Center manifolds for stochastic evolution equations

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Abstract

Stochastic invariant manifolds are crucial in modelling the dynamical behavior of stochastic dynamics. Under the assumption of exponential trichotomy, existence and smoothness of center manifolds for a class of stochastic evolution equations with linearly multiplicative noise are proved. The exponential attraction and approximation to center manifolds are also discussed.

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Contents

1 Introduction

Invariant manifolds are some of the most important invariant sets in nonlinear dynamical systems. Stable, unstable, center and inertial manifolds have been widely studied in deterministic systems. But in many applications the nonlinear dynamical systems are influenced by noise. Recently invariant manifolds for stochastic differential equations and stochastic partial differential equations have been explored. Mohammed and Scheutzow [24] focused on the existence of local stable and unstable manifolds for stochastic differential equations driven by semimartingales. Boxler [8] obtained a stochastic version of center manifold theorems for finite dimensional random dynamical systems by using the multiplicative ergodic theorem. Arnold [2] summarised various invariant manifolds on finite dimensional random dynamical systems. An unresolved problem is the existence and nature of stochastic invariant manifolds on infinite dimensional space.

Recently there are some theory developments about the problem. Duan and others [17, 18] presented stable and unstable invariant manifolds for a class of stochastic partial differential equations under the assumption of exponential dichotomy or pseudo exponential dichotomy. Inertial manifolds is a generalization of center-unstable manifolds on finite dimensional space to infinite dimensional space. Flandoli and others [5, 11, 27] constructed the stochastic inertial manifolds on infinite dimensional space by different methods. Center manifolds are important invariant manifolds. It has been investigated extensively for infinite dimensional deterministic systems [4, 9, 34]. But there are only few papers dealing specifically with infinite dimensional center manifolds. Gallay [20] proved the existence of infinite dimensional center-stable manifolds and local center manifolds for a class of deterministic evolution equations in Banach spaces.

In this paper, we treat the problems of stochastic center manifolds on infinite dimensional Hilbert space with multiplicative white noise in the following class of stochastic evolution equation

$$\frac{du}{dt} = Au + F(u) + u \circ \dot{W}(t), \tag{1}$$

where $u \in H$ is a Hilbert space typically defined on some spatial domain, $W(t)$ is the standard $\mathbb{R}$-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is only dependent on time. A simple example is the linear stochastic parabolic equation

$$u_t = u_{xx} + 4u + u \circ \dot{W}(t), \quad u(0,t) = u(\pi,t) = 0\,, \quad t \geq 0\,. \tag{2}$$

In this example the Hilbert space H is the function space $L^2(0, \pi)$. This example stochastic partial differential equation (SPDE) has stable center and unstable subspaces. Thus, the theory of Duan et al. [12, 17, 18] does not apply. Sections 2–4 establish the theory of center manifolds of such SPDEs. That is we prove the existence and smoothness of the center manifolds for stochastic evolution equation (1). The results generalize the work of Duan et al. [17, 18], who proved the existence and smoothness of center-stable and center-unstable manifolds. The stochastic center manifolds we obtain are generally infinite dimensional. Then Section 5 and 6 proves the principle of exponential attraction from the stochastic center manifolds and an approximation to the stochastic center manifolds.

Our main assumption is that there exists a spectral gap for the linear operator A. Such a gap allows us to construct the infinite dimensional center manifolds. Exponential dichotomy is one of the basic assumptions in nonlinear dynamical systems. The assumption plays a central role in the study of stable and unstable manifolds. But the assumption of exponential dichotomy [17, 18] is not a sufficient condition for proving the existence of center manifolds. For example, a generator of a strongly continuous semigroup has some of its spectrum on the imaginary axis, which implies the existence of center manifolds, but there is no exponential dichotomy as Prüss [28] showed. The pseudo-exponential dichotomy [17, 26] is also not a sufficient assumption to prove the existence of center manifolds since in the finite dimension case we need to ensure the existence of zero real part of the spectrum of a bounded operator. The concept of exponential trichotomy is important for center manifold theory in infinite dimensional dynamical systems and non-autonomous systems [3, 20, 26]. The existence of exponential trichotomy means that the space is split into three subspaces: center subspace, unstable subspace and stable subspace. In finite dimensional space the exponential trichotomy is satisfied automatically. Center manifold theory is based on the assumption of exponential trichotomy [26]. Section 2 introduces the exponential trichotomy and conjugated random evolution equations. Using the Lyapunov–Perron method, Section 3 proves the existence of stochastic center manifolds under the assumption of exponential trichotomy. We consider the existence of local center manifolds via a cut-off technique. When the stable subspace or the unstable subspace disappear, then our theorems reduce to the results by Duan and others [17, 18].

Deterministic center manifold theory is widely studied and further extended by others in various contexts [1, 4, 9, 14, 21, 22, 34, 35, e.g.]. The method of proving the existence of center manifolds is to show that the center manifold is the fixed point of a certain of operator. It is also the intersection of the center-unstable manifolds and center-stable manifolds [4, 15]. In this paper, we define the stochastic center manifold as a graph of Lipschitz map. We directly prove the existence of such map by contraction mapping theorem, which is the method of Lyapunov–Perron [35]. It is different from the Hadamard's graph transform method [8, 18], which is more directly geometrical. Section 4 proves the center manifolds are smooth by using the method of Lyapunov–Perron.

There are many applications of the theory of center manifolds such as the dimensional reduction [19, 37], bifurcation [9], discretisation [29, 31]. The stability of an equilibrium point on center manifolds has been considered in deterministic case [1, 25]. In the random case we show the equilibrium points are asymptotically stable when the equilibrium points restricted on the stochastic center manifolds are asymptotically stable (Section 5). We prove that there exists an approximation of the stochastic centre manifolds (Section 6). There have some recent results on approximation of the invariant manifolds. Wang and Duan [36] proved an asymptotic completeness property for a class of stochastic partial differential equations. Sun et al. [33] showed that an invariant manifold is approximated by a deterministic manifold when the noise is small. Chen et al. [7, 13] gave a geometric shape of invariant manifolds for a class of stochastic partial differential equations. Blömker et al. [6] considered a different approach to approximate stochastic partial differential equations by amplitude equations. In Section 6 we give a class of stochastic approximation to random evolution equations by stochastic center manifolds, which make some progress in the computation of stochastic center manifolds [30]. For this pathwise approximation version, the difficulty is to find a random variable to approximate the center manifolds. This is different from the random norm [8] when using Carr's result [9]. Section 7 uses some examples to illustrate our results.

2 Exponential trichotomy

We consider the stochastic evolution equation (1). We assume that the linear operator $A : D(A) \to H$ generates a strongly continuous semigroup $S(t) := \exp(At)$ on a Hilbert space $(H, |\cdot|)$, which satisfies the *exponential trichotomy* with exponents $\alpha > \gamma > 0 > -\gamma > -\beta$ and bound K.

First we introduce the exponential trichotomy, which generalizes the exponential dichotomy [16, 17, 18, 32].

Definition 1. *A strongly continuous semigroup $S(t) := \exp(At)$ on H is said to satisfy the exponential trichotomy with exponents $\alpha > \gamma > 0 > -\gamma > -\beta$, if the there exist continuous projection operators P^c, P^u and P^s on H such that the following four conditions hold.*

(i) $\mathrm{id} = P^c + P^s + P^u$, *where $P^iP^j = 0$ for $i \neq j$ and $i, j \in \{c, s, u\}$, and where* id *is the identity operator.*

(ii) $P^c \exp(At) = \exp(At)P^c$, $P^u \exp(At) = \exp(At)P^u$, $P^s \exp(At) = \exp(At)P^s$ *for $t \geq 0$.*

(iii) Denote the reducing subspaces $H^c := P^cH$, $H^u := P^uH$ and $H^s := P^sH$. We call H^c, H^u and H^s the center subspace, the unstable subspace and the stable subspace, respectively. The restriction $\exp(At)|_{H^i}$, $t \geq 0$ *are isomorphism from H^i onto H^i, and we define* $\exp(At)|_{H^i}$ *for $t < 0$ as the inverse map of* $\exp(-At)|_{H^i}$, $i \in \{c, s, u\}$.

(iv)

$$|\exp(At)P^c v| \le K\exp(\gamma|t|)|P^c v|, \quad t\in\mathbb{R}, v\in H, \tag{3}$$
$$|\exp(At)P^u v| \le K\exp(\alpha t)|P^u v|, \quad t\le 0\,, v\in H, \tag{4}$$
$$|\exp(At)P^s v| \le K\exp(-\beta t)|P^s v|, \quad t\ge 0\,, v\in H. \tag{5}$$

Remark 1. If H is a finite dimensional space and there exist eigenvalues with real part zero, less than zero, and greater than zero, then we have the exponential trichotomy. If H is an infinite dimensional space and the spectrum of A satisfies $\sigma(A)=\sigma^s\cup\sigma^c\cup\sigma^u$, where $\sigma^s=\{\lambda\in\sigma(A):\operatorname{Re}\lambda\le-\beta\}$, $\sigma^c=\{\lambda\in\sigma(A):|\operatorname{Re}\lambda|\le\gamma\}$, $\sigma^u=\{\lambda\in\sigma(A):\operatorname{Re}\lambda\ge\alpha\}$, and A generates a strong continuous semigroup, then we have the exponential trichotomy [20].

We assume the nonlinear term F satisfies $F(0)=0$, and assume it to be Lipschitz continuous on H, that is,

$$|F(u_1)-F(u_2)|\le \operatorname{Lip}F\,|u_1-u_2|$$

with the sufficiently small Lipschitz constant $\operatorname{Lip}F>0$. If the nonlinear term F is locally Lipschitz, let $F^{(R)}(u)=\chi_R(u)F(u)$, where $\chi_R(u)$ is a cut-off function, and then $F^{(R)}$ is globally Lipschitz with Lipschitz constant $R\operatorname{Lip}F$.

Second we use a coordinate transform to convert the stochastic partial differential equation (1) into a random evolution equation [10, 17, 18]. Consider the linear stochastic differential equation,

$$dz+z\,dt=dW. \tag{6}$$

A solution of this equation is called an Ornstein–Uhlenbeck process. Let $C_0(\mathbb{R},\mathbb{R})$ be continuous functions on $\mathbb{R}$, the associated distribution $\mathbb{P}$ is a Wiener measure defined on the Borel-σ-algebra $\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$. $\{\theta_t\}_{t\in\mathbb{R}}$ is the metric dynamical system generated by Wiener process $W(t)$. A random dynamical system on a metric space H over on $(\Omega,\mathcal{B},\mathbb{P},\theta_t)$ is a measurable map

$$\begin{aligned}\phi:\mathbb{R}\times H\times\Omega &\to H\\ (t,u,\omega) &\mapsto \phi(t,u,\omega)\end{aligned}$$

such that

(i) $\phi(0,u,\omega)=\mathrm{id}$;

(ii) $\phi(t+s,u,\omega)=\phi(t,u,\theta_s\omega)\phi(s,u,\omega)$, $\forall\, t,s\in\mathbb{R}$ for almost for $\omega\in\Omega$.

Caraballo et al. [10] and Duan et al. [17] established the following lemma, which is used in the proof of existence of stochastic invariant manifolds.

Lemma 2. *(i) There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\Omega\in\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ of full measure with sublinear growth:*

$$\lim_{t\to\pm\infty}\frac{|\omega(t)|}{|t|}=0\,,\quad \omega\in\Omega\,.$$

(ii) For $\omega\in\Omega$ the random variable $z(\omega)=-\int_{-\infty}^{0}e^{\tau}\omega(\tau)\,d\tau$ exists and generates a unique stationary solution of the Ornstein–Uhlenbeck SDE *(6) given by the convolutions*

$$\begin{aligned}\Omega\times\mathbb{R}\ni(\omega,t)\to z(\theta_t\omega) &= -\int_{-\infty}^{0}\exp(\tau)\theta_t\omega(\tau)\,d\tau\\ &= -\int_{-\infty}^{0}\exp(\tau)\omega(\tau+t)\,d\tau+\omega(t).\end{aligned}$$

The mapping $t\mapsto z(\theta_t\omega)$ is continuous.

(iii) In particular,

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|}=0\quad \textit{for } \omega\in\Omega\,.$$

(iv) In addition,

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_\tau\omega)\,d\tau=0\quad \textit{for } \omega\in\Omega\,.$$

We now replace the Borel-σ-algebra $\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ by

$$\mathcal{F} = \{\Omega \cap D : D \in \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))\}$$

for Ω given in Lemma 2. The probability measure is the restriction of the Wiener measure to this new σ-algebra, which is also denoted by $\mathbb{P}$. In the following we consider a random dynamical system over the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.

We show that the solution of equation (1) define a random dynamical system. For any $u^* \in H$ and $\omega \in \Omega$, we introduce the coordinate transform

$$u = T(u^*, \omega) = u^* \exp[-z(\omega)] \tag{7}$$

and its inverse transform

$$T^{-1}(u^*, \omega) = u^* \exp[z(\omega)]. \tag{8}$$

Without loss of generality, we drop the $*$. We transform the stochastic evolution equation (1) into the following partial differential equation with random coefficients

$$\frac{du}{dt} = Au + z(\theta_t\omega)u + G(\theta_t\omega, u), \quad u(0) = u_0 \in H, \tag{9}$$

where $G(\omega, u) = \exp[-z(\omega)]F(\exp[z(\omega)]u)$. We have

$$|G(\omega, u_1) - G(\omega, u_2)| \leq \operatorname{Lip}_u G\, |u_1 - u_2|,$$

where $\operatorname{Lip}_u G$ denotes the Lipschitz constant of $G(\cdot, u)$ with respect to u. For any $\omega \in \Omega$ the function G has the same global Lipschitz constant as F by the construction of G. Define

$$\Psi_A(t, s) = \exp\left[A(t-s) + \int_s^t z(\theta_r\omega)\, dr\right]$$

as a 'state transition operator' for the linear random partial differential equation (9) in the case when the nonlinearity is neglected, $G(\omega, u) = 0$. The solution is interpreted in a mild sense

$$u(t) = \Psi_A(t,0)u_0 + \int_0^t \Psi_A(t,s)G(\theta_s\omega, u(s))\, ds\,.$$

This equation has a unique measurable solution from the Lipschitz continuity of G and measurable property of $z(\omega)$. Hence the solution mapping $(t, v, \omega) \mapsto u(t, v, \omega)$ generates a random dynamical system.

Duan, Lu and Schmalfuß [18] proved that the solution of (9) is the coordinate transformation the solution of (1).

Lemma 3. *Suppose that u is the random dynamical system generated by (9). Then*

$$(t, v, \omega) \mapsto T^{-1}(\theta_t\omega, u(t, T(\omega, v), \omega)) =: \hat{u}(t, v, \omega) \tag{10}$$

is a random dynamical system. For any $v \in H$ this process $(t, \omega) \mapsto \hat{u}(t, v, \omega)$ is a solution to (1).

3 Stochastic center manifolds

We introduce the definition of stochastic center manifolds. A basic tool of proving the existence of stochastic center manifolds is to define an appropriate function space, which is a Banach space.

Definition 4. *A random set $M(\omega)$ is called an (forward) invariant set for a random dynamical system $\phi(t, \omega, x)$ if $\phi(t, M(\omega), \omega) \subset M(\theta_t\omega)$ for $t \geq 0$. If we can represent $M(\omega)$ by a graph of a (Lipschitz) mapping from the center subspace to its complement, $h^c(\cdot, \omega) : H^c \to H^u \oplus H^s$, such that $M(\omega) = \{v + h^c(v, \omega) : v \in H^c\}$, $h^c(0, \omega) = 0$, and the tangency condition that the derivative $Dh^c(0, \omega) = 0$, $h^c(v, \cdot)$ is measurable for every $v \in H^c$, then $M(\omega)$ is called a (Lipschitz) center manifold, often denoted as $M^c(\omega)$.*

$M(\omega)$ is called a local center manifold if it is a graph of (Lipschitz) mapping $\chi_R(v)h^c(\cdot,\omega)$.

We first show the existence of a Lipschitz center manifold for the random partial differential equation (9). Then, we apply the inverse transformation T^{-1} to get a center manifold for the stochastic evolution equation (1).

For each $\eta > 0$, we denote the Banach space

$$C_\eta = \left\{ \phi \in C(\mathbb{R}, H) : \sup_{t\in\mathbb{R}} \exp\left[-\eta|t| - \int_0^t z(\theta_r\omega)\,dr\right] |\phi(t)| < \infty \right\}$$

with the norm

$$|\phi|_{C_\eta} = \sup_{t\in\mathbb{R}} \exp\left[-\eta|t| - \int_0^t z(\theta_r\omega)\,dr\right] |\phi(t)|.$$

The set C_η is the set of 'slowly varying' functions. We know that the functions are controlled by $\exp\left[\eta|t| + \int_0^t z(\theta_r\omega)\,dr\right]$. Let

$$M^c(\omega) = \{u_0 \in H : u(\cdot, u_0, \omega) \in C_\eta\},$$

where $u(t, u_0, \omega)$ is the solution of (9) with the initial data $u(0) = u_0$.

We prove that $M^c(\omega)$ is invariant and is the graph of a Lipschitz function.

For simple the expression of the proof, define the operator $B : \mathbb{R} \to L(H)$ by

$$B(t) := \begin{cases} -\exp(At)P^u, & t \le 0\,; \\ \exp(At)P^s, & t \ge 0\,. \end{cases}$$

Different from the proof process of Duan et al. [17, 18], we need analysis the behavior of the solution on center subspace.

Theorem 5. *If $\gamma < \eta < \min\{\beta, \alpha\}$ such that the nonlinearity term is sufficiently small,*

$$K \operatorname{Lip}_u G \left(\frac{1}{\eta - \gamma} + \frac{1}{\beta - \eta} + \frac{1}{\alpha - \eta} \right) < 1\,, \tag{11}$$

then there exists a center manifold for the random partial differential equation (9), which is written as the graph

$$M^c(\omega) = \{v + h^c(v,\omega) : v \in H^c\},$$

where $h^c(\cdot,\omega) : H^c \to H^u \oplus H^s$ is a Lipschitz continuous mapping from the center subspace and satisfies $h^c(0,\omega) = 0$.

Proof. First we claim that $u_0 \in M^c(\omega)$ if and only if there exists a slowly varying function $u(\cdot,u_0,\omega) \in C_\eta$ with

$$\begin{aligned} u(t,u_0,\omega) =& \Psi_A(t,0)v + \int_0^t \Psi_A(t,s)P^cG(\theta_s\omega,u(s))\,ds \\ &+ \int_{-\infty}^{+\infty} \Psi_0(t,s)B(t-s)G(\theta_s\omega,u(s))\,ds\,, \end{aligned} \tag{12}$$

where $v = P^cu_0$, $\Psi_0(t,s) = \exp\left[\int_s^t z(\theta_r\omega)\,dr\right]$.

To prove this claim, first we let $u_0 \in M^c(\omega)$. By using the variation of constants formula, the solution on each subspace denoted as

$$P^cu(t,u_0,\omega) = \Psi_A(t,0)v + \int_0^t \Psi_A(t,s)P^cG(\theta_s\omega,u)\,ds\,. \tag{13}$$

$$P^uu(t,u_0,\omega) = \Psi_A(t,\tau)P^uu(\tau,u_0,\omega) + \int_\tau^t \Psi_A(t,s)P^uG(\theta_s\omega,u)\,ds\,. \tag{14}$$

$$P^su(t,u_0,\omega) = \Psi_A(t,\tau)P^su(\tau,u_0,\omega) + \int_\tau^t \Psi_A(t,s)P^sG(\theta_s\omega,u)\,ds\,. \tag{15}$$

Since the slowly varying function $u \in C_\eta$, we have for $t < \tau$ that the magnitude

$$\begin{aligned} |\Psi_A(t,\tau)P^uu(\tau,u_0,\omega)| &\le K\exp[\alpha(t-\tau)]\Psi_0(t,0)\exp(\eta\tau)|u|_{C_\eta} \\ &= K\Psi_\alpha(t,0)\exp[-(\alpha-\eta)\tau]|u|_{C_\eta} \\ &\to 0 \quad \text{as } \tau \to +\infty. \end{aligned}$$

For $t > \tau$,

$$\begin{aligned}|\Psi_A(t,\tau)P^s u(\tau,u_0,\omega)| &\le K\exp[-\beta(t-\tau)]\Psi_0(t,0)\exp(\eta\tau)|u|_{C_\eta}\\ &= K\Psi_{-\beta}(t,0)\exp[(\beta+\eta)\tau]|u|_{C_\eta}\\ &\to 0 \quad \text{as } \tau\to-\infty.\end{aligned}$$

Then, taking the two separate limits $\tau \to \pm\infty$ in (14) and (15) respectively,

$$P^u u(t,u_0,\omega) = \int_{\infty}^{t} \Psi_A(t,s)P^u G(\theta_s\omega, u(s))\,ds\,, \tag{16}$$

$$P^s u(t,u_0,\omega) = \int_{-\infty}^{t} \Psi_A(t,s)P^s G(\theta_s\omega, u(s))\,ds\,. \tag{17}$$

Combining (13), (16) and (17), we have (12). The converse follows from a direct computation.

Next we prove that for any given $v \in H^c$, the centre subspace, the integral equation (12) has a unique solution in the slowly varying functions space C_η. Let

$$\begin{aligned}J^c(u,v) :=& \Psi_A(t,0)v + \int_0^t \Psi_A(t,s)P^c G(\theta_s\omega,u(s))\,ds\\ &+ \int_{-\infty}^{+\infty} \Psi_0(t,s)B(t-s)G(\theta_s\omega,u(s))\,ds\,.\end{aligned} \tag{18}$$

J^c is well-defined from $C_\eta \times H^c$ to the slowly varying functions space C_η. For each pair of slowly varying functions $u, \bar{u} \in C_\eta$, we have that for $\gamma < \eta <$

$\min\{\beta, \alpha\}$,

$$
\begin{aligned}
&|J^c(u,v) - J^c(\bar{u},v)|_{C_\eta} \\
&\le \sup_{t\in\mathbb{R}} \left\{ \exp\left[-\eta|t| - \int_0^t z(\theta_s\omega)\,ds\right] \left| \int_0^t \Psi_A(t,s)P^c(G(\theta_s\omega,u) - G(\theta_s\omega,\bar{u}))\,ds \right.\right. \\
&\qquad \left.\left. + \int_{-\infty}^{+\infty} \Psi_0(t,s)B(t-s)(G(\theta_s\omega,u) - G(\theta_s\omega,\bar{u}))\,ds \right| \right\} \\
&\le \sup_{t\in\mathbb{R}} \left\{ K \operatorname{Lip}_u G |u-\bar{u}|_{C_\eta} \left| \int_0^t \exp[(\gamma-\eta)|t-s|]\,ds \right.\right. \\
&\qquad \left.\left. + \int_{-\infty}^t \exp[(\eta-\beta)(t-s)]\,ds + \int_t^{+\infty} -\exp[(\alpha-\eta)(t-s)]\,ds \right| \right\} \\
&\le K \operatorname{Lip}_u G \left(\frac{1}{\eta-\gamma} + \frac{1}{\beta-\eta} + \frac{1}{\alpha-\eta} \right) |u-\bar{u}|_{C_\eta}.
\end{aligned}
\tag{19}
$$

From equation (19), J^c is Lipschitz continuous in v. By the theorem's precondition (11), J^c is a uniform contraction with respect to the parameter v. By the uniform contraction mapping principle, for each $v \in H^c$, the mapping $J^c(\cdot, v)$ has a unique fixed point $u(\cdot, v, \omega) \in C_\eta$. Combining equation (18) and equation (19),

$$
|u(\cdot,v,\omega) - u(\cdot,\bar{v},\omega)|_{C_\eta} \le \frac{K}{1 - K \operatorname{Lip}_u G \left(\frac{1}{\eta-\gamma} + \frac{1}{\beta-\eta} + \frac{1}{\alpha-\eta} \right)} |v-\bar{v}|, \tag{20}
$$

for each fixed point $u(\cdot, v, \omega)$. Then for each time $t \ge 0$, $u(t, \cdot, \omega)$ is Lipschitz from the center subspace H^c to slowly varying functions C_η. $u(\cdot, v, \omega) \in C_\eta$ is a unique solution of the integral equation (12). Since $u(\cdot, v, \omega)$ can be an ω-wise limit of the iteration of contraction mapping J^c starting at 0 and J^c maps a $\mathcal{F}$-measurable function to a $\mathcal{F}$-measurable function, $u(\cdot, v, \omega)$ is

$\mathcal{F}$-measurable. Combining $u(\cdot, v, \omega)$ is continuous with respect to H, we have $u(\cdot, v, \omega)$ is measurable with respect to $(\cdot, v, \omega)$.

Let $h^c(v, \omega) := P^s u(0, v, \omega) \oplus P^u u(0, v, \omega)$. Then

$$h^c(v, \omega) = \int_{-\infty}^{+\infty} \Psi_0(0, s) B(-s) G(\theta_s \omega, u(s, v, \omega))\, ds\,.$$

We see that h^c is $\mathcal{F}$-measurable and $h^c(0, \omega) = 0$. From the definition of $h^c(v, \omega)$ and the claim that $u_0 \in M^c(\omega)$ if and only if there exists $u(\cdot, u_0, \omega) \in C_\eta$ with $u(0) = u_0$ and satisfies (12) it follows that $u_0 \in M^c(\omega)$ if and only if there exists $v \in H^c$ such that $u_0 = v + h^c(v, \omega)$, therefore,

$$M^c(\omega) = \{v + h^c(v, \omega) : v \in H^c\}.$$

Next we prove that for any $x \in H$, the function

$$\omega \to \inf_{y \in H^c} |x - (y + h^c(y, \omega))| \tag{21}$$

is measurable. Let H' be a countable dense subset of the separable space H. From the continuity of $h^c(\cdot, \omega)$,

$$\inf_{y \in H^c} |x - (y + h^c(y, \omega))| = \inf_{y \in H'} |x - P^c y - h^c(P^c y, \omega)|. \tag{22}$$

The measurability of (21) follows since $\omega \to h^c(P^c y, \omega)$ is measurable for any $y \in H'$.

Finally, we show that $M^c(\omega)$ is invariant, that is for each $u_0 \in M^c(\omega)$, $u(s, u_0, \omega) \in M^c(\theta_s \omega)$ for all $s \geq 0$. Since for $s \geq 0$, $u(t + s, u_0, \omega)$ is a solution of

$$\frac{du}{dt} = Au + z(\theta_t(\theta_s \omega))u + G(\theta_t(\theta_s \omega), u), \quad u(0) = u(s, u_0, \omega).$$

Thus $u(t, u(s, u_0, \omega), \theta_s \omega) = u(t + s, u_0, \omega)$ and $u(t, u(s, u_0, \omega), \theta_s \omega) \in C_\eta$. So we have $u(s, u_0, \omega) \in M^c(\theta_s \omega)$

$\square$

Theorem 6. *Suppose that u is the solution of random partial differential equation (9), then $\widetilde{M}^c(\omega) = T^{-1}(\omega, M^c(\omega))$ is a center manifold of the stochastic partial differential equation (1).*

Proof. Denote $\tilde{u}(t,\omega,x)$ the solution of (1). From Lemma 3, for $t \geq 0$,

$$\begin{aligned}\tilde{u}(t,\omega,\widetilde{M}^c(\omega)) &= T^{-1}(\theta_t\omega, u(t,\omega,T(\omega,\widetilde{M}^c(\omega)))) \\ &= T^{-1}(\theta_t\omega, u(t,\omega,M^c(\omega))) \subset T^{-1}(\theta_t\omega, M^c(\theta_t\omega)) = \widetilde{M}^c(\theta_t\omega).\end{aligned}$$

So $\widetilde{M}^c(\omega)$ is an invariant set. Note that

$$\begin{aligned}\widetilde{M}^c(\omega) &= T^{-1}(\omega, M^c(\omega)) \\ &= \left\{u_0 = T^{-1}(\omega, v + h^c(v,\omega) : v \in H^c\right\} \\ &= \left\{u_0 = e^{z(\omega)}(v + h^c(v,\omega)) : v \in H^c\right\} \\ &= \left\{u_0 = v + e^{z(\omega)}h^c(e^{-z(\omega)}v,\omega) : v \in H^c\right\},\end{aligned}$$

which implies that $\widetilde{M}^c(\omega)$ is a center manifold. □

Note that a local center manifolds is not unique [9].

4 Smoothness of center manifolds

In this section, we prove that for each $\omega \in \Omega$, $M^c(\omega)$ is a C^k center manifold.

Theorem 7. *Assume that G is C^k in u. If $\gamma < k\eta < \min\{\beta,\alpha\}$ and*

$$K \operatorname{Lip}_u G \left(\frac{1}{i\eta - \gamma} + \frac{1}{\beta - i\eta} + \frac{1}{\alpha - i\eta}\right) < 1 \quad \text{for all } 1 \leq i \leq k,$$

then $M^c(\omega)$ is a C^k center manifold for the random evolutionary equation (9), and $Dh^c(0,\omega) = 0$.

Proof. We prove this theorem by induction. First, we consider $k = 1$. Since

$$K \operatorname{Lip}_u G \left(\frac{1}{\eta - \gamma} + \frac{1}{\beta - \eta} + \frac{1}{\alpha - \eta} \right) < 1$$

there exists a small number $\delta > 0$ such that $\gamma < \eta - \eta' < \min\{\beta, \alpha\}$ and for all $0 \le \eta' \le 2\delta$,

$$K \operatorname{Lip}_u G \left[\frac{1}{(\eta - \eta') - \gamma} + \frac{1}{\beta - (\eta - \eta')} + \frac{1}{\alpha - (\eta - \eta')} \right] < 1 .$$

Thus, $J^s(\cdot, v, \omega)$ defined in the proof of Theorem 5 is a uniform contraction in $C_{\eta-\eta'} \subset C_\eta$ for any $0 \le \eta' \le 2\delta$. Therefore, $u(\cdot, v, \omega) \in C_{\eta-\eta'}$. For $v_0 \in H^c$, we define two operators: let

$$Sv_0 = \Psi_A(t, 0) v_0 ,$$

and for $v \in C_{\eta-\delta}$ let

$$\begin{aligned} Tv = &\int_0^t \Psi_A(t, s) P^c D_u G(\theta_s \omega, u(s, v_0, \omega)) v \, ds \\ &+ \int_{-\infty}^{+\infty} \Psi_0(t, s) B(t - s) D_u G(\theta_s \omega, u(s, v_0, \omega)) v \, ds . \end{aligned}$$

From the assumption, S is a bounded linear operator from center subspace H^c to slowly varying functions space $C_{\eta-\delta}$. Using the same arguments in the proof Theorem 5 that J^c is a contraction, we have that T is a bounded linear operator from $C_{\eta-\delta}$ to itself and

$$\|T\| \le K \operatorname{Lip}_u G \left(\frac{1}{\eta - \delta - \gamma} + \frac{1}{\beta - (\eta - \delta)} + \frac{1}{\alpha - (\eta - \delta)} \right) < 1 ,$$

which implies that the operator $(\mathrm{id} - T)$ is invertible in $C_{\eta-\delta}$. For $v, v_0 \in H^c$, we set

$$
\begin{aligned}
I = \int_0^t \Psi_A(t,s)P^c\Big[&G(\theta_s\omega, u(s,v,\omega)) - G(\theta_s\omega, u(s,v_0,\omega)) \\
&- D_uG(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\, ds \\
+ \int_{-\infty}^{+\infty} \Psi_0(t,s)B(t-s)\Big[&G(\theta_s\omega, u(s,v,\omega)) - G(\theta_s\omega, u(s,v_0,\omega)) \\
&- D_uG(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\, ds\,.
\end{aligned}
$$

We obtain

$$
u(\cdot,v,\omega) - u(\cdot,v_0,\omega) - T(u(\cdot,v,\omega) - u(\cdot,v_0,\omega)) = S(v-v_0) + I, \tag{23}
$$

which yields

$$
u(\cdot,v,\omega) - u(\cdot,v_0,\omega) = (\mathrm{id} - T)^{-1}S(v-v_0) + (\mathrm{id} - T)^{-1}I.
$$

If $|I|_{C_{\eta-\delta}} = o(|v-v_0|)$ as $v \to v_0$, then $u(\cdot,v,\omega)$ is differentiable in v and its derivative satisfies $D_vu(t,v,\omega) \in L(H^c, C_{\eta-\delta})$, where $L(H^c, C_{\eta-\delta})$ is the usual space of bounded linear operators and

$$
\begin{aligned}
D_vu(t,v,\omega) = \Psi_A(t,0)v &+ \int_0^t \Psi_A(t,s)P^cD_uG(\theta_s\omega, u(s,v,\omega))D_vu(s,v,\omega)\, ds \\
&+ \int_{-\infty}^{+\infty} \Psi_0(t,s)B(t-s)D_uG(\theta_s\omega, u(s,v,\omega))D_vu(s,v,\omega)\, ds\,.
\end{aligned} \tag{24}
$$

The tangency condition $Dh^c(0,\omega) = 0$ is from equation (24).

Now we prove that

$$
|I|_{C_{\eta-\delta}} = o(|v-v_0|) \tag{25}
$$

as $v \to v_0$. We divide I into several sufficient small parts. Let N be a large positive number to be chosen later and define the following ten integrals

$$I_1 = \exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_N^t \Psi_A(t,s)P^c\Big[G(\theta_s\omega, u(s,v,\omega)) - G(\theta_s\omega, u(s,v_0,\omega)) - D_uG(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|$$

for $t \geq N$.

$$I_1' = \exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_t^{-N} \Psi_A(t,s)P^c\Big[G(\theta_s\omega, u(s,v,\omega)) - G(\theta_s\omega, u(s,v_0,\omega)) - D_uG(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|$$

for $t \leq -N$.

$$I_2 = \exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_0^N \Psi_A(t,s)P^c\Big[G(\theta_s\omega, u(s,v,\omega)) - G(\theta_s\omega, u(s,v_0,\omega)) - D_uG(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|$$

for $0 \leq t \leq N$.

$$I_2' = \exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_{-N}^0 \Psi_A(t,s)P^c\Big[G(\theta_s\omega, u(s,v,\omega)) - G(\theta_s\omega, u(s,v_0,\omega)) - D_uG(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|$$

for $-N \leq t \leq 0$.

Let $\overline{N}$ be a large positive number to be chosen later. For $|t| \le \overline{N}$, we set

$$
\begin{aligned}
I_3 = &\exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_t^{\overline{N}} [-\Psi_A(t,s)] P^u \Big[G(\theta_s\omega, u(s,v,\omega)) \\
&- G(\theta_s\omega, u(s,v_0,\omega)) - D_u G(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|. \\
I_3' = &\exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_{-\overline{N}}^{t} \Psi_A(t,s) P^s \Big[G(\theta_s\omega, u(s,v,\omega)) \\
&- G(\theta_s\omega, u(s,v_0,\omega)) - D_u G(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|. \\
I_4 = &\exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_{\overline{N}}^{\infty} [\Psi_A(t,s)] P^u \Big[G(\theta_s\omega, u(s,v,\omega)) \\
&- G(\theta_s\omega, u(s,v_0,\omega)) - D_u G(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|. \\
I_4' = &\exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_{-\infty}^{-\overline{N}} \Psi_A(t,s) P^s \Big[G(\theta_s\omega, u(s,v,\omega)) \\
&- G(\theta_s\omega, u(s,v_0,\omega)) - D_u G(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|.
\end{aligned}
$$

For $|t| \ge \overline{N}$, we set

$$
\begin{aligned}
I_5 = &\exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_t^{+\infty} [\Psi_A(t,s)] P^u \Big[G(\theta_s\omega, u(s,v,\omega)) \\
&- G(\theta_s\omega, u(s,v_0,\omega)) - D_u G(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|. \\
I_5' = &\exp\left[-(\eta-\delta)|t| - \int_0^t z(\theta_s\omega)\,ds\right] \bigg| \int_{-\infty}^{t} \Psi_A(t,s) P^s \Big[G(\theta_s\omega, u(s,v,\omega)) \\
&- G(\theta_s\omega, u(s,v_0,\omega)) - D_u G(\theta_s\omega, u(s,v_0,\omega))(u(s,v,\omega) - u(s,v_0,\omega))\Big]\,ds\bigg|.
\end{aligned}
$$

It is sufficient to show that for any $\epsilon > 0$ there is a $\sigma > 0$ such that if $|v - v_0| \le \sigma$, then $|I|_{C_{\eta-\delta}} \le \epsilon |v - v_0|$. Note that

$$\begin{aligned}|I|_{C_{\eta-\delta}} \le \sup_{t\ge N} I_1 + \sup_{N\ge t\ge 0} I_2 + \sup_{t\le -N} I_1' + \sup_{-N\le t\le 0} I_2' + \sup_{|t|\le \overline{N}} I_3 + \sup_{|t|\le \overline{N}} I_4 + \sup_{|t|\ge \overline{N}} I_5 \\ + \sup_{|t|\le \overline{N}} I_3' + \sup_{|t|\le \overline{N}} I_4' + \sup_{|t|\ge \overline{N}} I_5'.\end{aligned}$$

A computation similar to (20) implies that

$$\begin{aligned}I_1 &\le 2K \operatorname{Lip}_u G \\ &\int_N^t \exp[(\gamma - (\eta - \delta))|t - s|] \exp(-\delta|s|)|u(\cdot, v, \omega) - u(\cdot, v_0, \omega)|_{C_{\eta-2\delta}}\, ds \\ &\le \frac{2K^2 \operatorname{Lip}_u G \exp(-\delta N)}{(\eta-\gamma-\delta)\left\{1 - K \operatorname{Lip}_u G \left[\frac{1}{(\eta-2\delta)-\gamma} + \frac{1}{\beta-(\eta-2\delta)} + \frac{1}{\alpha-(\eta-2\delta)}\right]\right\}}|v - v_0|.\end{aligned}$$

Choose N so large that

$$\frac{2K^2 \operatorname{Lip}_u G \exp(-\delta N)}{(\eta-\gamma-\delta)\left\{1 - K \operatorname{Lip}_u G \left[\frac{1}{(\eta-2\delta)-\gamma} + \frac{1}{\beta-(\eta-2\delta)} + \frac{1}{\alpha-(\eta-2\delta)}\right]\right\}} \le \frac{1}{8}\epsilon.$$

Hence for such N we have that

$$\sup_{t\ge N} I_1 \le \frac{1}{8}\epsilon|v - v_0|.$$

Fixing such N, for I_2 we have that

$$I_2 \le K \int_0^N \exp[(\gamma + (\eta-\delta))|t-s|]\Big\{ \int_0^1 \big|D_u G(\theta_s\omega, \tau u(s,v,\omega) + (1-\tau) \\ u(s,v_0,\omega)) - D_u G(\theta_s\omega, u(s,v_0,\omega))\big|\, d\tau \Big\} |u(\cdot,v,\omega) - u(\cdot,v_0,\omega)|_{C_{\eta-\delta}}\, ds$$

$$\le \frac{K^2|v-v_0|}{1 - K \operatorname{Lip}_u G \left[\frac{1}{\eta-\delta-\gamma} + \frac{1}{\beta-(\eta-\delta)} + \frac{1}{\alpha-(\eta-\delta)}\right]} \\ \int_0^N \exp[(\gamma + (\eta-\delta))|t-s|]\Big\{ \int_0^1 \big|D_u G(\theta_s\omega, \tau u(s,v,\omega) + (1-\tau) \\ u(s,v_0,\omega)) - D_u G(\theta_s\omega, u(s,v_0,\omega))\big|\, d\tau \Big\}\, ds\,.$$

From the continuity of the integrand in (s,v), the last integral is continuous at the point v_0. Thus, we have that there is a $\sigma_1 > 0$ such that if $|v-v_0| \le \sigma_1$, then

$$\sup_{N\ge t\ge 0} I_2 \le \frac{1}{8}\epsilon|v-v_0|.$$

Therefore, if $|v-v_0| \le \sigma_1$, then

$$\sup_{t\ge N} I_1 + \sup_{N\ge t\ge 0} I_2 \le \frac{1}{4}\epsilon|v-v_0|.$$

In the same way, there is a $\sigma_1' > 0$ such that if if $|v-v_0| \le \sigma_1'$, then

$$\sup_{t\le -N} I_1 + \sup_{-N\le t\le 0} I_2 \le \frac{1}{4}\epsilon|v-v_0|.$$

Similarly, by choosing $\overline{N}$ to be sufficiently large,

$$\sup_{|t|\le \overline{N}} I_4 + \sup_{|t|\ge \overline{N}} I_5 \le \frac{1}{8}\epsilon|v-v_0|,$$

$$\sup_{|t|\le \overline{N}} I_4' + \sup_{|t|\ge \overline{N}} I_5' \le \frac{1}{8}\epsilon|v-v_0|,$$

and for fixed such $\overline{N}$, there exists $\sigma_2 > 0$ such that if $|v - v_0| \leq \sigma_2$, then

$$\sup_{|t|\leq\overline{N}} I_3 \leq \frac{1}{8}\epsilon|v_1 - v_2| \quad \text{and} \quad \sup_{|t|\geq\overline{N}} I_3' \leq \frac{1}{8}\epsilon|v_1 - v_2|.$$

Taking $\sigma = \min\{\sigma_1, \sigma_1', \sigma_2\}$, we have that if $|v - v_0| \leq \sigma$, then

$$|I|_{C_{\eta-\delta}} \leq \epsilon|v - v_0|.$$

Therefore $|I|_{C_{\eta-\delta}} = o(|v - v_0|)$ as $v \to v_0$.

We now prove that $D_v u(t, v, \omega)$ is continuous from the center subspace H^c to slowly varying functions space C_η. For $v, v_0 \in H^c$, using (24),

$$\begin{aligned}
&D_v u(t, v, \omega) - D_v u(t, v_0, \omega) \\
&= \int_0^t \Psi_A(t, s) P^c \Big(D_u G(\theta_s\omega, u(s, v, \omega)) D_v u(s, v, \omega) \\
&\qquad - D_u G(\theta_s\omega, u(s, v_0, \omega)) D_v u(s, v_0, \omega) \Big)\, ds \\
&\quad + \int_{-\infty}^{\infty} \Psi_0(t, s) B(t - s) \Big(D_u G(\theta_s\omega, u(s, v, \omega)) D_v u(s, v, \omega) \\
&\qquad - D_u G(\theta_s\omega, u(s, v_0, \omega)) D_v u(s, v_0, \omega) \Big)\, ds \qquad (26) \\
&= \int_0^t \Psi_A(t, s) P^c \Big(D_u G(\theta_s\omega, u(s, v, \omega)) \\
&\qquad (D_v u(s, v, \omega) - D_v u(s, v_0, \omega)) \Big)\, ds \\
&\quad + \int_{-\infty}^{\infty} \Psi_0(t, s) B(t - s) \Big(D_u G(\theta_s\omega, u(s, v, \omega)) \\
&\qquad (D_v u(s, v, \omega) - D_v u(s, v_0, \omega)) \Big)\, ds + \bar{I},
\end{aligned}$$

where

$$\bar{I} = \int_0^t \Psi_A(t,s)P^c\big(D_uG(\theta_s\omega, u(s,v,\omega)) \\ - D_uG(\theta_s\omega, u(s,v_0,\omega))\big)D_vu(s,v_0,\omega)\,ds \\ + \int_{-\infty}^{+\infty} \Psi_0(t,s)B(t-s)\big(D_uG(\theta_s\omega, u(s,v,\omega)) \\ - D_uG(\theta_s\omega, u(s,v_0,\omega))\big)D_vu(s,v_0,\omega)\,ds\,.$$

Then from estimating $|D_vu(\cdot,v,\omega) - D_vu(\cdot,v_0,\omega)|_{L(H^c,C_\eta)}$, we have

$$|D_vu(\cdot,v,\omega) - D_vu(\cdot,v_0,\omega)|_{L(H^c,C_\eta)} \le \frac{|\bar{I}|_{L(H^c,C_\eta)}}{1 - K\,\mathrm{Lip}_u\,G\left(\frac{1}{\eta-\gamma} + \frac{1}{\beta-\eta} + \frac{1}{\alpha-\eta}\right)}.$$

Using the same argument we used for the last claim of equation (25), we obtain that $|\bar{I}|_{L(H^c,C_\eta)} = o(|v-v_0|)$ as $v \to v_0$. Hence $D_vu(t,v,\omega)$ is continuous from the center space H^c to bounded linear operators space $L(H^c,C_\eta)$. Therefore, $u(t,v,\omega)$ is C^1 from H^c to C_η.

Now we show that u is C^k from the center space H^c to slowly varying functions space $C_{k\eta}$ by induction for $k \ge 2$. By the induction assumption, we know that u is C^{k-1} from H^c to $C_{(k-1)\eta}$ and the $(k-1)$st derivative $D_v^{k-1}u(t,v,\omega)$ satisfies the following equation.

$$D_v^{k-1}u = \int_0^t \Psi_A(t,s)P^c(D_uG(\theta_s\omega,u)D_v^{k-1}u\,ds \\ + \int_{-\infty}^{+\infty} \Psi_0(t,s)B(t-s)D_uG(\theta_s\omega,u)D_v^{k-1}u\,ds \\ = \int_0^t \Psi_A(t,s)P^cR_{k-1}(s,v,\omega)\,ds \\ + \int_{-\infty}^{+\infty} \Psi_0(t,s)B(t-s)R_{k-1}(s,v,\omega)\,ds\,,$$

where

$$R_{k-1}(s,v,\omega)=\sum_{i=0}^{k-3}\binom{k-2}{i}D_v^{k-2-i}\big(D_uG(\theta_s\omega,u(s,v,\omega))\big)D_v^{i+1}u(s,v,\omega).$$

Note that $D_v^i u\in C_{i\eta}$ for $i=1,\dots,k-1$, from the induction hypothesis. Thus, using G is C^k, we verify that $R_{k-1}(\cdot,v,\omega)\in L^{k-1}\big(H^c,C_{(k-1)\eta}\big)$ and is C^1 with respect to v, where $L^{k-1}\big(H^c,C_{(k-1)\eta}\big)$ is the usual space of bounded $k-1$ linear forms. Since

$$K\,\mathrm{Lip}_u\,G\left(\frac{1}{i\eta-\gamma}+\frac{1}{\beta-i\eta}+\frac{1}{\alpha-i\eta}\right)<1\quad\text{for all } 1\le i\le k.$$

The same argument used in the case $k=1$ shows that $D_v^{k-1}u(\cdot,v,\omega)$ is C^1 from H^c to $L^k(H^c,C_{k\eta})$. This completes the proof. $\square$

Theorem 8. *Assume that the nonlinearity $F(u)$ in equation (1) is C^k smooth. If $\gamma<k\eta<\min\{\beta,\alpha\}$ and*

$$K\,\mathrm{Lip}\,F\left(\frac{1}{i\eta-\gamma}+\frac{1}{\beta-i\eta}+\frac{1}{\alpha-i\eta}\right)<1\quad\textit{for all } 1\le i\le k,$$

then $\widetilde{M}^c(\omega)=T^{-1}(\omega,M^c(\omega))$ is a C^k center manifold for the stochastic partial differential equation (1).

Proof. From the construction of center manifolds, we obtain

$$\widetilde{M}^c(\omega)=\big\{v+\tilde{h}^c(v,\omega):v\in H^c\big\},$$

where $\tilde{h}^c(v,\omega)=\exp[z(\omega)]h^c(\exp[-z(\omega)]v,\omega)$. Note that $h^c(\cdot,\omega)$ is C^k, then $\tilde{h}^c(\cdot,\omega)$ is C^k. We conclude $\widetilde{M}^c(\omega)$ is a C^k center manifold for the stochastic partial differential equation (1).

$\square$

5 Exponential attraction principle

This section proves the stability of the random evolution equation (9). Our main result is Theorem 10, which shows that the dynamic behavior of equation (9) is exponentially approximated by the solution of on stochastic center manifold. We do not assume $H^s = \emptyset$ or $H^u = \emptyset$. Instead, we project the solution of equation (9) to the stable and unstable subspaces. Lemma 9 is a week approach solutions version as it only concerns solutions approach $M^c(\omega)$. It does not assert solutions.

Lemma 9. *Let $M^c(\omega)$ be a locally stochastic center manifold for stochastic evolution equation (9), then there exist positive constants U and μ such that for $\omega \in \Omega$,*

$$|P^s u(t, u_0, \omega) - P^s h^c(P^c u(t, u_0, \omega), \omega)| \leq U \exp(-\mu t)|P^s u_0 - P^s h^c(P^c u_0, \omega)|,$$

$t \geq 0$ large enough.

Proof. Let $X(t, u_0, \omega) = P^s u(t, u_0, \omega) - P^s h^c(P^c u(t, u_0, \omega), \omega)$, then X satisfies the equation

$$\frac{dX(t, u_0, \omega)}{dt} = AX + z(\theta_t\omega)X + N(X, \omega), \tag{27}$$

where

$$\begin{aligned} N(X, \omega) =& P^s G(\theta_t\omega, (\mathrm{id} - P^s)u + X + P^s h^c(P^c u(t, u_0, \omega), \omega)) \\ & - P^s G(\theta_t\omega, (\mathrm{id} - P^s)u + P^s h^c(P^c u(t, u_0, \omega), \omega), \omega)) \\ & + DP^s h^c(P^c u(t, u_0, \omega), \omega), \omega)[P^c G(\theta_t\omega, (\mathrm{id} - P^s)u \\ & + P^s h^c(P^c u(t, u_0, \omega), \omega)) \\ & - P^c G(\theta_t\omega, (\mathrm{id} - P^s)u + X + P^s h^c(P^c u(t, u_0, \omega), \omega))]. \end{aligned}$$

Since the derivative of stochastic center manifold is less than Lipschitz constant $\mathrm{Lip}_u G$ and G is Lipschitz continuous, we have $|N(\omega, X)| < U_1|X|$. We obtain

$$X(t,u_0,\omega) \;=\; \Psi_A(t,0)X(0,u_0,\omega)+\int_0^t \Psi_A(t,s)N(X,\omega)\,ds\,.$$

The conclusion of Lemma 2 yields $\int_0^t z(\theta_r\omega)dr < \epsilon t\,,\ \int_s^t z(\theta_r\omega)dr < \epsilon(t-s)$ for t large enough and $\beta > \epsilon > 0$. Then

$$\begin{aligned}|X(t,u_0,\omega)\ \ | \le\ & K\exp[(-\beta+\epsilon)t]|X(0,u_0,\omega)| \\ & + U_1 K\int_0^t \exp[(-\beta+\epsilon)(t-s)|X(s,u_0,\omega)|\,ds\,.\end{aligned}$$

From Gronwall's inequality we have the lemma. □

Remark 2. Similarly we have the conclusion there exist positive constants V and ν such that for $\omega\in\Omega$,

$$|P^u u(t,u_0,\omega)-P^u h^c(P^c u(t,u_0,\omega),\omega)| \le V\exp(\nu t)|P^u u_0 - P^u h^c(v,\omega)|,$$

$t\le 0$ small enough. In the finite dimensional case [8, Theorem 7.1], the conclusions are expressed as the random norm.

Now we consider the asymptotic behavior of the stochastic evolution equation (9). The stochastic dynamical system on the stochastic center manifold is governed by the following random evolution equation:

$$\dot v = P^c A v + z(\theta_t\omega)v + P^c G(\theta_t\omega, v+h^c(v,\omega)). \tag{28}$$

If the subspace H^c is finite dimension, then equation (28) is an ordinary differential equation. The following theorem shows that the solution of stochastic evolution equation (9) is exponentially approximated by the solution of equation (28).

Theorem 10. *Let $M^c(\omega)$ be a locally stochastic center manifold for stochastic evolution equation (9), then there are positive random variables $L_1(\omega)$ and $L_2(\omega)$*

and constants l_1 and l_2 such that for initial values $u_0 \in H$ and $v_0 \in M^c$, the following two conditions hold:

$$|P^c u(t, u_0, \omega) - v(t, v_0, \omega)| \leq L_1(\omega) \exp(-l_1 t), \tag{29}$$
$$|P^s u(t, u_0, \omega) - P^s h^c(v(t, v_0, \omega), \omega)| \leq L_2(\omega) \exp(-l_2 t), \tag{30}$$

$t \geq 0$ large enough and $\omega \in \Omega$.

Proof. We construct a differential equation about the error of the two solutions on centre subspace. Let $X'(t, u_0, \omega) = P^c u(t, u_0, \omega) - v(t, v_0, \omega)$, which is the errors of the two solutions on centre subspace, then X' satisfies the equation

$$\frac{dX'(t, u_0, \omega)}{dt} = AX' + z(\theta_t \omega) X' + N'(X', \omega), \tag{31}$$

where the nonlinear term

$$N'(X', \omega) = P^c G(\theta_t \omega, u(t, u_0, \omega)) - P^c G(\theta_t \omega, v + h^c(v, \omega)).$$

Rewrite the nonlinearity N' as three steps, so that N' is bounded.

$$\begin{aligned} N'(X', \omega) = & P^c G(\theta_t \omega, u(t, u_0, \omega)) - P^c G(\theta_t \omega, P^c u + h^c(P^c u, \omega)) \\ & + P^c G(\theta_t \omega, P^c u + h^c(P^c u, \omega)) - P^c G(\theta_t \omega, P^c u + h^c(v, \omega)) \\ & + P^c G(\theta_t \omega, P^c u + h^c(v, \omega)) - P^c G(\theta_t \omega, v + h^c(v, \omega)). \end{aligned}$$

Rewrite equation (31) as an integral equation.

$$X'(t, u_0, \omega) = \Psi_A(t, 0) X'(0, u_0, \omega) + \int_0^t \Psi_A(t, s) N'(X', \omega)\, ds\,.$$

Since G is Lipschitz continuous, there exist positive constants V_1, v_1 and V_2 such that

$$|N'(X', \omega)| \leq V_1 \exp(-v_1 t) |(\mathrm{id} - P^c) u_0 - h^c(P^c u_0, \omega)| + V_2 |X'|.$$

Lemma 2 yields $\int_0^t z(\theta_r\omega)dr < \epsilon t$ and $\int_s^t z(\theta_r\omega)dr < \epsilon(t-s)$ for t large enough and $\gamma > \epsilon > 0$. It follows that

$$\begin{aligned}|X'(t,u_0,\omega)| \leq & K\exp[(\gamma+\epsilon)t]|X'(0,u_0,\omega)| \\ & + K(\gamma+\epsilon+v_1)^{-1}\exp[(\gamma+\epsilon)t]V_1|(\mathrm{id}-P^c)u_0 - h^c(P^c u_0,\omega)| \\ & + V_2 K\int_0^t \exp[(\gamma+\epsilon)(t-s)|X'(s,u_0,\omega)|\,ds\,.\end{aligned}$$

From Gronwall's inequality we have inequality (29).

Now we prove inequality (30),

$$\begin{aligned}& |P^s u(t,u_0,\omega) - P^s h^c(v(t,v_0,\omega),\omega)| \\ & \leq |P^s u(t,u_0,\omega) - P^s h^c(P^c u,\omega)| + |P^s h^c(P^c u,\omega) - P^s h^c(v(t,v_0,\omega),\omega)|.\end{aligned}$$

By Lemma 9 and inequality (29) we have inequality (30). □

Remark 3. Similarly we conclude there exist positive random variable $L_3(\omega)$ and positive constant l_3 such that

$$|P^c u(t,u_0,\omega) - v(t,v_0,\omega)| \leq L_3(\omega)\exp(l_3 t), \tag{32}$$
$$|P^u u(t,u_0,\omega) - P^u h^c(v(t,v_0,\omega),\omega)| \leq L_4(\omega)\exp(l_4 t), \tag{33}$$

$t \leq 0$ small enough and $\omega \in \Omega$.

Corollary 11. *Let $H^u = \emptyset$. Suppose that the zero solution of equation (28) is asymptotically stable and the initial value u_0 is sufficiently small. Then the zero solution of equation (9) is asymptotically stable.*

Proof. From Theorem 10, we conclude that the zero solution of equation (28) is asymptotically stable on centre and stable subspaces respectively. This establishes Corollary 11. □

6 Approximation of the center manifolds

For a $\mathcal{F}$-measurable function $g(v,\omega) : H^c \to H^u \oplus H^s$ which are C^1 in a neighborhood of origin in H^c, define

$$\begin{aligned} M(g)(v,\omega) = & g_v(v,\omega)[P^cAv + z(\theta_t\omega)v + P^cG(\theta_t\omega, v+g)] - (\mathrm{id} - P^cA)v \\ & - z(\theta_t\omega)v - P^sG(\theta_t\omega, v+v) - P^uG(\theta_t\omega, v+g), \end{aligned}$$

where $v \in H^c$.

Theorem 12. *Suppose that a $\mathcal{F}$-measurable function $g(0,\omega)=0$, $g_v(0,\omega)=0$ and that $M(g)(v,\omega) = \mathcal{O}(|v|^q)$ for some $q>1$. Then as $v\to 0$,*

$$|h^c(v,\omega) - g(v,\omega)| = \mathcal{O}(|v|^q).$$

Proof. Let a $\mathcal{F}$-measurable function $y(t,v,\omega) \in C_\eta$ and let $y(0,v,\omega) : H^c \to H^u \oplus H^s$ be continuous differential with compact support such that $y(0,0,\omega) = g(0,\omega) = 0$, $y_v(0,0,\omega) = g_v(0,\omega) = 0$ and $y_v(0,v,\omega) = g_v(v,\omega)$ for $|v|$ small enough. The function $y(t,v,\omega)$ connects the functions $h^c(v,\omega)$ and $g(v,\omega)$. Set

$$\begin{aligned} M(y)(0,v,\omega) = & y_v(0,v,\omega)[P^cAv + z(\theta_t\omega)v + P^cG(\theta_t\omega, y(0,v,\omega))] \\ & - (\mathrm{id} - P^cA)y(0,v,\omega) - z(\theta_t\omega)y(0,v,\omega) - P^sG(\theta_t\omega, y(0,v,\omega)) \\ & - P^uG(\theta_t\omega, y(0,v,\omega)), \end{aligned}$$

where $v\in H^c$. For a $\mathcal{F}$-measurable function $w(t,v,\omega)$, define the operator

$$Sw = J^c\left[(w(t,v,\omega) + y(t,v,\omega)\right] - y(t,v,\omega),$$

where the operator J^c is defined by equation (18). The domain Y

$$Y(\omega) = \{w(t,v,\omega) \in C_\eta : |w(0,v,\omega)| \le K(\omega)|v|^q \quad \text{for all } v \in H^c\}.$$

From the construction of $Y(\omega)$, $Y(\omega)$ is closed in C_η. Since J^c is a contraction mapping on C_η, S is a contracting mapping on $Y(\omega)$. We prove that there exist a random variable $K(\omega)$ such that S maps $Y(\omega)$ into $Y(\omega)$. Then by the uniqueness of fixed points and definition of h^c we conclude the theorem.

Now we prove there exists $Y(\omega)$. For $w \in C_\eta$, let $u_c(t, v, \omega)$ be the solution of

$$\begin{aligned} &\frac{du_c}{dt} = P^c A u_c + z(\theta_t\omega)u_c + P^c G(\theta_t\omega, y(0, u_c, \omega) + w(0, u_c, \omega)), \\ &u_c(0, v, \omega) = v. \end{aligned} \tag{34}$$

From equation (18)

$$\begin{aligned} &J^c(y(0, u_c, \omega) + w(0, u_c, \omega)) = \\ &\qquad \int_{-\infty}^{+\infty} \Psi_0(0, s)B(-s)G(\theta_s\omega, y(s, u_c, \omega) + w(s, u_c, \omega))\, ds\,. \end{aligned}$$

Since

$$\begin{aligned} &y(0, u_c, \omega) \\ &= \int_{-\infty}^{0} \frac{d}{ds}\left[\Psi_{-A}(0, s)P^s y(0, u_c(s, v, \omega), \omega)\right] ds \\ &\quad - \int_{0}^{\infty} \frac{d}{ds}\left[\Psi_{-A}(0, s)P^u y(0, u_c(s, v, \omega), \omega)\right] ds \\ &= \int_{-\infty}^{0} \Psi_{-A}(0, s)\Big[(-A - z(\theta_s\omega))P^s y(0, u_c(s, v, \omega), \omega) \\ &\quad + \frac{d}{ds}P^s y(0, u_c(s, v, \omega), \omega)\Big]\, ds - \int_{0}^{\infty} \Psi_{-A}(0, s) \\ &\quad \left[(-A - z(\theta_s\omega))P^u y(0, u_c(s, v, \omega), \omega) + \frac{d}{ds}P^u y(0, u_c(s, v, \omega), \omega)\right] ds\,. \end{aligned}$$

Here

$$
\begin{aligned}
&(-A-z(\theta_s\omega))P^s y(0,u_c(s,v,\omega),\omega)+\frac{d}{ds}P^s y(0,u_c(s,v,\omega),\omega)\\
=&-AP^s y(0,u_c(s,v,\omega),\omega)-z(\theta_s\omega)P^s y(0,u_c(s,v,\omega),\omega)\\
&+P^s y_v(0,u_c(s,v,\omega),\omega)[Au_c+z(\theta_t\omega)u_c+P^cG(\theta_t\omega,y(0,u_c,v)+w(0,u_c,v))]\\
=&M(P^s y)-P^s y_v(0,u_c(s,v,\omega),\omega)[Au_c+z(\theta_t\omega)u_c+P^cG(\theta_t\omega,y(0,u_c,v))]\\
&+P^sG(\theta_t\omega,y(0,u_c,v))+P^s y_v(0,u_c(s,v,\omega),\omega)\\
&[Au_c+z(\theta_t\omega)u_c+P^cG(\theta_t\omega,y(0,u_c,v)+w(0,u_c,v))]\\
=&M(P^s y)-P^s y_v(0,u_c(s,v,\omega),\omega)[P^cG(\theta_t\omega,y(0,u_c,v))-P^cG(\theta_t\omega,y(0,u_c,v)\\
&+w(0,u_c,v))]+P^sG(\theta_t\omega,y(0,u_c,v)).
\end{aligned}
$$

$$
\begin{aligned}
&(-A-z(\theta_s\omega))P^u y(0,u_c(s,v,\omega),\omega)+\frac{d}{ds}P^u y(0,u_c(s,v,\omega),\omega)\\
=&-AP^u y(0,u_c(s,v,\omega),\omega)-z(\theta_s\omega)P^u y(0,u_c(s,v,\omega),\omega)\\
&+P^u y_v(0,u_c(s,v,\omega),\omega)[Au_c+z(\theta_t\omega)u_c+P^cG(\theta_t\omega,y(0,u_c,v)+w(0,u_c,v))]\\
=&M(P^u y)-P^u y_v(0,u_c(s,v,\omega),\omega)[P^cG(\theta_t\omega,y(0,u_c,v))-P^cG(\theta_t\omega,y(0,u_c,v)\\
&+w(0,u_c,v))]+P^uG(\theta_t\omega,y(0,u_c,v)).
\end{aligned}
$$

From the above calculations

$$
\begin{aligned}
&J^c(y(0,u_c,\omega)+w(0,u_c,\omega))-y(0,u_c,\omega)\\
&\quad=\int_{-\infty}^0\Psi_{-A}(0,s)Q^s\,ds+\int_0^\infty\Psi_{-A}(0,s)Q^u\,ds\,,
\end{aligned}
$$

where Q^s and Q^u is

$$
\begin{aligned}
Q^s(y,w,\omega)=&P^s y_v(0,u_c(s,v,\omega),\omega)[P^cG(\theta_t\omega,y(0,u_c,\omega))-P^cG(\theta_t\omega,y(0,u_c,\omega)\\
&+w(0,u_c,\omega))]-M(P^s y)+[P^sG(\theta_t\omega,y(0,u_c,\omega)+w(0,u_c,\omega))\\
&-P^s(\theta_t\omega,y(0,u_c,\omega))],\\
Q^u(y,w,\omega)=&M(P^u y)-P^s y_v(0,u_c(s,v,\omega),\omega)[P^cG(\theta_t\omega,y(0,u_c,\omega))\\
&P^cG(\theta_t\omega,y(0,u_c,\omega)+w(0,u_c,\omega))]-[P^uG(\theta_t\omega,y(0,u_c,\omega)\\
&+w(0,u_c,\omega))-P^u(\theta_t\omega,y(0,u_c,\omega))].
\end{aligned}
$$

Since $g_v(0,\omega)=0$ and g is C^1 in a neighborhood of origin in H^c, for a small random variable $\epsilon(\omega)$, there exists a positive random variable $\delta(\omega)$ such that

$$|g_v(v,\omega)|<\epsilon(\omega)$$

if $|v|<\delta(\omega)$. From $M(y)(0,u_c,\omega)=C_1(\omega)|u_c|^q$ for a positive random variable $C_1(\omega)$ and all $u_c\in H^c$, $|Q^s|\le C_2(\omega)|u_c|^q$ and $|Q^u|\le C_3(\omega)|u_c|^q$. Using the Gronwall's inequality in equation (34) we conclude $|u_c|\le C_4(\omega)|v|$. $\square$

7 Examples

We present some examples to show our results. Example 7.1 shows that we directly use the computer algebra [29] to compute the stochastic center manifolds. We also do not need the convolution term. This is different from the results of Roberts [30], which assume the existence of theory of center manifolds in the stochastic partial differential equations.

7.1 Reaction-diffusion dynamics

Consider the stochastic parabolic equation

$$u_t=u_{xx}+u-au^3+\sigma u\circ\dot W \quad \text{with} \quad u(0,t)=u(\pi,t)=0\,, \quad t\ge 0\,. \tag{35}$$

We set $H:=L^2(0,\pi)$, $Au=\frac{d^2u}{dx^2}+u$ with domain $D(A):=H^2(0,\pi)\cap H_0^1(0,\pi)$. Then the spectrum of the operator A is

$$\sigma(A)=\{1-n^2:n=1,2,\dots\}.$$

The corresponding complete and orthogonal eigenfunctions are $\sin nx$, $n=1,2,\dots$. The operator A satisfies the exponent trichotomy. Denote $H^c=\operatorname{span}\{\sin x\}$, $H^s=\operatorname{span}\{\sin nx, n=2,3,\dots\}$.

Random variable $z(\theta_t\omega)$ is the stationary solution of the Ornstein–Uhlenbeck SDE (6). The transformed random parabolic equation is

$$u_t = u_{xx} + u + \sigma z(\theta_t\omega)u - a\exp[2\sigma z(\theta_t\omega)]u^3. \tag{36}$$

Select parameter a small enough such the condition of Theorem 5 is satisfied. Then we have the slow (center) manifold

$$M^c(\omega) = \{s\sin x + h^c(s\sin x,\omega)\} = \left\{ s\sin x + \sum_{n=2}^{\infty} c_n(s,\omega)\sin nx \right\},$$

where $c_n(s,\omega) = \mathcal{O}\big(s^3\big)$ as $s \to 0$ for $\omega \in \Omega$. The random dynamical system on the slow manifold $M^c(\omega)$ satisfies

$$\begin{aligned}\frac{ds}{dt}\sin x + \sum_{n=2}^{\infty} c_n'(s,\omega)\sin nx = \sum_{n=2}^{\infty}&[1-n^2+z(\theta_t\omega)]c_n(s,\omega)\sin nx\\ &+s\sigma z(\theta_t\omega)\sin x - \exp[2\sigma z(\theta_t\omega)]as^3\sin^3 x + \mathcal{O}\big(s^5\big).\end{aligned} \tag{37}$$

It follows that

$$\frac{ds}{dt} = s\sigma z(\theta_t\omega) - \frac{3}{4}\exp[2\sigma z(\theta_t\omega)]as^3 + \mathcal{O}\big(s^5\big).$$

We now calculate the slow manifold $h^c(s\sin x,\omega)$. Assume

$$h^c(s\sin x,\omega) = c_3(s,\omega)\sin 3x + \mathcal{O}\big(s^5\big).$$

Since

$$a\exp[2\sigma z(\theta_t\omega)](s\sin x + c_3\sin 3x)^3 = as^3\exp[2\sigma z(\theta_t\omega)]\sin^3 x + \mathcal{O}\big(s^5\big).$$

So

$$P^cG(\theta_t\omega, s\sin x + h^c) = \frac{3}{4}as^3\exp[2\sigma z(\theta_t\omega)]\sin x + \mathcal{O}\big(s^5\big),$$

$$P^s G(\theta_t\omega, s\sin x + h^c) = -\frac{1}{4}as^3 \exp[2\sigma z(\theta_t\omega)] \sin 3x + \mathcal{O}\big(s^5\big).$$

From

$$M(h^c)(v,\omega) = h^c_v(v,\omega)[P^c A + \sigma z(\theta_t\omega)v + P^c G(\theta_t\omega, v + h^c)] - (\mathrm{id} - P^c A)h^c \\ - \sigma z(\theta_t\omega)h^c - P^s G(\theta_t\omega, v + h^c) - P^u G(\theta_t\omega, v + h^c),$$

and $h^c(v,\omega) = c_3(s,\omega)\sin 3x$, we have $c_3(s,\omega) = \frac{1}{8-\sigma z(\theta_t\omega)} \times \frac{1}{4}as^3 \exp[2\sigma z(\theta_t\omega)]$. So

$$h^c(s\sin x,\omega) = \frac{1}{8 - \sigma z(\theta_t\omega)} \times \frac{1}{4}as^3 \exp[2\sigma z(\theta_t\omega)] \sin 3x + \mathcal{O}\big(s^5\big).$$

Note that $z(\theta_t\omega) = -\int_{-\infty}^0 \exp(\tau)\omega(\tau+t)\,d\tau + \omega(t) = \int_{-\infty}^t \exp(s-t)dW(s)$. In fact this is the convolution term [30]. Then the expectation of stochastic centre manifold is

$$Eh^c(s\sin x,\omega) = \frac{1}{32}as^3 \sin 3x + \mathcal{O}\big(s^5\big).$$

It is the corresponding center manifold of equation (35) without noise. The stochastic center manifold of stochastic parabolic equation (35) is

$$\tilde{h}^c(s\sin x,\omega) = \frac{1}{32}as^3 \sin 3x - \frac{1}{32}\sigma z(\theta_t\omega)as^3 \sin 3x + \mathcal{O}\big(s^5, \sigma^2\big).$$

$$\frac{ds}{dt} = s\sigma z(\theta_t\omega) - \frac{3}{4}as^3 + \mathcal{O}\big(s^5\big).$$

Since $z(\theta_t\omega)$ =phi(0, {}), this is the first and second terms of the follow stochastic slow manifold by the computer algebra [29].

```
Slow manifold to error O(a^3,sig^2):
noise := phi(0,{})
                                3      3      3    2  5
dsdt := s*sig*phi(0,{}) - ---*a*s  + -----*a *s
```

```
                         4          128
            3    2  5
        - -----*a *s *sig*phi(0,{})
           512
                   1      3
uslow := s*sin(x) + ----*a*s *sin(3*x)
                   32
            1      3
         - ----*a*s *sig*phi(0,{3})*sin(3*x)
            16
            2  5    3                 1                 2
         + a *s *(------*sin(3*x) + ------*sin(5*x)) + a
                  1024                1024
          5        3
        *s *sig*(-----*phi(0,{3})*sin(x)
                 512
                3
            - -----*phi(0,{3})*sin(3*x)
               256
               3
            - ----*phi(0,{3,3})*sin(3*x)
               64
                1
            - -----*phi(0,{5})*sin(5*x)
               256
               3
            - ----*phi(0,{5,3})*sin(5*x))
               64
```

7.2 Semilinear damped wave dynamics

Wang and Duan [36] presented a two dimension center–unstable manifold in a semilinear wave equation. We show that the semilinear damped wave equation has a one dimensional center manifold.

Consider the equation

$$u_{tt} + u_t = \frac{1}{4} u_{xx} + u + f(u) + u \circ \dot{W}, \quad x \in (0, 2\pi), \quad t \geq 0\,, \tag{38}$$

with

$$u(0,t) = u(2\pi, t) = 0\,.$$

$W(t)$ is the standard $\mathbb{R}$-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $H = H_0^1(0, 2\pi) \times L^2(0, 2\pi)$. Rewrite the equation (38) as the following first order stochastic evolution equations in H:

$$du = v\,dt \tag{39}$$

$$dv = \big[\frac{1}{4} u_{xx} + u - v + f(u)\big] dt + u \circ dW(t). \tag{40}$$

First we prove that the system (39)–(40) generates a continuous random dynamical system in H. Let $\Psi_1(t) = u(t)$, $\Psi_2(t) = v(t) - u(t) z(\theta_t \omega)$, where $z(\omega)$ is the stationary solution of (6). Then we have the following random evolution equation

$$d\Psi_1 = \big[\Psi_2 + \Psi_1 z(\theta_t \omega)\big] dt\,, \tag{41}$$

$$\begin{aligned} d\Psi_2 = {} & \big[\frac{1}{4} \frac{d^2 \Psi_1}{dx^2} + \Psi_1 - \Psi_2 + f(\Psi_1)\big] dt + \big[- z^2(\theta_t \omega) \Psi_1 \\ & - z(\theta_t \omega) \Psi_2\big] dt\,. \end{aligned} \tag{42}$$

Let $\Psi(t, \omega) = (\Psi_1(t, \omega), \Psi_2(t, \omega)) \in H$, then $\varphi(t, \Psi(0, \omega), \omega) = \Psi(t, \omega)$ defines a continuous random dynamical system in H. Notice that the stochastic system (39)–(40) is conjugated to the random system (41)–(42) by the homeomorphism

$$T(\omega,(u,v)) = (u, v + uz(\omega)), \quad (u,v) \in H$$

with inverse

$$T^{-1}(\omega,(u,v)) = (u, v - uz(\omega)), \quad (u,v) \in H\,.$$

Then $\hat{\varphi}(t,\omega,(u_0,v_0)) = T(\theta_t\omega, \varphi(t,\omega,T^{-1}(\omega,(u_0,v_0))))$ is the random dynamical system generated by (39)–(40).

Define

$$\mathcal{A} = \begin{bmatrix} 0, & \mathrm{id} \\ \frac{1}{4}\frac{\partial^2}{\partial x^2} + \mathrm{id}, & -\mathrm{id} \end{bmatrix}, \quad F(\Psi) = \begin{bmatrix} 0 \\ f(\Psi_1) \end{bmatrix},$$
$$Z(\theta_t\omega) = \begin{bmatrix} z(\theta_t\omega), & 0 \\ -z^2(\theta_t\omega), & -z(\theta_t\omega) \end{bmatrix},$$

where id is the identity operator on the Hilbert space $L^2(0,2\pi)$.

Then (41)–(42) can be written as

$$\frac{d\Psi}{dt} = \mathcal{A}\Psi + Z(\theta_t\omega)\Psi + F(\Psi) \tag{43}$$

Then the operator $\mathcal{A}$ has the eigenvalues

$$\delta_k^{\pm} = -\frac{1}{2} \pm \sqrt{\frac{5}{4} - \frac{1}{4}k^2}, \quad k = 1, 2, \dots\,,$$

and corresponding eigenvectors are $q_k^{\pm} = (1, \delta_k^{\pm}) \sin kx$. The operator $\mathcal{A}$ has one zero eigenvalue $\delta_2^+ = 0$, one negative eigenvalue $\delta_1^+ = \frac{1}{2}$ and the others are all complex numbers with negative real part. By using a new inner product [36], we define an equivalent norm on H with $\{(1, \delta_k^{\pm}) \sin kx\}$ are orthogonal.

A calculation yields that, in terms of the new norm, the semigroup $\mathcal{S}(t)$ generated by $\mathcal{A}_\nu$ satisfies exponent trichotomy with $\alpha = \frac{1}{2} + \epsilon$, $\gamma = \frac{1}{4} + \epsilon$ $\beta = \frac{1}{2} - \epsilon$ for ϵ small enough and $\dim H^c = \dim H^u = 1$.

Let $\Psi = rq_1^+ + sq_2^+ + y$, $y \in P^s H$. From equation (43) we have

$$\dot{s}q_2^+ = Z(\theta_t\omega)sq_2^+ + P^c F(\theta_t\omega, rq_1^+ + sq_2^+ + y). \tag{44}$$

From Theorem 5 we obtain a center manifold $M(\omega)$ which we denote as the graph of a random Lipschitz map $h^c(sq_2^+, \omega)$.

By Theorem 10, the asymptotic behavior of the equation (43) is governed by the one dimension equation

$$\dot{s}q_2^+ = Z(\theta_t\omega)sq_2^+ + P^c F(\theta_t\omega, sq_2^+ + h^c(sq_2^+, \omega)). \tag{45}$$

Suppose $f(u) = -u^3$, then

$$\dot{s}q_2^+ = Z(\theta_t\omega)sq_2^+ - s^3 q_2^+ + \mathcal{O}\big(s^3 q_2^+\big). \tag{46}$$

Or

$$\dot{s} = z(\theta_t\omega)s - s^3 + \mathcal{O}\big(s^3\big). \tag{47}$$

7.3 An infinite dimensional center manifold

We have not found some examples of infinite dimensional center manifolds except Mielke [23]. As a toy example let's consider the SPDEs

$$\frac{\partial u}{\partial t} = au - uv + \sigma u \circ \dot{W}, \quad \frac{\partial v}{\partial t} = -v + \frac{\partial^2 v}{\partial x^2} + u^2 - 2\mathcal{K}_a(u^2 v) + \sigma v \circ \dot{W}, \tag{48}$$

with Neumann boundary conditions $v_z = 0$ at $z = 0, \pi$, and the operator $\mathcal{K}_a = (1 + 2a - \partial_{zz})^{-1}$.

Transform the SPDEs (48) to systems of SPDEs with random coefficient.

$$\begin{aligned}\frac{\partial u}{\partial t} &= au + \sigma z(\theta_t\omega)u - \exp(\sigma z(\theta_t\omega))uv,\\ \frac{\partial v}{\partial t} &= -v + \frac{\partial^2 v}{\partial x^2} + \sigma z(\theta_t\omega)v + \exp(\sigma z(\theta_t\omega))u^2 - 2\exp(2\sigma z(\theta_t\omega))\mathcal{K}_a(u^2 v)\,,\end{aligned}$$

For the specific case $a = 0$,

$$\mathcal{K}_0 v = \int_0^\pi K_0(z,\zeta)v(\zeta)\,d\zeta \quad \text{where } K_0 = \begin{cases} \frac{\cos(\pi-\zeta)\cos z}{\sin\pi}, & z<\zeta\,,\\ \frac{\cos(\pi-z)\cos\zeta}{\sin\pi}, & z>\zeta\,,\end{cases}$$

for which $\csc\pi \le K_0 \le \cot\pi$ and $\int_0^\pi K_0\,d\zeta = 1\,.$

If the parameter σ is small enough, then the above system of SPDEs has an approximation stochastic slow manifold of $v = \mathcal{K}_0 u^2$.

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