Complex Analysis

zyl

Contents

Chapter 1

Preliminary

1.1 Complex numbers

IMAGINARY NUMBER

Let i be a number such that

$$i^2 = -1,$$

we call it *imaginary unit*. A *complex number* z is a number of the form $z = x + iy$ where i is the imaginary number defined previously and x and y are real numbers. Hence it is a vector space over $\mathbb{R}$, of dimension 2. We call x of z the real part, denoted by $x = \operatorname{Re} z$, and similarly, we call y the imaginary part of z, denoted by $y = \operatorname{Im} z$.

SUM, DIFFERENCE, MULTIPLICATION, INVERSE

$$\begin{aligned} z &:= x + iy \\ w &:= u + iv \\ z + w &= (x+u) + i(y+v) \\ z - w &= (x-u) + i(y-v) \\ zw &= (xu - yv) + i(xv + yu) \\ 1/z &= \frac{x - iy}{x^2 + y^2} \end{aligned}$$

WARNING When taking inverse, we require it is not zero.

ABSOLUTE VALUE/ MODULUS

The *absolute value* (or *modulus*) of a complex number $z = x + iy$ is defined by

$$|z| = \sqrt{x^2 + y^2}$$

CONJUGATION

The *conjugation* of $z = x + iy$ is defined by $\bar{z} = x - iy$. Easily finding that

$$\bar{\bar{z}} = z.$$

Especially, we have

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

EULER'S IDENTITY

Surprisingly, we have beautiful *Euler's identity*

$$e^{i\pi} + 1 = 0$$

It is a special case of *Euler's fomula*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Here, we require θ is a real number.

POLAR FORM

By the fact every complex number z of absolute value 1 is of the form $\cos\theta + i\sin\theta$ uniquely up to a multiple of 2π, we have for every z,

$$z = \rho e^{i\theta}, \quad \rho = |z|.$$

It is called the *polar form* of z.

Proposition 1.1. *Let $z \neq 0$, then, for every $n \in \mathbb{N}$, z has n roots of order n, i.e., n different numbers $z_0, \ldots, z_{n-1}$ such that*

$$z_j^n = z, \quad j = 0, \ldots, n-1$$

Proof. Clearly, any root of order n of z is nonzero with a polar form

$$w = re^{it}$$

with $z = \rho e^{i\theta}$. If $w^n = z$, then $r = \rho^{1/n}$, $nt = \theta \pmod{2\pi}$. i.e.

$$\boxed{t = \frac{\theta}{n} + \frac{2k\pi}{n}}, \quad k \in \mathbb{Z}$$

□

Exercise 1

Compute

$$\sum_{j=0}^{n-1} z_j^m, \quad m \in \mathbb{N}$$

1.2 Different views of complex numbers

algebraic view

There are different viewpoints of complex number, algebraically, it is isomorphic to the algebra

$$\mathbb{R}[x]/(x^2+1).$$

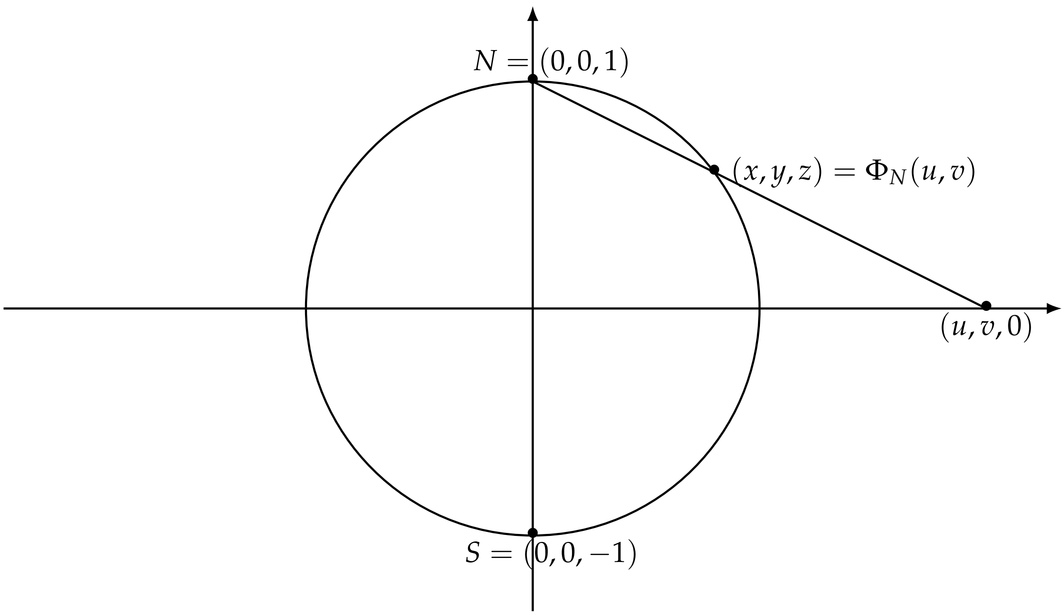
In this view, every number can uniquely as $a + bx$ where $a, b \in \mathbb{R}$. From this, we can induce that conjugation map $\mathbb{C} \to \mathbb{C}, z \mapsto \bar{z}$ is a field isomorphism.
Or we can embedded it into the matrix algebra,by

$$z = x + iy \mapsto \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

geometric view

STEREOGRAPHIC PROJECTION

The stereographic projection Φ_N is defined by the following graph.



The explicit formula of Φ_N is given by:

$$\Phi_N(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$
$$\Phi_N^{-1}(z) = \left(\frac{z + \bar{z}}{|z|^2 + 1}, \frac{z - \bar{z}}{i(|z|^2 + 1)}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

EXTENDED PLANE

$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the *extended plane*. The intersection of a plane (*nontangent*) with S^2 is a circle. The ordinary projection of a circle on S^2 on $\mathbb{C}$ is **NOT**a circle in general. However,using the stereographic projection, we have

Exercise 2

1. There is a one-to-one correspondence between circles on S^2 and lines and circles on $\mathbb{C}$.
2. Let C be a circle on S^2. Then $\Phi_N(C \setminus \{(0,0,1)\})$ is a circle iff $(0,0,1) \notin C$ and is a line otherwise.

Summarize: Via the map Φ_N^{-1}, the point at ∞ of the extended plane $\widehat{\mathbb{C}}$ should be seen as any other point of $\mathbb{C}$. Furthermore, there is no difference between lines and circles of $\mathbb{C}$. A line is a circle whose image under Φ_N^{-1} goes through $(0,0,1)$.

1.3 *Inequality*

TRIANGLE INEQUALITY

For arbitrary complex numbers, we have the following inequality

$$|z + w| \leq |z| + |w|.$$

From this ,we have *triangle inequality*,

$$|z_1 - z_3| \leq |z_1 - z_2| + |z_2 - z_3|.$$

Thus,

$$|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$$
$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$$

From this, we have

$$\left|a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n\right| \leq |a_0| + |a_1| |z| + |a_2| |z|^2 + \dots + |a_n| |z|^n$$

IDENTITY

$$\frac{w^n - z^n}{w - z} = w^{n-1} + z w^{n-2} + \dots + z^{n-1}$$

Then

$$\left|\frac{w^n - z^n}{w - z}\right| \leq n r^{n-1}, \quad \text{if } |w|, |z| < r$$

Chapter 2

Analytic functions

2.1 Power series

POWER SERIES

A *power series* is a series of function of the form

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}$$

Proposition 2.1. *Let $\rho = \sup\{r \in [0, +\infty) \mid \sum_{n=0}^{\infty} |a_n| r^n < \infty\}$, then*

(i) For all $r < \rho$, the series $\sum_{n=0}^{\infty} a_n z^n$ converges normally on the disc $B_r(0)$.

(ii) the series $\sum_{n=0}^{\infty} a_n z^n$ diverges for $|z| > \rho$.

WARNING

1. ρ exists, but can be 0, finite or infinite.

2. the case $|z| = \rho$ is uncertain.

3. the sum is continuous inside the disk $B_\rho(0)$.

RADIUS OF CONVERGENCE

We call such ρ the *radius of convergence*, denoted by R.C. The closed disc $\overline{B_\rho(0)}$ is called the *disc of convergence.*

Lemma 2.2 (Abel). *Let r, r_0 be real numbers such that $0 < r < r_0$. If there exists $M > 0$ such that*

$$|a_n| \, r_0^n \leq M, \quad \text{for all } n \geq 0$$

Then $\sum_{n=0}^{\infty} a_n z^n$ converges normally for $z \in \overline{B_r(0)}$.

Proof. For $|z| \leq r$,

$$|a_n z^n| \leq |a_n| r^n \leq M \left(\frac{r}{r_0}\right)^n.$$

Then use comparison criterion for convergence with the geometric series. □

Note. Let ρ be the R.C of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g : [0, +\infty) \to [0, +\infty]$ be a function defined by

$$g(r) = \max_{|z|=r} |f(z)|$$

Then $g(r) < \infty$ when $r < \rho$ and $g(r) = \infty$ when $r > \rho$.

compute R.C

If the sequence $\left|\frac{a_{n+1}}{a_n}\right|$ has a limit ℓ, then $R = \frac{1}{\ell}$:
we look at $\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right|$ which converges to $\ell\,|z|$, and then compare with a geometric series.
Recall: $\limsup a_n = \lim_{p\to\infty} \left(\sup_{n\geq p} a_n\right)$ (always exists, finite or infinite)
Then the R.C ρ satisfies

$$\frac{1}{\rho} = \limsup |a_n|^{1/n}$$

Ex. 1 — Find the R.C ρ for

$$\sum_{n=0}^{\infty} n! z^n, \qquad \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \sum_{n=0}^{\infty} 2^{-n} z^n, \qquad \sum_{n=0}^{\infty} 9^{n^2} z^n$$

$$\sum_{n=0}^{\infty} \frac{1}{n} z^n, \qquad \sum_{n=0}^{\infty} z^n, \qquad \sum_{n=0}^{\infty} \frac{1}{n^2} z^n.$$

What happend for $|z| = \rho$ for the last 3 series.

Ex. 2 — Find the R.C ρ for

$$\sum_{n=0}^{\infty} (\log n)^2 z^n, \qquad \sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} z^n, \qquad \sum_{n=0}^{\infty} n^2 z^n, \qquad \sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$$

$$\sum_{n=0}^{\infty} n^{(-1)^n} z^n, \qquad \sum_{n=0}^{\infty} z^{n!}, \qquad \sum_{n=0}^{\infty} z^{(2^n)}.$$

Exercise 3 Sum and Products

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where R.C of f is ρ_1, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ where R.C of g is ρ_2, $R = \min\{\rho_1, \rho_2\}$. $f(z) + g(z)$ and $f(z)g(z)$ have R.C at least equal to R where

$$f(z) + g(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{with } c_n = a_n + b_n$$
$$f(z)g(z) = \sum_{n=0}^{\infty} d_n z^n \quad \text{with } d_n = \sum_{i+j=n} a_j b_j$$

solutions to exercises

2.2 Analytic functions, first step

ANALYTIC FUNCTIONS

Let $U \subset \mathbb{C}$ be an open set and

$$f : U \to \mathbb{C}$$

be a mapping. For $z_0 \in U$, we say that f is *analytic at* z_0 if there exist

(i) $r > 0$ such that the disk $B_r(z_0)$ is included in U

(ii) A power series $\sum_{n=0}^{\infty} a_n w^n$ of radius of convergence $\rho \geq r$, such that for all $z : |z - z_0| < r$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

We say that f is *analytic* on U if it is analytic at every point of U. (It is a local property.)
EXAMPLE Polynomials are analytic in $\mathbb{C}$.

Proposition 2.3. *The set of analytic function defined over U is an algebra over* $\mathbb{C}$. *We denote it by* $\mathcal{O}(U)$.

EXAMPLE $f(z) = z$ is analytic on $\mathbb{C}$, $\dfrac{1}{f(z)} = \dfrac{1}{z}$ is **NOT**analytic on $\mathbb{C}$.

isolated property of zeros

ISOLATED PROPERTY OF ZEROS (AND POLES)

The zeros (or poles) of an analytic function has no convergent points over $\mathbb{C}$ unless it is zero itself.

Proposition 2.4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be the sum of a power series of radius of convergence* $\rho > 0$. *If at least one of the coefficients* a_n *is nonzero, there exists* $r \in (0, \infty)$ *such that* $f(z)$ *does not vanish for* $B_r(0)^* := \{z : 0 < |z| < r\}$.

Proof. Let p be the smallest nonzero integer such that $a_p \neq 0$. Then

$$f(z) = \sum_{n \geq p} a_n z^n = z^p g(z)$$

where

$$g(z) = a_p + a_{p+1} z + \dots, \quad g(0) = a_p \neq 0$$

$g(z)$ is the sum of a convergent power series, and so it is continuous inside the convergence disc and thus in a neighbourhood of 0, $g(z)$ does not vanish, and so $f(z)$ is not zero on this neighbourhood (except possibly at 0.) □

Remark. In this proof, if $p = 0$, $f = g$ and so f is nonzero in a neighbourhood of 0. If $p > 0$, then $f(z) = z^p g(z)$ vanishes at 0, but 0 is an isolated zero of $f(z)$.
The principal of isolated series says:
If not all the a_n's are zero. There exists $r' > 0$ such that $f(z)$ does not vanish for $B_{r'}(z_0)$.

Corollary. Every analytic function on U has a unique power series expansion in the neighbourhood of each point of U.

Proof. If two analytic function $f(z)$, $g(z)$ such that $f(z) = g(z)$, then all coefficients of their power series are the same. □

continuation principle

CONNECTED

Let $U \subset \mathbb{C}$ be an open set, U is called *connected* if: whenever $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and U_1, U_2 open, then $U_1 = \emptyset$ or $U_2 = \emptyset$.

CONTINUATION PRINCIPLE

From isolated property of zeros, we can conclude that

Theorem 2.5 (Analytic Continuation Principle). *Let W be an open set of U that is also connected (it is called* domain*). Let f and g be two analytic functions on W. If f and g coincide on a subset $\Sigma \subset W$ which has an accumulation point in W. Then f and g coincide every where on W.*

Proof. It is a corollary of the isolated property of zeros. We can consider $h := f - g$ at the accumulation point. Let

$$A = \{b \in W \mid \exists\, \Sigma_b, \text{a neighbourhood of } b, h|_{\Sigma_b} = 0\}$$

Then A is open by definition and close by continuity, A is nonempty, hence $h(z)$ is zero on its connected component of A. □

2.3 Derivative of analytic function

DERIVATIVE

We start from a proposition and also a definition.

Proposition 2.6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series of R.C ρ and let $f'(z) = \sum_{n\geq 1} n a_n z^{n-1}$. The R.C of $f'(z)$ is ρ and for $|z| < \rho$, we have*

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

$f'(z)$ is called the *derivative* of $f(z)$.

Proof. Let $\dfrac{1}{\rho} = \limsup_{n\to\infty} |a_n|^{1/n}$. Then

$$|na_n|^{1/n} = n^{1/n} |a_n|^{1/n}, \quad \lim_{n\to\infty} n^{1/n} = 1$$

which implies $\limsup_{n\to\infty} |na_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}$.
Then we have to show

$$\lim_{|h|\to 0} \frac{f(z+h) - f(z)}{h} - f'(z) = 0$$

To be well-defined, we focus on the open subset $\overline{B_r(0)}$ where $r < \rho$, which means, $z, w \in \overline{B_r(0)}$. Let $w = z + h$,

$$\begin{aligned}
\frac{f(w) - f(z) - hf'(z)}{h} &= \frac{\sum_{n=0}^{\infty} a_n \left(w^n - z^n - nhz^{n-1}\right)}{h} \\
&\leq \sum_{n=0}^{\infty} |a_n| \left| \sum_{i=0}^{n-1} z^{n-i} \left(w^i - z^i\right) \right| \\
&\leq \frac{|h|}{2} \sum_{n=0}^{\infty} n(n-1) |a_n| r^{n-1}
\end{aligned}$$

Since

$$\limsup_{n\to\infty} |n(n-1)a_n|^{1/n} = \rho > r$$

then $\sum_{n=0}^{\infty} n(n-1) |a_n| r^{n-1}$ converges. Thus we get

$$\lim_{h\to 0} \frac{f(w) - f(z)}{h} = f'(z).$$

□

Corollary. An analytic function on U has a derivative of any order.

Theorem 2.7. *The sum of a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

is analytic inside its disc of convergence.

To be more precise, we have

Proposition 2.8. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with R.C $\rho > 0$. Let z_0 be a point inside the disk of convergence. Then the series*

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0) w^n$$

has a radius of convergence at least equal to $\rho - |z_0|$ and we have

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n.$$

Proof. Set $r_0 = |z_0|$, $\alpha_n = |a_n|$

$$f^{(p)}(z_0) = \sum_{q=0}^{\infty} \frac{(p+q)!}{q!} a_{p+q} z_0^q$$

so that $\left| f^{(p)}(z_0) \right| \leq \sum_{q=0}^{\infty} \frac{(p+q)!}{q!} \alpha_{p+q} r_0^q$.
For $r_0 \leq r < \rho$, we have

$$\begin{aligned} |f(r - r_0)| &= \sum_{p=0}^{\infty} \frac{1}{p!} \left| f^{(p)}(z_0) \right| (r - r_0)^p \\ &\leq \sum_{p,q=0}^{\infty} \frac{(p+q)!}{p!q!} \alpha_{p+q} (r - r_0)^p r_0^q \\ &= \sum_{n=0}^{\infty} \alpha_n r^n \\ &< \infty \end{aligned}$$

Note. We have used the fact that a series of positive terms can be rearranged because it converges absolutely.

Therefore, the radius of convergence of $\sum_{n=0}^{\infty} \dfrac{1}{n!} f^{(n)}(z_0) z^n$ is at least $r - r_0$. We have chosen r arbitrary between r_0 and ρ, therefore, this radius of convergence is at least $\rho - |z_0|$.
The above calculation also use that the double series

$$\sum_{p,q=0}^{\infty} \frac{(p+q)!}{p!q!} a_{p+q} z_0^q (z - z_0)^p$$

converges absolutely. Therefore we can compute it by rearranging its terms in two ways:

- $p+q=n$:

$$\sum_{n=0}^{\infty} a_n \left(\sum_{p=0}^{n} \frac{n!}{p!(n-p)!} (z-z_0)^p z_0^{n-p} \right) = \sum_{n=0}^{\infty} a_n z^n = f(z).$$

- p, q

$$\sum_{p=0}^{\infty} \frac{(z-z_0)^p}{p!} \left(\sum_{q=0}^{\infty} \frac{(p+q)!}{q!} a_{p+q} z_0^q \right)$$

Thus,

$$f(z) = \sum_{p=0}^{\infty} \frac{f^{(p)}(z_0)}{p!} (z-z_0)^p.$$

□

2.4 *Exponential and logarithm*

COMPLEX EXPONENTIAL

The *complex exponential* is defined by

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

Remark. R.C $\rho = \infty$. $\exp(0) = 1$ and $\exp(a+b) = \exp(a)\exp(b)$.

Proposition 2.9. *(i)* $\exp : \mathbb{C} \to \mathbb{C}^*$ *is onto.*

(ii) $\exp' = \exp$.

(iii) the restriction of exp *to* $\mathbb{R}$ *is strictly increasing.*

(iv) there exists a positive real number π *such that* $\exp(i\pi/2) = i$ *and* $e^z = 1$ *iff* $z \in 2\pi i\mathbb{Z}$.

(v) exp *is periodic of period* $2\pi i$.

(vi) $t \mapsto e^{it}$ *sends the real line onto the unit circle.*

The exponential map is a group homomorphism.

$$\exp : \mathbb{C} \to \mathbb{C}^*, \quad \phi : \mathbb{R} \to S^1$$

where $\phi(t) = e^{it}$. $\ker\phi = 2\pi\mathbb{Z}$. Then we have a group isomorphism:

$$\mathbb{R}/\ker\phi \cong S^1$$

ARGUMENTS

The inverse map associate to each complex number w in S^1 a class modulo $2\pi\mathbb{Z}$ of real numbers: its *arguments*.
It is a multi-valued function, to define a single-valued function, the *principal value* of the argument (sometimes denoted $\operatorname{Arg} z$) is used. It is often chosen to be the unique value of the argument that lies within the interval $(-\pi, \pi]$.
Also, for $z \neq 0$, we define $\arg z = \arg(z/|z|)$ up to unique multiple of 2π.

COMPLEX LOGARITHM

We have to invert exp. It does not look easy because exp is not one-to-one.

Exercise 4

Let $t \in \mathbb{C}^*$, we look for z such that $e^z = t$. Note that $t = |t| \exp(i \arg t)$.

Answer of exercise 4

Look for $z = x + iy$, we have $e^z = e^x \cdot e^{iy} = |t| e^{i \arg t} \implies |t| = e^x$, that is $x = \log|t|, y = \arg t$. We wish we can write

$$|z| = \log|t| + i \arg t,$$

WARNING I will fix it.
but this is up to a multiple of 2π.

determination

DETERMINATION

We say that a continuous function f defined on a domain $U \subset \mathbb{C}$, not containing 0, is a *determination* of the logarithm on U if:

$$\forall t \in U, \exp(f(t)) = t.$$

Note. Such determination does not always exist. For instance when $U = \mathbb{C}^*$.
However, if once exists then many others exist else.

Proposition 2.10. *Let U be a domain, $0 \notin U$. If f is a determination of the* log *on U, every other determination of the* log *on U has the form $f + 2ik\pi$ for some integer k. Conversely, every $f + 2ik\pi$ is a determination of the* log *on U.*

Proof. If f and g are two determination of log,

$$h(t) = \frac{f(t) - g(t)}{2i\pi}$$

is continuous on the domain U and takes only integer values:

$$\exp(f(t)) = \exp(g(t)) = t \implies \exp(f(t) - g(t)) = 1 \implies f(t) - g(t) = 2i\pi k, \quad k \in \mathbb{Z}.$$

Then we get $h(t) = k_0$ for some integer k_0 thanks to h is continuous and U is connected.

Remark. The continuous image of a connected set is connected. And $\mathbb{Z}$ is *discrete*, which means every single point is a connected component.
The converse is clear.

□

Proposition 2.11. *There is no continuous determination of the logarithm on $\mathbb{C}^*$.*

Proof. Suppose such determination exists. Set $u(z) = \operatorname{Im} f(z)$. On S^1, define $v(\theta) = u(e^{i\theta})$, then $v : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic.

$$\begin{aligned} e^{iv(\theta)} &= e^{iu(e^{i\theta})} \\ &= e^{f(e^{i\theta})}/e^{\operatorname{Re} f(e^{i\theta})} \\ &= e^{i\theta}/\lambda \end{aligned}$$

Remark. Taking modulus, we have $\lambda = 1$.
For all θ, both θ and $v(\theta)$ are arguments for $e^{i\theta}$. Therefore, $\exists\, n(\theta) \in \mathbb{Z}$:

$$v(\theta) - \theta = 2n(\theta)\pi.$$

As v is continuous, n is a continuous function: $\mathbb{R} \to \mathbb{Z}$, so it is constant. Thus

$$v(\theta) = \theta + 2n\pi$$

As v is 2π-periodic,

$$\theta + 2\pi + 2n\pi = v(\theta + 2\pi) = v(\theta) = \theta + 2n\pi$$

Impossible. □

Proposition 2.12. *The power series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

converges for $|z| < 1$. The sum of the series

$$f(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(t-1)^n}{n}$$

is a determination of the logarithm on the open disk $|z - 1| < 1$.

Proof.

$$\begin{aligned} f'(t) &= \sum_{n=1}^{\infty} (-1)^{n-1}(t-1)^{n-1} \\ &= \sum_{n=0}^{\infty} (1-t)^n \\ &= \frac{1}{1-(1-t)} \\ &= \frac{1}{t} \end{aligned}$$

The restriction $f(t)$ to real t (on $(0,2)$) is $\log t$. The composition $\exp \circ f$ is $= t$ on $(0,2)$ and so it coincides with $g(t) = t$ on $(0,2)$ which has accumulation points. $\exp \circ f$ is the composition of 2 analytic functions and so it is analytic, thus $\exp \circ f(t) = t$ on $|t-1| < 1$ using the principle of analytic continuation. $\square$

Remark. In fact. we have a determination of the logarithm on every disc in $\mathbb{C}^*$.

Corollary. Let $t_0 \in \mathbb{C}, t_0 \neq 0$, θ_0 is an argument of $t_0, t_0 = |t_0|\, e^{i\theta_0}$. On the disc $B_{|t_0|}(t_0)$, the series

$$g(t) = \log|t_0| + i\theta_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{t-t_0}{t_0}\right)^n$$

is an analytic determination of the logarithm.

Proof. The above proposition says that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{t}{t_0} - 1\right)^n$$

converges for $\left|\frac{t}{t_0} - 1\right| < 1$ to $f(t/t_0)$. Hence

$$\exp(g(t)) = |t_0|\, e^{i\theta_0} \exp f(t/t_0) = t_0 \cdot \frac{t}{t_0} = t.$$

$\square$

principal determination of the logarithm

Proposition 2.13. *Let U_π be the complement in $\mathbb{C}$ of $\mathbb{R}_{\leq 0}$ (All complex numbers whose argument is not π.) The expression:*

$$f(t) = \log(|t|) + i \begin{cases} \arcsin \dfrac{y}{|t|} & \text{if } x \geq 0 \\ \pi - \arcsin \dfrac{y}{|t|} & \text{if } x \leq 0, y \geq 0 \\ = -\pi - \arcsin \dfrac{y}{|t|} & \text{if } x \leq 0, y \leq 0 \end{cases}$$

where $t = x + iy$ and $\arcsin : (-1,1) \to (-\pi/2, \pi/2) \nearrow$ *define a continuous function on U_π which is a determination of the logarithm on U_π.*

Proof. Continuous is easy to verify, and the reason why is it analytic is that it is a determination of logarithm, thereof it is a difference of $2k\pi$ with $g(t)$ in the previous corollary and $g(t)$ is analytic, so is f. $\square$

Corollary. If $\theta \in \mathbb{R}$, $U_\theta = \{z \in \mathbb{C} \mid \theta \text{ is not an argument of } z\}$. On U_γ, there exists a determination of the logarithm.

Proof. We get back to U_π by a rotation of angle $\phi = \pi - \theta : z \mapsto \exp -i\phi \cdot z$ sends the half line of argument θ to the half line argument π.

$$g(t) = f(e^{-i\phi t}) + i\phi$$

$$\implies \exp(g(t)) = \exp(f(e^{-i\phi}t))e^{i\phi} = e^{-i\phi} \cdot t \cdot e^{i\phi} = t$$

Then $g(t)$ is a determination of log on U_θ. □

THE SQUARE ROOT OF FUNCTION

We denote $z^{1/2}$ or $\sqrt{z}$ to be the square root of function z.
In polar coordinate, $z = re^{i\theta} = re^{i(\theta+2\pi)}$, then

$$z^{1/2} = r^{1/2}e^{i\theta} \neq z^{1/2} = r^{1/2}e^{i\theta+i\pi} = -r^{1/2}e^{i\theta/2}.$$

the same applies to m-th root of $z : z^{1/m}$.

Remark. For x real > 0, $x^{1/m} = e^{1/m \log x}$. Whenever there is a determination of the logarithm, there is also a determination of $z^{1/m}$. For instance, there is always a determination of $z^{1/m}$ over $U_\theta, \theta \in \mathbb{R}$.
$\sqrt{z}$ will defined a complex number since $w^2 = z$ has always a solution.

Exercise 5

If f analytic, nonzero over U,

$$f(a+b) = f(a)f(b)$$

Then $f(z) = e^{bz}$ for some constant $b \in \mathbb{C}$.

Chapter 3

Holomorphic functions

3.1 Holomorphic functions

HOLOMORPHIC

Here we go from definition,

Definitiion 3.1. Let U be an open set in $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is $\mathbb{C}$ differentiable at $z \in U$ if:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. If the limit exists at every $z \in U$, and $f' : U \to \mathbb{C}$ thus defined is continuous, we say that f is *holomorphic* on U.

In this case, we write

$$f(z+h) - f(z) = f'(z) \cdot h + \alpha(h) \cdot |h|$$

where $\lim_{h \to 0} \alpha(h) = 0$.
Identify $\mathbb{C}$ with $\mathbb{R}^2$, $x + iy \mapsto (x, y)$ and $f(z) \mapsto f(x, y)$.

CAUCHY RIEMANN EQUATION

$\mathbb{R}$-differentiability says:

$$f(x+k, y+l) - f(x, y) = a \cdot k + b \cdot l + \beta(k, l)\sqrt{k^2 + l^2}$$

where $\lim_{(k,l) \to (0,0)} \beta(k, l) = 0$ and $a = \dfrac{\partial f}{\partial x}(x, y)$, $b = \dfrac{\partial f}{\partial y}(x, y)$.
$\mathbb{C}$-differentiability says:

$$\frac{\partial f}{\partial x} \cdot k + \frac{\partial f}{\partial y} \cdot l = f'(z)(k + il)$$

here, $h \leftrightarrow k + il$. As they are both equal to the linear term in $f(z+h) - f(z) = f(x+k, y+l) - f(x,y)$ which implies

$$\begin{aligned} f'(z) &= \frac{\partial f}{\partial x}(x,y) \\ if'(z) &= \frac{\partial f}{\partial y} \end{aligned}$$

which implies

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \tag{3.1}$$

If $f(x,y) = P(x,y) + iQ(x,y)$

$$P, Q : \mathbb{R}^2 \to \mathbb{R}$$

then

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} \\ \frac{\partial f}{\partial y} &= \frac{\partial P}{\partial y} + i\frac{\partial Q}{\partial y} \end{aligned}$$

the equation (3.1) implies

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) = 0$$

We obtain the *Cauchy-Riemann equation*:

$$\begin{aligned} \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} &= 0 \\ \frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} &= 0 \end{aligned} \tag{3.2}$$

Proposition 3.1. *A function $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable at a point iff it is differentiable at a function of two real variables and its partial derivatives satisfies the Cauchy-Riemann equation* (3.2).

Remark. If f, g are holomorphic on U, then all the rules of differentiation apply. For example, the sum, the product, composition $\cdots$

examples

EXAMPLE Consider $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$ where

$$P = x^2 - y^2, \quad Q = 2xy$$

$$\frac{\partial P}{\partial x} = 2x = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -2y = -\frac{\partial Q}{\partial x}$$

Hence, Cauchy-Riemann equation satisfied, z^2 is holomorphic on $\mathbb{C}$.

EXAMPLE Consider $g(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$, where

$$P = e^x \cos y, \quad Q = e^x \sin y$$

$$\frac{\partial P}{\partial x} = e^x \cos y = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -e^x \sin y = -\frac{\partial Q}{\partial x}$$

hence, Cauchy-Riemann equation satisfied, e^z is holomorphic.

EXAMPLE Consider $f(z) = z\bar{z} = x^2 + y^2 = |z|^2$, where

$$P = x^2 + y^2, \quad Q = 0$$

$$\frac{\partial P}{\partial x} = 2x \neq \frac{\partial Q}{\partial y}$$

f is **NOT** $\mathbb{C}$-differentiable at any $z \neq 0$.

EXAMPLE Consider $\bar{z} = x - iy$,

$$\frac{\partial P}{\partial x} = 1, \quad \frac{\partial Q}{\partial y} = -1,$$

Cauchy-Riemann equation fails. $\bar{z}$ is **NOT** holomorphic.

HOLOMORPHIC FROM $\partial\bar{z}$

We have

$$f(z) = \text{Re}(z) = x, \quad g(z) = \text{Im}(z) = y, \quad Q = 0$$

Both are not $\mathbb{C}$-differentiable anywhere. Actually,

$$\text{Re}(z) = \frac{1}{2}(z + \bar{z}), \quad \text{Im}\, z = \frac{1}{2i}(z - \bar{z})$$

For $\mathbb{R}^2$, x, y are independent variables.

$$z = x + iy, \quad \bar{z} = x - iy$$

$z, \bar{z}$ are independent variables.

By *chain rule*,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$z = x + iy \implies \bar{z} = x - iy,$$

$$\frac{\partial \bar{z}}{\partial x} = 1, \frac{\partial \bar{z}}{\partial y} = -i$$

$$\implies \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}$$

From (3.1), f is $\mathbb{C}$-differentiable at z iff

$$\frac{\partial f}{\partial \bar{z}}(z) = 0$$

3.2 Analyticity of holomorphic functions

CAUCHY THEOREM

It is remarkable that the $\mathbb{C}$-differentiability implies analyticity. Here is the tools:

Theorem 3.2 (Cauchy). *Let f be a holomorphic function on the open disc $|z - z_0| < \rho$. The complex number*

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{it}) e^{-int} dt$$

is independent of the choice of $r < \rho$.
Furthermore, the power $\sum_{n=0}^{\infty} a_n z^n$ has R.C $\geq \rho$ and we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $|z - z_0| \leq \rho$.

Proof. We may suppose that $z_0 = 0$ and let z such that $|z| < \rho$ and $r > 0$ with $|z| < r < \rho$. We consider the function $g : [0, 1] \to \mathbb{C}$ defined by

$$g(\alpha) = \int_0^{2\pi} \frac{f\left((1-\alpha)z + \alpha re^{it}\right) - f(z)}{re^{it} - z} re^{it} dt$$

the integrand is continuous and differentiable (z fixed, and denominator $\neq 0$), hence g is differentiable.

$$\begin{aligned} g'(\alpha) = &\int_0^{2\pi} f'\left((1-\alpha)z + \alpha re^{it}\right) re^{it} dt \\ =&\frac{1}{\alpha i} f\left((1-\alpha)z + \alpha re^{it}\right) |_0^{2\pi} \\ =&0 \end{aligned}$$

which implies g is constant on $[0, 1]$.
As $g(0) = 0$, the constant is 0. In particular $g(1) = 0$ which yields

$$\int_0^{2\pi} \frac{f(re^{it}) - f(z)}{re^{it} - z} re^{it} = 0$$

$$\implies f(z) \cdot \int_0^{2\pi} \frac{re^{it}}{re^{it} - z} dt = \int_0^{2\pi} \frac{re^{it} f(re^{it}) dt}{re^{it} - z} \tag{3.3}$$

Now, for $r > |z|$:

$$\begin{aligned}\frac{re^{it}}{re^{it}-z} &= \frac{1}{1-z/(re^{it})} \\ &= 1+\frac{z}{re^{it}}+\cdots+\left(\frac{z}{re^{it}}\right)^n+\cdots\end{aligned}$$

with normal convergence for all $t \in \mathbb{R}$, we can integrate termwise:

$$\begin{aligned}\int_0^{2\pi} \frac{re^{it}}{re^{it}-z}dt &= \int_0^{2\pi} 1dt + \int_0^{2\pi} \frac{z}{re^{it}}dt + \cdots \\ &= 2\pi\end{aligned}$$

For $n > 0$:

$$\begin{aligned}\int_0^{2\pi} \left(\frac{z}{re^{it}}\right)^n dt &= \left(\frac{z}{n}\right)^n \int_0^{2\pi} e^{-int}dt \\ &= \left(\frac{z}{r}\right)^n \left(\frac{-1}{in}\right) e^{-int} \,|_0^{2\pi} \\ &=0\end{aligned}$$

$$\implies \int_0^{2\pi} \frac{re^{it}}{re^{it}-z}dt = 2\pi.$$

As $f(re^{it})$ is bounded, we can integrate termwise in the second integral.

$$2\pi f(z) = \int_0^{2\pi} \frac{re^{it}f(re^{it})}{re^{it}-z}dt = \sum_{n=0}^{\infty} z^n \int_0^{2n} \frac{f(re^{it})}{r^n e^{int}}dt = \sum_{n=0}^{\infty} a_n z^n$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The fact that a_n is independent of r is a consequence of the uniqueness of the power series expansion of f. □

3.3 Cauchy's integral formula

If f is a holomorphic function on a disc $B_((z_0)r)$, then for all z in this disc, we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0+re^{it})re^{it}}{z_0+re^{it}-z}dt$$

which takes the equivalent form:

$$f(z) = \frac{1}{2\pi i} \int_{|z_0-w|=r} \frac{f(w)}{w-z}dw$$

From the previous calculation of a_n, we have

$$\frac{1}{n!}f^{(n)}(z_0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(z_0+re^{it})}{r^n e^{int}}dt$$

or

$$f^n = \frac{n!}{2\pi i}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}dw$$

Summarize: If f is a holomorphic function on open disc $B_\rho(z_0)$. Then f is analytic on $B_\rho(z_0)$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for

$$a_n = \frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0+re^{it})e^{-int}dt$$

and $r \in (0,\rho)$.

$$f(z) = \frac{1}{2\pi i}\int_{|w-z_0|=r}\frac{f(w)}{w-z}dw$$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{|w-z|=r}\frac{f(w)}{(w-z)^{n+1}}dw$$

Corollary. If f is holomorphic on an open set U, then, for $z_0 \in U$, f has a power series expansion on the maximal disc $B_\rho(z_0)$ included in U.

For all $z \in B_\rho(z_0), f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. In particular, R.C $\geq \rho$.

Proposition 3.3 (Cauchy's inequality). *Let f be a holomorphic function on an open set U in $\mathbb{C}$. Let $z_0 \in U$ and $r > 0$ such that $B_r(z_0) \subseteq U$. For all $n \in \mathbb{N}$, we have:*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{r^n}\sup_{|w-z_0|=r}|f(w)|$$

Proof. We have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{|w-z_0|=r}\frac{f(w)}{(w-z)^{n+1}}dw$$

If $w = z_0 + re^{it}$, $dw = ire^{it}dt$ □