MIT 16.90 Problem Set 3

Spring 2016

Due March 2, 2016

1. **Implicit Time Stepping Schemes for Temperature Evolution**

   Recall the temperature evolution of the sauce prepared by Professor Radovitzky considered previously in Problem Set 1. The differential equation

   $$\frac{du}{dt} = f(u) = -\lambda u$$

   governs the evolution of a non-dimensional temperature $u(t)$ (where $\lambda = \frac{Ah}{\rho V c}$ in terms of physical constants). Previously, we considered using two explicit schemes in order to advance our solution (namely Forward Euler and the Midpoint Method). Now, we consider the use of implicit methods in advancing the numerical solution for this same problem.

   (a) Derive an equation for the numerical solution at time step $n+1$, given by v^{n+1}, in terms of the numerical solution at the previous time step v^n for the Backward Euler method. Repeat also for the implicit trapezoidal method.

   (b) Implement the Backward Euler scheme derived in part (a) for $\lambda = 0.5$ from time $t = 0$ to $t = T = 10s$ with initial condition $u(0) = v^0 = 1$. Attach your code, along with a plot of the analytical solution ($u(t) = e^{-\lambda t}$) and numerical solution from Backward Euler using two different time steps of your choice. Pick one small time step and one larger time step and note your selections.

   (c) Lastly, why would one choose to use an implicit method over an explicit method? What kind of solutions could be studied using implicit methods (transient vs. steady state)?

2. **Nonlinear Pendulum with Implicit Scheme and Newton-Raphson**

   Consider the nonlinear pendulum problem studied in class. The equation of motion for the pendulum is given by

   $$\theta_{tt} + \frac{g}{L} sin(\theta) = 0$$

   In class, we linearized the equation by assuming that $\sin\theta \approx \theta$, and obtained a linear system of the form $u_t = Au$.

Let us now consider the nonlinear case. We may define

$$\theta_t = \omega$$

so that

$$\omega_t + \frac{g}{L}\sin\theta = 0$$

resulting in a system of the form

$$\frac{du}{dt} = f(u)$$

where

$$u = \begin{pmatrix} \omega \\ \theta \end{pmatrix} \quad f(u) = \begin{pmatrix} -\frac{g}{L} sin\theta \\ \omega \end{pmatrix}$$

In this problem, we consider the use of the Backward Euler method in solving this system. This results in a nonlinear problem, as we require the value of $f(u^{n+1})$, so we must apply the Newton-Raphson method.

(a) Derive an equation for the residual vector $R(w)$, as well as an equation for the Jacobian matrix $\frac{\partial R}{\partial w}$ for the Backward Euler method.

(b) Apply Newton-Raphson in order to solve the nonlinear pendulum problem for $t = 0$ to $t = 10s$, using $\Delta t = 0.01s$. Use the initial condition $\omega = 0$ and $\theta = \pi/4$. For physical constants, use the values $g = 9.8m/s^2$ and $L = 1m$. Lastly, for each Newton-Raphson iteration, use a tolerance of 1e-12. Attach a plot of θ versus time, along with a printout of your code.

(c) What do you notice about the behavior of your solution? Is Backward Euler a good method for this pendulum problem?