A Daily Problem

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Mathematic

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目录

1 函数极限与连续

Problem 1. Let $f(x) = \prod_{k=2}^{n} \sqrt[k]{cosx}$, if $\lim_{x\to 0} \frac{1-f(x)}{x^2} = 10$, Find n.

Solution 1

We use the L'Hospital rule, we get:

$$\lim_{x\to 0} \frac{1-f(x)}{x^2} = \lim_{x\to 0} \frac{(tanx + tan2x + \cdots + tannx)f(x)}{2x} = \frac{1}{2}\sum_{k=2}^{n} k = \frac{(n-1)(n+2)}{4} = 10$$

thus $n = 6$.

Note: We can use that : if $f(x) = f_1(x)f_2(x)\cdots f_n(x)$, so $f'(x) = \sum_{k=1}^{n} \frac{f_k'(x)}{f_k(x)} f(x)$.

Problem 2. Evaluate the limit:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{6^k k!}{(2k+1)^k}}{\sum_{k=1}^{n} \frac{3^k k!}{k^k}}.$$

Solution 2

First use the ratio test to show that $\sum_{k=1}^{\infty} \frac{3^k k!}{k^k}$ diverges. Now you can apply the Stolz–Cesàro lemma. The limit should be $\frac{1}{\sqrt{e}}$.

2 一元函数微积分

2.1 一元函数微分学

Problem 3. 设 $f(x)$ 在 $(1, +\infty)$ 上连续可微, 且存在 $L > 0$ 使得对 $\forall x, y \in (0, +\infty)$ 都有

$$| f'(x) - f'(y) | < L | x - y |$$

证明: $(f'(x))^2 < 2Lf(x)$.

Solution Waiting...

2.2 一元函数积分学

Problem 3. Suppose $f : [0,1] \to \mathbb{R}$ is continuous in $[0,1]$ and differentiable in $(0,1)$. Supoppse $f(0) = 0$ and $0 < f'(x) \leq 1$ for all $x \in (0,1)$.

(a)Prove that the function $\Phi(x) = (\int_0^x f(t)dt)^2 - \int_0^x f^3(t)dt$ is monotone.

(b)Find all functions f such that

$$(\int_0^1 f(t)dt)^2 = \int_0^1 f^3(t)dt.$$

Solution 1

(a)

(b) For $x > 0$, $f(x) = \int_0^x f'(t)dt > 0$.
We can get $\Phi(0) = \Phi(1) = 0$, so $\Phi(x) = 0 \implies \Phi'(x) = 0 \implies 2\int_0^x f(t)dt = f^2(x)$
$\implies f'(x) = 1 \implies f(x) = x + C, forC \in \mathbb{R}.$

3 多元向量代数与空间解析几何

4 多元微积分

4.1 多元函数微分学

4.2 重积分

4.3 曲线积分与曲面积分

5 无穷级数

Problem 1. 设 $(\lambda_n)_{n=1,2,\cdots}$ 是严格单调递增趋于无穷大的正数列。证明：若级数 $\sum_{n=1}^{+\infty}\lambda_n a_n$

收敛，则 $\sum\limits_{n=1}^{+\infty} a_n$ 收敛。

Solution 1.

- **解法一(构造法):** 我们令 $A_n = \sum\limits_{k=1}^{n} \lambda_k a_k$, $B_n = \sum\limits_{k=1}^{n} a_k$, $A_0 = 0$, 其中$n \geq 1$, 则有:

$$a_k = \frac{A_k - A_{k-1}}{\lambda_k}(k \geq 1)$$

$$\begin{aligned} B_n &= \sum_{k=1}^{n} a_k \\ &= \sum_{k=1}^{n} \frac{A_k - A_{k-1}}{\lambda_k} \\ &= \sum_{k=1}^{n-1} (\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}) A_k + \frac{A_n}{\lambda_n} \end{aligned}$$

用为 $\sum\limits_{n=1}^{+\infty} \lambda_n a_n$ 收敛, 所以A_n有界, 又:

$$\sum_{n=1}^{+\infty} (\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}) = \frac{1}{\lambda_1}$$

所以级数$\sum\limits_{n=1}^{\infty} (\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}) A_n$收敛. 由已知条件可知$\lambda_n \to +\infty (n \to \infty)$, 我们可以得到:

$$\lim_{n\to\infty} \frac{A_n}{\lambda_n} = 0$$

所以

$$\sum_{n=1}^{+\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} (\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}) A_k + \frac{A_n}{\lambda_n} \quad = \sum_{n=1}^{+\infty} (\frac{1}{\lambda_n} - \frac{1}{\lambda n + 1}) A_n$$

故级数$\sum\limits_{n=1}^{+\infty} a_n$收敛, 得证.

- **解法二(Abel判别法):** 我们令$A_n = \frac{1}{\lambda_n} (n \geq 1)$, 由题意可知数列$A_n$单调递减且有界,

又级数$\sum\limits_{n=1}^{+\infty}$收敛, 所以级数

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \lambda_n a_n \cdot A_n$$

收敛.

Problem 2. 求级数 $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot \sqrt[n]{n}}(\frac{x}{2x+1})^n$ 的收敛域.

Solution 2.

令 $t = \frac{x}{2x+1}$,考察 $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot \sqrt[n]{n}} t^n$ 的收敛域. 于是我们由根值判别法可得:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{n \cdot \sqrt[a]{n}}\right|} &= \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}} \cdot n^{\frac{1}{n^2}}} \\
&= \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}+\frac{1}{n^2}}} \\
&= \exp(\lim_{n\to\infty} -(\frac{1}{n} + \frac{1}{n^2}) \ln n) \\
&= 1
\end{aligned}$$

当 $x = -1$ 时, $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot \sqrt[n]{n}}(-1)^n = \sum\limits_{n=1}^{n\to\infty} \frac{1}{n \cdot \sqrt[n]{n}}$, 而 $\lim\limits_{n\to\infty} \frac{\frac{1}{n \cdot \sqrt[n]{n}}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 1$, 从而该级数发散;

类似可以得到当 $x = 1$ 时, $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot \sqrt[n]{n}} \cdot (1)^n = \sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot \sqrt[n]{n}}$ 收敛.

所以级数 $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot \sqrt[n]{n}} t^n$ 的收敛域为 $-1 < x \leq 1$. 从而有 $-1 < \frac{x}{2x+1} \leq 1$, 解该不等式可得 $x \leq -1$ 或 $x > -\frac{1}{3}$. 所以原级数的收敛域为: $(-\infty, -1] \cup (-\frac{1}{3}, +\infty)$.

Problem 3. 设 $f(x)$ 为周期为 2π 的连续函数, 令

$$F(x)=\frac{1}{2h}\int_{x-h}^{x+h}f(t)dt.$$

设 $f(x)$ 的Fourier系数为 $a_0,a_n,b_n(n=1,2,\cdots)$, 试求 $F(x)$ 的Fourier系数 $A_0,A_n,B_n(n=1,2,\ldots)$.

Solution 3

Problem 4. 求 $\sum\limits_{n=1}^{10^9}n^{-\frac{2}{3}}$ 的整数部分.

Solution 4

记 $I=\sum_{n=1}^{10^9}n^{-\frac{2}{3}}$, 则由积分与级数之间的关系有:

$$I>\int_1^{10^9+1}x^{-\frac{2}{3}}dx=3(\sqrt[3]{1+10^9}-1)>3(10^3-1).$$

$$\begin{aligned}I-1=\sum_{2}^{10^9}n^{-\frac{2}{3}}dx&<\int_1^{10^9}x^{-\frac{2}{3}}dx\\&=3x^{\frac{1}{3}}\Big|_1^{10^9}=3(10^3-1).\end{aligned}$$

即 $I<3(10^3-1)+1$, 所以 $[I]=3(10^3-1)=2997$.

6 微分方程

7 Linear Algebra

8 真题

9 Others

Problem 1. 证明Cauthy-Schwarz不等式:

$$|<\boldsymbol{a},\boldsymbol{b}>|\leq\parallel\boldsymbol{a}\parallel\parallel\boldsymbol{b}\parallel$$

其中 $<\cdot,\cdot>$ 为内积运算.

Solution Cauthy-Schwarz不等式的另一种形式:

$$\sum_{k=1}^{n}a_kb_k\leq\sum_{k=1}^{n}a_k^2\sum_{k=1}^{n}b_k^2$$

我们令:

$$A=\sum_{k=1}^{n}a_k^2,B=\sum_{k=1}^{n}b_k^2,C=\sum_{k=1}^{n}a_kb_k$$

我们只需证明:

$$C^2\leq AB$$

考虑到: $(a+b)^2\geq 0$, 有:

$$0\leq\sum_{k=0}^{n}(a_k+tb_k)^2=A+2tC+t^2B$$

当 $B=0$ 时, 显然 $C=AB$. 当 $B\neq 0$ 时, 由 $\Delta<0$, 可得:

$$4C^2-4AB<0$$

$$C^2<AB$$

综上所述:

$$C^2\leq AB$$

得证.

Problem 6. a) Suppose an entire function f is bounded by M along $|z| = R$. Show that the coefficients C_k in its power series expansion about 0 satisfy

$$|C_k| \leq \frac{M}{R^k}.$$

b) Suppose a polynomial is bounded by 1 in the unit disc. Show that all its coefficients are bounded by 1.

Solution Part a): Since f is an entire function it can be expressed as an infinite power series, i.e.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} C_k z^k.$$

If we recall Cauchy's Integral we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw,$$

carefully notice that $\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}}$ can be written as a geometric series. We have

$$\begin{aligned}
\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw &= \frac{1}{2\pi i} \int_\gamma \left\{ \frac{f(w)}{w} \cdot \left(\frac{1}{1 - \frac{z}{w}} \right) \right\} dw \\
&= \frac{1}{2\pi i} \int_\gamma \left\{ \frac{f(w)}{w} \cdot \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \frac{z^3}{w^3} + \cdots \right) \right\} dw \\
&= \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w}\, dw \right) z^0 + \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^2}\, dw \right) z^1 + \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^3}\, dw \right) z^2 \cdots
\end{aligned}$$

Now take the modulus of C_k to get

$$|C_k| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^{k+1}}\, dw \right| \leq \frac{1}{2\pi} \int_\gamma \frac{|f(w)|}{|w^{k+1}|}\, |dw| \leq \frac{M}{2\pi} \int_\gamma \frac{|dw|}{|w^{k+1}|}$$

Then integrate along $\gamma(\theta) = Re^{i\theta}$ for $\theta \in [0, 2\pi]$ to get

$$|C_k| \leq \frac{M}{2\pi} \int_0^{2\pi} \frac{|iRe^{i\theta}\ d\theta|}{|R^{k+1}e^{ik\theta}|} = \frac{M}{2\pi \cdot R^k} \int_0^{2\pi}\ d\theta = \frac{M}{R^k}.$$

Hence, $|C_k| \leq \frac{M}{R^k}$.