Errors on projective integration of Ornstein–Uhlenbeck processes

Xiaopeng Chen* A. J. Roberts† Ioannis G. Kevrekidis‡

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Abstract

Projective integration has the potential to be an effective method to compute the long time dynamic behaviour of multiscale systems. However, there is some intrinsic difficulty when the behaviour is described by a stochastic process. The improved projective integration of stochastic processes is explored. The theory errors of projective integration of Ornstein–Uhlenbeck processes are obtained by deriving the bias of the maximum likelihood estimation. The errors estimation of parameters improves some results in either the statistic or econometrics literature. A number of examples are provided and Monte Carlo simulation demonstrate how the results is closed to the error formulas.

1 Introduction

Projective integration uses bursts of the microscale simulator, using microscale time steps, and computes an approximation to the system over a macroscale time step by extrapolation. Our recent work concerned about the projective integration influenced by noise [5]. We estimate a linear stochastic differential equation from short bursts of data. The analytic solution of the linear stochastic differential equation then estimates the solution over

*School of Mathematical Sciences, University of Adelaide, Adelaide, Australia. mailto:x.chen@adelaide.edu.au

†School of Mathematical Sciences, University of Adelaide, Adelaide, Australia. mailto:anthony.roberts@adelaide.edu.au

‡Department of Chemical and Biological Engineering and Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544, usa mailto: yannis@princeton.edu

a macroscale time step. We have explored how the noise affects the projective integration and presented two numerical algorithms in the empirical exploration. The simulation results shows that the two methods are stability and accuracy. We consider a third method which is called Adam-Moulton inspired method. It appears this version of an Adams–Moulton method is more robust to noise. Monte Carlo simulation results shows that the method is better in estimation of the errors of design parameters. We focus on the maximum likelihood estimation of the SDE

$$dX = (aX + b)dt + c\,dW, \quad X(0) = x_0. \tag{1.1}$$

The solution X is a continuous Markov process with mean, variance function $E[X(t)] = \exp(at)E[X(0)]$, $\operatorname{Var}(X(t)) = -\frac{c^2}{2a} + (\operatorname{Var}(X(0)) + \frac{c^2}{2a})\exp(2at)$. If $x_0 \sim N(0, -\frac{c^2}{2a})$, then the process is a stationary Ornstein–Uhlenbeck process. If the distribution of initial condition is not $N(0, -\frac{c^2}{2a})$, we denote the process a non-stationary Ornstein–Uhlenbeck process.

An early work of the projective integration simulate the errors of the designed parameters [5], which depends on the bursts and macroscale time step. We give an analysis errors of the parameters we estimation from the maximum likelihood in the present paper. First we analysis the errors of the estimation of the coefficients. There are some works about the estimation of the parameter from the stochastic differential equations [1, 3, 8, e.g.]. However, there are few works about the errors of parameters of non-stationary stochastic Ornstein–Uhlenbeck equations. The difficulty is to obtain a closed form of the parameter estimation. An discreteization approximation to the continuous time stochastic differential equations is used in estimation of the parameters by Kloeden et al. [6]. The method is easily to conduct and an explicit expression for the likelihood function is easy found. Some week convergence of the estimation parameters is obtained by Yoshida [10]. For approximation to the bias, a basic tool is by taking the stochastic Taylor expansions. Tang et al. [9, 7] gave an explicit expressions to the errors of the stationary Ornstein-Uhlenbeck processes, where the estimator is from the exact likelihood function. It is also important to consider the non-stationary Ornstein-Uhlenbeck processes. By using the Euler-like approximation, we present a bias and variance of the parameters estimation of non-stationary Ornstein-Uhlenbeck processes. We compute the errors from a burst. Two examples are given and the Monte Carlo simulation results shows that the power law is quite closed to the analysis errors we compute.

We also obtain a closed form expression of the parameters estimation from two bursts. Some relation of the results is the threshold autoregressive mode. The threshold autoregressive model [4] is from the unequally spaced data in practice. They maybe partially observed with some missing observations.

We then compute the error of projective integration of the Ornstein–Uhlenbeck equation (4.1). We show the errors depend on the number of data, the burst and the projective integration time step.

2 Adams-Moulton inspired method

In the former work we present two version of projective integration of the stochastic systems [5]. A variation on the theme of predictor corrector methods is analogous to deterministic Adam–Moulton methods. Here, instead of estimating the equation from just two bursts, the current and first predicted bursts, also use the burst from the preceding time step. We fit the burst with the Ornstein-Uhlenbeck equation (4.1).

- Execute the microscale simulator for a burst $[t_n, t'_n]$.
- For constant a and c and so take a projective integration time step of

  $$\hat{X}_{n+1} = X'_n + a\Delta t + c\Delta W \quad \text{for some } \Delta W \sim N(0, \Delta t). \tag{2.1}$$

  Then crudely predict to time t_{n+1} using (2.1).
- Compute a ‘predictor’ burst over $[t_{n+1}, t'_{n+1}]$;
- Fit the maximum likelihood estimation to the *three* bursts over $[t_{n-1}, t'_{n-1}]$, $[t_n, t'_n]$ and $[t_{n+1}, t'_{n+1}]$. Denote the estimation coefficients as $\hat{a}$, $\hat{b}$, $\hat{c}$.
- The analytic solution is following: given $X(t'_n, \omega) = X'_n$,

  $$X_t = e^{\hat{a}t}(X'_n + \hat{b}\int_0^t e^{-as}ds + c\int_0^t e^{-\hat{a}s}dW_s). \tag{2.2}$$

  Correct the estimate of $X(t_{n+1}, \omega)$ via solution (2.2).

We applied the numerical algorithm to microscale Ornstein–Uhlenbeck equation

$$dX = (1 - X)dt + 0.03dW, \quad X(0) = 1. \tag{2.3}$$

Assessing errors of this scheme is a little more difficult as the scheme depends upon the previous step as well. Here we record a partial assessment of errors by taking two projective integration steps: the first using one long burst of the microscale process from time t_0, which is then split into two

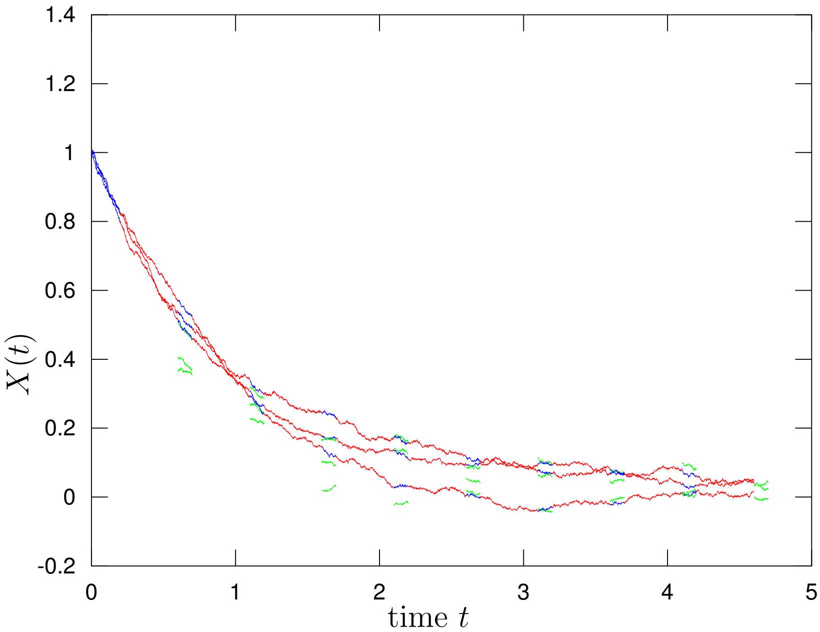
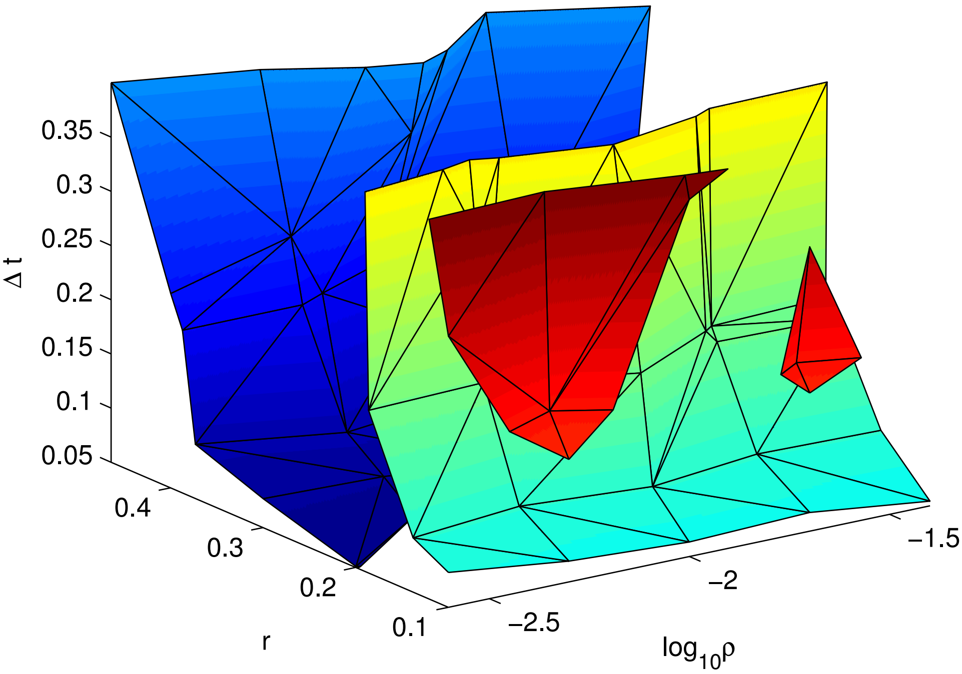
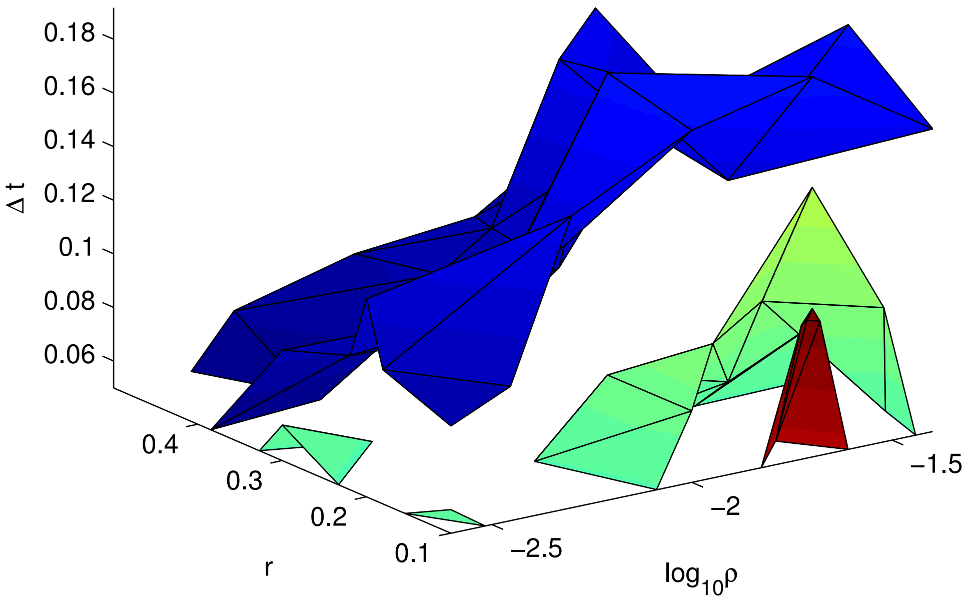


Figure 1: three realisations of the 'Adams-Moulton' projective integration applied to the Ornstein-Uhlenbeck equation (4.1) simulated with micro-time step $\mathfrak{d}t = 0.001$. In the projective integration: initial bursts (blue) lasted $\delta t = 0.1$; second bursts (green) were started following the simple prediction (2.1); and lastly using (2.2) the stochastic projective time step (red) was $\Delta t = 0.4$.



(errorsAdamsB03err.u3d)

Figure 2: isosurfaces of an error in projective Adam–Moulton-like integration: the surfaces are for median absolute error MAE $=0.01, 0.02, 0.03$ (in order blue to red, obtained from 300 realisations). Roughly, the error fits MAE $\approx 0.046\rho^{0.02}(1-r)^{2.1}(\Delta t)^{0.23}$.



(errorsAdamsB03smr.u3d)

Figure 3: isosurfaces of the error ratio RMS / MAE in projective Adam–Moulton-like integration: the surfaces are for ratios $2, 6, 20$ (in order blue to red). Note the change in the scale of the vertical axis.

and treated as two regular bursts; and the second step, as in the previous paragrpah, using a burst at time t_0 and a second burst from time t_1. Figure 1 shows the initial long burst from time $t = 0$ that is split into two for the initial step. We then assess the error in the second step which goes from $t \approx 0.5$ to $t \approx 1$. Figure 2 plots isosurfaces of the median absolute error when the algorithm, over a range of design parameters, was applied to the example (2.3). Roughly, the error fits

$$\text{MAE} \approx 0.046\rho^{0.02}(1-r)^{2.1}(\Delta t)^{0.23}. \tag{2.4}$$

Figure 2 plots the median absolute errors as these are robust to the outliers in any fat tailed error distribution. Figure 3 plots the ratio between the root mean square and the median absolute errors. Again the domain of design parameters above the blue surface indicates parameters for which there should be few, if any, outlier predictions.

3 The errors of the maximum likelihood estimation of the Ornstein–Uhlenbeck process

Consider

$$dX = (aX + b)dt + c\,dW, \quad X(0) = x_0.$$

Then the solution $X_t = e^{at}(x_0 + b\int_0^t e^{-as}ds + c\int_0^t e^{-as}dW_s)$. The Euler scheme to discrete the equation (4.1) is

$$X_{j+1} - X_j = (aX_j + b)\mathfrak{d}t_1 + c\mathfrak{d}W_j.$$

Then $X_{j+1} - X_j$ are independent Gaussian random variables with mean $(aX_j + b)\mathfrak{d}t_1$ and variance $c^2\mathfrak{d}t_1$. Therefore the transition density of the process is

$$p(y|x) = \frac{1}{\sqrt{2\pi c^2 \mathfrak{d}t_1}} \exp(-\frac{1}{2}\frac{(y - x - (ax + b)\mathfrak{d}t_j)^2}{c^2\mathfrak{d}t_j}).$$

The log-likelihood function for X_j, $j = 0, \ldots, n$, is

$$\log L_n(a, b, c) = -\frac{1}{2}n\log 2\pi c^2 - \frac{1}{2}\sum \log \mathfrak{d}t_j - \frac{1}{2}\sum \frac{(X_{j+1} - X_j - (aX_j + b)\mathfrak{d}t_j)^2}{c^2\mathfrak{d}t_j}.$$

We find

$$\hat{a} = \frac{n}{\delta t}\frac{n\sum X_{j+1}X_j - n\sum X_j^2 - \sum(X_{j+1} - X_j)\sum X_j}{n\sum X_j^2 - (\sum X_j)^2};$$

$$\hat{b} = \frac{\sum X_{j+1} - \sum X_j - \hat{a}\mathfrak{d}t_1 \sum X_j}{\delta t};$$

$$\hat{c}^2 = \frac{1}{\delta t} \sum (X_{j+1} - X_j - (\hat{a}X_j + \hat{b})\mathfrak{d}t_j)^2,$$

where the data is from

$$X_{j+1} = (1 + a\mathfrak{d}t_1)X_j + b\mathfrak{d}t_1 + c\mathfrak{d}W_j. \tag{3.1}$$

The microscale increments $\mathfrak{d}X_j = X_{j+1} - X_j$, $j = 0, \ldots, n-1$, from a burst of simulation of small time $\delta t = t_n - t_0$ and microscale time increments $\mathfrak{d}t_j = t_{j+1} - t_j$. We simply write $\sum_{j=0}^{n-1}$ as $\sum$ and consider the the same time span $\mathfrak{d}t_j$.

When compute the high order random variables, we use the Wick's theorem [2].

Lemma 3.1. *Let ξ_1, ξ_2, ξ_3, ξ_4 be a Gaussian random vector with zero mean. Then*

$$E(\xi_1\xi_2\xi_3\xi_4) = E\xi_1\xi_2 E\xi_3\xi_4 + E\xi_1\xi_3 E\xi_2\xi_4 + E\xi_1\xi_4 E\xi_2\xi_3.$$

Theorem 3.2. *For the Ornstein–Uhlenbeck equation* (4.1)*, as $n \to \infty$, we have the following errors of the parameter estimation.*

$$E\hat{a} = a + C + O(n^{-1}).$$

$$\operatorname{Var}\hat{a} = D + O(n^{-1}).$$

$$E\hat{b} = b + C\frac{b}{a}(\frac{\exp(a\delta t) - 1}{a\delta t} - 1) + O(n^{-1}).$$

$$\operatorname{Var}\hat{b} = \frac{C^2c^2}{a^2\delta t^2}[\delta t + \frac{\exp(2a\delta t) - 1}{2a} - \frac{2(\exp(a\delta t) - 1)}{a}] + O(n^{-1}).$$

$$E\hat{c^2} = c^2 + F + O(n^{-1}).$$

$$\operatorname{Var}\hat{c^2} = \frac{3c^4}{\delta tn} + G + O(n^{-2}).$$

The constants C, D, F, G are defined in the following.

Proof. Let $\phi := 1 + a\mathfrak{d}t_1$. Without loss of generality, we assume $x_0 = 0$. From the discrete data, we compute

$$S(i,j) = EX_iX_j = \phi^{|i-j|}[\frac{c^2\mathfrak{d}t_1(1-\phi^{2j})}{1-\phi^2} + \frac{b^2}{a^2}(1-\phi^j)^2].$$

Since

$$\sum EX_i = \frac{b}{a}(\frac{1-\phi^n}{1-\phi} - n).$$

$$
\begin{aligned}
\sum_{i>j} S(i,j) &= \left(\frac{c^2}{a}\frac{1}{1+\phi}+\frac{b^2}{a^2}\right)\left(\frac{(n-1)\phi}{1-\phi}+\phi^n\frac{\phi-\phi^{-(n-2)}}{(1-\phi)^2}\right)\\
&\quad+\left(\frac{b^2}{a^2}-\frac{c^2}{a}-\frac{1}{1+\phi}\right)\left(\frac{\phi-\phi^{2n-1}}{(1-\phi)^2(1+\phi)}-\frac{1-\phi^{n-1}}{(1-\phi)^2}\phi^n\right)\\
&\quad-\frac{2b^2}{a^2}\left(\frac{\phi-\phi^n}{(1-\phi)^2}-\frac{(n-1)\phi}{1-\phi}\right).
\end{aligned}
$$

$$
\begin{aligned}
n^{-1}\sum EX_j^2 &= -\frac{c^2}{2a}[1-\frac{\phi^{2n}-1}{2a\delta t}]+\frac{b^2}{a^2}[1+\frac{2(1-\phi^n)}{a\delta t}+\frac{\phi^{2n}-1}{2a\delta t}]\\
&\quad-\left(\frac{c^2}{8a\delta t}+\frac{b^2}{4a^3\delta t}\right)\mathfrak{d}t_1(\phi^{2n}-1)+O(n^{-2}).\\
n^{-2}E(\sum X_j)^2 &= \frac{c^2}{a^2\delta t^2}[\delta t+\frac{\phi^{2n}-\phi}{2a}-\frac{2(\phi^n-\phi)}{a}\\
&\quad+\frac{b^2}{a^2}[1+\frac{2(1-\phi^n)}{a\delta t}+\frac{(\phi^n-1)^2}{a^2\delta t^2}]\\
&\quad-\frac{c^2}{\delta t^2}\mathfrak{d}t_1[\phi^{2n}-\phi]+O(n^{-2}).
\end{aligned}
$$

$$
E(X_j\mathfrak{d}W_jX_i^2)=\begin{cases}0, & \text{if } i\le j;\\ 2c\mathfrak{d}t_1\phi^{i-j-1}(\phi^{2(i-j)-1}EX_j^2 & \\ \qquad+\frac{b}{a}(\phi^j-1)EX_j), & \text{if } i>j.\end{cases}
$$

Then

$$
\begin{aligned}
B_1: &= \frac{1}{\mathfrak{d}t_1}\lim_{n\to\infty} n^{-2}\sum E\mathfrak{d}W_jX_i = c\lim_{n\to\infty} n^{-2}[\frac{n-1}{1-\phi}+\phi^n\frac{\phi-\phi^{-(n-1)}}{(1-\phi)^2}] \\
&= \frac{c(\exp(a\delta t)-1)}{a^2\delta t^2}-\frac{c}{a\delta t}. \\
B_2: &= \frac{1}{\mathfrak{d}t_1}\lim_{n\to\infty} n^{-2}\sum_{i>j} E(X_j\mathfrak{d}W_jX_i^2) \\
&= \lim_{n\to\infty} 2cn^{-2}[\frac{c^2\phi^{-1}}{a(1-\phi^2)(1+\phi)}(\frac{\phi^2-\phi^{2n}}{1-\phi^2}-(n-1)\phi^{2n}-(n-1)\phi^2 \\
&\quad +\frac{\phi^2-\phi^{-2(n-2)}}{\phi^2-1}\phi^{2n})+\phi^{-1}\frac{b^2}{a^2(1-\phi^2)}((n-1)\phi^2 \\
&\quad -\frac{\phi^2-\phi^{-2(n-2)}}{\phi^2-1}\phi^{2n}-2\frac{\phi^2-\phi^{n+1}}{1-\phi}+2\frac{\phi-\phi^{-n+2}}{\phi-1}\phi^{2n} \\
&\quad +\frac{\phi^2-\phi^{2n}}{1-\phi^2}-(n-1)\phi^{2n})].
\end{aligned}
$$

$$
\begin{aligned}
B_3: &= \lim_{n\to\infty} n^{-2}E\sum X_j^2\sum X_i^2 \\
&= \frac{c^4}{a^2}(2\frac{\exp(2a\delta t)-1}{a^2\delta t^2}+1)+\frac{2b^2c^2}{a^3}(2\frac{1-\exp(2a\delta t)}{a^2\delta t^2}-1-2\frac{\exp(a\delta t)-1}{a^2\delta t^2} \\
&\quad -2\frac{(1-\exp(2a\delta t))(1-\exp(a\delta t))}{a^2\delta t^2})+\frac{b^4}{a^4}(1+2\frac{\exp(2a\delta t)-1}{a^2\delta t^2} \\
&\quad +4\frac{(\exp(a\delta t)-1)^2}{a^2\delta t^2}-4\frac{(1-\exp(2a\delta t))(1-\exp(a\delta t))}{a^2\delta t^2}) \\
&= \frac{2c^4}{a^2}(2\frac{1-\exp(2a\delta t)}{a^2\delta t^2}-\frac{1}{\delta t})+\frac{4b^2c^2}{a^3}(\frac{1}{a\delta t}+2\frac{\exp(a\delta t)+1}{a^2\delta t^2}) \\
&\quad +\frac{2b^4}{a^4}(-\frac{1}{a\delta t}+6\frac{1-\exp(2a\delta t)}{a^2\delta t^2}-8\frac{1-\exp(a\delta t)}{a^2\delta t^2})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{c^4}{a^2}(1+2\frac{1-\exp(2a\delta t)}{a\delta t}-\frac{1-\exp(4a\delta t)}{a\delta t})+\frac{2b^2c^2}{a^3}(-\frac{1-\exp(4a\delta t)}{a\delta t}-1 \\
&\quad +2\frac{1-\exp(3a\delta t)}{a\delta t}-2\frac{1-\exp(a\delta t)}{a\delta t})+\frac{b^4}{a^4}(1-\frac{1-\exp(4a\delta t)}{a\delta t} \\
&\quad -6\frac{1-\exp(2a\delta t)}{a\delta t}-4\frac{1-\exp(a\delta t)}{a\delta t}-4\frac{1-\exp(3a\delta t)}{a\delta t}).
\end{aligned}
$$

$$
\begin{aligned}
B_4 : &= \lim_{n\to\infty} n^{-2}\sum S(i,j) = \lim_{n\to\infty} n^{-2}E(\sum X_j)^2 \\
&= \frac{c^2}{a^2\delta t^2}[\delta t + \frac{\exp(2a\delta t)-1}{2a} - \frac{2(\exp(a\delta t)-a)}{a} \\
&\quad + \frac{b^2}{a^2}[1 + \frac{2(1-\exp(a\delta t))}{a\delta t} + \frac{(\exp(a\delta t)-1)^2}{a^2\delta t^2}]. \\
B_5 : &= \frac{1}{\mathfrak{d}t_1^2}\lim_{n\to\infty} n^{-2}\sum E(\mathfrak{d}W_i X_j)^2. \\
B_6 : &= \lim_{n\to\infty} n^{-2}(\sum EX_j)^2 = \lim_{n\to\infty} n^{-2}E(\sum X_j)^2 \\
&= \frac{c^2}{a^2\delta t^2}[\delta t + \frac{\exp(2a\delta t)-1}{2a} - \frac{2(\exp(a\delta t)-1)}{a} \\
&\quad + \frac{b^2}{a^2}[1 + \frac{2(1-\exp(a\delta t))}{a\delta t} + \frac{(\exp(a\delta t)-1)^2}{a^2\delta t^2}].
\end{aligned}
$$

Define

$$
\begin{aligned}
u_1 : &= \lim_{n\to+\infty} n^{-1}\sum EX_{j+1}X_j - \lim_{n\to+\infty} n^{-2}E\sum X_{j+1}\sum X_j \\
&= \lim_{n\to+\infty} n^{-1}\sum EX_j^2 - \lim_{n\to+\infty} n^{-2}E(\sum X_j)^2 \\
&= -\frac{c^2}{2a}[\frac{2}{a\delta t} + 1 + \frac{\exp(2a\delta t)-1}{a^2\delta t^2} - \frac{\exp(2a\delta t)-1}{2a\delta t} - \frac{4(\exp(a\delta t)-1)}{a^2\delta t^2}] \\
&\quad + \frac{b^2}{a^2}[\frac{\exp(2a\delta t)-1}{2a\delta t} - \frac{(\exp(a\delta t)-1)^2}{a^2\delta t^2}]. \\
u_2 : &= \lim_{n\to+\infty} n^{-1}E\sum X_j^2 - \lim_{n\to+\infty} n^{-2}E(\sum X_j)^2 = u_1.
\end{aligned}
$$

Note that

$$(1+\frac{x}{n})^n = \exp(x)(1-\frac{x^2}{2n}+O(n^{-2})).$$

We have

$$
\begin{aligned}
&E(n^{-1}\sum X_{j+1}X_j - n^{-2}\sum X_{j+1}\sum X_j - u_1) = O(\frac{1}{n}). \\
&E(n^{-1}\sum X_j^2 - n^{-2}(\sum X_j)^2 - u_2) = O(\frac{1}{n}).
\end{aligned}
$$

And

$$
\begin{aligned}
&E(u_1 - n^{-1}\sum X_{j+1}X_j - n^{-2}\sum X_{j+1}\sum X_j)^2 = O(\frac{1}{n}). \\
&E(u_2 - n^{-1}\sum X_j^2 - n^{-2}(\sum X_j)^2)^2 = O(\frac{1}{n}).
\end{aligned}
$$

Now we analysis the errors, the errors are expressed by the data. Denote $t_1 = n^{-1}\sum X_{j+1}X_j - u_1$, $t_2 = n^{-1}\sum X_j^2 - u_2$, $t_3 = n^{-1}\sum X_{j+1}$, $t_4 = n^{-1}\sum X_j$. Let $f(x,y) = \frac{u_1+x}{u_2+y}$, then by the multivariate Taylor expansion

$$\frac{n^{-1}\sum X_{j+1}X_j - n^{-2}\sum X_{j+1}\sum X_j}{n^{-1}\sum X_j^2 - n^{-2}(\sum X_j)^2} = \frac{u_1 + t_1 - t_3t_4}{u_2 + t_2 - t_4^2} = \frac{u_1}{u_2} + A_1 + A_2 + O_p(n^{-\frac{3}{2}}),$$

where

$$\begin{aligned} A_1 &= \frac{t_1}{u_2} - \frac{u_1 t_2}{u_2^2}, \\ A_2 &= -\frac{t_3t_4}{u_2} + \frac{u_1 t_4^2}{u_2^2} - \frac{t_1t_2}{u_2^2} + \frac{u_1 t_2^2}{u_2^3}. \end{aligned}$$

The computation gives

$$E(t_1 - t_2) = E(n^{-1}\sum X_{j+1}X_j - n^{-1}\sum X_j^2) = a\mathfrak{d}t_1 n^{-1}\sum EX_j^2 + b\mathfrak{d}t_1 n^{-1}\sum EX_j.$$

$$Et_3t_4 = n^{-2}\sum S(i, j+1).$$

$$Et_4^2 = n^{-2}\sum S(i,j).$$

From equation (3.7), we have

$$EX_{j+1}X_i = \phi EX_jX_i + b\mathfrak{d}t_1 EX_j + cE\mathfrak{d}W_jX_i.$$

So

$$S(i, j+1) = \phi S(i,j) + b\mathfrak{d}t_1 EX_j + cE\mathfrak{d}W_jX_i.$$

$$E(t_4^2 - t_3t_4) = -a\mathfrak{d}t_1 n^{-2}\sum S(i,j) - b\mathfrak{d}t_1 n^{-1}\sum EX_j - cn^{-2}\sum E\mathfrak{d}W_jX_i.$$

$$\begin{aligned} E(t_2^2 - t_1t_2) &= u_2 a\mathfrak{d}t_1 n^{-1}\sum EX_j^2 + u_2 b\mathfrak{d}t_1 n^{-1}\sum EX_j - (a\mathfrak{d}t_1 n^{-2}E\sum X_j^2\sum X_i^2 \\ &\quad + b\mathfrak{d}t_1 n^{-2}E\sum X_j\sum X_i^2 + cn^{-2}E\sum \mathfrak{d}W_jX_j\sum X_i^2). \end{aligned}$$

Denote

$$C = \frac{b^2}{au_1}\left(\frac{\exp(a\delta t) - 1}{a\delta t} - 1\right) - \frac{cB_1}{u_1} - \frac{cB_2}{u_1^2} - \frac{aB_3}{u_1^2}.$$

We have

$$E\hat{a} = a + C + O(n^{-1}). \tag{3.2}$$

$$B_3 - B_6 = \frac{1}{\delta t} n^{-1} \lim_{n\to\infty} \sum EX_j^2 - \lim_{n\to\infty} n^{-2} E(\sum X_j)^2.$$

$$B_5 - B_6 = \frac{1}{\delta t} n^{-1} \lim_{n\to\infty} \sum EX_j^2 - \lim_{n\to\infty} n^{-2} E(\sum X_j)^2.$$

$$D = \frac{a^2(B_3 - B_6)}{u_1^2} + \frac{b^2(B_5 - B_6)}{u_1^2} + \frac{c^2 B_4}{u_1^2} + \frac{2acB_2}{u_1^2}.$$

Then

$$\operatorname{Var} \hat{a} = D + O(n^{-1}). \tag{3.3}$$

$$\begin{aligned}
\hat{b} &= \frac{\sum X_{j+1} - \sum X_j - \hat{a}\mathfrak{d}t_1 \sum X_j}{\delta t} \\
&= \frac{\sum X_{j+1} - \sum X_j - a\mathfrak{d}t_1 \sum X_j}{\delta t} - Cn^{-1} \sum X_j + O_p(n^{-1}).
\end{aligned}$$

So

$$E\hat{b} = b + C\frac{b}{a}(\frac{\exp(a\delta t) - 1}{a\delta t} - 1) + O(n^{-1}).$$

$$\operatorname{Var} \hat{b} = \frac{C^2 c^2}{a^2 \delta t^2}[\delta t + \frac{\exp(2a\delta t) - 1}{2a} - \frac{2(\exp(a\delta t) - 1)}{a}] + O(n^{-1}).$$

Since

$$\hat{c}^2 = \frac{1}{\delta t} \sum (X_{j+1} - X_j - (aX_j + b)\mathfrak{d}t_j)^2 + Cn^{-1} \sum X_j^2 + O_p(n^{-1}).$$

Denote

$$F = C \lim_{n\to\infty} n^{-1} \sum EX_j^2;$$

$$G = \frac{1}{\mathfrak{d}t_1^2} \lim_{n\to\infty} n^{-2} \sum E(\mathfrak{d}W_i X_j)^2.$$

We have

$$E\hat{c^2} = c^2 + F + O(n^{-1}).$$

$$\operatorname{Var} \hat{c^2} = \frac{3c^4}{\delta t n} + G + O(n^{-2}).$$

□

Remark 3.3. *Phillips et al. [7] used the estimator*

$$\hat{c}^2 = \frac{1}{n} \sum (X_{j+1} - X_j)^2.$$

Example 3.4. *Consider the constant coefficient* SDE

$$dX = a\,dt + c\,dW, \quad X(0) = x_0. \tag{3.4}$$

The solution of (3.4) *is* $X(t) = x_0 + at + cW$*. The microscale increments* $\mathfrak{d}X_j = X_{j+1} - X_j$*,* $j = 0, \ldots, n-1$*, from a burst of simulation of small time* $\delta t = t_n - t_0$ *are distributed normally with mean* $a\mathfrak{d}t_j$ *and variance* $b^2\mathfrak{d}t_j$*,* $\mathfrak{d}t_j = t_{j+1} - t_j$*.*

Solve $\hat{a}$ *and* $\hat{b}$*.*

$$\hat{a} = \frac{\delta X}{\delta t}, \tag{3.5}$$

where $\delta X = X_{t_n} - X_{t_0}$*.*

$$\hat{c}^2 = \sum \frac{1}{\delta t}(\mathfrak{d}X_j - \hat{a}\mathfrak{d}t_j)^2. \tag{3.6}$$

The errors of parameter estimates depend on the number of data values and microscale time step.

$$\begin{aligned}
&E\hat{a} = a;\\
&\operatorname{Var}\hat{a} = \frac{c^2}{\delta t};\\
&E\hat{c}^2 = c^2(1 - \frac{1}{n});\\
\operatorname{Var}\hat{c}^2 \quad &= \operatorname{Var}\left[\sum \frac{1}{\delta t}(\mathfrak{d}X_j - (a + b\frac{\delta W}{\delta t})\mathfrak{d}t_j)^2\right]\\
&= \frac{c^4}{\delta t^2}\operatorname{Var}\sum\left[(\mathfrak{d}W_j)^2 - 2\mathfrak{d}W_j \times \frac{\delta W}{\delta t}\mathfrak{d}t_j + \frac{\delta W^2}{\delta t^2}\mathfrak{d}t_j^2\right]\\
&= \frac{c^4}{\delta t^2}\operatorname{Var}\left[\sum(\mathfrak{d}W_j)^2 - \frac{(\delta W)^2}{n}\right]\\
&= \frac{c^4}{\delta t^2}\left[\operatorname{Var}(\sum(\mathfrak{d}W_j)^2) + \operatorname{Var}\frac{(\delta W)^2}{n} - 2\operatorname{Cov}(\sum(\mathfrak{d}W_j)^2, \frac{(\delta W)^2}{n})\right]\\
&= \frac{c^4}{\delta t^2}\left[\operatorname{Var}(\sum(\mathfrak{d}W_j)^2) + \operatorname{Var}\frac{(\delta W)^2}{n} - \frac{2}{n}\operatorname{Cov}(\sum(\mathfrak{d}W_j)^2, \sum(\mathfrak{d}W_j)^2)\right]\\
&= \frac{c^4}{\delta t^2}\left[(1 - \frac{2}{n})\operatorname{Var}(\sum(\mathfrak{d}W_j)^2) + \frac{1}{n^2}\operatorname{Var}(\delta W)^2\right]\\
&= \frac{2}{n}c^4 - \frac{2}{n^2}b^4.
\end{aligned}$$

Example 3.5.

$$dX = aX dt + c\, dW, \quad X(0) = x_0. \tag{3.7}$$

The solution is $X_t = e^{at}(x_0 + c\int_0^t e^{-as} dW_s)$. *We use the Euler scheme to discrete the equation* (3.7)

$$X_{j+1} - X_j = aX_j \mathfrak{d} t_j + c\mathfrak{d} W_j \tag{3.8}$$

From the maximum likelihood estimation, we find

$$\hat{a} = \frac{\sum X_{j+1}X_j - \sum X_j^2}{\sum X_j^2 \mathfrak{d} t_j}. \tag{3.9}$$

$$\hat{c}^2 = \frac{1}{n}\sum \frac{(X_{j+1} - X_j - \hat{a}X_j\mathfrak{d} t_j)^2}{\mathfrak{d} t_j}. \tag{3.10}$$

Let $\phi := 1 + a\mathfrak{d} t_1$. *From the data, we compute*

$$EX_i^2 = c^2 \mathfrak{d} t_1 \frac{1-\phi^{2i}}{1-\phi^2}; \quad EX_iX_{i-1} = c^2\mathfrak{d} t_1 \phi \frac{1-\phi^{2i}}{1-\phi^2}.$$

Define

$$\begin{aligned} u_1 : &= \lim_{n\to+\infty} n^{-1} E\sum X_j^2 = -\frac{c^2}{2a}(1 - \frac{\exp(2a\delta t) - 1}{2a\delta t}). \\ u_2 : &= \lim_{n\to+\infty} n^{-1}\sum EX_{j+1}X_j = u_1. \end{aligned}$$

Now we analysis the errors, the errors are expressed by the data. Let $f(x,y) = \frac{x}{y}$, *then by the multivariate Taylor expansion*

$$\frac{n^{-1}\sum X_{j+1}X_j}{n^{-1}\sum X_j^2} = \frac{u_2}{u_1} + \frac{1}{u_1}(n^{-1}\sum X_{j+1}X_j - u_2) - \frac{u_2}{u_1^2}(n^{-1}\sum X_j^2 - u_1) - \frac{1}{u_1^2}$$
$$(n^{-1}\sum X_{j+1}X_j - u_2)(n^{-1}\sum X_j^2 - u_1) + \frac{u_2}{u_1^3}(n^{-1}\sum X_j^2 - u_1)^2 + O_p(n^{-\frac{3}{2}}).$$

Compute

$$\begin{aligned} n^{-2}\sum (EX_iX_j)^2 \;\; = \;\; & \frac{1}{u_1^2}\frac{2c^4}{a^2(1+\phi)^2} n^{-2}[\frac{(n-1)\phi^2}{1-\phi^2} - \frac{\phi^4 - \phi^{2n+2}}{(1-\phi^2)^2} + \frac{\phi^2 - \phi^{4n-2}}{(1-\phi^2)^2(1+\phi^2)} \\ & - \frac{\phi^{2n} - \phi^{4n-2}}{(1-\phi^2)^2}] - \frac{4c^4}{a^2(1+\phi)^2} n^{-1}\frac{1-\phi^{2n}}{1-\phi^2} \\ & + \frac{c^4}{a^2(1+\phi)^2} n^{-1}[n - \frac{2(1-\phi^{2n})}{1-\phi^2} + \frac{1-\phi^{4n}}{1-\phi^4}]. \end{aligned}$$

$$
\begin{aligned}
E\frac{n^{-1}\sum X_jX_{j-1}}{n^{-1}\sum X_j^2} &= \frac{2}{u_1}a\mathfrak{d}t_1n^{-1}\sum EX_j^2 - \frac{1}{u_1^2}a\mathfrak{d}t_1n^{-2}\sum(EX_j^2)^2 \\
&\quad -\frac{2a\mathfrak{d}t_1}{u_1^2}n^{-2}\sum(EX_iX_j)^2 + O(n^{-2}).
\end{aligned}
$$

We show that

$$
E\hat{a} = a - C + O(n^{-1}),
$$

where

$$
C = \frac{c^4}{u_1^2a}\left[\frac{1}{2} + \frac{7}{8a\delta t} - \frac{1}{8a^2\delta t^2} - \frac{3\exp(2a\delta t)}{2a\delta t} + \frac{\exp(4a\delta t)}{8a\delta t} + \frac{\exp(4a\delta t)}{8a^2\delta t^2}\right].
$$

$$
\begin{aligned}
\frac{n^{-1}\sum X_jX_{j-1}}{n^{-1}\sum X_j^2} &= \frac{1}{u_1^2}\Big[\operatorname{Cov}(n^{-1}\sum X_{j+1}X_j, n^{-1}\sum X_{i+1}X_i) \\
&\quad + \operatorname{Cov}(n^{-1}\sum X_i^2, n^{-1}\sum X_j^2) \\
&\quad -2\operatorname{Cov}(n^{-1}\sum X_{j+1}X_j, n^{-1}\sum X_i^2)\Big].
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Var}\frac{n^{-1}\sum X_jX_{j-1}}{n^{-1}\sum X_j^2} &= \frac{n^{-2}}{u_1^2}\Big[a^2\mathfrak{d}t_1^2\operatorname{Cov}(\sum X_i^2, \sum X_j^2) \\
&\quad + c^2\operatorname{Cov}(\sum X_j\mathfrak{d}W_j, \sum X_j\mathfrak{d}W_j)\Big] + O(n^{-3}) \\
&= \frac{c^4\mathfrak{d}t_1^2}{u_1^2}(-\frac{1}{2a\delta t} - \frac{1-\exp(2a\delta t)}{4a^2\delta t^2}) + O(n^{-3}) \\
&= \frac{1}{n}(1-\exp(2a\mathfrak{d}t_1)) + O(n^{-3}).
\end{aligned}
$$

We have

$$
\operatorname{Var}\hat{a} = -\frac{2a}{\delta t} + O(n^{-1}).
$$

From the result of $\hat{a}$, we have

$$
\hat{c}^2 = \frac{1}{\delta t}\sum(X_{j+1} - X_j - \hat{a}X_j\mathfrak{d}t_j)^2 = \frac{1}{\delta t}\sum(X_{j+1} - X_j - aX_j\mathfrak{d}t_j)^2 + O_P(\frac{1}{n}).
$$

Then

$$
E\hat{c}^2 = c^2 + O(\frac{1}{n});
$$

$$
\operatorname{Var}\hat{c}^2 = \frac{2c^4}{n} + O(\frac{1}{n^2}).
$$

4 Monte Carlo simulation results

Since the errors of the expression of the Theorem 3.2 is hard to application. For the stationary case [9], the expressions are also complicated. We give an simulation order of the parameters from the Monte Carlo simulation in this section. Firstly we simulate the parameters from the maximum likelihood estimation for some SDEs.

Table 1: We simulate the SDE $dX = a\,dt + c\,dW, \quad X(0) = 1$. The burst $\delta t = 2$. The numbers of data $n = 1000$. The maximum likelihood expressions are equations (3.5) and (3.6). The errors are expressed as the mean and variance of a and c^2.

Parameters	a	c	a	c	a	c
True values	1	0.2	2	0.5	3	0.8
Mean	0.9991	0.0400	1.9896	0.2493	3.0144	0.6386
Variance	0.0197	3.0320e-06	0.1139	1.2376e-04	0.2992	7.8635e-04

Table 2: We simulate the SDE $dX = a\,dt + c\,dW, \quad X(0) = 2$. The burst $\delta t = 5$. The numbers of data $n = 1500$. The maximum likelihood expressions are equations (3.5) and (3.6). The errors are expressed as the mean and variance of a and c^2.

Parameters	a	c	a	c	a	c
True values	1	0.2	2	0.5	3	0.8
Mean	0.9969	0.0399	2.0016	0.2494	2.9945	0.6387
Variance	0.0078	2.0527e-06	0.0488	7.9425e-05	0.1183	5.4581e-04

Secondly we set varied parameters to simulate the power law of the errors. Then we have the simulate errors. The power law we simulate equation (3.4) is quite closed to the analytic expression errors we compute.

$$\begin{aligned}
&E\hat{a} \propto a;\\
&\operatorname{Var}\hat{a} \propto \frac{c^2}{\delta t};\\
&E\hat{c}^2 \propto c^2;\\
&\operatorname{Var}\hat{c}^2 \propto \frac{1}{n}c^4.
\end{aligned}$$

Table 3: We simulate the SDE $dX = aX\,dt + c\,dW, \quad X(0) \sim N(0, -\frac{c^2}{2a})$. The burst $\delta t = 2$. The numbers of data $n = 500$. The maximum likelihood expressions are equations (3.9) and (3.10). The errors are expressed as the mean and variance of a and c^2.

Parameters	a	c	a	c	a	c
True values	-1	0.2	- 2	0.5	-3	0.8
Mean	-1.7954	0.0398	-2.8753	0.2493	-3.8746	0.6389
Variance	2.8198	6.5877e-06	3.4996	2.4565e-04	4.7498	0.0016

Table 4: We simulate the SDE $dX = aX\,dt + c\,dW, \quad X(0) \sim N(0, -\frac{c^2}{2a})$. The burst $\delta t = 4$. The numbers of data $n = 1000$. The maximum likelihood expressions are equations (3.9) and (3.10). The errors are expressed as the mean and variance of a and c^2.

Parameters	a	c	a	c	a	c
True values	-2.1	0.03	- 3.5	0.04	-4.5	0.05
Mean	-2.6047	8.9910e-04	-3.9577	0.0016	-4.8337	0.0025
Variance	1.5889	1.6351e-09	2.3352	4.9424e-09	2.4358	1.2806e-08

The simulation power law of simulation (3.7) is a little different from the analysis expression. This is because the errors of simulation from the maximum likelihood estimation. The errors depends on the burst time δt.

$$\begin{aligned}
&E\hat{a} \propto a;\\
&\operatorname{Var}\hat{a} \propto \frac{ac}{\delta t};\\
&E\hat{c}^2 \propto c^2;\\
&\operatorname{Var}\hat{c}^2 \propto \frac{1}{n}c^4.
\end{aligned}$$

Table 5: We simulate the SDE $dX = (aX + b)dt + c\,dW$. $X(0) = 2$. The burst $\delta t = 1$. The numbers of data $n = 1500$. The maximum likelihood expressions are equations (3.9) and (3.10). The errors are expressed as the mean and variance of a, b and c^2.

Parameters	a	b	c	a	b	c
Truev alues	3.5	1	0.3	4	2	0.4
Mean	3.5003	0.9832	0.08990	4.0000	2.0172	0.1597
Variance	2.4085e-04	0.1939	1.0845e-05	1.3803e-04	0.2961	3.2811e-05

Table 6: We simulate the SDE $dX = (aX + b)dt + c\,dW$. $X(0) = 2$. The burst $\delta t = 1.7$. The numbers of data $n = 5500$. The maximum likelihood expressions are equations (3.9) and (3.10). The errors are expressed as the mean and variance of a, b and c^2.

Parameters	a	b	c	a	b	c
True values	1.2	0.1	0.2	2.5	1	0.8
Mean	1.2007	0.1041	0.0400	2.5006	1.0060	0.6395
Variance	0.0016	0.0944	5.3574e-007	2.2978e-004	0.6926	1.4544e-004

The simulation equation (4.1) of the power law is following.

$$
\begin{aligned}
&E\hat{a} \propto a;\\
&\operatorname{Var}\hat{a} \propto \frac{c^2}{a^{3.7}b^{0.5}\delta t^{5.2}};\\
&E\hat{b} \propto b;\\
&\operatorname{Var}\hat{b} \propto \frac{c^2}{a\delta t^2};\\
&E\hat{c^2} \propto c^2;\\
&\operatorname{Var}\hat{c^2} \propto \frac{c^4}{n}.
\end{aligned}
$$

4.1 Two bursts

$$dX = (aX + b)dt + c\,dW, \quad X(0) = 0. \tag{4.1}$$

Suppose the data is from two same length bursts, denoted as δt. Denote the data is $X_0, X_1, \dots, X_n$, $X_{1,0}, \dots, X_{1,n}$. The Euler scheme to discrete the equation (4.1) are

$$X_{j+1} - X_j = (aX_j + b)\mathfrak{d}t_1 + c\mathfrak{d}W_j,$$

$j = 0, \ldots, n-1$, and

$$X_{1,j+1} - X_{1,j} = (aX_{1,j} + b)\mathfrak{d}t_1 + c\mathfrak{d}W_{1,j},$$

$j = 0, \ldots, n-1$. $X_{1,0} = \exp a(\delta t + \Delta t)(x_0 + b\int_0^{\delta t+\Delta t} \exp(-as)ds + c\int_0^{\delta t+\Delta t} e^{-as} dW_s)$. Denote $dX_j = X_{j+1} - X_j$, $dX_{1,j} = X_{1,j+1} - X_{1,j}$. Ignoring the $X_{1,0}$ component. We find

$$\hat{a} = \frac{n}{\delta t} \frac{2n\sum X_j dX_j - 2n\sum X_{1,j} dX_{1,j} - (\sum dX_j + \sum dX_{1,j})(\sum X_j + \sum X_{1,j})}{2n\sum X_j^2 + 2n\sum X_{1,j}^2 - (\sum X_j + \sum X_{1,j})^2}; \tag{4.2}$$

$$\begin{aligned}\hat{b} &= \frac{(\sum X_{j+1} - \sum X_j - \hat{a}\mathfrak{d}t_1 \sum X_j) + (\sum X_{1,j+1} - \sum X_{1,j} - \hat{a}\mathfrak{d}t_1 \sum X_{1,j})}{2\delta t} \\ &= \frac{n}{\delta t} \frac{(\sum dX_j + \sum dX_{1,j})(\sum X_j^2 + \sum X_{1,j}^2) - (\sum X_j dX_j + \sum X_{1,j} dX_{1,j})(\sum X_j + \sum X_{1,j})^2}{2n\sum X_j^2 + 2n\sum X_{1,j}^2 - (\sum X_j + \sum X_{1,j})^2};\end{aligned} \tag{4.3}$$

$$\begin{aligned}\hat{c}^2 &= \frac{1}{2\delta t}\Big\{ \Big[\sum (dX_j - (\hat{a}X_j + \hat{b})\mathfrak{d}t_j)^2\Big] \\ &\quad + \Big[\sum (dX_{1,j} - (\hat{a}X_{1,j} + \hat{b})\mathfrak{d}t_{1,j})^2\Big] \Big\}.\end{aligned} \tag{4.4}$$

The simulation equation (4.1) of the power law is following.

$$\begin{aligned}&E\hat{a} \propto a; \\ &\operatorname{Var}\hat{a} \propto \frac{c^2}{a^4 b \delta t^{3.4}}; \\ &E\hat{b} \propto b; \\ &\operatorname{Var}\hat{b} \propto \frac{c^2}{a^{0.5}\delta t^{1.2}}; \\ &E\hat{c} \propto c; \\ &\operatorname{Var}\hat{c} \propto \frac{c^2}{n}.\end{aligned}$$

Table 7: We simulate the SDE $dX = (aX + b)dt + c\,dW$. $X(0) = 2$. The two bursts $\delta t = 0.5$, $\Delta t = 1.2$. The numbers of data $n = 500$. The errors are expressed as the mean and variance of a, b and c.

Parameters	a	b	c	a	b	c
True values	5.2	2.1	0.2	3.2	1.3	0.1
One burst Mean	5.214	2.110	0.199	3.207	1.290	0.099
One burst Variance	0.035	0.474	0.006	0.053	0.329	0.003
Two bursts Mean	5.213	2.101	0.200	3.205	1.299	0.100
Two bursts Variance	0.0000	0.2441	0.004	0.000	0.130	0.002

5 Errors of projective integration

The maximum likelihood estimation equation is

$$d\hat{X} = (\hat{a}\hat{X} + \hat{b})dt + \hat{c}\,dW, \quad \hat{X}(0) = 0. \tag{5.1}$$

From equation (4.1) and equation (5.1) we have

$$d(\hat{X} - X) = \hat{a}(\hat{X} - X)dt + (\hat{a} - a)X dt + (\hat{b} - b)dt + (\hat{c} - c)\,dW. \tag{5.2}$$

From the analytic solution of equation (5.2),

$$\begin{aligned} E(\hat{X}_n - X_n) &= E(\hat{a} - a)\int_0^{\delta t} X(s)\exp(\hat{a}(\delta t - s))ds \\ &\quad + E(\hat{b} - b)\int_0^{\delta t} \exp(\hat{a}(\delta t - s))ds \\ &\le \delta t^{1/2} M(C^{1/2} + D^{1/2}) + \mathrm{O}(n^{-1}), \end{aligned}$$

where M is a constant. Consider the equations of the time step Δt.

$$d\hat{Y} = (\hat{a}\hat{Y} + \hat{b})dt + \hat{c}\,dW, \quad \hat{Y}(0) = \hat{X}_n. \tag{5.3}$$

$$d\hat{X} = (\hat{a}\hat{X} + \hat{b})dt + \hat{c}\,dW, \quad \hat{X}(0) = X_n. \tag{5.4}$$

The errors on projective integration time step Δt is

$$\begin{aligned} E(Y_{\delta t+\Delta t} - X_{\delta t+\Delta t}) &= E\exp(\hat{a}\Delta t)(\hat{X}_n - X_n) \\ &\le \delta t^{1/2} M(C^{1/2} + D^{1/2}) + \mathrm{O}(\Delta t) + \mathrm{O}(n^{-1}). \end{aligned}$$

Table 8: We simulate the SDE $dX = (aX + b)dt + c\,dW$. $X(0) = 2$. The two bursts $\delta t = 0.7$, $\Delta t = 1.8$. The numbers of data $n = 550$. The maximum likelihood expressions are equations (4.2), (4.4) and (**??**). The errors are expressed as the mean and variance of a, b and c^2.

Parameters	a	b	c	a	b	c
True values	5.2	2.1	0.2	3.2	1.3	0.1
Mean	5.2000	2.0270	0.0469	3.1999	1.1347	0.0210
Variance	8.6288e-016	6.6933e-006	1.2685e-010	2.2978e-004	0.0719	2.5659e-006

Table 9: We simulate the SDE $dX = (aX + b)dt + c\,dW$. $X(0) = 2$. The maximum likelihood expressions are equations (4.2), (4.4) and (**??**). The errors are expressed as the burst, the projective time step, the number of data.

Parameters	n	δt	Δt	n	δt	Δt
True values	550	0.1	0.6	650	0.2	1.1
Mean		-0.9980			-3.0684	
Variance		0.1958			1.3594	

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