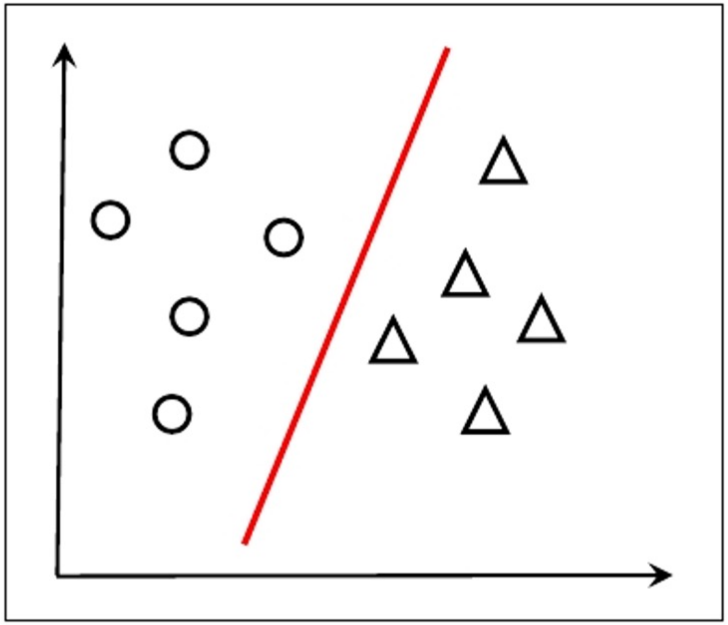
Linear model

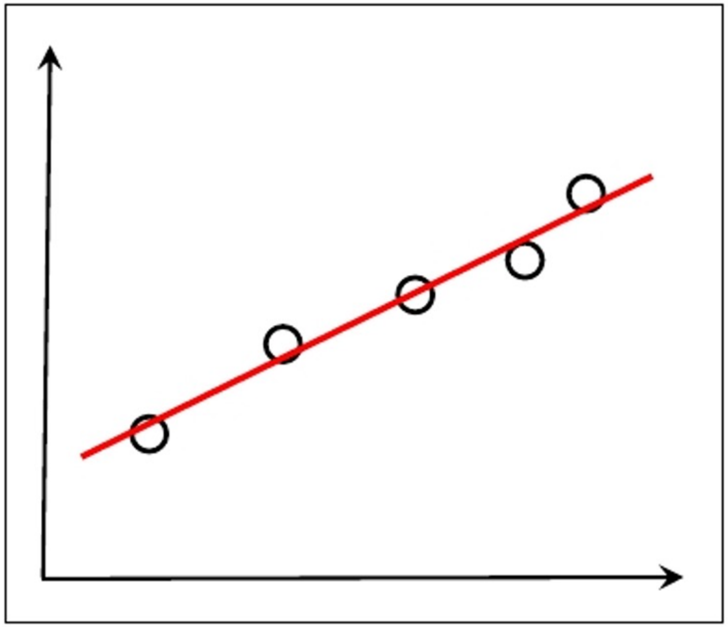
Alymzhan Toleu
alymzhan.toleu@gmail.com

Linear Model

Classification



Regression



A linear model specifies a linear relationship between a dependent variable (Y/f(x)) and *d* independent variables (X):

$$f(\boldsymbol{x}) = w_1 x_1 + w_2 x_2 + \ldots + w_d x_d + b$$

Vectorization: $f(\boldsymbol{x}) = \boldsymbol{w}^{\mathrm{T}} \boldsymbol{x} + b$

Linear Regression

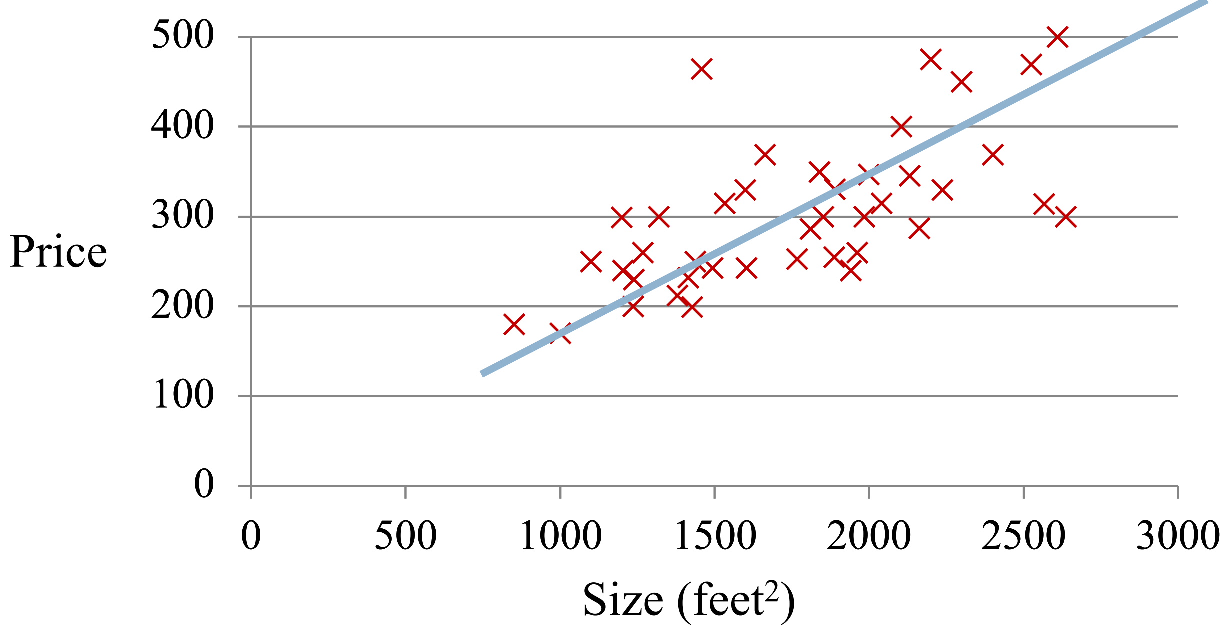
Trainning set

Size in feet² (x)	Price ($) in 1000's (y)
2104	460
1416	232
1534	315
852	178
…	…

Supervised Learning

Given the "real value" for each example.

$$f(x) = w_1 x_1 + b$$



Regression Problem

Predict real-valued output

Linear Regression with One Variable

Size in feet² (x)	Price ($) in 1000's (y)
2104	460
1416	232
1534	315
852	178
…	…

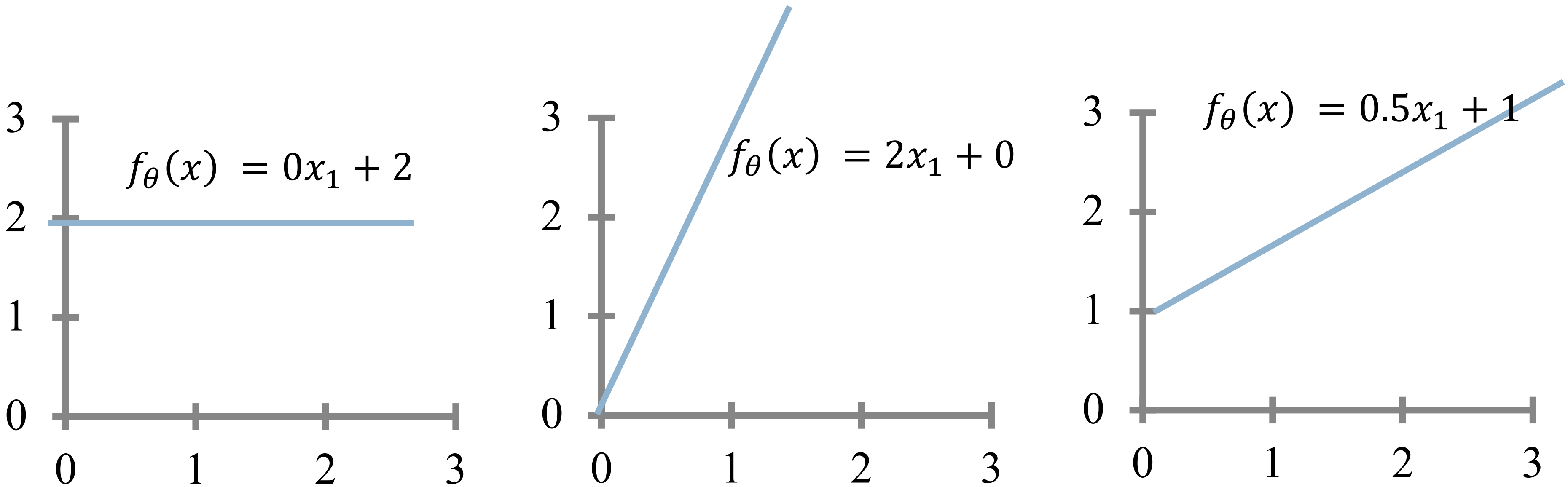
Hypothesis/Model: $y \approx f_\theta(x) = w_1 x_1 + b$

Parameters are: $\theta = \{w_1,\ b\}$

How to choose θ ?

Linear Regression: Example

Hypothesis/Model: $f_\theta(x) = w_1 x_1 + b$ Parameters: $\theta = \{w_1,\ b\}$



$w_1 = 0$ $b = 2$

$w_1 = 2$ $b = 0$

$w_1 = 0.5$ $b = 1$

Idea: Choose $\theta = \{w_1,\ b\}$ so that $f_\theta(x)$ is close to *y* for our training examples *(x, y)* .

Linear Regression: Example

Hypothesis/Model: $f_\theta(x) = w_1 x_1 + b$

Parameters: $\theta = \{w_1,\ b\}$

Cost Function:

	Size in feet² (x)	Price ($) in 1000's (y)
m	2104	460
	1416	232
	1534	315
	852	178
	…	…

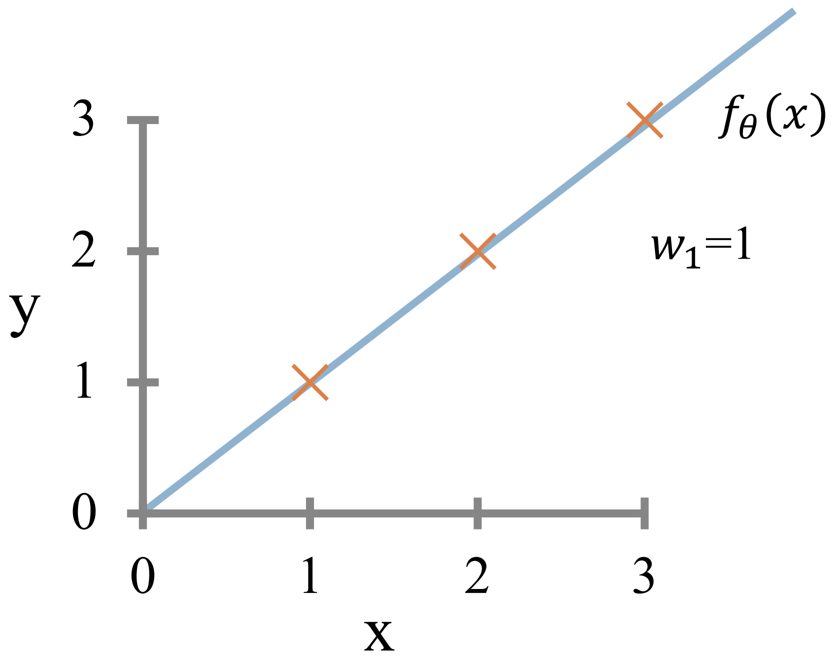
$$E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\left(f_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$

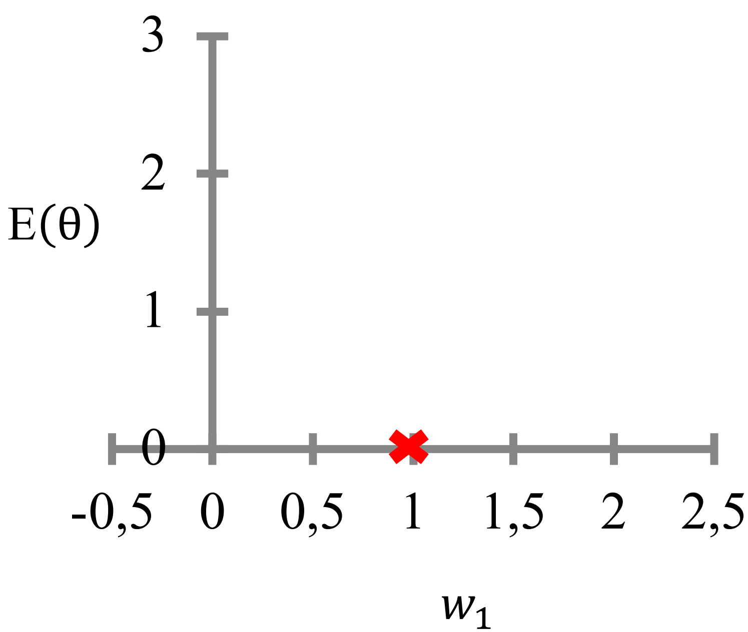
Goal: minimize $E(\theta)$

$f_\theta(x)$ Vs. $E(\theta)$

Simple hypothesis : $f_\theta(x) = w_1 x_1$
(ignoring parameter b)

$E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\left(f_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$



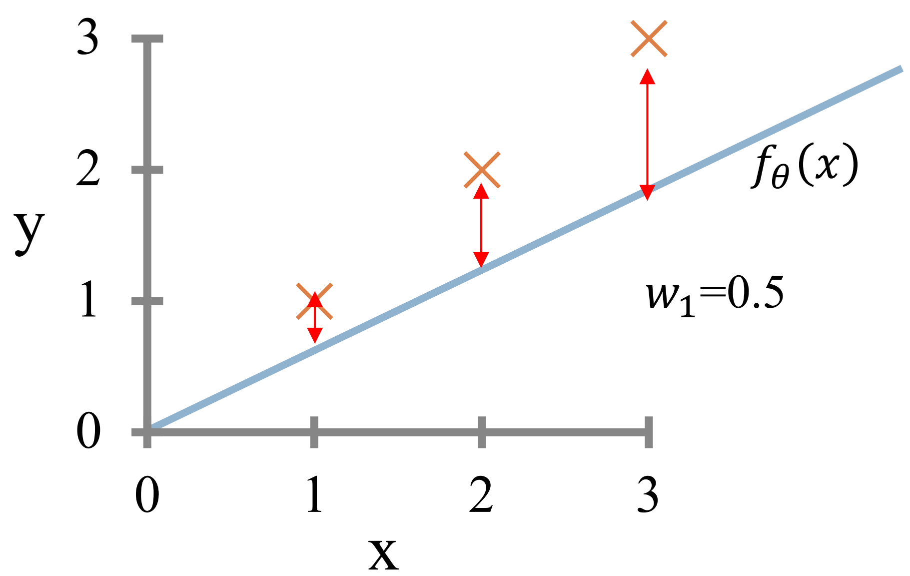


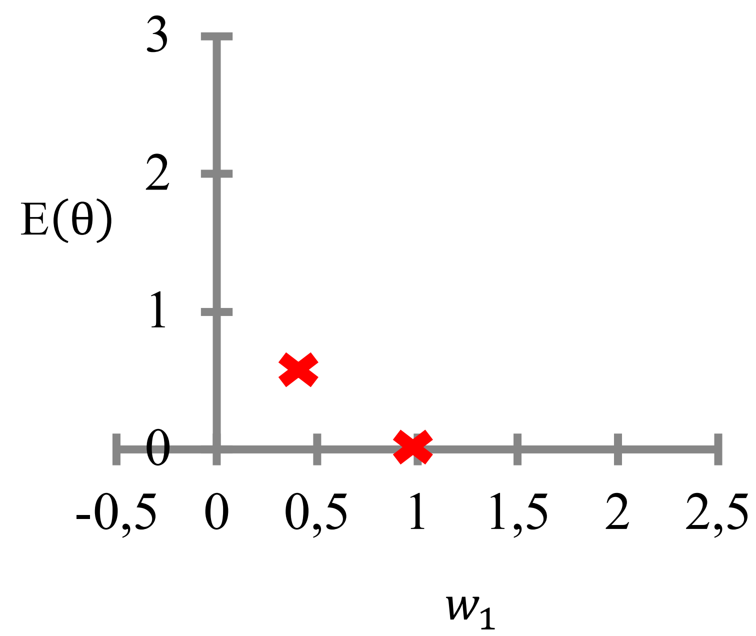
$E(w_1{=}1) = \frac{1}{2m}[(1-1)^2 + (2-2)^2 + (3-3)^2] = 0$

$f_\theta(x)$ Vs. $E(\theta)$

Simple hypothesis : $f_\theta(x) = w_1 x_1$
(ignoring parameter *b*)

$$E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\left(f_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$



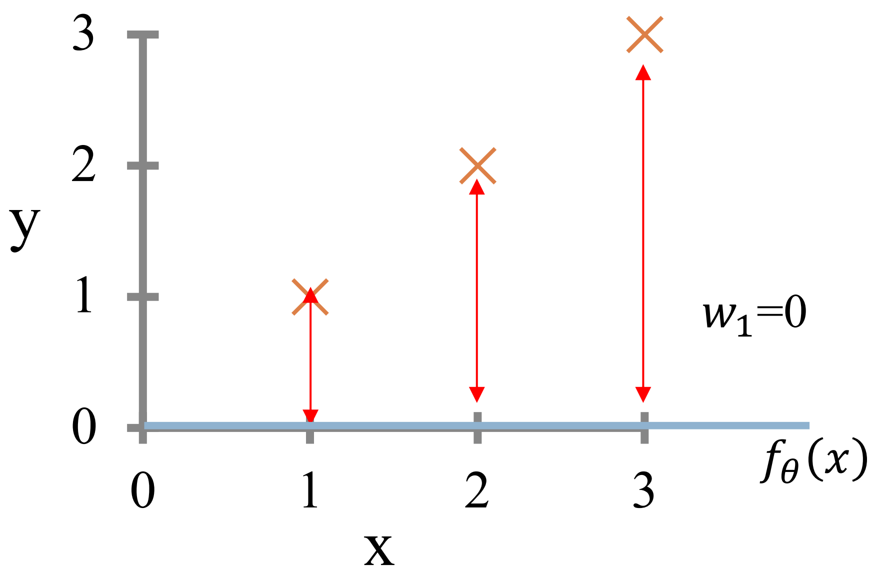


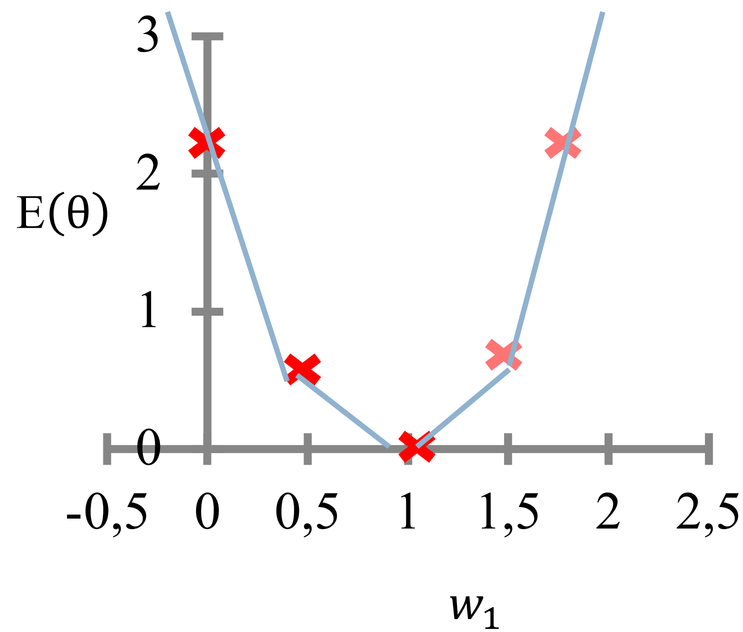
$E(w_1{=}0.5) = \frac{1}{2m}[\,(0.5-1)^2+(1-2)^2+(1.5-3)^2] =$
0.548

$f_\theta(x)$ Vs. $E(\theta)$

Simple hypothesis : $f_\theta(x) = w_1 x_1$
(ignoring parameter *b*)

$$E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\left(f_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$

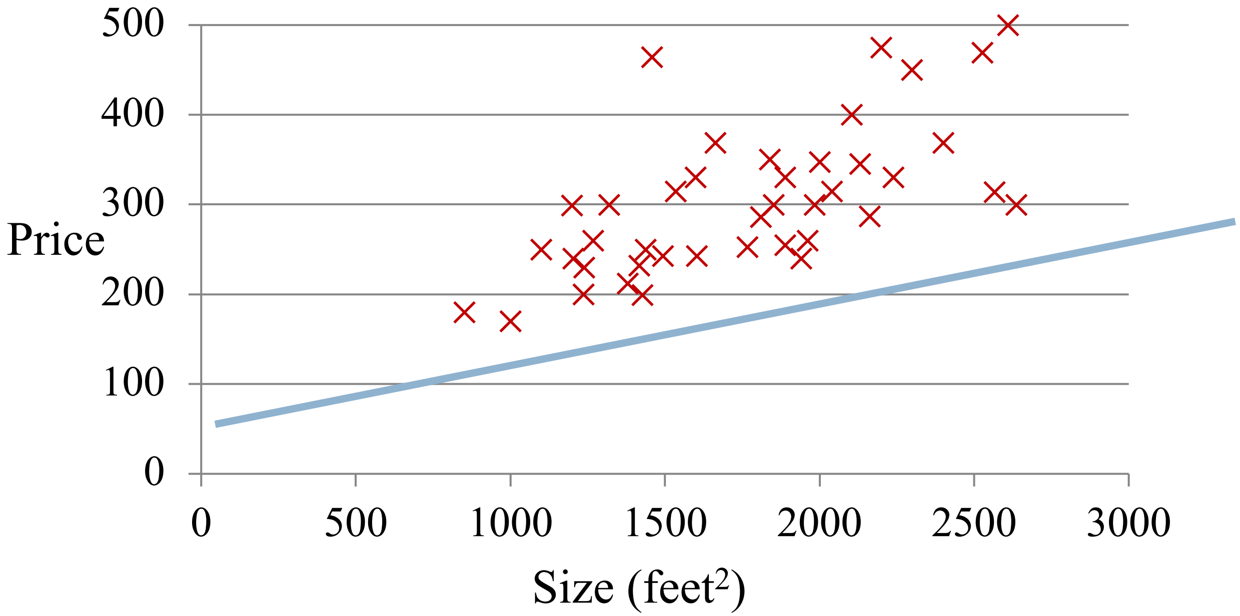




$E(w_1{=}0) = \frac{1}{2m}[\,(0-1)^2+(0-2)^2+(0-3)^2] \approx 2.3$

Choose parameter θ that minimizes cost function E(θ), which corresponds to finding a straight line that fits the data well.

Example: House price prediction



Hypothesis: $f_\theta(x) = 0.05x_1 + 50$

Parameters: $\theta = \{w_1,\ b\}$

Cost Function: $E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(f_\theta(x^{(i)}) - y^{(i)})^2$

Goal: minimize $E(\theta)$

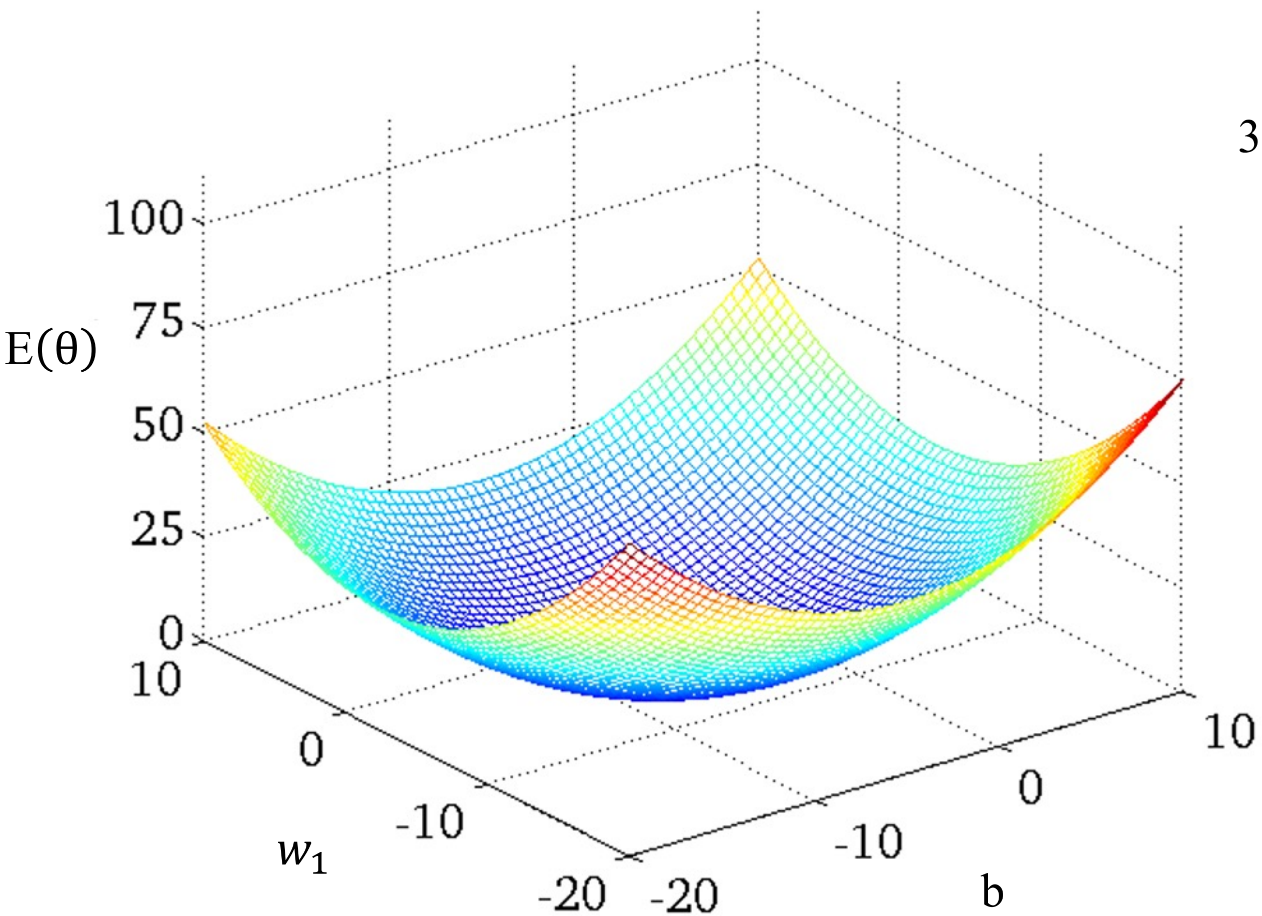
How to plot the E(θ) ?

Hypothesis: $f_\theta(x) = w_1 x_1 + b$

Parameters: $\theta = \{w_1,\ b\}$

Cost Function: $E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\left(f_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$

Goal: minimize $E(\theta)$



3-d surface plot

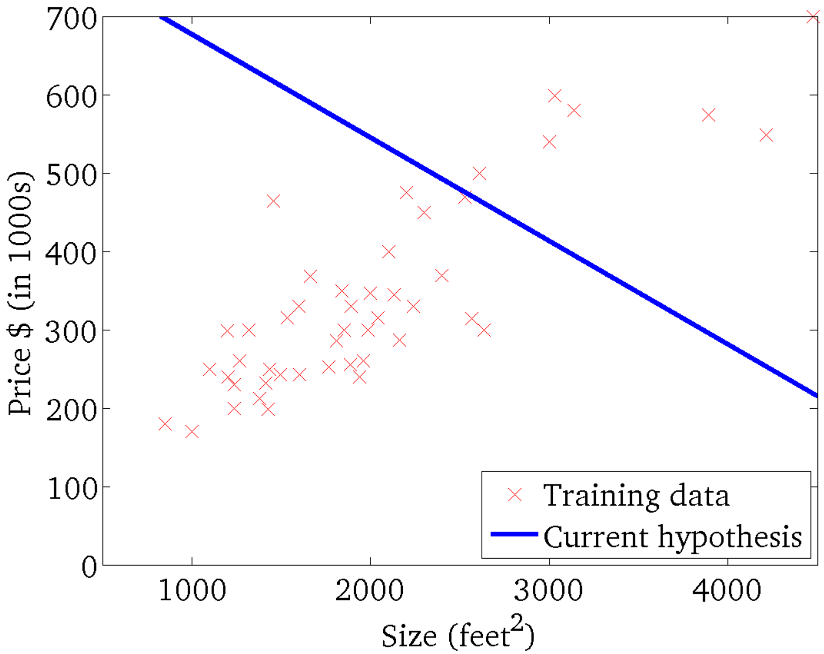
Contour figure

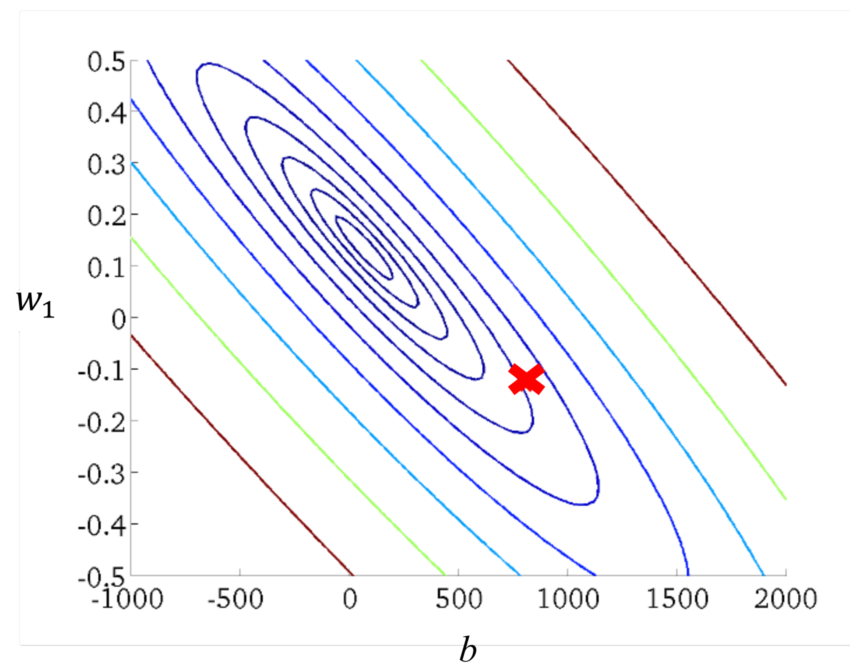
$f_\theta(x) = w_1 x_1 + b$

(for fixed $\theta = \{w_1,\ b\}$, this is a function of x)

$\mathrm{E}(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(f_\theta\left(x^{(i)}\right) - y^{(i)})^2$

(function of the parameters $\theta = \{w_1,\ b\}$)





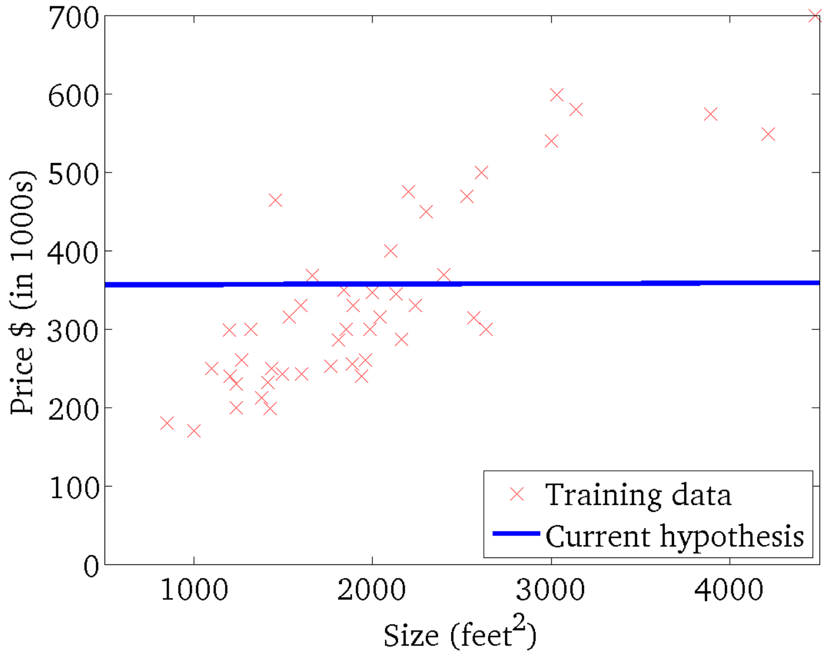
Contour figure

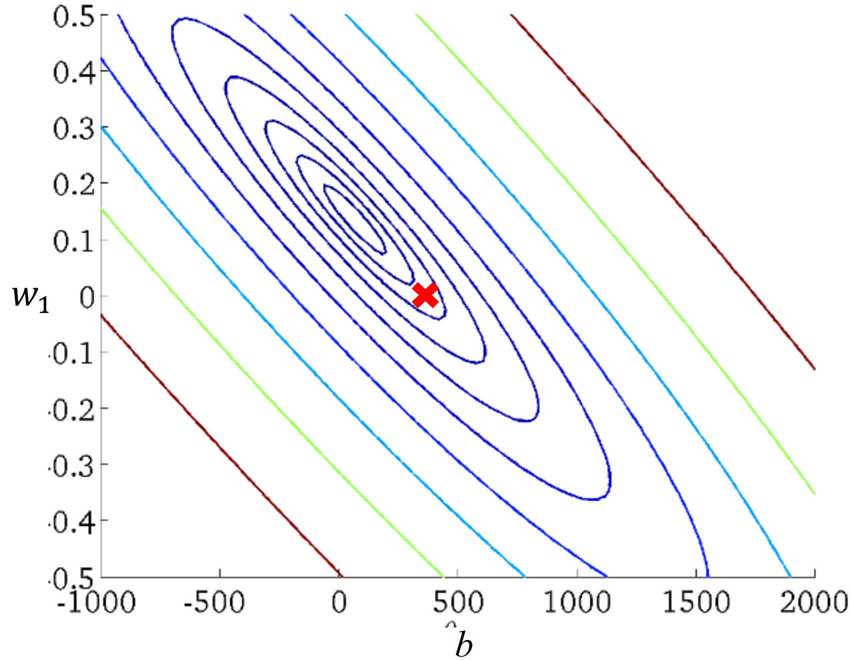
$$f_\theta(x) = w_1 x_1 + b$$

(for fixed $\theta = \{w_1,\ b\}$, this is a function of x)

$$E(\theta) = \frac{1}{2m} \sum_{i=1}^{m} \left(f_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$

(function of the parameters $\theta = \{w_1,\ b\}$)





$$f_\theta(x) = 0x_1 + 370$$

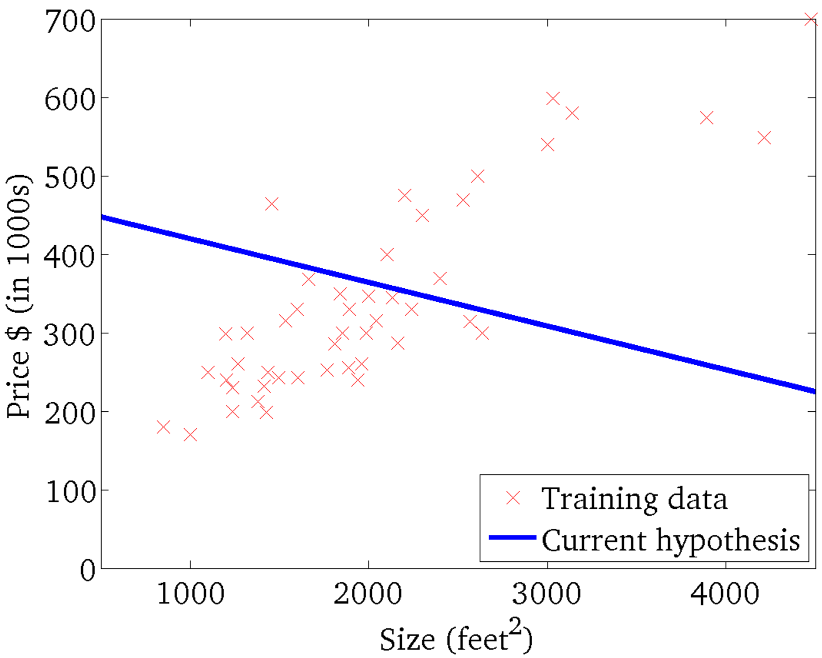
Contour figure

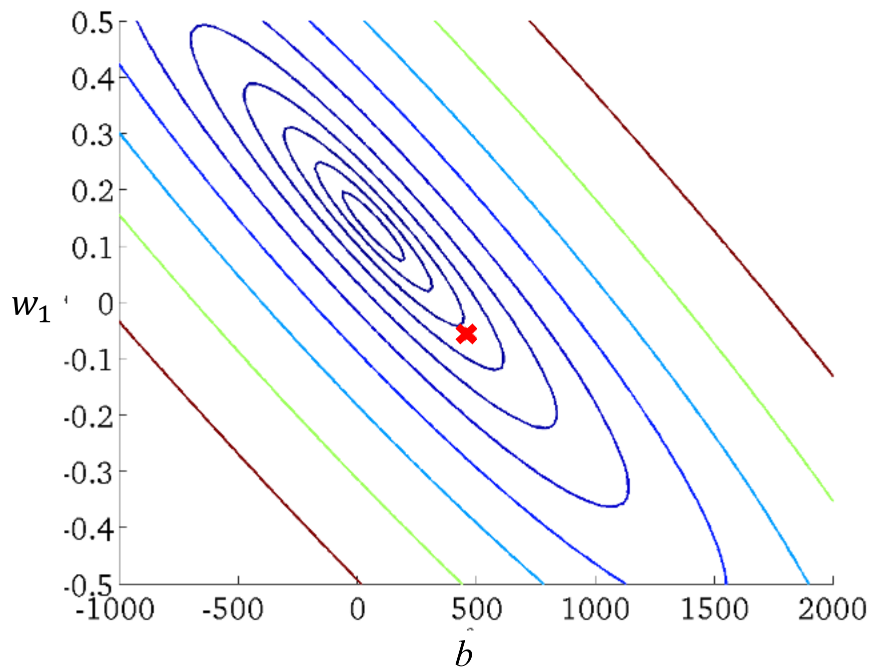
$$f_\theta(x) = w_1 x_1 + b$$

(for fixed $\theta = \{w_1,\ b\}$, this is a function of x)

$$\mathrm{E}(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\left(f_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$

(function of the parameters $\theta = \{w_1,\ b\}$)

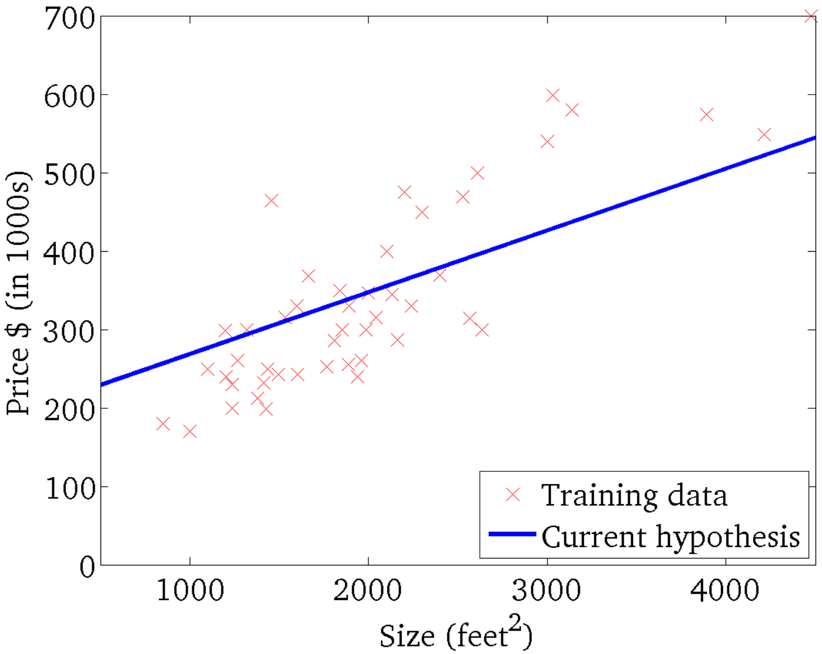




Contour figure

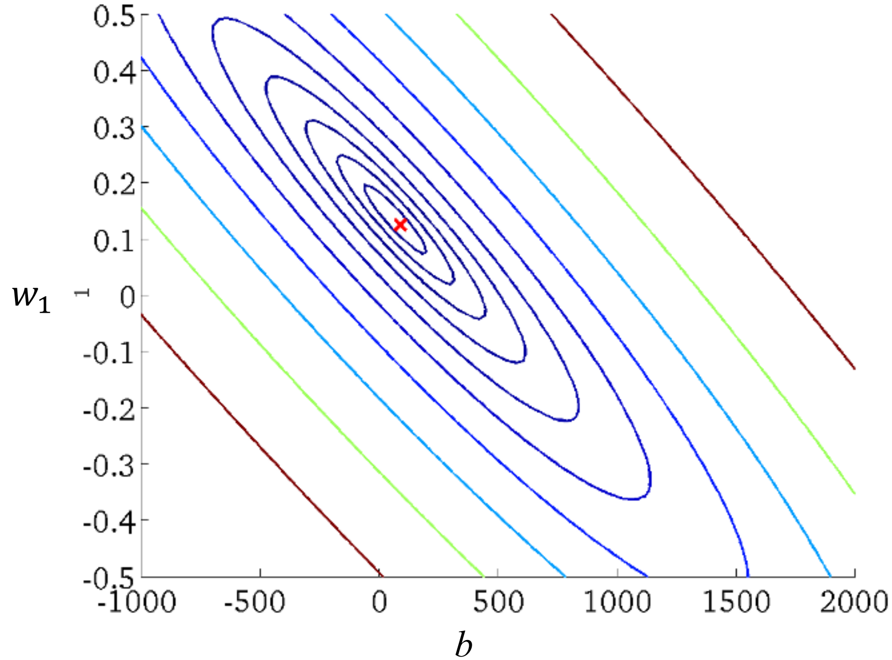
$$f_\theta(x) = w_1 x_1 + b$$

(for fixed $\theta = \{w_1,\ b\}$, this is a function of x)



$$E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\left(f_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$

(function of the parameters $\theta = \{w_1,\ b\}$)



Gradient descent

Hypothesis: $f_\theta(x) = w_1 x_1 + b$

Parameters: $\theta = \{w_1,\ b\}$

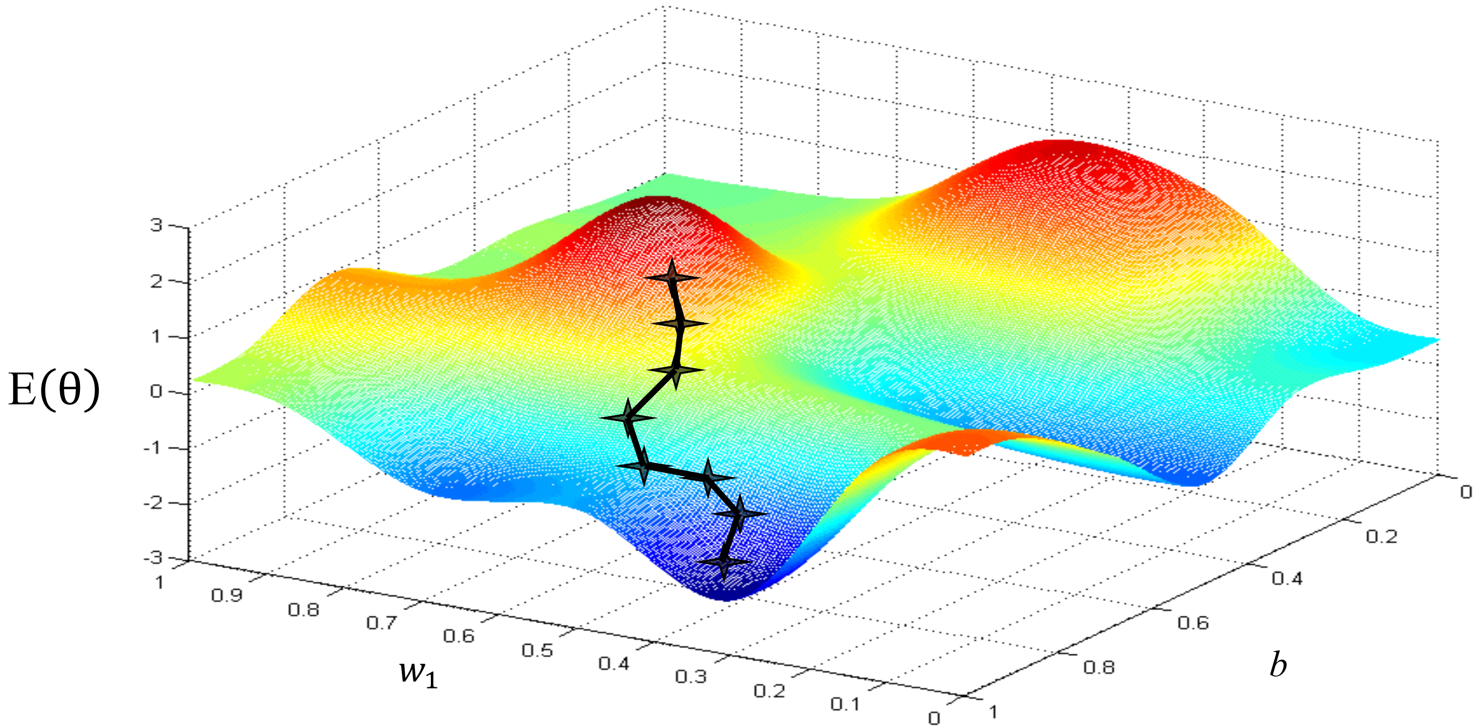
Cost Function: $E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(f_\theta\left(x^{(i)}\right) - y^{(i)})^2$

Goal: minimize $E(\theta)$

Steps:

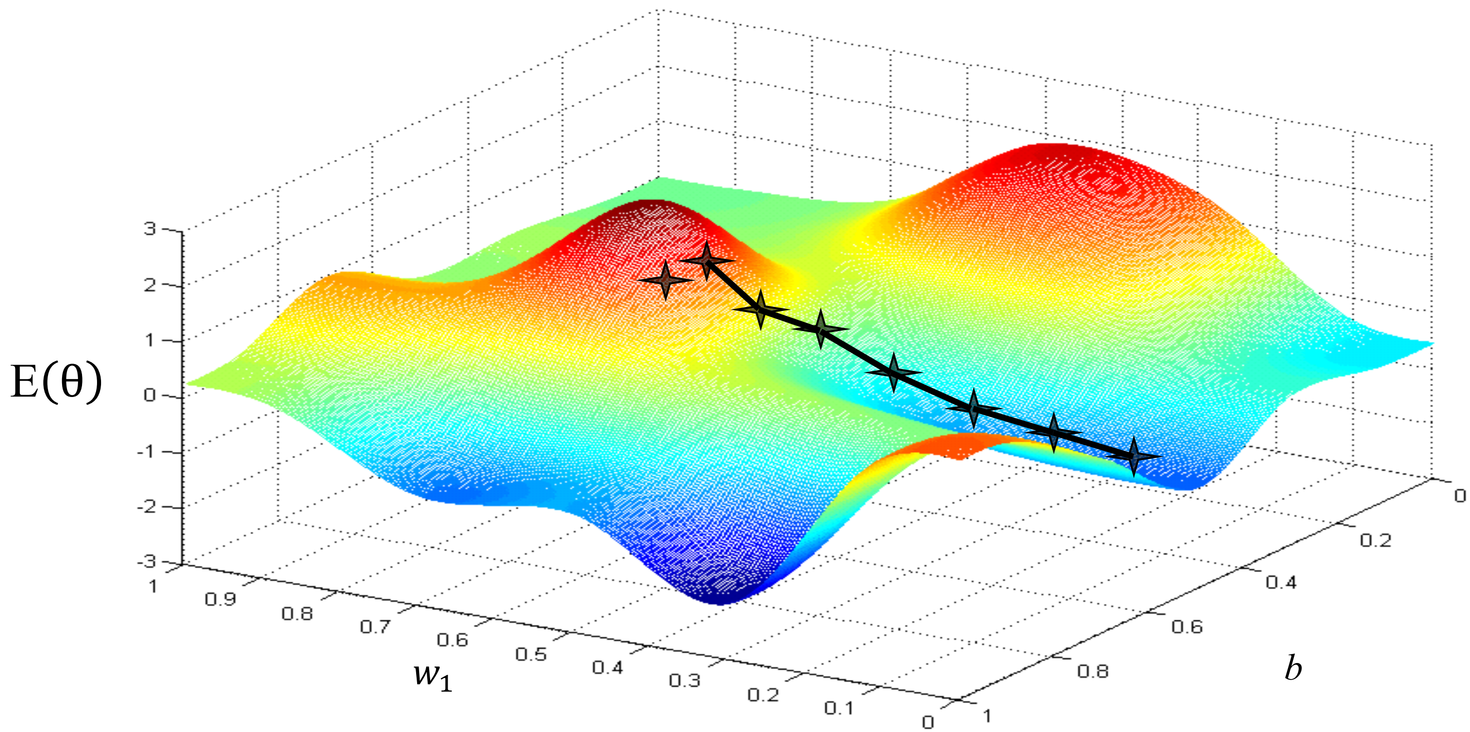
- Start with some $\theta = \{w_1,\ b\}$; *e.g.* $w_1 = 0,\ \ b = 0$.
- Keep changing θ to reduce $E(\theta)$ until end up at a minimum.

Intuition picture of gradient descent



Starting at some points on the surface of this fucntion.
Take a step in the direction of steepest descent.
Each step changes parameter θ to reduce E(θ) until end up at a local minimum.

Intuition picture of gradient descent



Starting at another point on the surface of this fucntion.
Each step changes parameter θ to reduce E(θ) until end up at a local minimum.
There are many local minimums.

Gradient descent algorithm

- Repeat until convergence

$$\theta_j := \theta_j - \alpha \boxed{\frac{\partial}{\partial \theta_j} E(\theta)} \qquad \theta = \{\theta_0 = w_1, \theta_1 = b\}$$

Learning rate

Gradient

- Simultaneous update
- Learning rate
  - α determines the step size at each iteration while moving toward a minimum of a cost function.
  - If α is too small, gradient descent can be slow;
  - If α is too large, gradient descent can overshoot the minimum. It may fail to converge, or even diverge.
- Gradient descent can converge to a local minimum, even with the learning rate α fixed.

Gradient descent for linear regression

Hypothesis:

$$\boxed{f(x) = wx_i + b \qquad f(x_i) \simeq y_i}$$

Cost Function:

$$\begin{aligned}(w^*, b^*) &= \arg\min_{(w,b)} \sum_{i=1}^{m} \left(f(x_i) - y_i\right)^2 \\ &= \arg\min_{(w,b)} \sum_{i=1}^{m} \left(y_i - wx_i - b\right)^2\end{aligned}$$

Minimize:

$$\boldsymbol{E(\theta)} = E_{(w,b)} = \sum_{i=1}^{m} \left(y_i - wx_i - b\right)^2$$

Derivates

Finding partial derivates of function $E_{(w,b)}$ with two variables: w and b

$$
\begin{aligned}
\frac{\partial E_{(w,b)}}{\partial w} &= \frac{\partial}{\partial w}\left[\sum_{i=1}^{m}(y_i - wx_i - b)^2\right] \\
&= \sum_{i=1}^{m}\frac{\partial}{\partial w}\left[(y_i - wx_i - b)^2\right] \\
&= \boxed{\sum_{i=1}^{m}[2\cdot(y_i - wx_i - b)\cdot(-x_i)]} \\
&= \sum_{i=1}^{m}\left[2\cdot\left(wx_i^2 - y_ix_i + bx_i\right)\right] \\
&= 2\cdot\left(w\sum_{i=1}^{m}x_i^2 - \sum_{i=1}^{m}y_ix_i + b\sum_{i=1}^{m}x_i\right) \\
&= 2\left(w\sum_{i=1}^{m}x_i^2 - \sum_{i=1}^{m}(y_i - b)\,x_i\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E_{(w,b)}}{\partial b} &= \frac{\partial}{\partial b}\left[\sum_{i=1}^{m}(y_i - wx_i - b)^2\right] \\
&= \sum_{i=1}^{m}\frac{\partial}{\partial b}\left[(y_i - wx_i - b)^2\right] \\
&= \boxed{\sum_{i=1}^{m}[2\cdot(y_i - wx_i - b)\cdot(-1)]} \\
&= \sum_{i=1}^{m}[2\cdot(b - y_i + wx_i)] \\
&= 2\cdot\left[\sum_{i=1}^{m}b - \sum_{i=1}^{m}y_i + \sum_{i=1}^{m}wx_i\right] \\
&= 2\left(mb - \sum_{i=1}^{m}(y_i - wx_i)\right)
\end{aligned}
$$

Gradient descent for linear regression

Hypothesis: $f_\theta(x) = w_1 x_1 + b$

Parameters: $\theta = \{w_1,\ b\}$

Cost Function: $E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(f_\theta(x^{(i)}) - y^{(i)})^2$

Goal: minimize $E(\theta)$

- Repeat until convergence

$\frac{\partial}{\partial \theta_j} E(\theta)$

$$w_1 := w_1 - \alpha \boxed{\frac{1}{m}\sum_{i=1}^{m}\left[f_\theta(x^{(i)}) - y^{(i)}\right] x^{(i)}}$$

$$b := b - \alpha \frac{1}{m}\boxed{\sum_{i=1}^{m}\left[f_\theta(x^{(i)}) - y^{(i)}\right]}$$

update w_1 and b simultaneously

Linear Regression: Example

Size in feet² (x)	**Price ($) in 1000's (y)**
2104	460
1416	232
1534	315
852	178

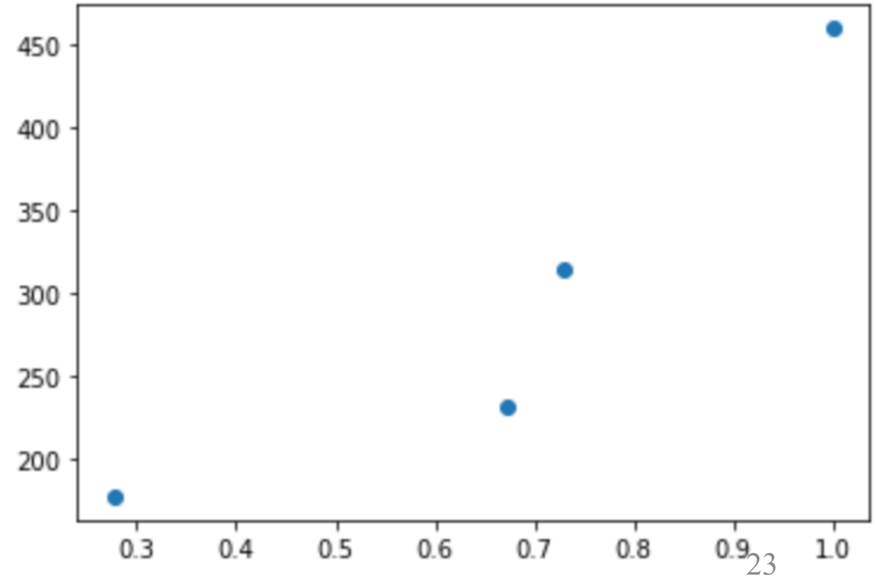
starting point(can be random):
$w = 0, \quad b = 0$

learning rate $a = 0.5$

max_inter = 10

Normalizing & add a column with value one

```
array([[1.        , 1.        ],
       [1.        , 0.6730038 ],
       [1.        , 0.72908745],
       [1.        , 0.27661597]])
```



Iteration 1

Hypothesis: $f_\theta(x) = w_1 x_1 + b$

Parameters: $\theta = \{w_1,\ b\}$

Cost Function: $E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(f_\theta(x^{(i)}) - y^{(i)})^2$

Estimation: (w = 0, b = 0)

$$f_\theta(1) = (0 * 1 + 0) = 0$$

$$f_\theta(0.67) = (0 * 67 + 0) = 0$$

$$f_\theta(0.72) = (0 * 0.72 + 0) = 0$$

$$f_\theta(0.27) = (0 * 0.27 + 0) = 0$$

Normalized input x

```
array([[1.        , 1.        ],
       [1.        , 0.6730038 ],
       [1.        , 0.72908745],
       [1.        , 0.27661597]])
```

Cost: $E(\theta) = \frac{1}{2*4}[(0-460)^2 + (0-232)^2 + (0-315)^2 + (0-178)^2] \approx 49541.62$

Gradient:

w' = $\frac{\partial E(\theta)}{\partial w} = \frac{1}{4}[(0-460)*1 + (0-232)*0.67 + (0-315)*0.72 + (0-178)*0.27] \approx$ - 222.575

b' = $\frac{\partial E(\theta)}{\partial b} = (\frac{1}{4})[(0-460) + (0-232) + (0-315) + (0-178)] \approx$- 296.25

Update:

$$w_1 := w_1 - \alpha * w' = 0 - 0.5 * (-222.575) \approx 111.2875$$
$$b := b - \alpha * b' = 0 - 0.5 * (-296.25) \approx 148.125$$

New parameters

Iteration 2

Hypothesis: $f_\theta(x) = w_1 x_1 + b$

Parameters: $\theta = \{w_1,\ b\}$

Cost Function: $E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(f_\theta(x^{(i)}) - y^{(i)})^2$

Estimation: (w = 111.28,   b = 148.12)

$$f_\theta(1) = (111.28 * 1 + 148.12) \approx 260$$

$$f_\theta(0.67) = (111.28 * 67 + 148.12) \approx 223$$

$$f_\theta(0.72) = (111.28 * 0.72 + 148.12) \approx 229$$

$$f_\theta(0.27) = (111.28 * 0.27 + 148.12) \approx 179$$

Normalized input x

```
array([[1.        , 1.        ],
       [1.        , 0.6730038 ],
       [1.        , 0.72908745],
       [1.        , 0.27661597]])
```

Cost: $E(\theta) = (\frac{1}{2*4})[(260 - 460)^2 + (223 - 232)^2 + (229 - 315)^2 + (179 - 178)^2] \approx 5918.73$

Gradient:

w' = $\frac{\partial E(\theta)}{\partial w} = (\frac{1}{4})\ [(260 - 460) * 1 + (223 - 232) * 0.67 + (229 - 315) * 0.72 + (179 - 178) * 0.27] \approx -66.92$

b' = $\frac{\partial E(\theta)}{\partial b} = (\frac{1}{4})[(260 - 460) + (223 - 232) + (229 - 315) + (179 - 178)] \approx -73.5$

Update:

$$w_1 := w_1 - \alpha\ * w' = 0 - 0.5\ * (-66.92) \approx 184.72$$

$$b := b - \alpha\ * b' = 0 - 0.5\ * (-73.5) \approx\ 145.33$$

New parameters

Continue…

Iteration 3

```
iter: 3 cost: 2778.85472457512
prediction: [330.06397483 282.53897763 290.69006727 224.92873394]
gradient: [-14.19456158 -25.16648269]
parameters: [191.8231828  157.92131416]
```

Iteration 4

```
iter: 4 cost: 2487.2205920097786
prediction: [349.74449696 298.10482769 306.96163143 235.50674024]
gradient: [  1.32942408 -13.9300234 ]
parameters: [191.15847076 164.88632586]
```

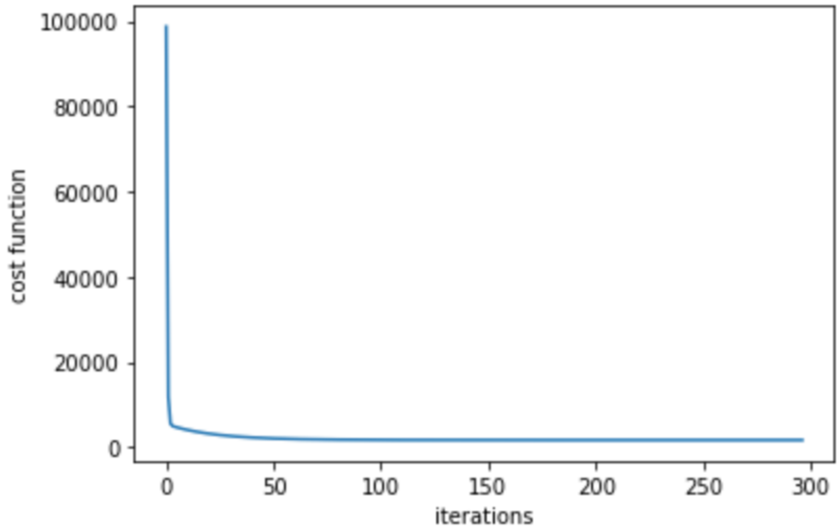
Iteration 5

```
iter: 5 cost: 2398.9324937073943
prediction: [356.04479662 302.12759501 311.37502203 236.76866166]
gradient: [  5.32901883 -10.78641023]
parameters: [188.49396135 170.27953098]
```

…..

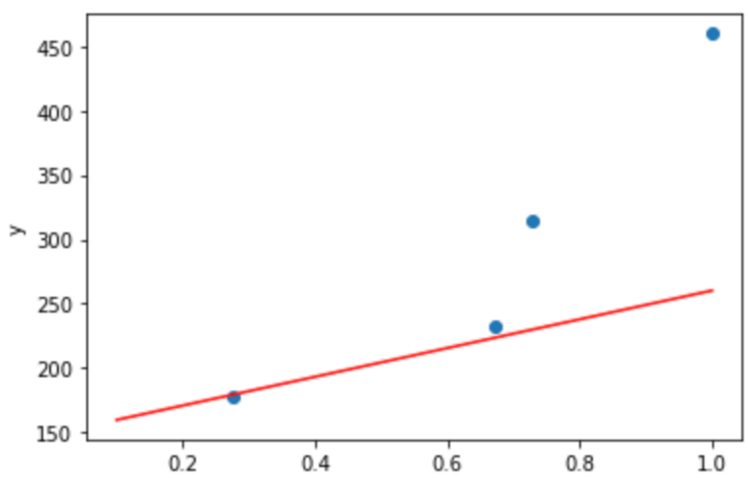
Until it converges

Linear Regression: Example

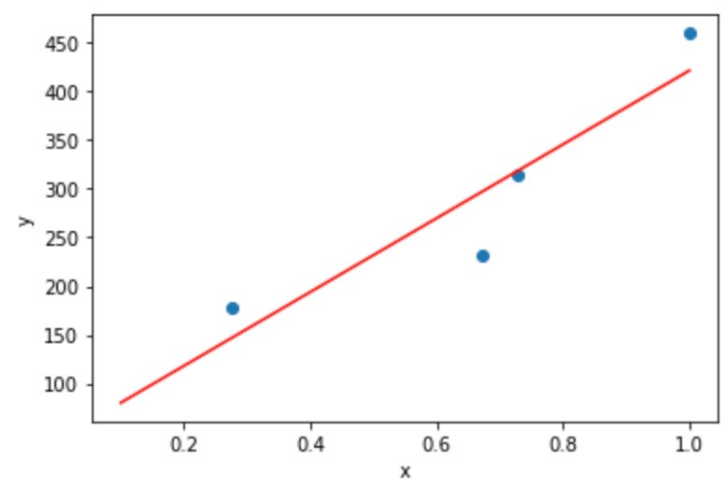


learning_rate = 0.5

```
iter: 0 cost: 98663.42912068815
```

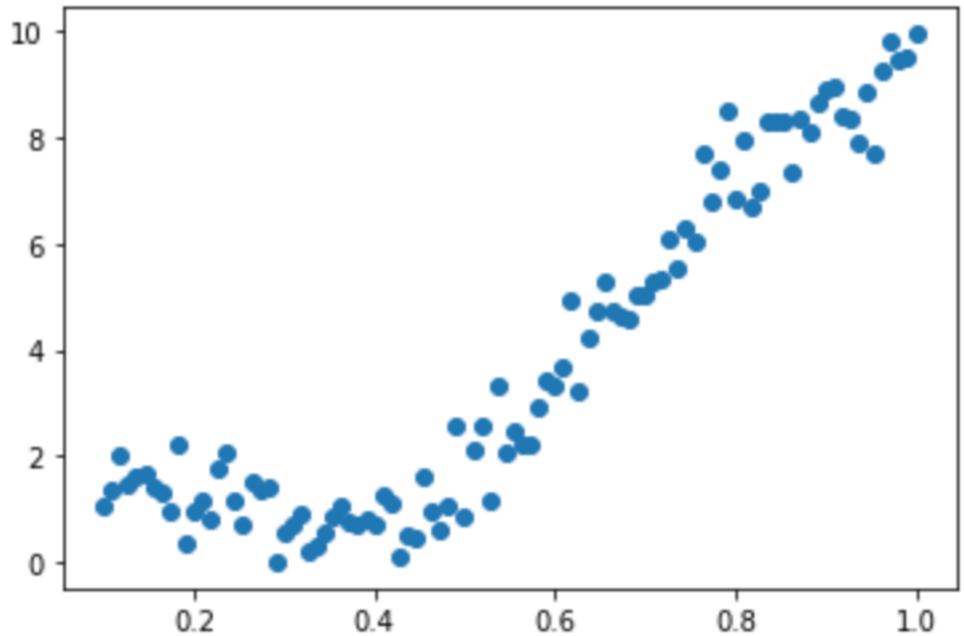



```
iter: 296 cost: 1684.0276765926678
```



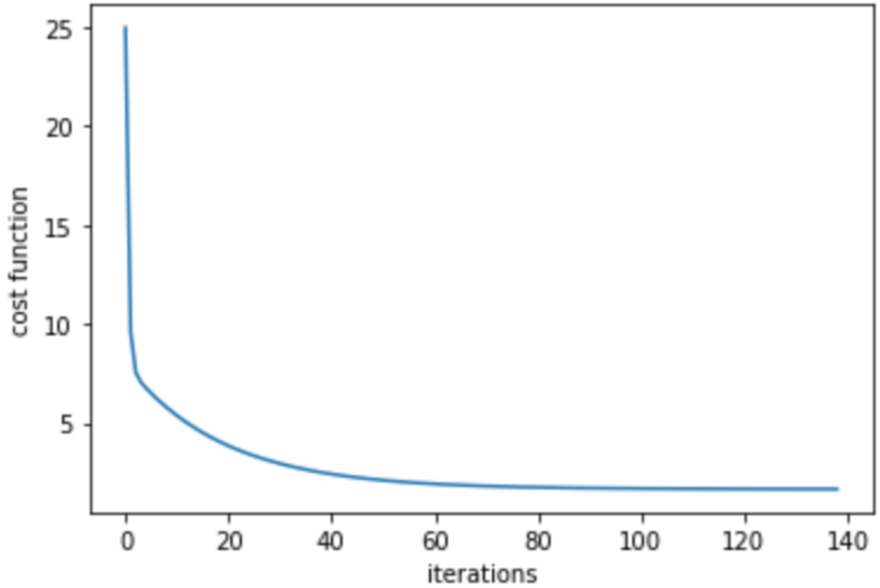

```
Converged.
```

Linear Regression: Example-2

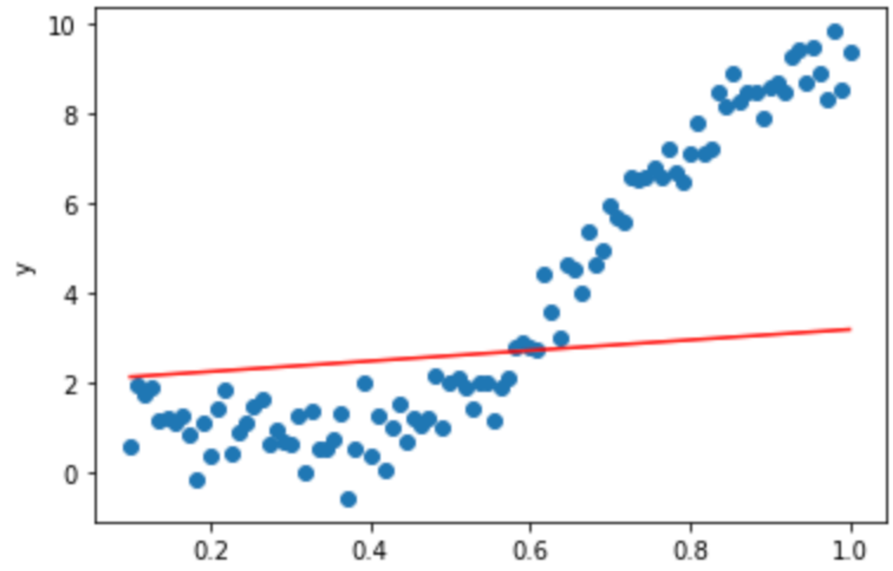


N = 100 samples

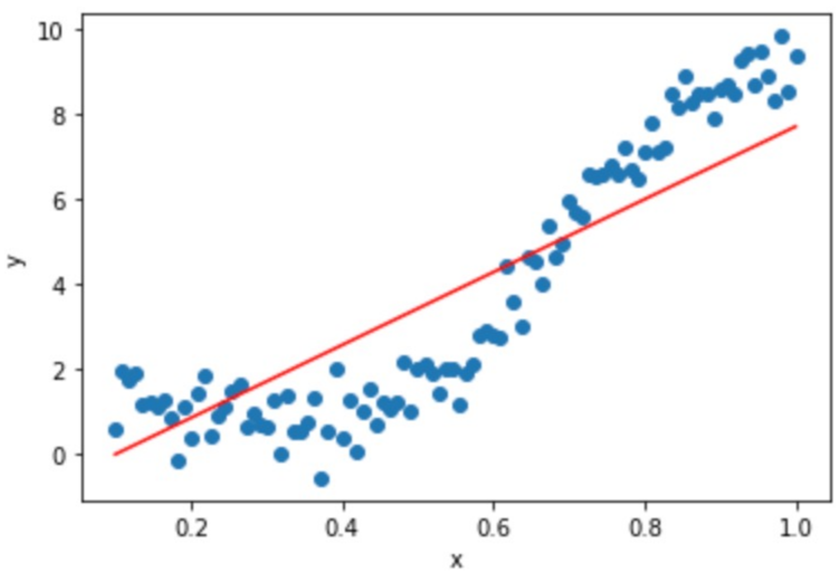
learning_rate = 0.5



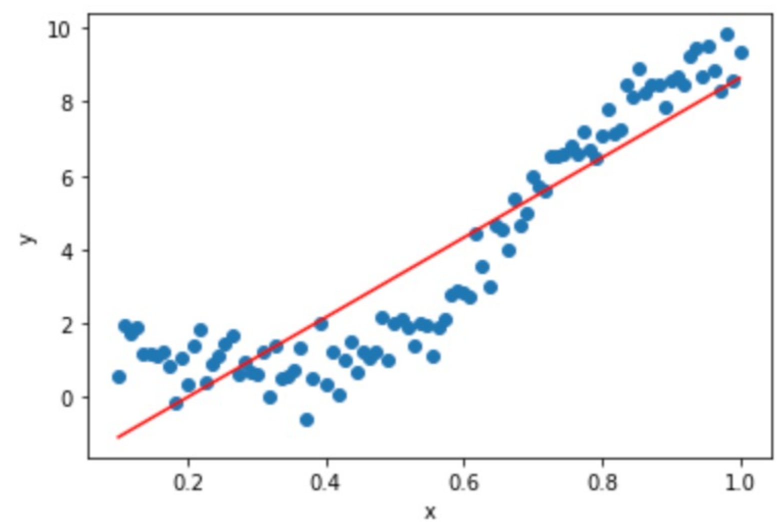
iter: 0 cost: 24.973176452747104



iter: 50 cost: 2.0982659858775956



iter: 138 cost: 1.6546338133174023



Converged.

Multivariate linear regression

Size in feet² (x)	Price ($) in 1000's (y)
2104	460
1416	232
1534	315
852	178

$$\cancel{f_\theta(x) = w_1 x_1} + b$$

linear regression with one variable

Size (feet²) x_1	Number of bedrooms x_2	Number of floors x_3	Age of home (years) x_4	Price ($1000) y
2104	5	1	45	460
1416	3	2	40	232
1534	3	2	30	315
852	2	1	36	178
…	…	…	…	…

linear regression with multiple variables

n – is number of features

$$f_\theta(x) = w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 + \cdots + w_n x_n + b$$

$x_0 = 1$

Hypothesis: $f_\theta(x) = \theta^T x = w_0 x_0 + w_1 x_1 + \cdots + w_n x_n$

Parameters: $\theta = \{w_0, \ldots, w_n\}$

Cost Function: $E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(f_\theta(x^{(i)}) - y^{(i)})^2$

Goal: minimize $E(\theta)$

- Repeat until convergence

$$w_0 := w_0 - \alpha \frac{1}{m}\sum_{i=1}^{m}[f_\theta(x^{(i)}) - y^{(i)}]x_0^{(i)}$$

$$w_1 := w_1 - \alpha \frac{1}{m}\sum_{i=1}^{m}[f_\theta(x^{(i)}) - y^{(i)}]x_1^{(i)}$$

update θ simultaneously

$$w_2 := w_1 - \alpha \frac{1}{m}\sum_{i=1}^{m}[f_\theta(x^{(i)}) - y^{(i)}]x_2^{(i)}$$

……

Feature scaling

Size (feet²) x_1	Number of bedrooms x_2	Number of floors x_3	Age of home (years) x_4	Price ($1000) y
2104	5	1	45	460
1416	3	2	40	232
1534	3	2	30	315
852	2	1	36	178
…	…	…	…	…

Get every feature into approximately a $-1 \leq x_i \leq 1$ range.

Normalization:

$$x_i = \frac{x_i - \mu_i}{S_i}$$

Standard deviation:

$$S_i = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_i)^2}$$

For example:

$$x_1 = \frac{size - 1000}{2000}$$

$$x_2 = \frac{\#bedrooms - 2}{4}$$

...

Multivariate linear regression: Vectorization

x_0	Size (feet²) x_1	Number of bedrooms x_2	Number of floors x_3	Age of home (years) x_4	Price ($1000) y
1	2104	5	1	45	460
1	1416	3	2	40	232
1	1534	3	2	30	315
1	852	2	1	36	178
…	…	…	…	…	…

$$X = \begin{bmatrix} 1 & 2104 & 5 & 1 & 45 \\ 1 & 1416 & 3 & 2 & 40 \\ 1 & 1534 & 3 & 2 & 30 \\ 1 & 852 & 2 & 1 & 36 \end{bmatrix} \qquad y = \begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}$$

$$\theta = (X^T X)^{-1} X^T y$$

Normal equation

$(X^T X)^{-1}$ is inverse of matrix $X^T X$.

Polynomial regression

Linear regression

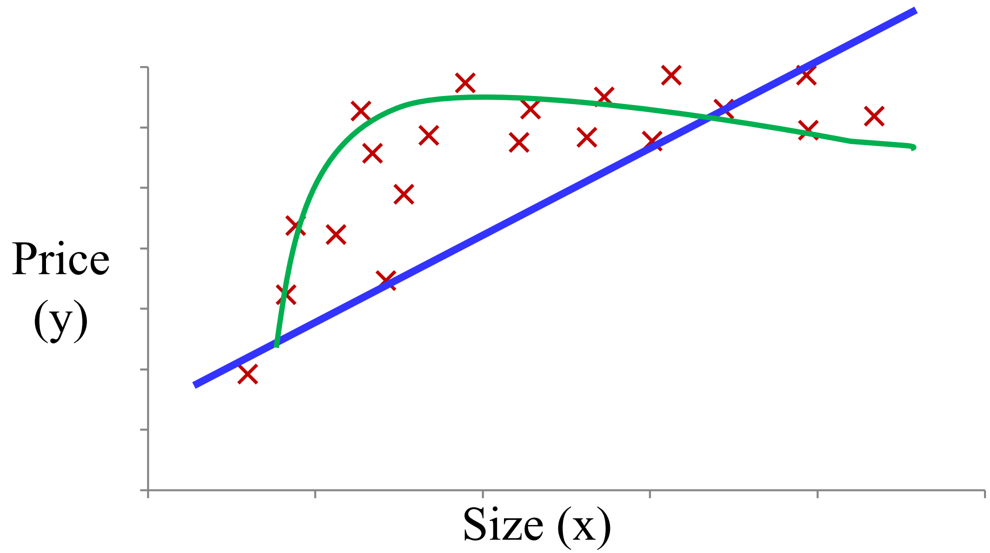
- House price prediction

	Size (feet²)	Number of bedrooms	Number of floors	Age of home (years)	Price ($1000)
x_0	x_1	x_2	x_3	x_4	y
1	2104	5	1	45	460
1	1416	3	2	40	232
1	1534	3	2	30	315
1	852	2	1	36	178
…	…	…	…	…	…

$$f_\theta(x) = \theta^T x = w_0 x_0 + w_1 size + w_2 \#bed + w_3 \#floor + w_4 \#year$$

Defining a new feature to get a better model. How?

Polynomial regression



Why polynomial regression?

If the relationship between the data is non-linear. Then Linear regression will not capable to draw a best-fit line.

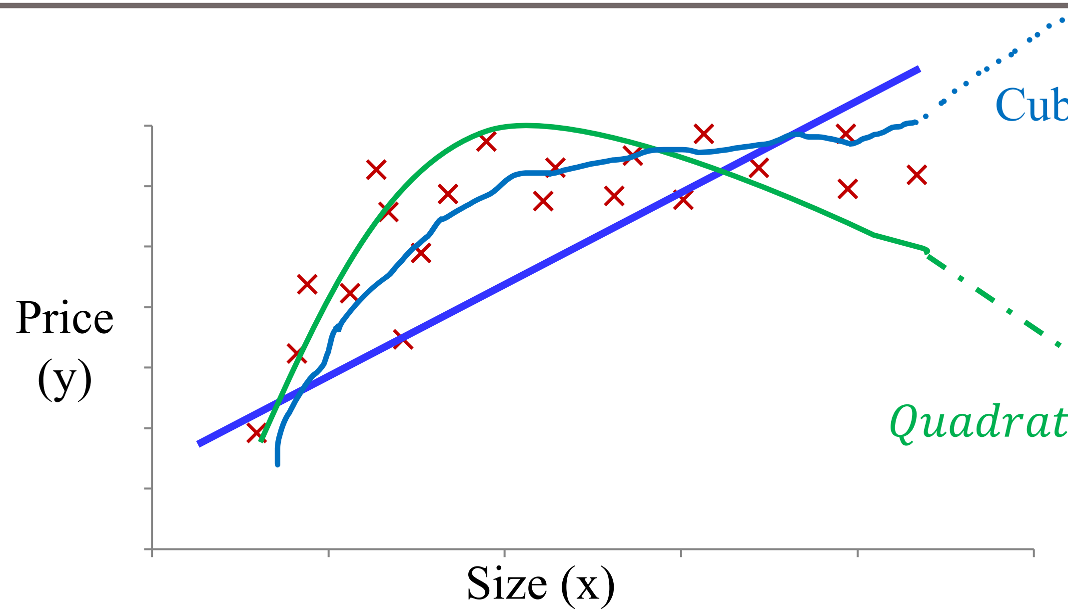
Linear hypothesis: $f_\theta(x) = \theta^T x = w_0 x + w_1 x + w_2 x$

add some polynomial terms to linear regression

non-linear relationship between dependent and independent variables

Non-linear hypothesis: $f_\theta(x) = w_0 + w_1 x + \; w_2 x^2 + \cdots + w_n x^n$

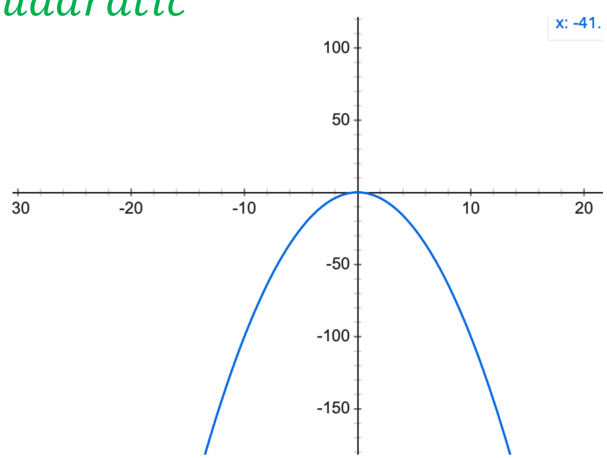
Polynomial regression



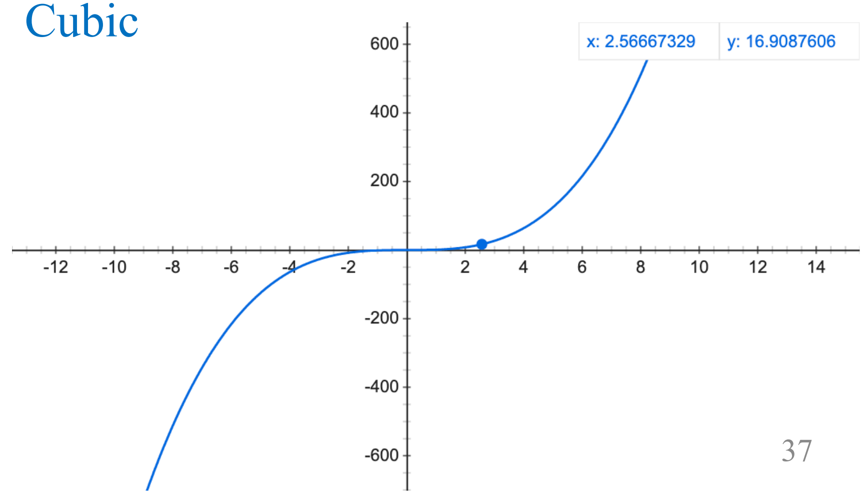
Cubic: $f_\theta(x) = w_0 + w_1 x + \ w_2 x^2 + \ w_2 x^3$

Quadratic: $f_\theta(x) = w_0 + w_1 x + \ w_2 x^2$

Quadratic



Cubic



Polynomial regression

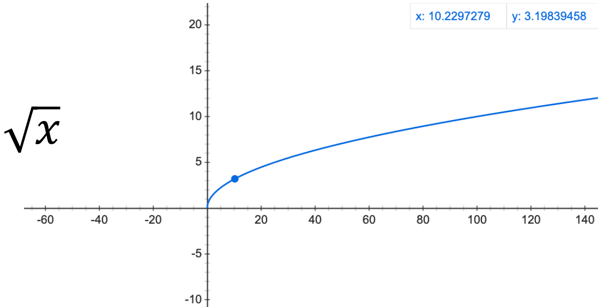
Hypothesis: $f_\theta(x) = w_0x_0 + w_1x_1 + w_2x_2^2 + w_3x_3^3$

$= w_0 + w_1\,(\text{size}) + w_2(size)^2 + w_3(size)^3$

The new hypothesis leads to a substantial increase in the value of each new feature.
Feature scaling becomes important.

Reasonable function:

$f_\theta(x) = w_0 + w_1\text{size} + w_2\sqrt{size}$ $\qquad f(x) = \sqrt{x}$



Logistic Regression

Binary Classification

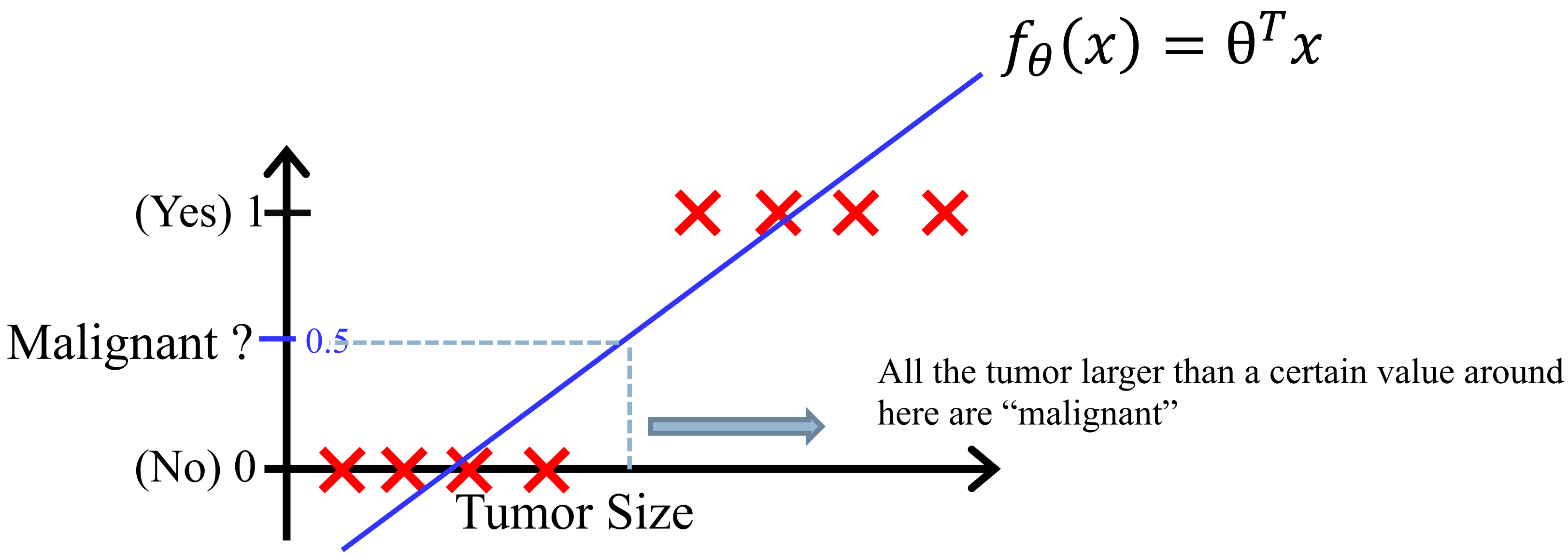
Email: Spam / Not Spam?
Watermelon: Good / not?
Tumor: Malignant / Benign ?

e.g. "tumor"

$$x \longrightarrow \boxed{\text{Model}} \longrightarrow y \in \{0, 1\}$$

0: "Negative Class" (e.g., benign tumor)
1: "Positive Class" (e.g., malignant tumor)

Classification with linear regression?



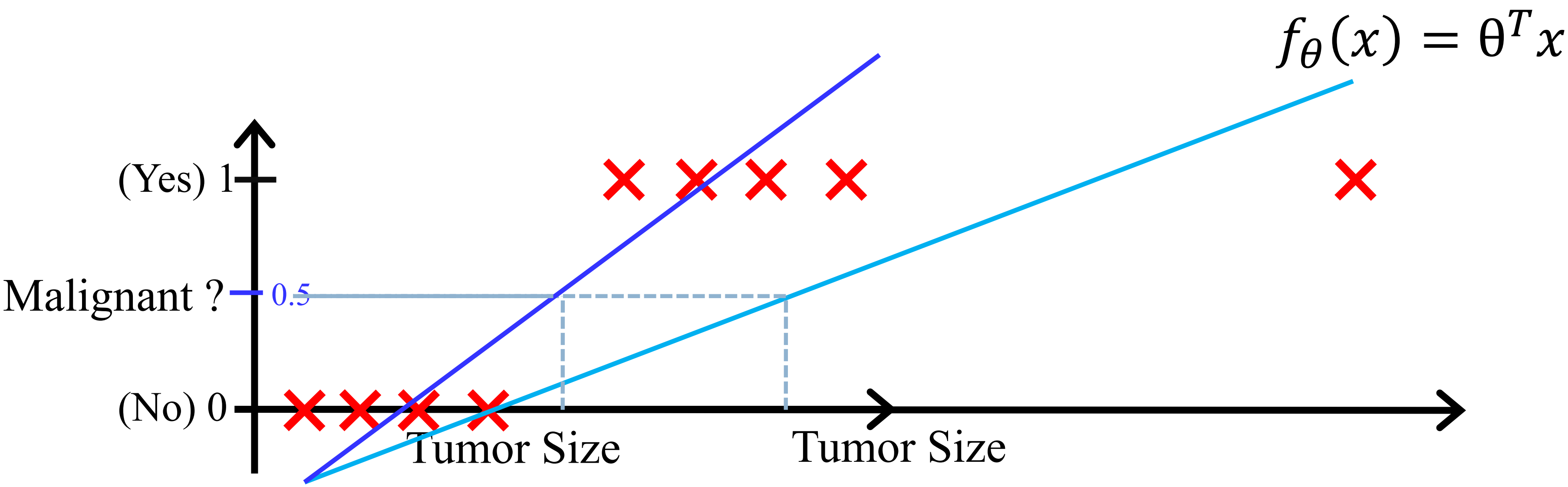
Hypothesis with linear regression : $f_\theta(x) = \theta^T x$

Threshold = 0.5

If $f_\theta(x) \geq 0.5$, output "Yes"

If $f_\theta(x) \leq 0.5$, output "No"

Classification with linear regression?



Hypothesis with linear regression : $f_\theta(x) = \theta^T x$

Threshold = 0.5

If $f_\theta(x) \geq 0.5$, output “Yes”

If $f_\theta(x) \leq 0.5$, output “No”

Add extra samples, linear regression may give you a wrose hypothesis.
Applying linear regression to a classification **is not a good idea**.

Logistic regression model

Linear regression model: $f_\theta(x) = \theta^T x$

Linear regression can be: $f_\theta(x) > 1 \; or. \; f_\theta(x) < 0$

Want: $0 \leq f_\theta(x) \leq 1$

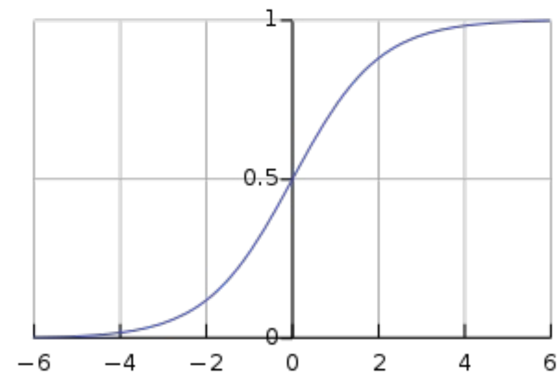
$$f_\theta(x) = g(\theta^T x)$$

$$f_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

where,

$$g(z) = \frac{1}{1 + e^{-z}}$$

Sigmoid/Logistic function



m examples

Training set: $\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \cdots, (x^{(m)}, y^{(m)})\}$

$$x \in \begin{bmatrix} x_0 \\ x_1 \\ \cdots \\ x_n \end{bmatrix} \qquad x_0 = 1, y \in \{0, 1\}$$

Hypothesis: $f_\theta(x) = \dfrac{1}{1 + e^{-\theta^T x}}$ where, $\theta^T x = w_0 x_0 + w_1 x_1 + \cdots + w_n x_n$

What cost function we choose?

How to choose parameters θ ?

Cost function

- Chose cost function same with linear regression

Cost Function: $E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(f_\theta\left(x^{(i)}\right) - y^{(i)})^2$

But the hypothesis looks like: $f_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$

Problem: it is non-convex function, there are many local minums. Gradient descent algorithm not guranteed to find a good local minima.

Want: the cost function should be convex.

Hypothesis: $f_\theta(x) = \dfrac{1}{1 + e^{-\theta^T x}}$ where， $\theta^T x = w_0 x_0 + w_1 x_1 + \cdots + w_n x_n$

Parameters: $\theta = \{w_0, \dots, w_n\}$

$y \in \{0, 1\}$ Real-value from data set

Cost function: $E(\theta) = -\frac{1}{m}[\sum_{i=1}^{m} y^{(i)} \log f_\theta\left(x^{(i)}\right) + \left(1 - y^{(i)}\right)\log(1 - f_\theta\left(x^{(i)}\right)]$

if $y^{(i)} = 0$:

$$E(\theta) = -\frac{1}{m}[\sum_{i=1}^{m}\left(1 - y^{(i)}\right)\log(1 - f_\theta\left(x^{(i)}\right)]$$

if $y^{(i)} = 1$:

$$E(\theta) = -\frac{1}{m}[\sum_{i=1}^{m} y^{(i)} \log f_\theta\left(x^{(i)}\right)]$$

Goal: minimize $E(\theta)$

Logistic regression: Gradient Descent

Cost function: $E(\theta) = -\frac{1}{m}[\sum_{i=1}^{m} y^{(i)} \log f_\theta(x^{(i)}) + (1 - y^{(i)})\log(1 - f_\theta(x^{(i)})]$

Parameters: $\theta = \{w_0, \ldots, w_n\}$

Goal: minimize $E(\theta)$

Repeat {

$$\theta_j := \theta_j - \alpha \boxed{\frac{\partial}{\partial \theta_j} E(\theta)}$$

(simultaneously update all θ)

Learning rate

Gradient

}

Logistic regression: Gradient Descent

Hypothesis: $f_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$ where，$\theta^T x = w_0 x_0 + w_1 x_1 + \cdots + w_n x_n$

Cost function: $E(\theta) = -\frac{1}{m}[\sum_{i=1}^{m} y^{(i)} \log f_\theta(x^{(i)}) + (1 - y^{(i)})\log(1 - f_\theta(x^{(i)})]$

Parameters: $\theta = \{w_0, \ldots, w_n\}$

Goal: minimize $E(\theta)$

Repeat {

$$\theta_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^{m} [f_\theta(x^{(i)}) - y^{(i)}]\, x^{(i)}$$

}

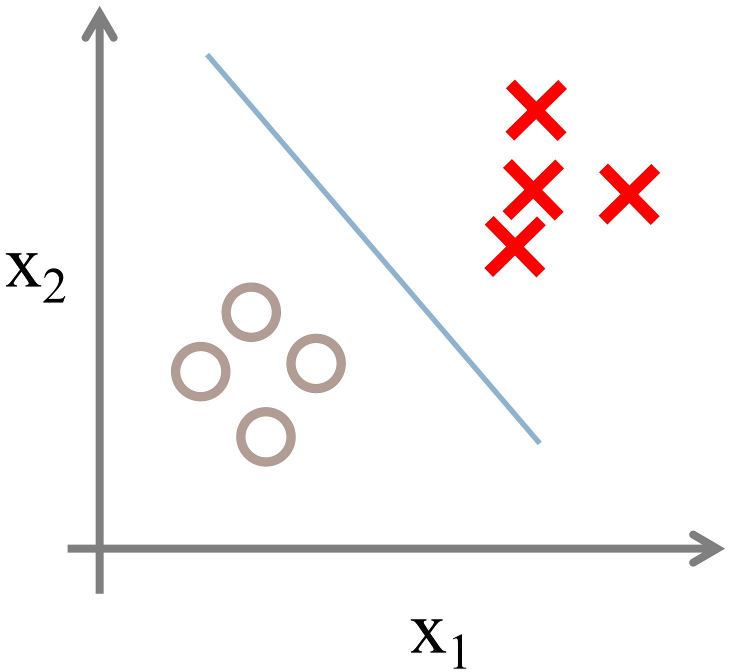
$$f_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$$

(simultaneously update all θ)

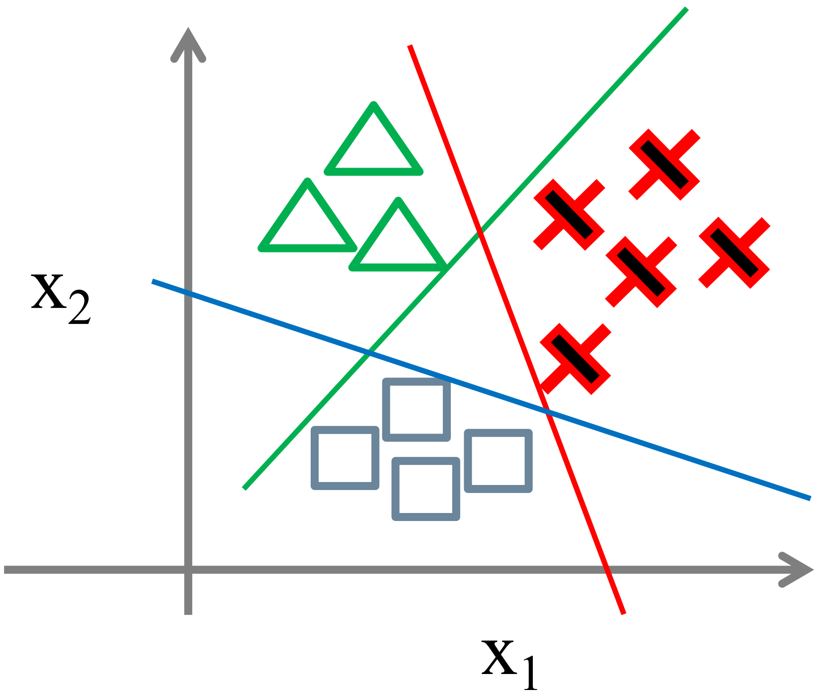
After taking the derivatives, the parameter's update functions are similar with linear regression.

Multiclass classification

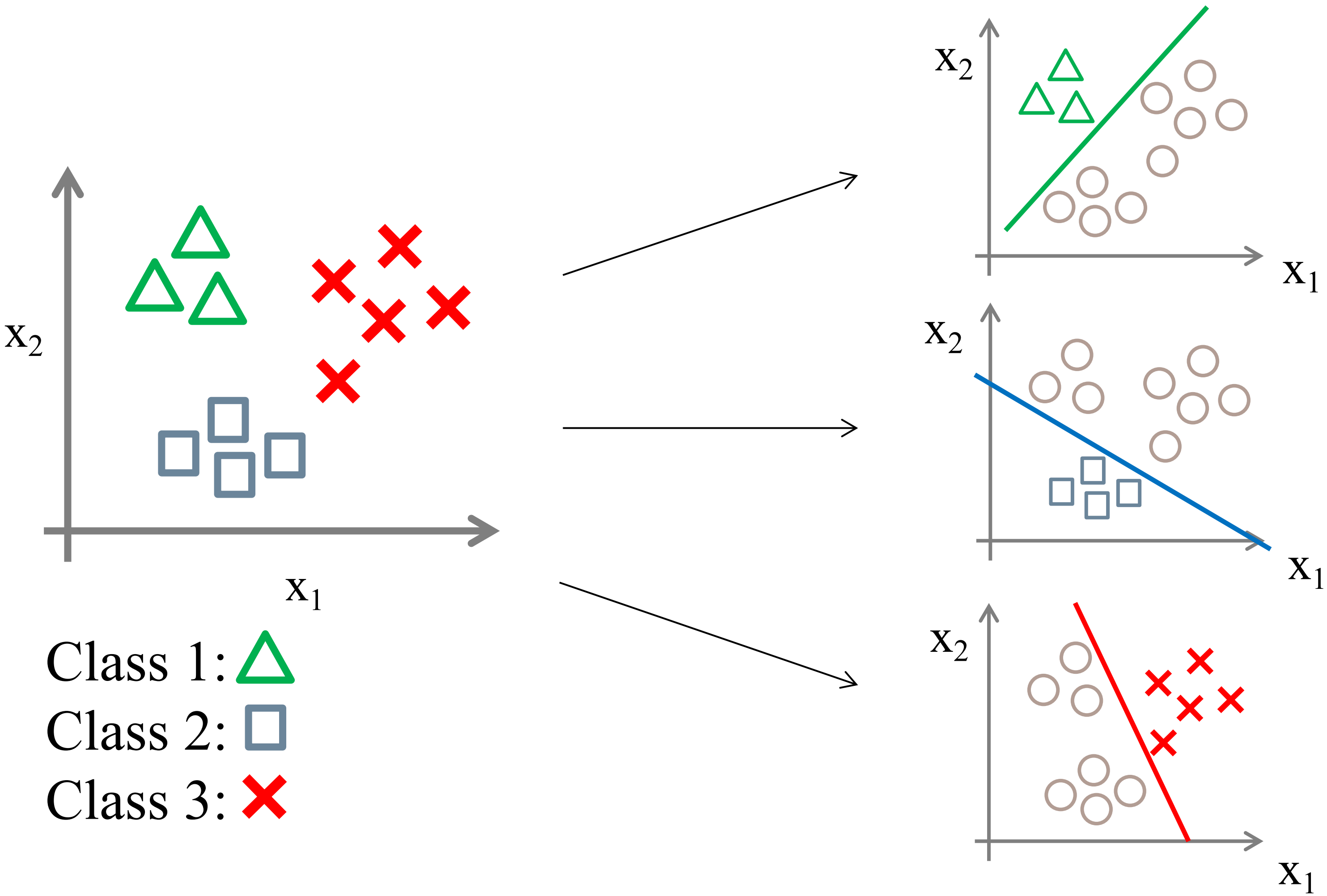
Binary classification:



Multi-class classification:

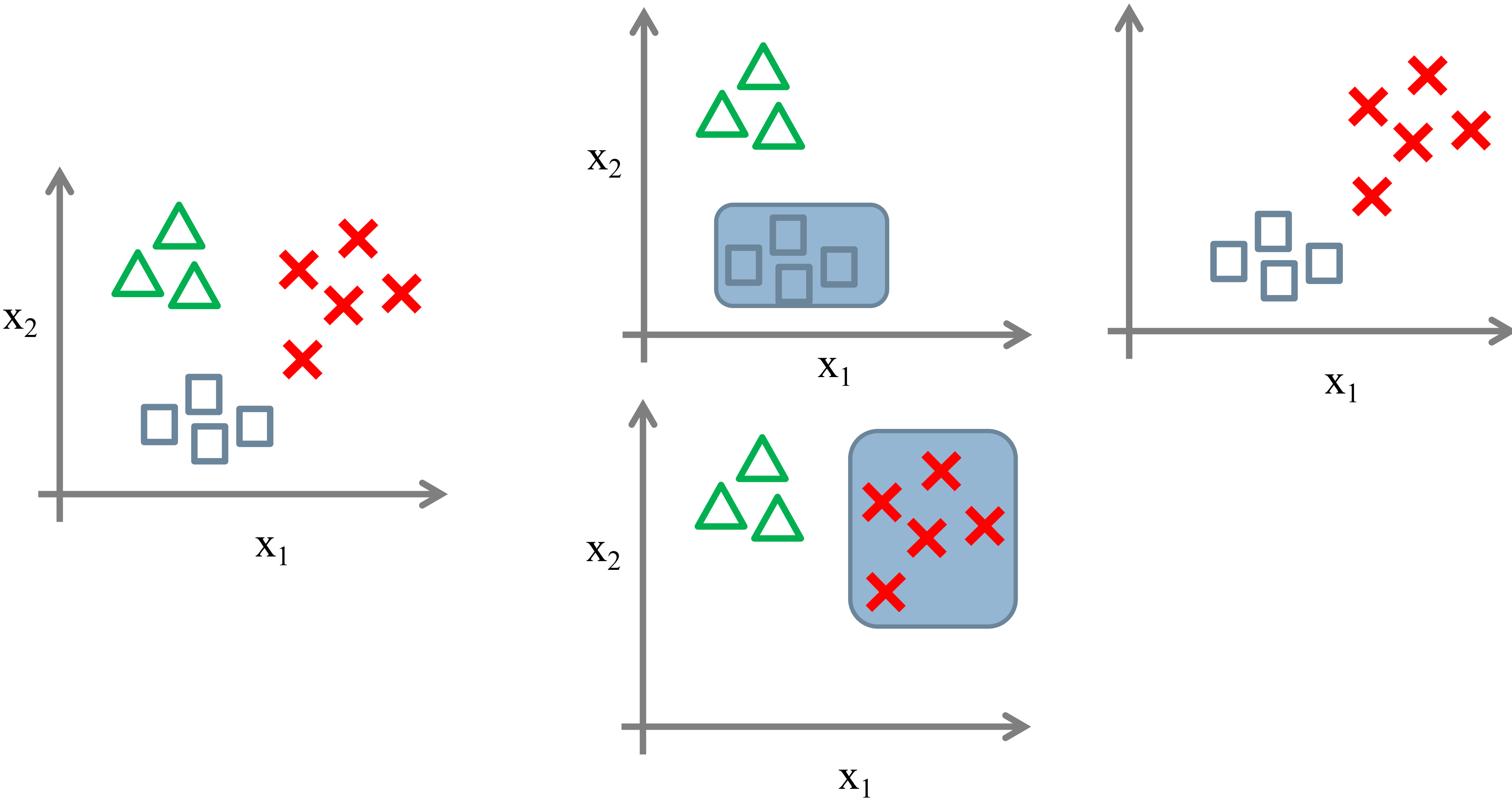


One-vs-rest



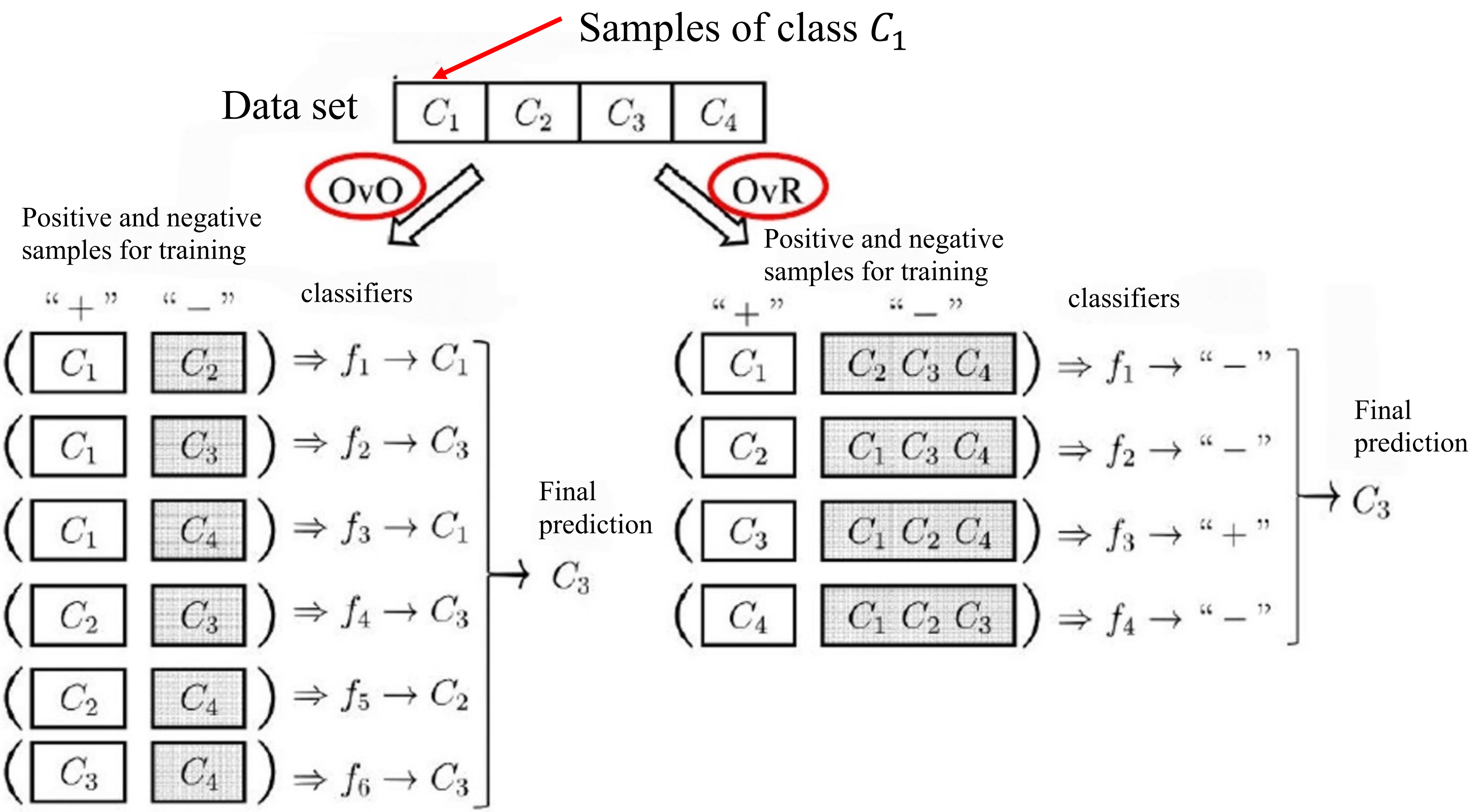
N classifier need to be trained

One-vs-one



N(N-1)/2 classifier need to be trained

OvR and OvO



- Need to train N(N-1)/2 classifiers;
- Requires large storage and test time;
- Training only uses examples of two classes, and the training time is short;

- Need to train N classifiers;
- Requires small storage and less test time;
- All training samples are used for training, and the training time is long;

Many to Many, MvM

- MvM: Treat some classes as positive examples and others as negative examples.

- Error Correcting Output Code (ECOC)

Encoding: divide N classes into two parts: positive and negative; repeat M times.

obtain M binary classifiers

Measure the distance;
Chose the closest one

Decoding: use M #classfiers to make predictions for a given test data.

Obtain a vector (length is M)

Distance calculations

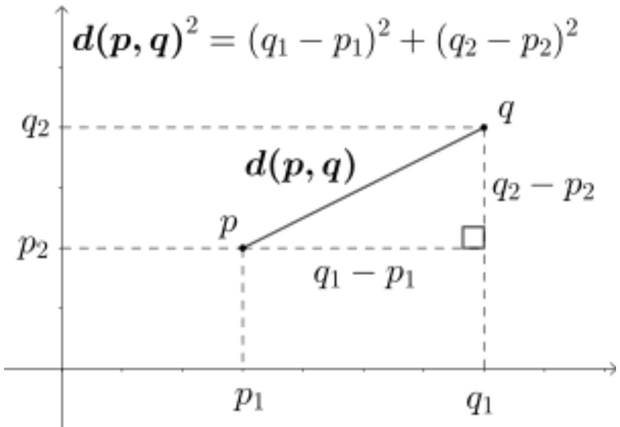
Hamming distance between two equal-length strings of symbols is the number of positions at which the corresponding symbols are different. In other words, it measures the minimum number of substitutions required to change one string into the other.

e.g.:

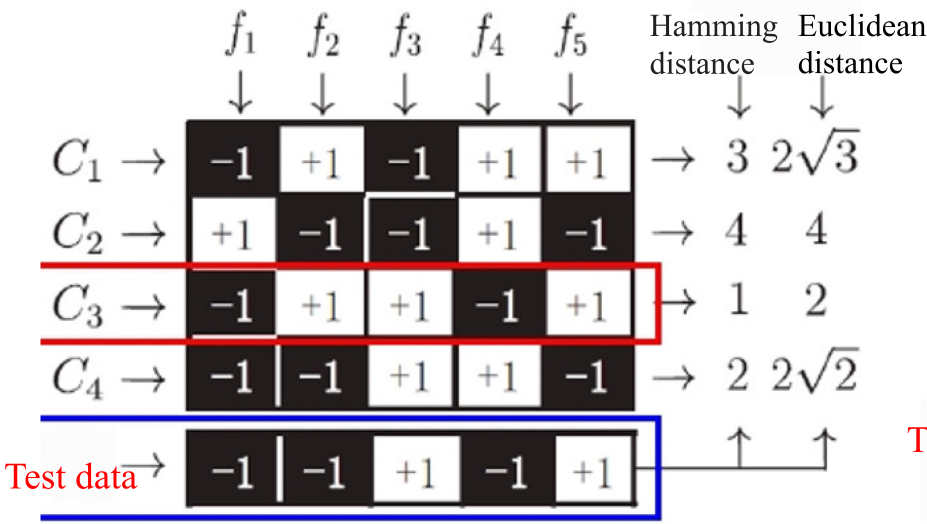
0100→1001 has distance 3;
0110→1110 has distance 1;

String similarity

Euclidean distance between two points in Euclidean space is the length of a line segment between the two points.

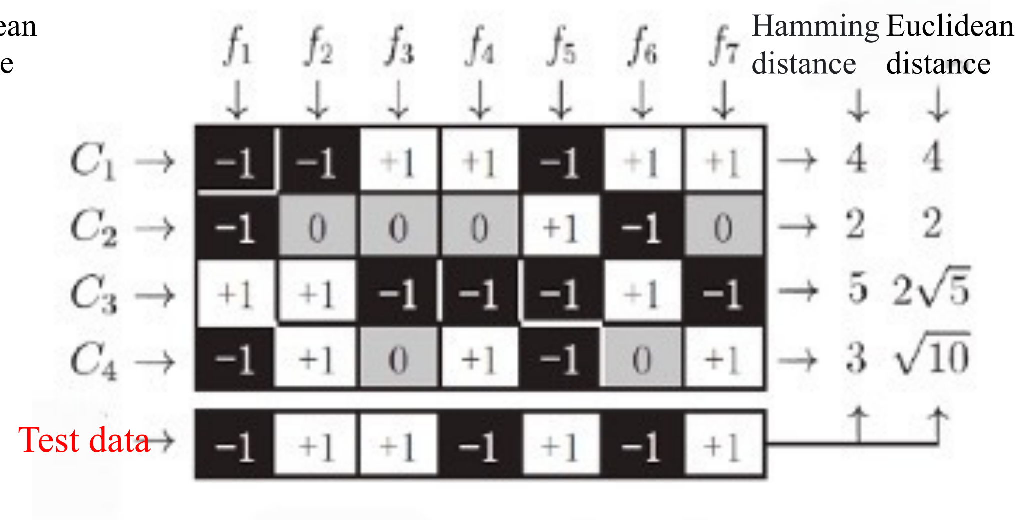


Error Correcting Output Code (ECOC)



Bigram ECOC

[Dietterich and Bakiri,1995]



Trigram ECOC

[Allwein et al. 2000]

ECOC codes have a tolerance and correction capacity for classifier's errors, with a longer code making it easier to correct errors and a stronger correction ability.

Class-Imbalance

Class-Imbalance

- A classification data set with skewed class proportions is called imbalanced.

Some classes may occupy a small portion of the data, but they are still important.

Threshold = 0.5
If $f_\theta(x) \geq 0.5$, output "positive (+)"
If $f_\theta(x) \leq 0.5$, output "negative (-)"

$f_\theta(x)$ - the probability of x is a "+" sample

$1 - f_\theta(x)$ - the probability of x is a "-" sample

If $\frac{f_\theta(x)}{1-f_\theta(x)} > 1$, indicates positive samples

Ratio between "+" and "-" samples

Class-Imbalance

- m^+ and m^- is the number of positive and negative samples in the dataset.

$Observation\ probability$: $\dfrac{m^+}{m^-}$

Generally, we consider the training set is obtained from the original dataset with unbiased sampling (Assumption).

Model's $probability$: $\dfrac{f_\theta(x)}{1 - f_\theta(x)}$

Then, $Observation\ probability$ can be treat as the true ratios.

However, accurately estimating m-/m+ is often challenging!

If $\dfrac{f_\theta(x)}{1-f_\theta(x)} > \dfrac{m^+}{m^-}$, it indicates the positive samples

$\boxed{\text{Rescalling/Rebalancing}}$

Class-Imbalance

- three ways:
  - Undersampling
    - Deleting some positive/negative samples from training set to ensure their number are approximately equal.
    - e.g.: SMOTE [Chawla et al., 2002]
  - Oversampling
    - Increasing the number of positive/negative samples from training set with sampling process.
    - e.g.: EasyEnsemble [Liu et al., 2009]
  - Threshold moving

- Thank you!