MIT 16.90 Spring 2013: Solution Set 3

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Due: Friday March 8, in class

Solution 2. 1 *Reading Assignment*

- See the notes for solutions

Solution 2. 2 *Finite Differences for the Steady Convection-Diffusion Equation*

- The equation is a second order boundary value problem (the diffusion term makes it second order). Thus, we need two boundary conditions to uniquely determine the solution.
- The steady convection diffusion equation can be written as

$$U\frac{du}{dx} - \nu\frac{d^2u}{dx^2} = 0$$

  This is a second order ODE with constant coefficients. We know the solution is of the form

$$u(x) = c_1 e^{kx} + c_2$$

  Plugging in the assumed form of the equation into the ODE, we have

$$Uc_1 k e^{kx} - \nu c_1 k^2 e^{kx} = 0$$

  which simplifies to

$$k(U - \nu k) = 0$$

  The two roots are $k = 0$ and $k = U/\nu$. Thus, the solution is

$$u(x) = c_1 e^{Ux/\nu} + c_2$$

  Plugging in the boundary conditions, we have

$$u(0) = 1 = c_1 + c_2$$

$$u(1) = 0 = c_1 e^{U/\nu} + c_2$$

  This gives $c_1 = 1/(1 - e^{U/\nu})$, $c_2 = 1 - 1/(1 - e^{U/\nu})$.

- Starting from the definition of the truncation error by plugging in the exact solution into the scheme:

$$\begin{aligned}
\tau &= U\delta_{2x}u_i - \nu\delta_x^2 u_i \\
&= U\left[u_{xi} + \frac{1}{6}\Delta x^2 u_{xxxi} + O(\Delta x^4)\right] - \nu\left[u_{xxi} + \frac{1}{12}\Delta x^2 u_{xxxxi} + O(\Delta x^4)\right] \\
&= (Uu_{xi} - \nu u_{xxi}) + \frac{1}{6}\Delta x^2 U u_{xxxi} - \frac{1}{12}\Delta x^2 \nu u_{xxxxi} + O(\Delta x^4)
\end{aligned}$$

  We can cancel the $(Uu_{xi} - \nu u_{xxi})$ since for the exact solution that satisfies the PDE, this will be zero. The scheme is thus second order accurate.

- We discretize with N points in our domain. We have to treat the first and last points differently since a Dirichlet boundary condition is imposed at these points. For $i = 2, ..., N$, the i^{th} row will only have three nonzero elements:

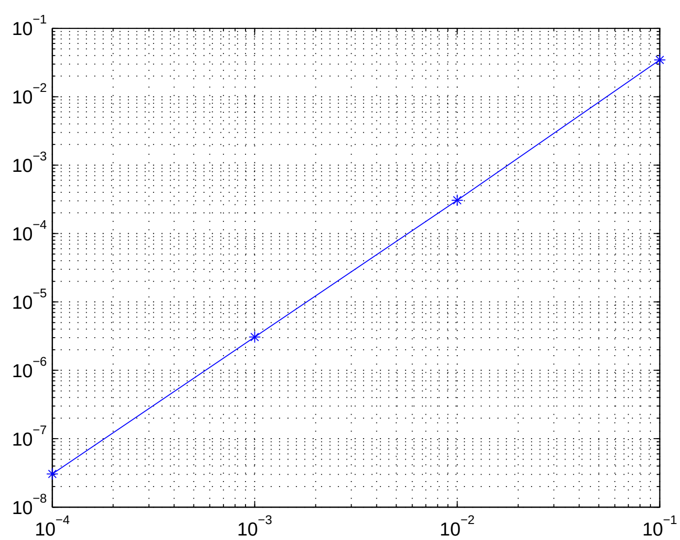
$$A_{i,i-1} = \frac{U}{2\Delta x} + \frac{\nu}{\Delta x^2}$$

$$A_{i,i} = -2\frac{\nu}{\Delta x^2}$$

$$A_{i,i+1} = -\frac{U}{2\Delta x} + \frac{\nu}{\Delta x^2}$$

  the b vector will have zeros for rows $2, ..., N$. In the first row, we set $A_{1,1} = 1$, and $b_1 = 1$. In the last row, we set $A_{N,N} = 1$, and $b_N = 0$. This enforces the Dirichlet boundary conditions. To see this, take the inner product of the first row of A and the vector u which gives u_1, which must equal $b_1 = 1$. This $u_1 = 1$, which is what we want.

- The plot of the error versus Δx is shown below. We observe the expected second order accuracy.



- The plot of the solution for various values of ν is shown in the plot below. We observe a numerical "boundary layer" forming on the right side of the domain. The ratio U/ν is essentially the Reynolds number, and as the ratio increases, the boundary layer becomes thinner.