16.90 Pset #6

Alan Wang

$$\frac{dU}{dt} + v\frac{dU}{dx} = cU$$

a.) Use Fourier Series w/ periodicity over Length L

$$U(x,t) = \sum_{m=-\infty}^{\infty} \hat{U}_m(t) e^{ik_m x} \quad \text{where } k_m = \frac{2\pi m}{L}$$

↳ plug into PDE → $\frac{d}{dt}\left[\sum_{m=-\infty}^{\infty} \hat{U}_m(t) e^{ik_m x}\right] + v\frac{d}{dx}\left[\sum_{m=-\infty}^{\infty} \hat{U}_m(t) e^{ik_m x}\right] = c\sum_{m=-\infty}^{\infty} \hat{U}_m(t) e^{ik_m x}$

$$\hookrightarrow \sum_{m=-\infty}^{\infty} \frac{d\hat{U}_m}{dt} e^{ik_m x} + v\sum_{m=-\infty}^{\infty} ik_m \hat{U}_m e^{ik_m x} = c\sum_{m=-\infty}^{\infty} \hat{U}_m e^{ik_m x}$$

$$\Rightarrow \sum_{m=-\infty}^{\infty}\left[\frac{d\hat{U}_m}{dt} + (vik_m - c)\hat{U}_m\right] e^{ik_m x} = 0$$

Utilize orthogonality of different Fourier modes over L: $\int_0^L e^{-ik_n x} e^{ik_m x}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$

$$\hookrightarrow \sum_{m=-\infty}^{\infty} \int_0^L \left[\frac{d\hat{U}_m}{dt} + (vik_m - c)\hat{U}_m\right] e^{ik_m x} e^{-ik_n x}\, dx = 0$$

$$\Rightarrow \frac{d\hat{U}_n}{dt} + (vik_n - c)\hat{U}_n = 0 \Rightarrow \boxed{\hat{U}_n(t) = \hat{U}_n(0)\, e^{-ivk_n t} e^{ct}}$$

The solutions will be bounded for $\boxed{c \leq 0}$

b.) $$\frac{U_j^{n+1} - U_j^n}{\Delta t} + v\frac{U_j^n - U_{j-1}^n}{\Delta x} = cU_j^n \qquad \text{FTBS}$$

$$\boxed{CFL \equiv \frac{|v|\Delta t}{\Delta x} \leq 1}$$

c.) Periodic BC's of convection problem w/ constant velocity v so we have a circulant matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_N & \cdots & & \\ \vdots & & & \\ a_2 & & & a_1 \end{pmatrix} \qquad a_1 = \frac{v}{\Delta x} \quad a_N = \frac{-v}{\Delta x}$$

Use formula $\lambda_n = \sum_{j=1}^{N} a_j e^{i2\pi(j-1)\frac{n}{N}}$

$e^{i2\pi n} = 1$ b/c n is integer

$$\hookrightarrow \lambda_n = \frac{v}{\Delta x} - \frac{v}{\Delta x} e^{i2\pi(N-1)\frac{n}{N}} = \frac{v}{\Delta x} - \frac{v}{\Delta x} e^{i2\pi n} e^{-i2\pi\frac{n}{N}}$$

$$\hookrightarrow \lambda_n = \frac{v}{\Delta x}\left(1 - e^{-i2\pi\frac{n}{N}}\right) = \frac{v}{\Delta x}\left(1 - \left[\cos\left(2\pi\frac{n}{N}\right) - i\sin\left(2\pi\frac{n}{N}\right)\right]\right)$$

$$\lambda_n \Delta t = \frac{v\Delta t}{\Delta x}\left(1 - \left[\cos\left(2\pi\frac{n}{N}\right) - i\sin\left(2\pi\frac{n}{N}\right)\right]\right) \Rightarrow \text{back}$$

c.) $\lambda_n \Delta t = \frac{v\Delta t}{\Delta x}\left[1 - \left(\cos(2\pi\frac{n}{N}) - i\sin(2\pi\frac{n}{N})\right)\right]$

For Forward Euler need $\lambda_n \Delta t$ to be inside unit circle at -1

$$\lambda_n \Delta t = \frac{v\Delta t}{\Delta x}\left[1 - e^{-i\alpha}\right]$$ where α is $[0, 2\pi)$ based on ratio $\frac{n}{N}$ where $n = 0, 1, 2 \ldots N-1$

$$\hookrightarrow \lambda_n \Delta t = \frac{v\Delta t}{\Delta x}\left[1 - e^{-i\alpha}\right] \le e^{i\theta} - 1$$

$$\frac{v\Delta t}{\Delta x} \le \frac{e^{i\theta} - 1}{1 - e^{-i\alpha}} \Rightarrow \boxed{\Delta t \le \frac{\Delta x}{v}\left(\frac{e^{i\theta}-1}{1-e^{-i\theta}}\right)}$$

d.) $CFL \le 1 \Rightarrow \lambda_n \Delta t \overset{max}{=} 1 - e^{-i\alpha} \le e^{i\theta} - 1$

Yes, it is possible for the scheme to be unstable while satisfying the CFL condition.

e.) No, it is not possible.