Contents

1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Objectives:

- Perform elementary row operations on a matrix.
- Solve systems of linear equations by using elementary row operations.
- Determine whether a linear system is consistent.

Why study systems of linear equations?

1. They are the central topic of linear algebra.
2. They plays a motivational role for introducing other important concepts of linear algebra.
3. They are one of the basic mathematical models for real world applications.

1.1.1 Basic Terminology

In elementary algebra, we learned how to solve systems of linear equations in 2 or 3 variables by *eliminating variables*. In linear algebra, we generalize this idea to a systematic procedure for solving systems of linear equations in any number of variables.

Definition 1.1: Linear Equation

A **linear equation** in the n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where the **coefficients** a_i and the **constant term** b are real (or complex) numbers. A **solution** of the equation (1) is an ordered n-tuple $(c_1, c_2, \ldots, c_n)$ of numbers that makes the equation a true statement when the values $c_1, \ldots, c_n$ are substituted for $x_1, \ldots, x_n$, respectively.

Definition 1.2: System of Linear Equations

A **system of linear equations** (or a **linear system**) in the n variables $x_1, x_2, \ldots, x_n$ is a collection of one or more linear equations in the same variables. A system of m linear equations in n variables can be written in the form

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases} \tag{2}$$

A **solution** of the system (2) is an ordered n-tuple $(c_1, c_2, \ldots, c_n)$ of numbers that is a solution of each equation in the system. The set of all solutions is called the **solution set** of the linear system. The process of finding all the solutions of a linear system is called **solving the linear system**.

Remark 1. In the system (2), m and n can be any positive integers, so it is possible that $m < n$, $m = n$, or $m > n$.

Definition 1.3: Equivalent Linear Systems

Two linear systems are called **equivalent** if they have the same solution set.

Definition 1.4: Consistent and Inconsistent Linear Systems

A system of linear equations is called **consistent** if it has at least one solutions. Otherwise, the system is called **inconsistent**.

Definition 1.5: Matrix

Let m and n be positive integers. An $m \times n$ **matrix** is a rectangular array of numbers with m rows and n columns

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where each number a_{ij} $(1 \leq m \leq m; 1 \leq j \leq n)$ is called an **entry** (or **element**) of the matrix. An $m \times n$ matrix is said to have the **dimensions** (or **size**) $m \times n$.

Definition 1.6: Augmented Matrix and Coefficient Matrix of a Linear System

For a linear system in m equations and n variables

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_n \end{cases}$$

1. The The $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

   is called the **coefficient matrix** of the system.

2. The $m \times (n+1)$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$$

   is called the **augmented matrix** of the system.

1.1.2 Solving Linear Systems

The basic idea for a solving linear system is to transform the system into a simpler system by eliminating variables systematically.

Example 1.1.1. Solve the following system of linear equations.

$$\begin{cases} \phantom{2x_1 + {}} x_2 + 2x_3 = 1 \\ x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + x_2 + x_3 = 1 \end{cases}$$

Definition 1.7: Elementary Row Operations

An **elementary row operation** on a matrix A is one of the following three operations.

1. **Type 1 (Row Interchange):** Interchange two rows R_i and R_j, denoted by $R_i \leftrightarrow R_j$.
2. **Type 2 (Row Scaling):** Multiply row R_i by a nonzero constant c, denoted by $cR_i \to R_i$.
3. **Type 3 (Row Addition):** Add a multiple of row R_i to row R_j, denoted by $cR_i + R_j \to R_j$.

Definition 1.8: Row Equivalence of Martices

Two matrices A and B are called **row equivalent**, denoted by $A \overset{r}{\sim} B$, if there is a sequence of elementary row operations that transforms A into B.

Proposition 1.1: Basic Properties of Row Equivalence

Let A, B, and C be $m \times n$ matrices.

1. $A \overset{r}{\sim} A$.
2. If $A \overset{r}{\sim} B$, then $B \overset{r}{\sim} A$.
3. If $A \overset{r}{\sim} B$ and $B \overset{r}{\sim} C$, then $A \overset{r}{\sim} C$.

Proposition 1.2

If the augmented matrices of two linear systems are row equivalent, then the two systems are equivalent; that is, they have the same solution set.

Two Fundamental Questions About a Linear System

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists, is it the only one; that is, is the solution unique?

Example 1.1.2. Determine whether the following linear system is consistent.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 + 3x_2 + 4x_3 = 1 \\ 3x_1 + 5x_2 + 7x_3 = 3 \end{cases}$$

1.2 Row Reduction and Echelon Forms

Objectives:

- Convert a matrix to row-echelon form and reduced row-echelon form.
- Solve linear systems by row reduction.

Definition 1.9: Row-Echelon Form and Reduced Row-Echelon Form

A matrix is said to be in **row-echelon form** (**REF**)if it satisfies the following conditions:

1. All zero rows (consisting entirely of zeros) occur below all nonzero rows.
2. The first nonzero entry (called a **leading entry**) of every nonzero row is in a column to the right of the leading entry of the row above it.

A matrix is said to be in **reduced row-echelon form** (**RREF**)if it is in row-echelon form and satisfies the following additional conditions:

3. The leading entry of every nonzero row is 1 (called a **leading 1**).
4. Each leading 1 is the only nonzero entry in its column.

Example 1.2.1. Determine whether each of the following matrices is in row-echelon form or reduced row-echelon form.

(a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Theorem 1.1: Uniqueness of Reduced Row-Echelon Form

Every matrix is row-equivalent to a unique reduced row-echelon form.

Definition 1.10: Pivot Position and Pivot Column

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Remark. The leading entries are always in the same positions in any echelon form obtained from a given matrix.

The Row Reduction Algorithm (Guass-Jordan Elimination)

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, use a Type 1 row operation to interchange rows to move this entry into the pivot position.
3. Use Type 3 row operations to create zeros in all positions below the pivot.
4. Cover the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.
5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a Type 2 row operation.

Remark. The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

Example 1.2.2. Row reduce the following matrix into REF and RREF.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 3 \\ 0 & 1 & 1 & 2 & 4 \end{bmatrix}$$

Definition 1.11: Basic Variables and Free Variables

In a linear system, the variables corresponding to pivot columns in the augmented matrix are called **basic variables**. The other variables are called **free variables**.

Definition 1.12: General Solution

The solution set of a linear system expressed in parametric form with the free variables as parameters is called the **general solution** of the system.

Example 1.2.3. Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 2 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Theorem 1.2: Existence and Uniqueness Theorem

1. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column–that is, if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \qquad (b \neq 0)$$

2. If a linear system is consistent, then the solution set contains either

   (i) a unique solution, when there are no free variables, or

   (ii) infinitely many solutions, when there is at least one free variable.

Procedure for Solving a Linear System Using Row Reduction

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.3 Vector Equations

Objective:

- Write a linear system as a vector equation.
- Write a vector as a linear combination of a set of vectors in $\mathbb{R}^n$.

1.3.1 Vectors in $\mathbb{R}^n$

Vectors in $\mathbb{R}^n$ are a generalization of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Let n be a positive integer. An ordered n-tuple $(a_1, \ldots, a_n)$ of real numbers is called n**-vector**. The set of all n-vectors is called n**-space**, denoted by $\mathbb{R}^n$. Two n-vectors u and v are **equal**, denoted by $u = v$, if they have the equal corresponding components. In practice, we usually write a n-vector as a $n \times 1$ matrix, or a **column vector**:

$$(a_1, \ldots, a_n) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Definition 1.13: Addition and Scalar Multiplication of n-Vectors

Let $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$ and let c be a scalar.

1. The **sum** of u and v is:

$$u + v = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

2. The **scalar product** of u by c is:

$$cu = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$$

The vector $0 = (0, \ldots, 0)$ is called the **zero vector**. For any vector u, the vector $-u = (-1)v$ is called the **opposite** of u. The vector $u - v = u + (-v)$ is called the **difference** of u and v.

Proposition 1.3: Algebraic Properties of $\mathbb{R}^n$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$
4. $\mathbf{v} + (-\mathbf{v}) = -\mathbf{v} + \mathbf{v} = \mathbf{0}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$
7. $c(d\mathbf{v}) = (cd)\mathbf{v}$
8. $1\mathbf{v} = \mathbf{v}$

1.3.2 Linear Combinations

Definition 1.14: Linear Combination

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbb{R}^n$ and let $c_1, c_2, \ldots, c_n$ be scalars. The vector

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

is called a **linear combination** of $v_1, v_2, \ldots, v_p$.

Example 1.3.1. Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$, and $v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. Determine whether v is a linear combination of v_1 and v_2.

Vector Form of a Linear System

A linear system

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_n \end{cases}$$

can be written as a **vector equation**

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$xa_1 + \cdots x_na_n = b$$

where $\begin{bmatrix} a_1 & \cdots & a_n & b \end{bmatrix}$ is the augmented matrix of the system.

Basic Fact

A linear system with augmented matrix $\begin{bmatrix} a_1 & \cdots & a_n & b \end{bmatrix}$ has a solution if and only if the constant term vector b can be written as a linear combination of the column vectors $a_1, \ldots, a_n$ of coefficient matrix of the system.

Definition 1.15: Linear Span

Let $S = \{v_1, v_2, \ldots, v_p\}$ be an indexed set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $v_1, v_2, \ldots, v_p$ is called the **subset of $\mathbb{R}^n$ spanned (or generated)** by S, or the **linear span** of S, denoted by Span (S). That is,

$$\text{Span}\,(S) = \{\, c_1v_1 + c_2v_2 + \cdots + c_nv_n \mid c_1, \ldots, c_n \in \mathbb{R} \,\}$$

Example 1.3.2. Let $v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. For what value(s) of λ is $v = \begin{bmatrix} -1 \\ 1 \\ \lambda \end{bmatrix}$ in Span $\{v_1, v_2, v_3\}$?

1.4 The Matrix Equation $Ax = b$

Objectives:

- Define the product of a matrix and a vector.
- Write a linear system as a matrix equation.

Definition 1.16: Matrix-Vector Product Ax

Let $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$ be an $m \times n$ matrix with columns $a_1, \ldots, a_n$ and let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$. Then **product of A and x** is defined to be

$$Ax = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + \cdots + x_n a_n$$

Example 1.4.1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Find Ax.

Example 1.4.2. Let v_1, v_2, v_3 be vectors in $\mathbb{R}^n$ and $3v_1 + 2v_2 - 5v_3$ as a matrix-vector product.

Row-Vector Rule for Computing Ax

If the product Ax is defined, then the ith entry in Ax is the sum of the products of corresponding entries from row i of A and from the vector x.

Theorem 1.3: Properties of Matrix-Vector Multiplication

Let A be an $m \times n$ matrix, u and v be vectors in $\mathbb{R}^n$, and c be a scalar. Then

1. $A(u+v) = Au + Av$
2. $A(cv) = c(Av)$

Matrix Form of a Linear System

A system of linear equations

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

can be written as a **matrix equation**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$Ax = b$$

where $\begin{bmatrix} A & b \end{bmatrix}$ is the augmented matrix of the system.

Basic Fact

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

One basic question: When does a linear system $Ax = b$ has a solution for all vectors $b \in \mathbb{R}^m$?

Theorem 1.4

Let A be an $m \times n$ matrix. Then the following statements are equivalent.

1. For each $b \in \mathbb{R}^n$, the equation $Ax = b$ has a solution.
2. Each $b \in \mathbb{R}^n$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^n$.
4. A has a pivot position in every row.

1.5 Solution Sets of Linear Systems

Objectives:

- Describe the condition for a homogeneous linear system to have a nontrivial solution.
- Describe the solution set of a nonhomogeneous linear system.

Definition 1.17: Homogeneous and Nonhomogeneous Linear Systems

Consider a linear system $Ax = b$.

1. If $b = 0$, then the system is called **homogeneous**.
2. If $b \neq 0$, then the system is called **nonhomogeneous**.

Remark. Any homogeneous linear system $Ax = 0$ always has at least one solution, namely, $x = 0$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**.

Proposition 1.4

The homogeneous linear system $Ax = 0$ has a nontrivial solution if and only if the system has at least one free variable.

Example 1.5.1. Describe the solution set of the homogeneous linear system in parametric vector form.

$$\begin{cases} x_1 - 2x_2 + x_3 + 3x_4 = 0 \\ 2x_1 - 3x_2 + 3x_3 + 4x_4 = 0 \\ x_1 - 3x_2 \phantom{{}+3x_3} + 5x_4 = 0 \end{cases}$$

Theorem 1.5: Solution Set of a Nonhomogeneous Linear System

Suppose the nonhomogeneous linear system $Ax = b$ is consistent and let x_p be a *particular* solution. Then the solution set of $Ax = b$ consists of all vectors of the form

$$x = x_h + x_p$$

where x_h is any solution of the associated homogeneous system $Ax = 0$.

Remark. Theorem 1.5 says that if $Ax = b$ has a solution, then the solution set is obtained by translating the solution set of $Ax = 0$, using any particular solution x_p of $Ax = b$ for the translation.

Example 1.5.2. Describe the solution set of the linear system in parametric vector form.

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ 4x_1 + 2x_2 + x_3 = 2 \\ 2x_1 + x_2 - x_3 = 1 \end{cases}$$

1.7 Linear Independence

Objective:

- Understand the concepts of linear independence and linear dependence.
- Determine whether a set of vectors in $\mathbb{R}^n$ is linearly independent.

Definition 1.18: Linear Dependence and Linear Independence

An indexed set of vectors $S = \{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is said to be **linearly dependent** if there exist scalars $c_1, \dots, c_p$ not all zero such that

$$c_1 v_1 + \cdots + c_p v_p = 0$$

Otherwise, the set S is said to be **linearly independent**. That is, the set S is linearly independent if and only if

$$c_1 v_1 + \cdots + c_p v_p = 0 \quad \text{implies} \quad c_1 = \cdots = c_p = 0 \tag{3}$$

Remark 1. The above definition is equivalent to saying that the set $S = \{v_1, \dots, v_p\}$ is linearly dependent if and only if the vector equation $x_1 v_1 + \cdots + x_p v_p = 0$ has a nontrivial solution.

Remark 2. The equation (3) is called a **linear dependence relation** among $v_1, \dots, v_p$ when the coefficients c_i are not all zero. For brevity, we may say that $v_1, \dots, v_p$ are linearly dependent (independent) when we mean that $v_1, \dots, v_p$ is a linearly dependent (independent) set.

Example 1.7.1. Determine whether the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ are linearly independent.

Basic Facts

1. A set of a single vector $\{v_1\}$ is linearly dependent if and only if $v_1 \neq 0$.
2. A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.
3. If a set $S = \{v_1, \ldots, v_p\}$ contains the zero vector 0, then S is linearly dependent.

Linear Independence of Matrix Columns

Let $A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ be any $m \times n$ matrix. Then the columns, $v_1, \ldots, v_n$, of A are linearly independent if and only if the homogeneous linear system $Ax = 0$ has only the trivial solution.

Theorem 1.6: Characterization of Linear Dependence

An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors in $\mathbb{R}^n$ is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $\{v_1, \ldots, v_{j-1}\}$.

Theorem 1.7

Any indexed set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$. That is, if a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

Example 1.7.2. Determine whether the vectors $v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are linearly dependent.

1.8 Linear Transformations

Objectives:

- Understand the concept of linear transformation.

1.8.1 Linear Transformations

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted $T : \mathbb{R}^n \to \mathbb{R}^m$, is a rule that assigns to each vector x in R^n a unique vector $T(x)$ in $\mathbb{R}^m$. The set R^n is called the **domain** of T, and R^m is called the **codomain** of T. For $x \in \mathbb{R}^n$, the vector $T(x) \in \mathbb{R}^m$ is called the **image** of x under T. The set of all images $T(x)$ is called the **range** (or **image**) of T .

Definition 1.19: Linear Transformation

A transformation (or mapping) $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$;
2. $T(cu) = cT(u)$ for all $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Basic Properties of Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

1. $T(0) = 0$
2. $T(cu + dv) = cT(u) + dT(v)$ for all $u, v \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

Example 1.8.1. Determine whether each transformation is linear.

1. $T : \mathbb{R}^3 \to \mathbb{R}^3,\ T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \left(\begin{bmatrix} 2x_1 \\ x_2 + x_3 \\ 0 \end{bmatrix}\right)$
2. $T : \mathbb{R}^3 \to \mathbb{R}^3,\ T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 \\ x_1 \\ x_2 + x_3 \end{bmatrix}\right)$

1.8.2 Matrix Transformations

Let A be an $m \times n$ be matrix. Then A induces a transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ as follows

$$T_A(x) = Ax$$

It can be easily shown that T_A is a linear transformation; that is, it preserves the addition and scalar multiplication. The linear transformation T_A is called the **matrix transformation** defined by A.

Example 1.8.2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then the matrix transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the **projection** onto the x_1-axis.

Example 1.8.3. Let $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. Then the matrix transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a **shear transformation**.

1.9 The Matrix of a Linear Transformation

Objective:

- Find the standard matrix of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Determine whether a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is one-to-one or onto.

1.9.1 The Matrix of a Linear Transformations

Theorem 1.8

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation. Then there exists a unique $m \times n$ matrix A such that

$$T(x) = Ax \quad \text{for all } x \in \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(e_j)$, where e_j is the jth column of the identity matrix in R^n:

$$A = [T(e_1) \ \cdots \ T(e_n)] \tag{4}$$

Remark. The matrix A in (4) is called the **standard matrix** for the linear transformation T.

Example 1.9.1. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle θ, with counterclockwise rotation for a positive angle. It can be shown geometrically that such a transformation is linear. Find the standard matrix of R_θ.

1.9.2 One-to-One and Onto Linear Transformations

Definition 1.20: One-to-One Mapping

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **one-to-one** if each $y \in \mathbb{R}^m$ is the image of *at most one* $x \in \mathbb{R}^n$, that is, for any $x_1, x_2 \in \mathbb{R}^n$, $T(x_1) = T(x_2)$ implies $x_1 = x_2$.

Definition 1.21: Onto Mapping

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **onto** if each $y \in \mathbb{R}^m$ is the image of *at least one* $x \in \mathbb{R}^n$, that is, for each $y \in \mathbb{R}^m$, there exists $x \in \mathbb{R}^n$ such that $T(x) = y$.

Example 1.9.2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, -x_1 - 2x_3)$. Show that T is an onto linear transformation. Is T one-to-one?

Theorem 1.9

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.

Theorem 1.10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then

1. T is onto if and only if the columns of A span $\mathbb{R}^m$;
2. T is one-to-one if and only if the columns of A are linearly independent.