MIT 16.90: Problem Set 7

Spring 2016

Due April 13th, 2016

Please attach a hardcopy of your code for this Problem

1. **Analysis of the 1D Nodal Basis Functions for Interpolation**

   In this problem, we consider interpolating three different functions as a linear combination of linear nodal basis functions, and analyze how close these interpolated solutions are to the actual function values.

   Consider the functions u_1, u_2, and u_3 below (plotted in Figure 1):

$$u_1 = \begin{cases} 5 & 0 < x < 0.5 \\ 1 & 0.5 < x < 1 \end{cases}$$

$$u_2 = \begin{cases} e^{x} & 0 < x < 0.5 \\ e^{2(x-0.25)} & 0.5 < x < 1 \end{cases}$$

$$u_3 = \quad 5(1 - x^3) \quad 0 < x < 1$$

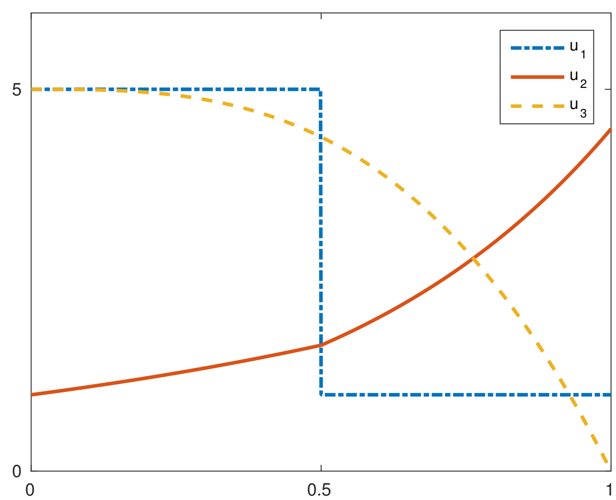

Figure 1: Plot of u_1, u_2, u_3; the exact functions to be interpolated

We wish to approximate these functions using a combination of linear nodal basis functions, given by

$$v(x) = \sum_{j=1}^{N} a_j \phi_j(x)$$

where N represents the number of nodes. In this problem, we consider equally spaced nodes, so that

$$\Delta x = \Delta x_{j-1} = \Delta x_j$$

For a nodal basis and linear elements, the j^{th} basis function is given by

$$\phi_j(x) = \begin{cases} 0 & x < x_{j-1} \\ \frac{x - x_{j-1}}{\Delta x_{j-1}} & x_{j-1} < x < x_j \\ \frac{x_{j+1} - x}{\Delta x_j} & x_j < x < x_{j+1} \\ 0 & x > x_{j+1} \end{cases}$$

and the coefficients are chosen to equal the exact values at the nodes so that

$$a_j = u(x_j)$$

We can measure the pointwise error between our interpolated solution and the actual function as

$$e(x) = |u(x) - v(x)|$$

In this problem, we have N nodal points. However, for obtaining the pointwise error, we will use 50 points within each element, so that there are 50 points between nodal points x_j and x_{j+1} (e.g. linspace(x_j,x_{j+1},50) in MATLAB). We will measure the error at each of these points within a given element, and do this for all elements to obtain the error for x $\in [0,1]$.

We may also measure the maximum error in our solution over x $\in [0,1]$ as

$$e_{max} = \max_x |u(x) - v(x)|$$

One way to obtain this value is to store the maximum pointwise error for each element, and find the maximum of these values.

(a) Let $N = 8$ so that we have 8 nodal grid points. Plot the pointwise error between your interpolated numerical solution and the actual function versus x for each of the three functions. Additionally, give the maximum error in each case.

(b) We may also analyze how quickly the error in our interpolation for each function decreases as the number of nodal points N is refined. Make a loglog plot of the maximum error versus Δx for $N = 2, 4, 8, 16, 32, 64$, for each of the three functions. How does the error behave as Δx gets smaller in each case? In other words, approximately what is the order of convergence p in each case if error scales with $\mathcal{O}(\Delta x^p)$?