A Daily Problem

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Mathematic

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目录

1 函数极限与连续

Problem 1. Let $f(x) = \prod_{k=2}^{n} \sqrt[k]{cosx}$, if $\lim_{x \to 0} \frac{1 - f(x)}{x^2} = 10$, Find n.

Solution 1

We use the L'Hospital rule, we get:

$$\lim_{x \to 0} \frac{1 - f(x)}{x^2} = \lim_{x \to 0} \frac{(tanx + tan2x + \cdots + tannx)f(x)}{2x} = \frac{1}{2}\sum_{k=2}^{n} k = \frac{(n-1)(n+2)}{4} = 10$$

thus $n = 6$.

Note: We can use that : if $f(x) = f_1(x)f_2(x)\cdots f_n(x)$, so $f'(x) = \sum_{k=1}^{n} \frac{f_k'(x)}{f_k(x)} f(x)$.

Problem 2. Evaluate the limit:

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \frac{6^k k!}{(2k+1)^k}}{\sum_{k=1}^{n} \frac{3^k k!}{k^k}}.$$

Solution 2

First use the ratio test to show that $\sum_{k=1}^{\infty} \frac{3^k k!}{k^k}$ diverges. Now you can apply the Stolz–Cesàro lemma. The limit should be $\frac{1}{\sqrt{e}}$.

2 一元函数微积分

2.1 一元函数微分学

Problem 1. 设 $f(x)$ 在 $(1, +\infty)$ 上连续可微, 且存在 $L > 0$ 使得对 $\forall x, y \in (0, +\infty)$ 都有

$$| f'(x) - f'(y) | < L | x - y |$$

证明: $(f'(x))^2 < 2Lf(x)$.

Solution Waiting...

Problem 2. 已知$\sinh x = x\cosh y$, $x, y \in (0,1)$, 证明$y < x < 2y$.

Solution 2

由拉格朗日中值定理:

$$\sinh x - \sinh 0 = x\cosh\xi, \xi \in (0,x)$$

又$\sinh x = x\cosh y$, 有:

$$x\cosh y = x\cosh\xi, x \in (0,1)$$

$$\cosh y = \cosh\xi, x \in (0,1)$$

又$\cosh x$在$(0,1)$上递增, 所以有$y = \xi < x$.

由$\cosh x = \dfrac{e^x + e^{-x}}{2} > \dfrac{2\sqrt{e^x \cdot e^{-x}}}{2} = 1$, 所以有:

$$\int_0^x \cosh x \mathrm{d}x > \int_0^x 1\mathrm{d}x = x$$

用$\dfrac{x}{2}$代替x可得到$\sinh\dfrac{x}{2} > \dfrac{x}{2}$, 两边同时乘上一个$\cosh x$, 得:

$$\cosh x \sinh x > \frac{x}{2}\cosh x$$

又$\sinh x = 2\sinh\frac{x}{2}\cosh\frac{x}{2}$, 有:

$$\sinh x > x\cosh\frac{x}{2}$$

又$\sinh x = x\cosh y$, 有

$$x\cosh y > x\cosh\frac{x}{2}, x \in (0,1)$$

即

$$\cosh y > \cosh\frac{x}{2}$$

又$\cosh x$在$(0,1)$上严格单增, 所以有$y > 2x$.
综上, $y < x < 2y$.

2.2 一元函数积分学

Problem 1. Suppose $f : [0,1] \to \mathbb{R}$ is continuous in $[0,1]$ and differentiable in $(0,1)$. Supoppse $f(0) = 0$ and $0 < f'(x) \leq 1$ for all $x \in (0,1)$.

(a)Prove that the function $\Phi(x) = (\int_0^x f(t)dt)^2 - \int_0^x f^3(t)dt$ is monotone.

(b)Find all functions f such that

$$(\int_0^1 f(t)dt)^2 = \int_0^1 f^3(t)dt.$$

Solution 1

(a)

(b) For $x > 0$, $f(x) = \int_0^x f'(t)dt > 0$.

We can get $\Phi(0) = \Phi(1) = 0$, so $\Phi(x) = 0 \implies \Phi'(x) = 0 \implies 2\int_0^x f(t)dt = f^2(x)$

$\implies f'(x) = 1 \implies f(x) = x + C, forC \in \mathbb{R}.$

Problem 2. 若函数 f 在 $[a,b]$ 上单调增加, 证明:

$$\int_a^b xf(x)\mathrm{d}x \geq \frac{a+b}{2}\int_a^b f(x)\mathrm{d}x.$$

Solution 2

- **解法一:** 令 $F(x) = \int_a^x tf(t)\mathrm{d}t - \frac{a+b}{2}\int_a^x f(t)\mathrm{d}t$. 则$F(a) = 0$, 于是有

$$\begin{aligned}
F'(x) &= \frac{1}{2}\Big[(x-a)f(x) - \int_a^x f(x)\mathrm{d}x\Big] \\
&= \frac{1}{2}\Big[f(x)\int_a^x \mathrm{d}t - \int_a^x f(t)\mathrm{d}t\Big] \\
&= \frac{1}{2}\int_a^x \Big[f(x) - f(t)\Big]\mathrm{d}t
\end{aligned}$$

因为 f 在$[a,b]$上单调增加, $f(x)-f(t)-x\geq t$, 所以$f(x)-f(t)\geq 0$, 所以$F'(x)\geq 0$, 即 $F(x)$单调增加, 所以

$$F(x)\geq F(0)=0(x\in[a,b])$$

从而有$F(b)\geq 0$, 即原不等式成立.

- **解法二:** 由于f在$[a,b]$上单调增加, 从而对$\forall x,y\in[a,b]$恒有

$$(f(x)-f(\frac{a+b}{2}))(x-\frac{a+b}{2})\geq 0$$

即:

$$xf(x)-\frac{a+b}{2}f(x)-xf(\frac{a+b}{2})+\frac{a+b}{2}f(\frac{a+b}{2})\geq 0$$

对两边积分可得:

$$\int_a^b xf(x)\mathrm{d}x-\frac{a+b}{2}\int_a^b f(x)\mathrm{d}x-f(\frac{a+b}{2})\int_a^b x\mathrm{d}x-\frac{a+b}{2}f(\frac{a+b}{2})\int_a^b \mathrm{d}x\geq 0$$

$$\int_a^b xf(x)\mathrm{d}x-\frac{a+b}{2}\int_a^b f(x)\mathrm{d}x\geq 0$$

即原不等式成立.

- **解法三(积分第二中值定理):** 将不等式的右边移到左边, 然后用积分第二中值定理变形即可.

Problem 3. 设函数$f(x)$为区间$[a,b]$上的正值连续函数, 且单调递减, 证明:

$$\frac{\int_0^1 xf^2(x)\mathrm{d}x}{\int_0^1 xf(x)\mathrm{d}x}\leq\frac{\int_0^1 f^2(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}.$$

Solution 3

- **解法一:** 由于函数$f(x)$为区间$[0,1]$上的正值连续函数, 则

$$\int_0^1 f(x)\mathrm{d}x>0,\int_0^1 xf(x)\mathrm{d}x>0$$

这样只需证明

$$\int_0^1 f^2(x)\mathrm{d}x \int_0^1 xf(x)\mathrm{d}x - \int_0^1 f(x)\mathrm{d}x \int_0^1 xf^2(x)\mathrm{d}x \geq 0,$$

即证

$$\int_0^1 f^2(x)\mathrm{d}x \int_0^1 yf(y)\mathrm{d}y - \int_0^1 f(x)\mathrm{d}x \int_0^1 yf^2(y)\mathrm{d}y \geq 0,$$

亦即证

$$\int_0^1 \int_0^1 yf(x)f(y)[f(x)-f(y)]\mathrm{d}x\mathrm{d}y \geq 0.$$

考虑二重积分

$$I = \int_0^1 \int_0^1 f(x)f(y)(y-x)[f(x)-f(y)]\mathrm{d}x\mathrm{d}y.$$

因为函数$f(x)$在区间$[0,1]$上单调减少, 则对于$\forall x, y \in [0,1]$有

$$(y-x)[f(x)-f(y)] \geq 0.$$

又函数$f(x)$为区间$[0,1]$上的正值函数, 则由二重积分的保号性知$I \geq 0$, 又

$$I = \int_0^1 \int_0^1 f(x)f(y)y[f(x)-f(y)]\mathrm{d}x\mathrm{d}y - \int_0^1 \int_0^1 f(x)f(y)x[f(x)-f(y)]\mathrm{d}x\mathrm{d}y,$$

将上式右边第二项中的x,y对调, 可得

$$\begin{aligned} I &= \int_0^1 \int_0^1 f(x)f(y)y[f(x)-f(y)]\mathrm{d}x\mathrm{d}y - \int_0^1 \int_0^1 f(x)f(y)y[f(y)-f(x)]\mathrm{d}x\mathrm{d}y \\ &= 2\int_0^1 \int_0^1 f(x)f(y)y[f(x)-f(y)]\mathrm{d}x\mathrm{d}y. \end{aligned}$$

则由$I \geq 0$知原不等式成立.

- **解法二:** 令

$$F(t) = \int_0^1 f^2(x)\mathrm{d}x \int_0^1 xf(x)\mathrm{d}x - \int_0^1 f(x)\mathrm{d}x \int_0^1 xf^2(x)\mathrm{d}x, 0 \leq t \leq 1.$$

则

$$
\begin{aligned}
F'(t) &= f^2(t)\int_0^1 xf(x)\mathrm{d}x + tf(t)\int_0^1 f^2(x)\mathrm{d}x - f(t)\int_0^1 xf^2(x)\mathrm{d}x - tf^2(t)\int_0^1 f(x)\mathrm{d}x \\
&= f(t)\int_0^1 (x-t)[f(t)-f(x)]f(x)\mathrm{d}x
\end{aligned}
$$

因$f(x)$为区间$[0,1]$上的正值单调减函数, 有$(x-t)[f(t)-f(x)] \geq 0$, 由积分的保号性知$F'(t) \geq 0$, 即$F(t)$在区间$[0,1]$上单调增加, 即$F(1) \geq F(0) = 0$, 即原不等式成立.

Problem 4. 设$f(x)$在$[0,1]$上连续且递减, 证明:

$$
\text{当}0 < \lambda < 1\text{时}, \int_0^\lambda f(x)\mathrm{d}x \geq \lambda \int_0^1 f(x)\mathrm{d}x.
$$

Solution 4

- **解法一:** 令$F(x) = \int_0^x f(t)\mathrm{d}t - x\int_0^1 f(t)\mathrm{d}t, x \in (0,1)$, 则$F(0) = 0, F(1) = 0$.
  有
  $$
  \begin{aligned}
  F'(x) &= f(x) - \int_0^1 f(t)\mathrm{d}t \\
  &= f(x) - f(\phi), \phi \in (0,1)
  \end{aligned}
  $$
  于是当$0 < x < \phi$时, $F'(x) = f(x) - f(\phi) > 0$, 当$\phi < x < 1$时, $F'(x) = f(x) - f(\phi) < 0$. 即$F(x)$在$(0,\phi)$内单调增加, 在$(\phi,1)$内单调减少, 所以$F(x) \geq minF(0), F(1) = 0$. 即
  $$
  \int_0^x f(t)\mathrm{d}t - x\int_0^1 f(t)\mathrm{d}t \geq 0, x \in (0,1)\,.
  $$
  得证.

- **解法二:** 对$\int_0^\lambda f(x)\mathrm{d}x$, 令$x = \lambda t$, 则原不等式可化为
  $$
  \int_0^\lambda f(x)\mathrm{d}x = \int_0^1 f(\lambda t)\mathrm{d}\lambda t \geq \lambda \int_0^1 f(x)\mathrm{d}x.
  $$

即证

$$\lambda \int_0^1 f(\lambda t)\mathrm{d}t \geq \lambda \int_0^1 f(x)\mathrm{d}x$$

即证$f(\lambda t) \geq f(t)$, 因为$0 < \lambda < 1$, 所以$\lambda t < t$, 又$f(x)$单调减少, 故结论成立.

Problem 5. 计算$\int_0^{2\pi} \frac{\sin(2n+1)x}{\sin x}\mathrm{d}x$.

Solution 5

由欧拉公式$e^{ix} = \cos x + i\sin x$, 则$e^{-ix} = \cos x - i\sin x$. 由以上两个公式, 相减可得

$$sinx = \frac{e^{ix} - e^{-ix}}{2} \tag{1}$$

因此, $\sin(2n+1) = \dfrac{e^{(2n+1)ix} - e^{-(2n+1)ix}}{2}$. 考虑到被积函数为$\dfrac{\sin(2n+1)}{\sin x}$, 所以

$$\begin{aligned}\frac{\sin(2n+1)x}{\sin x} &= \frac{\frac{e^{(2n+1)ix} - e^{-(2n+1)ix}}{2}}{\frac{e^{ix} - e^{-ix}}{2}} \\ &= \frac{e^{(2n+1)ix - e^{-(2n+1)ix}}}{e^{ix} - e^{-ix}}\end{aligned}$$

利用恒等式

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

所以$e^{(2n+1)ix} - e^{-(2n+1)ix}$可以展开为:

$$(e^{ix} - e^{-ix})(e^{2nix} + e^{(2n-1)ix} \cdot e^{-ix} + \cdots + e^{-2nix}).$$

因此, 原积分可以化为:

$$\begin{aligned}&\int_0^{2\pi} \frac{\sin(2n+1)x}{\sin x}\mathrm{d}x \\ &= \int_0^{2\pi} \left(e^{2nix} + e^{(2n-1)ix} \cdot e^{-ix} + \cdots + e^{-2nix}\right)\mathrm{d}x \\ &= 2\pi\end{aligned}$$

Problem 6. 证明:$\displaystyle\int_1^a f(x^2 + \frac{a^2}{x^2})\frac{dx}{x} = \int_1^a f(x + \frac{a^2}{x})\frac{dx}{x}$.

Solution 6

$$\begin{aligned}
\int_1^a f(x^2+\frac{a^2}{x^2})\frac{dx}{x} &= \int_1^{\sqrt{a}} f(x^2+\frac{a^2}{x^2})\frac{dx}{x} + \int_{\sqrt{a}}^1 f(x^2+\frac{a^2}{x^2})\frac{dx}{x} \\
&\xRightarrow{u=\frac{a}{x}} \int_1^{\sqrt{a}} f(x^2+\frac{a^2}{x^2})\frac{dx}{x} + \int_1^{\sqrt{a}} f(u^2+\frac{a^2}{u^2})\frac{du}{u} \\
&= 2\int_1^{\sqrt{a}} f(x^2+\frac{a^2}{x^2})\frac{dx}{x} \\
&\xRightarrow{t=x^2} \int_1^a f(t+\frac{a^2}{t})\frac{dt}{t} \\
&= \int_1^a f(x+\frac{a^2}{x})\frac{dx}{x}
\end{aligned}$$

Problem 7. 计算 $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x$.

Solution 7

- **解法一(含参积分):** 考虑积分$I(\alpha) = \displaystyle\int_0^1 \frac{\ln(1+\alpha x)}{1+x^2}\mathrm{d}x$.
- **解法二(换元法):** 令$x = \tan t$, 则$dx = \sec^2 t\mathrm{d}t$.

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(1+\tan t)\mathrm{d}t$$

又

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\mathrm{d}x &\xRightarrow{t=\frac{\pi}{4}-x} \int_0^{\frac{\pi}{4}} \ln(1+\tan(\frac{\pi}{4}-t))\mathrm{d}t \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\frac{1-\tan x}{1+\tan x})\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\frac{2}{1+\tan x})\mathrm{d}x
\end{aligned}$$

所以有

$$
\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x &= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\frac{2}{1+\tan x}\mathrm{d}x)\right] \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\mathrm{d}x \\
&= \frac{\pi}{8}\ln 2
\end{aligned}
$$

3 多元向量代数与空间解析几何

4 多元微积分

4.1 多元函数微分学

4.2 重积分

4.3 曲线积分与曲面积分

5 无穷级数

Problem 1. 设 $(\lambda_n)_{n=1,2,\cdots}$ 是严格单调递增趋于无穷大的正数列。证明：若级数 $\sum\limits_{n=1}^{+\infty}\lambda_n a_n$ 收敛，则 $\sum\limits_{n=1}^{+\infty}a_n$ 收敛。

Solution 1.

- **解法一(构造法):** 我们令 $A_n = \sum\limits_{k=1}^{n}\lambda_k a_k$, $B_n = \sum\limits_{k=1}^{n} a_k$, $A_0 = 0$, 其中$n \geq 1$, 则有:

$$
a_k = \frac{A_k - A_{k-1}}{\lambda_k}(k \geq 1)
$$

$$
\begin{aligned}
B_n &= \sum_{k=1}^{n} a_k \\
&= \sum_{k=1}^{n} \frac{A_k - A_{k-1}}{\lambda_k} \\
&= \sum_{k=1}^{n-1} (\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}) A_k + \frac{A_n}{\lambda_n}
\end{aligned}
$$

用为 $\sum\limits_{n=1}^{+\infty}\lambda_n a_n$ 收敛, 所以A_n有界, 又:

$$
\sum_{n=1}^{+\infty}(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}) = \frac{1}{\lambda_1}
$$

所以级数$\sum\limits_{n=1}^{\infty}(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}})A_n$收敛. 由已知条件可知$\lambda_n \to +\infty(n \to \infty)$, 我们可以得到:

$$
\lim_{n\to\infty} \frac{A_n}{\lambda_n} = 0
$$

所以

$$
\sum_{n=1}^{+\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} (\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}) A_k + \frac{A_n}{\lambda_n} \quad = \sum_{n=1}^{+\infty} (\frac{1}{\lambda_n} - \frac{1}{\lambda n + 1}) A_n
$$

故级数$\sum\limits_{n=1}^{+\infty}a_n$收敛, 得证.

- **解法二(Abel判别法):** 我们令$A_n = \frac{1}{\lambda_n}(n \geq 1)$, 由题意可知数列$A_n$单调递减且有界, 又级数$\sum\limits_{n=1}^{+\infty}$收敛, 所以级数

$$
\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \lambda_n a_n \cdot A_n
$$

收敛.

Problem 2. 求级数 $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n\cdot\sqrt[n]{n}}(\frac{x}{2x+1})^n$ 的收敛域.

Solution 2.

令 $t=\frac{x}{2x+1}$,考察 $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n\cdot\sqrt[n]{n}}t^n$ 的收敛域. 于是我们由根值判别法可得:

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{n\cdot\sqrt[a]{n}}\right|} &= \lim_{n\to\infty}\frac{1}{n^{\frac{1}{n}}\cdot n^{\frac{1}{n^2}}} \\
&= \lim_{n\to\infty}\frac{1}{n^{\frac{1}{n}+\frac{1}{n^2}}} \\
&= \exp(\lim_{n\to\infty}-(\frac{1}{n}+\frac{1}{n^2})\ln n) \\
&= 1
\end{aligned}$$

当 $x=-1$ 时, $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n\cdot\sqrt[n]{n}}(-1)^n=\sum_{n=1}^{n\to\infty}\frac{1}{n\cdot\sqrt[n]{n}}$, 而 $\lim_{n\to\infty}\frac{\frac{1}{n\cdot\sqrt[n]{n}}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}=1$, 从而该级数发散;

类似可以得到当 $x=1$ 时, $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n\cdot\sqrt[n]{n}}\cdot(1)^n=\sum_{n=1}^{+\infty}\frac{(-1)^n}{n\cdot\sqrt[n]{n}}$ 收敛.

所以级数 $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n\cdot\sqrt[n]{n}}t^n$ 的收敛域为 $-1<x\le 1$. 从而有 $-1<\frac{x}{2x+1}\le 1$, 解该不等式可得 $x\le -1$ 或 $x>-\frac{1}{3}$. 所以原级数的收敛域为: $(-\infty,-1]\cup(-\frac{1}{3},+\infty)$.

Problem 3. 设 $f(x)$ 为周期为 2π 的连续函数, 令

$$F(x)=\frac{1}{2h}\int_{x-h}^{x+h}f(t)dt.$$

设 $f(x)$ 的Fourier系数为 $a_0,a_n,b_n(n=1,2,\cdots)$, 试求 $F(x)$ 的Fourier系数 $A_0,A_n,B_n(n=1,2,\ldots)$.

Solution 3

Problem 4. 求 $\sum_{n=1}^{10^9} n^{-\frac{2}{3}}$ 的整数部分.

Solution 4

记 $I=\sum_{n=1}^{10^9} n^{-\frac{2}{3}}$, 则由积分与级数之间的关系有:

$$I > \int_1^{10^9+1} x^{-\frac{2}{3}} dx = 3(\sqrt[3]{1+10^9}-1) > 3(10^3-1).$$

$$\begin{aligned} I-1=\sum_2^{10^9} n^{-\frac{2}{3}} dx &< \int_1^{10^9} x^{-\frac{2}{3}} dx \\ &= 3x^{\frac{1}{3}}\Big|_1^{10^9} = 3(10^3-1). \end{aligned}$$

即 $I<3(10^3-1)+1$, 所以 $[I]=3(10^3-1)=2997.$

6 微分方程

7 Linear Algebra

8 真题

9 Others

Problem 1. 证明Cauthy-Schwarz不等式:

$$|< \boldsymbol{a}, \boldsymbol{b} >| \leq \| \boldsymbol{a} \| \| \boldsymbol{b} \|$$

其中 $< \cdot, \cdot >$ 为内积运算.

Solution Cauthy-Schwarz不等式的另一种形式:

$$\sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$$

我们令:

$$A = \sum_{k=1}^{n} a_k^2, B = \sum_{k=1}^{n} b_k^2, C = \sum_{k=1}^{n} a_k b_k$$

我们只需证明:

$$C^2 \leq AB$$

考虑到: $(a+b)^2 \geq 0$, 有:

$$0 \leq \sum_{k=0}^{n} (a_k + t b_k)^2 = A + 2tC + t^2 B$$

当 $B = 0$ 时, 显然 $C = AB$. 当 $B \neq 0$ 时, 由 $\Delta < 0$, 可得:

$$4C^2 - 4AB < 0$$

$$C^2 < AB$$

综上所述:

$$C^2 \leq AB$$

得证.

Problem 2. a) Suppose an entire function f is bounded by M along $|z| = R$. Show that the coefficients C_k in its power series expansion about 0 satisfy

$$|C_k| \leq \frac{M}{R^k}.$$

b) Suppose a polynomial is bounded by 1 in the unit disc. Show that all its coefficients are bounded by 1.

Solution Part a): Since f is an entire function it can be expressed as an infinite power series, i.e.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} C_k z^k.$$

If we recall Cauchy's Integral we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, dw,$$

carefully notice that $\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}}$ can be written as a geometric series. We have

$$\begin{aligned}
\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, dw &= \frac{1}{2\pi i} \int_\gamma \left\{ \frac{f(w)}{w} \cdot \left(\frac{1}{1 - \frac{z}{w}} \right) \right\} \, dw \\
&= \frac{1}{2\pi i} \int_\gamma \left\{ \frac{f(w)}{w} \cdot \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \frac{z^3}{w^3} + \cdots \right) \right\} \, dw \\
&= \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w} \, dw \right) z^0 + \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^2} \, dw \right) z^1 + \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^3} \, dw \right) z^2 \cdots
\end{aligned}$$

Now take the modulus of C_k to get

$$|C_k| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^{k+1}} \, dw \right| \leq \frac{1}{2\pi} \int_\gamma \frac{|f(w)|}{|w^{k+1}|} \, |dw| \leq \frac{M}{2\pi} \int_\gamma \frac{|dw|}{|w^{k+1}|}$$

Then integrate along $\gamma(\theta) = Re^{i\theta}$ for $\theta \in [0, 2\pi]$ to get

$$|C_k| \leq \frac{M}{2\pi} \int_0^{2\pi} \frac{|iRe^{i\theta}\ d\theta|}{|R^{k+1}e^{ik\theta}|} = \frac{M}{2\pi \cdot R^k} \int_0^{2\pi}\ d\theta = \frac{M}{R^k}.$$

Hence, $|C_k| \leq \frac{M}{R^k}$.