16.90 Project 1

$$M_{hh}\frac{d^2h}{dt^2} + M_{h\alpha}\frac{d^2\alpha}{dt^2} + D_h\frac{dh}{dt} + K_h h + L = 0$$
$$M_{\alpha\alpha}\frac{d^2\alpha}{dt^2} + M_{\alpha h}\frac{d^2h}{dt^2} + D_\alpha\frac{d\alpha}{dt} + K_\alpha(1 + k_{NL}h^2)\alpha + M = 0$$

Setup

$$v = \frac{dh}{dt}$$
$$p = \frac{d\alpha}{dt}$$
$$M_{hh}\frac{dv}{dt} + M_{h\alpha}\frac{dp}{dt} + D_h v + K_h h + L = 0$$
$$M_{\alpha\alpha}\frac{dp}{dt} + M_{\alpha h}\frac{dv}{dt} + D_\alpha p + K_\alpha(1 + k_{NL}h^2)\alpha + M = 0$$

$$\frac{d\alpha}{dt} = p \tag{1}$$
$$\frac{dh}{dt} = v \tag{2}$$
$$\frac{dp}{dt} = \frac{-K_h M_{\alpha h}h - M_{\alpha h}L + M_{hh}K_\alpha(1 + k_{NL}h^2)\alpha + M_{hh}M + D_\alpha M_{hh}\ p - D_h M_{\alpha h}v}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} \tag{3}$$
$$\frac{dv}{dt} = \frac{K_h M_{\alpha\alpha}h + M_{\alpha\alpha}L - K_\alpha M_{h\alpha}(1 + k_{NL}h^2)\alpha - M_{h\alpha}M - D_\alpha M_{h\alpha}\ p + D_h M_{\alpha\alpha}v}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} \tag{4}$$

$$U \equiv \begin{pmatrix} \alpha \\ h \\ p \\ v \end{pmatrix} \tag{5}$$

$$\frac{d}{dt}\begin{pmatrix} \alpha \\ h \\ p \\ v \end{pmatrix} = f(U) = \begin{pmatrix} p \\ v \\ \frac{-K_h M_{\alpha h}h + M_{\alpha h}L + M_{hh}K_\alpha(1+k_{NL}h^2)\alpha - M_{hh}M + D_\alpha M_{hh}\ p - D_h M_{\alpha h}v}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} \\ \frac{K_h M_{\alpha\alpha}h - M_{\alpha\alpha}L - K_\alpha M_{h\alpha}(1+k_{NL}h^2)\alpha + M_{h\alpha}M - D_\alpha M_{h\alpha}\ p + D_h M_{\alpha\alpha}v}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} \end{pmatrix} \tag{6}$$

$$\frac{\partial f}{\partial U} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{M_{hh}\left(c_m + K_\alpha(1+k_{NL}h^2)\right) - M_{\alpha h}c_l}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} & \frac{2\alpha h K_\alpha k_{NL}M_{hh} - K_h M_{\alpha h}}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} & \frac{D_\alpha M_{hh}}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} & \frac{-D_h M_{\alpha h}}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} \\ \frac{-M_{h\alpha}\left(c_m + K_\alpha(1+k_{NL}h^2)\right) + M_{\alpha\alpha}c_l}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} & \frac{-2\alpha h K_\alpha k_{NL}M_{h\alpha} + K_h M_{\alpha\alpha}}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} & \frac{-D_\alpha M_{h\alpha}}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} & \frac{D_h M_{\alpha\alpha}}{M_{\alpha h}M_{h\alpha} - M_{\alpha\alpha}M_{hh}} \end{pmatrix} \tag{7}$$

1

1.1 Solutions

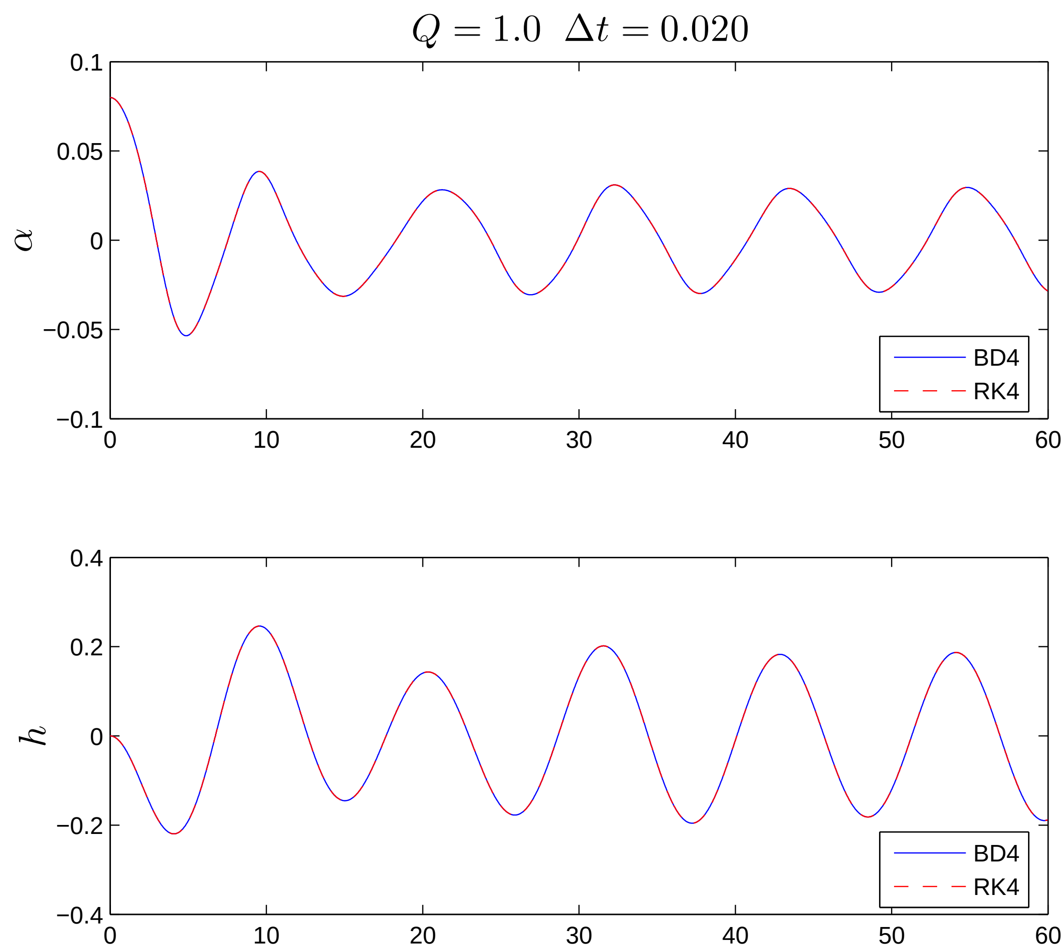

Figure 1: Solutions using RK4 and BD4 for $\Delta t = 0.02$ at $Q = 1.0$

Figures 1 and 2 shows solutions for $h(t)$ and $\alpha(t)$ using both methods with a time step of $\Delta t = 0.02$. At $Q = 1$ there are regular oscillations with at a roughly constant frequency. At $Q = 1.5$ the oscillations are much more irregular and α and h oscillate at different and varying frequencies.
Error:
The errors in both RK4 and BD4 are proportional to Δt^4 however BD4 bottoms out sooner and has a larger absolute error for any given Δt.
Efficiency:
RK4 requires four evaluations of our ODE each time step. BD4, however, is implicit and requires some kind of iterative method to solve (e.g. Netwon-Raphson). Each step of BD4 will likely take much longer than a step of RK4. Add to that the fact the RK4 has a much lower error for any given value of Δt and it is clear that its much more efficient to use RK4 if we are trying to solve this ODE to a certain accuracy.

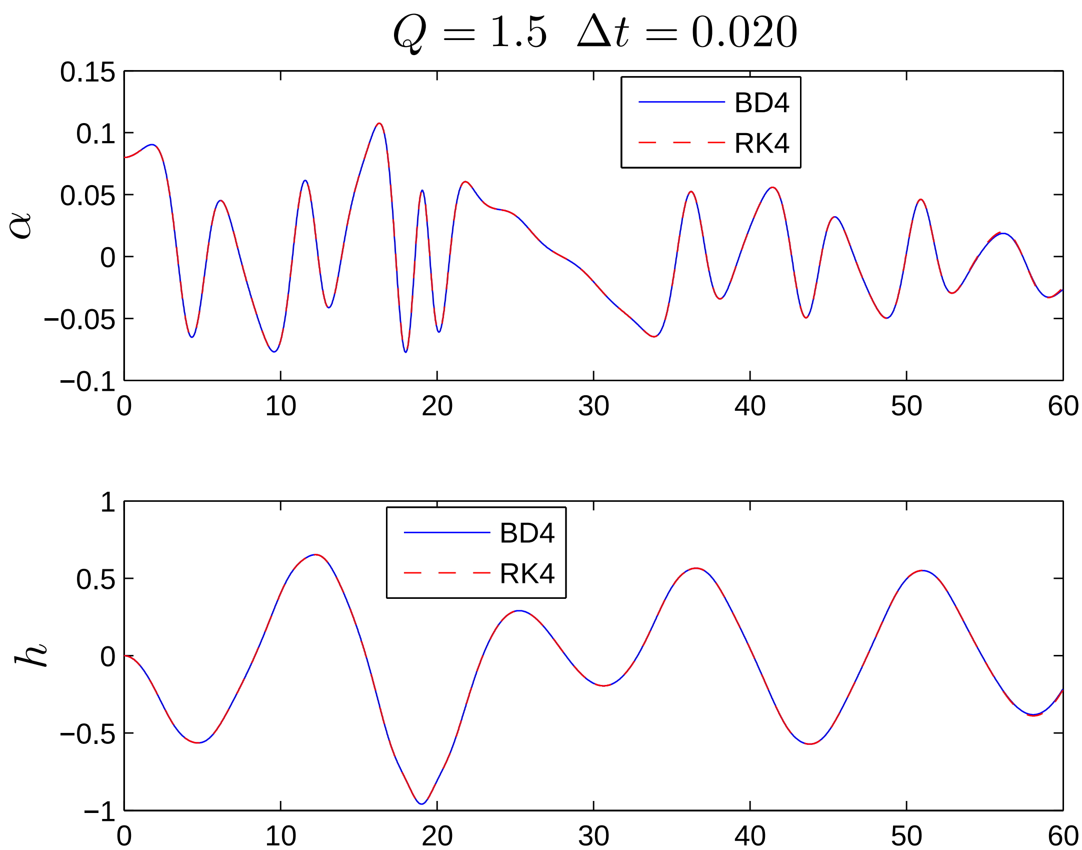

Figure 2: Solutions using RK4 and BD4 for $\Delta t = 0.02$ at $Q = 1.5$

1.2 RK4

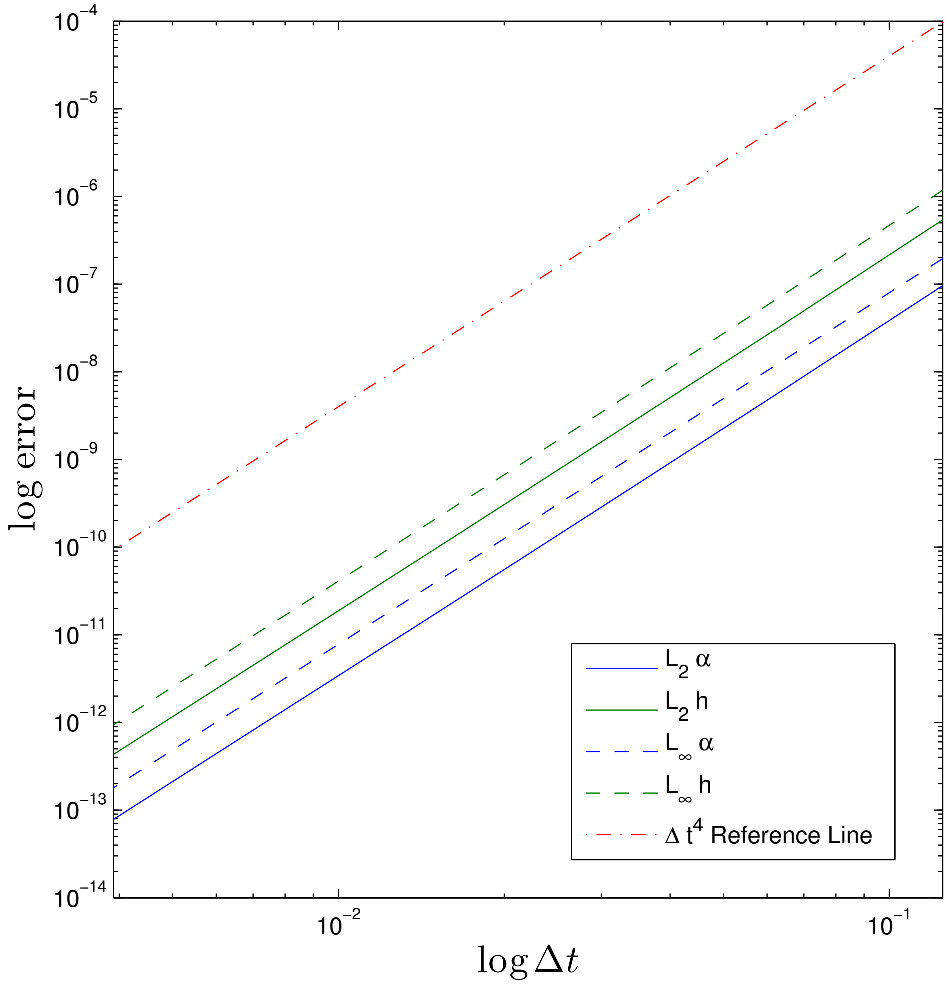

Figure 3: Error in ODE Solution Using RK4 Scheme

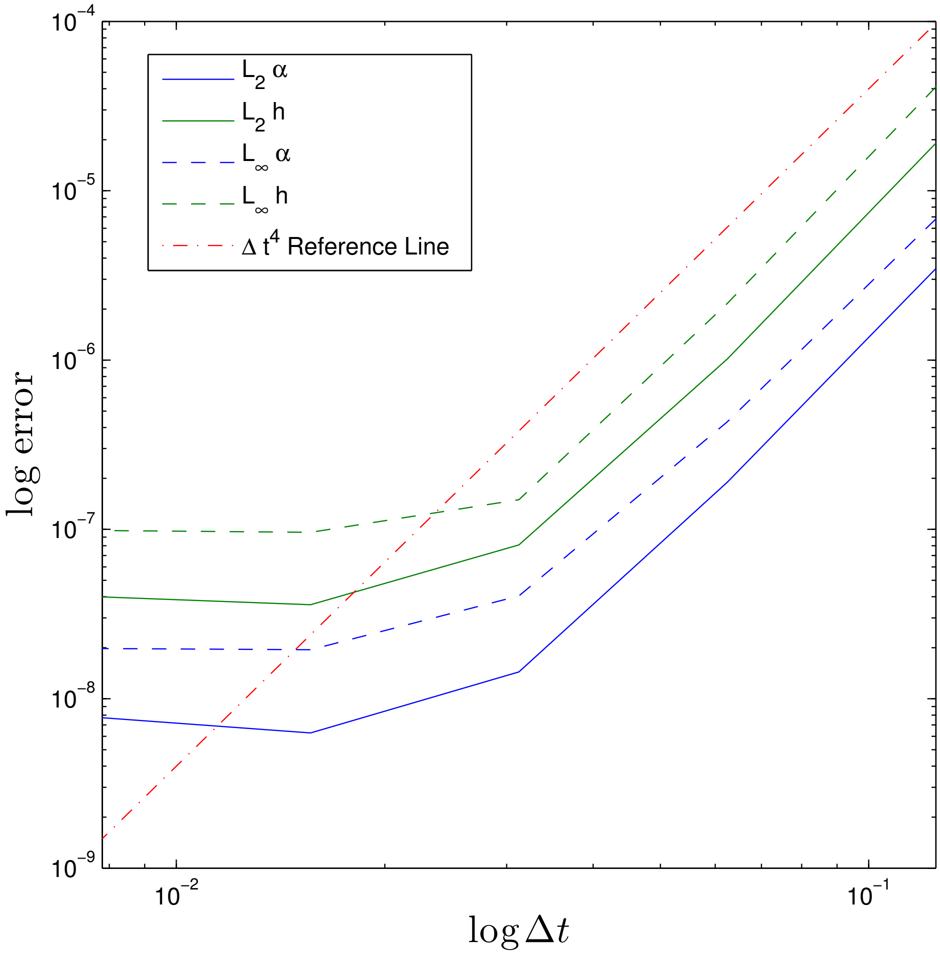

Figure 4: Error in ODE Solution Using BD4 Scheme

1.3 BD4

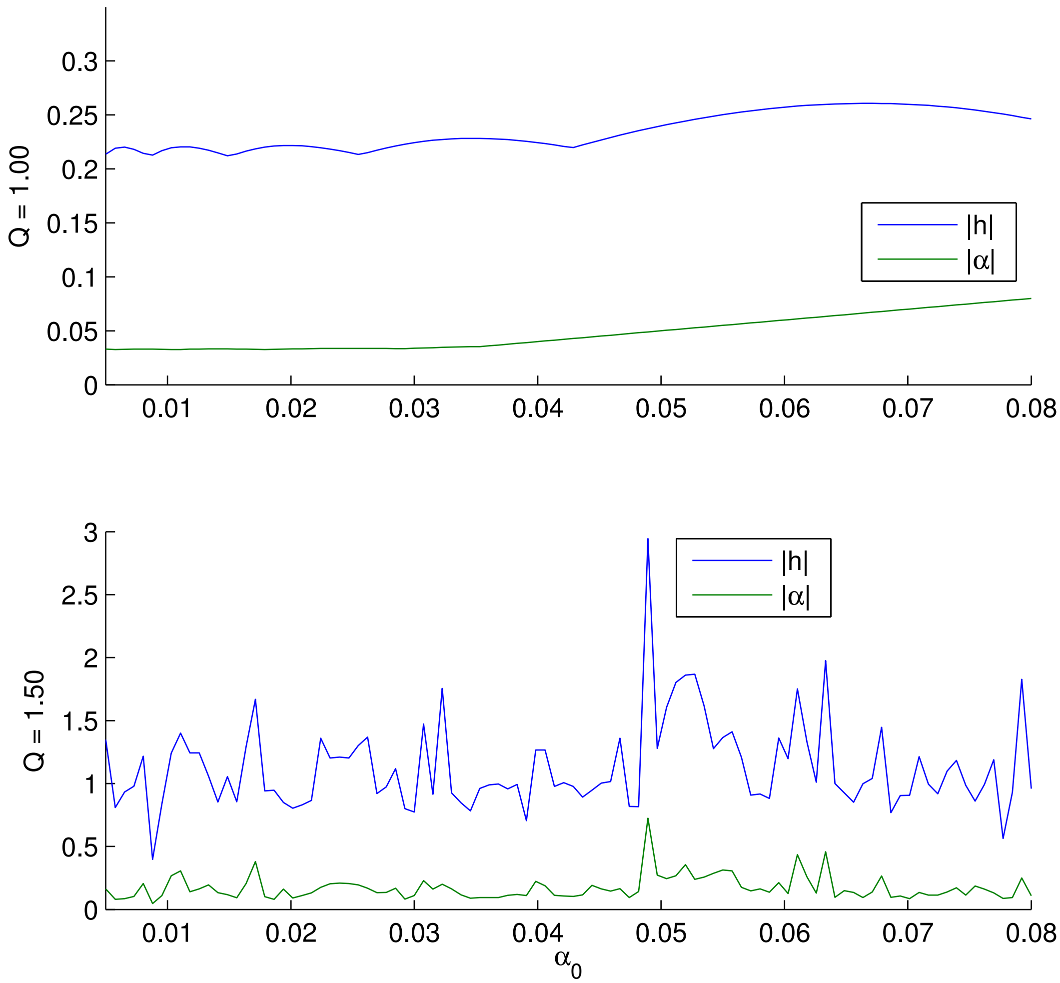

Figure 5: Maximum Absolute h and α on $t = [0, 60]$ vs Initial Pitch Angle α_0

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Figure 5 shows the maximum absoulte value of h and α for $t = [0, 60]$.

- At $Q = 1$ the magnitude of h and α change only slightly and smoothly when changing α_0. There is a slight peak in $|h|$ around $\alpha_0 = 0.065$ while $|\alpha|$ simply increase with α_0.
- At $Q = 1.5$ the maximum of each does not vary smoothly. However there is a clear peak near $\alpha_0 = 0.05$, ($|h|_{\max} \approx 2.9$, $|\alpha|_{\max} \approx 0.73$).
- The chaotic nature of the solution at $Q = 1.5$ makes it difficult to be confident in these maximum plots.

3

Over $t = [0, 20]$ the only area where h is too big is near $\alpha_0 = 0.06$ while α exceeds the required maximimum pitch from before $\alpha_0 = 0.05$ to $\alpha_0 = 0.055$.

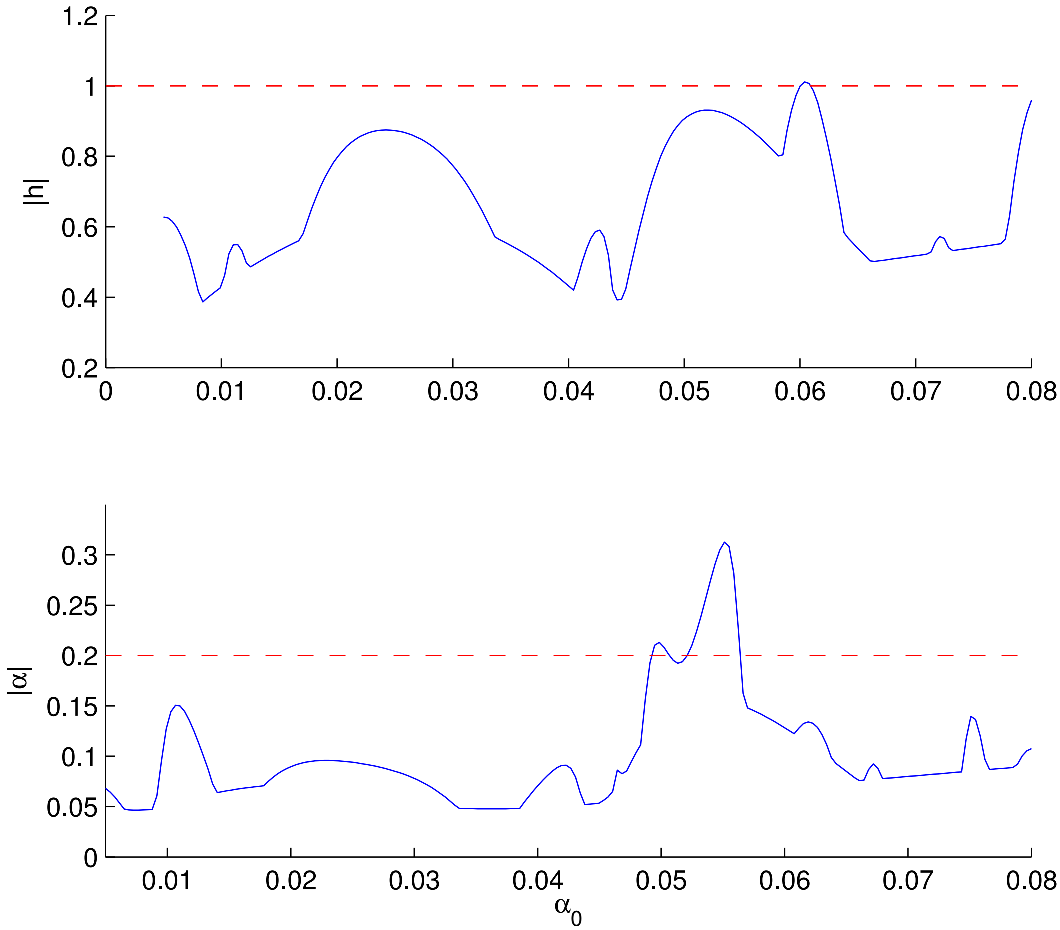

Figure 6: Maximum Absolute h and α on $t = [0, 20]$ vs Initial Pitch Angle $\alpha_0$at $Q = 1.5$

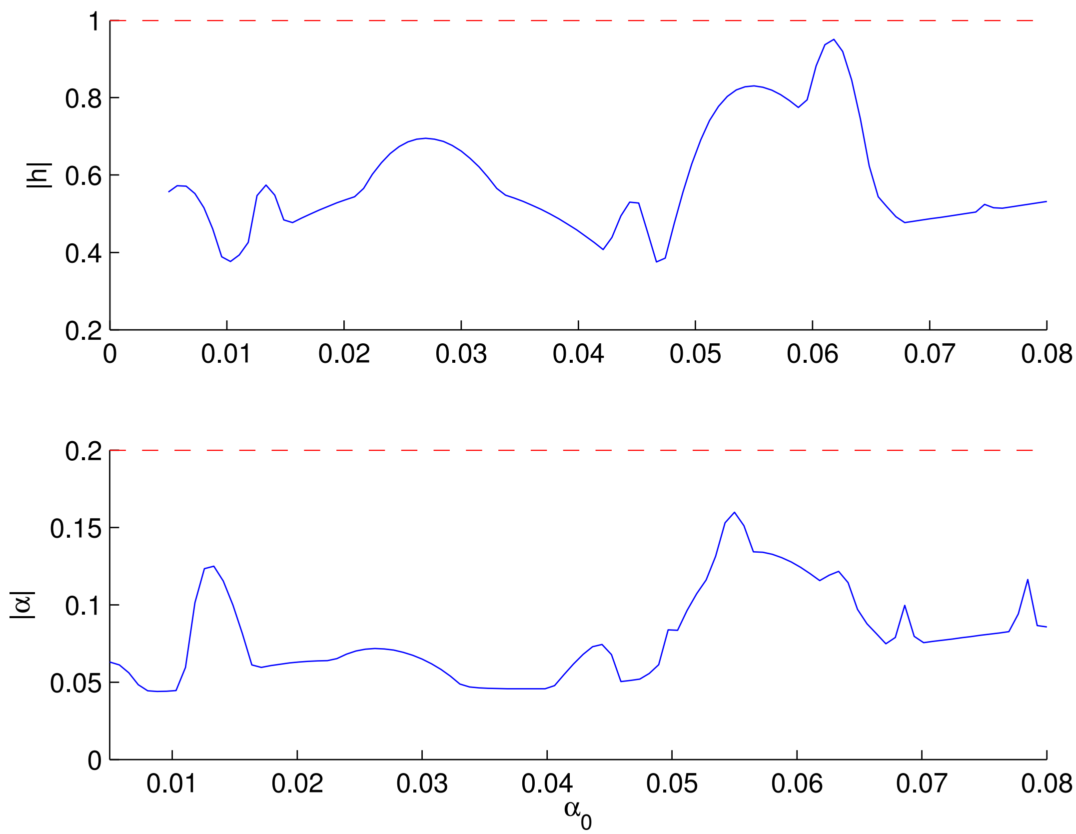

Figure 7: Maximum Absolute h and α with $K_\alpha = 1.30$

3.1 Optimization

Varying K_h does not seem to help bring these maximum curves within the limits. However, increasing K_α, D_h, and D_α all seem to help bring these maximums down.

We can meet the limit by only increasing K_α to 1.30 as seen if Figure 7. Or we can meet the limit by only increasing D_h to 0.20 as seen in Figure 8. And finally we can meet the limit by increasing D_α to 0.30 as seen in Figure 9. Each of these individual changes cost 10% in weight. A true optimum may be found by changing these parameters simultaneously.

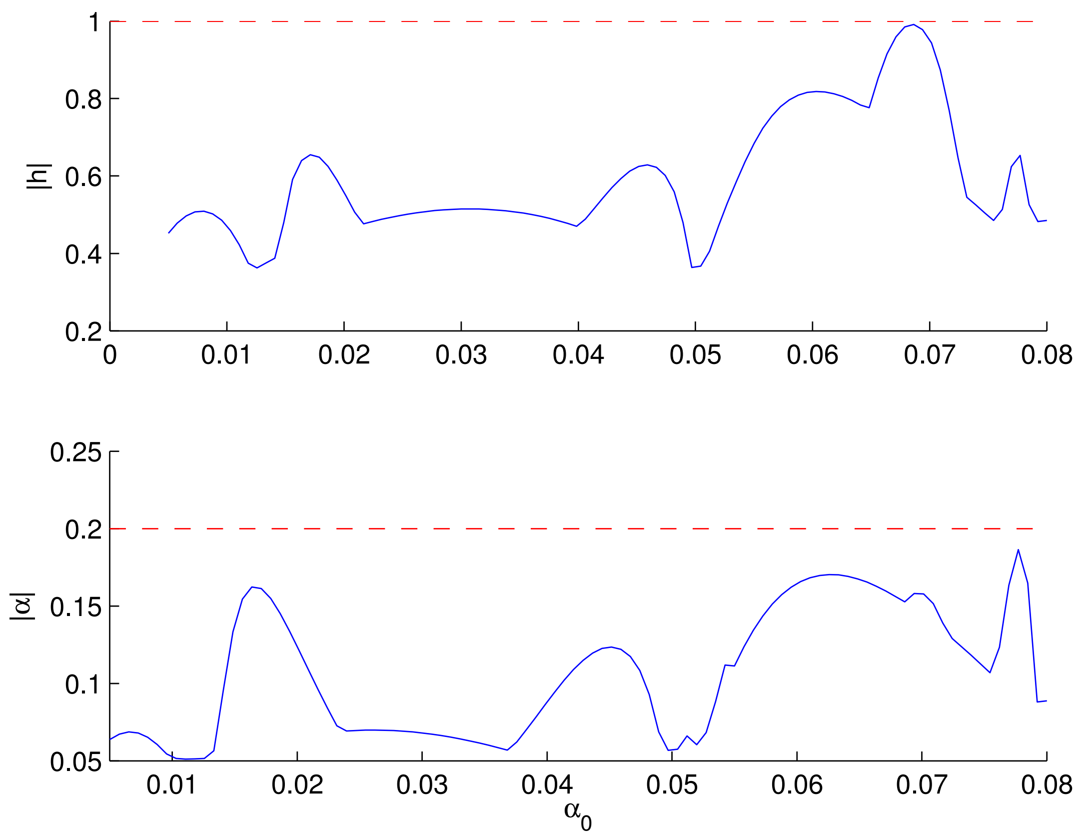

Figure 8: Maximum Absolute h and α with $D_h = 0.20$

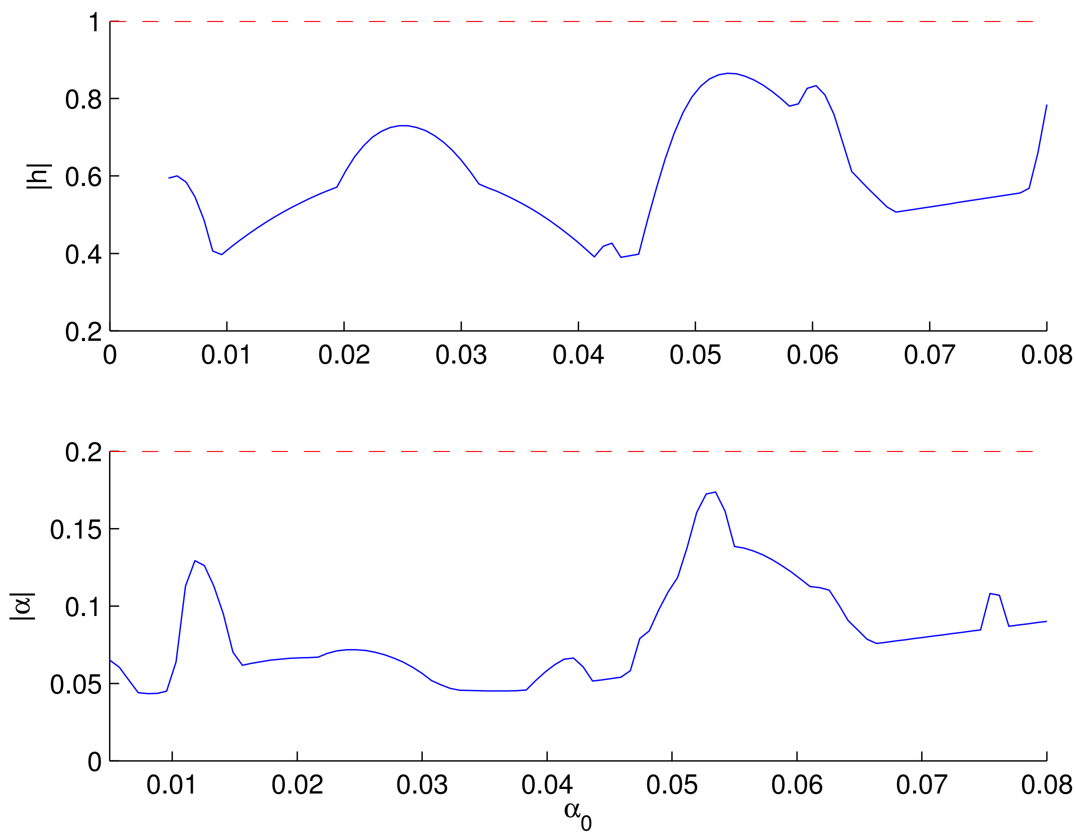

Figure 9: Maximum Absolute h and α with $D_\alpha = 0.30$