

# Fast Matrix Multiplication Attempts

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# Overview

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# Brent's Equations \*)

Substituting (5.01) in (5.02), equating coefficients, and using the definition of matrix multiplication, gives the set of equations

$$\sum_{p=1}^T \alpha_{ijp} \beta_{kLp} \gamma_{mnp} = \delta_{ni} \delta_{jk} \delta_{Lm} , \quad (5.03)$$

where  $\delta$  is Kronecker's delta. (The subscripts on the  $c_{nm}$  were reversed to increase the symmetry of (5.03).) For the multiplication of  $M \times N$  matrices by  $N \times P$  matrices, (5.03) gives  $(MNP)^2$  equations as  $i, j, k, L, m$ , and  $n$  range over the integers  $1 \leq i, n < M, 1 \leq j, k < N, 1 \leq L, m < P$ .

Set of non-linear equations.

Each equation is a sum of „triples“, products of three coefficients.

Approach is to solve the equations modulo 2 in a first phase

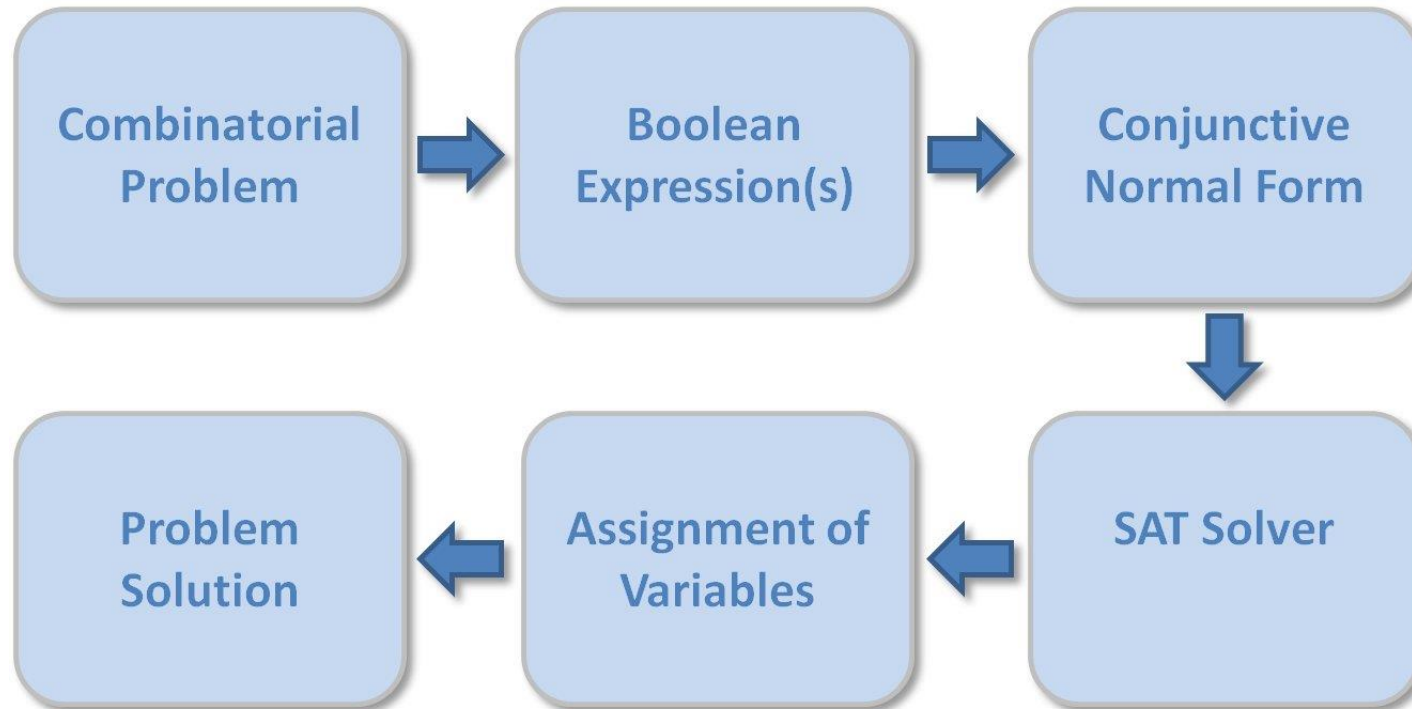
\*) R.P. Brent, Stanford University, 1970 <http://maths-people.anu.edu.au/~brent/pub/pub002.html>

# Attempts to solve Brent's Equations

- Encode in solver-specific format:

CNF	Boolean Satisfiability
SMT	Satisfiability Modulo Theories
ASP	Answer Set Programming
PBS	Pseudo Boolean Satisfaction
MiniZinc	Constraint Programming
Linear Program	Linear Integer Programming
- Add constraints to break symmetries
- Fix certain variables („odd triples“)
- Assume properties of products and triples
- Use 0 as initial value for variables
- Partition set of equations
- Use common subexpressions
- Shuffle constraints and variables

# Solution Flow



# Observations with Brent's Equations

- Small cases can be solved easily:  
2x2x2 with 7 products in 50ms  
2x3x3 with 11 products in 1-2s  
3x3x3 with 27 products in 15s
- Solutions for 3x3x2 and 3x3x3 take excessive runtime
- Quite a few SAT solvers have problems even with small cases
- SAT solvers tend to beat all other solvers tried so far
- All known solutions for 3x3x3 with 23 products have the same 376 „all-zero“ equations
- Product columns of odd triples can be used as symmetry breaking sort criteria
- Composition of EXOR<sub>n</sub> from EXOR<sub>3</sub> yields good results
- Solutions of equations are unambiguous
- Overlap of odd triples is essential

# Brent Equation Explorer

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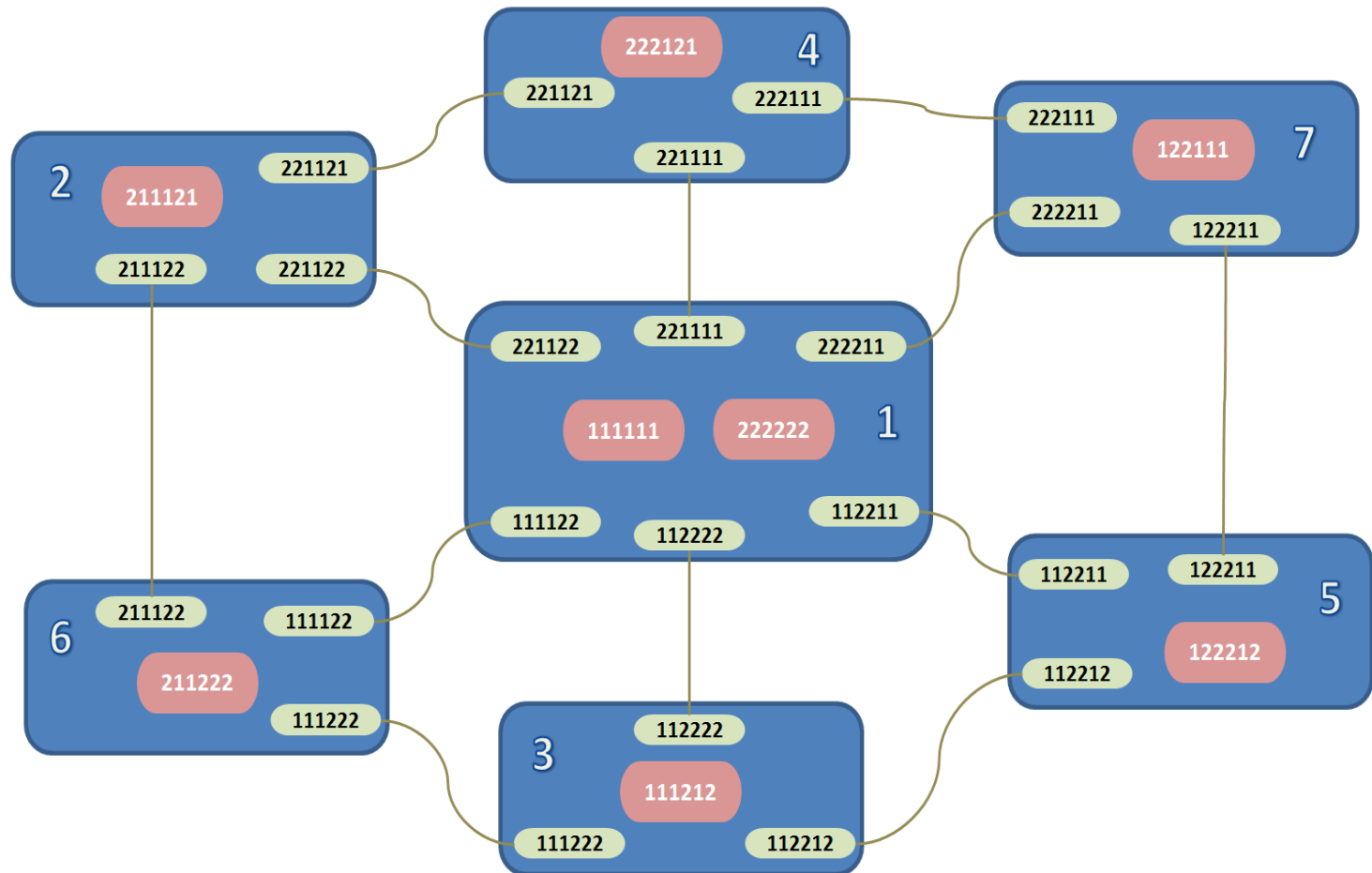
# Encoding of Brent's Equations

- Mod 2: every equation is an XOR or XNOR constraint
- Summands of the equations are AND3 “triples”
- Common AND2 (“dyads”) subexpressions are used
- Tseitin encoding to prevent exponential blow-up
- Big XOR/XNOR expressions are broken down using XOR cut proposed by Gregory V. Bard in his dissertation
- Random selection of triples in “odd” equations set to 1 (one per equation, one or two per product)
- Fruitless experiments with symmetry breaking constraints





# Analysis of Strassen's Algorithm 2x2x2



# Other Results

- Detailed algorithm for  $5 \times 5 \times 5$  with 100 products  
(Makarov showed the feasibility but omitted the algorithm)
- Solution for  $3 \times 3 \times 2$  for  $a_{33} = 0$  with 14 products
- An improved solution for  $3 \times 3 \times 3$  would improve  $5 \times 5 \times 5$
- Implementation of Šýkora's method  
(yields Laderman's solution for  $3 \times 3 \times 3$ )
- Tools to analyze, visualize and convert solutions
- Implemented search algorithm of Jinsoo Oh, et al. \*)
- Tools to convert Boolean circuits to CNF
- Collection of small-factor solutions
- Tried to derive  $3 \times 3 \times 3$  solutions from larger solutions

\*) <http://www.sciencedirect.com/science/article/pii/S0020019013001531>

# Questions

- Is there any hope to find a solution for  $3 \times 3 \times 3$  with 22 or fewer products?
- How to exploit symmetries?
- How did J. Laderman find his solution?

# Contact



Greetings!

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