11.7

6) $f(x,y) = x^3y + 12x^2 - 8y$

$\frac{df}{dx} = 3x^2y + 24x = 0$ $\quad$ $3x^2y + 24x = 0$

$\frac{df}{dy} = x^3 - 8 = 0$ $\quad$ $x^3 = 8 \quad x = 2$

$\downarrow$

plug into first equation

$3(4)y + 48 = 0$

$y = -4$.

check for saddle point:

(2, -4) is the critical point

$\frac{d^2f}{dx^2} = 6xy + 24$

$\frac{d^2f}{dxdy} = 3x^2$

$\frac{d^2f}{dy^2} = 0$

plug in (2,-4) to the formula: $D(x,y) = \frac{d^2f}{dx^2} \cdot \frac{d^2f}{dy^2} - \left(\frac{d^2f}{dxdy}\right)^2$

$D(2,-4) = (6(2)(-4) + 24)(3(2)^2) - 0^2$

$= -288$

-288<0 so there is a saddle point and (2,-4) is the critical point

10) $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$

$\frac{df}{dx} = y + (-1)\frac{1}{x^2} = y - \frac{1}{x^2} = 0$ $\quad$ $y = \frac{1}{x^2}$

$\frac{df}{dy} = x - \frac{1}{y^2} = 0$

$x - \frac{1}{(\frac{1}{x^2})^2} = 0$

$x - \frac{1}{\frac{1}{x^4}} = 0$

$x - x^4 = 0 \quad x=0$ and $x = 1$.

$\downarrow$

plug into $y = \frac{1}{x^2}$. $x=0$ DNE

$x=1, y=1$.

(1, 1) is the critical point

$\frac{d^2f}{dx^2} = 2\left(\frac{1}{x^3}\right)$

$\frac{d^2f}{dy^2} = 2\left(\frac{1}{y^3}\right)$

$\frac{d^2f}{dxdy} = 1$.

$D(1,1) = \frac{2}{(1)^3} \cdot \frac{2}{(1)^3} - (1)^2 = 1 - 1 = 0$.

The saddle point test is inconclusive and (1,1) is a critical point

12) $f(x,y) = y\cos x$

$\frac{df}{dx} = -y\sin x \quad = 0$

$\frac{df}{dy} = \cos x \quad = 0$ $\quad x = n\frac{\pi}{2}$ w/n being any real integer. $(n \in \mathbb{Z})$

substitute into $\frac{df}{dx}$ and $y=0$.

$\frac{d^2f}{dx^2} = -y\cos x.$

$\frac{d^2f}{dy^2} = 0$

$\frac{d^2f}{dxdy} = -\sin x.$

$D(x,y) = (-y\cos x)(0) - (-\sin x)^2$

$= -\sin^2 x$ where $x = n\frac{\pi}{2}$ for all $n \in \mathbb{Z}$

so $D = -1 < 0$

$\boxed{\text{Therefore, saddle points at } (n\pi, 0) \text{ for all } n \in \mathbb{Z}}$

14) $f(x,y) = e^y(y^2 - x^2) = e^y y^2 - e^y x^2$

$\frac{df}{dx} = -2e^y x \quad = 0$ $\quad x = 0.$

plug in.

$\frac{df}{dy} = e^y y^2 + 2e^y y - e^y x^2 = 0$ $\quad e^y y^2 + 2e^y y = 0 \rightarrow \cancel{e^y} y^{\cancel{2}} = -2\cancel{e^y}\cancel{y} \quad y = -2.$

$\frac{d^2f}{dx^2} = -2e^y$

$\frac{d^2f}{dy^2} = e^y y^2 + 2e^y y + 2e^y y + 2e^y - e^y x^2 = e^y y^2 + 4e^y y + 2e^y - e^y x^2$

$\frac{d^2f}{dxdy} = -2e^y x$

$D(0,-2) = (-2e^y)(e^y y^2 + 4e^y y + 2e^y - e^y x^2) - (-2e^y x)^2$

$= (-2e^{-2})(e^{-2}(4) - 8e^{-2} + 2e^{-2})$

$= (-2e^{-2})(-2e^{-2})$

$= 4e^{-2} > 0$

$\frac{d^2f}{dx^2}(0,-2) = -2e^{-2} < 0$

$\boxed{\text{(0,-2) is a critical point and f(x,y) has a local maximum at (0,-2)}}$

28) $f(x,y)=3+xy-x-2y$

$\frac{df}{dx}=y-1=0 \rightarrow y=1$ critical point at $(2,1)$

$\frac{df}{dy}=x-2=0 \rightarrow x=2$

$\frac{d^2f}{dx^2}=0$

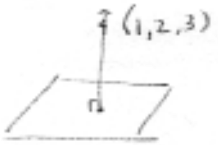
$\frac{d^2f}{dy^2}=0$ $D(2,1)=(0^2)(0^2)-1^2=-1<0$ so there is a saddle point at $(2,1)$

$\frac{d^2f}{dxdy}=1$

To test for absolute max, min we need to plug in endpoints as well.

$f(1,0)=2$
$f(5,0)=-2$
$f(1,4)=-2$

$\boxed{\text{There is an absolute max of 2 at (1,0), and absolute min value of -2 at (5,0) and (1,4)}}$

36) 

equation of line: $\vec{n}=\langle 1,-1,1\rangle$ and $P(1,2,3)$.

$\left.\begin{array}{l} x=1+t \\ y=2-t \\ z=3+t \end{array}\right\}$ plug into $x-y+z=4$.

$1+t-2+t+3+t=4$

$3t=2$

at $t=\frac{2}{3}$ $\boxed{\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)}$ is the point on the plane closest to $(1,2,3)$

38) $d=\sqrt{x^2+y^2+z^2}$ and $y^2=9+xz$

$d=\sqrt{x^2+(9+xz)+z^2}$

$=\sqrt{x^2+9+xz+z^2}$

$d^2=x^2+9+xz+z^2$

$\frac{df}{dx}=2x+z=0$ $z=-2x$

$\frac{df}{dz}=x+2z=0$ $x+2(-2x)=x-4x=-3x=0$ $x=0$

$\xrightarrow{\text{plug back in}}$ $z=0$.

$\frac{d^2f}{dx^2}=2$

second deriv. test.

$\frac{d^2f}{dz^2}=2$ $D(0,0)=(2)(2)-1^2=4-1=3>0$ and $\frac{d^2f}{dx^2}=2>0$ so this is

$\frac{d^2f}{dxdz}=1$ a local minimum.

Plug into $f(x,)$ to get y-value:

$y^2=9+0$ $y=\pm 3$ so

$\boxed{\text{(0,3,0) and (0,-3,0) are the points on the surface closest to the origin}}$

39) $a+b+c=100$ $c=100-a-b$ AND abc is a maximum.

$\frac{df}{da} = 100b-2ab-b^2$ set $=0$ ① $b=0$ or ② $b=100-2a$. $b=100$

$\frac{df}{db} = 100a-a^2-2ba$ set $=0$ plug in plug in

$100a-a^2=0$ or $100a-a^2-2a(100-2a)=0$ or $a=0$

$a=0$ or $a=100$. $100a-a^2-200a+4a^2=0$.

$a(3a-100)=0$.

① $(100, 0)$ $a=0$ or $a=\frac{100}{3}$ and $b=100-\frac{200}{3}=\frac{100}{3}$

4 critical points ② $(0, 0)$

③ $(\frac{100}{3}, \frac{100}{3})$

④ $(0, 100)$.

$\frac{d^2f}{da^2} = -2b$.

$\frac{d^2f}{db^2} = -2a$

$\frac{d^2f}{dadb} = 100-2a-2b$.

$D(a,b) = 4ab-(100-2a-2b)^2$. plug in 4 critical points

① $-10,000$ <0

② $-10,000$ <0

③ $\frac{40,000}{9} - \frac{10,000}{9} = \frac{30,000}{9} > 0$

④ $-10,000$ <0

Therefore $(100,0)$, $(0,0)$, and $(0,100)$ are saddle points.

$(\frac{100}{3}, \frac{100}{3})$ is a maximum. Plugging in to $a+b+c=100$, $c=\frac{100}{3}$

$\boxed{(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}) \text{ is a maximum}}$

43) $V = xyz$ and $x + 2y + 3z = 6$

$$z = \frac{6 - x - 2y}{3}$$

$$V = xy\left(\frac{6 - x - 2y}{3}\right) = \frac{1}{3}(6xy - x^2y - 2xy^2)$$

$\frac{dV}{dx} = \frac{1}{3}(6y - 2xy - 2y^2) = 0$ $\quad y(6 - 2x - 2y) = 0 \quad$ $y = 0$ and $y = \frac{6-2x}{2} = 3 - x$

$\frac{dV}{dy} = \frac{1}{3}(6x - x^2 - 4xy) = 0$ $\quad x(6 - x - 4y) = 0. \quad$ $x = 0$ and $x = 6 - 4y$

1. $(0,3)$
2. $(6,0)$
3. $(0,0)$
4. $(2,1)$
5. ~~$\left(\frac{42}{5}, -\frac{27}{5}\right)$~~ ✗ not in first octant

$$\frac{d^2V}{dx^2} = \frac{1}{3}(-2y)$$

$$\frac{d^2V}{dy^2} = \frac{1}{3}(-4x)$$

$$\frac{d^2V}{dxdy} = \frac{1}{3}(6 - 2x - 4y)$$

$$D(x,y) = \left(-\frac{2y}{3}\right)\left(-\frac{4x}{3}\right) - \left(\frac{1}{3}(6 - 2x - 4y)\right)^2$$

1. $-4 \quad < 0$
2. $-4 \quad < 0$
3. $-4 \quad < 0$
4. $\left(-\frac{2}{3}\right)\left(-\frac{8}{3}\right) - \left(-\frac{2}{3}\right)^2 = \frac{16}{9} - \frac{4}{9} = 4 \quad > 0$

Therefore $f(2,1)$ is the maximum Volume and $z = \frac{6-2-2}{3} = \frac{2}{3}$

$$\boxed{\text{so } \left(2, 1, \tfrac{2}{3}\right) \text{ is max Volume and } V = (2)(1)\left(\tfrac{2}{3}\right) = \tfrac{4}{3}}$$

44) $V = xyz$ and $SA = 2xy + 2xz + 2yz = 64$.

$$xy + xz + yz = 32$$
$$xz + yz = 32 - xy$$
$$z = \frac{32 - xy}{x+y}$$

$$V = xy\left(\frac{32-xy}{x+y}\right) = \frac{32xy - x^2y^2}{x+y}$$

$$\frac{dV}{dx} = \frac{(32y - 2xy^2)(x+y) - (32xy - x^2y^2)}{(x+y)^2} = \frac{\cancel{32xy} - 2x^2y^2 + 32y^2 - 2xy^3 - \cancel{32xy} + x^2y^2}{(x+y)^2} = \frac{32y^2 - 2xy^3 - x^2y^2}{(x+y)^2}$$

$$\frac{dV}{dy} = \frac{(32x - 2x^2y)(x+y) - (32xy - x^2y^2)}{(x+y)^2} = \frac{32x^2 - 2x^3y + 32xy - 2x^2y^2 - 32xy + x^2y^2}{(x+y)^2} = \frac{32x^2 - 2x^3y - x^2y^2}{(x+y)^2}$$

$\frac{dV}{dx} = 0$ at $y^2(32 - 2xy - x^2) = 0$ $\quad y = 0$ or $\rightarrow x^2 = y^2$ (note- $x+y \neq 0$) because $x>0$ and $y>0$.

$\frac{dV}{dy} = 0$ at $x^2(32 - 2xy - y^2) = 0$ $\quad x = 0$ or $\rightarrow$ or $x = y$.

We can see from the original equations that x, y, z can all be written in terms of one another with equivalent relationships so $x = y = z$.

therefore $64 = 6x^2 \quad \therefore \frac{32}{3} = x^2 \quad \sqrt{\frac{32}{3}} = x = y = z$.

$$\boxed{x = \sqrt{\frac{32}{3}} \quad y = \sqrt{\frac{32}{3}} \text{ and } z = \sqrt{\frac{32}{3}}}$$

46) $V = xyz \quad C = 5xy + 2xz + 2yz$

$z = \frac{V}{xy} \quad C = 5xy + 2\frac{V}{y} + 2\frac{V}{x}$.

$\frac{dC}{dx} = 5y - \frac{2V}{x^2} = 0 \qquad 5y = \frac{2V}{x^2} \quad y = \frac{2V}{5x^2}$

$\frac{dC}{dy} = 5x - \frac{2V}{y^2} = 0 \qquad 5x - \frac{2V}{\left(\frac{2V}{5x^2}\right)^2} = 5x - \frac{2V}{\frac{4V^2}{25x^4}} = 5x - \frac{50\cancel{V}x^4}{4V^{\cancel{2}}} = 5x - \frac{50x^4}{4V}$

$0 = 5x\left(1 - \frac{10x^3}{4V}\right) \quad x = 0$ or $x = \sqrt[3]{\frac{2V}{5}}$ but $x = 0$ DNE b/c can't divide by 0 so $x = \sqrt[3]{\frac{2V}{5}}$

$y = \frac{2V}{5\left(\sqrt[3]{\frac{2V}{5}}\right)^2} = \sqrt[3]{\frac{2V}{5}}$

$\frac{d^2C}{dx^2} = +\frac{4V}{x^3}$

$\frac{d^2C}{dy^2} = \frac{4V}{y^3}$

$\frac{d^2C}{dxdy} = 5$

$$\left(\frac{4V}{\left(\sqrt[3]{\frac{2V}{5}}\right)^3}\right)\left(\frac{4V}{\left(\sqrt[3]{\frac{2V}{5}}\right)^3}\right) - 25 = \frac{16V^{\cancel{2}}}{\frac{4V^{\cancel{2}}}{25}} - 25 = 100 - 25 = 75 > 0$$

and $\frac{4V}{\sqrt[3]{\frac{2V}{5}}}$ will be >0 so this is a minimum

$$\boxed{x = y = \sqrt[3]{\frac{2V}{5}} \text{ and } z = \sqrt[3]{\frac{25V}{4}}} \rightarrow z = \frac{V}{\left(\sqrt[3]{\frac{2V}{5}}\right)^2} = \frac{(V^3)^{\frac{1}{3}}}{1} \cdot \left(\frac{25}{4V^2}\right)^{\frac{1}{3}} = \left(\frac{25V}{4}\right)^{\frac{1}{3}}$$

52) $a(x-1)+b(y-2)+c(z-3)=0$ $\quad \vec{n}=\langle a,b,c\rangle$ any (parallel) normal vector will work

$ax-a+by-2b+cz-3c=0$

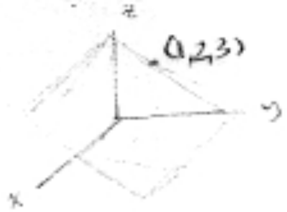
$ax+by+cz-(a+2b+3c)=0$ to simplify find a normal vector where $a+2b+3c=1$.

so, $ax+by+cz=1$.

x-intercept $=\frac{1}{a}$

y-intercept $=\frac{1}{b}$

z-intercept $=\frac{1}{c}$



$V=\frac{1}{3}(A_b)h$. $\quad A_b=$ area of base $=\frac{1}{2}(ab)$ $\quad h=c$

$V=\frac{1}{6}abc \rightarrow 6V=abc$. $\quad$ since $a+2b+3c=1$

$a=1-2b-3c$.

$V=\frac{1}{6}(1-2b-3c)bc$.

$V=\frac{1}{6}(bc-2b^2c-3bc^2)$

$\frac{dV}{db}=\frac{1}{6}(c-4bc-3c^2)=0 \quad c=0$ or $0=1-4b-3c$. $\quad c=\frac{1-4b}{3}$

$\frac{dV}{dc}=\frac{1}{6}(b-2b^2-6bc)=0 \quad b=0$ or $0=1-2b-6c \quad b=\frac{1-6c}{2}$.

① $(0,0)$

② $(0,\frac{1}{3})$

③ $(\frac{1}{2},0)$

④ $(\frac{1}{6},\frac{1}{9})$.

$\frac{d^2V}{db^2}=\frac{1}{6}(-4c)$

$\frac{d^2V}{dc^2}=\frac{1}{6}(-6b)$

$\frac{d^2V}{db\,dc}=\frac{1}{6}(1-4b-6c)$.

$D(b,c)=\frac{2cb}{3}-\frac{1}{36}(1-4b-6c)^2$

① $-\frac{1}{36}<0$
② $-\frac{1}{36}<0$
③ $-\frac{1}{36}<0$
} saddle points.

④ $\frac{1}{81}-\frac{1}{324}>0$

$\frac{1}{6}(-4\cdot\frac{1}{9})=\frac{-2}{27}$ so this is a maximum point. (?)

$a=1-2(\frac{1}{6})-3(\frac{1}{9})=\frac{1}{3}$

$\langle\frac{1}{3},\frac{1}{6},\frac{1}{9}\rangle$

$$\boxed{\tfrac{1}{3}x+\tfrac{1}{6}y+\tfrac{1}{9}z=1 \quad \text{or} \quad 6x+3y+2z=18}$$

11.8

4) $x^2+y^2-13 = g(x,y)$

$\frac{df}{dx} = 4$ $\quad \frac{dg}{dx} = 2x \rightarrow 2x\lambda = 4$. $\quad \lambda = \frac{2}{x}$

$\frac{df}{dy} = 6$ $\quad \frac{dg}{dy} = 2y \rightarrow 2y\lambda = 6$ $\quad 2y\left(\frac{2}{x}\right) = 6$

$\frac{d^2f}{dx^2} = 0$ $\quad \frac{4y}{x} = 6$

$\frac{d^2f}{dy^2} = 0$ $\quad {}^{2}\cancel{4}y = \cancel{6}^{3}x$

$\frac{d^2f}{dxdy} = 0$ $\quad 2y = 3x$

$y = \frac{3}{2}x$

$x^2 + \left(\frac{3}{2}x\right)^2 = 13$

$\frac{4x^2}{4} + \frac{9x^2}{4} = 13$

$13x^2 = 13(4)$

$x^2 = 4$

$x = \pm 2$. and $y = \pm 3$.

critical points

① $(2,3)$

② $(-2,-3)$.

$\boxed{f(2,3) = 8+18 = 26 \quad f(-2,-3) = -26 \text{ so } (2,3) \text{ is the maximum and } (-2,-3) \text{ is the minimum at } -26}$

8) $g(x,y,z) = x^2+10y^2+z^2-5$.

$\frac{df}{dx} = 8$ $\quad \frac{dg}{dx} = 2x$ $\quad 2x\lambda = 8$. $\quad x = \frac{4}{\lambda}$

$\frac{df}{dy} = 0$ $\quad \frac{dg}{dy} = 20y$ $\quad 20y\lambda = 0$ $\quad y = 0$

$\frac{df}{dz} = -4$. $\quad \frac{dg}{dz} = 2z$. $\quad 2z\lambda = -4$. $\quad z = \frac{-2}{\lambda}$

$\left(\frac{4}{\lambda}\right)^2 + 0 + \left(\frac{-2}{\lambda}\right)^2 = 5$

$\frac{16}{\lambda^2} + \frac{4}{\lambda^2} = 5$

$\frac{20}{5} = \lambda^2$

$4 = \lambda^2 \quad \lambda = \pm 2. \rightarrow$ critical points $(2,0,-1)$ and $(-2,0,1)$

$\boxed{\begin{array}{ll} f(2,0,-1) = 20 & \text{maximum} \\ f(-2,0,1) = -20 & \text{minimum} \end{array}}$

28). $f(x,y,z)=(x-1)^2+(y-2)^2+(z-3)^2$; $x-y+z=4 \rightarrow g(x)=x-y+z-4$.

$\frac{df}{dx}=2(x-1)$ $\quad$ $\frac{dg}{dx}=1$ $\quad$ $2x-2=\lambda$ $\quad$ $x=\frac{\lambda+2}{2}$

$\frac{df}{dy}=2(y-2)$ $\quad$ $\frac{dg}{dy}=-1$ $\quad$ $2y-4=-\lambda$ $\quad$ $y=\frac{4-\lambda}{2}$

$\frac{df}{dz}=2(z-3)$ $\quad$ $\frac{dg}{dz}=1$ $\quad$ $2z-6=\lambda$ $\quad$ $z=\frac{\lambda+6}{2}$

$$\left(\frac{\lambda+2}{2}\right)-\left(\frac{4-\lambda}{2}\right)+\left(\frac{\lambda+6}{2}\right)=4.$$

$3\lambda+4=8$

$3\lambda=4$

$\lambda=\frac{4}{3} \rightarrow$ $\boxed{\left(\frac{5}{3},\frac{4}{3},\frac{11}{3}\right)}$ is the point on the plane closest to $(1,2,3)$

31) $a+b+c=100$ $\quad$ $f(a,b,c)=abc$.

$g(a,b,c)=a+b+c-100$

$\frac{df}{da}=bc$ $\quad$ $\frac{dg}{da}=1$ $\quad$ $bc=\lambda$ $\quad$ $b=\frac{\lambda}{c}$ $\quad$ $\lambda^3=a^2b^2c^2$.

$\frac{df}{db}=ac$ $\quad$ $\frac{dg}{db}=1$ $\quad$ $ac=\lambda$ $\quad$ $c=\frac{\lambda}{a}$ $\quad$ $\lambda^{3/2}=abc$.

$\frac{df}{dc}=ab$ $\quad$ $\frac{dg}{dc}=1$ $\quad$ $ab=\lambda$. $\quad$ $a=\frac{\lambda}{b}$

$\frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c}=100$ $\quad$ $b=\frac{\lambda}{\frac{\lambda}{\frac{\lambda}{b}}} \rightarrow b^2=\lambda$. and $a=b=c=\sqrt{\lambda}$.

$$\frac{\lambda(ab+bc+ac)}{abc}=100$$

$$\frac{3\lambda^2}{\lambda^{3/2}}=100$$

$$\lambda^{\frac{1}{2}}=\frac{100}{3}$$

$\lambda=\left(\frac{100}{3}\right)^2 \rightarrow \boxed{a=b=c=\sqrt{\lambda}=\frac{100}{3}}$

35). $V = xyz$; $x + 2y + 3z = 6$

$f(x,y,z) = xyz$ $\quad g(x) = x + 2y + 3z - 6$

$\frac{df}{dx} = yz$ $\quad \frac{dg}{dx} = 1$ $\quad yz = \lambda$ $\quad y = \frac{\lambda}{z}$

$\frac{df}{dy} = xz$ $\quad \frac{dg}{dy} = 2$ $\quad xz = 2\lambda$ $\quad z = \frac{2\lambda}{x}$ $\quad xyz = \left(\frac{3\lambda}{y}\right)\left(\frac{\lambda}{z}\right)\left(\frac{2\lambda}{x}\right)$

$= \frac{6\lambda^3}{xyz}$

$\frac{df}{dz} = xy$ $\quad \frac{dg}{dz} = 3$ $\quad xy = 3\lambda$ $\quad x = \frac{3\lambda}{y}$ $\quad x^2y^2z^2 = 6\lambda^3$

$\frac{3\lambda}{y} + 2\left(\frac{\lambda}{z}\right) + 3\left(\frac{2\lambda}{x}\right) = 6$ $\quad yz = \frac{xz}{2} = \frac{xy}{3}$

$\frac{\lambda(\overset{(2\lambda)}{3xz} + \overset{(3\lambda)}{2xy} + \overset{(\lambda)}{6yz})}{xyz} = 6$

$\frac{\lambda(18\lambda)}{\sqrt{6\lambda^3}} = 6$

$\frac{18\lambda^2}{\sqrt{6}\,\lambda^{\frac{3}{2}}} = 6$

$\lambda^{\frac{1}{2}} = \frac{6\sqrt{6}}{18} = \frac{\sqrt{6}}{3}$

$\lambda = \frac{6}{9} = \frac{2}{3}$

$y = \frac{\frac{2}{3}}{z} = \frac{2}{3z}$

$z = \frac{\frac{4}{3}}{x} = \frac{4}{3x}$ $\quad z = \frac{4}{\frac{6}{y}} = \frac{2y}{3}$ $\quad y = \frac{2}{2y} = \frac{1}{y} \Rightarrow y^2 = 1 \quad y = 1$

$x = \frac{2}{y}$

$x = 2$

$z = \frac{2}{3}$

$\boxed{V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}}$