Problem Set #1

Reading Assignment

- Unit I: 1. Introduction to Numerical Methods for ODEs.
- Unit I: 2. Convergence of Numerical Methods.
- Unit I: 3. Accuracy of Numerical Methods.

Problem 1. 1 *Truncation error analysis*

Consider the following numerical integration schemes:

1. $$u^{n+1} = 2u^n - u^{n-1} + \Delta t F(u^n)$$

2. $$u^{n+1} = u^n + \frac{1}{4}\Delta t F(u^n) + \frac{3}{4}\Delta t F(u^{n-1})$$

- Determine the leading term of the local truncation error and identify the local order of accuracy.
- For the ODE
$$\frac{du}{dt} = c - u\ , \qquad 0 \leq t \leq 1$$
with conditions $c = u = 0$ for $t < 0$, and $c = 1$ for $0 \leq t \leq 1$. Use numerical experiments to determine the global order of accuracy of these two schemes.

2 Solutions

Taylor expansion of necessary factors

$$
\begin{aligned}
u^{n+1} &= u^n + \Delta t \left.\frac{du}{dt}\right|_{t^n} + \frac{1}{2}\Delta t^2 \left.\frac{d^2u}{dt^2}\right|_{t^n} + O(\Delta t^3) \\
u^{n-1} &= u^n - \Delta t \left.\frac{du}{dt}\right|_{t^n} + \frac{1}{2}\Delta t^2 \left.\frac{d^2u}{dt^2}\right|_{t^n} + O(\Delta t^3) \\
F(u^n) &= F^n \\
F(u^{n-1}) &= \left.\frac{du}{dt}\right|_{t^{n-1}} = \left.\frac{du}{dt}\right|_{t^n} - \Delta t \left.\frac{d^2u}{dt^2}\right|_{t^n} + O(\Delta t^2)
\end{aligned}
$$

2.1 Scheme 1

$$
\begin{aligned}
e &= u^{n+1} - 2u^n + u^{n-1} - \Delta t F(u^n) \\
&= u^n + \Delta t \left.\frac{du}{dt}\right|_{t^n} + \frac{1}{2}\Delta t^2 \left.\frac{d^2u}{dt^2}\right|_{t^n} + O(\Delta t^3) \\
&\quad - 2u^n \\
&\quad + \left(u^n - \Delta t \left.\frac{du}{dt}\right|_{n} + \frac{1}{2}\Delta t^2 \left.\frac{d^2u}{dt^2}\right|_{t^n} + O(\Delta t^3)\right) \\
&\quad - \Delta t F^n \\
&= (u^n - 2u^n + u^n) + \Delta t(F^n - F^n - F^n) + O(\Delta t^2) \\
&= -\Delta t F^n \\
&= o(\Delta t)
\end{aligned}
$$

$$
e = O(\Delta t^{p+1}) = O(\Delta t) \quad \Rightarrow \quad p = 0 \tag{1}
$$

The local truncation error is $O(\Delta t)$ and the local order of accuracy is $p = 0$.

2.2 Scheme 2

$$
\begin{aligned}
e &= u^{n+1} - u^n - \frac{1}{4}\Delta t \left(F^n + 3F^{n-1}\right) \\
&= u^n + \Delta t \left.\frac{du}{dt}\right|_{t^n} + \frac{1}{2}\Delta t^2 \left.\frac{d^2u}{dt^2}\right|_{n} + O(\Delta t^3) \\
&\quad - u^n \\
&\quad - \frac{\Delta t}{4}\left(F^n + 3F^n - 3\Delta t \left.\frac{d^2u}{dt^2}\right|_{t^n} + O(\Delta t^2)\right) \\
&= (u^n - u^n) + \Delta t\left(F^n - \frac{1}{4}4F^n\right) + \Delta t^2\left(\frac{1}{2}\frac{d^2u}{dt^2} + \frac{3}{4}\frac{d^2u}{dt^2}\right) \\
&= O(\Delta t^2)
\end{aligned}
$$

$$
e = O(\Delta t^{p+1}) = O(\Delta t^2) \quad \Rightarrow \quad p = 1 \tag{2}
$$

The local truncation error is $O(\Delta t^2)$ and the local order of accuracy is $p = 1$

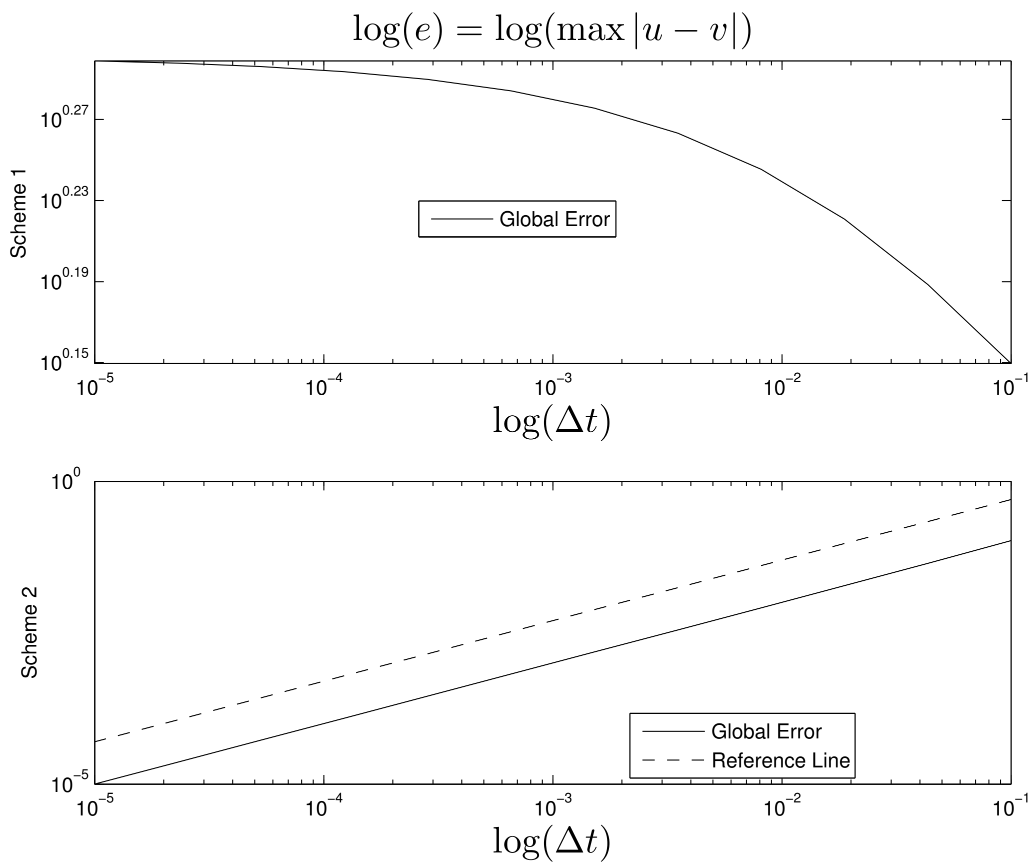

Figure 1: Log-Log Plots of Global Error with Δt

2.3 Numerical Experiments

Figure 1 shows the maximum absolute global error as a function of the time step Δt for schemes 1 and 2. Scheme 2 has an order of accuracy of 1, as you'd expect, the global error is proportional to Δt. Scheme 1 has rather large errors and the error is basically constant for small Δt (i.e. $p = 0$).

3 Code Example

```matlab
clear all; close all; clc;


T = 1.0;
u0 = 0;
c = 1;

%Define ODE
F = @(u) c - u;

%True ODE solution
u = @(t) 1 - exp(-t);

dt_s = logspace(-1, -5, 12);

e1 = zeros( size(dt_s) );
e2 = zeros( size(dt_s) );

for i = 1:length(dt_s);
    dt = dt_s(i);

    t = -dt:dt:1;
    N = length(t);

    v1 = zeros(1,N); v2 = zeros(1,N);
    F1 = zeros(1,N); F2 = zeros(1,N);

    v1(1) = u0; v2(1) = u0;  % j=1 is at t=-Δt, before ode starts
    v1(2) = u0; v2(2) = u0;  % b/c scheme 1 needs a previous step
    F1(1) = 0 - u0; F2(1) = 0 - u0; % Remember c = 0 for t < 0
    F1(2) =  F(u0); F2(2) =  F(u0);

    for j = 2:N-1;
        v1(j+1) = 2*v1(j) - v1(j-1) + dt*F1(j) ;
        v2(j+1) = v2(j) + .25*dt*( F2(j) + 3*F2(j-1) );

        F1(j+1) = F( v1(j+1) );
        F2(j+1) = F( v2(j+1) );
    end;

    % Use max absoulte error as measure of global error
    % Other measures are possible
    e1(i) = max(abs( u(t) - v1)); e2(i) = max(abs( u(t) - v2));

end;
```