MIT 16.90 Spring 2013: Solution Set 2

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Due: Friday Feb 22, in class

Solution 2. 1 *Reading Assignment*

- See the notes for solutions.

Solution 2. 2 *Accuracy and stability analysis*

1. To check if the scheme is zero stable, we plug in $v^n = v^0 z^n$ into the scheme with $\Delta t = 0$:
$$z^{n+1} - \frac{3}{2}z^n + \frac{1}{2}z^{n-1} = 0$$
Factoring this expression, we have that
$$z^{n-1}(z^2 - 3/2z + 1/2) = z^{n-1}(z - 1/2)(z - 1) = 0$$
Thus, the roots of the recurrence relation are $z = 0,\ 1/2,\ 1$. Since all of the roots satisfy $|z| < 1$, the scheme is zero stable.

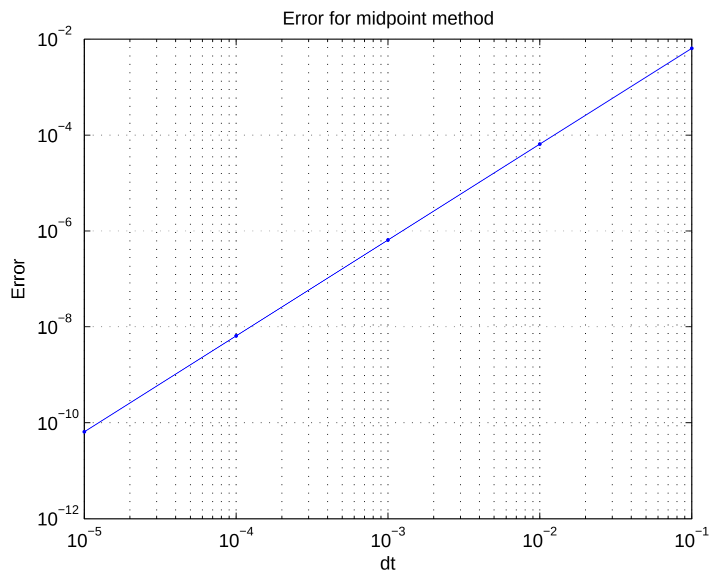
To determine the most accurate scheme, we compute the truncation error
$$\begin{aligned}\tau &= \alpha_1 u^n + \alpha_2[u^n - \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \mathcal{O}(\Delta t^3)] \\ &+ \beta_1 \Delta t u_t^n - [u^n + \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \mathcal{O}(\Delta t^3)]\end{aligned}$$
We would like to cancel as many terms in τ as possible. This results in the following system of equations:
$$\begin{array}{rcccccccc} u^n: & \alpha_1 & + & \alpha_2 & & & - & 1 & = & 0 \\ \Delta t u_t^n: & & - & \alpha_2 & + & \beta_1 & - & 1 & = & 0 \\ \Delta t^2 u_{tt}^n: & & & \frac{\alpha_2}{2} & & & - & \frac{1}{2} & = & 0 \end{array}$$
We solve this system and obtain the coefficients $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 2$, leading to the scheme
$$v^{n+1} = v^{n-1} + 2\Delta t F(v^n)$$
This is the familiar midpoint method.

To check if the resulting scheme is zero stable, we factor the recurrence relation

$$z^{n+1} - z^{n-1} = z^{n-1}(z^2-1) = z^{n-1}(z+1)(z-1) = 0$$

The roots are $z = 0,\ -1,\ 1$, so the scheme is zero stable.

The exact solution at $t = 1$ is e^{-1}. The error between the exact solution and the result using the scheme derived above is shown below. Note that the slope (in log-log space) is two, which is consistent with the truncation error we performed earlier.



2. For the second scheme, the recurrence relation can be factored as

$$z^n(z-1) = 0$$

The roots are $z = 0,\ 1$ and satisfy $|z| < 1$, so the scheme is zero stable.

The most accurate scheme is determined by considering the truncation error:

$$\begin{aligned} \tau &= \alpha_1 u^n + \beta_1 \Delta t u_t^n \\ &+ \beta_2 \Delta t [u_t^n - \Delta t u_{tt}^n + \frac{1}{2}\Delta t^2 u_{ttt}^n + \mathcal{O}(\Delta t^3)] \\ &- [u^n + \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \mathcal{O}(\Delta t^3)] \end{aligned}$$

The resulting system is equations that determines the most accurate scheme is

$$\begin{array}{rcccccccc} u^n: & \alpha_1 & + & & & & - & 1 & = & 0 \\ \Delta t u_t^n: & & & \beta_1 & + & \beta_2 & - & 1 & = & 0 \\ \Delta t^2 u_{tt}^n: & & - & \beta_2 & & & - & \frac{1}{2} & = & 0 \end{array}$$

We solve this system and obtain the coefficients $\alpha_1 = 1$, $\beta_1 = 3/2$, $\beta_2 = -1/2$, leading to the scheme

$$v^{n+1} = v^n + \frac{3}{2}\Delta t F(v^n) - \frac{1}{2}\Delta t F(v^{n-1})$$

This is the two-step Adams Bashforth method.

The resulting scheme is zero stable, and the analysis follows from the zero stability of the scheme given in the first problem set.

The error between the exact solution and the result using the scheme derived above is shown below. Note that the slope (in log-log space) is two, which is consistent with the truncation error we performed earlier.