Closed Cap Condition under the Cap Construction Algorithm

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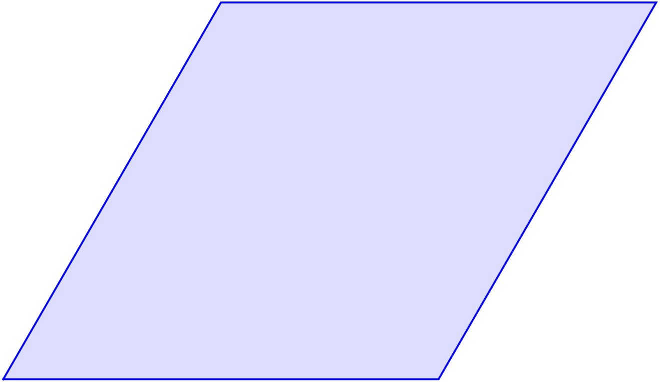
Parallelogram to Tetrahedron



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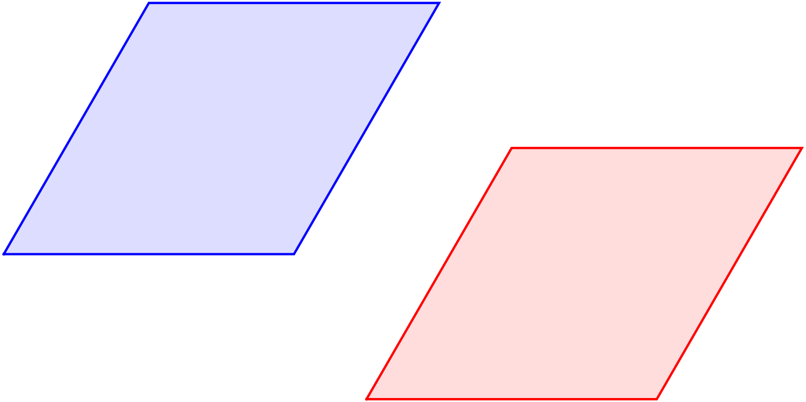

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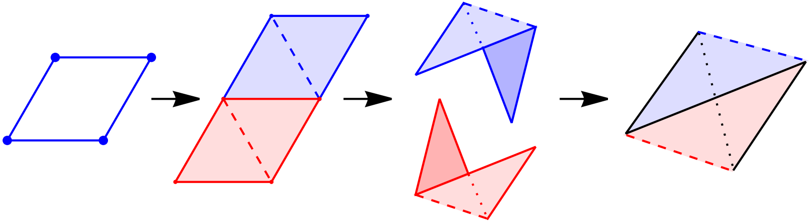
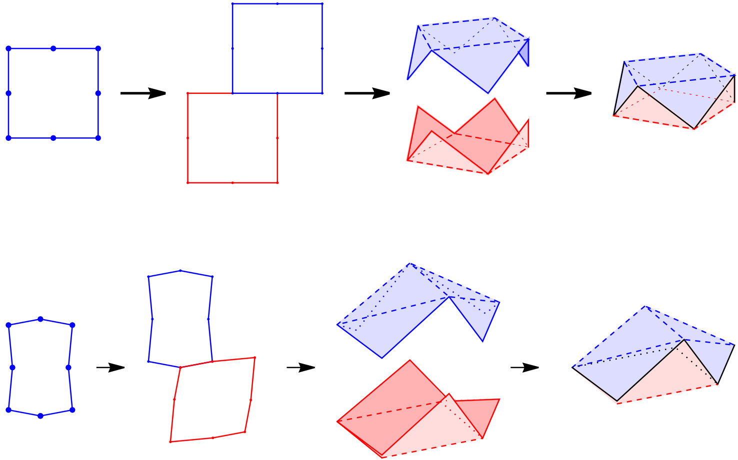


Other Examples



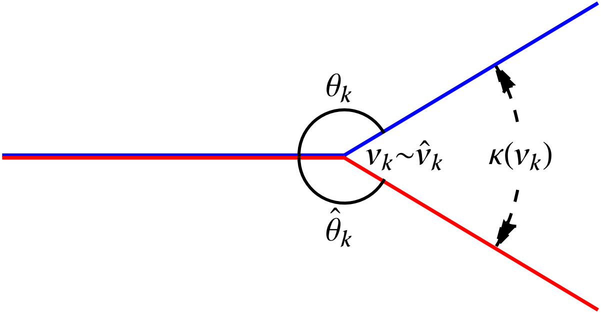
Main Result

$$\sum_{k=1}^{n} \omega^k v_k = 0, \qquad \omega = e^{-4\pi i/n}$$

$$\Updownarrow$$

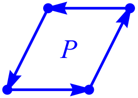
"closed cap condition"

Angular Defect



$$\kappa(v_k) = 2\pi - \theta_k - \hat{\theta}_k$$

Parallelogram Example



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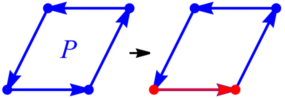

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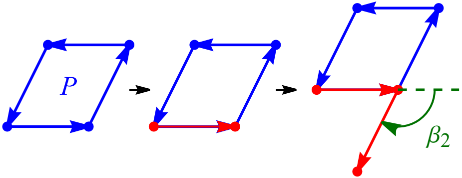

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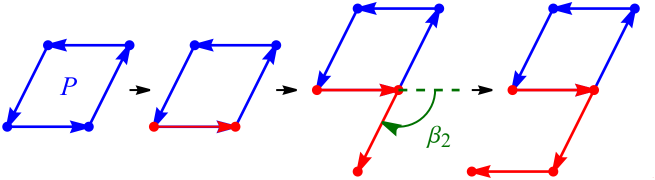

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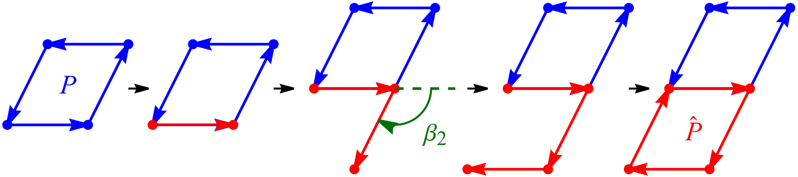

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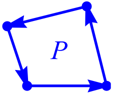

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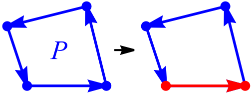

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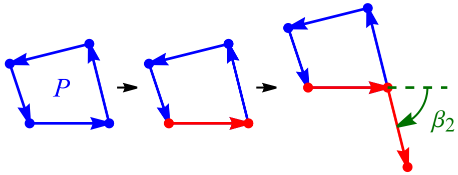

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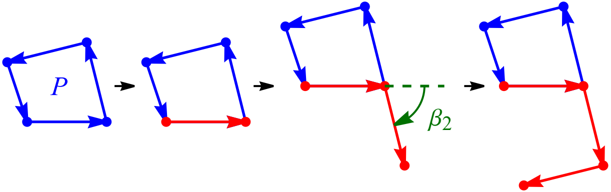

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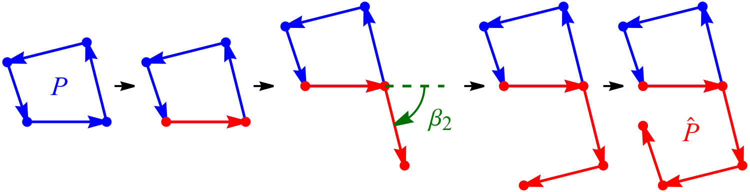
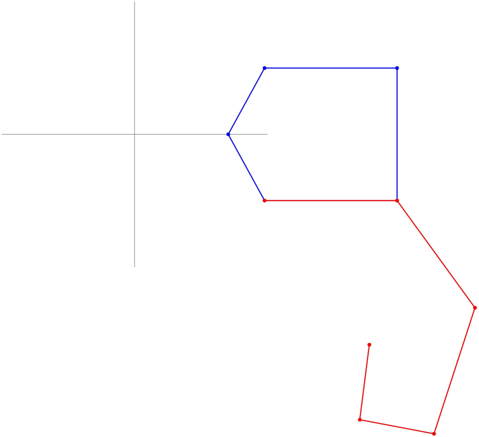


The Closed Cap Condition (CCC)

Definition

An n-sided polygon P, with an enumeration of its vertices $v_1, v_2, \cdots, v_n$ in counterclockwise order, is said to satisfy the closed cap condition if the resulting vertices $\hat{v}_{n+1} = \hat{v}_1$ under the cap construction algorithm.

Pentagon Example



Pentagon Example Proof Outline

Proposition

For an arbitrary pentagon P, the endpoint $\hat{v}_6$ of the polygonal cap curve is affine dependent on the fifth vertex v_5 of P.

Pentagon Example Proof Outline

We notice that

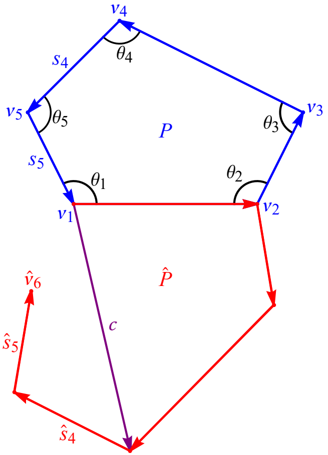
$$\hat{v}_6 = c + \hat{s}_4 + \hat{s}_5.$$

For pentagons,

$$\hat{s}_k = e^{-4\pi(k-1)i/5} s_k = \omega^{k-1} s_k.$$

Thus,

$$\hat{v}_6 = c + \omega^3 s_4 + \omega^4 s_5$$



Affine Dependence of Vertices

Theorem (S.-Weng-Zhang)

For an n-sided polygon P, it satisfies the closed cap condition if and only if

$$\sum_{k=1}^{n} \omega^k v_k = 0,$$

where $\omega = e^{-4\pi i/n}$.