

Truncated infinite dimensional stochastic slow manifold

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December 22, 2011

As a toy example let's consider the PDEs

$$\frac{\partial u}{\partial t} = au - uv + u\dot{W}, \quad \frac{\partial v}{\partial t} = -v + \frac{\partial^2 v}{\partial z^2} + u^2 - 2\mathcal{K}_a(u^2 v) + v\dot{W},$$

with Neumann boundary conditions $v_z = 0$ at $z = 0, \pi$, and where operator $\mathcal{K}_a = (1 + 2a - \partial_{zz})^{-1}$. For example, for the specific case $a = 0$

$$\mathcal{K}_a v = \int_0^\pi K_0(z, \zeta) v(\zeta) d\zeta \quad \text{where } K_0 = \begin{cases} \frac{\cosh(\pi - \zeta) \cosh z}{\sinh \pi}, & z < \zeta, \\ \frac{\cosh(\pi - z) \cosh \zeta}{\sinh \pi}, & z > \zeta, \end{cases}$$

for which $\operatorname{csch} \pi \leq K_0 \leq \coth \pi$ and $\int_0^\pi K_0 d\zeta = 1$. The above system of PDEs has an exact deterministic slow manifold of $v = \mathcal{K}_a u^2$.

The challenge is to approximate the stochastic slow manifold.

Expand the fields in a Fourier cosine series with coefficients which are the x_k and y_k , for slow and fast variables respectively, that I use in my web service.

```
1 on div; off allfac;
2 operator x; operator y;
3 o:=3;
4 u:=for k:=0:o sum x(k+1)*cos(k*z);
5 v:=for k:=0:o sum y(k+1)*cos(k*z);
```

Define the operator \mathcal{K}_a , setting parameter $a = 0$ for simplicity.

```
6 a:=0;
7 operator kk; linear kk;
8 let {kk(1,z)=>1, kk(cos(~k*z),z)=>cos(k*z)/(1+k^2)};
```

Substitute into the governing differential equations, defining a scalar noise term:

```
9 operator w;
10 uuv:=trigsimp(u^2*v,combine);
11 dudt:=trigsimp(a*u-u*v+u*w(1),combine);
12 dvdt:=trigsimp(-v+df(v,z,2)+u^2-2*kk(uuv,z)+v*w(1),combine);
```

Check the deterministic slow manifold.

```
13 sm:=(trigsimp(kk(u^2,z),combine) where w(1)=>0)$
14 sm:=(y(1)=sm where cos(~z)=>0).
15 (for k:=1:o collect (y(k+1)=coeffn(sm,cos(k*z),1)));
16 resv:=sub(sm,dvdt-kk(trigsimp(2*u*dudt,combine),z))$
17 resv:=(resv where {w(1)=>0,cos(~k*z)=>0 when k>o});
```

Get the components of each PDE up to the order resolved:

```
18 dudt:=(dudt where cos(~z)=>0).
19 (for k:=1:o collect coeffn(dudt,cos(k*z),1));
20 dvdt:=(dvdt where cos(~z)=>0).
21 (for k:=1:o collect coeffn(dvdt,cos(k*z),1));
```

Write these out for input to the web service.

```
22 off nat; linelength 60$
23 dudt:=dudt;
24 dvdt:=dvdt;
25 end;
```

Modelling the above gives the following.

```
26 The stochastic slow manifold
27          2      1      2      1      2      1      2
```

```

28 y(1) := xx(1) + ---*xx(2) + ---*xx(3) + ---*xx(4) + sig*
29           2           2           2
30           2           1           2
31       ( - xx(1) *z(w(1),tt,-1) - ---*xx(2) *z(w(1),tt,-1)
32                                   2
33           1           2
34       - ---*xx(3) *z(w(1),tt,-1)
35           2
36           1           2
37       - ---*xx(4) *z(w(1),tt,-1))
38           2
39           1           1
40 y(2) := xx(2)*xx(1) + ---*xx(3)*xx(2) + ---*xx(4)*xx(3) +
41           2           2
42       sig*( - xx(2)*xx(1)*z(w(1),tt,-2)
43             1
44             - ---*xx(3)*xx(2)*z(w(1),tt,-2)
45             2
46             1
47             - ---*xx(4)*xx(3)*z(w(1),tt,-2))
48             2
49 y(3) := -----*xx(2) + ---*xx(3)*xx(1) + ---*xx(4)*xx(2) +
50           10           5           5
51           1           2
52       sig*( - -----*xx(2) *z(w(1),tt,-5)
53             10
54             2
55             - ---*xx(3)*xx(1)*z(w(1),tt,-5)
56             5
57             1
58             - ---*xx(4)*xx(2)*z(w(1),tt,-5))
59             5
60
61           1           1
62 y(4) := -----*xx(3)*xx(2) + ---*xx(4)*xx(1) + sig*(

```

```

63          10          5
64          1
65          - ----*xx(3)*xx(2)*z(w(1),tt,-10)
66          10
67          1
68          - ---*xx(4)*xx(1)*z(w(1),tt,-10))
69          5
70 x(1) := xx(1)
71 x(2) := xx(2)
72 x(3) := xx(3)
73 x(4) := xx(4)
74 The slow SDEs
75          3          2          3          2
76 dx(1)/dt = - xx(1) - xx(2) *xx(1) - ----*xx(3)*xx(2)
77          10
78          7          2          2
79          - ----*xx(3) *xx(1) - ---*xx(4)*xx(3)*xx(2)
80          10          5
81          3          2
82          - ---*xx(4) *xx(1) + sig*(w(1)*xx(1)
83          5
84          3          3          2
85          + w(1)*xx(1) + ----*w(1)*xx(2) *xx(1)
86          4
87          27          2
88          + ----*w(1)*xx(3)*xx(2)
89          200
90          27          2
91          + ----*w(1)*xx(3) *xx(1)
92          50
93          3
94          + ----*w(1)*xx(4)*xx(3)*xx(2)
95          20
96          51          2          2
97          + ----*w(1)*xx(4) *xx(1)) + sig *(

```

```

98          100
99          3
100 w(1)*xx(1) *z(w(1),tt,-1)
101          1          2
102 + ---*w(1)*xx(2) *xx(1)*z(w(1),tt,-2)
103          4
104          1          2
105 + ---*w(1)*xx(2) *xx(1)*z(w(1),tt,-1)
106          2
107          1          2
108 + -----*w(1)*xx(3)*xx(2) *z(w(1),tt,-5)
109          100
110          1          2
111 + ---*w(1)*xx(3)*xx(2) *z(w(1),tt,-2)
112          8
113          1          2
114 + ----*w(1)*xx(3) *xx(1)*z(w(1),tt,-5)
115          25
116          1          2
117 + ---*w(1)*xx(3) *xx(1)*z(w(1),tt,-1) +
118          2
119          1
120 -----*w(1)*xx(4)*xx(3)*xx(2)*z(w(1),tt,-10)
121          200
122          1
123 + ----*w(1)*xx(4)*xx(3)*xx(2)*z(w(1),tt,-5)
124          50
125          1
126 + ---*w(1)*xx(4)*xx(3)*xx(2)*z(w(1),tt,-2)
127          8
128          1          2
129 + -----*w(1)*xx(4) *xx(1)*z(w(1),tt,-10)
130          100
131          1          2
132 + ---*w(1)*xx(4) *xx(1)*z(w(1),tt,-1))

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```

133          2
134          2      11      3
135 dxx(2)/dt = - 2*xx(2)*xx(1) - ----*xx(2)
136                    20
137          6          4      2
138 - ----*xx(3)*xx(2)*xx(1) - ----*xx(3) *xx(2)
139          5          5
140          3          2      4
141 - ----*xx(4)*xx(2) - ----*xx(4)*xx(3)*xx(1)
142          20          5
143          1          2      3      2
144 - ----*xx(4)*xx(3) - ----*xx(4) *xx(2) + sig*(
145          4          5
146          3          2
147 w(1)*xx(2) + ----*w(1)*xx(2)*xx(1)
148                2
149          51          3
150 + ----*w(1)*xx(2)
151          100
152          27
153 + ----*w(1)*xx(3)*xx(2)*xx(1)
154          50
155          63          2
156 + ----*w(1)*xx(3) *xx(2)
157          100
158          3          2
159 + ----*w(1)*xx(4)*xx(2)
160          100
161          3
162 + ----*w(1)*xx(4)*xx(3)*xx(1)
163          10
164          1          2
165 + ----*w(1)*xx(4)*xx(3)
166          8
167          13          2          2

```

```

168      + ----*w(1)*xx(4) *xx(2)) + sig *(
169          25
170      1          2
171  ---*w(1)*xx(2)*xx(1) *z(w(1),tt,-2)
172      2
173          2
174      + w(1)*xx(2)*xx(1) *z(w(1),tt,-1)
175          1          3
176      + ----*w(1)*xx(2) *z(w(1),tt,-5)
177          100
178          1          3
179      + ---*w(1)*xx(2) *z(w(1),tt,-1)
180          2
181          1
182      + ----*w(1)*xx(3)*xx(2)*xx(1)*z(w(1),tt,-5)
183          25
184          1
185      + ---*w(1)*xx(3)*xx(2)*xx(1)*z(w(1),tt,-2)
186          2
187          1          2
188      + ----*w(1)*xx(3) *xx(2)*z(w(1),tt,-10)
189          200
190          1          2
191      + ---*w(1)*xx(3) *xx(2)*z(w(1),tt,-2)
192          8
193          1          2
194      + ---*w(1)*xx(3) *xx(2)*z(w(1),tt,-1)
195          2
196          3          2
197      + ----*w(1)*xx(4)*xx(2) *z(w(1),tt,-5) +
198          100
199          1
200  ----*w(1)*xx(4)*xx(3)*xx(1)*z(w(1),tt,-10)
201      100
202          1

```

$$\begin{aligned}
& + \frac{1}{25} w(1) xx(4) xx(3) xx(1) z(w(1), tt, -5) \\
& + \frac{1}{4} w(1) xx(4) xx(3) xx(1) z(w(1), tt, -2) \\
& + \frac{1}{8} w(1) xx(4) xx(3) z(w(1), tt, -2) \\
& + \frac{1}{50} w(1) xx(4) xx(2) z(w(1), tt, -5) \\
& + \frac{1}{2} w(1) xx(4) xx(2) z(w(1), tt, -1) \\
& - \frac{3}{5} xx(2) xx(1) - \frac{7}{5} xx(3) xx(1) \\
& - \frac{4}{5} xx(3) xx(2) - \frac{1}{2} xx(3) \\
& - \frac{4}{5} xx(4) xx(2) xx(1) - \frac{1}{2} xx(4) xx(3) xx(2) \\
& - \frac{3}{4} xx(4) xx(3) + \text{sig} * (\\
& \quad \frac{27}{100} w(1) xx(2) xx(1) + w(1) xx(3) \\
& + \frac{27}{25} w(1) xx(3) xx(1) \\
& + \frac{63}{100} w(1) xx(3) xx(2) + \frac{1}{2} w(1) xx(3)
\end{aligned}$$

203
204
205
206
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208
209
210
211
212
213
214
215
216
217
218 dxx(3)/dt =
219
220
221
222
223
224
225
226
227
228
229
230
231
232
233
234
235
236
237


```

238          3
239      + ----*w(1)*xx(4)*xx(2)*xx(1)
240          10
241          1
242      + ---*w(1)*xx(4)*xx(3)*xx(2)
243          4
244          5          2          2
245      + ---*w(1)*xx(4) *xx(3)) + sig *(
246          8
247          1          2
248      ----*w(1)*xx(2) *xx(1)*z(w(1),tt,-5)
249          50
250          1          2
251      + ---*w(1)*xx(2) *xx(1)*z(w(1),tt,-2)
252          4
253          2          2
254      + ----*w(1)*xx(3)*xx(1) *z(w(1),tt,-5)
255          25
256          2
257      + w(1)*xx(3)*xx(1) *z(w(1),tt,-1)
258          1          2
259      + ----*w(1)*xx(3)*xx(2) *z(w(1),tt,-10)
260          200
261          1          2
262      + ---*w(1)*xx(3)*xx(2) *z(w(1),tt,-2)
263          8
264          1          2
265      + ---*w(1)*xx(3)*xx(2) *z(w(1),tt,-1)
266          2
267          1          3          1
268      + ---*w(1)*xx(3) *z(w(1),tt,-1) + ----*w(1)
269          2          100
270      *xx(4)*xx(2)*xx(1)*z(w(1),tt,-10)
271          1
272      + ----*w(1)*xx(4)*xx(2)*xx(1)*z(w(1),tt,-5)

```

```

273          25
274          1
275      + ---*w(1)*xx(4)*xx(2)*xx(1)*z(w(1),tt,-2)
276          4
277          1
278      + ---*w(1)*xx(4)*xx(3)*xx(2)*z(w(1),tt,-2)
279          4
280          1          2
281      + ---*w(1)*xx(4) *xx(3)*z(w(1),tt,-2)
282          8
283          1          2
284      + ---*w(1)*xx(4) *xx(3)*z(w(1),tt,-1))
285          2
286          1          3          4
287 dxx(4)/dt = - ----*xx(2) - ----*xx(3)*xx(2)*xx(1)
288          20          5
289          1          2          6          2
290      - ----*xx(3) *xx(2) - ----*xx(4)*xx(1)
291          4          5
292          3          2          3          2
293      - ----*xx(4)*xx(2) - ----*xx(4)*xx(3)
294          5          4
295          1          3          1          3
296      - ----*xx(4) + sig*(-----*w(1)*xx(2)
297          2          100
298          3
299      + ----*w(1)*xx(3)*xx(2)*xx(1)
300          10
301          1          2
302      + ---*w(1)*xx(3) *xx(2) + w(1)*xx(4)
303          8
304          51          2
305      + ----*w(1)*xx(4)*xx(1)
306          50
307          13          2

```

```

308      + ----*w(1)*xx(4)*xx(2)
309          25
310          5          2      1          3
311      + ---*w(1)*xx(4)*xx(3)  + ---*w(1)*xx(4) ) +
312          8          2
313      2      1          3          1
314  sig * (-----*w(1)*xx(2) *z(w(1),tt,-5) + -----
315          100          100
316      *w(1)*xx(3)*xx(2)*xx(1)*z(w(1),tt,-10)
317          1
318      + ----*w(1)*xx(3)*xx(2)*xx(1)*z(w(1),tt,-5)
319          25
320          1
321      + ---*w(1)*xx(3)*xx(2)*xx(1)*z(w(1),tt,-2)
322          4
323          1          2
324      + ---*w(1)*xx(3) *xx(2)*z(w(1),tt,-2)
325          8
326          1          2
327      + ----*w(1)*xx(4)*xx(1) *z(w(1),tt,-10)
328          50
329          2
330      + w(1)*xx(4)*xx(1) *z(w(1),tt,-1)
331          1          2
332      + ----*w(1)*xx(4)*xx(2) *z(w(1),tt,-5)
333          50
334          1          2
335      + ---*w(1)*xx(4)*xx(2) *z(w(1),tt,-1)
336          2
337          1          2
338      + ---*w(1)*xx(4)*xx(3) *z(w(1),tt,-2)
339          8
340          1          2
341      + ---*w(1)*xx(4)*xx(3) *z(w(1),tt,-1)
342          2

```

```

343          1          3
344      + ---*w(1)*xx(4) *z(w(1),tt,-1))
345          2
346
347 Sometimes we only want the mean drift effects from the
348 quadratic noises. Extract the mean drift and neglect
349 other sig^2 effects.
350          3          2          3          2
351 dxx(1)/dt = - xx(1) - xx(2) *xx(1) - ----*xx(3)*xx(2)
352                      10
353          7          2          2
354      - ----*xx(3) *xx(1) - ---*xx(4)*xx(3)*xx(2)
355          10          5
356          3          2
357      - ---*xx(4) *xx(1) + sig*(w(1)*xx(1)
358          5
359          3          3          2
360      + w(1)*xx(1) + ---*w(1)*xx(2) *xx(1)
361                      4
362          27          2
363      + -----*w(1)*xx(3)*xx(2)
364          200
365          27          2
366      + ----*w(1)*xx(3) *xx(1)
367          50
368          3
369      + ----*w(1)*xx(4)*xx(3)*xx(2)
370          20
371          51          2          2
372      + ----*w(1)*xx(4) *xx(1)) + sig *(
373          100
374          1          3          3          2
375      ----*xx(1) + ----*xx(2) *xx(1)
376          2          8
377          27          2          27          2

```

```

378          + -----*xx(3)*xx(2)  + -----*xx(3) *xx(1)
379              400                      100
380              3
381          + ----*xx(4)*xx(3)*xx(2)
382              40
383              51          2
384          + -----*xx(4) *xx(1))
385              200
386              2      11      3
387 dxx(2)/dt = - 2*xx(2)*xx(1) - ----*xx(2)
388                      20
389              6              4      2
390          - ----*xx(3)*xx(2)*xx(1) - ----*xx(3) *xx(2)
391              5              5
392              3              2      4
393          - ----*xx(4)*xx(2) - ----*xx(4)*xx(3)*xx(1)
394              20              5
395              1              2      3      2
396          - ----*xx(4)*xx(3) - ----*xx(4) *xx(2) + sig*(
397              4              5
398              3              2
399          w(1)*xx(2) + ----*w(1)*xx(2)*xx(1)
400                      2
401              51              3
402          + -----*w(1)*xx(2)
403              100
404              27
405          + ----*w(1)*xx(3)*xx(2)*xx(1)
406              50
407              63          2
408          + -----*w(1)*xx(3) *xx(2)
409              100
410              3              2
411          + -----*w(1)*xx(4)*xx(2)
412              100

```

$$\begin{aligned}
& + \frac{3}{10} w(1) xx(4) xx(3) xx(1) \\
& + \frac{1}{8} w(1) xx(4) xx(3) \\
& + \frac{13}{25} w(1) xx(4) xx(2) + \frac{2}{25} \text{sig} * (\\
& \frac{3}{4} xx(2) xx(1) + \frac{51}{200} xx(2) \\
& + \frac{27}{100} xx(3) xx(2) xx(1) \\
& + \frac{63}{200} xx(3) xx(2) + \frac{3}{200} xx(4) xx(2) \\
& + \frac{3}{20} xx(4) xx(3) xx(1) + \frac{1}{16} xx(4) xx(3) \\
& + \frac{13}{50} xx(4) xx(2) \\
& dx(3)/dt = - \frac{3}{5} xx(2) xx(1) - \frac{7}{5} xx(3) xx(1) \\
& - \frac{4}{5} xx(3) xx(2) - \frac{1}{2} xx(3) \\
& - \frac{4}{5} xx(4) xx(2) xx(1) - \frac{1}{2} xx(4) xx(3) xx(2) \\
& - \frac{3}{2} xx(4) xx(3) + \text{sig} * (
\end{aligned}$$

```

448      4
449      27      2
450      -----*w(1)*xx(2) *xx(1) + w(1)*xx(3)
451      100
452      27      2
453      + ----*w(1)*xx(3)*xx(1)
454      25
455      63      2      1      3
456      + -----*w(1)*xx(3)*xx(2) + ---*w(1)*xx(3)
457      100      2
458      3
459      + ----*w(1)*xx(4)*xx(2)*xx(1)
460      10
461      1
462      + ---*w(1)*xx(4)*xx(3)*xx(2)
463      4
464      5      2      2
465      + ---*w(1)*xx(4) *xx(3)) + sig *(
466      8
467      27      2      27      2
468      -----*xx(2) *xx(1) + ----*xx(3)*xx(1)
469      200      50
470      63      2      1      3
471      + -----*xx(3)*xx(2) + ---*xx(3)
472      200      4
473      3
474      + ----*xx(4)*xx(2)*xx(1)
475      20
476      1      5      2
477      + ---*xx(4)*xx(3)*xx(2) + ----*xx(4) *xx(3))
478      8      16
479      1      3      4
480 dxx(4)/dt = - ----*xx(2) - ----*xx(3)*xx(2)*xx(1)
481      20      5
482      1      2      6      2

```

```

483 - ----*xx(3) *xx(2) - ----*xx(4)*xx(1)
484      4              5
485      3              2      3              2
486 - ----*xx(4)*xx(2) - ----*xx(4)*xx(3)
487      5              4
488      1              3      1              3
489 - ----*xx(4) + sig*(-----*w(1)*xx(2)
490      2              100
491      3
492 + ----*w(1)*xx(3)*xx(2)*xx(1)
493      10
494      1              2
495 + ----*w(1)*xx(3) *xx(2) + w(1)*xx(4)
496      8
497      51              2
498 + ----*w(1)*xx(4)*xx(1)
499      50
500      13              2
501 + ----*w(1)*xx(4)*xx(2)
502      25
503      5              2      1              3
504 + ----*w(1)*xx(4)*xx(3) + ----*w(1)*xx(4) ) +
505      8              2
506      2      1      3      3
507 sig * (-----*xx(2) + ----*xx(3)*xx(2)*xx(1)
508      200      20
509      1      2      51      2
510 + ----*xx(3) *xx(2) + ----*xx(4)*xx(1)
511      16      100
512      13      2      5      2
513 + ----*xx(4)*xx(2) + ----*xx(4)*xx(3)
514      50      16
515      1      3
516 + ----*xx(4) )
517      4

```