Exercise 7.1 (a) Find the degree of the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$ (Ex. 2.12). [Answer: d^n]

(b) Find the degree of the Segre embedding of $\mathbf{P}^r \times \mathbf{P}^s$ in $\mathbf{P}^N$ (Ex. 2.14). [Answer: $\binom{r+s}{r}$]

(a) Let the homogeneous coordinate ring of its image be S, then we have k-vector space isomorphisms $S_l \cong k[x_0, \ldots, x_n]_{dl}$ (every degree l homogeneous polynomial in the degree d monomials of $x_0, \ldots, x_n$ is a degree dl homogeneous polynomial, and vice versa), hence $\varphi_S(l) = \varphi_{k[x_0,\ldots,x_n]}(dl) = \binom{dl+n}{n}$ which by definition shows the degree is d^n.

(b) We denote the homogeneous coordinate ring as S. Similarly we have $S_l \cong k[x_0, \ldots, x_r]_l \times k[y_0, \ldots, y_s]_l$, hence $\varphi_S(l) = \varphi_{k[x_0,\ldots,x_r]}(l)\varphi_{k[y_0,\ldots,y_s]}(l)$ whose leading term is $\frac{1}{r!s!}x^{r+s}$, hence the degree is $(r+s)!/r!s! = \binom{r+s}{r}$.

Exercise 7.2 Let Y be a variety of dimension r in $\mathbf{P}^n$, with Hilbert polynomial P_Y. We define the *arithmetic genus* of Y to be $p_a(Y) = (-1)^r(P_Y(0) - 1)$. This is an important invariant which (as we will see later in (III, Ex. 5.3)) is independent of the projective embedding of Y.

(a) Show that $p_a(\mathbf{P}^n) = 0$.

(b) If Y is a plane curve of degree d, show that $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

(c) More generally, if H is a hypersurface of degree d in $\mathbf{P}^n$, then $p_a(H) = \binom{d-1}{n}$.

(d) If Y is a complete intersection (Ex. 2.17) of surfaces of degrees a, b in $\mathbf{P}^3$, then $p_a(Y) = \frac{1}{2}ab(a+b-4) + 1$.

(e) Let $Y^r \subseteq \mathbf{P}^n$, $Z^s \subseteq \mathbf{P}^m$ be projective varities, and embed $Y \times Z \subseteq \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^N$ by the Segre embedding. Show that

$$p_a(Y \times Z) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z)$$

(a) This follows from $P_{\mathbf{P}^n}(z) = \binom{z+n}{n}$.

(b) (c) We have $P_H(z) = \binom{z+n}{n} - \binom{z-d+n}{n}$ (see (7.6 d)), hence

$$p_a(H) = (-1)^n(-d+n)(-d+n-1)\cdots(-d+1)/n! = \binom{d-1}{n}$$

(d) Similar to the solution of (7.7) we have

$$P_Y(z) = \binom{z+3}{3} - \binom{z-a+3}{3} - \binom{z-b+3}{3} + \binom{z-a-b+3}{3} = ab(z+2-\frac{a+b}{2})$$

The rest follows by definition.

(e) Let $S(Y), S(Z), S(Y \times Z)$ denote the corresponding homogeneous coordinate rings, then $S(Y \times Z)_l = S(Y)_l \otimes_k S(Z)_l$ for any degree l.[1] Hence $P_{Y\times Z}(z) = P_Y(z)P_Z(z)$, and the result follows from simple algebra.

[1] This is true for $Y = \mathbf{P}^n$ and $Z = \mathbf{P}^m$ as the defining relations for the Segre embedding forces the map $k[x_{ij}]_l \to k[y_i]_l \otimes_k k[z_j]_l, x_{ij} \mapsto y_i \otimes z_j$ to be an isomorphism. This argument descends to any varieties, as $I(Y \times Z)_l$ is precisely $I(Y)_l \otimes_k k[z_j]_l + k[y_i]_l \otimes_k I(Z)_l$.

Exercise 7.3 *The Dual Curve.* Let $Y \subseteq \mathbf{P}^2$ be a curve. We regard the set of lines in $\mathbf{P}^2$ as another projective space, $(\mathbf{P}^2)^*$, by taking (a_0, a_1, a_2) as homogeneous coordinates of the line $L : a_0x_0 + a_1x_1 + a_2x_2 = 0$. For each nonsingular point $P \in Y$, show that there is a unique line $T_P(Y)$ whose intersection multiplicity with Y at P is > 1. This is the *tangent line* to Y at P. Show that the mapping $P \mapsto T_P(Y)$ defines a *morphism* of $\operatorname{Reg} Y$ (the set of nonsingular points of Y) into $(\mathbf{P}^2)^*$. The closure of the image of this morphism is called the dual curve $Y^* \subseteq (\mathbf{P}^2)^*$ of Y.

As the intersection multiplicity for curves coincide with the local definition in exercise 5.4, we may take affine cover and assume Y is an affine curve defined by $f(x, y)$ in $\{x_2 = 0\} \cong \mathbf{A}^2$, and without loss of generality we assume P is $(0, 0)$. By exercise 5.3 we know that $f(x, y) = x - ay + g(x, y)$ (by a suitable coordinate change). Consider a line $l : y = bx$ (the line $x = 0$ is dealt with similarly), then the intersection multiplicity of l with Y is $\mu_P = \dim_k \mathcal{O}_P/(f, x-by) \cong \dim_k k[x]_{(x)}/(f(x, bx))$ which is the lowest degree of x in $f(x, bx)$ as polynomials with nonzero constant term is invertible in $k[x]_{(x)}$. The only b to make $\mu_P > 1$ is $b = a$.

More generally, a line l has > 1 intersection multiplicity with an affine curve $Y : f = 0$ at P if and only if l is the linear term of f (at P). In particular, it is defined by $(\partial f/\partial x(P))(x - x_P) + (\partial f/\partial y(P))(y - y_P) = 0$. Translating into the projective variety $Y : f = 0$, the desired map on P is

$$P \mapsto ((\partial f/\partial x(P)), (\partial f/\partial y(P)), (\partial f/\partial z(P))) \in (\mathbf{P}^2)^*$$

whose coordinates are all polynomials in P, i.e. it is a morphism.

Exercise 7.4 Given a curve Y of degree d in $\mathbf{P}^2$, show that there is a nonempty open subset U of $(\mathbf{P}^2)^*$ in its Zariski topology such that for each $L \in U, L$ meets Y in exactly d points. [*Hint*: Show that the set of lines in $(\mathbf{P}^2)^*$ which are either tangent to Y or pass through a singular point of Y is contained in a proper closed subset.] This result shows that we could hace defined the degree of Y to be the number d such that almost all lines in $\mathbf{P}^2$ meet Y in d points, where "almost all" refers to a nonempty open set of the set of lines, where this set is identified with the dual projective space $(\mathbf{P}^2)^*$.

There are only finite nonsingular points on Y (as $\operatorname{Sing} Y$ is closed), hence the lines in $(\mathbf{P}^2)^*$ that pass through a singular point is a finite union of proper closed sets. Those tangent to Y but don't pass through any singular points are contained in Y^* the dual curve, hence it suffices to prove that the dual curve is proper. If not, then the map of exercise 7.3 gives a dominant rational map from a curve Y to $(\mathbf{P}^2)^*$, which in turn corresponds to a map between function fields $k(x, y) \to K(Y)$ which doesn't exists because $K(Y)$ has transcendence degree 1.

Exercise 7.5 (a) Show tht an irreducible curve Y of degree $d > 1$ in $\mathbf{P}^2$ cannot have a point of multiplicity $\geq d$ (Ex. 5.3).

(b) If Y is an irreducible curve of degree $d > 1$ having a point of multiplicity $d - 1$, then Y is a rational curve (Ex. 6.1).

(a) Consider any line through $P \in Y$, by exercise 5.4 (a) and (c) we have $d = (L \cdot Y) \geq (L \cdot Y)_P \geq \mu_P(Y)$.

(b) Let point P be such that $\mu_P(Y) = d - 1$, and assume $P = (0, 0, 1)$. We know the number of lines l that $l \cap Y = \{P\}$ is finite (exercise 5.4 (b)). Consider the projection from P to the plane $\{z = 0\}$: it maps $(x, y, z) \in Y$ to $(x, y, 0)$. The defining polynomial f of Y must satisfy $\deg_z f = 1$ as $\mu_P(Y) = d - 1$, hence the inverse of the projection is well-defined and has rational coordinates, which induces a birational equivalence from Y to $\mathbf{P}^1$.

Exercise 7.6 *Linear Varieties.* Show that an algebraic set Y of pure dimension r (i.e., every irreducible component of Y has dimension r) has degree 1 if and only if Y is a linear variety (Ex. 2.11). [*Hint*: First, use (7.7) and treat the case $\dim Y = 1$. Then do the general case by cutting with a hyperplane and using induction.]

By (7.6 b) we know that Y is irreducible, i.e. a variety. If $\dim Y = 1$, take $x_1 \neq x_2 \in Y$, and consider any hyperplane H that pass through x_1, x_2; applying (7.7) we know Y is contained in such a hyperplane. Hence Y is contained in the intersection of all such hyperplanes, i.e. the line through x_1, x_2, hence Y is exactly the line.

If $\dim Y > 1$, by induction we know that every intersection of Y with a hyperplane $H \not\supseteq Y$ is linear, hence its intersection with any linear variety is linear. Therefore its intersection with the line x_1x_2 (where $x_1 \neq x_2 \in Y$) is linear, which must be the line itself, i.e. Y contains all lines containing two points on it. Thus it is easy to conclude that Y is linear.

Exercise 7.7 Let Y be a variety of dimension r and degree $d > 1$ in $\mathbf{P}^n$. Let $P \in Y$ be a nonsingular point. Define X to be the closure of the union of all lines PQ, where $Q \in Y, Q \neq P$.

(a) Show that X is a variety of dimension $r + 1$.

(b) Show that $\deg X < d$. [*Hint*: Use induction on $\dim Y$.]

We assume $P = (0, \ldots, 0, 1)$. Let the projection from P maps Y to $Y' \subseteq \{x_n = 0\} \cong \mathbf{P}^{n-1}$, then X can be naturally identified with the projective cone $\overline{C(\overline{Y'})}$ (exercise 2.10) of $\overline{Y'}$. We have $\dim Y' \leq \dim Y$ because the projection induces a map between function fields $K(Y') \to K(Y)$. In addition, Y' must be irreducible (otherwise two closed family of lines cover Y, which is irreducible), hence X is a variety; let Y' be defined by the polynomials $f_i(x_0, \ldots, x_{n-1}) = 0$, then X is also defined exactly by these polynomials.

(a) We have $Y \subsetneq X$, hence $\dim Y < \dim X = \dim Y' + 1 \leq \dim Y + 1$ so $\dim X = r + 1$.

(b) By the above analysis we have $S(X) \cong S(Y')[x_n]$, hence $P_X(l) = P_{Y'}(0) + \cdots + P_{Y'}(l)$. If $P_{Y'}$ begins with $\frac{d'}{r!}z^r$, then P_X begins with $\frac{d'}{(r+1)!}z^{r+1}$, hence $\deg X = \deg Y'$. As $\dim Y' = r$, we can find a generic linear varity of dimension $n - 1 - r$ in $\mathbf{P}^{n-1}$ that intersects Y' at $\deg Y'$ generic points. Its projective cone intersects Y at $\geq \deg Y' + 1$ (i.e. the $\deg Y'$ points corresponding to those on Y' and P), henc $\deg Y > \deg Y' = \deg X$.[2]

Exercise 7.8 Let $Y^r \subseteq \mathbf{P}^n$ be a variety of degree 2. Show that Y is contained in a linear subspace L of dimension $r+1$ in $\mathbf{P}^n$. Thus Y is isomorphic to a quadric hypersurface in $\mathbf{P}^{r+1}$ (Ex. 5.12).

Consider the construction in exercise 7.7; the variety X then must have dimention $r + 1$ and degree 1, which is the desired linear subspace.

[2]This is a very ambiguous answer, mainly because I couldn't find any answers on the Internet that is satisfactory. The closest I can find is this. Do let me know if you have any.