Project #1: ODEs and Finite Difference

Note: *Projects are meant to be open-ended and to allow some flexibility and creativity. Therefore it is important for you to show all relevant steps, numerical plots, and justifications for the choices made in your work.*

Problem 1

A next generation airplane design uses very light-weight structures, and relies on a computerized feedback controller to stabilize aeroelastic oscillations of its wing. In order to pass a certification test, the new design must demonstrate that no structural damage could occur if the aeroelastic control computer is forced to reboot in-flight, e.g., caused by a lightning strike. This project uses computational simulation to predict whether the current design is able to pass this certification test.

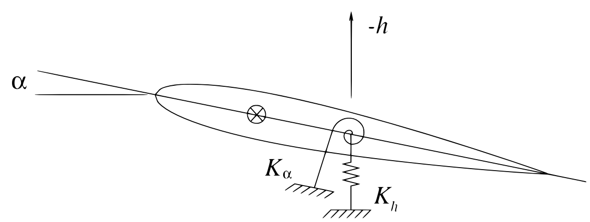


Figure 1: The 2-degree-of-freedom aeroelastic model

We model the aeroelastic vibration of the wing in the absence of the controller with two physical variables: the pitch, $\alpha(t)$, and plunge, $h(t)$. The pitch is in radians and the plunge is in nondimensional units of chord lengths. Figure 1 describes these two variables. The plunging degree of freedom is governed by a linear stiffness and damping forces, but the pitch degree of freedom includes both the linear components and a nonlinear coupling term between the pitch and plunge motions. Specifically, the motion is governed by the following system of ODEs

$$
\begin{aligned}
&M_{hh}\frac{d^2h}{dt^2} + M_{h\alpha}\frac{d^2\alpha}{dt^2} + D_h\frac{dh}{dt} + K_h h + L = 0 \\
&M_{\alpha\alpha}\frac{d^2\alpha}{dt^2} + M_{\alpha h}\frac{d^2h}{dt^2} + D_\alpha\frac{d\alpha}{dt} + K_\alpha(1 + k_{NL}h^2)\alpha + M = 0
\end{aligned}
\tag{1}
$$

where the parameters for our airplanes are

$$\begin{aligned}
M_{hh} &= 1 \\
M_{h\alpha} &= 0.625 \\
M_{\alpha\alpha} &= 1.25 \\
M_{\alpha h} &= 0.25 \\
D_h &= 0.1\ s^{-1} \\
D_\alpha &= 0.25\ s^{-1} \\
K_h &= 0.2\ s^{-2} \\
K_\alpha &= 1.25\ s^{-2} \\
k_{NL} &= 10
\end{aligned} \tag{2}$$

In the right hand side of Equation (1), L and M are the aerodynamic lift and moment, respectively. We model both the aerodynamic lift and moment as linear with respect to the angle of attack:

$$\begin{aligned}
L &= 1s^{-2}\, Q\, \alpha \\
M &= -0.7s^{-2}\ Q\, \alpha
\end{aligned} \tag{3}$$

Here Q is the aerodynamic pressure, which depends on the airspeed. It is nondimensionalized so that $Q = 1$ at a design airspeed V_{NO}. At the moment of the controller failure $t = 0$, the initial conditions are

$$\begin{aligned}
&h(0) = 0 \\
&\frac{dh}{dt}(0) = 0 \\
&\frac{d\alpha}{dt}(0) = 0 \\
&0 < \alpha(0) < 0.08 \text{ radians}
\end{aligned} \tag{4}$$

Tasks

1. Consider two design airspeeds, V_{NO} and V_{NE}, corresponding to $Q = 1$ and $Q = 1.5$, respectively. For the maximum initial pitch $\alpha = 0.08$ radians, solve the equations of motion for 60 seconds using (i) forward Euler, (i) midpoint rule, and (iii) a second-order backwards differentiation (BDF-2) scheme.

   Comment on the accuracy and efficiency of the ODE solution using each of the three schemes. Which scheme is the best one to use for this system? Also comment on the behavior of the oscillation at the two different airspeeds.

2. To determine the possibility of structural failure, the designer of the airplane is concerned about the maximum plunge and pitch motions experience during $0 < t < 60$ seconds, for a range of possible initial pitch angles $0 < \alpha < 0.08$ radians. Estimate the maximum values of $|h(t)|$ and $|\alpha(t)|$ during that time period, for both $Q = 1$ and $Q = 1.5$.

   How confident are you in the results you obtain? Why?

Problem 2

In this problem, we consider the linear advection equation in 2D,

$$\frac{\partial u}{\partial t} + \frac{\partial c_x(x,y)u}{\partial x} + \frac{\partial c_y(x,y)u}{\partial y} = 0$$

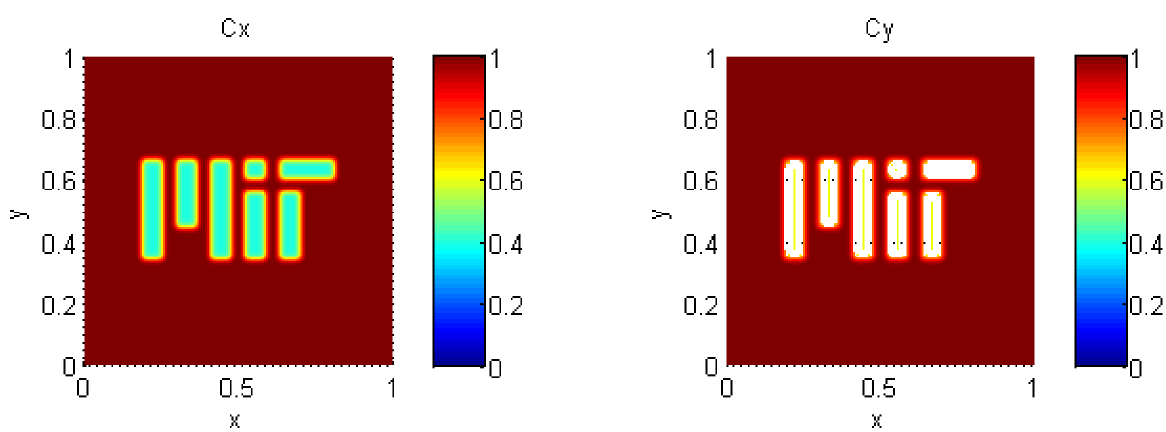
This equation describes how the solution $u(x, y, t)$, describing the density of a conserved quantity, evolve in a flow field (c_x, c_y). Here $c_x(x, y)$ and $c_y(x, y)$ are functions of space. We solve this equation in the square domain $0 \le x \le 1, 0 \le y \le 1$. The initial condition is $u(x, y, 0) = 0$. The boundary condition at the $x = 0$ edge and the $y = 0$ edge is

$$u(0, y, t) = u(x, 0, t) = \sin \pi t$$

Tasks

1. We solve the equation using finite difference method. We discretize the domain into a 513×513 uniform grid. The numerical solution $U^n_{i,j}$ denote the solution at $x_i = \frac{i-1}{512}, y_j = \frac{j-1}{512}$ and $t_n = n\Delta t$, where $i = 1, \ldots, 513, j = 1, \ldots, 513$ and Δt is the time step size.

   The nonuniform velocity field at x_i, y_j is denoted as $C_{x\ i,j}$ and $C_{y\ i,j}$. Their values are visualized in the following figure, and are stored in the text files Cx.txt and Cy.txt, respectively.

   Write a code to solve the differential equation using the Forward Time-Backward Space (FTBS) method, with $\Delta t = 1./1024$. Plot the solution at $t = 0.25, t = 0.5$ and $t = 1$.

   The solution can be visualized by the following Matlab code:

```
surf(x, x, u);
shading interp;
caxis([-2,2]);
view(2);
axis equal;
axis([0,1,0,1])
```

```
colorbar;
drawnow;
```

In answering the questions above, please include informative plots of all your numerical results, accompanied by clear written arguments and explanations. If you're making an assertion about how well a method performs or why the system behaves in a certain way, please support this assertion with numerical results and plots.