Exercise 2.1 Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbf{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$. [Hint: Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz, (1.3A).]

Following the hint, we consider the zero set of $\mathfrak{a}$ in $\mathbf{A}^{n+1}$, which can be identified as

$$\bigcup_{\substack{P \in Z(\mathfrak{a}) \text{ with homogeneous} \\ \text{coordinates } (x_0,\ldots,x_n)}} \{t(x_0, \ldots, x_n) : t \in k\}$$

Thus by homogeneity f also vanishes on this set. The rest is the usual Nullstellensatz.

Exercise 2.2 For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \varnothing$ (the empty set).

(ii) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$.

(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

(i)$\Longrightarrow$(ii) By the "homogeneous Nullstellensatz," we have $\sqrt{a} \supseteq S_+$. Then notice that S_+ is a maximal ideal in S.

(ii)$\Longrightarrow$(iii) As $S_1 \subseteq \sqrt{\mathfrak{a}}$, there are integers $a_0, \ldots, a_n$ such that $x_0^{a_0+1}, \ldots, x_n^{a_n+1} \in \mathfrak{a}$. Then $\mathfrak{a} \supseteq S_{a_0+\cdots+a_n+1}$.

(iii)$\Longrightarrow$(i) If $\mathfrak{a} \supseteq S_d$, then for $P \in Z(\mathfrak{a})$ with homogeneous coordinates $(x_0, \ldots, x_n)$, we have $x_0^d, \ldots, x_n^d = 0$, so $P = (0, \ldots, 0)$, which is not a point in $\mathbf{P}^n$.

Exercise 2.3 (a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

(c) For any two subsets Y_1, Y_2 of $\mathbf{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \varnothing$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \overline{Y}$.

The proofs are nearly identical to those in the affine case, with the only difference being that the ideal $I(Y)$ is 'generated' by homogeneous elements, whereas in the affine case it is exactly all the polynomials vanishing on Y.

Exercise 2.4 (a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$ and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note*: Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.

(b) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

(c) Show that $\mathbf{P}^n$ itself is irreducible.

The proofs are nearly identical to those in the affine case. The 'irrelevance' of S_+ comes from exercise 2.2.

Exercise 2.5 (a) $\mathbf{P}^n$ is a noetherian topological space.

(b) Every algebraic set in $\mathbf{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

The proofs are identical to those in the affine case.

Exercise 2.6 If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\varphi_i : U_i \to \mathbf{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\varphi_i(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]

Following the hint, we let $\varphi_i : U_i \to \mathbf{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\varphi_i(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring. Now we have an isomorphism

$$\begin{array}{ccc} S_{x_i} = k[x_0, \ldots, x_n, x_i^{-1}] & \cong & k[x_0, \ldots, \hat{x}_i{}^{1}, \ldots, x_n, x_i, x_i^{-1}] = A[x_i, x_i^{-1}] \\ x_j & \mapsto & x_i x_j \quad (j \neq i) \end{array}$$

Under which the ideal $I(Y)_{x_i}$ is mapped exactly to $I(Y_i)[x_i, x_i^{-1}]$: a generator $f/x_i^k \in I(Y)_{x_i}$ where $f \in I(Y)$ is a homogeneous generator gets mapped to

$$\frac{f(x_0 x_i, \ldots, x_{i-1} x_i, x_i, x_{i+1} x_i, \ldots, x_n x_i)}{x_i^k} = x_i^{\deg f - k} g(x_0, \ldots, \hat{x}_i, \ldots, x_n)$$

for some g; for all $(y_0, \ldots, y_n) \in Y \cap U_i$ we have $g(\frac{y_0}{y_i}, \ldots, \frac{y_n}{y_i}) = \frac{f(y_0, \ldots, y_n)}{y_i^{\deg f}} = 0$, i.e. $g \in I(Y_i)$, which is to say $I(Y)_{x_i}$ gets mapped into $I(Y_i)[x_i, x_i^{-1}]$. The opposite argument is similar. Hence the quotient rings are isomorphic, i.e.

$$S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$$

If $Y_i \neq \varnothing$ we have $\operatorname{Frac} S(Y)_{x_i} = \operatorname{Frac} S(Y)$ and $\operatorname{Frac}(A(Y_i)[x_i, x_i^{-1}]) = (\operatorname{Frac} A(Y_i))(x_i)$. By (1.8A) we have $\dim S(Y) = \dim A(Y_i) + 1$, hence $\dim Y \geq \dim Y_i = \dim A(Y_i) = \dim S(Y) - 1$. On the other hand any chain of irreducible closed subsets of Y corresponds to a chain of homogeneous prime ideals in $S(Y)$ which is not the image of S_+, so we can always insert S_+ into it; thus $\dim Y \leq \dim S(Y) - 1$. Those two inequalities give us the desired results.

Exercise 2.7 (a) $\dim \mathbf{P}^n = n$.

(b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.
[*Hint*: Use (Ex. 2.6) to reduce to (1.10).]

(a) $\dim \mathbf{P}^n = \dim k[x_0, \ldots, x_n] - 1 = n$.

(b) We have $\dim Y = \sup_i \dim(Y \cap U_i) = \sup_i \dim \varphi(Y \cap U_i)$ by exercise 1.10 (b), but every $\varphi(Y \cap U_i)$ is a quasi-affine variety which by (1.10) has the same dimension as its closure $\overline{\varphi(Y \cap U_i)} = \varphi(\overline{Y} \cap U_i)$, so $\dim Y = \sup_i \dim \varphi(\overline{Y} \cap U_i) = \dim \overline{Y}$ by exercise 2.6.

Exercise 2.8 A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n - 1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbf{P}^n$.

The proof is identical to that in the affine case.

[1] The hat denotes a missing term.

Exercise 2.9 *Projective Closure of an Affine Variety.* If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify $\mathbf{A}^n$ with an open set $U_0 \subseteq \mathbf{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbf{P}^n$, which is called the *projective closure* of Y.

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).

(b) Let $Y \in \mathbf{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\overline{Y} \subseteq \mathbf{P}^3$ is called the *twisted cubic curve* in $\mathbf{P}^3$. Find generators for $I(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

(a) We first notice that

$$I(\overline{Y}) = I(\overline{\varphi_0(Y)}) = I(Z(I(\varphi_0(Y)))) = \sqrt{I(\varphi_0(Y))} \subseteq I(\varphi_0(Y)) \subseteq I(\overline{\varphi_0(Y)}) = I(\overline{Y})$$

Hence $I(\overline{Y}) = I(\varphi_0(Y))$. By definition the latter is generated by homogeneous polynomials f which vanish on $\varphi_0(Y)$, which we can further assume are not multiples of x_0; therefore $f = \beta(\alpha(f))$. But f vanishes on $\varphi_0(Y)$ means $\alpha(f)$ vanishes on Y, so $f = \beta(\alpha(f)) \in \beta(I(Y))$, i.e. $I(\varphi_0(Y))$ is contained in the ideal generated by $\beta(I(Y))$. On the other hand all elements of $\beta(I(Y))$ are clearly in $I(\varphi_0(Y))$, so $I(\varphi_0(Y))$ must be exactly the ideal generated by $\beta(I(Y))$.

(b) In exercise 1.2 we have already proven that $I(Y) = (y - x^2, z - x^3) = (y - x^2, z - xy)$. We then prove that $I(\overline{Y}) = (x^2 - uy, xy - uz, y^2 - xz)$, where u is the 0-th variable. For a homogeneous polynomial $f \in k[u, x, y, z]$ we identify it with $f \in k[u, z][x, y]$ and continuously modulo it with $x^2 - uy$ and $y^2 - ux$; in the process $\deg_x + \deg_y$ is strictly decreasing so it must stop. The result is a polynomial of the form $f_1(u, z)xy + f_2(u, z)x + f_3(u, z)y + f_4(u, z)$. Then we modulo it with $xy - uz$ and we get $f_2(u, z)x + f_3(u, z)y + f_5(u, z)$. Plugging in $(u, x, y, z) = (1, t, t^2, t^3)$ we get a polynomial $p(t^3) + q(t^3)t + r(t^3)t^2$, which vanishes on all t iff $p = q = r = 0$. Thus $I(\overline{Y}) = (x^2 - uy, xy - uz, y^2 - xz)$. It is clear that $\beta(y - x^2) = uy - x^2$ and $\beta(z - x^3) = u^2z - x^3$ cannot generate $y^2 - xz$, because the first two are zero when $x = u = 0$ but the last one is not.

Exercise 2.10 *The Cone Over a Projective Variety.* Let $Y \subseteq \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta : A^{n+1} - \{(0, \ldots, 0)\} \to \mathbf{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $(a_0, \ldots, a_n)$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \ldots, 0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \ldots, x_n]$.

(b) $C(Y)$ is irreducible if and only if Y is.

(c) $\dim C(Y) = \dim Y + 1$.

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbf{P}^{n+1}$. This is called the *projective cone* over Y.

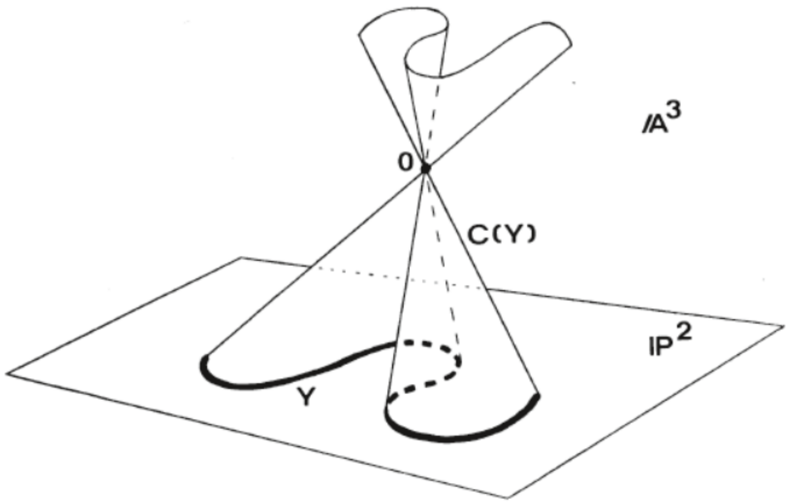

Figure 1. The cone over a curve in $\mathbf{P}^2$.

(a) We know $Z_{\text{affine}}(I(Y)) = C(Y)$ hence

$$I_{\text{affine}}(C(Y)) = I_{\text{affine}}(Z_{\text{affine}}(I(Y))) = \sqrt{I(Y)} = I(Y)$$

by the Nullstellensatz and the fact that $I(Y)$ is radical.

(b) They are all equivalent to saying that $I(Y)$ is prime.

(c) $\dim Y = \dim k[x_0, \ldots, x_n]/I(Y) - 1 = \dim C(Y) - 1$.

Exercise 2.11 *Linear Varieties in* $\mathbf{P}^n$. A hypersurface defined by a linear polynomial is called a *hyperplane.*

(a) Show that the following two conditions are equivalent for a variety Y in $\mathbf{P}^n$:

(i) $I(Y)$ can be generated by linear polynomials.

(ii) Y can be written as an intersection of hyperplanes.

In this case we say that Y is a *linear variety* in $\mathbf{P}^n$.

(b) If Y is a linear variety of dimension r in $\mathbf{P}^n$, show that $I(Y)$ is minimally generated by $n - r$ linear polynomials.

(c) Let Y, Z be linear varieties in $\mathbf{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r + s - n \geq 0$, then $Y \cap Z \neq \varnothing$. Furthermore, if $Y \cap Z \neq \varnothing$, then $Y \cap Z$ is a linear variety of dimension $\geq r + s - n$. (Think of $\mathbf{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

(a) The only nontrivial part is to prove that any ideal generated by linear polynomials is radical. In fact, such an ideal $(f_1, \ldots, f_r)$ is prime, because $k[x_0, \ldots, x_n]/(f_1)$ is isomorphic to $k[y_0, \ldots, y_{n-1}]$ (imagine quotienting f_1 as 'plugging in one of the variables that occurred in f_1 as a linear combination of the other variables'), hence quotienting $(f_1, \ldots, f_r)$ also gives a polynomial ring, which is integral.

(b) If an ideal $\mathfrak{a}$ is generated by the linear polynomials $\sum_j a_{ij}x_j$ $(1 \leq i \leq t)$, then by repeatedly quotienting one of such generators (sometimes quotienting the zero ideal) we find $k[x_0, \ldots, x_n]/\mathfrak{a}$ is a polynomial ring on $n + 1 - \operatorname{rank}(a_{ij})$ variables; thus height $\mathfrak{a} = \operatorname{rank}(a_{ij})$. By dimension we see that height $I(Y) = n - r$, so picking the $n - r$ linearly independent polynomials among the linear generators of $I(Y)$ suffices. It is minimal because a matrix with $n - r - 1$ rows cannot have rank $n - r$.

(c) Saying that Y is a linear variety of dimension t is equivalent to saying that $C(Y)$, cf. exercise 2.10, is a $t+1$-dimensional subspace of $\mathbf{A}^{n+1}$. Then this claim reduces to the elementary linear algebra fact that the intersection of a $r+1$-dimensional subspace and a $s+1$-dimensional subspace in an $n+1$-dimensional vector space has dimension at least $(r+1)+(s+1)-(n+1)=r+s-n+1$.

Exercise 2.12 *The d-Uple Embedding.* For given $n, d > 0$, let $M_0, M_1, \dots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \dots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbf{P}^n \to \mathbf{P}^N$ by sending the point $P = (a_0, \dots, a_n)$ to the point $\rho_d(P) = (M_0(a), \dots, M_N(a))$ obtained by substituting the a_i in the monomials M_i. This is called the d-uple *embedding* of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, if $n=1, d=2$, then $N=2$, and the image Y of the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ is a conic.

(a) Let $\theta : k[y_0, \dots, y_N] \to k[x_0, \dots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbf{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

(c) Now show that ρ_d is a homeomorphism of $\mathbf{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbf{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

To clear up thing, we also use the variables M_S to represent the monomial $\prod_{i\in S} x_i$, where S is a multiset[2] consisting of integers from 0 to n. We will use $[i]_d$ to denote the multiset $\{\underbrace{i, i, \dots, i}_{d \text{ times}}\}$ and $[i,j]_d$ to denote the multiset $\{\underbrace{i, i, \dots, i}_{d-1 \text{ times}}, j\}$, hence $M_{[i]_d} = x_i^d$ and $M_{[i,j]_d} = x_i^{d-1} x_j$.

(a) It is prime because $k[y_0, \dots, y_N]/\mathfrak{a}$ is a subring of $k[x_0, \dots, x_n]$, which is integral; we shall only prove that $\mathfrak{a}$ can be generated by homogeneous elements, i.e. a polynomial $f \in \mathfrak{a}$ has all its homogeneous parts in $\mathfrak{a}$. Let $f = \sum_i f_i$ where f_i is of degree i, then for a variable t, the degree d monomials in the variables tx_i are t^d times the degree d monomials in the x_i, hence $f(t^d M_0, \dots, t^d M_N) = 0$ (as a polynomial in t). The *di*-th degree coefficient of the left-hand side is $f_i(M_0, \dots, M_N)$, so $f_i(M_0, \dots, M_N) = 0$ for all i, i.e. $f_i \in \mathfrak{a}$.

(b) The only nontrivial part is to prove that if $N+1$ elements $L_S \in k$, labelled by d-element multisets S, vanish on all of $\mathfrak{a}$'s elements, then they must be of the form M_S for a suitable choice of $x_0, \dots, x_n$. Without loss of generality we can assume all L_S are nonzero, because $x_i = 0 \Leftrightarrow M_{[i]_d} = 0$. We arbitrarily choose a d-th root of $L_{[0]_d}$ and call it x_0, then we define recursively $x_i = L_{[i-1,i]_d}/x_{i-1}^{d-1}$. It is easily shown that the monomials M_S of this set of x_i are rational functions of L_S. The other monomials M_S agree with L_S, because the L_S have to satisfy the polynomials in $\mathfrak{a}$.[3]

(c) We first prove that $\rho_d : \mathbf{P}^n \to Z(\mathfrak{a})$ is a bijection: by (b) it is already surjective. For any point in $Z(\mathfrak{a})$ we can construct its inverse images by the procedure described in (b); the

[2]A set where elements can have multiplicities.

[3]For example,

$$x_0 x_1^{d-1} = \frac{L_{[1,0]_d}^{d-1}}{x_0^{d^2-2d}} = \frac{L_{[1,0]_d}^{d-1}}{L_{[0]}^{d-2}}$$

and this is equal to $L_{[0,1]}$ because $M_{[1,0]_d}^{d-1} - M_{[0]}^{d-2} M_{[0,1]} \in \mathfrak{a}$.

only ambiguity is the part 'choosing an arbitrary d-th root of $L_{[0]_d}$', but this only causes $(x_0, \dots, x_n)$ to multiply a d-th root of unity simultaneously, which results in the same point $(x_0, \dots, x_n) \in \mathbf{P}^n$. Hence ρ_d is injective.
Recall the definition of $\theta : k[y_0, \dots, y_N] \to k[x_0, \dots, x_n]$ in (a). It is clear that $\operatorname{Im}\theta$ is the set of all polynomials whose homogeneous parts' degrees are multiples of d. Pick arbitrarily a map $\varphi : \operatorname{Im}\theta \to k[y_0, \dots, y_N]$ such that $\theta\varphi = \mathrm{id}$. A closed subset of $Z(\mathfrak{a})$ is of the form $Z(\mathfrak{a}) \cap Z(g_1, \dots, g_r)$, which gets mapped to the closed subset $Z(\theta(g_1), \dots, \theta(g_r)) \in \mathbf{P}^n$. Conversely, a closed subset of $\mathbf{P}^n$ is of the form $Z(f_1, \dots, f_r)$, where $f_1, \dots, f_r$ is homogeneous; consider the closed subset $Z(\mathfrak{a}) \cap Z(\varphi(f_1^d), \dots, \varphi(f_r^d))$. Under ρ_d, it gets mapped to $Z(f_1^d, \dots, f_r^d) = Z(f_1, \dots, f_r)$. Hence ρ_d is a homeomorphism.

(d) The image of the 3-uple embedding is

$$\{(x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3) : (x_0, x_1) \in \mathbf{P}^1\} = \{(1, t, t^2, t^3) : t \in k\} \cup \{(0,0,0,1)\}$$

which is exactly the twisted cubic curve.

Exercise 2.13 Let Y be the image of the 2-uple embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$. This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbf{P}^5$ such that $V \cap Y = Z$.

The curve is the zero set of a single homogeneous polynomial f. Following exercise 2.12 (d) we denote $\theta : k[y_0, \dots, y_5] \to k[x_0, x_1, x_2]$, $\mathfrak{a} = \ker\theta$ and φ a right-inverse of θ on $\operatorname{Im}\theta$. Then by the same argument $Y \cap Z(\varphi(f^2)) = Z(\mathfrak{a}) \cap Z(\varphi(f^2)) = Z(f)$, where we identify Y with $\mathbf{P}^2$.

Exercise 2.14 *The Segre Embedding.* Let $\psi : \mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_i b_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a *subvariety* of $\mathbf{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbf{P}^N$ be $\{z_{ij} | i = 0, \dots, r, j = 0, \dots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \dots, x_r, y_0, \dots, y_s]$ which sends z_{ij} to $x_i y_j$. Then show that $\operatorname{Im}\psi = Z(\mathfrak{a})$.]

The only nontrivial part is to prove that if some numbers z_{ij} vanish on all of $\mathfrak{a}$'s elements, then they must be of the form $x_i y_j$. We arrange these numbers in a matrix $\mathcal{Z} = (z_{ij})$, and it is easy to see that z_{ij} vanish on $\mathfrak{a}$ means that the rank of $\mathcal{Z}$ is at most 1 (e.g. $z_{ik}z_{jl} = z_{il}z_{jk}$ implies that every 2×2 submatrix has determinant 0), thus can be written as the product of a column vector and a row vector, i.e. $z_{ij} = x_i y_j$.

Exercise 2.15 *The Quadric Surface in* $\mathbf{P}^3$. Consider the surface Q (a *surface* is a variety of dimension 2) in $\mathbf{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that Q is equal to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

(b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbf{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \varnothing$; if $M_t \neq M_u$, then $M_t \cap M_u = \varnothing$, and for all t, u, $L_t \cap M_u =$ one point.

(c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbf{P}^1 \times \mathbf{P}^1$ (where each $\mathbf{P}^1$ has its Zariski topology).

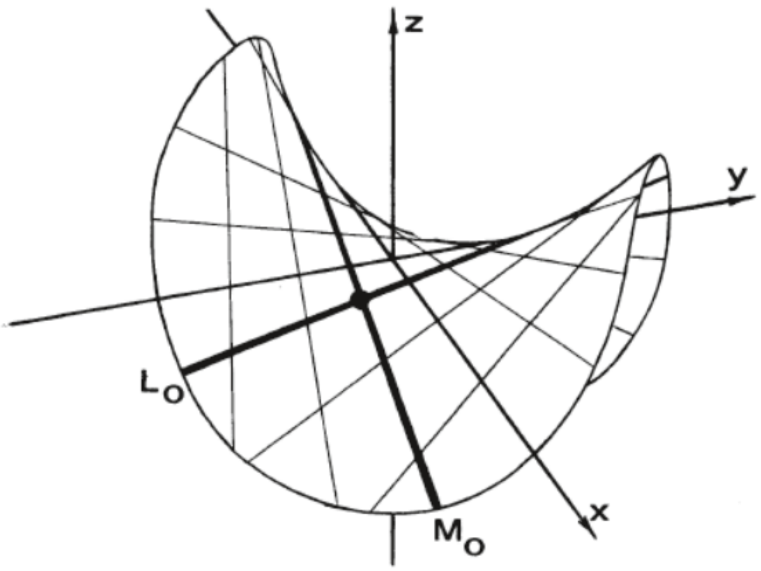

Figure 2. The quadric surface in $\mathbf{P}^3$.

(a) For any point $(x, y, z, w) \in \mathbf{P}^3$ such that $xy = zw$ there always exists $(a_0, a_1), (b_0, b_1) \in \mathbf{P}^1$ such that $(x, y, z, w) = (a_0b_0, a_1b_1, a_0b_1, a_1b_0)$; under suitable coordinates, this shows that Q is contained in the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ after a suitable coordinate change. The other direction is trivial.

(b) The two families of lines are the images of the two families of lines $\{t\} \times \mathbf{P}^1$ and $\mathbf{P}^1 \times \{t\}$ under the Segre embedding.

(c) For example the twisted cubic curve (coordinates modified) $\{(1, t^3, t, t^2) : t \in k\} \cup \{(0, 1, 0, 0)\}$ is a curve in Q. To show that Q is not homeomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, we consider the following condition on a topological space X:

For any three irreducible closed proper subsets A, B, C,
if $A \cap B, B \cap C, C \cap A \neq \varnothing$, then $A \cap B \cap C \neq \varnothing$.

This condition is satisfied by $\mathbf{P}^1 \times \mathbf{P}^1$ as the irreducible closed proper subsets are of the form $\{t\} \times \mathbf{P}^1$, $\mathbf{P}^1 \times \{t\}$ or single points. This condition is not satisfied by Q: take A to be the twisted cubic curve mentioned above, take $B = L_{(1,0)} = \{(b_0, 0, b_1, 0) : (b_0, b_1) \in \mathbf{P}^1\}$ and $C = M_{(1,1)} = \{(a_0, a_1, a_0, a_1) : (a_0, a_1) \in \mathbf{P}^1\}$.

Exercise 2.16 (a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbf{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.

(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbf{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

(a) The solutions to these equations are $\underbrace{\{(0, y, z, 0)\}}_{\text{line}} \cup \underbrace{\{(t^2, t, 1, t^3)\} \cup \{(0, 0, 0, 1)\}}_{\text{twisted cubic curve}}$.

(b) The point P is $(0, 0, 1)$, hence $I(P) = (x, y)$. But $I(C) + I(L) = (x^2, y) \neq (x, y)$.

Exercise 2.17 *Complete intersections.* A variety Y of dimension r in $\mathbf{P}^n$ is a *(strict) complete intersection* if $I(Y)$ can be generated by $n-r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n-r$ hypersurfaces.

(a) Let Y be a variety in $\mathbf{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geq n - q$.

(b) Show that a strict complete intersection is a set-theoretic complete intersection.

(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbf{P}^3$ (Ex. 2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees 2,3 respectively, such that $Y = H_1 \cap H_2$.

(a) We have $\operatorname{height} \mathfrak{a} \leq q$ (see solution to exercise 1.9), hence $\dim Y = \dim S(Y) - 1 = n - \operatorname{height} \mathfrak{a} \geq n - q$, by (1.8B).

(b) Let $I(Y) = (f_1, \dots, f_r)$, then $Y = Z(I(Y)) = Z(f_1, \dots, f_r) = \bigcap_{i=1}^r Z(f_i)$.

(c) We have already seen that $I(Y) = (x^2 - uy, xy - uz, y^2 - xz)$ with coordinates (u, x, y, z). The part of the ideal with degree 2 (i.e. $I(Y) \cap S_2$) is a 3 dimensional vector space over $\mathbb{C}$. If $I(Y) = (f_1, f_2)$, then $\deg f_1, \deg f_2 \geq 2$. For all $f \in I(Y) \cap S_2$, there should exist polynomials g_1, g_2 such that $f_1 g_1 + f_2 g_2 = f$, hence we could take the homogeneous part with degree 2 of both sides and reduce g_1, g_2 to scalars; thus $\dim_{\mathbb{C}}(I(Y) \cap S_2) \leq 2$, contradiction. To show that Y is the intersection of a quadric and a cubic, we can take $H_1 = Z(x^2 - yz)$ and $H_2 = Z(uz^2 - y^3)$.