MIT 16.90: Problem Set 8

Spring 2016

Due April 27th, 2016

1. **Finite Element Method**

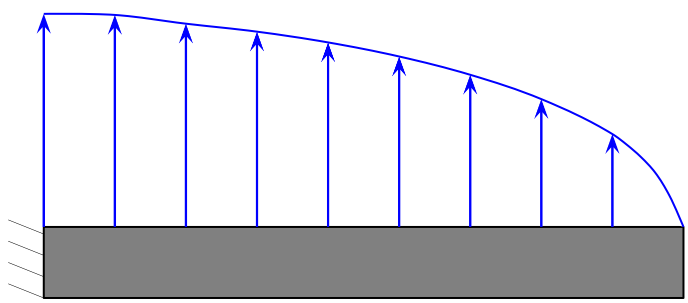
   In this problem we will solve a 1D beam bending problem using the Finite Element Method.

   In Euler Bernoulli beam theory, the transverse deflection, w, of the beam is governed by the fourth order differential equation:

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = p(x) \qquad 0 \leq x \leq L \tag{1}$$

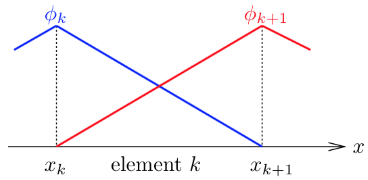
   where E is the modulus of elasticity, I is the moment of inertia, L is the length of the beam, and $p(x)$ is the distributed load.

   For this question we will analyze a wing with an elliptical lift distribution given by $p(x) = p_0\left(1 - \left(\frac{x}{L}\right)^2\right)^{0.5}$. We will model this wing as a tapered cantilever beam with a rectangular cross section. We split the beam into N elements of length l.



   (a) Derive the weak form for an approximate solution $\nu(x)$ and a weight function $\phi(x)$.

So far we have mainly considered the linear nodal basis functions (our tent functions).



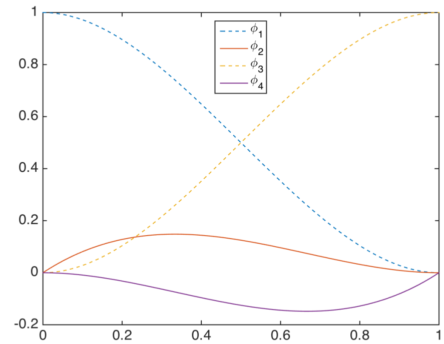
However, the governing equation for this problem has higher order derivatives, so we need higher order basis functions. To solve this problem we will therefore use the following set of basis functions:

$$\phi_1 = 1 - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3$$

$$\phi_2 = x\left(1 - \frac{x}{l}\right)^2$$

$$\phi_3 = 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3$$

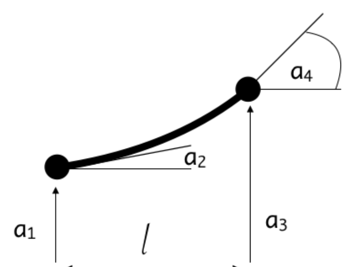
$$\phi_4 = x\left(\left(\frac{x}{l}\right)^2 - \frac{x}{l}\right)$$



We therefore assume that the solution within a given element has the form:

$$\nu^{(e)}(x) = \sum_{j=1}^{4} a_j \phi_j.$$

Instead of having only two degrees of freedom as we did in the diffusion problems, we now have 4 degrees of freedom for each element. a_1, a_2, a_3, and a_4 represent deflection and slope at each node as shown in the figure below.



(b) Given that

$$K_{ij}^{(e)} = EI \int_0^l \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\,dx$$

determine the local stiffness matrix (the stiffness matrix for a single beam element). Assume that each element has constant thickness and cross sectional area. The first row of the stiffness matrix is provided for you.

$$K^{(e)} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

.

We also need to construct our local forcing vector, which is given by:

$$F_i^{(e)} = \int_0^l p(x)\phi_i dx$$

where $p(x)$ is our elliptical lift disctribution given by $p(x) = p_0 \left(1 - \left(\frac{x}{L}\right)^2\right)^{0.5}$ and ϕ_i are our higher order basis functions. Expanding gives:

$$F^{(e)} = \begin{bmatrix} \int_0^l p(x)\phi_1 dx \\ \int_0^l p(x)\phi_2 dx \\ \int_0^l p(x)\phi_3 dx \\ \int_0^l p(x)\phi_4 dx \end{bmatrix}$$

Since these integrals are difficult to do analytically, we will apply Gaussian quadrature later on in this problem to numerically evaluate these integrals. Note that these integrals are over an individual element.

(c) For N=3 (recall N is the number of elements), construct the global stiffness matrix and forcing vector without applying the boundary conditions yet. The elements have decreasing cross sectional area, as show in the figure below:

Be sure to describe the mapping from the local degrees of freedom to the global degrees of freedom. Additionally, leave your global forcing vector in terms of integrals of $p(x)$ and ϕ over an element (i.e. the first entry of the global forcing vector is $\int_0^{L/3} p(x)\phi_1 dx$).

(d) Recall that the deflection and the slope of a cantilevered beam are both zero at the location where it is clamped. Apply these boundary conditions for a cantilevered beam to determine the final matrix system of equations.

(e) Finally write code to implement the FEM for this problem. Plot the FEM solution for deflection, ν, as a function of x, using $EI = 10$, $L = 3$, and $p_0 = 1$. Use the function lgwt.m provided in Stellar under Problem Set 8 (credit goes to Greg von Winckel for writing this function, available through Mathworks) to help you perform Gaussian quadrature to evaluate the forcing vector.

2. **Analysis of the Second Order 1D Nodal Basis Functions for Interpolation**

This problem is an extension of Pset 7 using second order nodal basis functions rather than the linear nodal basis functions. You do not need to submit anything for this problem; just give it a try if you are interested!

A nodal basis has the requirement that

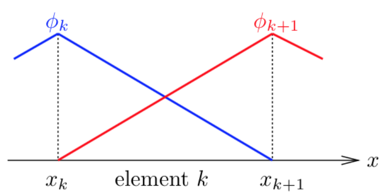
$$\phi_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

where ϕ_i denotes the i^{th} basis function and x_j is the j^{th} nodal point.

In Problem Set 7, we used the set of linear nodal basis functions (the tent functions) to interpolate several functions $u(x)$. You found that within each element, the interpolated solution $v(x)$ was a linear combination of only two nonzero basis functions, which we will denote as ϕ_1 and ϕ_2. Thus,

$$v(x) = \sum_{j=1}^{N} a_j \phi_j(x) = \ a_1\phi_1 + a_2\phi_2$$

where ϕ_1 and ϕ_2 (represented as ϕ_k and ϕ_{k+1}) within an element look like



Rather than expressing ϕ_1 and ϕ_2 in the x-coordinate (as was done in Pset 7), we may express them in the reference coordinate ξ, where $\xi \in [-1, 1]$. If x_1 and x_2 are the left and right x-coordinates of the nodes of an element, we may map from x to ξ within the element using the transformation rule

$$\xi = \frac{2(x - \frac{1}{2}(x_1 - x_2)}{x_2 - x_1}$$

As a result, we may ultimately write ϕ_1 and ϕ_2 in terms of ξ as

$$\phi_1(\xi) = \frac{1}{2}(1 - \xi)$$
$$\phi_2(\xi) = \frac{1}{2}(1 + \xi)$$

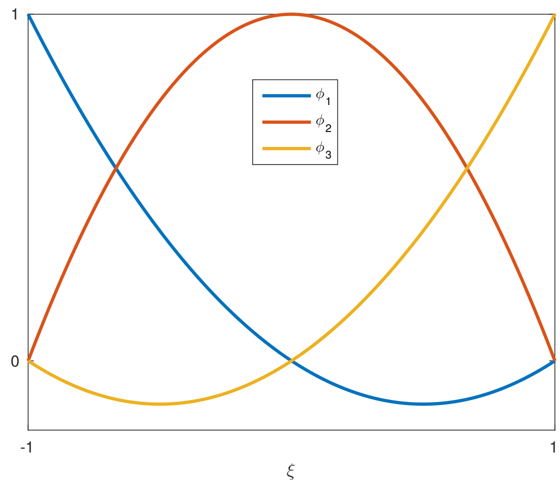
Now, let's consider the use of higher order nodal basis functions for interpolation, namely the second order nodal basis functions. For the second order nodal basis functions, there are three nonzero basis functions within an element so that the interpolated solution is given by

$$v(x) = \sum_{j=1}^{N} a_j \phi_j(x) = a_1\phi_1 + a_2\phi_2 + a_3\phi_3$$

where

$$\phi_1(\xi) = -\frac{1}{2}(1 - \xi)\xi$$
$$\phi_2(\xi) = (1 - \xi)(1 + \xi)$$
$$\phi_3(\xi) = \frac{1}{2}(1 + \xi)\xi$$

These second order basis functions within an element are depicted below.



The coefficients a_i in $v(x)$ for this case are related to those in the linear case. Namely, a_1 here corresponds to a_1 from the linear case and a_3 here corresponds to a_2 from the linear case. However, for each element, we must now determine another coefficient a_2 corresponding to the basis function at the midpoint of an element.

Task

Modify either your own Pset 7 code, or the Pset7Solutions.m code provided in Stellar under Problem Set 7 and implement the second order nodal basis functions for interpolation. In the given code, the linear basis functions are already computed using ξ coordinates, so you will simply need to change the basis functions to the second order basis functions. You will also need to obtain the coefficient a_2 for each element in order to construct your interpolated solution $v(x)$. *(Hint: Recall how we found the coefficients previously, and use the same idea.)*

Does the order of convergence of the maximum error change with the use of the second order nodal basis functions? If so, for which case(s) does it change, and how so?