore, from problem 32 we know F_x, F_y, F_z so we can plug in:

$$\frac{x}{\frac{1}{x+z}} = \frac{y}{-z} = \frac{z-1}{\frac{1}{x+z}-y}$$

and $F_x(0,0,1) = 1$

$F_y(0,0,1) = -1$

$F_z(0,0,1) = 1$

and $\frac{x}{1} = \frac{y}{-1} = \frac{z-1}{1}$

$$\boxed{x = -y = z-1}$$

50) $\nabla f = \langle 2x, -1, 2z \rangle$ which is normal to the tangent plane

Given plane has a normal vector $= \langle 1, 2, 3 \rangle$

Since the two planes are ∥ their $\vec{n}$ are ∥. and their ratios are equal.

$$\frac{2x}{1} = \frac{-1}{2} = \frac{2z}{3}$$

$x = -\frac{1}{4} \qquad z = -\frac{3}{4}$

$y = \left(\frac{-1}{4}\right)^2 + \left(\frac{-3}{4}\right)^2 = \frac{10}{16} = \frac{5}{8}$

The paraboloid's tangent plane is parallel to the given plane at

$$\boxed{\left(-\frac{1}{4}, \frac{5}{8}, \frac{-3}{4}\right)}$$

52) ∇f of the ellipsoid $= \langle 6x, 4y, 2z \rangle$

∇f of the sphere $= \langle 2x-8, 2y-6, 2z-8 \rangle$

at the point $(1,1,2)$ the gradient of ellipsoid is $\langle 6,4,4 \rangle$ and of the sphere is $\langle -6,-4,-4 \rangle$.

These vectors are the normal vectors to the tangent plane at that point. The vectors are equal (they point in opposite directions). This means the ellipsoid and the sphere have a common tangent plane and are therefore tangent to each other.