4.2 Tests on spatial dependence in the errors

We now present several tests on spatial dependence in the error terms of a standard regression model. If the disturbances are spatially correlated, the assumption of a spherical error covariance matrix,

(4.19) $$\mathrm{Cov}(\boldsymbol{\varepsilon}) = \mathrm{E}(\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}') = \sigma^2 \cdot \mathbf{I}$$

is violated.

The special form of the error covariance matrix depends on the spatial process the disturbances are generated from. The simplest spatial error process is a spatially autocorrelated process of first order [SAR(1) error process]:

(4.20) $$\boldsymbol{\varepsilon} = \lambda \mathbf{W} \boldsymbol{\varepsilon} + \mathbf{v}$$

that is defined analogous to Markov process in time-series analysis. λ is termed **spatial autoregressive coefficient**. For the error term v the classical assumptions are assumed to hold:

$$\mathrm{E}(\mathbf{v}) = \mathbf{o} \quad \text{and} \quad \mathrm{Cov}(\mathbf{v}) = \mathrm{E}(\mathbf{v} \cdot \mathbf{v}') = \sigma^2 \cdot \mathbf{I}$$

The covariance matrix of the spatially autocorrelated errors ε is of the form

(4.21)
$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{\varepsilon}) = \mathrm{E}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}') &= \mathrm{E}\left\{\left[(\mathbf{I}-\lambda\mathbf{W})^{-1}\mathbf{v}\right]\cdot\left[(\mathbf{I}-\lambda\mathbf{W})^{-1}\mathbf{v}\right]'\right\} \\
&= \mathrm{E}\left\{(\mathbf{I}-\lambda\mathbf{W})^{-1}\mathbf{v}\mathbf{v}'\left[(\mathbf{I}-\lambda\mathbf{W})^{-1}\right]'\right\} \\
&= (\mathbf{I}-\lambda\mathbf{W})^{-1}\cdot\mathrm{E}(\mathbf{v}\mathbf{v}')\cdot\left[(\mathbf{I}-\lambda\mathbf{W})^{-1}\right]' \\
&= \sigma^2\cdot(\mathbf{I}-\lambda\mathbf{W})^{-1}\cdot(\mathbf{I}-\lambda\mathbf{W}')^{-1}
\end{aligned}$$

In the following we present three tests for detecting spatial dependence in the error terms:

1. The Moran test ,
2. a Lagrange Multiplier test for spatial error dependence [LM(err)],
3. a Lagrange Multiplier test for spatial lag dependence [LM(lag)]

While the Moran test for spatial error autocorrelation is a general test, the LM tests are more specific. They provide a basis for choosing an appropriate spatial regression model. Significance of LM(err) points to a spatial error model, while significance of LM(lag) points to a spatial lag model.

4.2.1 The Moran test

We have introduced the Moran I statistic for establishing spatial autocorrelation of a georeferenced variable X. It can be, however, also straightforwardly applied for testing spatial autocorrelation in the regression residuals.

When using an unstandardized weight matrix **W***, Moran's I reads

(4.22) $$I = \frac{n}{S_0} \frac{\mathbf{e}'\mathbf{W}^*\mathbf{e}}{\mathbf{e}'\mathbf{e}} \quad \text{with} \quad S_0 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}^*$$

e: nx1 vector of OLS residuals

When the standardardized weight matrix **W** is used, formula (4.22) simplifies to

(4.23) $$I = \frac{\mathbf{e}'\mathbf{W}\mathbf{e}}{\mathbf{e}'\mathbf{e}}$$

because of $S_0 = n$.

I is interpretable as the coefficient of an OLS regression of **W*e** on **e** or **We** on **e**, respectively.

● **Significance test of Moran's I**

The standardized Moran coefficient follows a standard normal distribution under the null hypothesis of no spatial dependence.

Null hypothesis H_0: Absence of spatial dependence
Alternative hypothesis H_1: Presence of spatial dependence

The cause of spatial dependence under H_1 is unspecified, i.e. the underlying spatial process is not specified. Thus the Moran test is a general test for detecting spatial autocorrelation.

Test statistic: (4.24) $Z(I) = \dfrac{I - E(I)}{\sqrt{Var(I)}}$ ~ N(0,1)

Expected value: (4.25) $E(I) = tr(\mathbf{MW})/(n-k)$

Projection matrix **M**: $\mathbf{M} = \mathbf{I} - \underbrace{\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'}_{=\mathbf{H}\text{ (hat matrix)}}$

tr(**A**): trace of matrix **A**

Variance: (4.26) $Var(I) = \dfrac{tr(\mathbf{MWMW}') + tr(\mathbf{MWMW}) + [tr(\mathbf{MW})]^2}{(n-k)(n-k+2)} - [E(I)]^2$

Example:

We conduct the Moran test for residual spatial autocorrelation for the estimated Verdoorn relationship with the standardized weight matrix.

Vector of residuals: $$\mathbf{e} = \begin{bmatrix} -0.0503 \\ -0.0062 \\ 0.0600 \\ -0.1297 \\ 0.1262 \end{bmatrix}$$

Standardized weight matrix: $$\mathbf{W} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- **Calculation of Moran‘s I**

Numerator:

$$\mathbf{e'We} = \begin{bmatrix} -0.0503 & -0.0062 & 0.0600 & -0.1297 & 0.1262 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -0.0503 \\ -0.0062 \\ 0.0600 \\ -0.1297 \\ 0.1262 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0503 & -0.0062 & 0.0600 & -0.1297 & 0.1262 \end{bmatrix} \begin{bmatrix} 0.0269 \\ -0.0400 \\ -0.0621 \\ 0.0600 \\ -0.1297 \end{bmatrix} = -0.0290$$

Denominator:

$$\mathbf{e}'\mathbf{e} = \begin{bmatrix} -0.0503 & -0.0062 & 0.0600 & -0.1297 & 0.1262 \end{bmatrix} \begin{bmatrix} -0.0503 \\ -0.0062 \\ 0.0600 \\ -0.1297 \\ 0.1262 \end{bmatrix} = 0.0389$$

Moran's I: $$\mathrm{I} = \frac{\mathbf{e}'\mathbf{We}}{\mathbf{e}'\mathbf{e}} = \frac{-0.0290}{0.0389} = -0.7455$$

- **Significance test of Moran's I**

Observation matrix **X**: $\mathbf{X} = \begin{bmatrix} 1 & 0.6 \\ 1 & 1.0 \\ 1 & 1.6 \\ 1 & 2.6 \\ 1 & 2.2 \end{bmatrix}$

Inverse product matrix:

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 1.1412 & -0.5882 \\ -0.5882 & 0.3676 \end{bmatrix}$$

Projection matrix **M**: $\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0.6 \\ 1 & 1.0 \\ 1 & 1.6 \\ 1 & 2.6 \\ 1 & 2.2 \end{bmatrix} \begin{bmatrix} 1.1412 & -0.5882 \\ -0.5882 & 0.3676 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.6 & 1.0 & 1.6 & 2.6 & 2.2 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.7883 & 0.3676 \\ 0.5530 & -0.2206 \\ 0.2001 & -0.0000 \\ -0.3881 & 0.3676 \\ -0.1528 & 0.2205 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.6 & 1.0 & 1.6 & 2.6 & 2.2 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5677 & 0.4206 & 0.2001 & -0{,}1676 & -0.0205 \\ 0.4206 & 0.3324 & 0.2000 & -0.0206 & 0.0677 \\ 0.2001 & 0.2000 & 0.2000 & 0.2000 & 0.2000 \\ -0.1676 & -0.0206 & 0.2000 & 0.5675 & 0.4205 \\ -0.0205 & 0.0677 & 0.2000 & 0.4205 & 0.3323 \end{bmatrix}
$$

$$
\mathbf{M} = \begin{bmatrix} 0.4323 & -0.4206 & -0.2001 & 0{,}1676 & 0.0205 \\ -0.4206 & 0.6676 & -0.2000 & 0.0206 & -0.0677 \\ -0.2001 & -0.2000 & 0.8000 & -0.2000 & -0.2000 \\ 0.1676 & 0.0206 & -0.2000 & 0.4325 & -0.4205 \\ 0.0205 & -0.0677 & -0.2000 & -0.4205 & 0.6677 \end{bmatrix}
$$

Expected value: $E(I) = tr(\mathbf{MW})/(n-k)$

$$\mathbf{MW} = \begin{bmatrix} 0.4323 & -0.4206 & -0.2001 & 0{,}1676 & 0.0205 \\ -0.4206 & 0.6676 & -0.2000 & 0.0206 & -0.0677 \\ -0.2001 & -0.2000 & 0.8000 & -0.2000 & -0.2000 \\ 0.1676 & 0.0206 & -0.2000 & 0.4325 & -0.4205 \\ 0.0205 & -0.0677 & -0.2000 & -0.4205 & 0.6677 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2069 & 0.2053 & 0.1318 & -0.1864 & 0.0559 \\ 0.1559 & -0.2701 & 0.0191 & 0.0882 & -0.0069 \\ 0.2000 & 0.1000 & -0.2334 & -0.0000 & -0.0667 \\ -0.0598 & 0.1613 & 0.2348 & -0.4803 & 0.1442 \\ -0.0892 & -0.1966 & -0.1525 & 0.5785 & -0.1402 \end{bmatrix}$$

Trace of matrix product **MW**: tr(**MW**) = -1.3309

$$E(I) = tr(\mathbf{MW})/(n-k) = -1.3309/(5-2) = -0.4436$$

Variance of Moran‘s I:

$$\mathrm{Var}(\mathrm{I}) = \frac{\mathrm{tr}(\mathbf{MWMW}') + \mathrm{tr}(\mathbf{MWMW}) + [\mathrm{tr}(\mathbf{MW})]^2}{(\mathrm{n}-\mathrm{k})(\mathrm{n}-\mathrm{k}+2)} - [\mathrm{E}(\mathrm{I})]^2$$

$$\mathbf{MW} = \begin{bmatrix} -0.2069 & 0.2053 & 0.1318 & -0.1864 & 0.0559 \\ 0.1559 & -0.2701 & 0.0191 & 0.0882 & -0.0069 \\ 0.2000 & 0.1000 & -0.2334 & -0.0000 & -0.0667 \\ -0.0598 & 0.1613 & 0.2348 & -0.4803 & 0.1442 \\ -0.0892 & -0.1966 & -0.1525 & 0.5785 & -0.1402 \end{bmatrix}$$

$$\mathbf{MW}' = \begin{bmatrix} -0.3103 & 0.1333 & 0.0597 & -0.2001 & 0.1676 \\ 0.2338 & -0.2000 & 0.0892 & 0.1333 & -0.0206 \\ 0.3000 & 0.1333 & -0.2000 & 0.1333 & -0.2000 \\ -0.0897 & 0.1334 & 0.2069 & -0.2000 & 0.4325 \\ -0.1338 & -0.2000 & -0.1559 & 0.1333 & -0.4205 \end{bmatrix}$$

$$\mathbf{MWMW}' = \begin{bmatrix} 0.1610 & -0.0871 & -0.0677 & 0.1310 & -0.1609 \\ -0.1146 & 0.0877 & -0.0014 & -0.0814 & 0.0520 \\ -0.0998 & -0.0111 & -0.0014 & -0.0814 & 0.0520 \\ 0.1505 & -0.1018 & -0.1580 & 0.1800 & -0.3220 \\ -0.0971 & 0.1123 & 0.1492 & -0.1630 & 0.3206 \end{bmatrix}$$

Trace of **MWMW**‘: tr(**MWMW**‘) **=** 0.8273

$$\mathbf{MWMW} = \begin{bmatrix} 0.1073 & -0.1258 & -0.1064 & 0.1785 & -0.0536 \\ -0.0764 & 0.1198 & 0.0306 & -0.0913 & 0.0173 \\ -0.0665 & 0.0038 & 0.0929 & -0.0670 & 0.0368 \\ 0.1003 & -0.1382 & -0.1944 & 0.3395 & -0.1073 \\ -0.0648 & 0.1404 & 0.1773 & -0.3596 & 0.1069 \end{bmatrix}$$

Trace of **MWMW**: tr(**MWMW**) = 0.7663

$$\mathrm{Var}(I) = \frac{\mathrm{tr}(\mathbf{MWMW}') + \mathrm{tr}(\mathbf{MWMW}) + [\mathrm{tr}(\mathbf{MW})]^2}{(n-k)(n-k+2)} - [E(I)]^2$$

$$= \frac{0.8273 + 0.7663 + (-1.3309)^2}{(5-2)(5-2+2)} - 0.4436^2$$

$$= \frac{3.3649}{15} - 0.1968 = 0.2243 - 0.1968 = 0.0275$$

Test statistic:

$$z(I) = \frac{I - E(I)}{\sqrt{\mathrm{Var}(I)}} = \frac{-0.7455 - (-0.4436)}{\sqrt{0.0275}} = \frac{-0.3019}{0.1658} = -1.821$$

Critical value (α=0.05, two-sided test): $z_{0.975} = 1.96$

Testing decision: $|z(i)=-1.821| < (z_{0.975} = 1.96)$ => Accept H_0

4.2.2 Lagrange multiplier test for spatial error dependence

Unlike the Moran test Lagrange multiplier tests rely on well structured hypotheses The **Lagrange multiplier test for spatial dependence** (**LM error test**) is based on the estimation of the regression model (4.1) with spatially autocorrelated errors (4.20)

$$\boldsymbol{\varepsilon} = \lambda \mathbf{W} \boldsymbol{\varepsilon} + \mathbf{v}$$

under the null hypothesis

$$H_0: \lambda = 0.$$

This means that OLS estimation of the model (4.1)

$$\mathbf{y} = \mathbf{X} \cdot \boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

suffices for conducting the LM error test. The alternative hypothesis claims a spatial autoregressive coefficient λ unequal to zero:

$$H_1: \lambda \neq 0.$$

Test statistic:

(4.27) $$\mathrm{LM}_e = \frac{(\mathbf{e}'\mathbf{W}\mathbf{e}/s^2)^2}{T}$$

with $s^2 = \mathbf{e}'\mathbf{e}/n$ and $T = \mathrm{tr}[(\mathbf{W}+\mathbf{W}')\mathbf{W}]$

The test statistic is distributed as χ^2 (chi-square) with one degree of freedom.

Critical value (significance level α): $\chi^2(1;1-\alpha)$

Testing decision:

$\mathrm{LM}_e > \chi^2(1;1-\alpha)$ => Reject H_0

Example:

The LM error test is conducted for the relationship between productivity growth and output growth with the standardized weight matrix.

Vector of residuals: $\mathbf{e} = \begin{bmatrix} -0.0503 \\ -0.0062 \\ 0.0600 \\ -0.1297 \\ 0.1262 \end{bmatrix}$

Standardized weight matrix: $\mathbf{W} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Quantities of the numerator:

$$\mathbf{e'We} = \begin{bmatrix}-0.0503 & -0.0062 & 0.0600 & -0.1297 & 0.1262\end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -0.0503 \\ -0.0062 \\ 0.0600 \\ -0.1297 \\ 0.1262 \end{bmatrix}$$

$$= \begin{bmatrix}-0.0503 & -0.0062 & 0.0600 & -0.1297 & 0.1262\end{bmatrix} \begin{bmatrix} 0.0269 \\ -0.0400 \\ -0.0621 \\ 0.0600 \\ -0.1297 \end{bmatrix}$$ = -0.0290

$$\mathbf{e'e} = \begin{bmatrix}-0.0503 & -0.0062 & 0.0600 & -0.1297 & 0.1262\end{bmatrix} \begin{bmatrix} -0.0503 \\ -0.0062 \\ 0.0600 \\ -0.1297 \\ 0.1262 \end{bmatrix}$$ = 0.0389

$s^2 = \mathbf{e'e}/n$ = 0.0389 / 5= 0.00778

Denominator:

$$T = \mathrm{tr}[(\mathbf{W} + \mathbf{W}')\mathbf{W}]$$

$$\mathbf{W} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{W}' = \begin{bmatrix} 0 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1/3 & 1/3 & 0 \\ 1/2 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1 \\ 0 & 0 & 0 & 1/3 & 0 \end{bmatrix}$$

$$\mathbf{W} + \mathbf{W}' = \begin{bmatrix} 0 & 5/6 & 5/6 & 0 & 0 \\ 5/6 & 0 & 2/3 & 2/3 & 0 \\ 5/6 & 2/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 2/3 & 0 & 4/3 \\ 0 & 0 & 0 & 4/3 & 0 \end{bmatrix}$$

$$(\mathbf{W}+\mathbf{W}')\mathbf{W} = \begin{bmatrix} 0 & 5/6 & 5/6 & 0 & 0 \\ 5/6 & 0 & 2/3 & 2/3 & 0 \\ 5/6 & 2/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 2/3 & 0 & 4/3 \\ 0 & 0 & 0 & 4/3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5556 & 0.2778 & 0.2778 & 0.5556 & 0 \\ 0.2222 & 0.8611 & 0.6389 & 0.2222 & 0.2222 \\ 0.2222 & 0.6389 & 0.8611 & 0.2222 & 0.2222 \\ 0.4444 & 0.2222 & 0.2222 & 1.7778 & 0 \\ 0 & 0.4444 & 0.4444 & 0 & 0.4444 \end{bmatrix}$$

$$\mathrm{T} = \mathrm{tr}[(\mathbf{W}+\mathbf{W}')\mathbf{W}] = 4.5$$

Test statistic:

$$\mathrm{LM}_e = \frac{(\mathbf{e}'\mathbf{We}/s^2)^2}{T} = \frac{(-0.0290/0.00778)^2}{4.5} = \frac{13.8943}{4.5} = 3.0876$$

Critical value (α=0.05): $\chi^2(1;0.95) = 3.841$

Testing decision:

($LM_e = 3.0876$) < ($\chi^2(1;0.95) = 3.841$) => Accept H_0

4.2.2 Lagrange multiplier test for spatial lag dependence

Spatial dependence in regression models may not only be reflected in the error. Instead it may be accounted by entering a spatial lag **Wy** in the endogenous variable Y. In this case the regression model reads

(4.28) $$\mathbf{y} = \rho \cdot \mathbf{W} \cdot \mathbf{y} + \mathbf{X} \cdot \boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

Under the null hypothesis

$$H_0: \rho = 0$$

the standard regression model (4.1)

$$\mathbf{y} = \mathbf{X} \cdot \boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

holds, while under the alternative hypothesis

$$H_1: \rho \neq 0$$

the extended regression model (4.28) would be valid.

For conducting the Lagrange multiplier test for spatial lag dependence (LM lag test) again only the standard regression model (4.1) is to be estimated.

Test statistic:

(4.29) $$\mathrm{LM}_{\ell} = \frac{(\mathbf{e}'\mathbf{W}\mathbf{y}/s^2)^2}{(nJ)}$$

with $$nJ = T + (\mathbf{W}\mathbf{X}\boldsymbol{\beta})'\mathbf{M}(\mathbf{W}\mathbf{X}\boldsymbol{\beta})/s^2$$

$$T = \mathrm{tr}[(\mathbf{W} + \mathbf{W}')\mathbf{W}]$$

$$\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$$

The test statistic is distributed as χ^2 (chi-square) with one degree of freedom.

Critical value (significance level α): $\chi^2(1;1-\alpha)$

Testing decision:

$\mathrm{LM}_{\ell} > \chi^2(1;1-\alpha)$ => Reject H_0

Example:

The LM lag test is conducted for the relationship between productivity growth and output growth with the standardized weight matrix.

Vector of residuals: $$\mathbf{e} = \begin{bmatrix} -0.0503 \\ -0.0062 \\ 0.0600 \\ -0.1297 \\ 0.1262 \end{bmatrix}$$

Vector of endogenous variable: $$\mathbf{y} = \begin{bmatrix} 0.4 \\ 0.6 \\ 0.9 \\ 1.1 \\ 1.2 \end{bmatrix}$$

Observation matrix:

$$\mathbf{X} = \begin{bmatrix} 1 & 0.6 \\ 1 & 1.0 \\ 1 & 1.6 \\ 1 & 2.6 \\ 1 & 2.2 \end{bmatrix}$$

OLS estimator of Regression coefficients:

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} 0.2165 \\ 0.3897 \end{bmatrix}$$

Standardized weight matrix:

$$\mathbf{W} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Quantities of numerator:

$$\mathbf{e}'\mathbf{W}\mathbf{y} = \begin{bmatrix} -0.0503 & -0.0062 & 0.0600 & -0.1297 & 0.1262 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \\ 0.9 \\ 1.1 \\ 1.2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0503 & -0.0062 & 0.0600 & -0.1297 & 0.1262 \end{bmatrix} \begin{bmatrix} 0.75 \\ 0.8 \\ 0.7 \\ 0.9 \\ 1.1 \end{bmatrix} = 0.0214$$

$$s^2 = \mathbf{e}'\mathbf{e} / n = 0.0389 / 5 = 0.00778$$

Quantities of denominator:

$$\mathrm{nJ} = \mathrm{T} + (\mathbf{WX\beta})'\mathbf{M}(\mathbf{WX\beta}) / \mathrm{s}^2$$

$$\mathrm{T} = \mathrm{tr}[(\mathbf{W} + \mathbf{W}')\mathbf{W}] = 4.5$$

$$\mathbf{M} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \begin{bmatrix} 0.4323 & -0.4206 & -0.2001 & 0,1676 & 0.0205 \\ -0.4206 & 0.6676 & -0.2000 & 0.0206 & -0.0677 \\ -0.2001 & -0.2000 & 0.8000 & -0.2000 & -0.2000 \\ 0.1676 & 0.0206 & -0.2000 & 0.4325 & -0.4205 \\ 0.0205 & -0.0677 & -0.2000 & -0.4205 & 0.6677 \end{bmatrix}$$

$$\mathbf{WX\beta} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.6 \\ 1 & 1.0 \\ 1 & 1.6 \\ 1 & 2.6 \\ 1 & 2.2 \end{bmatrix} \begin{bmatrix} 0.2165 \\ 0.3897 \end{bmatrix} = \begin{bmatrix} 1 & 1.3 \\ 1 & 1.6 \\ 1 & 1.4 \\ 1 & 1.6 \\ 1 & 2.6 \end{bmatrix} \begin{bmatrix} 0.2165 \\ 0.3897 \end{bmatrix} = \begin{bmatrix} 0.7231 \\ 0.8400 \\ 0.7621 \\ 0.8400 \\ 1.2297 \end{bmatrix}$$

$$(\mathbf{WX\beta})'\mathbf{M}(\mathbf{WX\beta}) = \begin{bmatrix}0.7231 & 0.8400 & 0.7621 & 0.8400 & 1.2297\end{bmatrix}$$

$$\begin{bmatrix} 0.4323 & -0.4206 & -0.2001 & 0{,}1676 & 0.0205 \\ -0.4206 & 0.6676 & -0.2000 & 0.0206 & -0.0677 \\ -0.2001 & -0.2000 & 0.8000 & -0.2000 & -0.2000 \\ 0.1676 & 0.0206 & -0.2000 & 0.4325 & -0.4205 \\ 0.0205 & -0.0677 & -0.2000 & -0.4205 & 0.6677 \end{bmatrix} \begin{bmatrix} 0.7231 \\ 0.8400 \\ 0.7621 \\ 0.8400 \\ 1.2297 \end{bmatrix}$$

$$= \begin{bmatrix}-0.0269 & 0.0384 & -0.1169 & -0.1679 & 0.2734\end{bmatrix} \begin{bmatrix} 0.7231 \\ 0.8400 \\ 0.7621 \\ 0.8400 \\ 1.2297 \end{bmatrix} = 0.1188$$

$$\mathrm{nJ} = \mathrm{T} + (\mathbf{WX\beta})'\mathbf{M}(\mathbf{WX\beta}) / s^2 = 4.5 + 0.1188 / 0.00778 = 19.7699$$

Test statistic:

$$\mathrm{LM}_{\ell} = \frac{(\mathbf{e}'\mathbf{W}\mathbf{y}/s^2)^2}{(nJ)} = \frac{(0.0214/0.00778)^2}{19.7699} = \frac{7.5660}{19.7699} = 0.3827$$

Critical value (α=0.05): χ2(1;0.95) = 3.841

Testing decision:

(LM_{ℓ} = 0.3827) < (χ^2(1;0.95) = 3.841) => Accept H_0