- Deadline: 10/27, 請於實習課講解作業前繳交。
- 可以分組討論並標記組員姓名, 但一人需繳交一份 (如果參考組員請重新寫過, 不要直接照抄)。
- 收完作業後會公布解答, 因此逾時不候, 若當天真的無法到場, 請提前將作業掃描並寄到 r07323047@ntu.edu.tw。

Problem

1. Let X be a discrete random variable

$$f_X(x) = p(1-p)^{x-1},\ x = 1, 2, \ldots, \infty \quad (0 < p < 1)$$

   (a) Show that $f_X(x)$ is a well define p.m.f..
   (b) Compute $E(X)$、$Var(X)$.
   (c) Find the MGF of X.

2. Given $Y|X \sim \text{Binom}(x, p)$, and $X \sim \text{Poisson}(\lambda)$ (ie. the pmf of X is $f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \cdots$)
   (a) Compute $E(X)$.
   (b) Compute $Var(X)$.
   (c) Show that $Y \sim \text{Poisson}(\lambda p)$.
   (d) Compute $E(X|Y)$.

3. (不計分) Let the distribution of Z be the standard normal distribution. Then
   (a) $P(Z > -0.6) = 0.5 + P(Z > 0.6)$
   (b) $P(Z < -0.6) = 1 - P(Z > 0.6)$
   (c) $P(Z = 0) = 0.5$
   (d) $P(-2 < Z < 2) > 0.99$

4. In 1982 Bob's mother scored at 93^{rd} percentile in the math SAT exam. In 1982 the mean score was 503 and the variance of the scores was 9604. In 2008 Bob took the math SAT and got the same numerical score as his mother has received 26 years

before. In 2008 the mean score was 521 and the variance of the score was 10201. Math SAT scores are normally distributed. Calculate the percentile for Bob's score.

5. Let $X \sim \mathrm{U}[a, b]$, then the p.d.f. of X as following

$$f_X(x) = \frac{1}{b-a}, \ a \le x \le b \quad (a, b \text{ are constant.})$$

   (a) Show that $f_X(x)$ is a well define p.d.f..

   (b) Compute $E(X)$、$Var(X)$.

6. Given $X \sim \mathrm{U}[-1, 2]$, and let $Y = X^2$.

   (a) Compute $E(X)$ and $Var(X)$.

   (b) Find the p.d.f. of Y.

參考解答

1a. We have to show the following two things

- (nonnegative):

$$\because p、(1-p)^{x-1} \text{ are greater than } 0 \quad \therefore f_X(x) \geq 0$$

- (sum is equal to 1):

$$\sum_{x=1}^{\infty} p(1-p)^{x-1} = p \cdot \sum_{x=1}^{\infty} (1-p)^{x-1} = p \cdot \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1$$

1b. Let $q = 1 - p$

$$E(X) = \sum_{x=1}^{\infty} x \cdot pq^{x-1} = p \cdot \sum_{x=0}^{\infty} \frac{\mathrm{d}q^x}{\mathrm{d}q} = p \cdot \frac{\mathrm{d}}{\mathrm{d}q}(\sum_{x=0}^{\infty} q^x) = p \cdot \frac{\mathrm{d}}{\mathrm{d}q}(\frac{1}{1-q}) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E[X(X-1)] = \sum_{x=1}^{\infty} x(x-1) \cdot pq^{x-1} = pq \cdot \sum_{x=1}^{\infty} x(x-1) \cdot q^{x-2} = pq \cdot \sum_{x=1}^{\infty} \frac{\mathrm{d}^2 q^x}{\mathrm{d}q^2} = pq \cdot \frac{\mathrm{d}^2}{\mathrm{d}q^2}(\sum_{x=1}^{\infty} q^x)$$

$$= pq \cdot \frac{\mathrm{d}^2}{\mathrm{d}q^2}(\frac{1}{1-q}) = pq \cdot \frac{2(1-q)}{(1-q)^4} = \frac{2q}{(1-q)^2}$$

$$Var(X) = E[X(X-1)] + E(X) - [E(X)]^2 = \frac{2q}{(1-q)^2} + \frac{1}{1-q} - (\frac{1}{1-q})^2 = \frac{1-p}{p^2}$$

2a.

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} = \lambda$$

2b.

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} = \lambda \sum_{y=0}^{\infty} (y+1) \frac{e^{-\lambda}\lambda^y}{y!} = \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - [E(X)]^2 = \lambda$$

2c.

$$f_{Y|X}(y) = \binom{x}{y} p^y (1-p)^{x-y}, \quad y = 0, 1, \cdots, x$$

$$\begin{aligned} f_{XY}(x,y) &= f_{Y|X}(y) \cdot f_X(x) \\ &= \binom{x}{y} p^y (1-p)^{x-y} \cdot \frac{e^{-\lambda}\lambda^x}{x!}, \quad y = 0, 1, \cdots x;\ x = y, y+1, \cdots \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \sum_{x=y}^{\infty} f_{XY}(x,y) = \frac{p^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{\lambda^x (1-p)^{x-y}}{(x-y)!} \\ &= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!} = \frac{(\lambda p)^y e^{-\lambda p}}{y!}, \quad y = 0, 1, \cdots \end{aligned}$$

2d.

$$f_{X|Y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{e^{-\lambda p}[\lambda(1-p)]^{x-y}}{(x-y)!}, \quad x = y, y+1, \cdots$$

另 $z = x - y$, 可以得到

$$f_{Z|Y}(z) = \frac{e^{-\lambda(1-p)}[\lambda(1-p)]^z}{z!}, \quad z = 0, 1, \cdots$$

可看出 $Z|Y \sim Poisson(\lambda(1-p))$, 因此 $E(X-Y|Y) = \lambda(1-p)$, 故

$$E(X|Y) = E(X-Y|Y) + Y = \lambda(1-p) + Y$$

3a. False,

$$\begin{aligned} P(Z > -0.6) &= 1 - P(Z < -0.6) = 1 - 0.2743 = 0.7257 \\ P(Z > 0.6) &= 1 - P(Z < 0.6) = 1 - 0.7257 = 0.2743 \\ 0.5 + 0.2743 &= 0.7743 \neq 0.7257 \end{aligned}$$

3b. False, $1 - P(Z > 0.6) = P(Z < 0.6) \neq P(Z < -0.6)$

3c. False, Z is continuous random variable, so $P(Z = 0) = 0$.

3d. False, $P(-2 < Z < 2) = P(Z < 2) - P(Z < -2) = 0.9772 - 0.0228 < 0.99$

4. $P(Z \leq 1.475) \approx 0.93$, therefore the score Bob's mom got is

$$503 + 1.475\sqrt{9604} = 647.55$$

Let $X \sim \mathrm{N}(521, 10201)$, then

$$P(X \leq 647.55) = P(Z \leq \frac{647.55 - 521}{\sqrt{10201}}) \approx P(Z < 1.25) = 0.8944$$

Thus, Bob scored at 89.44 percentile.

5a. We have to show the following two things

- (nonnegative):

$$b - a > 0 \Rightarrow \frac{1}{b-a} = f_X(x) \geq 0 \quad \forall x \in [a, b]$$

- (sum is equal to 1):

$$\int_a^b \frac{1}{b-a}\,\mathrm{d}x = [\frac{x}{b-a}]_{x=a}^b = 1$$

5b.

$$E(X) = \int_a^b x \cdot f_X(x)\,\mathrm{d}x = \frac{1}{b-a} \cdot \int_a^b x\,\mathrm{d}x = \frac{1}{b-a} \cdot [\frac{x^2}{2}]_{x=a}^b = \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{a+b}{2}$$

$$E(X^2) = \int_a^b x^2 \cdot f_X(x)\,\mathrm{d}x = \frac{1}{b-a} \cdot \int_a^b x^2\,\mathrm{d}x = \frac{1}{b-a} \cdot [\frac{x^3}{3}]_{x=a}^b = \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3} = \frac{a^2 + ab + b^2}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{(b-a)^2}{3} - [\frac{b-a}{2}]^2 = \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$$

6a.

$$E(X) = \int_{-1}^2 x\frac{1}{3}\,\mathrm{d}x = \frac{1}{2}$$

$$E(X^2) = \int_{-1}^2 x^2\frac{1}{3}\,\mathrm{d}x = 1$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{4}$$

6b. Case1: $|X| \leq 1$

$$F_Y(y) = P(Y \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \frac{2\sqrt{y}}{3}$$

Case2: $1 < X < 2$

$$F_Y(y) = P(Y \leq y) = \frac{2}{3} + P(1 < X \leq \sqrt{y}) = \frac{1}{3}\sqrt{y} + \frac{1}{3}$$

合併可得

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{2\sqrt{y}}{3}, & 0 \leq y \leq 1 \\ \frac{1}{3}\sqrt{y} + \frac{1}{3}, & 1 < y \leq 4 \\ 1, & \text{otherwise} \end{cases}$$

因此

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}}, & 0 \leq y \leq 1 \\ \frac{1}{6\sqrt{y}}, & 1 < y \leq 4 \\ 0, & \text{otherwise} \end{cases}$$