SOME ASPECTS OF HOMOLOGICAL ALGEBRA

CONTENTS

§1. GENERALITIES ON ABELIAN CATEGORIES

1.1. Categories

(1.1.1). Recall that a *category* consists of a non-empty class $\mathtt{C}$ of objects together with, for $A, B \in \mathtt{C}$, a set $\mathrm{Hom}(A, B)$ collectively called morphisms of A into B, and for three objects $A, B, C \in \mathtt{C}$ a function (called composition of morphisms) $(u, v) \mapsto vu$ of $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C)$ to $\mathrm{Hom}(A, C)$, which satisfy the following two axioms:

(1) the composition of morphisms is *associative*;
(2) for $A \in \mathtt{C}$, there is an element $i_A \in \mathrm{Hom}(A, A)$ (called the identity morphism of A) which is a right and left unit for the composition of morphisms. (The element i_A is then unique.)

Finally, it will be prudent to suppose that a morphism u determines its source and target. In other words, if A, B and A', B' are two distinct pairs of objects of $\mathtt{C}$, then $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(A', B')$ are disjoint sets.

(1.1.2). If $\mathtt{C}$ is a category, we define the *dual category* $\mathtt{C}^{\mathtt{op}}$ as the category with the same objects as $\mathtt{C}$, and where the set $\mathrm{Hom}(A, B)^{\mathtt{op}}$ of morphisms of A into B is identical to $\mathrm{Hom}(B, A)$, with the composite of u and v in $\mathtt{C}^{\mathtt{op}}$ being identified as the composite of v and u in $\mathtt{C}$.

Any concept or statement about an arbitrary category admits a dual concept or statement (the process of reversing arrows), which will be just as useful in the applications. Making this more explicit is usually left to the reader.

(1.1.3). Suppose we are given a category $\mathtt{C}$ and a morphism $u : A \to B$ in $\mathtt{C}$. For any $C \in \mathtt{C}$, we define a function $v \mapsto uv : \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)$ and a function $w \mapsto wu : \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$.

We say that u is a *monomorphism* or that u is *injective* (respectively, u is an *epimorphism* or u is *surjective*) if the first (respectively, the second) of the two preceding function is always injective; u is called *bijective* if u is both injective and surjective.

We call a *left inverse* (respectively, a *right inverse*) of u a $v \in \mathrm{Hom}(B, A)$ such that $vu = i_A$ (respectively $uv = i_B$); v is called the *inverse of* u if it is both a left inverse and a right inverse of u (in which case it is uniquely determined). u is called an *isomorphism* if it has an inverse. If u has a left inverse (respectively, a right inverse) it is injective (respectively surjective). Thus an isomorphism is bijective (the converse being, in general, false).

(1.1.4). The composite of two monomorphisms (respectively, epimorphisms) is monomorphism (respectively, epimorphisms), hence the composite of two bijections is a bijection; similarly the composite of two isomorphisms is an isomorphism. If the composite vu of two morphisms u, v is a monomorphism (respectively, an epimorphism), then u (respectively, v) is as well.

Although the development of such trivialities is clearly necessary, we will subsequently refrain from setting them forth explicitly, feeling it sufficient to indicate the definitions carefully.

(1.1.5). Consider two monomorphisms $u : B \to A$ and $u' : B' \to A$. We say that u' *majorizes* or *contains* u and we write $u \leqslant u'$ if we can factor u as $u'v$ where v is a morphism from B to B' (which is then uniquely determined). That is a *preorder* in the class of monomorphism with target A. We will say that two such monomorphisms u, u' are *equivalent* if each one contains the other. Then the corresponding morphisms $B \to B'$ and $B' \to B$ are inverse isomorphism.

Choose (for example, using Hilbert's all-purpose symbol τ) a monomorphism in each class of equivalent monomorphisms: the selected monomorphisms will be called *subobjects* of A. Thus a subobject of A is not simply an object of $\mathtt{C}$, but an object B, together with a monomorphism $u : B \to A$ called the *canonical injection* of B into A. (Nonetheless, by abuse of language, we will often designate a subobject of A by the name B of the corresponding object of $\mathtt{C}$.) The containment relation defines an *order* relation (not merely a preorder relation) on the class of subobjects of A.

It follows from the above that the subobjects of A that are contained in a subobject B are identified with the subobjects of B, this corredpondence respecting the natural order. (This does not mean, however, that a subobject of B is equal to a subobject of A, which would require $A = B$.)

(1.1.6). Dually, consideration of a preorder on the class of epimorphisms of A makes it possible to define the ordered class of *quotient objects* of A.

(1.1.7). Let $A \in \mathtt{C}$ and let $(u_i)_{i \in I}$ be a non-empty family of morphisms $u_i : A \to A_i$. Then for any $B \in \mathtt{C}$, the functions $v \mapsto u_i v : \mathrm{Hom}(B, A) \to \mathrm{Hom}(B, A_i)$ define a natural transformation

$$\mathrm{Hom}(B, A) \longrightarrow \prod_{i \in I} \mathrm{Hom}(B, A_i) \tag{1.1.7.1}$$

We say that the u_i define a *representation of A as a direct product of the A_i* if for any B, the preceding displayed function is bijective.

If this holds and if A' is another object of $\mathtt{C}$ represented as a product of the A'_i by morphisms $u_i : A' \to A'_i$ (the set of indices being the same), then for any family (v_i) of morphisms $v_i : A_i \to A'_i$, there is a unique morphism $v : A \to A'$ such that $u_i v = v_i u_i$ for each $i \in I$. From this we conclude that if the v_i are equivalences, this holds for v: in particular, if the v_i are identity I_{A_i} we see that two objects A, A' represented as products of the family A_i are canonically isomorphic. It is therefore natural to select among all the $(A, (u_i))$, as above, a particular system, for example using Hilbert's τ symbol that will be called the *product* of the family of objects $(A_i)_{i \in I}$. It is therefore not a simple object A of $\mathtt{C}$, but such an object equipped with a family (u_i) of morphisms to A_i, called the *canonical projections from the product* to its factors A_i. We indicate the product of the A_i (if it exists) by $\prod_{i \in I} A_i$. If I is reduced to a single element i, then the product can be identified with A_i itself.

We say that $\mathtt{C}$ is a *category with products* if the product of two objects of $\mathtt{C}$ always exists (then it holds for the product of any nonempty finite family of objects of $\mathtt{C}$). We say that $\mathtt{C}$ is a *category with infinite products* i the product of any non-empty family of objects of $\mathtt{C}$ always exists. We have seen that if there are two products, $A = \prod_{i \in I} A_i$ and $B = \prod_{i \in I} B_i$ corresponding to the same set U if indices, then a family (v_i) of morphisms A_i to B_i canonically defines a morphism v from A to B, called *product of the morphisms v_i* and sometimes denoted $\prod_{i \in I} v_i$. If the v_i are monomorphisms, so is their product. But the analogous statement for epimorphisms fails in general (as we see, for example, in the category of sheaves over a fixed topological space).

(1.1.8). Dual considerations of the preceding can be used to define the notions of a *representation of an object as a sum* of a family of objects A_i by means of morphisms $u_i : A_i \to A$ (for any $B \in \mathtt{C}$, the natural transformation

$$\mathrm{Hom}(A,B) \longrightarrow \prod_{i\in I}\mathrm{Hom}(A_i,B) \tag{1.1.8.1}$$

is bijective), of a *direct sum* $\oplus_{i\in I}A_i$, equipped with *canonical injections* $A_i \to \oplus_{i\in I}A_i$ (which, however are not necessarily monomorphisms, despite their name), and also equipped with a *sum morphism* $\oplus_{i\in I}u_i$ of a family of morphisms $u_i : A_i \to B_i$. If the u_i are epimorphisms, their sum is as well.

1.2. Functors

(1.2.1). Let $\mathtt{C}$ and $\mathtt{C}'$ be categories. Recall that a covariant functor from $\mathtt{C}$ to $\mathtt{C}'$ consists of a "function" F which associates an object $F(A) \in \mathtt{C}'$ to each $A \in \mathtt{C}$ and a morphisms $F(u) : F(A) \to F(B)$ in $\mathtt{C}'$ to each morphism $u : A \to B$ in $\mathtt{C}$, such that we have $F(I_A) = I_{F(A)}$ and $F(vu) = F(v)F(u)$. There is an analogous definition of *contravariant functor* from $\mathtt{C}$ to $\mathtt{C}'$ (which are also covariant functors from $\mathtt{C}^{\mathtt{op}}$ to $\mathtt{C}'$ or from $\mathtt{C}$ to $\mathtt{C}'^{\mathtt{op}}$). We similarly define functors of several variables (or *multifunctors*), covariant in some variables and contravariant in others. In order to simplify, we will generally limit ourselves to functors of one variable. Functors are composed in the same way as functions are, this composition is associative and "identify functors" play the role of units.

(1.2.2). Let $\mathtt{C}$ and $\mathtt{C}'$ be fixed categories, F and G covariant functors from $\mathtt{C}$ to $\mathtt{C}'$. A *functorial morphism* f from F to G (also called a "natural transformation" from F to G by some authors) is a "function" that associates a morphism $f(A) : F(A) \to G(A)$ to any $A \in \mathtt{C}$, such that for any morphism $u : A \to B$ in $\mathtt{C}$ the following diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(u)"] \arrow[d, "f(A)"'] & F(B) \arrow[d, "f(B)"] \\ G(A) \arrow[r, "G(u)"'] & G(B) \end{tikzcd} \tag{1.2.2.1}$$

commutes. Natural transformations $F \to G$ and $G \to H$ are composed in the usual way. Such composition is associative, and the "identity transformation" of the functor F is a unit for composition of natural transformations. (Therefore, if $\mathtt{C}$ is a set the functors from $\mathtt{C}$ to $\mathtt{C}'$ again form a category.) Note that the composite GF of two functors $F : \mathtt{C} \to \mathtt{C}'$ and $G : \mathtt{C}' \to \mathtt{C}''$ is, in effect, a bifunctor with respect to the arguments G and F: a natural transformation $G \to G'$ (respectively, $F \to F'$) defines a natural transformation $GF \to G'F$ (respectively, $GF \to GF'$).

(1.2.3). An *equivalence* of a category $\mathtt{C}$ with a category $\mathtt{C}'$ is a system (F, G, ϕ, ψ) consisting of covariant functors:

$$F : \mathtt{C} \longrightarrow \mathtt{C}' \quad G : \mathtt{C}' \longrightarrow \mathtt{C} \tag{1.2.3.1}$$

and of natural equivalences

$$\phi : 1_{\mathtt{C}} \longrightarrow GF \quad \psi : 1_{\mathtt{C}'} \longrightarrow FG \tag{1.2.3.2}$$

(where $1_{\mathtt{C}}$ and $1_{\mathtt{C}'}$ are the identity functors of $\mathtt{C}$ and $\mathtt{C}'$, respectively) such that for any $A \in \mathtt{C}$ and $A' \in \mathtt{C}'$, the composites

$$\begin{gathered} F(A) \xrightarrow{F(\phi(A))} FGF(A) \xrightarrow{\psi^{-1}(F(A))} F(A) \\ G(A') \xrightarrow{G(\psi(A'))} GFG(A') \xrightarrow{\phi^{-1}(G(A'))} G(A') \end{gathered} \tag{1.2.3.3}$$

are the identities of $F(A)$ and $G(A')$, respectively. Then for any pair A, B of objects of $\mathtt{C}$, the functions $f \mapsto F(f)$ from $\mathrm{Hom}(A,B)$ to $\mathrm{Hom}(F(A),F(B))$ is a bijection whose inverse is the function $g \mapsto G(g)$ from $\mathrm{Hom}(F(A),F(B))$ to $\mathrm{Hom}(GF(A),GF(B))$, which is identified with $\mathrm{Hom}(A,B)$ thanks to the isomorphisms $\phi(A) : A \to GF(A)$ and $\phi(B) : B \to GF(B)$. Equivalences between categories compose like functors. Two categories are called *equivalent* if there is an equivalence between them. Current usage will not distinguish between equivalent categories. It is important, however, to observe the difference between this notion and the stricter notion of isomorphism (which applies if we wish to compare categories that are sets). Let $\mathtt{C}$ be a non-empty set. For any pair of objects $A, B \in \mathtt{C}$, suppose that $\mathrm{Hom}(A, B)$ consists of one element. Then under the unique composition laws $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, $\mathtt{C}$ becomes a category, and two categories constructed by this procedure are always equivalent, but they are isomorphic only if they have the same cardinality. None of the equivalences of categories that we encounter in practice is an isomorphism.

1.3. Additive Categories

(1.3.1). An *additive category* is a category $\mathtt{C}$ for which is given, for any pair A, B of objects of $\mathtt{C}$ an abelian group law in $\mathrm{Hom}(A, B)$ such that the composition of morphisms is a bilinear operation. We suppose also that the sum and the product of any two objects A, B of $\mathtt{C}$ exist. It is sufficient, moreover, to assume the existence of the sum or the product of A and B exists; the existence of the other can be easily deduced and, in addition, $A \oplus B$ is canonically isomorphic to $A \times B$. (Supposing, for example, that $A \times B$ exists, we consider the morphisms $A \to A \times B$ and $B \to A \times B$ whose components are $(i_A, 0)$, respectively, $(0, i_B)$, we check that we obtain thereby a representation of $A \times B$ as a direct sum of A and B.) Finally, we assume the existence of an object A such that $i_A = 0$; we call it a *zero object* of $\mathtt{C}$. It comes to the same thing to say that $\mathrm{Hom}(A, A)$ is reduced to 0, or that for any $B \in \mathtt{C}$, $\mathrm{Hom}(A, B)$ (or $\mathrm{Hom}(B, A)$) is reduced to 0. If A and A' are zero objects, there exists an unique isomorphism of A to A' (that is, the unique zero element of $\mathrm{Hom}(A, A')$!). We will identify all zero objects of $\mathtt{C}$ to a single one, denoted 0 by abuse of notation.

(1.3.2). The dual category of an additive category is still additive.

(1.3.3). Let $\mathtt{C}$ be an additive category and $u : A \to B$ a morphism in $\mathtt{C}$. For u to be injective (respectively, surjective) it is necessary and sufficient that there not exist a non-zero morphism whose left, respectively, right, composite with u is 0. We call a *generalized kernel* of u any monomorphism $i : A' \to A$ such that morphisms from $C \to A$ which are right zero divisors of u are exactly the ones that factor through $C \to A' \xrightarrow{i} A$. Such a monomorphism is defined up to equivalence (cf. (1.1)), so among the generalized kernels of u, if any, there is exactly one that is a subobject of A. We call it *the kernel of* u and denote it by $\mathrm{Ker}\, u$. Dually we define the *cokernel of* u (which is a quotient object of B, if it exists), denoted $\mathrm{Coker}\, u$. We call *image* (respectively, *coimage*) of the morphism u the kernel of its cokernel (respectively, the cokernel of its kernel) if it exists. It is thus a subobject of B (a quotient object of A). We denote them as $\mathrm{Im}\, u$ and $\mathrm{Coim}\, u$. If u has an image and a coimage, there exists a unique morphism $\bar{u} : \mathrm{Coim}\, u \to \mathrm{Im}\, u$ such that u is the composite $A \to \mathrm{Coim}\, u \xrightarrow{\bar{u}} \mathrm{Im}\, u \to B$, the extreme morphisms being the canonical ones.

(1.3.4). A functor F from one additive category $\mathtt{C}$ to another additive category $\mathtt{C}'$ is called an *additive functor* if for morphisms $u, v : A \to B$ in $\mathtt{C}$, we have that $F(u + v) = F(u) + F(v)$. The definition for multifunctors is analogous. The composite of additive functors is additive. If F is an additive functor, F transforms a finite direct sum of objects A_i into the direct sum of $F(A_i)$.

1.4. Abelian Catgories

(1.4.1). We call an *abelian category* an additive category $\mathtt{C}$ that satisfies the following two additional conditions (which are self-dual):

(AB 1). Any morphism admits a kernel and a cokernel (cf. (1.3))

(AB 2). Let u be a morphism in $\mathtt{C}$. Then the canonical morphism $\bar{u} : \mathrm{Coim}\, u \to \mathrm{Im}\, u$ (cf. (1.3)) is an isomorphism.

In particular, it follows that *a bijection is an isomorphism*. Note that there are numerous additive categories that satisfy (AB 1) and for which the morphism $\bar{u} : \mathrm{Coim}\, u \to \mathrm{Im}\, u$ is bijective without being an isomorphism. This is true, for example, for the additive category of separated topological modules over some topological ring, taking as morphisms the continuous homomorphisms as well as for the category of *filtered* abelian groups. A less obvious example: the additive category of holomorphic fibered spaces with vector fibers over a holomorphic variety of complex dimension 1. These are some non-abelian additive categories.

(1.4.2). If $\mathtt{C}$ is an abelian category, then the entire usual formalism of diagrams of homomorphisms between abelian groups can be carried over if we replace homomorphisms by morphisms in $\mathtt{C}$, insofar as we are looking at properties of finite type, i.e., not involving infinite direct sums or products (for which special precautions must be taken–see N°5). We coontent ourselves here with indicating a few particularly important facts, referring the reader to [Buc55] for additional details.

(1.4.3). In what follows we restrict ourselves to a fixed abelian category $\mathtt{C}$. Let $A \in \mathtt{C}$. To any subobject of A there corredponds the cokernel of its inclusion (which is thereby a quotient of A), and to each quotient object of A there corresponds the kernel of its projection (which is thereby a subobject of A). We thus obtain *one-one correspondence between the class of subjects of A and the class of quotient objects of A*. This correspondence is an *anti-isomorphism* between natural order relations. Moreover, the subobjects of A form a *lattice* (therefore so do the quotient objects): if P and Q are subobjects of A, their sup is the image of the direct sum $P \oplus Q$ under the morphism whose components are the canonical injections of P and Q into A, and their inf is the kernel of the morphism of A to the product $(A/P) \times (A/Q)$, whose components are the canonical surjections to A/P and A/Q. (In accordance with usage, we indicate by A/P the quotient of A corresponding to the cokernel of the inclusion of P into A; it seems natural to use a dual notation such as $A \setminus R$ for the subobject of A that corresponds to the quotient object R. There are dual interpretations for the inf and sup of a pair of quotients objects of A.)

(1.4.4). Let $u : A \to B$ be a morphism. If A' is a subobject of A, we define the image of A' under u, denoted $u(A')$, as $\operatorname{Im} ui$, where i is the canonical injection $A' \to A$. Dually, we deine the inverse image $u^{-1}(B')$ of a quotient B' of B; it is a quotient of A. If B' is now a subobject of B, we define the inverse image of B' under u, denoted $u^{-1}(B')$, as the kernel of ju, where j is the canonical surjection $B \to B/B'$. We define dually, the direct image $u(A')$ of a quotient A' of A; it is a quotient of B. We show all the usual formal properties for these notions.

(1.4.5). Recall that a pair $A \xrightarrow{u} B \xrightarrow{v} C$ of two consecutive morphisms is said to be *exact* if $\operatorname{Ker} u = \operatorname{Im} u$; more generally, we can define the notion of an *exact sequence* of morphisms. For a sequence $0 \to A \to B \to C$ to be exact it is necessary and sufficient that for $X \in \mathtt{C}$, the following sequence of homomorphisms of abelian groups be exact:

$$0 \longrightarrow \operatorname{Hom}(X, A) \longrightarrow \operatorname{Hom}(X, B) \longrightarrow \operatorname{Hom}(X, C) \tag{1.4.5.1}$$

There is a dual criterion for $A \to B \to C \to 0$ to be exact. Finally, a necessary and sufficient condition that the sequence $0 \to A' \xrightarrow{u} A \xrightarrow{v} A'' \to 0$ be exact is that u is a monomorphism and that v is a generalized cokernel of u.

(1.4.6). Let F be a covariant functor of one abelian category $\mathtt{C}$ to another $\mathtt{C}'$. Following the terminology introduced in [CE56], we say that F is a *half exact* functor (respectively, *left exact*, respectively, *right exact*) if for any exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathtt{C}$, the corresponding sequence of morphisms $0 \to F(A') \to F(A) \to F(A'') \to 0$ is exact at $F(A)$ (respectively, exact at $F(A)$ and $F(A')$, respectively, exact at $F(A)$ and $F(A'')$). F is called an *exact functor* if it is both left exact and right exact, i.e., transforms an exact sequence of the preceding type into an exact sequence; then F transforms *any* exact sequence into an exact sequence. If F is left exact, F transforms an exact sequence $0 \to A \to B \to C$ into an exact sequence $0 \to F(A) \to F(B) \to F(C)$. There is a dual statement for the right exact functors. If F is a contravariant functor, we say that F is a half exact (respectively, F is left exact, etc.) if F has the corresponding property, as a covariant functor $\mathtt{C}^{\mathrm{op}}$ to $\mathtt{C}'$. The composite of left exact, respectively, right exact, covariant functors is of the same type. We refer back to [CE56] for further trivialities and for the study of exactness properties of multifunctors. As a significant example, we note that $\operatorname{Hom}(A, B)$ is an additive bifunctor on $\mathtt{C}^{\mathrm{op}} \times \mathtt{C}$, with values in the abelian category of abelian groups, contravariant in A and covariant in B, and left exact with the respect to each argument (that is, in the terminology of [CE56], a left exact bifunctor).

1.5. Infinite Sums and Products

(1.5.1). In some constructions we will require the existence and certain properties of both infinite direct sums and infinite direct products. Here, in order of increasing strength, are the most commonly used axioms

(AB 3). for any family, $(A_i)_{i\in I}$ of objects of $\mathtt{C}$, the direct sum of the A_i exists (cf. N° 1).

(1.5.2). This axiom implies that for any family of subobjects (A_i) of an $A \in \mathtt{C}$ the sup of the A_i exists. It suffices to take the image of the direct sum $\oplus A_i$ under the morphism whose components are the canonical injections $A_i \to A$. We have seen that the direct sum of any family of surjective morphisms is surjective, (N° 1); in fact, we even see that the functor $(A_i)_{i\in I} \mapsto \oplus_{i\in I} A_i$, defined over the "product category", $\mathtt{C}^I$ with values in $\mathtt{C}$, is *right exact*. It is even exact if I is finite, but not necessarily if I is infinite, for the direct sum of an infinite family of monomorphisms is not necessarily a monomorphism, as we have noted in N° for the dual situation. Consequently we introduce the following axiom

(AB 4). Axiom (AB 3) is satisfied and a direct sum of a family of monomorphisms is a monomorphism.

The following axiom is strictly stronger than (AB 4).

(AB 5). Axiom (AB 3) is satisfied, and if $(A_i)_{i\in I}$ is an increasing directed family of subobjects of $A \in \mathtt{C}$, and B is any subobject of A, we have

$$\left(\sum_{i\in I} A_i\right) \cap B = \sum_{i\in I} (A_i \cap B)\,. \tag{1.5.2.1}$$

(1.5.3). (In accordance with normal usage we have denoted by $\sum A_i$ the sup of the A_i, and by $P \cap Q$ the inf of the subobjects P and Q of A.) (AB 5) can also be expressed thus:

(1) (AB 3) is satisfied, and if $A \in \mathtt{C}$ is the sup of an increasing directed family of subobjects A_i, and if for any $i \in I$ we are given a morphism $u_i : A_i \to B$ such that when $A_i \subseteq A_j$, $u_i = u_j|A_i$, then there is a morphism u (obviously unique) from $A \to B$ such that $u_i = u|A_i$.

We mention the following axiom that strengthens (AB 5), which we will not require in this memoir:

(AB 6). Axiom (AB 3) holds and for any $A \in \mathtt{C}$ and any family $(B^j)_{j\in J}$ of increasing directed families of $B^i = (B^j_i)_{i\in I_j}$ of subobjects B^j of A, we have:

$$\bigcap_{j\in J}\left(\sum_{i\in I_j} B^i_j\right) = \sum_{(i_j)\in\prod I_j}\left(\bigcap_{j\in J} B^i_{i_j}\right) \tag{1.5.3.1}$$

(This axiom implicitly assumes the existence of the inf of any family of subobjects of A.)

(1.5.4). We leave it to the reader to state the dual axioms (AB 3*), (AB 4*), (AB 5*), (AB 6*), pertaining to infinite products. By way of example, let us point out that the category of abelian groups (or more generally the category of modules over a unital ring), satisfies, with respect to direct sums, the strongest axiom (AB 6); it also satisfies axioms (AB 3*) and (AB 4*), but not (AB 5*). The situation is reversed for the dual category, which by the Pontrjagin duality is equivalent to the category of compact topological abelian groups. (This shows that (AB 5*) is not a consequence of (AB 4*) and hence neither in (AB 5) a consequence of (AB 4). The abelian category o sheaves of abelian groups over a given topological space X satisfies axiom (AB 5) and (AB 5*),then $\mathtt{C}$ is reduced to the zero object (for we then easily see that for $A \in \mathtt{C}$, the canonical morphism $A^{(I)} \to A^I$ is an isomorphism and we may verify that that is possible only when A is zero.)

(1.5.5). The preceding axioms will be particularly useful for the study of inductive and projective limit that we will need to provide usable existence conditions for "injective" and "projective" objects (see N° 10). To avoid repetition we will first study a very general and widely used procedure for forming new categories using diagrams.

1.6. Caegories of Diagrams and Permanence Properties

(1.6.1). A *diagram scheme* is a triple (I, Φ, d) made up of two sets I and Φ and a function d from Φ to $I \times I$. The elements of I are *vertices*, the elements of Φ are *arrows* of the diagram and if ϕ is an arrow of the diagram, $d(\phi)$ is called the *direction*, characterized as the *source* and *target* of the arrow (these are therefore vertices of the scheme). A *composite arrow* with source i and target j is, by definition, a non-empty finite sequence of arrows of the diagram, the source of the first being i, the target of each being the source of the next and the target of the last one being j. If $\mathtt{C}$ is a category, we call *diagram in* $\mathtt{C}$ *from the scheme* S a function D which associated to each $i \in I$ an object $D(i) \in \mathtt{C}$ and to any arrow $\phi \in \Phi$ with source i and target j, a morphism $D(\phi) : D(i) \to D(j)$. The class of such diagrams will be denoted by $\mathtt{C}^S$; it will be considered a category, taking as morphisms from D to D' a family of morphisms $v_i : D(i) \to D'(i)$ such that for any arrow ϕ with source i and target j the following diagram commutes:

(1.6.1.1)
$$\begin{tikzcd}
D(i) \arrow[r, "v_i"] \arrow[d, "D(\phi)"'] & D'(i) \arrow[d, "D'(\phi)"] \\
D(j) \arrow[r, "v(j)"'] & D'(j)
\end{tikzcd}$$

Morphisms of diagrams compose in the obvious way, and it is trivial to verify the category axioms. If D is a diagram on the scheme S, then for any composite arrow $\phi = (\phi_1, \ldots, \phi_k)$ in S, we define $D(\phi) = D(\phi_1) \cdots D(\phi_k)$; it is a morphism from $D(i) \to D(j)$ if i and j are, respectively, the source and the target of ϕ. We call D a *commutative diagram* if we have $D(\phi) = D(\phi')$ whenever ϕ and ϕ' are two composite arrows with the same source and same target. More generally, if R is a set consisting of pairs (ϕ, ϕ') of composite arrows having the same source and target, and of composite arrows whose source equals its target, we consider the subcategoy $\mathtt{C}^{S,R}$ of $\mathtt{C}^S$ consisting of diagrams satisfying the *commutativity conditions* $D(\phi) = D(\phi')$ for $(\phi, \phi') \in R$ and $D(\phi)$ is the identity morphism of $D(i)$ if $\phi \in R$ has i as its source and target.

(1.6.2). We have to consider still other types of commutation for diagram, whose nature varies according to the category in question. What follows seems to cover the most important casses. For any $(i, j) \in I \times I$ we take a set R_{ij} of formal linear combinations with integer coefficients of composite arrow with source i and target j, and if $i = j$, of an auxiliary element e_i. Then if D is a diagram with values in an additive category $\mathtt{C}$, then, for any $L \in R_{ij}$, we can define the morphism $D(L) : D(i) \to D(j)$, by replacing, in he expression of L, a composite arrow ϕ by $D(\phi)$ and e_i by the identity element of $D(i)$. If we denote by R the union of the R_{ij}, we will say that D is R-commutative if all the $D(L)$, $L \in R$, are 0. We call *a diagram scheme with commutativity* conditions a pair $(S, R) = \Sigma$ consisting of a diagram scheme S and a set R as above. For any additive category $\mathtt{C}$, we can then consider the subcategory $\mathtt{C}^\Sigma$ of $\mathtt{C}^S$ consisting of the R-commutative diagrams.

Proposition (1.6.1). — *Let Σ be a diagram scheme with commutativity conditions and $\mathtt{C}$ an additive category. Then the category $\mathtt{C}^\Sigma$ is an additive category and if $\mathtt{C}$ has infinite direct (respectively, infinite direct sums), so does $\mathtt{C}^\Sigma$. Moreover, if $\mathtt{C}$ satisfies any one of the axioms (AB 1)-(AB 6) or the dual axioms (AB 3*)-(AB 6*), so does $\mathtt{C}^\Sigma$.*

(1.6.3). Moreover, if $D, D' \in \mathtt{C}^\Sigma$, and if u is a morphism from D to D', then its kernel (respectively, cokernel, image, coimage) is a diagram formed by the kernel (respectively, …) of the components u_i, the morphism in this diagram (corresponding to the arrows of he scheme) being obtained from those of D (respectively, those of D',…) in the usual way by restriction (respectively, passage to the quotient). We interpret analogously the direct sum or the direct product of a family of diagrams. Subobjects D' of the diagram D are identified as families $(D'(i))$ of subobjects of $D(i)$ such that for any arrow ϕ with source i and target j we have $D(\phi) : D'(i) \hookrightarrow D'(j)$; then $D'(\phi)$ is defined as the morphism $D'(i) \to D'(j)$ defined by $D(\phi)$. The quotient objects of D are defined dually.

(1.6.4). If S is a diagram scheme, we call the *dual scheme* and denote it by S^{op}, the scheme with the same vertices and the same sets of arrows as S, but with the source and target of the arrows of S interchanged. If, moreover, we give a set R of commutativity conditions for S, we will keep the same set for S^{op}. Using this convention, for an additive category $\mathtt{C}$, the dual category of $\mathtt{C}^{\Sigma}$ can be identified as $(\mathtt{C}^{\text{op}})^{\Sigma^{\text{op}}}$.

(1.6.5). Let $\mathtt{C}$ and $\mathtt{C}'$ be two additive categories and Σ be a diagram scheme with commutativity conditions. For any functor F from $\mathtt{C}$ to $\mathtt{C}'$, we define in the obvious way the functor F^{Σ} from $\mathtt{C}^{\Sigma}$ to $\mathtt{C}'^{\Sigma}$, called *the canonical extension of F to the diagram*. F^{Σ} behaves formally like a functor with respect to the argument F, in particular, a natural transformation $F \to F'$ induces a natural transformation $F^{\Sigma} \to F'^{\Sigma}$. Finally we note that for a composite functor, we have $(GF)^{\Sigma} = G^{\Sigma}F^{\Sigma}$, and the exactness properties of a functor are preserved by extension to a class of diagrams.

1.7. Examples of Categories Defined by Diagram Schemes

(1.7.1). (a) Take I reduced to a singl element and the empty set of arrows. Then the commitativity relations are of the form $n_i e = 0$, and thus can be reduced to a unique relation $ne = 0$. Then $\mathtt{C}^{\Sigma}$ is the subcategory of $\mathtt{C}$ consisting of objects annihilated by the integer n. If $n = 0$, we recover $\mathtt{C}$.

(b) Take any I, no arrows, no commutativity relations. Then $\mathtt{C}^{\Sigma}$ ca be identified with the product category $\mathtt{C}^{I}$. If we suppose we are given commutativity relations, then we get the kind of product category described in (1.5).

(c) Take I reduced to two elements a and b, with a single arrow with source a and target b; we find the category of morphisms $u : A \to B$ between objects of $\mathtt{C}$. The introduction of commutativity relations would leave those (a, b, u) that are annihilated by a given integer.

(d) *Categories of functors*. Let $\mathtt{C}'$ be another category, and suppose that it is small. Then the covariant functors from $\mathtt{C}'$ to $\mathtt{C}$ form a category, taking as morphisms the natural transformations (cf (1.1)). This category can be interpreted as a category $\mathtt{C}^{\Sigma}$, where we take $I = \mathtt{C}'$, the arrows with source A' and target B', being by definition the elements of $\mathrm{Hom}(A', B')$ and the commutativity relations being those that express the two functorial axioms. If $\mathtt{C}'$ is also an additive category, the *additive* functors from $\mathtt{C}'$ to $\mathtt{C}$ can also be interpreted as a category $\mathtt{C}^{\Sigma}$ (we add the necessary commutativity relations).

(e) *Complexes with values in* $\mathtt{C}$. $I = \mathbf{Z}$ (the set of integers), the set of arrows being $(d_n)_{n\in\mathbf{Z}}$ where d_n has source n the target $n+1$, the commutativity relations being $d_{n+1}d_n = 0$. We can also add relations of the form $e_n = 0$ if we want to limit ourselves to complexes of positive degrees or to those of negative degrees. We obtain bicomplexes, etc. analogously.

(f) *The category* $\mathtt{C}^{G}$, *where G is a group*. Let G be a group and $\mathtt{C}$ a category (not necessarily additive). We call an *object with a group G of operators* in $\mathtt{C}$, a pair (A, r) consisting of an object $A \in \mathtt{C}$ and a representation r of G into the group of automorphisms of A. If (A', r') is a second such pair, we call a morphism from the first to the second a morphism A to A' which commutes with the operations of G. The class $\mathtt{C}^{G}$ of objects in $\mathtt{C}$ with group G of operators thus becomes a category. We can interpret it as a class $\mathtt{C}^{\Sigma}$ in which we take for $\Sigma = \Sigma(G)$ the following scheme with relations: the set of vertices is reduced to one element i_G, the set of arrows is G, the commutativity relations are $(s)(t) = (st)$ (where the left hand side denotes a composed arrow), and $(e) = e_{i_o}$ (where e denotes the unit element of G). In particular, if $\mathtt{C}$ is additive, the same is true of $\mathtt{C}^{G}$; in that case our construction is contained in the one that follows (by considering the algebraic relations in the group G).

(g) *The category* $\mathtt{C}^{U}$ *where U is a unitary ring*. We consider the additive category consisting of pairs (A, r) of an object A of $\mathtt{C}$ and a unitary representation of U into the ring $\mathrm{Hom}(A, A)$, the morphisms in this category being defined in the obvious way. It is interpreted as above as a category $\mathtt{C}^{\Sigma(U)}$, where $\Sigma(U)$ is the scheme with relations having a single vertex, with U as a set of arrows, and with commutativity relations that we omit.

(h) *Inductive systems and projective systems*. We take as a set of vertices a *preordered* set I, with arrows being pairs (i, j) of vertices with $i \leqslant j$, the source and target of (i, j) being i and j, respectively. The commutativity relations are $(i, j)(j, k) = (i, k)$ and $(i, i) = e_i$. The corresponding diagrams (for a give category $\mathtt{C}$, not necessarily additive) are known as *inductive systems* over I with values in $\mathtt{C}$. If we change I to the opposite preordered set, or change $\mathtt{C}$ to $\mathtt{C}^{\text{op}}$ we get *projective systems* over I with values in $\mathtt{C}$. An important case is the one in which I is the lattice of open sets of a topological space X, ordered by containment: we then obtain the notions of *presheaf* over X with values in $\mathtt{C}$.

1.8. Inductive and Projective Limits

(1.8.1). We will discuss only inductive limits, since the notion of projective limit is dual. Let $\mathtt{C}$ be a category I be a preordered set and $\mathbf{A} = (A_i, u_{i,j})$, be an inductive system over I with values in $\mathtt{C}$ (u_{ij} is a morphism $A_j \to A_i$, defined for $i \geqslant j$). We call a (generalized) *inductive limit* of $\mathbf{A}$ a system consisting of $A \in \mathtt{C}$ and a family (u_i) of morphisms $u_i : A_i \to A$, satisfying the following conditions:

(a) for $i \leqslant j$, we have $u_i = u_j u_{ji}$;
(b) for every $B \in \mathtt{C}$ and every family (v_i) of morphisms $v_i : A_i \to B$, such that for any pair $i \leqslant j$, then relation $v_i = v_j u_{ji}$ holds, we can find a unique morphism $v : A \to B$ such that $v_i = vu_i$ for all $i \in I$.

If $(A, (u_i))$ is an inductive limit of $\mathbf{A} = (A_i, u_{ij})$, and if $(B, (v_i))$ is an inductive limit of a second inductive system, $\mathbf{B} = (B_i, v_{ij})$ and finally if $\mathbf{w} = (w_i)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$, then there exists a unique morphism $w : A \to B$ such that for all $i \in I$, $wu_i = v_i w_i$. In particular, two inductive limits of the same inductive system are canonically isomorphic (in an obvious way), so it is natural to choose, for every inductive system that admits an inductive limit, a fixed inductive limit (for example, using Hilbert's τ symbol) which we will denote by $\varinjlim \mathbb{A}$ or $\varinjlim_{i \in I} A_i$ and which we will call *the* inductive limit of the given inductive system. If I and $\mathtt{C}$ are such that $\varinjlim \mathbb{A}$ exists for *every* system $\mathbb{A}$ over I with values in $\mathtt{C}$, it follows from the preceding that $\varinjlim \mathbb{A}$ is a *covariant functor* defined over the category of inductive systems on I with values in $\mathtt{C}$.

Proposition (1.8.1). — *Let $\mathtt{C}$ be an abelian category satisfying Axiom (AB 3) (existence of arbitrary direct sums) and let I be an increasing filtered preordered set. Then for every inductive system $\mathbf{A}$ over I with values in $\mathtt{C}$, the $\varinjlim \mathbf{A}$ exists, and it is a right exact additive functor on $\mathbf{A}$. If $\mathtt{C}$ satisfies Axiom (AB 5) (cf. (1.5)), this functor is even exact, and then the kernel of the canonical morphism $u_i : A_i \to \varinjlim \mathbf{A}$ is the sup of the kernels of the morphisms $u_{ji} : A_i \to A_j$ for $j \geqslant i$ (in particular, if the u_{ji} are injective, so are the u_i).*

(1.8.2). To construct an inductive limit of $\mathbf{A} = (A_i, u_{ij})$ we consider $S = \oplus_{i \in I} A_i$ and for every pair $i \leqslant j$, $T = \oplus_{i \leqslant j} A_i$. If $v_i : A_i \to S$ and $w_{ij} : A_i \to T$ are the inclusions into those coproducts, there are two maps $d, e : T \to S$ defined as the unique maps for which $dw_{ij} = v_i$ and $ew_i = v_j u_{ij} s$, for all $i \leqslant j$. Then $\varinjlim \mathbf{A}$ is the coequalizer of d and e. We leave the rest of the proof this proposition to the reader.

1.9. Generators and Cogenerators

(1.9.1). Let $\mathtt{C}$ be a category, and let $(U_i)_{i \in I}$ be a family of objects of $\mathtt{C}$. We say that it is *a family of generators* of $\mathtt{C}$ if for any object $A \in \mathtt{C}$ and any subobject $B \neq A$, we can find an $i \in I$ and a morphism $u : U_i \to A$ which does not come from a morphism $U_i \to B$. Then for any $A \in \mathtt{C}$ the subobjects of A form a *set*: in fact, a subobject B of A is completely determined by the set of morphisms of objects $U_i \to A$ that factor through B. We say that an object $U \in \mathtt{C}$ is a *generator* of $\mathtt{C}$ if the family $\{U\}$ is a family generators.

Proposition (1.9.1). — *Suppose that $\mathtt{C}$ is an abelian category satisfying Axiom (AB 3) (existence of infinite direct sums). Let $(U_i)_{i \in I}$ be a family of objects of $\mathtt{C}$ and $U = \oplus U_i$ its direct sum. The following conditions are equivalent:*

(a) $(U_i)_{i \in I}$ is a family of generators of $\mathtt{C}$.
(b) U is a generator of $\mathtt{C}$.
(c) Any $A \in \mathtt{C}$ is isomorphic to a quotient of a direct sum $U^{(I)}$ of objects that are all identical to U.

(1.9.2). The equivalence of (a) and (b) is an almost immediate consequence of the definition. (b) implies (c). for it is sufficient to take for I the set $\mathrm{Hom}(U, A)$ and to consider the morphism from $U^{(I)}$ to A whose component corresponding to $u \in I$ is u itself: the image B of this morphism is A since otherwise there would exist a $u \in \mathrm{Hom}(U, A) = I$ that does not factor through B, which would be absurd. Thus A is isomorphic to a quotient of $U^{(I)}$. (c) implies (b), for it is immediate that if A is a quotient of $U^{(I)}$, then for any subobject B of A, distinct from A there exists $i \in I$ such that the canonical image in A of the i-th factor of $U^{(I)}$ is not contained in B, whence a morphism from

U to A that does not factor through B (it can be noted that the additive structure of $\mathtt{C}$ has not been used here).

Example (1.9.2). — If $\mathtt{C}$ is the abelian category of unitary left modules on a unital ring U, then U (considered as a left module over itself) is a generator. If $\mathtt{C}$ is the category of sheaves of abelian groups over a fixed topological space X, and if for any open $U \subseteq X$, we denote by $\mathscr{Z}_U$ the sheaf on X which is 0 over $\complement U$ and the constant sheaf of integers over U, the family of $\mathscr{Z}_U$ forms a system of generators of $\mathtt{C}$. This example can be immediately generalized to the case in which there is a sheaf $\mathscr{O}$ of rings given over X, and in which we consider the category of sheaves of $\mathscr{O}$-modules over X. There are other examples in the following proposition:

Proposition (1.9.3). — *Let Σ be a diagram scheme with commutativity relations (cf. (1.6)), and let $\mathtt{C}$ be an abelian category and $(U_i)_{i\in I}$ a family of generators of $\mathtt{C}$. Assume that for any arrow of Σ, the source and target of the arrow are distinct, and that the commutativity relations do not involve any identity morphism e_s (where s is a vertex of Σ). Then for any $A \in \mathtt{C}$ and any vertex s of the scheme, the diagram $\mathrm{E}_s(A)$ whose value is A at the vertex s and 0 at all other vertices, and whose value at each arrow is 0, belongs to $\mathtt{C}^\Sigma$. In addition, the family of $\mathrm{E}_s(U_i)$, (where s and i are variables) is a system of generators of $\mathtt{C}$.*

(1.9.3). The proof is immediate; it suffices to note, for the last assertion, that the morphisms of $\mathrm{E}_s(A)$ in a diagram D can be identified with the morphisms of $A \in D(s)$.

(1.9.4). We leave it to the reader to develop the dual notion of *family of cogenerators* of an abelian category. We can show that if $\mathtt{C}$ is an abelian category that satisfies Axiom (AB 5) (cf. (1.5)), then the existence of a generator implies the existence of a cogenerator. (We will not make use of this result.) Thus the category of unitary left modules over a unital ring U always admits a cogenerator: if for example, $U = \mathbf{Z}$, we can take as a cogenerator the group of rational numbers (or the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$).

1.10. Injective and Projective Objects

(1.10.1). Recall that an object M of an abelian category $\mathtt{C}$ is said to be *injective* if the functor $A \mapsto \mathrm{Hom}(A, M)$ (which in any case is left exact) is exact, i.e., if for any morphism $u : B \to M$ of a subobject B of an $A \in \mathtt{C}$, there is a morphism of $A \to M$ that extends it. A morphism $A \to M$ is called an *injective effacement* of A if it is a monomorphism, and if for any monomorphism $B \to C$ and any morphism $B \to A$, we can find a morphism $C \to M$ making the diagram

(1.10.1.1)
$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

commute. Thus, for the identity arrow of M to be an injective effacement, it is necessary and sufficient that M be injective. Any monomorphism into an injective object is an injective effacement.

Theorem (1.10.1). — *If $\mathtt{C}$ satisfies Axiom (AB 5) (cf (1.5)) and admits a generator (cf. (1.9)), then any $A \in \mathtt{C}$ has a monomorphism into an injective object.*

(1.10.2). We will even construct a functor $M : A \mapsto M(A)$ (non-additive in general!) from $\mathtt{C}$ into $\mathtt{C}$ and a natural transformation f of the identity functor into M such that for any $A \in \mathtt{C}$, $M(A)$ is injective and $f(A)$ is a monomorphism of A into $M(A)$. Since the proof is essentially known, we will sketch only the main points.

Lemma (1). — *If $\mathtt{C}$ satisfies Axiom (AB 5), then the object $M \in \mathtt{C}$ is injective if and only if for any subobject V of the generator U, any morphism $V \to M$ can be extended to a morphism $U \to M$.*

Proof. It suffices to prove that the condition is sufficient. Thus let U be a morphism from a subobject B of $A \in \mathtt{C}$ to M. We show that there is a morphism of A to M that extends u. We consider the set P of extensions of u to subobjects of A that contain B (it is certainly a set, because by virtue of the existence of a generator, the subobjects of A form a set). We order it by the extension relation. By virtue of the second formulation of Axiom (AB 5) (cf. (1.5)), this set is inductive. It therefore admits a maximal element; we are thus reduced to the case that u is itself maximal, and to showing that then

$B = A$. We prove then that if $B \neq A$, there is an extension of U to a $B' \neq B$. In fact, let $j : U \to A$ be a morphism such that $j(U) \not\subseteq B$; set $B' = j(U) + B$ (therefore $B' \neq B$). Let $V = j^{-1}(B)$ be the inverse image of B under j, let $j' : V \to B$ be the morphism induced by j, considr the sequence of morphisms $V \xrightarrow{\phi'} U \times B \xrightarrow{\phi} B' \to 0$, where the morphism ϕ' has as components the inclusion function of $V \hookrightarrow U$ and $-j'$, and ϕ has as components j and the inclusion function $B \hookrightarrow B'$. We can see immediately that this sequence is exact, so to define a morphism $v : B' \to M$, it suffices to define a morphism $w : U \times B \to M$ such that $w\phi' = 0$. Now let k be an extension to U of $uj' : V \to M$. We take for w the morphism from $U \times V$ to M whose components are k and u. We show immediately that $w\phi' = 0$ and that the morphism $v : B' toM$ induced by w extends u, which completes the proof of Lemma (L.1.10.1). □

Proof. Let $A \in \mathtt{C}$ and let $I(A)$ be the set of all the morphisms u_i from subobjects $V_i(U)$ to A. Consider the morphism $\oplus V_i \to A \times U^{(I(A))}$ whose restriction to V_i has as components $-u_i : V_i \to A$ and the canonical injection of V_i into the i-th factor of the direct sum $U^{(I(A))}$. Let $M_1(A)$ be the cokernel of the desired morphism, $f(A) : A \to M_1(A)$ be the morphism induced by the canonical epimorphism of $A \times U(I(A))$ over its quotient. Then $f(A)$ is a monomorphism (easily proved thanks to the fact that the canonical morphism $\oplus V_i \to U^{(I(A))}$ is a monomorphism by (AB 4)) and, in addition, any morphism $u_i : V_i \to A$ "can be extended" to a morphism $U \to M_1(A)$ (in other words, the morphism induced on the i-th factor of $U^{(I(A))}$ by the canonical epimorphism of $A \times U^{(I(A))}$ onto its quotient $M_1(A)$). We define by transfinite induction, for any ordinal number i the object $M_i(A)$, and for two ordinal number $i \leqslant j$, an *injective* morphism $M_i(A) \to M_j(A)$, such that the $M_i(A)$, for $i < i_0$ (i_0 being a fixed ordinal number) form an inductive system. For $i = 0$, we will take $M_0(A) = A$; for $i = 1$, $M_1(A)$ and $M_0(A) \to M_1(A)$ are already defined. If the construction has been carried out for the ordinals less than i, and if i has the form $j + 1$, we set $M_i(A) = M_1(M_j(A))$ and the morphism $M_j(A) \to M_{j+1}(A)$ will be $f(M_j(A))$ (which defines at the same time the morphisms $M_k(A) \to M_i(A)$ for $k \leqslant i$). If i is a limit ordinal, we will set $M_i(A) = \lim_{j<i} M_j(A)$, and we will take as morphisms $M_j(A) \to M_i(A)$, for $j < i$, the canonical morphisms, which are certainly injective (Proposition (P.1.8.1)). Now let k be the smallest ordinal whose cardinality is large than that of the set of subobjects of U (we take $M(A) = M_k(A)$); everything comes down to proving that M_k is injective, i.e satisfies the condition of Lemma (L.1.10.1). With the notation of this lemma, we prove that $v(V)$ is contained in one of the M_i with $i < k$ (which will complete the proof). In fact, from $M_k = \sup M_i$ we get $V = \sup_{i<k} v^{-1}(M_i)$ (by virtue of (AB 5)). Thus since the set of subobjects of V has cardinality less than k, and since any set of ordinal numbers less than k and having k as its limit must itself have cardinality k (since k is not a limit ordinal), it follows that $v^{-1}(M_i)$ stays constant starting from some $i_0 < k$, whence $V = v^{-1}(M_{i_0})$, which completes the proof. □

Remark (1). — Variant of Theorem (T.1.10.1): if $\mathtt{C}$ satisfies Axiom (AB 3), (AB 4), and (AB 3*) and admits a cogenerator T, then any $A \in \mathtt{C}$ admits an injective effacement. We will not have to use this result.

Remark (2). — The fact that $M(A)$ is functorial in A may be convenient, for example, to prove that any $A \in \mathtt{C}^G$ (i.e. an object $A \in \mathtt{C}$ with a group G of operators - cf (1.7), Example (f)) which is injective in $\mathtt{C}^G$ is also injective in $\mathtt{C}$.

Remark (3). — In many cases, the existence of a monomorphism of A into an injective object can be seen directly in a simple way. Theorem (T.1.10.1) has the advantage of applying to many different cases. Moreover, the conditions of the theorem are stable under passage to certain categories of diagrams (cf. Proposition (P.1.6.1) and Example (E.1.9.2)), in which the existence of sufficiently many injective is not always visible to the naked eye.

Remark (4). — We leave it to the reader to provide the dual statements relative to the projective objectives and projective effacements.

1.11. Quotient Categories

(1.11.1). Although they will not be used in the remainder of this work, the considerations of this section, which systematize and make more flexible, the “language modulo $\mathtt{C}$" of Serre, [Ser53], are convenient in various applications.

(1.11.2). Let $\mathtt{C}$ be a category. We call a *subcategory* of $\mathtt{C}$ a category $\mathtt{C}'$ whose objects are objects of $\mathtt{C}$, such that for $A, B \in \mathtt{C}'$, the set $\mathrm{Hom}_{\mathtt{C}'}(A, B)$ of morphisms from A to B in $\mathtt{C}'$ is a subset of $\mathrm{Hom}_{C}(A, B)$ of morphisms from A to B in $\mathtt{C}$, the composition of morphisms in $\mathtt{C}'$ being induced by the composition of morphisms in $\mathtt{C}$, and the identity morphisms in $\mathtt{C}'$ being the identity morphisms in $\mathtt{C}$. The last two conditions mean that the function that assigns to each object or morphism of $\mathtt{C}$; the same object or morphism of $\mathtt{C}$ is a covariant functor from $\mathtt{C}'$ to $\mathtt{C}$ (call *the canonical injection from* $\mathtt{C}'$ *to* $\mathtt{C}$). If $\mathtt{C}, \mathtt{C}'$ are additive categories, $\mathtt{C}'$ is called an *additive subcategory* if, in addition to the preceding conditions, the groups $\mathrm{Hom}_{\mathtt{C}'}(A, B)$ are subgroups of $\mathrm{Hom}_{\mathtt{C}}(A, B)$. Suppose that $\mathtt{C}$ is an abelian category. Then $\mathtt{C}'$ is called a *complete. It is actually a full abelian subcategory.* if

(i) for $A, B \in \mathtt{C}'$, we have $\mathrm{Hom}_{\mathtt{C}'}(A, B) = \mathrm{Hom}_{\mathtt{C}}(A, B)$;
(ii) if in an exact sequence $A \to B \to C \to D \to E$, the four extreme terms belong to $\mathtt{C}'$, so does the middle term C.

In accordance with (i) the subcategory $\mathtt{C}'$ is completely determined by its objects. (ii) is equivalent to saying the for any morphism $P \to Q$ with $P, Q \in \mathtt{C}'$, the kernel and the cokernel belong to $\mathtt{C}'$ and for any exact sequence $0 \to R' \to R \to R'' \to 0$, whenever $R', R'' \in \mathtt{C}'$, then $R \in \mathtt{C}'$. We see immediately that then $\mathtt{C}'$ is itself an abelian category and that for a morphism $u : A \to B$ in $\mathtt{C}'$, the kernel, cokernel, (and thus the image, and coimage) are identical to the corresponding constructions in $\mathtt{C}$.

(1.11.3). The subcategory $\mathtt{C}'$ of $\mathtt{C}$ is called *thick* if it satisfies Condition (i) above and the following strengthening of Condition (ii):

(iii) If in an exact sequence $A \to B \to C$, the outer terms A and C belong to $\mathtt{C}'$, so does B.

If $\mathtt{C}$ is the abelian category of abelian groups, we find the notion of “class of abelian groups" of [Ser53]. We see how in [Ser53], (iii) is equivalent to the conjunction of the following three conditions: any zero object belongs to $\mathtt{C}'$; any object that is isomorphic to a subobject or quotient object of $\mathtt{C}'$, belongs to $\mathtt{C}'$; any extension of an object of $\mathtt{C}'$ by an object of $\mathtt{C}'$ belongs to $\mathtt{C}'$.

(1.11.4). Let $\mathtt{C}$ be an abelian category and $\mathtt{C}'$ a Serre subcategory. We will define a new abelian category, denoted $\mathtt{C}/\mathtt{C}'$ and called the *quotient category* of $\mathtt{C}$ by $\mathtt{C}'$. The objects of $\mathtt{C}/\mathtt{C}'$ are, by definition, the objects of $\mathtt{C}$. We will define the morphisms in $\mathtt{C}/\mathtt{C}'$ from A to B, called “morphisms mod $\mathtt{C}'$ from A to B". We say that a subobject A' of A is equal mod $\mathtt{C}'$ or quasi-equal to A ifA/A' belongs to $\mathtt{C}'$; then any subobject of A containing A' is also quasi-equal to A; moreover, the inf of two subobjects of A that are quasi-equal to A is also quasi-equal to A. Dually, we introduce the notion of quotient of B quasi-equal to B: such a quotient B/N is quasi0equal to B if $N \in \mathtt{C}'$. A morphism mod $\mathtt{C}'$ from A to B is then defined by a morphism f' from a subobject A' of A, quasi-equal to A, to a quotient B' of B quasi-equal to B, it being understood that a morphism $f'' : A'' \to B''$ (satisfying the same conditions) defines the same morphism mod $\mathtt{C}'$ if and only if we can find $A''' \leqslant (A' \wedge A'')$, $B''' \leqslant (B' \wedge B'')$, A''' quasi-equal to A, B''' quasi-equal to B such that the morphisms $A''' \to B'''$ induced by f' and f'' are the same. This last relation between f' and f'' is certainly an equivalence elation and the preceding definition of mophisms mod $\mathtt{C}'$ is therefore coherent. Suppose that for any $A \in \mathtt{C}$, thee subobjects of A form a set (which is true for all known categories). Then we can consider the set of morphisms mod $\mathtt{C}'$ from A to B, denoted $\mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(A, B)$. $\mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(A, B)$ appears as the inductive limit of the abelian groups $\mathrm{Hom}_{\mathtt{C}}(A', B')$ where A' ranges over the subobjects of A quasi-equal to A, and B' ranges over the quotient of B quasi-equal to B and is consequently an abelian group. We similarly define a pairing $\mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(A, B) \times \mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(B, C) \to \mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(A, C)$ as follows. Let $u \in \mathrm{Hom}_{C}(A', B')$ represent $u' \in \mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(A, B)$ and $v \in \mathrm{Hom}_{\mathtt{C}}(B'', C)$ represent $v' \in \mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(B, C)$. Let Q be the image of the canonical morphism $B'' \to B$ of the subobject B'' of B in the quotient B'; Q is also isomorphic to the coimage of this morphism and is therefore both a subobject of B' and a quotient object of B''. By decreasing, if necessary, the subobject A' of A and the quotient object $\mathtt{C}'$ of $\mathtt{C}$, we can assume that u and v come from morphisms (denoted by the same letters), $A' \to Q$ and $Q \to C'$. We can now take the composite $vu \in \mathrm{Hom}_{\mathtt{C}}(A', C')$ and we verify that the element of $\mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(A, C)$ that this defines depends only on u' and v'. We denote it $v'u'$. There is no difficulty in proving that the law of composition thus defined is bilinear and associative, and the that the class in $\mathrm{Hom}_{\mathtt{C}/\mathtt{C}'}(A, A)$ of the identity morphism i_A is a universal unit, so $\mathtt{C}/\mathtt{C}'$ is an additive category, and finally that it is even an abelian category. We will not complete the extremely tedious proof. Thus $\mathtt{C}/\mathtt{C}'$ *appears as an abelian category; moreover the identity functor* $F : \mathtt{C} \to \mathtt{C}/\mathtt{C}'$ *is exact* (and, in particular, commutes with kernels, cokernels, images , and coimages), $F(A) = 0$ *if and only if* $A \in \mathtt{C}'$, *and any object of* $\mathtt{C}/\mathtt{C}'$ *has the form* $F(A)$ *for some* $A \in \mathtt{C}$. These are the facts (which essentially characterize the quotient category) which allow us to safely apply the "mod $\mathtt{C}'$" language, since this language signifies simply that we are in the quotient abelian category. It is particularly convenient to use, when we have a spectral sequence (cf. (2.4)) in $\mathtt{C}$, the fact that some terms of the spectral sequence belong to $\mathtt{C}'$: reducing mod $\mathtt{C}'$ (i.e. applying the functor F), we find a spectral sequence in $\mathtt{C}/\mathtt{C}'$ in which the corresponding terms vanish, whence we have exact sequences mod $\mathtt{C}'$, with the help of the usual criteria for obtaining exact sequences from spectral sequences in which certain terms have vanished.

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