# Decision Tree

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# Entropy and Decision Tree

# Types of classifier

- We can divide the large variety of classification approaches into roughly two main types
  - Discriminative
    - directly estimate a decision rule/boundary e.g., decision tree
  - Instance based classifiers
    - Use observation directly (no models) e.g. K nearest neighbors
  - Generative
    - build a generative statistical model e.g., Bayesian networks

### **Decision Trees**

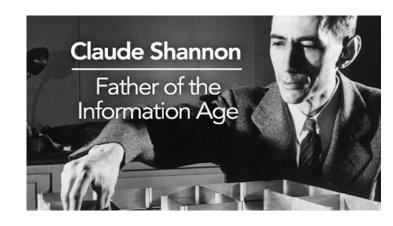
- One of the most intuitive classifiers
- Easy to understand and construct
- Surprisingly, also works very well

# Entropy

# Entropy

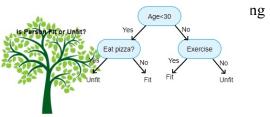
- Quantifies the amount of uncertainty associated with a specific probability distribution.
- The higher the entropy, the less confident we are in the outcome.
- Definition

$$H(X) = \sum_{c} -p(X=c)\log_2 p(X=c)$$

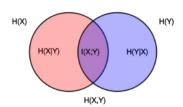


Claude Shannon (1916 – 2001), most of the work was done in Bell labs

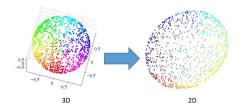
Entropy can be used to build classification tree that used to classify things.



Entropy is also the basis of mutual information that quantifies the relationship between two things. (e.g. in NLP, measure the relationship between two words)

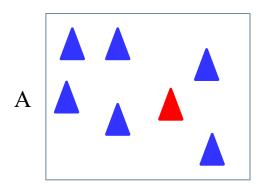


Entropy is basis of relative entropy (Kullback–Leibler divergence, KL) and cross entropy. (e.g. used in dimension reduction algorithms)



These three things all uses **entropy** to quantify **similarities** and differences.

#### Surprise -> Entropy

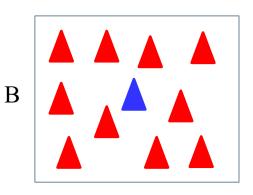


Randomly pick up a triangle.

There are 5 blue triangles, and 1 red triangle, then there is a higher probability that blue one will be picked up.

Since there is higher probability to pick up blue, it would be **not** surprising if it happened.

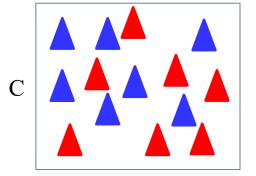
In contrast, if the red was picked up, it would be relatively surprised.



**Area B** has a lot of red triangle than blue.

High probability to pick up a red triangle, it is not surprised.

Low probability to pick up a blue triangle, it is relatively surprising.

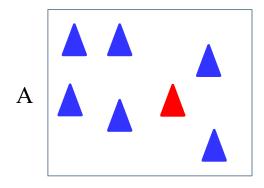


**Area** C has equal number of red and blue triangles.

Regardless what color triangle we picked up, we would be equally surprised.

In some way, surprise is inversely related to probability.

#### Surprise -> Entropy



When the probability of picking up a blue triangle is **high**, the surprise is low.

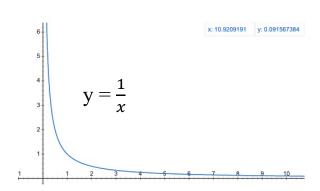
B A A A

When the probability of picking up a blue triangle is **low**, **the surprise** is **high**.

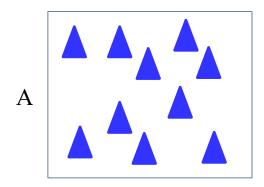
We may calculate the **surprise** use the inverse of the probability.

1

probability



There is a problem associated with inverse of the probability to calculate the surprise  $\approx \frac{1}{probability}$ 



For example, we have **Area A**, which only contains blue triangles.

The probability to picking up a blue triangle is 1.

Then we want the **surprise** to picking up a blue triangle to be 0.

However, when we calculate the inverse of the probability:

$$\frac{1}{probability} = \frac{1}{1} = 1$$
 We want 0

Instead of just using the inverse of the probability, we could add a log function:

$$Log(\frac{1}{probability})$$

Now, let's calculate the surprise with the inverse of the probability:

Picking up the blue: 
$$\log\left(\frac{1}{probability}\right) = \log\left(\frac{1}{1}\right) = \log(1) = 0$$

Picking up the blue: 
$$\log\left(\frac{1}{probability}\right) = \log\left(\frac{1}{1}\right) = \log(1) = 0$$
 Picking up the red:  $\log\left(\frac{1}{probability}\right) = \log\left(\frac{1}{0}\right) = \log(0) = undefined$ 

# Now, surprise $\approx \log(\frac{1}{nrohahility})$

For example, we have **Area A** 

The probability to picking up a blue triangle is 0.9.

The probability to picking up a red triangle is 0.1.

**Surprise** for picking up a blue triangle: 
$$log_2\left(\frac{1}{probability}\right) = log_2\left(\frac{1}{0.9}\right) = log_2(1) - log_2(0.9) = 0.15$$

Surprise for picking up a red triangle: 
$$log_2\left(\frac{1}{probability}\right) = log_2\left(\frac{1}{0.1}\right) = log_2(1) - log_2(0.1) = 3.32$$

For example, we want to calculate surprise for three times picking events:

after three times we got:









→ The probability of this sequence happens is 0.9\*0.9\*0.1

Calculate the total surprise:

$$total \ surprise = log_2\left(\frac{1}{0.9*0.9*0.1}\right) = log_2(1) - log_2(0.9*0.9*0.1)$$

$$= log_2(1) - [log_2(0.9) + log_2(0.9) + log_2(0.1)]$$

$$= 0 - log_2(0.9) - log_2(0.9) - log_2(0.1)$$

$$= 0.15 + 0.15 + 3.32$$

sum of the surprise for each individual event

Now, imagine we do 100 times of pickings. What will be the total surprise for these 100 times?

Probability $P(x)$	0.9	0.1
Surprise	0.15	3.32
$log_2\left(\frac{1}{P(x)}\right)^{-1}$		

Entropy of the triangle - the expected **surprise** every time we pick up the triangle.

average surprise for per time : 
$$\frac{(0.9*100)*0.15 + (0.1*100)*3.32}{100} = 0.467$$

average surprise for per time : 
$$(0.9 * 0.15) + (0.1 * 3.32) = 0.467$$

$$\sum_{x} x P(X = x)$$

Plug the surprise and probability function in

Entropy equation: 
$$\sum log_2\left(\frac{1}{P(x)}\right) P(x)$$

$$\sum log_2\left(\frac{1}{P(x)}\right)P(x)$$

Surprise value

the probability of surprise

#### Standard entropy equation:

$$Entropy = \sum log_2 \left(\frac{1}{P(x)}\right) P(x)$$

$$= \sum P(x) \left[log_2(1) - log_2(P(x))\right]$$

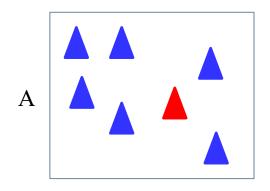
$$= \sum P(x) \left[0 - log_2(P(x))\right]$$

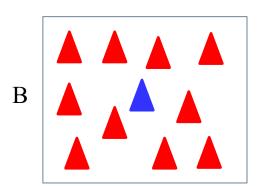
$$= -\sum P(x) log_2(P(x))$$

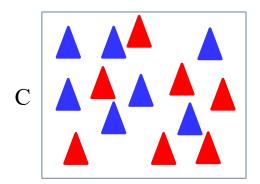
This is the original version, but not easy to understand.

### Calculate the Entropy

$$Entropy = \sum log_2\left(\frac{1}{P(x)}\right)P(x)$$







$$E_A = log_2 \left(\frac{1}{1/7}\right) \frac{1}{7} + log_2 \left(\frac{1}{6/7}\right) \frac{6}{7} = 0.59$$

As result, we can use entropy to **quantify the** similarities and differencs in the number of blue and red triangels in each area.

$$E_B = log_2 \left(\frac{1}{1/11}\right) \frac{1}{11} + log_2 \left(\frac{1}{10/11}\right) \frac{10}{11} = 0.44$$

Entropy is **high**, when we have **the same number** of both colors of triangle.

Entropy is **low**, when we **increase** the difference in the number of blue and red triangles.

$$E_c = log_2 \left(\frac{1}{7/14}\right) \frac{1}{14} + log_2 \left(\frac{1}{7/14}\right) \frac{7}{14} = 1$$

### Conditional entropy

$$Entropy = \sum log_2\left(\frac{1}{P(x)}\right)P(x)$$

Movie length	Liked?
Short	Yes
Short	No
Medium	Yes
long	No
Long	No
Medium	Yes
Short	Yes
Long	Yes
Medium	Yes

Entropy measures the uncertainty in a specific distribution.

What is the entropy of "Liked"?

$$E(Liked) = \frac{6}{9}log_2\left(\frac{1}{\frac{6}{9}}\right) + \frac{3}{9}log_2\left(\frac{1}{\frac{3}{9}}\right) = 0.92$$

What is the entropy of "Liked" among movies with a certain value of length?

This becomes a conditional entropy problem:

$$E(Liked \mid Length = "Short") = ?$$

$$E(Liked \mid Length = "Short") = \frac{2}{3}log_2(\frac{1}{\frac{2}{3}}) + \frac{1}{3}log_2(\frac{1}{\frac{1}{3}}) = 0.92$$

#### Conditional entropy: Examples for specific values

Movie length	Liked?
Short	Yes
Short	No
Medium	Yes
long	No
Long	No
Medium	Yes
Short	Yes
Long	Yes
Medium	Yes

$$E(Liked \mid len = "S") = \frac{2}{3}log_2\left(\frac{1}{\frac{2}{3}}\right) + \frac{1}{3}log_2\left(\frac{1}{\frac{1}{3}}\right) = 0.92$$

$$E(Liked \mid len = "M") = 0$$

$$H(Y | X) = \sum_{i} P(X = i) H(Y | X = i)$$

$$E(Liked | len = "L") = 0.92$$

Conditional entropy equation

Lets compute

$$E(Y|X) = \sum P(X = i)E(Y|X = i)$$

$$E(Liked | len) = P(len = "S")E(Liked | len = "S")$$

$$+ P(len = "M")E(Liked | len = "M")$$

$$+ P(len = "L")E(Liked | len = "L")$$

$$= \frac{3}{9} * 0.92 + \frac{3}{9} * 0 + \frac{3}{9} * 0.92 = 0.62$$

#### Information Gain

- How much do we gain (in terms of reduction in entropy) from knowing one
  of the attributes.
- In other words, what is the reduction in entropy from this knowledge.
- Definition:

$$IG(Y|X) = E(Y) - E(Y|X)$$

For example,

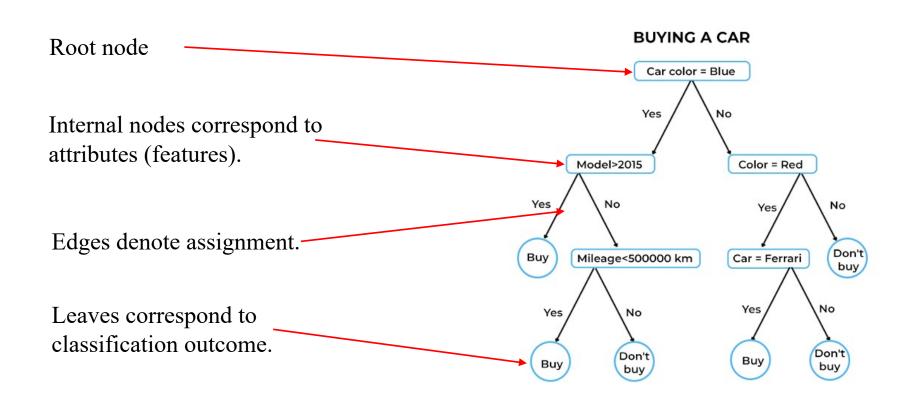
$$E(Liked) = 0.92$$

$$E(Liked \mid len) = 0.62$$

$$IG(Liked|len) = E(Liked) - E(Liked|len)$$
  
= 0.92-0.62 = 0.3

Movie length	Liked?
Short	Yes
Short	No
Medium	Yes
long	No
Long	No
Medium	Yes
Short	Yes
Long	Yes
Medium	Yes

#### Structure of a decision tree



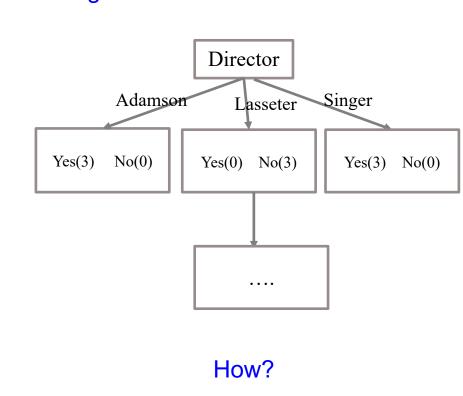
#### Netflix



#### Data set

#### Building a decision tree from the dataset!

	Attributes (features)							
Movie	Туре	Length	Director	Famous actors	Liked?			
m1	Comedy	Short	Adamson	No	Yes			
m2	Animated	Short	Lasseter	No	No			
m3	Drama	Medium	Adamson	No	Yes			
m4	animated	long	Lasseter	Yes	No			
m5	Comedy	Long	Lasseter	Yes	No			
m6	Drama	Medium	Singer	Yes	Yes			
m7	animated	Short	Singer	No	Yes			
m8	Comedy	Long	Adamson	Yes	Yes			
m9	Drama	Medium	Lasseter	No	Yes			

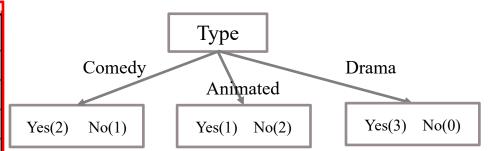


- defined the entropy, conditional entropy and information gain.
- looking for a good criteria for selecting the best attribute for a node split.
- use Gini impurity, entropy and information gain as our criteria for a good split.

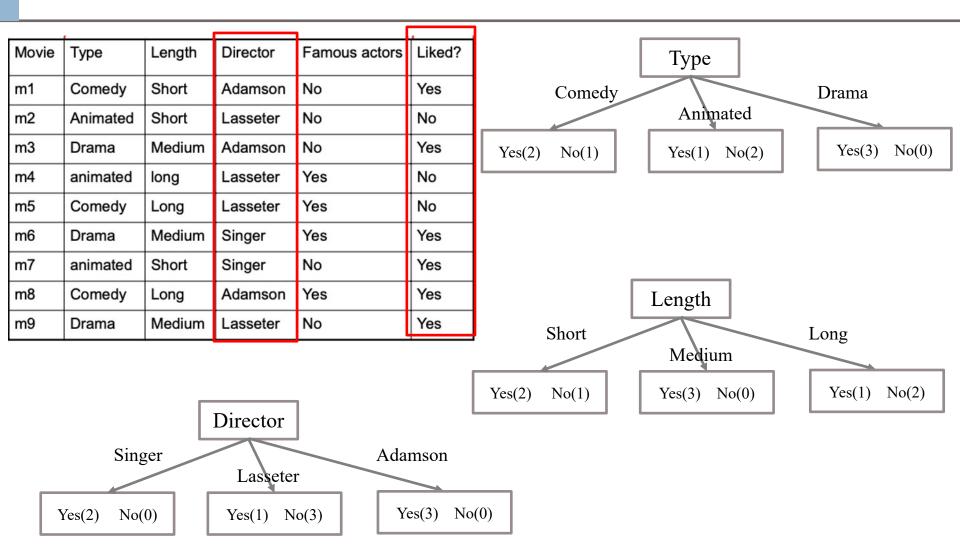
#### There are three ways to build a decision tree

- Gini impurity (minimizes the Gini impurity )
- Entropy (minimizes the entropy)
- Information gain (maximizes the information gain at each node)

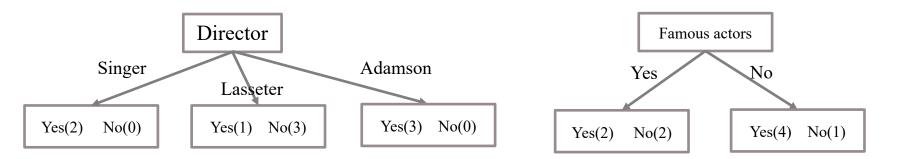
Movie	Туре	Length	Director	Famous actors	Liked?
m1	Comedy	Short	Adamson	No	Yes
m2	Animated	Short	Lasseter	No	No
m3	Drama	Medium	Adamson	No	Yes
m4	animated	long	Lasseter	Yes	No
m5	Comedy	Long	Lasseter	Yes	No
m6	Drama	Medium	Singer	Yes	Yes
m7	animated	Short	Singer	No	Yes
m8	Comedy	Long	Adamson	Yes	Yes
m9	Drama	Medium	Lasseter	No	Yes

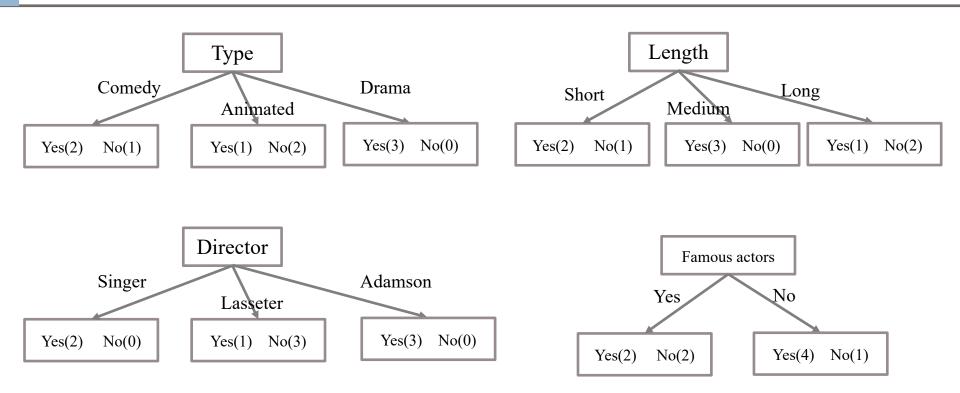


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Movie	Туре	Length	Director	Famous actors	Liked?	Туре
m1	Comedy	Short	Adamson	No	Yes	Comedy Drama
m2	Animated	Short	Lasseter	No	No	Animated
m3	Drama	Medium	Adamson	No	Yes	Yes(2) No(1) Yes(1) No(2) Yes(3) No(0)
m4	animated	long	Lasseter	Yes	No	
m5	Comedy	Long	Lasseter	Yes	No	
m6	Drama	Medium	Singer	Yes	Yes	
m7	animated	Short	Singer	No	Yes	
m8	Comedy	Long	Adamson	Yes	Yes	Length
m9	Drama	Medium	Lasseter	No	Yes	Short Long
						Medium
						Yes(2) No(1) Yes(3) No(0) Yes(1) No(2)



Movie	Туре	Length	Director	Famous actors	Liked?	Туре
m1	Comedy	Short	Adamson	No	Yes	Comedy Drama
m2	Animated	Short	Lasseter	No	No	Animated
m3	Drama	Medium	Adamson	No	Yes	Yes(2) No(1) Yes(1) No(2) Yes(3) No(0)
m4	animated	long	Lasseter	Yes	No	
m5	Comedy	Long	Lasseter	Yes	No	Length
m6	Drama	Medium	Singer	Yes	Yes	Short Long
m7	animated	Short	Singer	No	Yes	Medium
m8	Comedy	Long	Adamson	Yes	Yes	Yes(2) No(1) Yes(3) No(0) Yes(1) No(2)
m9	Drama	Medium	Lasseter	No	Yes	163(2) 110(1)

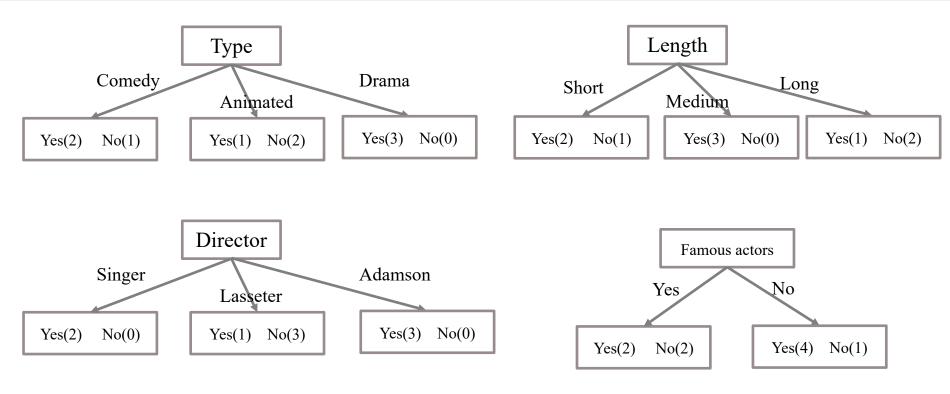




Looking at these four little trees, we see **neither one** does a pefect job predicting the movies will be liked or will not be liked.

Specifically, these leaves contain mixtures of movies that are liked and not liked.

They are called **impure leaves**.



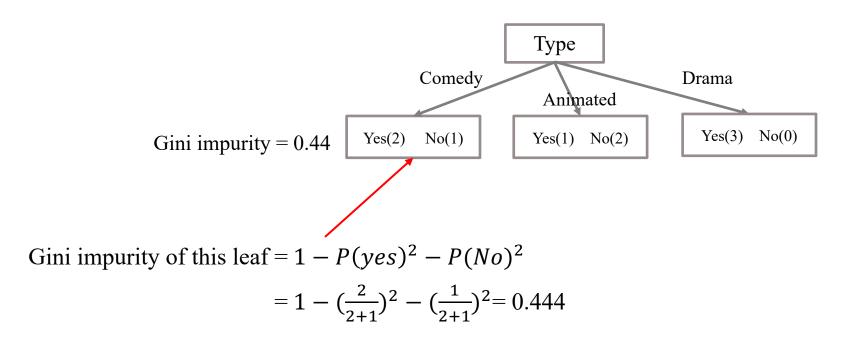
Both leaves of Famou actors tree are **impure**.

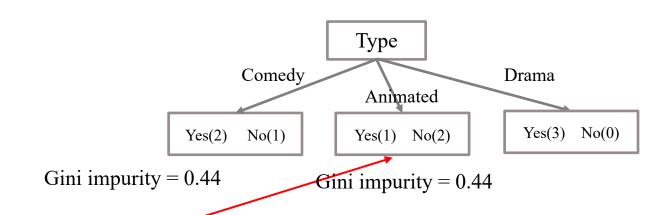
Type and Length trees contain two impure leaves in each of them.

Only one leaf in Director tree is **impure**.

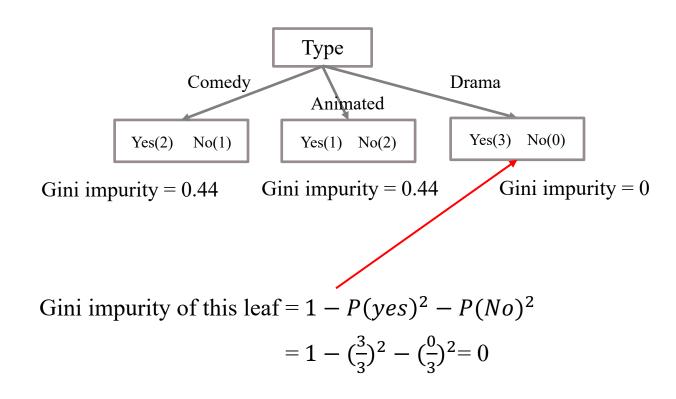
It seems that **Director does a better job** at predicting if the movies will be liked or will not be liked.

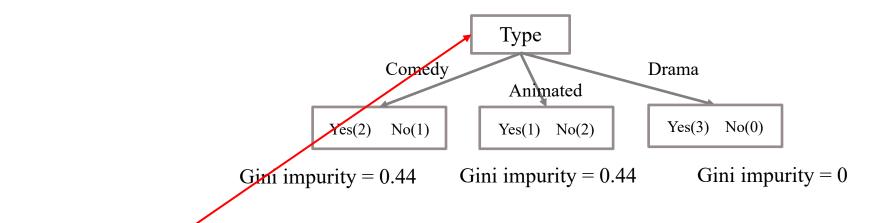
How to quantifies the differences?





Gini impurity of this leaf = 
$$1 - P(yes)^2 - P(No)^2$$
  
=  $1 - (\frac{1}{1+2})^2 - (\frac{2}{1+2})^2 = 0.444$ 



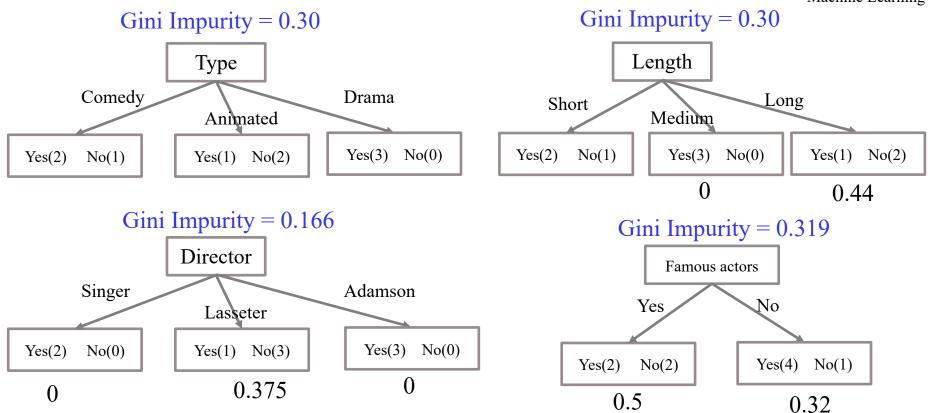


Total Gini impurity = weighted average of Gini impurity for the leaves

The total number of movies the first leaf. 
$$= \frac{3}{9} * 0.44 + \frac{3}{9} * 0.44 + \frac{3}{9} * 0 \approx 0.30$$

The total number of movies on the all leaves

So, the Gini impurity for Type features is 0.30



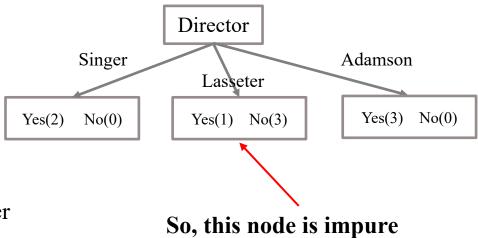
Likewise, Gini impurity for all of the features' tree are calculated.

Compare Gini impurity values for Type, Length, Director and Famous actor.

To decide which feature should be at very top of the tree.

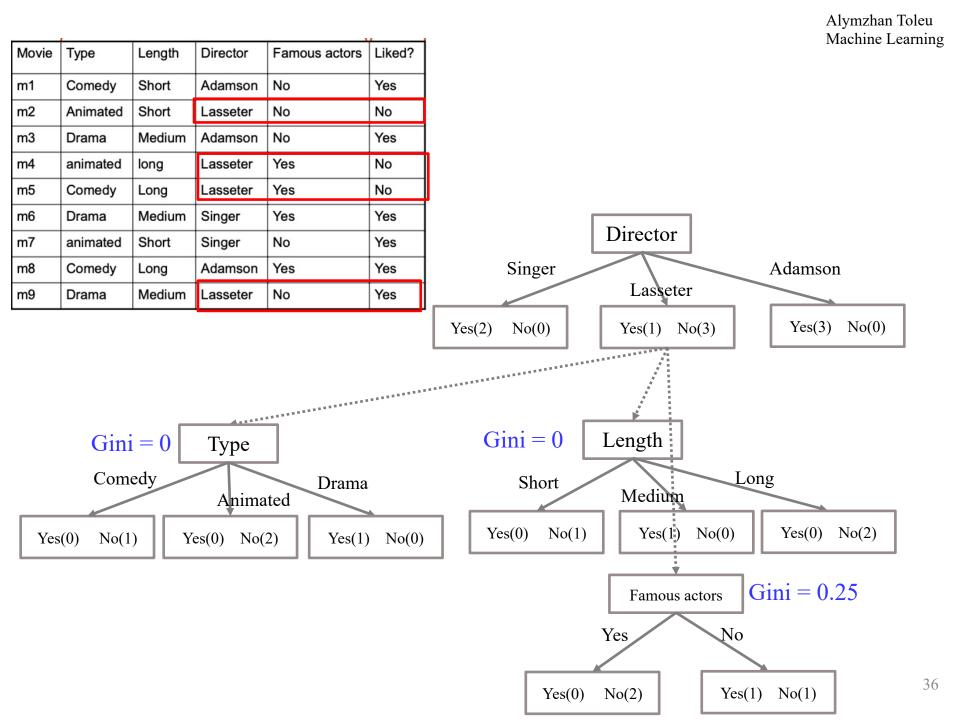
We know that **Director feature** has **the lowest Gini impurity**, so we put it on the top of tree.

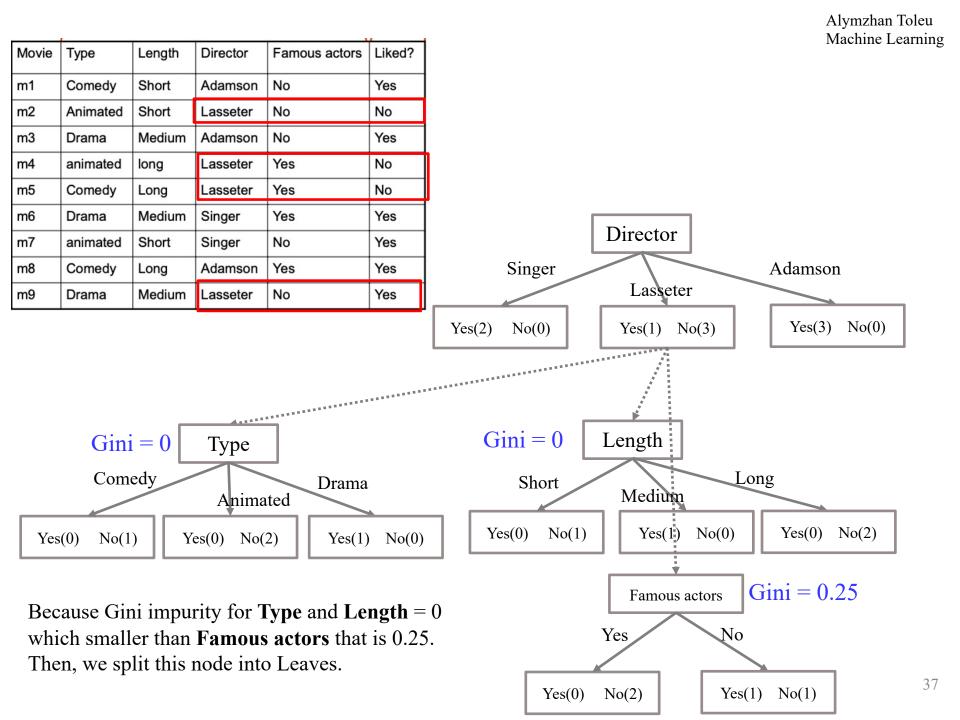
Movie	Туре	Length	Director	Famous actors	Liked?
m1	Comedy	edy Short Adamson		No	Yes
m2	Animated	Short	Lasseter	No	No
m3	Drama	Medium	Adamson	No	Yes
m4	animated	long	Lasseter	Yes	No
m5	Comedy	Long	Lasseter	Yes	No
m6	Drama	Medium	Singer	Yes	Yes
m7	animated	Short	Singer	No	Yes
m8	Comedy	Long	Adamson	Yes	Yes
m9	Drama	Medium	Lasseter	No	Yes

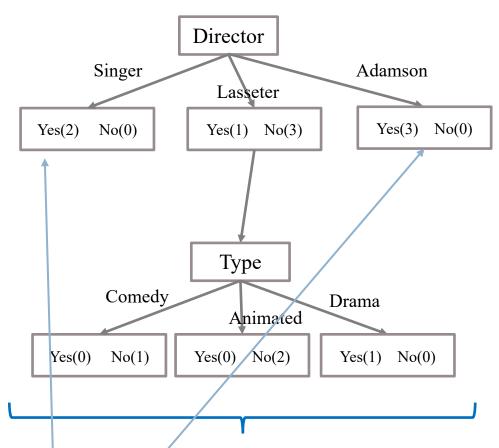


Let's us see if we can reduce the impurity by splitting this node by other features.

Movie	Туре	Length	Director	Famous actors	Liked?			
m1	Comedy	Short	Adamson	No	Yes			
m2	Animated	Short	Lasseter	No	No			
m3	Drama	Medium	Adamson	No	Yes			
m4	animated	long	Lasseter	Yes	No			
m5	Comedy	Long	Lasseter	Yes	No			
m6	Drama	Medium	Singer	Yes	Yes			
m7	animated	Short	Singer	No	Yes			
m8	Comedy	Long	Adamson	Yes	Yes	_		
m9	Drama	Medium	Lasseter	No	Yes		Director	
					Yes(2)	No(0)	Lasseter  Yes(1) No(3)	Adamson Yes(3) No(0)
	Gini = (	Typ		Drama	C	Short	Length Lo	ng
Yes(0)	No(1)	Yes(0)	No(2)	Yes(1) No(0)	Y	Yes(0) No(1)	Yes(1) No(0)	Yes(0) No(2)



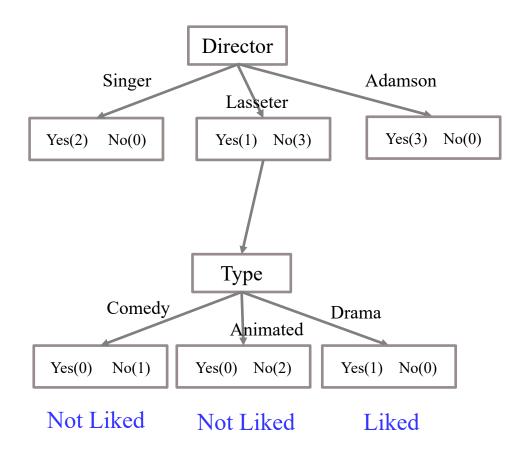


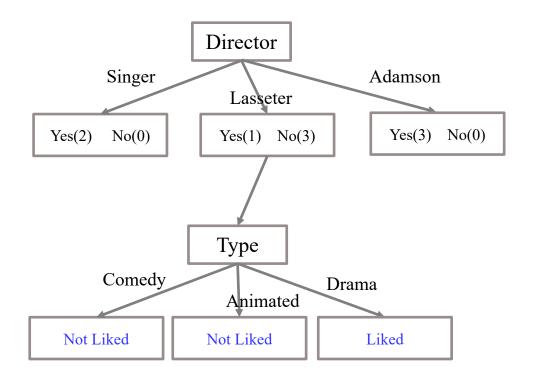


These are the leaves, because there is no reason to continue to splitting them into smaller groups.

Likewise, these nodes are leaves.

Last thing is that we need to assign output value for each **leaf**.





#### New samples:

Type	Length	Director	Fanous actor
Animated	Short	Lasseter	?
Drama	Medium	Adamson	?

# Building a decision tree

- There are three ways to build a decision tree
  - Gini impurity
  - Entropy (minimizes the entropy)
  - Information gain (maximizes the information gain at each node)

Movie	Туре	Length	Director	Famous actors	Liked?
m1	Comedy	Short	Adamson	No	Yes
m2	Animated	Short	Lasseter	No	No
m3	Drama	Medium	Adamson	No	Yes
m4	animated	long	Lasseter	Yes	No
m5	Comedy	Long	Lasseter	Yes	No
m6	Drama	Medium	Singer	Yes	Yes
m7	animated	Short	Singer	No	Yes
m8	Comedy	Long	Adamson	Yes	Yes
m9	Drama	Medium	Lasseter	No	Yes

$$Entropy = \sum log_2\left(\frac{1}{P(x)}\right)P(x)$$

$$E(Liked) = \frac{6}{9}log_2\left(\frac{1}{\frac{6}{9}}\right) + \frac{3}{9}log_2\left(\frac{1}{\frac{3}{9}}\right) = 0.92$$

$$E(Liked \mid len) = ?$$

$$E(Liked \mid Type) = ?$$

$$E(Liked \mid Director) = ?$$

$$E(Liked \mid Famous \ actor) = ?$$

$$E(Liked | len) = P(len = "S")E(Liked | len = "S")$$

$$+ P(len = "M")E(Liked | len = "M")$$

$$+ P(len = "L")E(Liked | len = "L")$$

$$= \frac{3}{9} * 0.92 + \frac{3}{9} * 0 + \frac{3}{9} * 0.92 = 0.62$$

Movie	Туре	Length	Director	Famous actors	Liked?
m1	Comedy	Short	Adamson	No	Yes
m2	Animated	Short	Lasseter	No	No
m3	Drama	Medium	Adamson	No	Yes
m4	animated	long	Lasseter	Yes	No
m5	Comedy	Long	Lasseter	Yes	No
m6	Drama	Medium	Singer	Yes	Yes
m7	animated	Short	Singer	No	Yes
m8	Comedy	Long	Adamson	Yes	Yes
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$$Entropy = \sum log_2\left(\frac{1}{P(x)}\right)P(x)$$

$$E(Liked) = \frac{6}{9}log_2\left(\frac{1}{\frac{6}{9}}\right) + \frac{3}{9}log_2\left(\frac{1}{\frac{3}{9}}\right) = 0.92$$

$$E(Liked \mid len) = 0.62$$

$$E(Liked \mid Type) = 0.62$$

$$E(Liked \mid Director) = 0.36$$

$$E(Liked \mid Famous \ actor) = 0.85$$

Information gain: IG(Liked | len) = E(Liked) - E(Liked | len) = 0.92 - 0.62 = 0.3

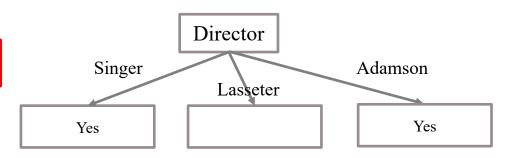
$$IG(Liked | Type) = E(Liked) - E(Liked | Type) = 0.92 - 0.62 = 0.3$$

$$IG(Liked | Director) = E(Liked) - E(Liked | Director) = 0.92 - 0.36 = 0.55$$

 $IG(Liked | Famous \ actor) = E(Liked) - E(Liked | Famous \ actor) = 0.92 - 0.85 = 0.06$ 

				_	
Movie	Туре	Length	Director	Famous actors	Liked?
m1	Comedy	Short	Adamson	No	Yes
m2	Animated	Short	Lasseter	No	No
m3	Drama	Medium	Adamson	No	Yes
m4	animated	long	Lasseter	Yes	No
m5	Comedy	Long	Lasseter	Yes	No
m6	Drama	Medium	Singer	Yes	Yes
m7	animated	Short	Singer	No	Yes
m8	Comedy	Long	Adamson	Yes	Yes
m9	Drama	Medium	Lasseter	No	Yes

We only need to focus on the records (samples) associated with this node.



We eliminated the 'director' attribute. All samples have the same director

Movie	Туре	Length	Famous actors	Liked ?
m2	Animated	Short	No	No
m4	animated	Long	Yes	No
m5	Comedy	Long	Yes	No
m9	Drama	Medium	No	Yes

$$E(Liked \mid en) = 0$$

$$E(Liked \mid len) = 0$$

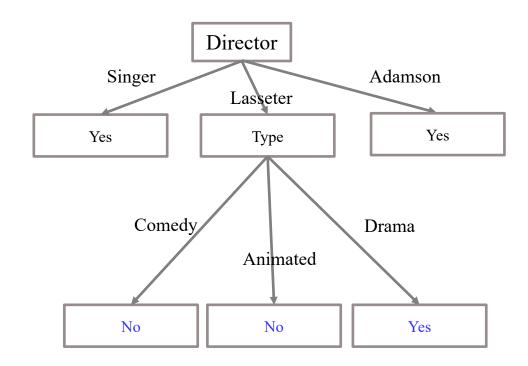
$$IG = 0.81$$

$$E(Liked \mid Type) = 0$$

$$IG = 0.81$$

$$E(Liked \mid Famous \ actor) = 0.5$$
  $IG = 0.31$ 

Movie	Туре	Length	Famous actors	Liked ?
m2	Animated	Short	No	No
m4	animated	long	Yes	No
m5	Comedy	Long	Yes	No
m9	Drama	Medium	No	Yes



This is final tree!

### Additional points

- The algorithm we gave reaches homogonous nodes (or runs out of attributes)
- This is dangerous: For datasets with many (non relevant) attributes the algorithm will continue to split nodes.
- This will lead to overfitting!

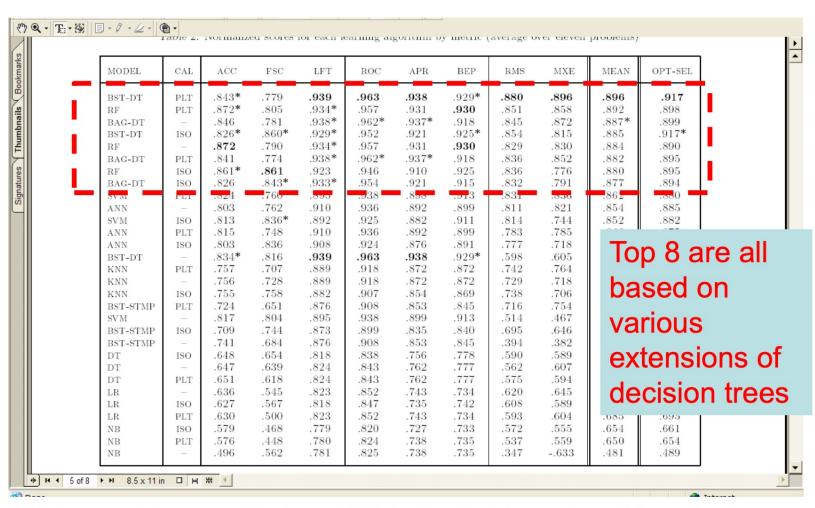
### Avoiding overfitting: Tree pruning

- Split data into train and test set
- Build tree using training set
  - for all internal nodes (starting at the root);
  - remove sub tree rooted at node;
- assign class to be the most common among training set check test data error;
  - if error is lower, keep change;
  - otherwise restore subtree, repeat for all nodes in subtree;
- Alternatively, we can **put limit on how trees grow**, for example, by requiring a certain number (evaluated empirically) or more movies per leaf.

### Important points

- Discriminative classifiers
- Building decision trees
  - Gini impurity (minimizes the Gini impurity )
  - Entropy (minimizes the entropy)
  - Information gain (maximizes the information gain at each node)

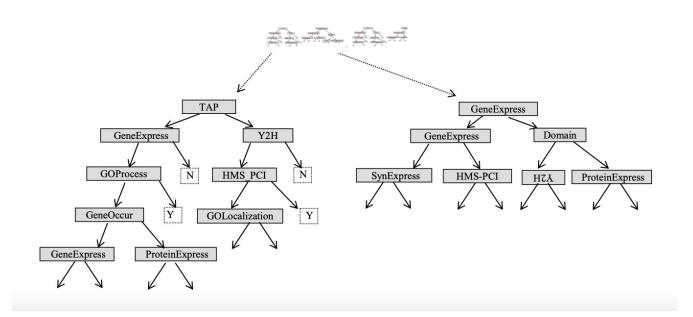
## Ranking classifiers



Rich Caruana & Alexandru Niculescu-Mizil, An Empirical Comparison of Supervised Learning Algorithms, ICML 2006

### Random forest

- A collection of decision trees.
- For each tree we select a subset of the attributes and build tree using just these attributes.
- An input sample is classified using majority voting.



• Thank you!