Maths IA - Written 5

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Algebra

To show that V is a subspace of $\mathbf{R}^3$, three conditions need to be shown true:

1) The zero vector must be in V
2) If vectors $\mathbf{u}, \mathbf{v}$ are in V, then $\mathbf{u} + \mathbf{v} \in V$
3) If $\mathbf{u} \in V$, and $c \in \mathbf{R}$, then $c\mathbf{u} \in V$

Given the set $V = \{(x, y, z) \in \mathbf{R}^3 | x - y - z = 0\}$, it is clear that $(0, 0, 0) \in V$, as $(0 - 0 - 0 = 0)$, $\therefore$ the first condition is true. Now if we take two solutions $\mathbf{u}, \mathbf{v}$, then:

$$u_1 - u_2 - u_3 = 0 \text{ and } v_1 - v_2 - v_3 = 0$$

Adding these two equations together, we get:

$$(u_1 + v_1) - (u_2 + v_2) - (u_3 + v_3) = 0$$

As required, and so the second condition is met. Finally, if $c \in \mathbf{R}$, then:

$$cu_1 - cu_2 - cu_3 = c(u_1 - u_2 - u_3) = 0$$

So $c\mathbf{u}$ is a solution. This shows that V satisfies the three defining properties of a subspace.

Calculus

Taking the derivative of the equation given for ΔA with respect to time t yields the following:

$$\Delta A = \frac{1}{\pi}(\pi - 2arccos(\frac{a}{2r}) + \frac{a}{2r}\sqrt{4r^2 - a^2}$$

$$\therefore \frac{d}{dt}\Delta A = \frac{1}{\pi}(\frac{da}{dt}\frac{2}{\sqrt{1 - \frac{a}{2r}^2}} + \frac{a}{2r^2}\frac{da}{dt}\frac{1}{2}(4r^2 - a^2)^{\frac{-1}{2}}(-2a) + \frac{da}{dt}\frac{1}{2r^2}\sqrt{4r^2 - a^2})$$

Which is many things, short sharp and shiny not being ones that spring to mind. Simplifying this expression leads us to:

$$\therefore \frac{d}{dt}\Delta A = \frac{1}{\pi}(\frac{da}{dt}\frac{2}{\sqrt{1 - \frac{a}{2r}^2}} + \frac{da}{dt}(\frac{-2a^2}{4r^2\sqrt{4r^2 - a^2}} + \frac{\sqrt{4r^2 - a^2}}{2r^2}$$

$$= \frac{1}{\pi}(\frac{da}{dt}\frac{2}{\sqrt{1-\frac{a}{2r}^2}} + \frac{da}{dt}\frac{4r^2-2a^2}{2r^2\sqrt{4r^2-a^2}})$$

$$= \frac{1}{\pi r}\frac{da}{dt}(\frac{1}{\sqrt{1-\frac{a}{2r}^2}} + \frac{2r^2-a^2}{r^2\sqrt{4r^2-a^2}})$$

(b)

At the moment of totality, $a = r$ by definition, and it is given that the apparent radius $r = 4$, which means the equation $\frac{d}{dt}\Delta A$ simplifies again significantly to:

$$\frac{d}{dt}\Delta A = \frac{1}{\pi}\frac{da}{dt}(\frac{1}{\sqrt{1-\frac{4}{2\times 4}^2}} + \frac{2\times 4^2-4^2}{4^2\sqrt{4\times 4^2-4^2}})$$

$$= \frac{1}{\pi}\frac{da}{dt}(\frac{3\sqrt{3}}{4})$$

Defining t as hours since the moment when the edges of the sun and the moon first touch, consider how the distance between the centre of each circle is at its greatest at $t = 0$ and $t = 3$, and that at the midpoint of the eclipse, $t = 1.5$, a is 0. This is enough to define $a(t)$, and consequently $a'(t)$ as follows:

$$a(t) = \frac{4r}{3}t - 2r$$

$$a'(t) = \frac{4r}{3}$$

$$a'(t) = \frac{16}{3}$$

Bearing in mind that velocity is negligibly close to constant, for this case.

Which is all the information needed to claim that

$$\Delta A = \frac{4\sqrt{3}}{3\pi}$$