MIT 16.90 Spring 2013: Problem Set 3

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Due: Friday March 8, in class

Problem 2. 1 *Reading Assignment*

- Unit II: 1. Partial Differential Equations.
- Unit II: 2. Introduction to Finite Difference Methods.
- Unit II: 3. More on Finite Difference Methods.
- Unit II: 4. Introduction to Finite Volume Methods.
- Unit II: 5. Method of Weighted Residuals.

Problem 2. 2 *Finite Differences for the Steady Convection-Diffusion Equation*

Consider the steady convection-diffusion equation for the domain $x \in [0, 1]$:

$$U \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0) = 1 \qquad u(1) = 0$$

where U is the convection velocity and ν is the diffusivity.

- Why do we need two boundary conditions to make the solution unique?
- Determine the analytical solution $u_{\text{ex}}(x)$ that satisfies the specified boundary conditions. (Hint: the equation is written with partial derivatives, but is this really a PDE? 18.03 is your friend...)
- To solve this equation numerically, we use the following central difference approximation:

  $$U\delta_{2x}u_i - \nu\delta_x^2 u_i = 0$$

  Determine the local truncation error for this approximation.
- Implement the finite difference scheme in MATLAB. Do this in matrix form, i.e. construct a linear system $Au = b$ that can be solved to give $u(x)$. What does the i^{th} row of this matrix look like? How do you handle the boundary conditions at the left and right end of the domains?

- Using the values $U = 1.0$, $\nu = 0.1$, plot the error versus Δx in a log-log plot for $\Delta x \in \{1.0 \times 10^{-1}, 1.0 \times 10^{-2}, 1.0 \times 10^{-3}, 1.0 \times 10^{-4}\}$. Use the infinity norm of the difference between the numerical and exact solutions to quantify the error. The infinity norm is defined as

  $$||e||_\infty = \max\{|e_1|, ..., |e_n|\}$$

  The MATLAB `norm` command might be useful (type `help norm` to see how to use it). Does your error plot agree with the analysis performed earlier?

- For $\Delta x = 0.01$ and $U = 1.0$, plot the solution for $\nu = 0.5, 0.1$, and 0.05 on the same plot. What do you see forming at the right end of the solution as we decrease ν? Does this remind you of anything you have seen in fluids? What is the significance of the ratio U/ν?