Algebraic Geometry

zyl

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Abstract

1 affine variety

Ex 1.1. *1.* $A(Y) = k[x,y]/(y-x^2) \cong k[x]$

2. $A(Z) = k[x,y]/(xy-1) \neq A[Y]$. $A(Z) = k[x,1/x]$ *which is not isomorphic to* $k[x]$*. Because all invertible section is* k *in* $k[x]$.

3. What is conic? conic section.

Ex 1.2. $A(Y) = k[x,y,z]/(y-x^2, y-x^3) \cong k[x]$.

Ex 1.3. $xz - x = 0 \iff x = 0 \vee z = 1$*, then we can find the three components are* $(x,y), (x,z), (z-1, x^2-y)$.

Ex 1.4. $\boldsymbol{A}^2 \neq \boldsymbol{A}^1 \times \boldsymbol{A}^1$.

Ex 1.5. *By corollary 1.4, we have that algebraic set is one-to-one correspondence to the radical ideals.*

Ex 1.6. Y *is not irreducible, then* $Y = Y_1 \cup Y_2$ *where* $Y_i = \overline{Y_i} \cap Y$*. Then* $\overline{Y} = \overline{Y_1} \cup \overline{Y_2}$*. If* Y *is not irreducible, then* $\overline{Y} = Z_1 \cup Z_2$ *then* $Y = (Z_1 \cap Y) \cup (Z_2 \cap Y)$.

Ex 1.7. *(a)* X *is noetherian, then it satisfies DCC, then every nonempty family of closed subsets has a minimal element, otherwise, we have a infinitely many descending chain. DCC also means ACC for open subset. In other word, every nonempty collection of open subset has a maximal one. Also, the DCC for closed subsets hold.*

(b) *Each open cover has a finite cover, if it does not have finite cover, then we can find an infinite sequence of open subset, such that $\bigcup_{i\leq n} U_i \subsetneq \bigcup_{i\leq n+1} U_i$. Then it can construct an infinitely ascending chain of open subset.*

(c) *In the induced topology, every closed subset in this subset is the intersection of closed subset and this subset. Hence it is DCC.*

(d) *Hausdorff space means every two points have non-intersect open nbhd. Then we can construct an infinitely many sequence of disjoint open set.*

Ex 1.8. *Let A be the coordinate ring of Y, then that of $H \cap Y$ is $A/(f)$ where f is the hypersurface. The irreducible component of $Y \cap H$ corresponds to the minimal prime ideal p.*

Theorem 1.8 tells us that

$$\text{height } p + \dim A/p = \dim A = r$$

Theorem 1.9 tells us that

$$\text{height } p = 1$$

Ex 1.9. *Using induction, every irreducible component corresponds to a prime ideal p. Then $p \supset (f_1, f_2, \ldots, f_r)$, then*

$$\dim A/p + \text{height } p = \dim A$$

We have to show that $n - r \leq \dim A - \text{height } p$

Ex 1.10. (a)

(b)

(c)

2 Projective Varieties

Ex 2.1. *Prove "homogeneous Nullstellensatz"*