Problem 1.a. Let expression E: $e \to 0 \mid 3 \mid e{+}e \mid e{*}e$, and a semantic function $[\![\cdot]\!] \in E \to \mathbb{Z}$. Then, $[\![0]\!] = 0$, $[\![3]\!] = 3$, $[\![e_1 + e_2]\!] = [\![e_1]\!] + [\![e_2]\!]$, $[\![e_1 * e_2]\!] = [\![e_1]\!] \times [\![e_2]\!]$.

Problem 1.b. Theorem: $\forall e \in E$, $[\![e]\!] = 3k$ where $k \in \mathbb{Z}$.
(1) Use structional induction on e.
(2) Base Case:
  (2.1) $e \to 0$: $[\![e]\!] = [\![0]\!] = 0 = 3 \cdot 0$.
  (2.2) $e \to 3$: $[\![e]\!] = [\![3]\!] = 3 = 3 \cdot 1$.
(3) $e \to e_1 + e_2$:
  (3.1) $[\![e]\!] = [\![e_1 + e_2]\!] = [\![e_1]\!] + [\![e_2]\!]$.
  (3.2) By induction hypothesis, $[\![e_1]\!] = 3k_1$, $[\![e_2]\!] = 3k_2$ where $k_1, k_2 \in \mathbb{Z}$.
  (3.3) $\therefore [\![e]\!] = [\![e_1]\!] + [\![e_2]\!] = 3k_1 + 3k_2 = 3 \cdot (k_1 + k_2)$, and $(k_1 + k_2) \in \mathbb{Z}$.
(4) $e \to e_1 * e_2$:
  (4.1) $[\![e]\!] = [\![e_1 * e_2]\!] = [\![e_1]\!] \times [\![e_2]\!]$.
  (4.2) By induction hypothesis, $[\![e_1]\!] = 3k_1$, $[\![e_2]\!] = 3k_2$ where $k_1, k_2 \in \mathbb{Z}$.
  (4.3) $\therefore [\![e]\!] = [\![e_1]\!] \times [\![e_2]\!] = 3k_1 \times 3k_2 = 3 \cdot (3k_1k_2)$, and $3k_1k_2 \in \mathbb{Z}$.

Problem 1.c. $\dfrac{}{\langle 0 \rangle \to 0}$ $\dfrac{}{\langle 3 \rangle \to 3}$ $\dfrac{\langle e_1 \rangle \to v_1, \langle e_2 \rangle \to v_2}{\langle e_1 + e_2 \rangle \to v_1 + v_2}$ $\dfrac{\langle e_1 \rangle \to v_1, \langle e_2 \rangle \to v_2}{\langle e_1 * e_2 \rangle \to v_1 \times v_2}$

Problem 2.a. Let $d_i = i$. Obviously, $\{d_i\}$ is a chain.
(1) $f(\bigsqcup_i d_i) = f(1 \sqcup 2 \sqcup \cdots) = f(\infty) = \top$
(2) $\bigsqcup_i f(d_i) = f(1) \sqcup f(2) \sqcup \cdots \sqcup f(n) \sqcup \cdots = \bot \sqcup \bot \sqcup \cdots \bot \sqcup \cdots = \bot$
(3) $\therefore f(\bigsqcup_i d_i) \neq \bigsqcup_i f(d_i)$, hence f is not continuous.

Problem 2.b. f_k is continuous.
(1) $x \sqsubseteq x' \Rightarrow f(x) \sqsubseteq f(x')$ (i.e f is monotonic)
  (1.1) $x' \leq k \Rightarrow f(x) = f(x') = \bot$ ($\because x \leq x'$)
  (1.2) $x \leq k \wedge k < x' \Rightarrow f(x) = \bot \sqsubseteq \top = f(x')$
  (1.3) $k < x \Rightarrow f(x) = f(x') = \top$
(2) $\forall \{d_i\}$, $f(\bigsqcup_i d_i) = \bigsqcup_i f(d_i)$.
  (2.1) Assume $k \neq \infty$. (Otherwise, $\forall x$ $f_\infty(x) = \bot$ and obviously satisfies (2))
  (2.2) Let $x \in \{d_i\}$ and assume that $x > k$.
  (2.3) If x exists, $\bigsqcup_i d_i > k$ so $f_k(\bigsqcup_i d_i) = \top$, $\bigsqcup_i f_k(d_i) = \cdots \sqcup f_k(x) \sqcup \cdots = \cdots \sqcup \top \sqcup \cdots = \top$.
  (2.4) Otherwise, $\bigsqcup_i d_i \leq k$ so $f_k(\bigsqcup_i d_i) = \bot$, $\bigsqcup_i f_k(d_i) = \bot \sqcup \bot \sqcup \cdots \sqcup \bot = \bot$.

Problem 2.c. $f_0, f_1, \cdots$ constitute a chain, and f_∞ is the least element.
(1) $n \leq m \Rightarrow f_m \sqsubseteq f_n$
  (1.1) $x \leq n \Rightarrow f_m(x) = f_n(x) = \bot$
  (1.2) $n < x \leq m \Rightarrow f_m(x) = \bot \sqsubseteq \top = f_n(x)$
  (1.3) $m < x \Rightarrow f_m(x) = f_n(x) = \top$
  (1.4) $\therefore \forall x \in P$ $f_m(x) \sqsubseteq f_n(x)$
(2) By (1), $\forall n \in P$ $f_\infty \sqsubseteq f_n$. So, f_∞ is the least element.

Problem 3. Use the fixed point theorem.

(0) Let $f^{\perp}(x) = \begin{cases} \perp & \text{if } x = \perp \\ f(x) & \text{otherwise} \end{cases}$ and $[\![\cdot]\!] : \mathbb{Z}_{\perp} \to \mathbb{Z}_{\perp}$.

(1) In plain English, f is a function returns 1 if $z \geq 0$, otherwise returns $-z+1$.

(2) $[\![f]\!] = fix(F)$ where $F = (\lambda f.\lambda z.\text{ if } z \geq 0 \text{ then } 1 \text{ else } (1 + f^{\perp}(z+1)))$.

(3) $[\![f]\!] = fix(F) = \bigsqcup\{F^0(\perp_f), F^1(\perp_f), \cdots\}$ where $\perp_f = \lambda z.\perp$.

(3.1) $F^0(\perp_f) = \lambda z.\perp$.

(3.2) $F^1(\perp_f) = \lambda z.\text{ if } z \geq 0 \text{ then } 1 \text{ else } (1 + (\lambda z.\perp)^{\perp}(z+1))) = \begin{cases} 1 & \text{if } z \geq 0 \\ \perp & \text{otherwise} \end{cases}.$

(3.3) $F^2(\perp_f) = \lambda z.\text{ if } z \geq 0 \text{ then } 1 \text{ else } (1 + (F^1(\perp_f))^{\perp}(z+1))) = \begin{cases} 1 & \text{if } z \geq 0 \\ 2 & \text{if } z = -1 \\ \perp & \text{otherwise} \end{cases}.$

(3.4) $\therefore F^n(\perp_f) = \lambda z.\text{ if } z \geq 0 \text{ then } 1 \text{ else } (1 + (F^{n-1}(\perp_f))^{\perp}(z+1))) = \begin{cases} 1 & \text{if } z \geq 0 \\ -z+1 & \text{if } 0 > z \geq -n+1 \\ \perp & \text{otherwise} \end{cases}.$

(4) $\therefore [\![f]\!] = \begin{cases} 1 & \text{if } z \geq 0 \\ -z+1 & \text{otherwise} \end{cases}$ and matches to the plain meaning that is described in (1).