B.-Construction of moduli space

With the notations of A, we no longer distinguish between R and $R(k)$. Let R^{ss} (resp. R^s) be the set of points q of R such that the vector bundles $\mathcal{U}_q$ on X is semi-stable (resp. stable). It is clear that according to proposition 22 that they are $PGL(p)$-invariant subsets. We know from lemma 20 that for any vector bundle E on X whose isomorphism class belongs to $S(r, d)$, there is an element q of R^{ss} such that $\mathcal{U}_q$ is isomorphic to E. We can therefore say that R^{ss} (resp. R^s) "contains" all semi-stable (resp. stable) bundles of rank r and degree d on X.

In fact, according to proposition 22, we have

$$S(r, d) = R^{ss}/PGL(p), \quad \text{and} \quad S'(r, d) = R^s/PGL(p)$$

To obtain an "algebraic" quotient, it is necessary to use the theory of Mumford (see Mumford [16] and Newstead [29] for an introduction). We cannot get a quotient of an open $PGL(p)$-invariant of R, consisting of the points called "semi-stable" of R under the action of $PGL(p)$ (for the definition to see the further). Obviously the semi-stable points of R correspond to semi-stable bundles. Unfortunately, this seems difficult to prove directly and we resort to a stratagem:

We construct a projective variety Z on which $PGL(p)$ acts, much easier to handle than R. In particular, we can calculate the semi-stable points of Z. Then we define an injective $PGL(p)$-morphism

$$R \to Z$$

possessing sufficient properties so that the known information about Z to transfer to R.

Below is an overview of some results of Mumford's theory.

Some results of Mumford's theory

The following results are only valid for geometrically reductive algebraic groups:

An algebraic group G is said to be *geometrically reductive* if for any representation

$$G \to GL(n)$$

and any point v of k^n, other than 0, there exists a non-homogeneous polynomial constant with n variables f, G-invariant for the action of G on k^n induced by the previous representation, such that $f(v) \neq 0$.

The fact that in non-zero characteristic a group such that $PGL(p)$ is geometrically reductive has only recently been demonstrated by Haboush ([10]).

Let G be a geometrically reductive algebraic group operating on an algebraic variety Y.

A good quotient of Y by G is a couple (M, f), where $f : Y \to M$ is a morphism of algebraic variety such that:

1. f is affine, G-invariant, surjective.
2. If U is an open set of M,

$$f : H^0(U, \mathcal{O}_M) \to H^0(f^{-1}(U), \mathcal{O}_Y)$$

   induces an isomorphism of $H^0(U, \mathcal{O}_M)$ on the section space of $\mathcal{O}_Y$ on $f^{-1}(U)$ which are G-invariant.
3. If W is a closed G-invariant sub-variety of Y, $f(W)$ is a closed sub-variety of M.
4. If W_1 and W_2 are two closed G-invariant disjoint sub-variety of Y, the sub-varieties $f^{-1}(W__1) $ and $f^{-1}(W_2)$ of M are also disjoint.

A *good quotient* (M, f) of Y by G has the following universal property: if M is a finite type k-scheme and $f' : Y \to M'$ a G-invariant morphism, there exists a unique morphism $g : M \to M'$ such that $f' = g \circ f$.

A good quotient (M, f) of Y by G is called geometric quotient if it is an orbital space, in other words, if the map $Y/G \to M$ induced by f is bijective.

In general, there is no good quotient of Y by G. For this, there must first be a *linearization* of the action of G on Y, that is to say there exists a line bundle L on Y and a linear action of G on L, inducing that on Y. We can therefore only consider the existence of a good quotient ton some open G-invariants of Y.

A point y of Y (not necessarily closed) is called *semi-stable* if there is a positive integer n and a G-invariant section s of L^n such that $s(y) \neq 0$, and that the set Y_s of the points of Y where s does not vanish is an affine open set of Y.

y is *stable* if $\dim(G.y) = \dim(G)$ and if there is a section s as before and in addition the action of G on Y_s is closed, i.e. the orbits of closed point is a closed sub-variety of Y_s.

We denote by $Y^{ss}(L)$ the set of semi-stable points of Y, and $Y^s(L)$ the set of stable points. If the line bundle is fixed, we will use the simpler notation Y^{ss} and Y^s.

The subset $Y^s(L)$ and $Y^{ss}(L)$are open in set of Y, which could be empty.

An important special case" Y is a closed sub-variety of $\mathbb{P}_n$ and the action of G extends into an action of G on k^{n+1} which is linear, i.e. coming from representation of $G \to GL(n+1)$, and linearization resulting from the obvious action of $GL(n+1)$ on the line bundle $\mathcal{O}(1)$.

The most important result is the

Theorem 25 Let Y be a quasi-projective variety on which a geometrically reductive group G acts. We assume that given a linearization of the action of G on Y.

1. There exists a good quotient (M, f) of Y^{ss} by G, and M is an quasi-projective variety.
2. If Y is a closed sub-variety of $\mathbb{P}_n$ and if the action of G on Y comes from a linear action of G on k^{n+1}, the linearization is obviously G acts on $\mathcal{O}(1)$, M is a projective variety.
3. There is an open set M^s of M such that $f^{-1}(M^s) = Y^s$ and $(M^s, f|_{Y_s})$ is geometric quotient of Y^s by G.
4. If y_1 and y_2 are closed points of Y^{ss}, we have $f(y_1) = f(y_2)$ if and only if
$$\overline{G.y_1} \cap \overline{G.y_2} \cap Y^{ss} \neq \emptyset.$$
5. If Y is an element of Y^{ss}, y is stable if and only if $(\dim(G.y) = \dim(G))$ and if $G.y$ is closed in Y^{ss}.

We will use the following result, from Ramanathan [32]:

Proposition 26 Let G be an algebraic geometrically reductive group acting on the variety Y, Y' and $f : Y \to Y'$ an affine G-morphism. Then if Y' has a good quotient, it is the same for Y.

To apply theorem 25 in concrete cases, we have a useful numerical criterion for determining the (semi-) stable points of Y, closed sub-variety of $\mathbb{P}_n$, such that the action of G on Y comes from a linear action of G on k^{n+1}, the linearization is the obvious action of G on $\mathcal{O}(1)$. We set $L = \mathcal{O}(1)_Y$.

We call a *subgroup with one parameter* of G a non-trivial morphism of groups
$$c : k^* \to G.$$

We can show that there exists a basis $(e_0, \ldots, e_n)$ of $k^{n+1}$and integers $r_0, \ldots, r_n$ such that for any element t of k, we have
$$c(t).e_i = t^{r_i}.e_i, \qquad 0 \leq i \leq n$$

Let y be a point of Y and $y_0, \ldots, y_n$ the components in the base $(e_0, \ldots, e_n)$ from a point $\hat{y}$ of k^{n+1} above y. We set
$$\mu(y, c) = \max\left(\{-r_i,\, y_u \neq 0\}\right)$$

We can give an intrinsic definition of $\mu(y, c)$: it is the least integer μ such that the morphism
$$k^* \to \hat{Y}, \qquad t \mapsto t^\mu.c(t).\hat{y}$$
is extendable into a morphism $k \to \hat{Y}$, $\hat{Y}$ denoting the cone of Y.

The following theorem makes it possible to reduce the research for the (semi-)stable points of Y:

Theorem 27 One point y of Y is semi-stable (resp. stable) if and only if for any subgroup with one parameter c of G, we have

$$\mu(y,c) \geq 0 \qquad \text{(resp. } > 0).$$

For any integer $m > 0$, there exists a canonical linearization of the action of G on Y whose associated line bundle is L^m: for that it is enough to consider the Veronese embedding of Y in $\mathbb{P}_{\binom{n+m}{n}-1}$.

Let

$$c : k^* \to G$$

be a subgroup with one parameter. For any point y of Y we denote by $\mu_m(y,c)$ the number $\mu(y,c)$ corresponding to the linearization of the action of G on Y defined by L^m. It is easy to see that

$$\mu_m(y,c) = m.\mu(y,c).$$

Let q be an integer > 0, and for $1 \leq i \leq q$,, Y_i a closed sub-variety of $\mathbb{P}_{n_i}$ on which G acts, as before. Let L_i be the line bundle $\mathcal{O}(1)|_{Y_i}$. Each strictly positive integers sequence $(m_1, \ldots, m_q)$ defines a linearization of the action of G on $\prod_{i=1}^q Y_i$, whose associated line bundle on $\prod_{i=1}^q Y_i$ is

$$p_1^*(L_1)^{m_1} \otimes \ldots \otimes p_q^*(L_q)^{mq},$$

(denoting by p_j the projection $\prod_{i=1}^q Y_i \to Y_j$). To see this we use a Serge's embeddings.

Let $(y_1, \ldots, y_q)$ be a point of $\prod_{i=1}^q Y_i$. Then we can easily see that

$$\mu((y_1, \ldots, y_q), c) = \sum_{i=1}^q m_i.\mu(y_i, c).$$

We immediately deduce the

Proposition 28 One point $(y_1, \ldots, y_q)$ of $\prod_{i=1}^q Y_i$ is semi-stable (resp. stable) if and only if for any subgroup of one parameter c of G, we have

$$\sum_{i=1}^q m_i.\mu(y_i, c) \geq 0 \qquad \text{(resp. } > 0).$$

We can easily see that for any integer $p > 0$, the semi-stable (resp. stable) points of $\prod_{i=1}^q Y_i$, provided with (m_i) are the same as those of $\prod_{i=1}^q Y_i$, provided with $(p.m_i)$.

Consequently any multiple of sequence $(h_1, \ldots, h_q)$ of strictly positive rational numbers defines a family of linearization of the action of G on $\prod_{i=1}^q Y_i$ which gives the same (semi-)stable points.

The sequence $(h_1, \ldots, h_q)$ is called a polarization of the action of G on $\prod_{i=1}^q Y_i$.

Now let p be an integer such that $p \geq 2$, q an integer such that $1 \leq q \leq p-1$. We denote by $Gr(p,q)$ the Grassmannian of vector subspaces of k^p of dimension q, on which $SL(p)$ obviously acts. The variety $Gr(p,q)$ is immersed in $\mathbb{P}(\bigwedge^q k^p)$, and the action of $SL(p)$ comes from the representation

$$SL(p) \to GL\left(\bigwedge^q k^p\right), \qquad A \mapsto A \wedge \ldots \wedge A.$$

Let c be a one-parameter subgroup of $SL(p)$, defined by a base $(e_1, \ldots, e_p)$ of k^p, and integers $r_1, \cdots, r_p$ not all zero, such that

$$r_1 \geq \ldots \geq r_p \qquad \sum_{i=1}^p r_i = 0.$$

For $1 \leq i \leq p$ we have

$$c(t)(e_i) = t^{r_i}.e_i.$$

We denote by M_i the vector subspace of k^p generated by $e_1, \ldots, e_i$. Then we can show that we have, for any vector subspace E of k^p of dimension q,

$$\mu(E,c) = -q.r_p + \sum_{i=1}^{p-1} \dim(E \cap M_1).(r_{i+1} - r_i).$$

Let N be a strictly positive integer. Applying the above to $Gr(p,q)^N$, endowed with the polarization $(1, \ldots, 1)$, we get the

Theorem 29 An element $(E_1, \ldots, E_N)$ of $G(p,q)^N$ is semi-stable (resp. stable) if and only if, for any proper vector space E of k^p, we have

$$p.\sum_{i=1}^n \dim(E_j \cap E) \leq N.q.\dim(E)$$

(resp. $<$).

Let r be an integer such that $1 \leq r \leq p-1$.

Let $H_{p,r}$ be the Grassmannian of k-vector spaces of dimensional r quotients of k^p, and N a strictly positive integer. We set

$$Z = H_{p,r}^N.$$

On Z, $SL(p)$ acts in an obvious way. In view of the fact that $H_{p,r}$ is identified with $Gr(p,p-r)$, we deduce from the preceding theorem the

Theorem 30 Let $E_1, \dots, E_N$ be vector sub-spaces of dimension $p-r$ of k^p. For any vector space F of k^p we denote by F_j the image of F in k^p/E_j.

The point $(k^p/E_1, \dots, k^p/E_N)$ of Z is semi-stable (resp. stable) if and only if for any proper vector subspace F of k^p we have

$$\frac{r}{p} \leq \frac{1}{N.\dim(F)}.\sum_{i=1}^{N} \dim(F_j) \quad (\text{resp. } <).$$

Application of these results to the construction of moduli spaces

Let q be a point of R and x a point of X. The bundle $\mathcal{U}_q$ on X is a quotient of $\mathcal{O} \otimes k^p$, then the fiber $(\mathcal{U}_q)_x$ of $\mathcal{U}_q$ is a k-vector space of dimension r quotient of k^p.

Let N be a strictly positive integer and $x_1, \dots, x_N$ the distinct points of X. We can therefore define a map

$$\tau : R \to H^N_{p,r} = Z$$

by

$$\tau(q) = ((\mathcal{U}_q)_{x_1}, \dots, (\mathcal{U}_q)_{x_N}),$$

for any point q of R.

It is easy to see that τ is a $PGL(p)$-morphism.

The main tool for proving theorem 17 and 18 is the

Theorem 31 If the integers d and N are large enough, it is possible to choose the points $x_1, \dots, x_N$ so that \$we have the following properties:

1. τ is injective.
2. $R^{ss} = \tau^{-1}(Z^{ss})$
3. $R^s = \tau^{-1}(Z^s)$
4. $\tau : R^{ss} \to Z^{ss}$ is proper.

For a proof of this theorem, see Seshadri ([36], p.232) or Newstead ([29], p. 151). We obviously use theorem 30. For a general realization (demonstrated) of this theorem, see the seventh part.

Choose the points $x_1, \dots, x_N$ as in the previous theorem.

Morphism τ is proper and injective, therefore it is affine. There is a good quotient (Y, ϕ) of Z^{ss}, according to the theorem 25, therefore according to the proposition 26 there exists a good quotient $(U(r,d), f)$ of R^{ss}. The morphism τ is proper and injective, it is the same for the induced morphism

$$U(r,d) \to Y,$$

as we can easily see by using the properties of the good quotient.

Therefore $U(r,d)$ is a projective manifold, Y is one.