MIT 16.90 Problem Set 4

Spring 2016

Solutions to be released March 9, 2016
(Will not be collected or graded)

1. **Conductive Heat Transfer & Truncation Error**

   Consider a rod with circular cross-section of area A and length L. We will assume the rod is significantly longer than its diameter so that we can model the temperature in the rod as varying in x (the distance along the rod), but not radially. The rod is initially heated such that the temperature is highest at $L/2$ and decreases towards its ends. Specifically, the temperature distribution at time $t = 0$ is,

   $$T(x, t = 0) = 20 + 100 \sin\left(\pi \frac{x}{L}\right)$$

   where the rod is from $0 \leq x \leq L$ and the temperature is in degrees Celsius. In this problem, we will consider the analytic behavior of the temperature when the heat source is no longer applied and the temperature at the ends is maintained, i.e.

   $$T(0) = T(L) = 20$$

   .

   We will also consider the error made in approximating the relevant PDE using a finite difference method.

   The conservation of energy for this one-dimensional heat transfer problem reduces to the following PDE for diffusion,

   $$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

   where ρ, c, and k are the density, thermal heat capacity, and thermal conductivity of the material.

(a) Prove that the solution to this one-dimensional heat transfer problem is,

$$T(x,t) = 20 + 100 \exp\left(-\pi^2 \frac{k}{\rho c L^2} t\right) \sin\left(\pi \frac{x}{L}\right).$$

Specifically, do this by substituting this into the governing PDE and showing that it satisfies this PDE.

Solution:

$$\begin{aligned} T_t &= -100\pi^2 \frac{k}{\rho c L^2} \exp\left(-\pi^2 \frac{k}{\rho c L^2} t\right) \sin\left(\frac{\pi x}{L}\right) \\ T_{xx} &= -100(\pi/L)^2 \exp\left(-\pi^2 \frac{k}{\rho c L^2} t\right) \sin\left(\frac{\pi x}{L}\right) \end{aligned}$$

Substituting into $\rho c T_t = k T_{xx}$ shows that the PDE is satisfied.

(b) Determine an expression in terms of the rod properties for the time, t_{cool}, at which the temperature is within 10 degrees of the final temperature ($T \leq 30°C$ for all x).

Solution: Max temp occurs in the middle of the rod, at $x = L/2$.

$$T_{max}(t) = 20 + 100 \exp\left(-\pi^2 \frac{k}{\rho c L^2} t\right)$$

For $T_{max} = 30$, solve to find t_{cool}:

$$\begin{aligned} 10 &= 100 \exp\left(-\pi^2 \frac{k}{\rho c L^2} t\right) \\ \log(0.1) &= -\pi^2 \frac{k}{\rho c L^2} t \\ t_{cool} &= -\log(0.1) \frac{\rho c L^2}{\pi^2 k} \end{aligned}$$

(c) For the remainder of this problem, the bar is made of aluminum with the following properties,

- $L = 20$ cm,
- $\rho = 2.7$ g/cm^3, $c = 0.90$ J/g-$°C$, $k = 167$ W/m-$°C$

Consider the following finite difference discretization

$$\rho c \frac{T_i^{n+1} - T_i^n}{\Delta t} = k \delta_x^2 T_i^n$$

Develop a short Matlab script that implements this discretization to determine the evolution of the temperature in the aluminum rod starting from the given initial condition and running until a final time of $t = t_{cool}$. Calculate the mean of the absolute value of the error at the nodes at the final time, i.e.,

$$\text{Error} = \frac{1}{N_x + 1} \sum_{i=1}^{N_x+1} |T_i^{N_t} - T(x_i, t_{cool})|$$

where N_x is the number of spatial divisions (i.e. $N_x \Delta x = L$) and N_t is the final temporal iteration index (i.e. $N_t \Delta t = t_{cool}$). Fill in the error in following table:

	$N_x = 10$	$N_x = 20$	$N_x = 40$	$N_x = 80$
$N_t = 50$				
$N_t = 100$				
$N_t = 200$				
$N_t = 400$				
$N_t = 800$				
$N_t = 1600$				
$N_t = 3200$				

Note: you should see some very large error for some of these combinations.

Solution:

Any mean error greater than 1000 recorded as ∞.

	$N_x = 10$	$N_x = 20$	$N_x = 40$	$N_x = 80$
$N_t = 50$	0.1970	∞	∞	∞
$N_t = 100$	0.0435	∞	∞	∞
$N_t = 200$	0.0330	0.0516	∞	∞
$N_t = 400$	0.0712	0.0115	∞	∞
$N_t = 800$	0.0903	0.0086	0.0132	∞
$N_t = 1600$	0.0998	0.0187	0.0029	∞
$N_t = 3200$	0.1046	0.0237	0.0022	0.0033

See code below:

```
function [mean_err, dtmax] = ps4_1c(Nx,Nt)

% Set bar parameters
L = 0.2; % m
rho = 2700; % kg/m^3
c = 900; % J/kg-C
k = 167; % W/m-C

% Determine temporal coefficient
Ct = pi^2*k/rho/c/L^2;

% Calculate tcool
tcool = -log(0.1)/Ct;

% Set grid size
x = linspace(0,L,Nx+1);
dx = L/(Nx);

dt = tcool/(Nt);

% Set initial condition
T = 20 + 100*sin(pi/L*x);
for n = 1:Nt,
    t = n*dt;
    Txx = (T(3:Nx+1) - 2*T(2:Nx) + T(1:Nx-1))/dx^2;
    T(2:Nx) = T(2:Nx) + k*dt/rho/c*Txx;
       Tex = 20 + 100*exp(-Ct*t)*sin(pi/L*x);
       plot(x,T,'b*',x,Tex,'r*');
       xlabel('x');
       ylabel('T');
       axis([0,L,20,120]);
       drawnow;
end

% Calculate error
Tex = 20 + 100*exp(-Ct*t)*sin(pi/L*x);
Terr = T - Tex;
mean_err = mean(abs(Terr));


% Calculate max dt
dtmax = 0.5*dx^2/(k/rho/c);

end
```

(d) The very large errors for some combinations are because the Δt chosen is not eigenvalue stable for the chosen Δx. Based on your results, how do you think the maximum eigenvalue stable timestep scales with the mesh size? Specifically, does the maximum timstep scale linearly with the mesh size, i.e. $\Delta t_{\text{max}} = K\Delta x$ where K is some constant. If not, based on your results, can you determine a power of Δx that the maximum timestep scales with?

Solution:

The maximum stable timestep scales with Δx^2.

To see this, consider the threshold for which the scheme becomes unstable.

e.g. Consider $N_x = 20$. From our results, the first stable result is when $N_t = 200$ (it's actually more like 167, but lets stick to the table). If we then increase to $N_x = 40$ (decreasing Δx by a factor of 2), the first stable result is for $N_t = 800$ (decreasing Δt by a factor of 4).

Thus, we conclude that $\Delta t_{max} \propto \Delta x^2$.