Gambler's Ruin Subproblem

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The Gambler's Ruin problem has a few key rules. You start with some amount n of money, and you can go bankrupt and lose at 0 money. Each round you play, you either gain 1 or lose 1. We will look specifically at the case where the probability of winning and losing is equal.

Denote $X_{k,l} : 0 \leq k, l$ as the probability of reaching state l from k in this problem, where the state indexed represents the money you have, i.e. 0 is an absorbing state in the equivalent Markov chain. We trivially deduce:

$$
\begin{aligned}
&X_{0,0} = 1 \\
&X_{1,0} = 0.5 + 0.5X_{2,0} \\
&X_{2,0} = 0.5X_{1,0} + 0.5X_{3,0}
\end{aligned}
$$

Or more generally:

$$
\begin{aligned}
&\forall n \geq 2 : X_{n,0} = 0.5X_{n-1,0} + 0.5X_{n+1,0} \\
\iff &X_{n+1,0} - X_{n,0} = X_{n,0} - X_{n-1,0}, \forall n \geq 2
\end{aligned}
$$

This result was proved in lectures in Warwick's ST227 Stochastic Processes module. Solving this system for all n will result in the trivial answer that it is certain we will go bankrupt given a large enough number of rounds. This is not dissimilar from the standard result in a Simple Random Walk, that we will cross the positive and negative square root of n an infinite number of times.

The more interesting, computational problem, we can produce for ourselves here, is to answer what if there was an upper bound, i.e our gambler was in fact **not insane**, and stopped with some number m of money? Is it a result that the difference between 2 consecutive terms in the sequence $(X_{n,0} : 0 \leq n \leq m)$ is still constant? This is the problem that my program intended to solve.

In hindsight, the proof for this is also pretty trivial. Nevertheless, I write a program (O(n)) to solve this and figure out that in fact my theory was correct, and therefore write a second program for the same purpose as the first, with constant runtime and space complexity. This was an idea that intuition first provided me with, given the uniform probability distribution of going up vs going down, but anyway, its nice to prove properly and be certain of a result!