MIT 16.90 Spring 2014: Solution Set 2

Qiqi Wang, Karen Willcox, Vikram Garg

Solution 2. 1 *Reading Assignment*

- See the notes for solutions.

Solution 2. 2 *Accuracy and stability analysis*

1. To check if the scheme is zero stable, we plug in $v^n = v^0 z^n$ into the scheme with $\Delta t = 0$:

$$z^{n+1} + 4z^n - 5z^{n-1} = 0$$

Factoring this expression, we have that

$$z^{n-1}(z^2 + 4z - 5) = 0$$

Thus, the roots of the recurrence relation are $z = 0, 1, -5$. The last solution makes the scheme NOT zero-stable. The second scheme has

$$z^n(z - 1) = 0$$

Thus, the roots of the recurrence relation are $z = 0, 1$. The scheme is zero stable.

2. The first scheme is not zero stable. It therefore has no global accuracy. Through Taylor series analysis, the second scheme has first order local accuracy. Because it is zero-stable, we use the Dalquist equivalence theorem to determine that it has first order global accuracy.

3. To check if the scheme is eigenvalue stable, we plug in $v^n = v^0 z^n$ into the scheme:

$$z^{n+1} + 4z^n - 5z^{n-1} + 4\lambda\Delta t z^n + 2\lambda\Delta t z^{n-1} = 0$$

Factoring this expression, we have that

$$z^{n-1}(z^2 + (4 + 4\lambda\Delta t)z - 5 + 2\lambda\Delta t) = 0$$

Thus

$$z = -2 - 2\lambda\Delta t \pm \sqrt{9 + 6\lambda\Delta t + 4(\lambda\Delta t)^2}$$

When $\pm$ takes $-$, $z < -1$ for all possible $\lambda\Delta t$. Therefore the scheme is always unstable.

For the second scheme, we get

$$z^{n-1}\left(\left(1 + \frac{3}{2}\lambda\Delta t\right)z^2 - z - \frac{1}{2}\lambda\Delta t\right) = 0$$

and

$$z = \frac{1 \pm \sqrt{1 + 2\lambda\Delta t + 3(\lambda\Delta t)^2}}{2 + 3\lambda\Delta t}$$

Because the inside of the square root is positive, the + eigenvalue is more dangeous (has larger magnitude).

$$z^+ - 1 = \frac{-1 - 3\lambda\Delta t + \sqrt{1 + 2\lambda\Delta t + 3(\lambda\Delta t)^2}}{2 + 3\lambda\Delta t}$$

Because

$$1 + 2\lambda\Delta t + 3(\lambda\Delta t)^2 < (-1 - 3\lambda\Delta t)^2 = 1 + 6\lambda\Delta t + 9(\lambda\Delta t)^2$$

we conclude that

$$z^+ - 1 < 0$$

and therefore the scheme is stable for all nonnegative λ.

Solution 2. 3 *Backward Euler for a nonlinear equation*

$$\frac{v^{n+1} - v^n}{\Delta t} = -(v^{n+1})^2$$

One needs to solve a quadratic equation for v^{n+1}. This can either be done analytically or via Newton-Raphson.
