Center Manifolds for infinite dimensional random dynamical systems

Xiaopeng Chen [*] A. J. Roberts[†] Jinqiao Duan[‡]

April 14, 2013

Abstract

Stochastic center manifolds theory are crucial in modelling the dynamical behavior of complex systems under stochastic influences. A multiplicative ergodic theorem on Hilbert space is proved to be satisfied to the exponential trichotomy condition. Then the existence of stochastic center manifolds for infinite dimensional random dynamical systems is shown under the assumption of exponential trichotomy. The result is then applied to a nonlinear stochastic partial differential equation with linear multiplicative noise.

Mathematics Subject Classifications (2010) Primary 37L55; Secondary 37D10, 34D35.

Keywords Multiplicative ergodic theorem, exponential trichotomy, random dynamical systems, center manifolds, stochastic partial differential equation.

1 Introduction

The theory of centre manifolds plays an important role in the deterministic dynamical systems and has been proved a tremendous applications such as bifurcation [18, 13]. It has been developed by many people(e.g. Kelly [14],

[*]Beijing International Center for Mathematical Research, Peking University, Beijing, P. R. China. mailto:chenxiao002214336@yahoo.cn

[†]School of Mathematical Sciences, University of Adelaide, Adelaide, Australia. mailto:anthony.roberts@adelaide.edu.au

[‡]Institute for Pure and Applied Mathematics, University of California, Los Angeles, USA. mailto:jduan@ipam.ucla.edu

Carr [7], Vanderbauwhede [30]). It is also important to study the stochastic center manifolds since in many applications the dynamical systems are influenced by noise and now call the random dynamical system theory[1]. Arnold [1] summarised various invariant manifolds on finite dimensional random dynamical systems. Mohammed and Scheutzow [20] focused on the existence of local stable and unstable manifolds for stochastic differential equations driven by semimartingales. Boxler [4] proved the existence of stochastic center manifold for finite dimensional random dynamical systems by using the multiplicative ergodic theorem and discrete random map. Roberts [24, 25, 26] assumed existence of stochastic center manifolds for infinite stochastic partial differential equations in exploring the interactions of microscale noises and their macroscale modelling. The natural problem is to show the nature of stochastic center manifolds on infinite dimensional spaces.

In the present paper, we study the stochastic center manifolds for infinite dimensional random dynamical systems. We prove the existence of stochastic center manifolds on infinite dimensional random dynamical systems. We extend the results of the stochastic center manifold theory on finite dimensional random dynamical systems [1, 4] to the infinite dimensional random dynamical systems. However, Boxer's prove is not suitable in infinite dimension case since the Ascoli's theorem ([4], Lemma 4.4) cannot directly extend to the general infinite dimensional space. Recently we explain the existence and properties of stochastic center manifolds for a class of stochastic evolution equations [8] by the existence of exponential trichotomy. However it is not suitable for general stochastic partial differential equations.

There are some results about the invariant manifolds for random dynamical systems that generated from stochastic partial differential equations. Duan and others [6, 10, 11] presented stable and unstable invariant manifolds for a class of stochastic partial differential equations under the assumption of exponential dichotomy or pseudo exponential dichotomy. Stochastic inertial manifolds which generalized from center-unstable manifolds on finite dimensional spaces are constructed by different methods [3, 5, 23]. Chen et al.[8] proved the existence and its properties of center manifolds for a class of stochastic evolution equations with linearly multiplicative noise. It seems it is a natural way to consider the random dynamical systems generated by general stochastic partial differential equations. However, the problem is still open since one cannot apply Kolmogorovs theorem to ensure the stochastic flow property [10, 17]. For the infinite dimensional random dynamical systems, a classic result is the multiplicative ergodic theorem(MET). Ruelle [28] proved the stable and unstable manifolds in Hilbert space for differentiable dynamical systems by the technique of multiplicative ergodic theorem. Mohammed [21] gave a details of the extension of Ruelle [28] to prove the stable and unstable manifold of a class of semilinear stochastic evolution equations. For the discussion of stable and unstable manifolds in discrete random dynamical systems we refer to Li and Lu [15], Lian and Lu [16].

The concept of exponential trichotomy is important for center manifold theory in infinite dimensional dynamical systems and non-autonomous systems [2, 12, 22]. The existence of exponential trichotomy means that the space is split into three subspaces: center subspace, unstable subspace and stable subspace. Center manifold theory is based on the assumption of exponential trichotomy [22]. We first introduce the exponential trichotomy for random dynamical systems. Then we change the random dynamical systems under the context of the stochastic partial differential equations which we consider. Our basic tool is the multiplicative ergodic theorem. We introduce the multiplicative ergodic theorem(MET) on two side time in an infinite dimensional Hilbert space. Then the condition of exponential trichotomy holds if the Lyapunov exponents satisfies some gap condition. Various versions of multiplicative ergodic theorem for random dynamical systems on finite dimension space have been summarized by Arnlod [1]. A discrete version on multiplicative ergodic theorem for random dynamical systems on Hilbert space have recently been proved by Ruelle [28]. Mohammed [19, 21] gave a details of the extension to one side continuous time from a discrete version of multiplicative ergodic theorem [28].

Section 2 show how to adapt the results to two side time continuous random dynamical systems. By the multiplicative ergodic theorem we have proved, we split the Hilbert space into three subspaces: center subspace, unstable subspace and stable subspace. Then we introduce the exponential trichotomy and give a condition of existence of exponential trichotomy in Section 3. Used the Lyapunov–Perron method, Section 4 proves the existence of stochastic center manifolds under the assumption of exponential trichotomy. This result applied to the random dynamical systems generated by stochastic or random partial differential equations in Section 5.

2 Preliminaries

We recall below the definition of a cocycle in Hilbert space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose $\theta : \mathbb{R} \times \omega \to \omega$ is a group of $\mathbb{P}$-preserving ergodic transformations on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let H be a real separable Hilbert space with norm $|\cdot|$ and Borel σ-algebra $\mathcal{B}(H)$.

A random dynamical system (U, θ) on H is a $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}\otimes\mathcal{B}(H)$- measurable random mapping $U : \mathbb{R} \times \Omega \times H \to H$ with the following properties.

(i) $U(t_1+t_2,\omega) = U(t_2,\theta(t_1,\omega))\circ U(t_1,\omega)$ for all $t_1,t_2\in\mathbb{R}$, all $\omega\in\Omega$.

(ii) $U(0,\omega)x = x$ for all $x\in H, \omega\in\Omega$.

A random variable

$$X : (\Omega,\mathcal{F}) \to (\mathbb{R}^+\setminus\{0\}, \mathcal{B}(\mathbb{R}^+\setminus\{0\}))$$

is called *tempered from above* if

$$\limsup_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = 0$$

for ω contained in a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant set of full measure($t\to-\infty$ applies only to two-sided time). Such a random variable X is called *tempered from below* if X^{-1} is tempered from above. X is call *tempered* if both tempered from above and tempered from below. Arnold [1] proved that the random variable is tempered if and if it is ε-slowly varying for some $\varepsilon\geq 0$.

Boxler [4] used the random norm in order to obtain the random variables $K_1(\omega)$, $K_2(\omega)$, $K_3(\omega)$ are constants. However, for comparing convenience in the same metric, we used the Hilbert norm and prove the random variables are slowly varying.

We describe an infinite dimensional version of the multiplicative ergodic theorem(MET) with two side continuous time. The following result is mentioned by Ruelle in Hilbert space [28] for differentiable dynamical systems.

Theorem 1. *Let U be a linear random dynamical system of compact operators on H satisfying the following integrability condition*

$$\mathbb{E}\sup_{0\leq t\leq 1}\log^+\|U^{\pm}(t,\omega)\| + \mathbb{E}\sup_{0\leq t\leq 1}\log^+\|U^{\pm}(1-t,\theta_t\omega)\| < \infty. \tag{1}$$

Then there is a measurable set $\Omega_0\in\mathcal{F}$ such that $\theta_t(\Omega_0)\subset\Omega_0$ for all $t\in\mathbb{R}$, and for each $\omega\in\Omega_0$, the limit

$$\Lambda(\omega) := \lim_{t\to\pm\infty}[U(t,\omega)^*\circ U(t,\omega)]^{1/(2t)}$$

exists in the uniform operator norm. Each linear operator $\Lambda(\omega)$ is compact, non-negative and self-adjoint with a discrete spectrum

$$exp(\lambda_1) > exp(\lambda_2) > exp(\lambda_3) > \dots,$$

where the λ_i's are distinct and non-random. Then there exist linear spaces $H = W_1(\omega)\oplus\cdots\oplus W_\infty(\omega)$, $\dim W_i(\omega) = d_i$, $i = 1,2,\dots$, such that

$$\lim_{t\to\pm\infty}\frac{1}{t}\log\|U(t,\omega)x\| = \begin{cases} \lambda_i & \text{if } x\in W_i(\omega), \\ -\infty & \text{if } x\in W_\infty(\omega), \end{cases}$$
$$U(t,\omega)W_i(\omega)\subset W_i(\theta_t\omega).$$

for $t\in\mathbb{R}$ and all $\omega\in\Omega_0$.

Remark 1. The numbers λ_1, $\lambda_2, \cdots$ are called the *Lyapunov exponents* associated to random dynamical system U. The set of these numbers forms the *Lyapunov spectrum*.

Proof. Mohammed [19, 21] gave a details of the extension to one side continuous time from a discrete version of multiplicative ergodic theorem [28]. We show how to adapt to the Mohammed [19, 21]'s results to two side continuous time. The technique is from the discrete result [1, 27, 28].

From the integration condition (1) and Theorem 2.1.1 of Mohammed [21], we know that the Lyapunov exponents associated to random dynamical system $U^-(t,\omega) = U^{-1}(t,\theta_t^{-1}\omega)$ is $-\lambda_1 < -\lambda_2 < -\lambda_3 < \cdots$, and the corresponding eigenspace $F_{-i}(\omega)$ with $d_i = \dim F_{-i}(\omega)$, $\omega \in \Omega_1$. The Lyapunov exponents associated to random dynamical system $U^+(t,\omega) = U(t,\omega)$ is $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$, and the corresponding eigenspace $F_i(\omega)$ with $d_i = \dim F_i(\omega)$, $\omega \in \Omega_2$. Define

$$\begin{aligned} V_r(\omega) &= [\oplus_{j=1}^{r-1} F_j(\omega)]^{\perp}, \\ V_{-r}(\omega) &= \oplus_{j=1}^{r} F_{-j}(\omega), \end{aligned}$$

for $r = 1, 2, \ldots$. Then

$$V_{-1}(\omega) \subset V_{-2}(\omega) \subset V_{-3}(\omega) \subset \cdots \subset V_{-\infty}(\omega) = H$$

for $\Omega_1 \in \mathcal{F}$ such that for all $t \in \mathbb{R}^+$, $\theta_t^{-1}(\Omega_1) \subset \Omega_1$, codim $V_{r+1}(\omega) = \dim V_{-r}(\omega)$, and

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log \|U^{-1}(t,\theta_t^{-1}\omega)x\| = \begin{cases} -\lambda_i & \text{if } x \in V_{-i-1}(\omega) \setminus V_{-i}(\omega), \\ \infty & \text{if } x \in V_{-1}(\omega), \end{cases} \\ &U^{-1}(t,\theta_t^{-1}\omega)V_{-i}(\omega) \subset V_{-i}(\theta_t^{-1}\omega). \end{aligned}$$

That is $\theta_t(\Omega_1) \subset \Omega_1$, and

$$\begin{aligned} &\lim_{t\to-\infty} \frac{1}{t} \log \|U(t,\omega)x\| = \begin{cases} \lambda_i & \text{if } x \in V_{-i-1}(\omega) \setminus V_{-i}(\omega), \\ -\infty & \text{if } x \in V_{-1}(\omega), \end{cases} \\ &U(t,\omega)V_{-i}(\omega) \subset V_{-i}(\theta_t\omega), \end{aligned}$$

for all $t \in \mathbb{R}^-$.

We show that for almost $\omega \in \Omega_1 \cap \Omega_2 =: \Omega_0$, $V_{r+1}(\omega) \cap V_{-r}(\omega) = \emptyset$, $V_{r+1}(\omega) \oplus V_{-r}(\omega) = H$.

Let $B := \{\omega \in \Omega_0 : V_{r+1}(\omega) \cap V_{-r}(\omega) \neq \emptyset\}$. Select $\omega \in B$ such that $v \in V_{r+1}(\omega) \cap V_{-r}(\omega)$. Given $\delta > 0$, let B_n be the subset of B such that if $\omega \in B_n$, for all $v \in V_{r+1}(\omega) \cap V_{-r}(\omega)$,

$$\|U(n,\omega)v\| \leq \|v\| \exp[n(\lambda_{r+1} + \delta)], \tag{2}$$

$$\|U^{-1}(n,\omega)v\| \leq \|v\| \exp[n(-\lambda_r + \delta)], \tag{3}$$

If $v \in V_{r+1}(\omega) \cap V_{-r}(\omega)$, then $U(n,\omega)v \in V_{r+1}(\theta_n\omega) \cap V_{-r}(\theta_n\omega)$. If $\omega \in \theta_n^{-1}B_n \cap B_n$, we obtain

$$\|v\| = \|U^{-1}(n,\omega)U(n,\omega)v\| \le \|U(n,\omega)v\| \exp(n(-\lambda_r + \delta)). \tag{4}$$

Equations (2) and (4) implies $\lambda_r - \lambda_{r+1} \le 2\delta$. Since $\mathbb{P}(B_n \cap \theta_n^{-1}B_n) \to \mathbb{P}(B)$ and δ is arbitrary, we conclude $\mathbb{P}(B) = 0$. $V_{r+1}(\omega) \oplus V_{-r}(\omega) = H$ follows $\operatorname{codim} V_{r+1} = \dim V_{-r}$.

Let $W_i(\omega) = V_i(\omega) \cap V_{-i}(\omega)$, we obtain $H = V_1(\omega) \cap (V_{-1}(\omega) \oplus V_2(\omega)) \cap \cdots \cap V_{-\infty}(\omega) = W_1(\omega) \oplus \cdots W_\infty(\omega)$. □

3 MET implies exponential trichotomy

In this section, we consider the exponential trichotomy for random dynamical systems $U(t,\omega)$ in a Hilbert space.

First we introduce the exponential trichotomy, which generalizes the exponential dichotomy [9, 10, 11, 29].

Definition 2. *$U(t,\omega)$ is said to be exponential trichotomy if there exists a θ_t-invariant set $\tilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$, the phase space H splits into*

$$H = E^s(\omega) \oplus E^c(\omega) \oplus E^u(\omega)$$

of closed subspaces satisfying

(i) This splitting is invariant under $U(t,\omega)$:

$$U(t,\omega)E^s(\theta_t\omega) \subset E^s(\theta_t\omega),$$
$$U(t,\omega)E^c(\theta_t\omega) \subset E^c(\theta_t\omega),$$
$$U(t,\omega)E^u(\theta_t\omega) \subset E^u(\theta_t\omega);$$

(ii) There are θ-invariant random variables $\alpha(\omega) > \beta(\omega)$, and tempered variables $K^s(\omega) : \tilde{\Omega} \to (0,\infty)$, $K^c(\omega) : \tilde{\Omega} \to (0,\infty)$ and $K^u(\omega) : \tilde{\Omega} \to (0,\infty)$ such that

$$\|U(t,\omega)\| \le K^s(\omega) \exp\left[-\alpha(\omega)t\right] \quad \text{for } t \ge 0, \tag{5}$$
$$\|U(t,\omega)\| \le K^c(\omega) \exp \gamma(\omega)t \quad \text{for } t \in \mathbb{R}, \tag{6}$$
$$\|U(t,\omega)\| \le K^u(\omega) \exp \beta(\omega)t \quad \text{for } t \le 0, \tag{7}$$

where $\Pi^i(\omega)$, $i = s,c,u$, are the measurable projections associated with the splitting.

Lemma 3. *Suppose that the following exponential integrability condition is satisfied*

$$\mathbb{E} \log^+ \sup_{0 \le t \le 1} \|U^{\pm}(t,\omega)\| + \mathbb{E} \log^+ \sup_{0 \le t \le 1} \|U^{\pm}(1-t,\theta_t\omega)\| < \infty. \tag{8}$$

Then there exists a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant set of full $\mathbb{P}$-measure and a constant $H(\omega)$ such that

$$\|U^u(t,\omega)\| \ge H(\omega)\exp(at) \quad \text{for } t \le 0,\ \omega \in \tilde{\Omega}$$

for a sufficiently large a.

Proof. Since

$$\|U^u(t,\omega)\| \le \|U(t,\omega)\|,$$

we conclude that $\mathbb{E}\log^+\|U^u(-1,\omega)\| < \infty$. By subadditive ergodic theorem, there exists a set of measure one such that

$$\lim_{i\to-\infty} \frac{1}{i} \log \|U^u(i,\omega)\| = a. \tag{9}$$

Set

$$\Omega_n^1 := \{\omega \in \Omega : \lim_{i\to-\infty} \frac{1}{i} \log \|U^u(i,\theta_n\omega)\| = \tilde{a}\} \in \mathcal{F},$$

$$\Omega^1 := \bigcap_{n\in\mathbb{Z}} \Omega_n^1.$$

The set Ω^1 is $\{\theta_t\}_{t\in\mathbb{Z}}$–invariant and has probability one.

Let $D_1(\omega) = \log\inf_{-1\le t\le 0}\|U^u(t,\omega)\|$. Since $\mathbb{E}D_1^+(\omega) < \infty$, by Borel-Cantelli lemma, there exists a full measurable set Ω^2 so that

$$\limsup_{i\to-\infty} \frac{1}{i} D_1(\theta_{i+n}\omega) = 0, \qquad n \in \mathbb{N}.$$

For $\omega \in \Omega^1 \cap \Omega^2$,

$$\log\|U^u(t,\omega)\| = \log\|U^u(t-[t],\theta_{[t]}\omega)\| + \log\|U^u([t],\omega)\|. \tag{10}$$

Thus $\limsup_{t\to-\infty} \log\|U^u_\lambda(t,\omega)\|/t \ge a$. We obtain the conclusion. $\square$

Now we show that the multiplicative ergodic theorem (MET) in Section 2 implies exponential trichotomy in Hilbert spaces.

Theorem 4. *Suppose that the following exponential integrability condition is satisfied*

$$\mathbb{E}\log^+ \sup_{0\le t\le 1} \|U^{\pm}(t,\omega)\| + \mathbb{E}\log^+ \sup_{0\le t\le 1} \|U^{\pm}(1-t,\theta_t\omega)\| < \infty. \tag{11}$$

and suppose that $\alpha, \beta, \gamma \in \mathbb{R}$ *is not contained in the Lyapunov spectrums such that* $\cdots < \lambda_i < -\beta < -\gamma < \lambda_j < \gamma < \alpha < \cdots < \lambda_2 < \lambda_1$*. Then there exists a* $\{\theta_t\}_{t\in\mathbb{R}}$ *invariant set* $\tilde{\Omega}$ *of full measure such that for* $\omega \in \tilde{\Omega}$ *we have the following properties: There exist linear spaces* $E^u(\omega)$*,* $E^c(\omega)$*,* $E^s(\omega)$ *such that*

$$H = E^u(\omega) \oplus E^c(\omega) \oplus E^s(\omega).$$

There exist tempered random variables $K^s(\omega)$*,* $K^c(\omega)$ *and* $K^u(\omega)$ *such that,*

$$\begin{aligned} \|U^s(t,\omega)\| &\le K^s(\omega)\ \exp(-\alpha t), \quad t \ge 0; \\ \|U^c(t,\omega)\| &\le K^c(\omega)\ \exp(\gamma|t|), \quad t \in \mathbb{R}; \\ \|U^u(t,\omega)\| &\le K^u(\omega)\ \exp(\beta t), \quad t \le 0. \end{aligned}$$

Proof. Let the Lyapunov spectrums $\lambda_1 > \lambda_2 > \cdots > \lambda_+ > \lambda_j = 0 > \lambda_- > \lambda_{i+1} > \cdots$, $E^s(\omega) = W_{i+1} \oplus W_{i+2} \oplus \ldots$. We start with $K^s(\omega)$. Define

$$K^s(\omega) := \sup_{t\in\mathbb{R}^+} \frac{\|U^s(t,\omega)\|}{\exp[(\lambda_- + \varepsilon)t]},$$

where $-\alpha = \lambda_- + \varepsilon$. Then by the multiplicative ergodic theorem the random variable is finite. Let $F_n(\omega) = \log\|U^s(n,\omega)\|$. Since

$$\mathbb{E}F_1^+(\omega) = \mathbb{E}\log^+ \|U^s(1,\omega)\| < +\infty$$

and $F_{m+n}(\omega) = \log\|U^s(m+n,\omega)\| \le \log\|U^s(m,\theta_n\omega)\|\|U^s(n,\omega)\| = F_n(\omega) + F_m(\theta_n\omega)$. By subadditive ergodic theorem, there exists a $\{\theta_t\}_{t\in\mathbb{Z}}-$invariant measurable function $F(\omega)$ such that

$$F(\omega) = \lim_{n\to\infty} \frac{1}{n} F_n(\omega) = \lim_{n\to\infty} \frac{1}{n} \log\|U^s(n,\omega)\| \le \lambda_-. \tag{12}$$

By a consequence of Kingman's subadditive ergodic theorem[28, Corollary A.2], for every $\epsilon > 0$, there is a finite-valued random variable $K_\epsilon(\omega)$ such that when $n > m$,

$$\log\|U^s(n-m,\theta_m\omega)\| \le (n-m)\lambda_- + n\epsilon + K_\epsilon(\omega), \quad a.s. \tag{13}$$

Let $D(\omega) = \log^+ \sup_{0\le t\le 1} \|U(t,\omega)\| + \log^+ \sup_{0\le t\le 1} \|U(1-t,\theta_t\omega)\|$, then $\mathbb{E}D(\omega) < \infty$. From the Borel-Cantelli lemma, there is a measurable set Ω_1 such that $\mathbb{P}(\Omega_1) = 1$, and

$$\lim_{i\to\infty} \frac{1}{i} D(\theta_{i+n}\omega) = \limsup_{i\to\infty} \frac{1}{i} D(\theta_{i+n}\omega) = 0, \quad \omega \in \Omega_2, n \in \mathbb{N}.$$

From

$$\begin{aligned}\|U^s(t,\theta_r\omega)\| \le{}& \|U^s(1+t-[t]-1-[r]+r,\theta_{[r]+[t]}\omega)\| \\ & \times \|U^s([t]-1,\theta_{1+[r]}\omega)\| \times \|U^s(1-r+[r],\theta_r\omega)\| \\ \le{}& \exp[D(\theta_{[r]+[t]-1}\omega)] \exp[K_\varepsilon(\omega) + \epsilon([r]+1) + (\lambda_- + \epsilon)([t]-1)] \\ & \exp[D(\theta_{[r]}\omega)].\end{aligned}$$

From the definition of $K^s(\omega)$,

$$\lim_{r\to\infty} \frac{\log^+ K^s(\theta_r\omega)}{r} = 0.$$

So the random variable $K^s(\omega)$ is tempered.

Next we prove $K^u(\omega)$ is tempered. Let $E^u(\omega) = W_1(\omega) \oplus W_2(\omega) \oplus \ldots \oplus W_k(\omega) \ldots$ for $\lambda_k \ge \lambda_+$. Since

$$\lim_{t\to-\infty} \frac{1}{t} \log \|U^u(t,\omega)x\| \ge \lambda_+ > 0, \quad x \in E^u(\omega), \tag{14}$$

then there exists a $\varepsilon > 0$ and $t_1(x,\varepsilon,\omega) < 0$ such that $\|U(t,\omega)x\| \ge e^{t(\lambda_+-\varepsilon)}$ for $t < t_1$. Let $\beta = \lambda_+ - \varepsilon$. From Lemma 3, assume that

$$\|U^u(t,\omega)\| \ge H(\omega)\exp[at] \quad \text{for} \quad t \le 0,\, \omega \in \Omega, \tag{15}$$

for a measurable function $H(\omega)$ We show that

$$K^u(\omega) := \sup_{t\le 0} \frac{\|U^u(t,\omega)\|}{\exp[(\lambda_+ - \varepsilon)t]}$$

is a tempered random variable in $(-\infty, 0)$. Since

$$\exp(at) \le \frac{\|U^u(t,\omega)\|}{H(\omega)} \quad \text{for any } t \le 0.$$

We then see that for $s < 0$

$$
\begin{aligned}
K^u(\theta_s\omega)\exp(as) &= \sup_{t\le 0}\frac{\|U^u(t,\theta_s\omega)\|}{\exp[(\lambda_+-\varepsilon)t]}\exp(as)\\
&\le \sup_{t\le 0}\frac{\|U^u(t,\theta_s\omega)\|\|U^u(s,\omega)\|}{\exp[(\lambda_+-\varepsilon)(t+s)]}\frac{1}{H(\omega)}\exp[(\lambda_+-\varepsilon)s]\\
&\le \sup_{t\le 0}\frac{\|U^u(t+s,\omega)\|}{\exp[(\lambda_+-\varepsilon)(t+s)]}\frac{1}{H(\omega)}\exp[(\lambda_+-\varepsilon)s]\\
&= \frac{K^u(\omega)}{H(\omega)}\exp[(\lambda_+-\varepsilon)s]
\end{aligned}
$$

which goes to zero for $s \to -\infty$.

Similarly there exists a tempered random variable $K^c(\omega)$.

□

4 Stochastic center manifolds

We introduce the definition of stochastic center manifolds. A basic tool of proving the existence of stochastic center manifolds is to define an appropriate function space, which is a Banach space.

Definition 5. *A random set $M(\omega)$ is called an (forward) invariant set for a random dynamical system $\phi(t,\omega,x)$ if $\phi(t,M(\omega),\omega) \subset M(\theta_t\omega)$ for $t \ge 0$. If we can represent $M(\omega)$ by a graph of a (Lipschitz) mapping from the center subspace to its complement, $h^c(\cdot,\omega) : H^c \to H^u \oplus H^s$, such that $M(\omega) = \{v + h^c(v,\omega) \mid v \in H^c\}$, $h^c(0,\omega) = 0$, and the tangency condition that the derivative $Dh^c(0,\omega) = 0$, $h^c(v,\cdot)$ is measurable for every $v \in H^c$, then $M(\omega)$ is called a (Lipschitz) center manifold, often denoted as $M^c(\omega)$.*

Let $U(t,x,\omega)$ is the linearization of random dynamical system $\varphi(t,x,\omega)$, i.e. the Fréchet derivative $D\varphi(t,x,\omega)$ at point x. Then $U(t,x,\omega)$ is also a random dynamical system. Arnold [1] proved the random differential equation could be generated from a random dynamical systems. We prove a similar result in the Hilbert space.

Lemma 6. *Let $U(t,x,\omega) = D\varphi(t,x,\omega)$. Suppose the random dynamical systems $\varphi(t,x,\omega)$ and $U(t,x,\omega)$ are differentiable at $t = 0$,*

$$
\begin{aligned}
f(\omega,x) &= \frac{d}{dt}\varphi(t,x,\omega)|_{t=0},\\
A(\omega)x &= \frac{d}{dt}U(t,x,\omega)|_{t=0}.
\end{aligned}
$$

Then $\varphi(t,x,\omega)$ is the solution of

$$\begin{aligned}&\frac{du}{dt}=f(\theta_t\omega,u),\\&u(0,x,\omega)=x.\end{aligned}$$

$U(t,x,\omega)$ is the solution of

$$\begin{aligned}&\frac{dv}{dt}=A(\theta_t\omega)v,\\&v(0,x,\omega)=x.\end{aligned}$$

Proof. Since $\varphi(s+t,x,\omega)=\varphi(s,\cdot,\theta_t\omega)\circ\varphi(t,x,\omega)$, then

$$\frac{\varphi(s+t,x,\omega)-\varphi(t,x,\omega)}{s}=\frac{\varphi(s,\cdot,\theta_t\omega)\circ\varphi(t,x,\omega)-\varphi(t,x,\omega)}{s}.$$

Let $s\to 0$ yields

$$\frac{d\varphi}{dt}=f(\theta_t\omega,\varphi).$$

Similar

$$\frac{D\varphi(s+t,x,\omega)-D\varphi(t,x,\omega)}{s}=\frac{D\varphi(s,\cdot,\theta_t\omega)\circ D\varphi(t,x,\omega)-D\varphi(t,x,\omega)}{s}.$$

Let $s\to 0$ yields

$$\frac{dU}{dt}=A(\theta_t\omega)U.$$

□

Denote $B(\theta_t\omega)x=f(\theta_t\omega,x)-A(\theta_t\omega)x$, then φ is the solution of

$$\begin{aligned}&\frac{du}{dt}=A(\theta_t\omega)u+B(\theta_t\omega)u,\\&u(0,x,\omega)=x.\end{aligned}\tag{16}$$

We assume the nonlinear term $B(\theta_t\omega)$ satisfies $B(\theta_t\omega)(0)=0$, and assume it to be Lipschitz continuous on H, that is,

$$|B(\theta_t\omega)u_1-B(\theta_t\omega)u_2|\le \operatorname{Lip}B|u_1-u_2|$$

with the sufficiently small Lipschitz constant $\operatorname{Lip}B>0$. We show the existence of a center manifold for the random partial differential equation (16).

For each $\eta(\omega) > 0$, we denote the Banach space

$$C_\eta(\omega) = \left\{ \phi \in C(\mathbb{R}, H) \mid \sup_{t\in\mathbb{R}} \exp\left[-\eta(\omega)|t|\right] |\phi(t)| < \infty \right\}$$

with the norm

$$|\phi|_{C_\eta(\omega)} = \sup_{t\in\mathbb{R}} \exp\left[-\eta(\omega)|t|\right] |\phi(t)|.$$

The set $C_\eta(\omega)$ is the set of 'slowly varying' functions. We know that the functions are controlled by $\exp\left[\eta(\omega)|t|\right]$. Let

$$M^c(\omega) = \{u_0 \in H \mid u(\cdot, u_0, \omega) \in C_\eta(\omega)\},$$

where $u(t, u_0, \omega)$ is the solution of (16) with the initial data $u(0) = u_0$.

We prove that $M^c(\omega)$ is invariant and is the graph of a Lipschitz function.

Different from the proof process of Duan et al. [10, 11], we need analysis the behavior of the solution on center subspace.

Theorem 7. *Suppose $U(t, \omega)$ satisfies the exponential trichotomy. If $\gamma(\omega) < \eta(\omega) < \min\{\beta(\omega), \alpha(\omega)\}$ such that the nonlinearity term is sufficiently small,*

$$K(\omega)\,\mathrm{Lip}\,B\left(\frac{1}{\eta(\omega)-\gamma(\omega)} + \frac{1}{\beta(\omega)-\eta(\omega)} + \frac{1}{\alpha(\omega)-\eta(\omega)}\right) < 1, \tag{17}$$

then there exists a center manifold for the random differential equation (16), which is written as the graph

$$M^c(\omega) = \{v + h^c(v, \omega) \mid v \in H^c\},$$

where $h^c(\cdot, \omega) : H^c \to H^u \oplus H^s$ is a Lipschitz continuous mapping from the center subspace and satisfies $h^c(0, \omega) = 0$.

Proof. First we claim that $u_0 \in M^c(\omega)$ if and only if there exists a slowly varying function $u(\cdot, u_0, \omega) \in C_\eta(\omega)$ with

$$\begin{aligned} u(t, u_0, \omega) = {} & U^c(t, v, \omega) + \int_0^t U^c(t-s, \omega) P^c B(\theta_r\omega) u(r)\, dr \\ & + \int_{-\infty}^t U^s(t-r, \omega) P^s B(\theta_r\omega) u(r)\, dr \\ & - \int_t^{+\infty} U^u(t-r, \omega) P^u B(\theta_r\omega) u(r)\, dr, \end{aligned} \tag{18}$$

where $v = P^c u_0$.

To prove this claim, first we let $u_0 \in M^c(\omega)$. By using the variation of constants formula, the solution on each subspace denoted as

$$P^c u(t, u_0, \omega) = U^c(t, \omega)v + \int_0^t U^c(t-r, \omega) P^c B(\theta_r \omega) u(r)\, dr\,. \tag{19}$$

$$P^u u(t, u_0, \omega) = U^u(t-\tau, \omega) P^u u(\tau, u_0, \omega) + \int_\tau^t U^u(t-r, \theta_r \omega) P^u B(\theta_r \omega) u(r)\, dr\,. \tag{20}$$

$$P^s u(t, u_0, \omega) = U^s(t-\tau, \omega) P^s u(\tau, u_0, \omega) + \int_\tau^t U^s(t-r, \theta_r \omega) P^s B(\theta_r \omega) u(r)\, dr\,. \tag{21}$$

Since the slowly varying function $u \in C_\eta(\omega)$, we have for $t < \tau$ that the magnitude

$$\begin{aligned}
|U^u(t-\tau, \omega) P^u u(\tau, u_0, \omega)| &\le K^u(\omega) \exp[\alpha(\omega)(t-\tau)] \exp(\eta(\omega)\tau) |u|_{C_{\eta(\omega)}} \\
&= K^u(\omega) \exp[\alpha(\omega)t] \exp[-(\alpha(\omega) - \eta(\omega))\tau] |u|_{C_{\eta(\omega)}} \\
&\to 0 \quad \text{as } \tau \to +\infty.
\end{aligned}$$

For $t > \tau$,

$$\begin{aligned}
|U^s(t-\tau, \omega) P^u u(\tau, u_0, \omega)| &\le K^s(\omega) \exp[-\beta(\omega)(t-\tau)] \exp(\eta(\omega)\tau) |u|_{C_{\eta(\omega)}} \\
&= K^s(\omega) \exp[-\beta(\omega)t] \exp[(\beta(\omega) + \eta(\omega))\tau] |u|_{C_{\eta(\omega)}} \\
&\to 0 \quad \text{as } \tau \to -\infty.
\end{aligned}$$

Then, taking the two separate limits $\tau \to \pm\infty$ in (20) and (21) respectively,

$$P^u u(t, u_0, \omega) = \int_\infty^t U^u(t-r, \theta_r \omega) P^u B(\theta_r \omega) u(r)\, dr, \tag{22}$$

$$P^s u(t, u_0, \omega) = \int_{-\infty}^t U^s(t-r, \theta_r \omega) P^s B(\theta_r \omega) u(r)\, dr\,. \tag{23}$$

Combining (19), (22) and (23), we have (18). The converse follows from a direct computation.

Next we prove that for any given $v \in H^c$, the centre subspace, the integral equation (18) has a unique solution in the slowly varying functions space $C_\eta(\omega)$. Let

$$
\begin{aligned}
J^c(u,v) := U^u(t,\omega)v &+ \int_0^t U^c(t-r,\omega)P^c B(\theta_r\omega)u(r)\,dr \\
&+ \int_{-\infty}^t U^s(t-r,\omega)P^s B(\theta_r\omega)u(r)\,dr \\
&- \int_t^{+\infty} U^u(t-r,\omega)P^u B(\theta_r\omega)u(r)\,dr\,.
\end{aligned}
\tag{24}
$$

J^c is well-defined from $C_\eta(\omega)\times H^c$ to the slowly varying functions space $C_\eta(\omega)$. For each pair of slowly varying functions $u, \bar{u} \in C_\eta(\omega)$, we have that for $\gamma(\omega) < \eta(\omega) < \min\{\beta(\omega), \alpha(\omega)\}$, $K(\omega) = \max\{K^s(\omega), K^c(\omega), K^u(\omega)\}$,

$$
\begin{aligned}
&|J^c(u,v) - J^c(\bar{u},v)|_{C_\eta(\omega)} \\
&\le \sup_{t\in\mathbb{R}} \Bigg\{ \exp\left[-\eta(\omega)|t|\right] \Bigg| \int_0^t U^c(t-r,\omega)P^c(B(\theta_r\omega)u - B(\theta_r\omega)\bar{u})\,dr \\
&\quad + \int_{-\infty}^t U^s(t-r,\omega)(P^s B(\theta_r\omega)u - P^s B(\theta_r\omega)\bar{u})\,dr \\
&\quad + \int_t^{+\infty} U^u(t-r,\omega)(P^u B(\theta_r\omega)u - P^u B(\theta_r\omega)\bar{u})\,dr \Bigg| \Bigg\} \\
&\le \sup_{t\in\mathbb{R}} \Bigg\{ K(\omega)\,\mathrm{Lip}\,B|u-\bar{u}|_{C_\eta(\omega)} \Bigg| \int_0^t \exp[(\gamma(\omega)-\eta(\omega))|t-r|]\,dr \\
&\quad + \int_{-\infty}^t \exp[(\eta(\omega)-\alpha(\omega))(t-r)]\,dr + \int_t^{+\infty} \exp[(\beta(\omega)-\eta(\omega))(t-r)]\,dr \Bigg| \Bigg\} \\
&\le K(\omega)\,\mathrm{Lip}\,B\left(\frac{1}{\eta(\omega)-\gamma(\omega)} + \frac{1}{\alpha(\omega)-\eta(\omega)} + \frac{1}{\beta(\omega)-\eta(\omega)}\right)|u-\bar{u}|_{C_\eta(\omega)}.
\end{aligned}
\tag{25}
$$

From equation (25), J^c is Lipschitz continuous in v. By the theorem's precondition (29), J^c is a uniform contraction with respect to the parameter v. By the uniform contraction mapping principle, for each $v \in H^c$, the mapping $J^c(\cdot,v)$ has a unique fixed point $u(\cdot,v,\omega) \in C_\eta(\omega)$. Combining equation (24) and equation (25),

$$
\begin{aligned}
&|u(\cdot,v,\omega) - u(\cdot,\bar{v},\omega)|_{C_\eta(\omega)} \\
&\le \frac{K(\omega)}{1 - K(\omega)\,\mathrm{Lip}\,B\left(\frac{1}{\eta(\omega)-\gamma(\omega)} + \frac{1}{\beta(\omega)-\eta(\omega)} + \frac{1}{\alpha(\omega)-\eta(\omega)}\right)}|v-\bar{v}|,
\end{aligned}
\tag{26}
$$

for each fixed point $u(\cdot, v, \omega)$. Then for each time $t \geq 0$, $u(t, \cdot, \omega)$ is Lipschitz from the center subspace H^c to slowly varying functions $C_\eta(\omega)$. $u(\cdot, v, \omega) \in C_\eta(\omega)$ is a unique solution of the integral equation (18). Since $u(\cdot, v, \omega)$ can be an ω-wise limit of the iteration of contraction mapping J^c starting at 0 and J^c maps a $\mathcal{F}$-measurable function to a $\mathcal{F}$-measurable function, $u(\cdot, v, \omega)$ is $\mathcal{F}$-measurable. Combining $u(\cdot, v, \omega)$ is continuous with respect to H, we have $u(\cdot, v, \omega)$ is measurable with respect to $(\cdot, v, \omega)$.

Let $h^c(v, \omega) := P^s u(0, v, \omega) \oplus P^u u(0, v, \omega)$. Then

$$\begin{aligned} h^c(v, \omega) = &\int_{-\infty}^{0} U^s(-r, \omega) P^s B(\theta_r \omega) u(r, v, \omega)\, dr \\ &- \int_{0}^{+\infty} U^u(-r, \omega) P^u B(\theta_r \omega) u(r, v, \omega)\, dr\,. \end{aligned}$$

We see that h^c is $\mathcal{F}$-measurable and $h^c(0, \omega) = 0$. From the definition of $h^c(v, \omega)$ and the claim that $u_0 \in M^c(\omega)$ if and only if there exists

$$u(\cdot, u_0, \omega) \ \in \ C_\eta(\omega)$$

with $u(0) = u_0$ and satisfies (18) it follows that $u_0 \in M^c(\omega)$ if and only if there exists $v \in H^c$ such that $u_0 = v + h^c(v, \omega)$, therefore,

$$M^c(\omega) = \{v + h^c(v, \omega) \mid v \in H^c\}.$$

Next we prove that for any $x \in H$, the function

$$\omega \to \inf_{y \in H^c} |x - (y + h^c(y, \omega))| \tag{27}$$

is measurable. Let H' be a countable dense subset of the separable space H. From the continuity of $h^c(\cdot, \omega)$,

$$\inf_{y \in H^c} |x - (y + h^c(y, \omega))| = \inf_{y \in H'} |x - P^c y - h^c(P^c y, \omega)|. \tag{28}$$

The measurability of (27) follows since $\omega \to h^c(P^c y, \omega)$ is measurable for any $y \in H'$.

Finally, we show that $M^c(\omega)$ is invariant, that is for each $u_0 \in M^c(\omega)$, $u(s, u_0, \omega) \in M^c(\theta_s \omega)$ for all $s \geq 0$. Since for $s \geq 0$, $u(t + s, u_0, \omega)$ is a solution of

$$\frac{du}{dt} = A(\theta_t(\theta_s \omega))u + B(\theta_t(\theta_s \omega))u, \quad u(0) = u(s, u_0, \omega).$$

Thus $u(t, u(s, u_0, \omega), \theta_s \omega) = u(t + s, u_0, \omega)$ and $u(t, u(s, u_0, \omega), \theta_s \omega) \in C_\eta(\omega)$. So we conclude $u(s, u_0, \omega) \in M^c(\theta_s \omega)$.

□

Corollary 8. *Suppose the linear random dynamical systems $U(t,\omega)$ satisfies the multiplicative ergodic theorem (*MET*) and $\alpha, \beta, \gamma \in \mathbb{R}$ is not contained in the Lyapunov spectrums such that $\cdots < \lambda_i < -\beta < -\gamma < \lambda_j < \gamma < \alpha < \cdots < \lambda_2 < \lambda_1$. If $\gamma < \eta < \min\{\beta, \alpha\}$ such that the nonlinearity term is sufficiently small,*

$$K(\omega)\,\mathrm{Lip}\,B\left(\frac{1}{\eta-\gamma}+\frac{1}{\beta-\eta}+\frac{1}{\alpha-\eta}\right)<1\,, \tag{29}$$

then there exists a center manifold for the random random dynamical system $\varphi(t,x,\omega)$, which is written as the graph

$$M^c(\omega)=\{v+h^c(v,\omega)\mid v\in H^c\},$$

where $h^c(\cdot,\omega): H^c \to H^u \oplus H^s$ is a Lipschitz continuous mapping from the center subspace and satisfies $h^c(0,\omega)=0$.

5 An application

In this section, we show the above center manifold theory by illustrating a stochastic evolution equation

$$\frac{du}{dt}=Au+F(u)+u\circ\dot{W}(t), \tag{30}$$

where $u \in H$ is a Hilbert space typically defined on some spatial domain, $W(t)$ is the standard $\mathbb{R}$-valued Wiener process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, which is only dependent on time. Suppose the spectrum of A satisfies

$$\mu_1>\cdots>\mu_j>0>\mu_{j+1}>\mu_{j+2}>\cdots\quad(\text{with}\ \ \mu_j\to-\infty\ \ \text{as}\ \ j\to\infty),$$

and A generates a strong continuous semigroup $S_A(t)$, being $S_A(t)$ compact for all $t \geq 0$. We assume the nonlinear term F satisfies $F(0)=0$, and assume it to be Lipschitz continuous on H, that is,

$$|F(u_1)-F(u_2)|\leq \mathrm{Lip}\,F\,|u_1-u_2|$$

with the sufficiently small Lipschitz constant $\mathrm{Lip}\,F>0$.

The above example has been show that there exist a stochastic center manifold by Chen et al. [8]. Now we only verify the equation (30) satisfies the MET in Section 2, then there exists a stochastic center manifold.

Let $C_0(\mathbb{R},\mathbb{R})$ be continuous functions on $\mathbb{R}$, the associated distribution $\mathbb{P}$ is a Wiener measure defined on the Borel-σ-algebra $\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$. Define

$\{\theta_t\}_{t\in\mathbb{R}}$ to be the metric dynamical system generated by the Wiener process $W(t)$.

We transform the stochastic evolution equation (30) into the following partial differential equation with random coefficients

$$\frac{du}{dt} = Au + z(\theta_t\omega)u + G(\theta_t\omega, u), \quad u(0) = u_0 \in H, \tag{31}$$

where $G(\omega, u) = \exp[-z(\omega)]F(\exp[z(\omega)]u)$, $z(\theta_t\omega)$ is the solution of Ornstein–Uhlenbeck equation,

$$dz + z\,dt = dW. \tag{32}$$

And

$$|G(\omega, u_1) - G(\omega, u_2)| \le \operatorname{Lip}_u G\,|u_1 - u_2|,$$

where $\operatorname{Lip}_u G$ denotes the Lipschitz constant of $G(\cdot, u)$ with respect to u. For any $\omega \in \Omega$ the function G has the same global Lipschitz constant as F by the construction of G. The linearization equation is

$$\frac{du}{dt} = Au + z(\theta_t\omega)u, \quad u(0) = u_0 \in H, \tag{33}$$

$U(t,\omega)$ is compact since $S_A(t)$ is compact. We prove that the assumption (1) is satisfied. For $t \in [-1, 1]$,

$$\|U(t,\omega)\| \le \|S_A(t)\|| \exp \int_0^t z(\theta_r\omega)dr|,$$

$$\|U(1-t,\theta_t\omega)\| \le \|S_A(1-t)\|| \exp \int_0^{1-t} z(\theta_{t+r}\omega)dr|,$$

and

$$\log^+ \|U(t,\omega)\| \le \log^+ \|S_A(t)\| + \int_{-1}^1 |z(\theta_r\omega)|dr,$$

$$\log^+ \|U(1-t,\theta_t\omega)\| \le \log^+ \|S_A(1-t)\| + \int_{-2}^1 |z(\theta_r\omega)|dr,$$

Therefore,

$$\mathbb{E} \sup_{0\le t\le 1} \log^+ \|U^\pm(t,\omega)\| + \mathbb{E} \sup_{0\le t\le 1} \log^+ \|U^\pm(1-t,\theta_t\omega)\| < \infty.$$

Acknowledgement This work was supported by the Australian Research Council grants DP0774311 and DP0988738, and by the NSF grant 1025422.

References

[1] L. Arnold. *Random Dynamical Systems*, Springer, New York, 1998.

[2] L. Barreira and C. Valls, Smooth center manifolds for nonuniformly partially hyperbolic trajectories. *J. Differential Equations*, **237**, 307–342, 2007.

[3] A. Bensoussan and F. Flandoli, Stochastic inertial manifold. *Stochastics Stochastics Rep.*, **53**, 13 – 39, 1995.

[4] P. Boxler, A stochastic version of center manifold theory. *Probab. Theory Related Fields*, **83**, 509–545, 1989.

[5] T. Caraballo, I. Chueshov, and J. A. Langa, Existence of invariant manifolds for coupled parabolic and hyperbolic stochastic partial differential equations. *Nonlinearity*, **18**, 747–767, 2005.

[6] T. Caraballo, J. Duan, K. Lu and B. Schmalfuss, Invariant manifolds for random and stochastic partial differential equations. *Adv. Nonlinear Stud.*, **10**, 23–52, 2010.

[7] J. Carr, Applications of Center Manifold Theory, Springer, New York, 1981.

[8] X. Chen, A. J. Roberts and J. Duan, Center manifolds for stochastic evolution equations. arXiv:1210.5924.

[9] W. A. Coppel, *Dichotomies in stability theory.* Springer-Verlag, 1978.

[10] J. Duan, K. Lu and B. Schmalfuß, Smooth stable and unstable manifolds for stochastic evolutionary equations. *J. Dynam. Differential Equations*, **16**, 949–972, 2004 .

[11] J. Duan, K. Lu and B. Schmalfuß, Invariant manifolds for stochastic partial differential equations. *Ann. Probab.*, **31**, 2109–2135, 2003.

[12] T. Gallay, A center-stable manifold theorem for differential equations in Banach spaces. *Comm. Math. Phys.*, **152**, 249–268, 1993.

[13] M. Haragus and G. Iooss, Local bifurcations, center manifolds, and normal forms in infinite dimensional dynamical systems, Springer, 2010.

[14] A. Kelly, The stable, center-stable, center, center-unstable and unstable manifolds. *J. Differential Equations*, **3**, 546–570, 1967.

[15] W. Li and K. Lu, Sternberg theorems for random dynamical systems. *Comm. Pure Appl. Math.*, **58**, 941 - 988, 2005.

[16] Z. Lian and K. Lu, Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space. *Mem. Amer. Math. Soc.* **206**, 2010.

[17] K. Lu and B. Schmalfuß, Invariant manifolds for stochastic wave equations. *J. Differential Equations*, **236**, 460-C492, 2007.

[18] J. E. Marsden and M. McCracken, The Hopf bifurcation and its application, Springer-Verlag, Berlin, 1976.

[19] S. -E. A. Mohammed, The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, *Stochastics and Stochastic Reports*, **29**, 89–131, 1990.

[20] S.-E. A. Mohammed and M. K. R. Scheutzow, The stable manifold theorem for stochastic differential equations. *Ann. Probab.*, **27**, 615–652, 1999.

[21] S. -E. A. Mohammed, T. Zhang and H. Zhao, The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations. *Mem. Amer. Math. Soc.*, **196**, 1–105, 2008.

[22] V. Pliss and G. Sell, Robustness of exponential dichotomies in infinite dimensional dynamical systems. *J. Dynam. Differential Equations*, **11**, 471–513, 1999.

[23] G. Da Prato and A. Debussche, Construction of stochastic inertial manifolds using backward integration. *Stochastics Stochastics Rep.*, **59**, 305–324, 1996.

[24] A. J. Roberts, Low-dimensional modelling of dynamics via computer algebra. *Comput. Phys. Comm.*, **100**, 215–230, 1997.

[25] A. J. Roberts, Resolving the multitude of microscale interactions accurately models stochastic partial differential equations. *LMS J. Comput. Math.*, **9**, 193–221, 2006.

[26] A. J. Roberts, Model dynamics on a multigrid across multiple length and time scales. *Multiscale model. simul.*, **7**, 1525–1548, 2009.

[27] D. Ruelle, Ergodic theory of differentiable dynamical systems, *Publ. Math. Inst. Hautes Etud. Sci.*, **50**, 275–306, 1979.

[28] D. Ruelle, Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics*, **115**, 243–290, 1982.

[29] R. J. Sacker and G. R. Sell, Existence of dichotomies and invariant splittings for linear differential systems. III. *J. Differential Equations*, **22**, 497–522, 1976.

[30] A. Vanderbauwhede and G. Iooss, Center manifold theory in infinite dimensions. *Dynam. Report*, **1**, 125–163, 1992.