Chapter 1

Introduction

Here, I will do the exercises in the book *from calculus to cohomology.*

Exercise 1.1. *Is there a smooth function $F : U \to \mathbb{R}$, such that*

$$\frac{\partial F}{\partial x_1} = f_1,\ \frac{\partial F}{\partial x_2} = f_2,\ f = (f_1, f_2)$$

Proof.

$$\frac{\partial^2 F}{\partial x_1 \partial x_2} = \frac{\partial^2 F}{\partial x_2 \partial x_1} \implies \frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}$$

□

Is the above condition sufficient?

Exercise 1.2. *The answer is no.*

For example, we set $f = (f_1, f_2)$ as follows:

$$f_1 = \frac{-x_2}{x_1^2 + x_2^2},\ f_2 = \frac{x_1}{x_1^2 + x_2^2}$$

Proof.

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}$$

□

Definition 1.3. A subset $X \subset \mathbb{R}^n$ is said to be *star-shaped* with respect to the point $x_0 \in X$ is the line segment $\{tx_0 + (1-t)x \mid t \in [0, 1]\}$ is contained in X for all $x \in X$.

Exercise 1.4. *If X is a star-shaped space, then the solution of Exercise 1 is affirmative.*

Proof. WLOG, $x_0 = 0$. Let $G(t) = F(tx_1, tx_2)$, then

$$G(t) = \int_0^t \frac{\partial G(s)}{\partial s} ds = \int_0^t x_1 f_1(sx_1, sx_2) + x_2 f_2(sx_1, sx_2) ds$$

as we desired. $\square$

Remark 1.5. Star-shaped space is "contractible" (topological property).

Definition 1.6. Let $U \subset \mathbb{R}^k$ and $C^\infty(U, \mathbb{R}^k)$ be the set of smooth functions $\phi : U \to \mathbb{R}^k$. For $k = 2$, we define the *gradient* and *rotation*

$$\text{grad} : C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^k), \qquad \text{rot} : C^\infty(U, R^2) \to C^\infty(U, \mathbb{R})$$

Proposition 1.7.

$$\text{rot} \circ \text{grad} = 0$$

Definition 1.8.

$$H^1(U) = \ker(\text{rot})/\text{im}(\text{grad})$$

Remark 1.9. We have the following fact:

$$H^1\left(\mathbb{R}^2 - \bigcup_{i=1}^k \{x_i\}\right) \cong \mathbb{R}^k$$

$$\implies h^1\left(\mathbb{R}^2 - \bigcup_{i=1}^k \{x_i\}\right) = \#\{\text{holes}\}$$

Definition 1.10. We can define the *gradient* for $U \subset \mathbb{R}^k$ with $k \geq 1$ as follows

$$\text{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

Definition 1.11.

$$H^0(U) = \ker(\text{grad})$$

Remark 1.12.

$$0 \longrightarrow C^\infty(U, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(U, \mathbb{R})$$

Definition 1.13. An open set $U \subset \mathbb{R}^k$ is connected if and only if $H^0(U) = \mathbb{R}$.

Proof. If $\operatorname{grad}(f) = 0$, then f is locally constant. Hence, $f^{-1}(f(x_0))$ is open and closed, hence, the connected component of U, which is exactly U itself. $\square$

Remark 1.14.

$$h^0(U) = \#\{\text{the component of } U\}$$

Definition 1.15. When $k = 3$, we define the concept of *gradient, rotation, divergence.*

$$\begin{aligned}
\operatorname{grad} &: C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^3) \\
\operatorname{rot} &: C^\infty(U, \mathbb{R}^3) \to C^\infty(U, \mathbb{R}^3) \\
\operatorname{div} &: C^\infty(U, \mathbb{R}^3) \to C^\infty(U, \mathbb{R})
\end{aligned}$$

which is defined by

$$\begin{aligned}
\operatorname{grad}(f) &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \\
\operatorname{rot}(f_1, f_2, f_3) &= \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \\
\operatorname{div}(f_1, f_2, f_3) &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.
\end{aligned}$$

Proposition 1.16. *The sequence below*

$$0 \longrightarrow C^\infty(U, \mathbb{R}) \xrightarrow{\operatorname{grad}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\operatorname{rot}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\operatorname{div}} C^\infty(U, \mathbb{R})$$

is a complex, i.e.

$$\operatorname{rot} \circ \operatorname{grad} = 0, \qquad \operatorname{div} \circ \operatorname{rot} = 0$$

Definition 1.17.

$$H^2(U) = \ker(\operatorname{div}) / \operatorname{im}(\operatorname{rot}).$$

Theorem 1.18. *For an open star-shaped set in $\mathbb{R}^3$ we have that $H^0(U) = \mathbb{R}$, $H^1(U) = 0$ and $H^2(U) = 0$.*

Proof. First, $\operatorname{grad}(f) = 0$ implies that f is locally constant.

Assume $x_0 = 0$ again, then we can define a function

$$G(t) = F(tx_1, tx_2, tx_3)$$

$$G(t) = \int_0^t (x_1 f_1 + x_2 f_2 + x_3 f_3)\, ds$$

Then $F(x_1, x_2, x_3) = G(1)$

$$\begin{aligned}
\left.\frac{\partial F}{\partial x_1}\right|_{(x_1,x_2,x_3)} &= \int_0^1 \left(f_1 + t x_1 \frac{\partial f_1}{\partial x_1} + t x_2 \frac{\partial f_2}{\partial x_1} + t x_3 \frac{\partial f_3}{\partial x_1} \right) dt \\
&= \int_0^1 \left(f_1 + t x_1 \frac{\partial f_1}{\partial x_1} + t x_2 \frac{\partial f_1}{\partial x_2} + t x_3 \frac{\partial f_1}{\partial x_3} \right) dt \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \left(t f_1(t x_1, t x_2, t x_3) \right) dt \\
&= \left. t f_1(t x_1, t x_2, t x_3) \right|_{t=0}^1 \\
&= f(x_1, x_2, x_3)
\end{aligned}$$

For $H^2(U)$, $\square$

Chapter 2

Simplicial sets

2.1 Triangulated spaces

2.1.1 Main Definition