FVM and FEM Project 2 Solutions

1 Fluid part

1.1 2.1.1

A general conservation law is strong form is written as

$$\frac{d\rho}{dt} + \nabla \cdot F(\rho) = S \tag{1}$$

where ρ is the conserved quantity, $F(\rho)$ is the flux term, and S is the source term. We can write equation (5) of the problem statement, noting the steady state condition, as

$$\frac{d\left(c_p \dot{m} T_{t,cool}\right)}{dx} = 2\vec{q} \cdot \vec{n} \tag{2}$$

Where the flux term is $c_p \dot{m} T_{t,cool}$ and the source term is the right hand side. Additionally note that convection is occuring in the positive direction and therefore the upwind direction will be to the right.

1.2 2.1.2

When discretizing we make sure that for each volume i the integral of the conservation law is satisfied

$$\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{F(T)}{dx} dx = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} 2\vec{q} \cdot \vec{n} dA$$
$$F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} 2\vec{q} \cdot \vec{n} dA$$

Now, since we know that upwind is always to the right we can replace the flux terms with their expression

$$c_p \dot{m} \left(T_{t,cool}^{i} - T_{t,cool}^{i-1}\right) = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} 2\vec{q} \cdot \vec{n} dA \tag{3}$$

1.3 2.1.3

The analytic solution to the equation

$$c_p \dot{m} \frac{dT_{t,cool}}{dx} = 2\left(1 - 100x\right) 10^6 \frac{J}{m^2 s} \tag{4}$$

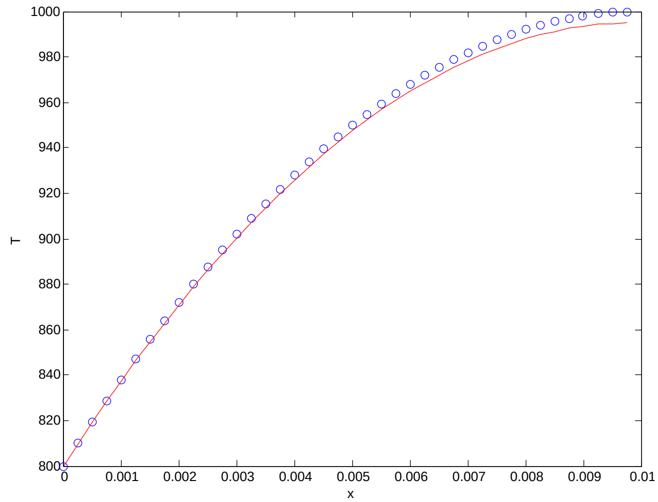

Figure 1: Analytic solution - blue circles, FV solution with 40 control volumes - red line

is obtained through integration, obtaining

$$T_{t,cool}(x) = \frac{2}{c_p \dot{m}} \left(x - 50x^2\right) 10^6 + T_{t,cool}(0) \quad (5)$$

$$T_{t,cool}(x) = \frac{2}{c_p \dot{m}} \left(x - 50x^2\right) 10^6 + 800 \quad (6)$$

The numerical scheme solves for $T^i_{t,cool}$ in each control volume $i = 1 \ldots N$. This can be written in matrix format as $\mathbf{A}\hat{T} = \hat{S}$ where $\hat{T}_i = T^i_{t,cool}$ and

$$\hat{S}_1 = 800 \quad (7)$$

$$\hat{S}_{i \neq 1} = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} 2\left(1 - 100x\right) 10^6 dx \quad (8)$$

$$= 2\left(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} - 50\left(x^2_{i+\frac{1}{2}} - x^2_{i-\frac{1}{2}}\right)\right) 10^6 \quad (9)$$

Finally since $\mathbf{A}_{[i,:]}\hat{T} = \hat{S}_i$ indicates the conservation law for each control volume we have

$$\mathbf{A}_{1,1} = 1, \quad \mathbf{A}_{1,j>1} = 0 \quad (10)$$

for the boundary condition and using equation 3 for the equation in each column we have

$$\mathbf{A}_{i,i} = c_p \dot{m}, \quad \mathbf{A}_{i,i-1} = -c_p \dot{m}, \quad \text{for } i = 2 \ldots N \quad (11)$$

40 control volumes are required to have maximum error less than $5K$. The convergence of the maximum error is seen in the figure below:

The resulting scheme is first order accurate as seen from the log-log convergence plot.

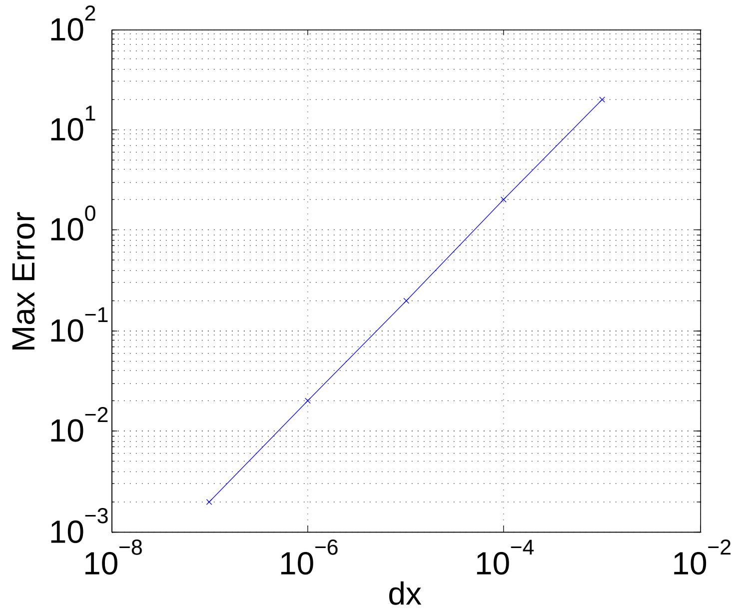

Figure 2: First order convergence as seen by slope of 1 of the plot

1.4 2.1.4

If the control volumes are non-uniform and the heat flux is a piecewise linear then the integration of the source term will change. Specifically, if we suppose control volume i has width Δx_i then integrating a linear heat flux $ax + b$ over the control volume:

$$\begin{aligned}
\int_{CV_i} ax + bdx &= \left[\frac{a}{2}x^2 + bx\right] \Big|_{x_i - \frac{\Delta x_i}{2}}^{x_i + \frac{\Delta x_i}{2}} \\
&= \frac{a}{2}\left[\left(x_i + \frac{\Delta x_i}{2}\right)^2 - \left(x_i - \frac{\Delta x_i}{2}\right)^2\right] + b\left(x_i + \frac{\Delta x_i}{2} - \left(x_i - \frac{\Delta x_i}{2}\right)\right) \\
&= \frac{a}{2}\left[\left(x_i + \frac{\Delta x_i}{2}\right)^2 - \left(x_i - \frac{\Delta x_i}{2}\right)^2\right] + b\Delta x_i \\
&= \Delta x_i(ax_i + b)
\end{aligned}$$

2 Thermal Part

2.1 Boundary conditions

- **Boundary 1** is insulating bounday condition $\vec{q} \cdot \vec{n} = 0$
- **Boundary 2** is Robin boundary conditions $\vec{q} \cdot \vec{n} = h(T - T_{t,hot})$
- **Boundary 3** is Robin boundary conditions $\vec{q} \cdot \vec{n} = h(T - T_{t,hot})$
- **Boundary 4** is Robin boundary conditions $\vec{q} \cdot \vec{n} = h(T - T_{t,cool})$

2.2 Weak form derivation

2.2.1 Bilinear form

The bilinear form is obtained by projecting the residual onto a weighting function ϕ, and using integration by parts followed by the divergence identity

$$\begin{aligned}
0 = r(T,\phi) &= \int \nabla \cdot (k\nabla T)\, \phi dA \\
&= \int \nabla \cdot (\phi k \nabla T) - k\nabla T \cdot \nabla \phi dA \\
&= -\int k\nabla\phi \cdot \nabla T dA + \int_{\partial A} \phi k \nabla T \cdot \vec{n} ds
\end{aligned}$$

2.2.2 Incorporation of boundary conditions

For each boundary condition we can plug in the appropriate expression for $k\nabla T \cdot \vec{n}$

- **Boundary 1:** $\int_{\partial A_1} \phi k \nabla T \cdot \vec{n} ds = 0$
- **Boundary 2 and 3:**

$$\begin{aligned}
\int_{\partial A_{2,3}} \phi k \nabla T \cdot \vec{n} ds &= \int_{\partial A_{2,3}} \phi h \left(T_{t,hot} - T\right) ds \\
&= \int_{\partial A_{2,3}} \phi h T_{t,hot} ds - \int_{\partial A_{2,3}} \phi h T ds
\end{aligned}$$

- **Boundary 4:**

$$\begin{aligned}
\int_{\partial A_4} \phi k \nabla T \cdot \vec{n} ds &= \int_{\partial A_4} \phi h \left(T_{t,cool} - T\right) ds \\
&= \int_{\partial A_4} \phi h T_{t,cool} ds - \int_{\partial A_4} \phi h T ds
\end{aligned}$$

Plugging these boundary conditions into the bilinear form we obtain the **weak form**

$$\int k\nabla\phi \cdot \nabla T dA + \int_{\partial A_{2,3,4}} \phi h T ds = \int_{\partial A_{2,3}} \phi h T_{t,hot} ds + \int_{\partial A_4} \phi h T_{t,cool} ds \tag{12}$$

Where the terms on the left hand side are symmetric and contain both ϕ and T and thus go into the *stiffness* matrix. The terms on the right hand side contain only ϕ and thus are in the right hand side vector.

2.2.3 Evaluation of boundary integrals on reference triangle

The nodal base on each triangle of the mesh are ϕ_1, ϕ_2, and ϕ_3 We can approximate T in the reference element i as $\hat{T}_i = \sum_{j=1}^{3} a_j \phi_j$. In the Galerkin form our weighting functions are also ϕ_i Thus, on the boundaries we need to compute all integrals $\int_{\partial A_{2,3,4}} \phi_i h \phi_j ds$ for $i, j = 1 \ldots 3$. We perform all operations on a reference triangle whose coordinate axis are denoted as ξ_1 and ξ_2, and the basis function evaluate to $\phi_1 = \xi_1$, $\phi_2 = \xi_2$, and $\phi_3 = 1 - \xi_1 - \xi_2$.

The integral evaluations we need to perform on the boundary are then determined by integrating $\int \phi_i \phi_j d\xi_1 d\xi_2$ along one boundary of the reference triangle ($\xi_2 = 0$)

$$\int_0^1 \phi_1^2 d\xi_1 = \frac{1}{3}$$
$$\int_0^1 \phi_1 \phi_2 d\xi_1 = 0$$
$$\int_0^1 \phi_1 \phi_3 d\xi_1 = \int_0^1 \xi_1 - \xi_1^2 d\xi_1 = \frac{1}{6}$$
$$\int_0^1 \phi_2 \phi_3 d\xi_1 = 0$$
$$\int_0^1 \phi_3^2 d\xi_1 = \frac{1}{3}$$

One should note that the physical edge can lie along any of the three boundaries of the reference triangle. The reason we don't have to evaluate these integrals along the other boundaries is that for boundary $\xi_1 = 0$ we get the same values by symmetry. Additionally, the integrals along the hypotenuse of the reference triangle also turn out to be the same. These facts are taken care of in the code.

2.2.4 Results

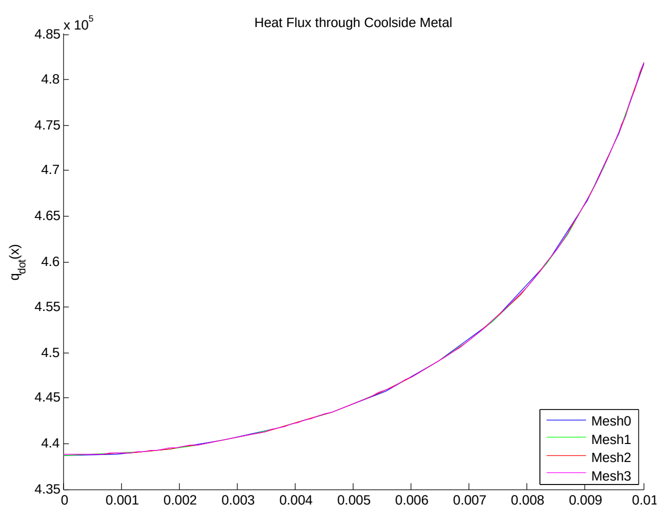

Figure 3: Temperature distribution on domain exposed to cooling air

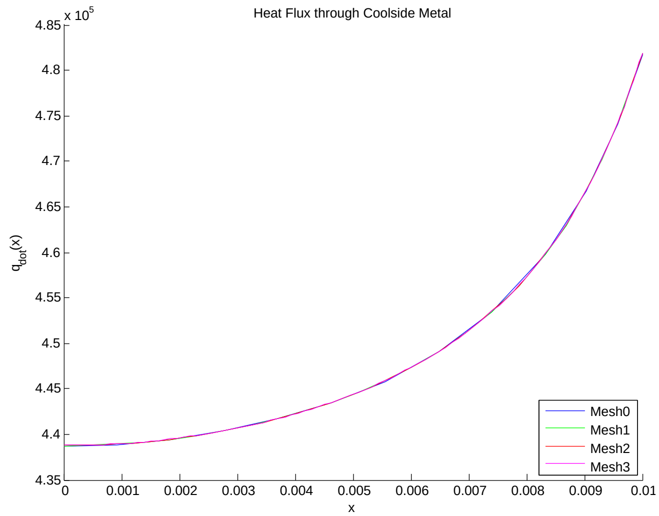

Figure 4: Heat flux on domain exposed to cooling air

3 Fluid - Thermal coupling

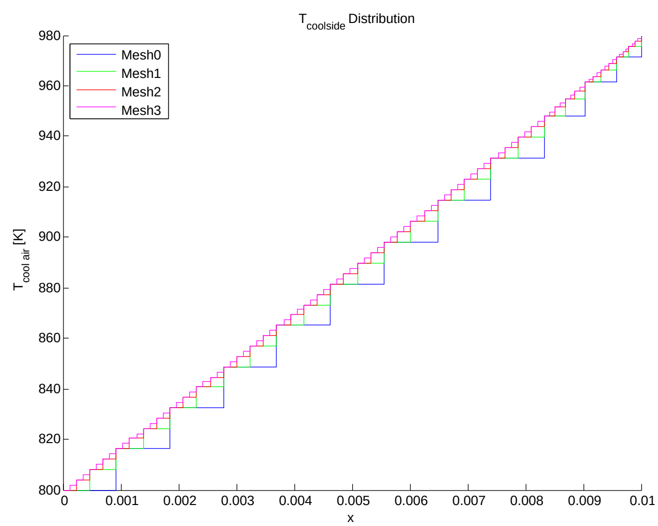

Figure 5: Solution For 2.3.8

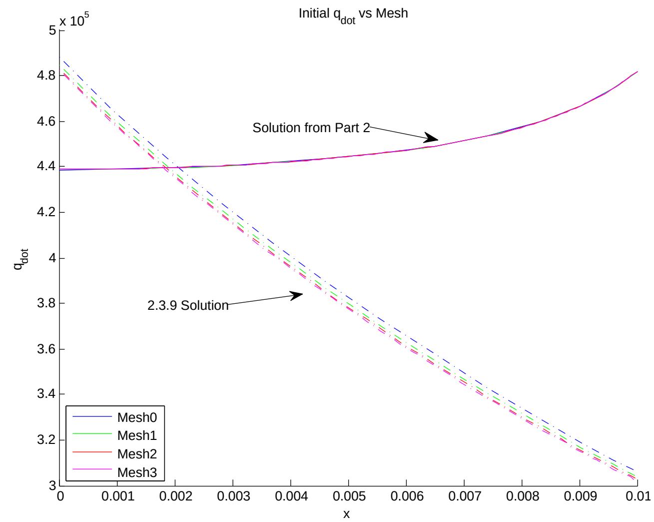

Figure 6: Solution For 2.3.9

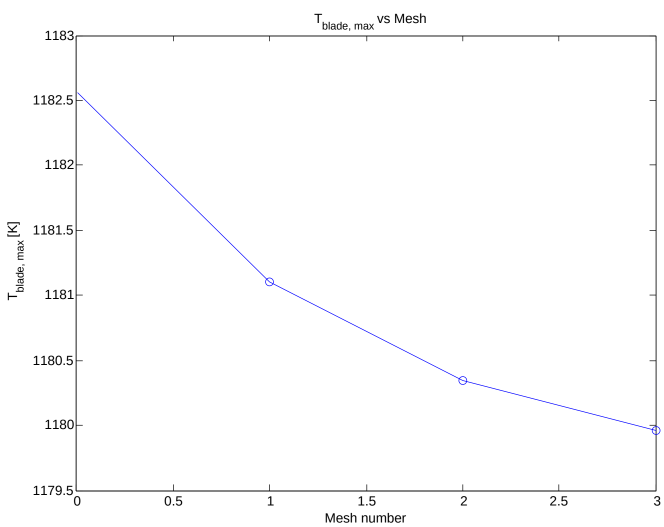

Figure 7: Maximum Temperature distribution as a function of mesh resolution

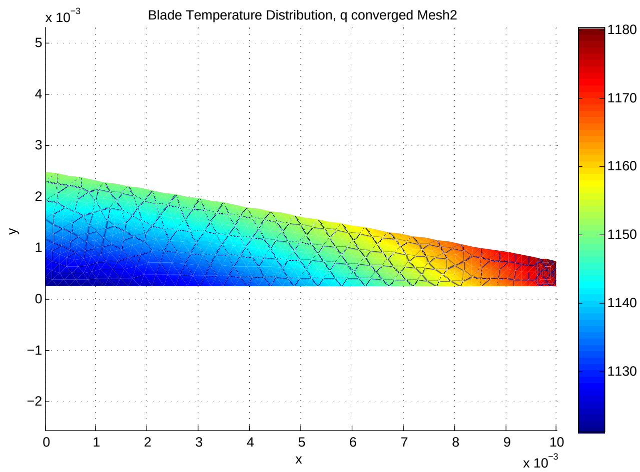

Figure 8: Temperature distribution for mesh 0

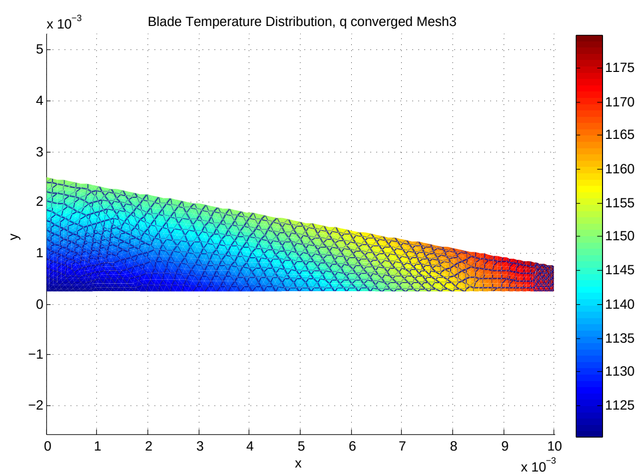

Figure 9: Temperature distribution for mesh 3

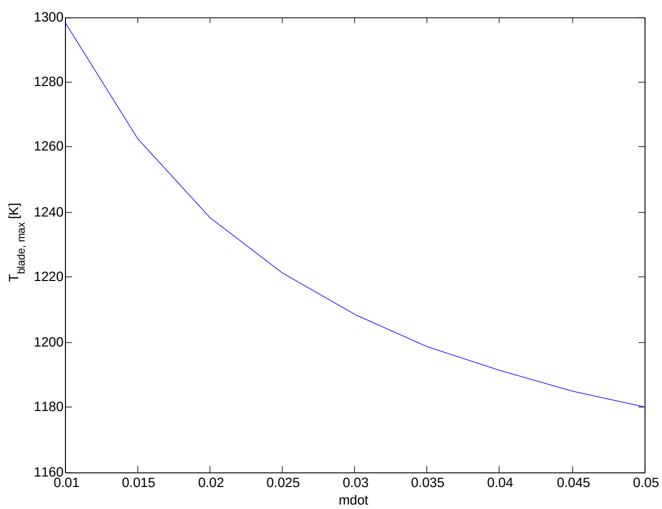

Figure 10: Max temperature on blade vs mass flow rate