Exercise 4.1 If f and g are regular functions on open sets U and V of a variety X, and if $f = g$ on $U \cap V$, show that the function which is f on U and g on V is a regular function on $U \cup V$. Conclude that if f is a *rational* function on X, then there is a largest open subset U of X on which f is represented by a regular function. We say that f *is defined* at the points of U.

Open sets in U and V are also open sets in $U \cap V$, so f on U and g on V is a regular function by definition. For the second claim, consider the set of open sets on which f is represented by a regular function. The union of them is the largest open set on which f is represented by a regular function.

Exercise 4.2 Same problem for rational maps. If φ is a rational map of X to Y, show there is a largest open set on which φ is represented by a morphism. We say the rational map *is defined* at the points of that open set.

We should prove that if φ and ψ are morphisms from open sets U and V of a variety X to Y, and if $\varphi = \psi$ on $U \cap V$, then the function φ' which is φ on U and ψ on V is a morphism on $U \cup V$. For all regular function f on an open set W of Y, $f \circ \varphi' : \varphi'^{-1}(W) \to k$ is a function which is $f \circ \varphi$ on $\varphi^{-1}(W)$ and $g \circ \psi$ on $\psi^{-1}(W)$. By exercise 4.1 we know this is regular, hence φ' is a morphism. The proof to the second claim is exactly as in exercise 4.1.

Exercise 4.3

(a) Let f be the rational function on $\mathbf{P}^2$ given by $f = x_1/x_0$. Find the set of points where f is defined and describe the corresponding regular function.

(b) Now think of this function as a rational map from $\mathbf{P}^2$ to $\mathbf{A}^1$. Embed $\mathbf{A}^1$ in $\mathbf{P}^1$, and let $\varphi : \mathbf{P}^2 \to \mathbf{P}^1$ be the resulting rational map. Find the set of points where φ is defined, and describe the corresponding morphism.

(a) f is defined on $x_0 \neq 0$, and the corresponding regular function is x_1/x_0. It can also be thought of taking the isomorphism to $\mathbf{A}^2$ then taking its first coordinate.

(b) φ is defined on $\mathbf{P}^2 - \{(0,0,1)\}$, and the corresponding morphism is given by $(x_0, x_1, x_2) \mapsto (x_0, x_1)$.

Exercise 4.4 A variety Y is *rational* if it is birationally equivalent to $\mathbf{P}^n$ for some n (or, equivalently by (4.5), if $K(Y)$ is a pure transcendental extension of k).

(a) Any conic in $\mathbf{P}^2$ is a rational curve.

(b) The cuspidal cubic $y^2 = x^3$ is a rational curve.

(c) Let Y be the nodal cubic curve $y^2 z = x^2(x+z)$ in $\mathbf{P}^2$. Show that the projection φ from the point $P = (0,0,1)$ to the line $z = 0$ (Ex. 3.14) induces a birational map from Y to $\mathbf{P}^1$. Thus Y is a rational curve.

(a) Any conic in $\mathbf{P}^2$ is isomorphic to $\mathbf{P}^1$ (exercise 3.1(c)), hence birationally equivalent to $\mathbf{P}^1$.

(b) The cuspidal cubic has function field $\operatorname{Frac}(k[x,y]/(y^2 - x^3)) \cong \operatorname{Frac} k[t^2, t^3] = k(t)$, a pure transcendental extension of k.

(c) We identify the line $z = 0$ with $\mathbf{P}^1$, then the projection induces morphisms

$$\begin{aligned} &\varphi : Y - \{(0,0,1)\} \to \mathbf{P}^1, (x,y,z) \mapsto (x,y) \\ &\psi : \mathbf{P}^1 - \{(1,1),(1,-1)\} \to Y, (x,y) \mapsto \left(x, y, \frac{x^3}{y^2 - x^2}\right) \end{aligned}$$

which obviously satisfy that $\varphi\psi$ and $\psi\varphi$ are equivalent to the identity rational map.

Exercise 4.5 Show that the quadric surface $Q : xy = zw$ in $\mathbf{P}^3$ is birational to $\mathbf{P}^2$, but not isomorphic to $\mathbf{P}^2$ (cf. Ex. 2.15).

By intersecting with $x \neq 0$ we know $K(Q) = \operatorname{Frac}(k[y,z,w]/(y - zw)) = \operatorname{Frac} k[z,w] = K(\mathbf{P}^2)$, hence Q is birational to $\mathbf{P}^2$. They are not isomorphic because any two curves on $\mathbf{P}^2$ must intersect (exercise 3.7a), while exercise 2.15 shows that Q contains two disjoint lines.

Exercise 4.6 *Plane Cremona Transformations.* A birational map of $\mathbf{P}^2$ to itself is called a *plane Cremona transformation.* We give an example, called a *quadratic transformation.* It is the rational map $\varphi : \mathbf{P}^2 \to \mathbf{P}^2$ given by $(a_0, a_1, a_2) \to (a_1a_2, a_0a_2, a_0a_1)$ when no two of a_0, a_1, a_2 are 0.

(a) Show that φ is birational, and is its own inverse.

(b) Find open sets $U, V \subseteq \mathbf{P}^2$ such that $\varphi : U \to V$ is an isomorphism.

(c) Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms. See also (V, 4.2.3).

(a) To make things clearer, Let $A = \{(a_0, a_1, a_2) \in \mathbf{P}^2 : a_0 \neq 0, a_1 \neq 0, a_2 \neq 0\}$. Then $\varphi(A) \subseteq A$, $\varphi : A \to A$ is a morphism, and obviously $\varphi^2 = \mathrm{id}_A$.

(b) We already see $\varphi : A \to A$ is an isomorphism.

(c) φ and φ^{-1} are both defined on the open set $\mathbf{P}^2 - \{(0,0,1), (0,1,0), (1,0,0)\}$, while the corresponding morphism is $(a_0, a_1, a_2) \mapsto (a_1a_2, a_0a_2, a_0a_1)$.

Exercise 4.7 Let X and Y be two varieties. Suppose there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as k-algebras. Then show that there are open sets $P \in U \subseteq X$ and $Q \in V \subseteq Y$ and an isomorphism of U to V which sends P to Q.

Varieties have affine covers, and taking closure does not change the result; hence we may assume X and Y are affine varieties. By a translation we can assume $P = 0, Q = 0$. Let $f : \mathcal{O}_{P,X} \leftrightarrow \mathcal{O}_{Q,Y} : g$ be the isomorphism. The coordinate function y_j is in $\mathcal{O}_{Q,Y}$, hence we define $\varphi = (g(y_j))_j$ from an open set U_0 of X to Y. We must have $g(y_j)(0) = 0$ or else $\frac{1}{g(y_j)} \in \mathcal{O}_{P,X}$ but $\frac{1}{y_j} \notin \mathcal{O}_{Q,Y}$, thus $\varphi(P) = Q$. Similarly we define $\psi = (f(x_i))_i$ from an open set V_0 of Y to X. On where they are defined, $\psi\varphi$ and $\varphi\psi$ are both identities, hence they induce an isomorphism from $\varphi^{-1}(\psi^{-1}(U_0)) \subseteq X$ to $\psi^{-1}(\varphi^{-1}(V_0)) \subseteq Y$ (see proof of (4.5)).

Exercise 4.8 (a) Show that any variety of positive dimension over k has the same cardinality as k. [*Hints*: Do $\mathbf{A}^n$ and $\mathbf{P}^n$ first. Then for any X, use induction on the dimension n. Use (4.9) to make X birational to a hypersurface $H \subseteq \mathbf{P}^{n+1}$. Use (Ex. 3.7) to show that the projection of H to $\mathbf{P}^n$ from a point not on H is finite-to-one and surjective.]

(b) Deduce that any two curves over k are homeomorphic (cf. Ex. 3.1).

(a) It is obvious for $\mathbf{A}^n$ and $\mathbf{P}^n$ because $|k|$ is infinite implies $|k|^n = |k|$. Any variety X can be embedded into one of $\mathbf{A}^n$ or $\mathbf{P}^n$, so $|X| \leq |k|$.
If X is a quasi-affine variety of dimension 1, Noether normalization gives a geometric fact that there exists a subspace k and a linear surjection $f : \overline{X} \to k$ (see Atiyah-Macdonald Exercise 16 of Chapter 5), hence has cardinality $\geq |k|$. The irreducible closed sets of $\overline{X}$ containing a point P corresponds to prime ideals of $\mathcal{O}_{P,\overline{X}}$ which has dimension 1, hence

the only proper irreducible closed subset containing P is $\{P\}$. Thus the closed sets of $\overline{X}$ are finite union of irreducible components, i.e. finite points, hence open subsets of $\overline{X}$ has cardinality $|k|$.
As every variety contains a quasi-affine curve, they must have cardinality at least $|k|$, hence equal to $|k|$.

(b) Any two curves are sets of cardinality $|k|$ with cofinite topology, hence homeomorphic.

Exercise 4.9 Let X be a projective variety of dimension r in $\mathbf{P}^n$ with $n \geq r+2$. Show that for suitable choice $P \notin X$, and a linear $\mathbf{P}^{n-1} \subseteq \mathbf{P}^n$, the projection from P to $\mathbf{P}^{n-1}$ (Ex. 3.14) induces a birational morphism of X onto its image $X' \subseteq \mathbf{P}^{n-1}$. You will need to use (4.6A), (4.7A), and (4.8A). This shows in particular that the birational map of (4.9) can be obtained by a finite number of such operations.

[1]Assume $X \cap U_0 \neq \emptyset$, then the rational functions $\frac{x_i}{x_0}$ generate $K(X)$. By (4.7A) and (4.8A) these generators contain a transcendence base. We assume $\frac{x_1}{x_0}, \dots, \frac{x_r}{x_0}$ are algebraically independent, and by (4.6A) we can take a k-linear combination of $\frac{x_{r+1}}{x_0}, \dots, \frac{x_n}{x_0}$ to be the other generator. After composing a linear transformation we may assume $K(X) = k(\frac{x_1}{x_0}, \dots, \frac{x_r}{x_0}, \frac{x_{r+1}}{x_0})$.

We take $P = (0, p_1, \dots, p_r, 0, 1, p_{r+3}, \dots) \notin X$, whose existence is guaranteed by $r < n-1$, and the hyperplane $\{x_{r+2} = 0\}$. The projection π sends point $(x_0, \dots, x_n)$ to $(x_i - x_{r+2}p_i)_i$ where $p_0 = p_{r+1} = 0, p_{r+2} = 1$. Hence the k-algebra homomorphism $K(\pi(X)) \to K(X)$ has image containing

$$k\left(\frac{x_1 - x_{r+2}p_1}{x_0}, \dots, \frac{x_r - x_{r+2}p_r}{x_0}, \frac{x_{r+1}}{x_0}\right) = k\left(\frac{x_1}{x_0}, \dots, \frac{x_r}{x_0}, \frac{x_{r+1}}{x_0}\right) = K(X)$$

as $\frac{x_{r+2}}{x_0} \in k(\frac{x_1}{x_0}, \dots, \frac{x_r}{x_0}, \frac{x_{r+1}}{x_0})$. Hence $K(\pi(X)) = K(X)$, thus π is birational.

Exercise 4.10 Let Y be the cuspidal cubic curve $y^2 = x^3$ in $\mathbf{A}^2$. Blow up the point $O = (0,0)$, let E be the exceptional curve, and let $\widetilde{Y}$ be the strict transform of Y. Show that E meets $\widetilde{Y}$ in one point, and that $\widetilde{Y} \cong \mathbf{A}^1$. In this case the morphism $\varphi : \widetilde{Y} \to Y$ is bijective and bicontinuous, but it is not an isomorphism.

Similarly to example 4.9.1 we find $\widetilde{Y}$ is defined by $u^2 = x, y = xu$ where t, u are the homogeneous coordinates and we assumed $t = 1$. $\widetilde{Y}$ intersects with E at $u = 0$, and it clearly does not pass through $t = 1, u = 0$ in E, hence $|\widetilde{Y} \cap E| = 1$.

We have $\widetilde{Y} \subseteq \mathbf{A}^2 \times (\mathbf{P}^1 - \{(1,0)\}) \cong \mathbf{A}^3$, where it has coordinates (x, y, u) and is defined by $u^2 = x, y = xu$. The map $\widetilde{Y} \leftrightarrow \mathbf{A}^1 : (u^2, u^3, u) \leftrightarrow u$ is clearly seen to be an isomorphism. Using this coordinate, $\varphi : \widetilde{Y} \to Y$ maps (u^2, u^3, u) to (u^2, u^3), which is clearly bijective and continuous; the continuity of its inverse is guaranteed by the fact that any $f = 0$ has the same solutions as $f^2 = 0$, where f^2 is easily written as a polynomial in u^2, u^3. It is not an isomorphism because $\widetilde{Y}$ is regular but Y is not.

[1]This solution is based on this math stackexchange answer, which needs the following stronger version of (4.6A): In the condition of (4.6A), for all infinite set $S \subseteq K$ we can take $\alpha = \sum_{i=1}^{n} c_i\beta_i$ where $c_i \in S$.