Exercise 5.1 Locate the singular points and sketch the following curves in $\mathbf{A}^2$ (assume $\operatorname{char} k \neq 2$). Which is which in Figure 4?

(a) $x^2 = x^4 + y^4$;
(b) $xy = x^6 + y^6$;
(c) $x^3 = y^2 + x^4 + y^4$;
(d) $x^2y + xy^2 = x^4 + y^4$.

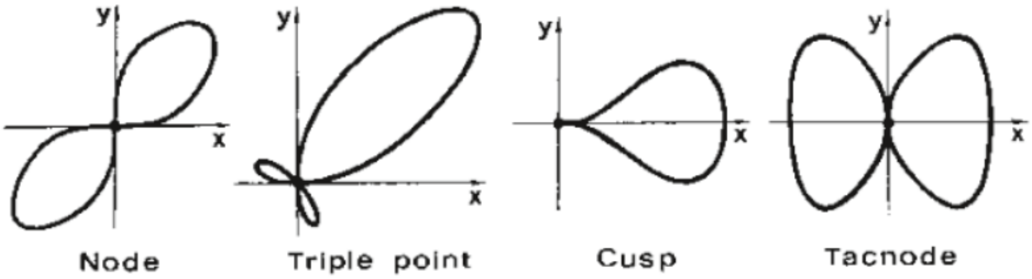

Figure 4. Singularities of plane curves.

By definition, these curves $f = 0$ are singular on (x, y) if and only if $\partial f/\partial x = \partial f/\partial y = 0$; by solving this equation we find that all of them have singular point $(0, 0)$.

(a) There is an automorphism of $k[[x, y]]$ sending x to $x(1 - x^2)^{1/2}$, hence around $(0, 0)$ this curve looks like the curve $x^2 = y^4$. This is a tacnode.

(b) Analogous to example 5.6.3 we have $xy - x^6 - y^6 = (x - y^5 + \cdots)(y - x^5 + \cdots)$ so around $(0, 0)$ this curve looks like $xy = 0$. This is a node.

(c) We have an automorphism of $k[[x, y]]$ sending y to $y(1+y^2)^{1/2}$ and x to $x(1-x)^{1/3}$; hence around $(0, 0)$ this curve looks like $y^2 = x^3$. This is a cusp.

(d) The polynomial $x^2y + xy^2 - x^4 - y^4$ can be factorized to $(x+p_2+\cdots)(y+q_2+\cdots)(x+y+p_2+q_2+\cdots)$ (see exercise 5.14 (c)), hence around $(0, 0)$ this curve looks like $xy(x+y) = 0$. This is a triple point.

Exercise 5.2 Locate the singular points and describe the singularities of the following surfaces in $\mathbf{A}^3$ (assume $\operatorname{char} k \neq 2$). Which is which in Figure 5?

(a) $xy^2 = z^2$;
(b) $x^2 + y^2 = z^2$;
(c) $xy + x^3 + y^3 = 0$.

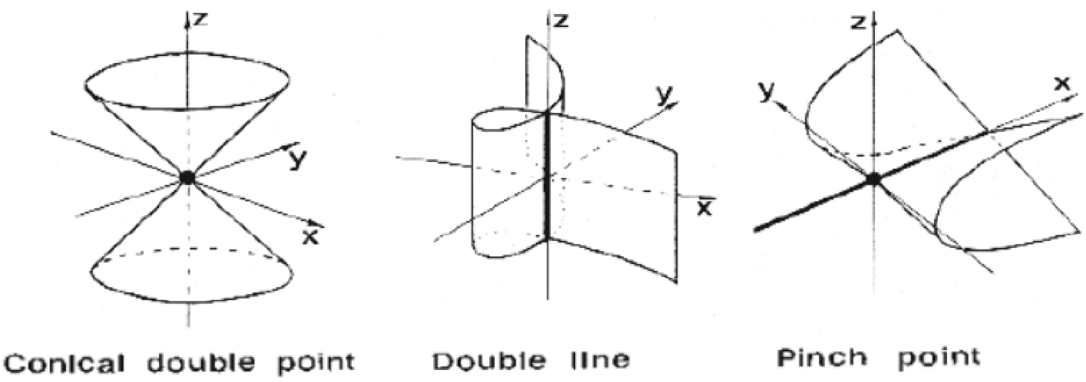

Figure 5. Surface singularities.

Again, the singular points has all of its partial deriatives 0.

(a) Its singular points are on the line $y = z = 0$. For $a \neq 0$ we know $(x-a)y^2 - z^2$ can be factored as $(-z + \sqrt{-a}y + \cdots)(z + \sqrt{-a}y + \cdots)$, so the singularity at $(a, 0, 0)$ is a double point (the intersection of two planes). On the other hand $z^2 - xy^2$ isn't factorizable in $k[[x, y, z]]$, so $(0, 0, 0)$ is a different singularity with the others. It is a pinch point.

(b) It has only one singular point $(0, 0, 0)$, and it is a conical double point. It is different from a double point or a pinch point: $x^2 + y^2 - z^2$ cannot be factorized, nor can it be written in the form $fg^2 = h^2$ where f, g, h has no constant terms.

(c) The singular points are on the line $x = y = 0$, and they are all double points as $xy+x^3+y^3$ is factorizable in $k[[x, y, z]]$. The line $x = y = 0$ is a double line.

Exercise 5.3 *Multplicities.* Let $Y \subseteq \mathbf{A}^2$ be a curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbf{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \cdots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the multiplicity of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y \Leftrightarrow \mu_P(Y) > 0$.) The linear factors of f_r are called the *tangent directions* at P.

(a) Show that $\mu_P(Y) = 1 \Leftrightarrow P$ is a nonsingular point of Y.

(b) Find the multiplicity of each of the singular points in (Ex. 5.1) above.

(a) As $f(0, 0) = 0$ we necessarily have $f_0 = 0$. It is easy to see $(\partial f/\partial x, \partial f/\partial y)(0, 0) = (\partial f_1/\partial x, \partial f_1/\partial y)(0, 0)$. As f_1 is linear, this matrix has rank 1 if and only if $f_1 \neq 0$, i.e. $\mu_P(X) = 1$.

(b) In exercise 5.1, the node, cusp, and the tacnode have multiplicity 2, while the triple point has multiplicity 3.

Exercise 5.4 *Intersection Multiplicity.* If $Y, Z \subseteq \mathbf{A}^2$ are two distinct curves, given by equations $f = 0, g = 0$, and if $P \in Y \cap Z$, we define the *intersection multiplicity* $(Y \cdot Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_P$-module $\mathcal{O}_P/(f, g)$.

(a) Show that $(Y \cdot Z)_P$ is finite, and $(Y \cdot Z)_P \geq \mu_P(Y) \cdot \mu_P(Z)$.

(b) If $P \in Y$, show that for almost all lines L through P (i.e., all but a finite number), $(L \cdot Y)_P = \mu_P(Y)$.

(c) If Y is a curve of degree d in $\mathbf{P}^2$, and if L is a line in $\mathbf{P}^2$, $L \neq Y$, show that $(L \cdot Y) = d$. Here we define $(L \cdot Y) = \sum (L \cdot Y)_P$ taken over all points $P \in L \cap Y$, where $(L \cdot Y)_P$ is defined using a suitable affine cover of $\mathbf{P}^2$.

Wthout loss of generality we assume $P = (0, 0)$. Let $r = \mu_P(Y)$ and $s = \mu_P(Z)$. We use f_r and g_s to denote the lowest nonzero homogeneous parts of f and g.

(a) Here $\mathcal{O}_P$ is the local ring of $\mathbf{A}^2$ at P. As submodules of $\mathcal{O}_P/(f, g)$ corresponds to submodules (ideals) of $\mathcal{O}_P$ containing (f, g), it suffices to prove that the ring $\mathcal{O}_P/(f, g)$ is Noetherian and Artinian, or that it is Noetherian of dimension 0. Its prime ideals have at least height 2 (properly containing (f)) but have height at most $\dim \mathcal{O}_P = 2$, hence every prime ideal is maximal, i.e. $\dim \mathcal{O}_P/(f, g) = 0$.
A composition series $0 \subset M_1 \subset \cdots \subset M_n = \mathcal{O}_P/(f, g)$ has subquotients isomorphic to $\mathcal{O}_P/\mathfrak{m}$ for some maximal ideal of $\mathcal{O}_P$, hence the subquotients are all isomorphic to k,

i.e. $(Y \cdot Z)_P = \dim_k \mathcal{O}_P/(f,g)$. In the polynomial ring $k[x,y]$, the homogeneous polynomials of degree n form a $n+1$ dimensional k-subspace S_n, and $\dim(S_n \cap (f,g)) \le \max\{0, n-r\} + \max\{0, n-s\} =: d_n$. Therefore there are at least $n+1-d_n$ linearly independent elements in S_n such that they are not $k[x,y]$-linear combinations of f_r and g_s, which gives $n+1-d_n$ linearly independent elements in $\mathcal{O}_P/(f,g)$. Taking sum gives $\dim_k \mathcal{O}_P/(f,g) \ge \sum\limits_{d_n \le n+1} (n+1-d_n) = rs$ by elementary algebra.

(b) By a linear transformation we may assume the line is given by $g : x = 0$. By the analysis in (a) we find $(L \cdot Y)_P = \dim_k \mathcal{O}_P/(f,g) = \dim_k k[x,y]_{(x,y)}/(f,x)$. Denote $h(y) = f(0,y)$, then the latter ring is isomorphic to $k[y]_{(y)}/(h)$. Let the lowest degree term of h be of degree r', then $h(y)/y^{r'}$ is invertible in $k[y]_{(y)}$, hence $k[y]_{(y)}/(h) \cong k[y]_{(y)}/(y^{r'}) \cong k[y]/(y^{r'})$, which has dimension r' over k. But $\mu_P(Y)$ is the lowest degree that occurred in $f(x,y)$, $k \neq \mu_P(Y)$ if and only if the lowest homogeneous part of f is a multiple of x. By undoing the linear transformation at the beginning, we see $(L \cdot Y)_P \neq \mu_P(Y)$ if and only if L is a factor of f_r, which can only occur for a finite number of times.

(c) By a linear transformation we may assume $L : x = 0$ and that Y does not pass through $(0,1,0)$, therefore we may work on the affine plane $z \neq 0$. The curve is given by $f(x,y) = 0$ where f is a degree d polynomial and contains the term y^d with a nonzero coefficient. Its intersection with L is given by solutions to $f(0,y) = 0$, and by the same analysis in (b) we find that the intersection multiplicity at a solution y_0 is exactly its algebraic multiplicity, which sums to $\deg_y f(0,y) = d$. Hence all of $(L \cdot Y)_P$ sums to d.

Exercise 5.5 For every degree $d > 0$, and every $p = 0$ or a prime number, give the equation of a nonsingular curve of degree d in $\mathbf{P}^2$ over a field k of characteristic p.

By exercise 5.9 we shall only look for f where $f = \partial f/\partial x = \partial f/\partial y = \partial f/\partial z = 0$ has no solution in $\mathbf{P}^2$. The easiest curve one can think of is the curve $x^d + y^d + z^d = 0$ which works for $p \nmid d$. For $3 \neq p | d$ we can choose the curve $xy^{d-1} + yz^{d-1} + zx^{d-1} = 0$, and for $3 = p|d$ we choose the curve $x^d + xy^{d-1} + yz^{d-1} + zx^{d-1} = 0$. They all satisfy the condition above.

Exercise 5.6 *Blowing Up Curve Singularities.*

(a) Let Y be the cusp or node of (Ex. 5.1). Show that the curve $\widetilde{Y}$ obtained by blowing up Y at $O = (0,0)$ is nonsingular (cf. (4.9.1) and (Ex. 4.10)).

(b) We define a *node* (also called *ordinary double point*) to be a double point (i.e., a point of multiplicity 2) of a plane curve with distinct tangent directions (Ex. 5.3). If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on the blown-up curve $\widetilde{Y}$. We say that "blowing up P resolves the singularity at P".

(c) Let $P \in Y$ be the tacnode of (Ex. 5.1). If $\varphi : \widetilde{Y} \to Y$ is the blowing-up at P, show that $\varphi^{-1}(P)$ is a node. Using (b) we see that the tacnode can be resolved by two successive blowings-up.

(d) Let Y be the plane curve $y^3 = x^5$, which has a "higher order cusp" at O. Show that O is a triple point; that blowing up O give rise to a double point (what kind?) and that one further blowing up resolves the singularity.

Note: We will see later (V, 3.8) that any singular point of a plane curve can can be resolved by a finite sequence of successive blowings-up.

(a) Example 4.9.1 blew up a node, while exercise 4.10 blew up a cusp. The results of the blowings-up are both nonsingular.

(b) $\widetilde{Y}$ intersects with the exceptional curve E at the points corresponding to the tangent directions, which there are two.

(c) The set $\widetilde{Y} \cup E$ is given by $x^2 = x^4 + y^4, x = yt$ (where $t \in \mathbf{A}^1$). By ruling out the other points on the exceptional curve, we see that the blow-up $\widetilde{Y}$ is defined by $y^2(1+t^4) = t^2$ and $x = yt$. The singularity at $(0,0,0)$ is a node because it has tangent directions $y = \pm t$.

(d) O is triple because the lowest degree of $y^3 = x^5$ is 3. Blowing up O gives the curve $y^3 = x^5, y = xu$, which shows $\widetilde{Y}$ is given by $u^3 = x^2, y = xu$, $(0,0,0)$ as a double point (actually a cusp).

Exercise 5.7 Let $Y \subseteq \mathbf{P}^2$ be a nonsingular plane curve of degree > 1, defined by the equation $f(x,y,z) = 0$. Let $X \subseteq \mathbf{A}^3$ be the affine variety defined by f (this is the cone over Y; see (Ex. 2.10)). Let P be the point $(0,0,0)$, which is the *vertex* of the cone. Let $\varphi : \widetilde{X} \to X$ be the blowing-up of X at P.

(a) Show that X has just one singular point, namely P.

(b) Show that $\widetilde{X}$ is nonsingular (cover it with open affines).

(c) Show that $\varphi^{-1}(P)$ is isomorphic to Y.

(a) Exercise 5.8 tells us that X is nonsingular at all points except P, and indeed P is singular.

(b) The blown-up curve is given by $f(1,s,t) = 0, y = xs, z = tx$ with coordinates (x,y,z,s,t) in $\mathbf{A}^5 = \mathbf{A}^3 \times \{(r,s,t) : r \neq 0\} \subseteq \mathbf{A}^3 \times \mathbf{P}^2$. We shall only verify

$$\operatorname{rank} \begin{pmatrix} 0 & 0 & 0 & \frac{\partial f}{\partial y}(1,s,t) & \frac{\partial f}{\partial z}(1,s,t) \\ s & -1 & 0 & x & 0 \\ t & 0 & -1 & 0 & x \end{pmatrix} = 3$$

which is equivalent to saying that $\partial f/\partial y, \partial f/\partial z$ cannot both vanish on $(1,s,t)$. This is because of Euler's lemma, $\partial f/\partial x(1,s,t) + s\partial f/\partial y(1,s,t) + t\partial f/\partial z(1,s,t) = \deg f \cdot f$, and f is nonsingular. The other open affines can be verified similarly.

(c) In fact, $\varphi^{-1}(P) = \{(P,x) : x \in \mathbf{P}^2, f(x) = 0\}$, which is clearly isomorphic to Y.

Exercise 5.8 Let $Y \subseteq \mathbf{P}^n$ be a projective variety of dimension r. Let $f_1, \ldots, f_t \in S = k[x_0, \ldots, x_n]$ be homogeneous polynomials which generate the ideal of Y. Let $P \in Y$ be a point, with homogeneous coordinates $P = (a_0, \ldots, a_n)$. Show that P is nonsingular on Y if and only if the rank of the matrix $\|(\partial f_i/\partial x_j)(a_0, \ldots, a_n)\|$ is $n - r$. [*Hint*: (a) Show that this rank is independent of the homogeneous coordinates chosen for P; (b) pass to an open affine $U_i \subseteq \mathbf{P}^n$ containing P and use the affine Jacobian matrix; (c) you will need Euler's lemma, which says that if f is a homogeneous polynomial of degree d, then $\sum x_i(\partial f/\partial x_i) = d \cdot f$.]

The partial derivatives are still homogeneous so this rank does not depend on the choice of homogeneous coordinates as the rows differ at most by a nonzero factor. Without loss of generality we assume $a_0 \neq 0$ in the homogeneous coordinates of P, thus $Y \cap \{x_0 \neq 0\}$ is an affine variety defined by $f_i(1, x_1, \ldots, x_n)$. Hence the 'projective' Jacobian matrix is the 'affine' Jacobian matrix with another column consisting of $(\partial f/\partial x_0)(1, a_1, \ldots, a_n)$. But Euler's lemma tells us that this column is a linear combination of the other columns (remember $f(P) = 0$), hence the two Jacobian matrices have the same rank. The rest is clear.

Exercise 5.9 Let $f \in k[x, y, z]$ be a homogeneous polynomial, let $Y = Z(f) \subseteq \mathbf{P}^2$ be the algebraic set defined by f, and suppose that for every $P \in Y$, at least one of $(\partial f/\partial x)(P), (\partial f/\partial y)(P), (\partial f/\partial z)(P)$ is nonzero. Show that f is irreducible (and hence that Y is a nonsingular variety). [*Hint*: Use (Ex. 3.7).]

If $f = gh$, then g and h must both be homogeneous. $Z(g)$ and $Z(h)$ must intersect at some point P, but the three partial derivtives all vanish on P, which is not possible by assumption.

Exercise 5.10 For a point P on a variety X, let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_P$. We define the *Zariski tangent space* $T_P(X)$ of X at P to be the dual k-vector space of $\mathfrak{m}/\mathfrak{m}^2$.

(a) For any point $P \in X$, $\dim T_P(X) \geq \dim X$, with equality if and only if P is nonsingular.

(b) For any morphism $\varphi : X \to Y$, there is a natural induced k-linear map $T_P(\varphi) : T_P(X) \to T_{\varphi(P)}(Y)$.

(c) If φ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map $T_0(\varphi)$ of tangent spaces at the origin is the zero map.

(a) Atiyah-Macdonald Corollary 11.15 with $\dim T_P(X) = \dim_k \mathfrak{m}/\mathfrak{m}^2$.

(b) A morphism induces a local map $\mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$, which induces $\mathfrak{m}_{\varphi(P)}/\mathfrak{m}^2_{\varphi(P)} \to \mathfrak{m}_P/\mathfrak{m}^2_P$, which induces the dual map $T_P(X) \to T_{\varphi(P)}(Y)$.

(c) The local ring of the x-axis is $k[x]_{(x)}$ while the local ring of the parabola at $(0, 0)$ is $(k[x, y]/(x - y^2))_{(x,y)} \cong k[y]_{(y)}$. The induced map on the local rings sends x to y^2, so the maximal ideal of $k[x]_{(x)}$ gets mapped to the square of the maximal ideal in the latter, hence induces the zero map on $\mathfrak{m}/\mathfrak{m}^2$, whose dual map is still the zero map.

Exercise 5.11 *The Elliptic Quartic Curve in* $\mathbf{P}^3$. Let Y be the algebraic set in $\mathbf{P}^3$ defined by the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Let P be the point $(x, y, z, w) = (0, 0, 0, 1)$, and let φ denote the projection from P to the plane $w = 0$. Show that φ induces an isomorphism of $Y - P$ with the plane cubic curve $y^2z - x^3 + xz^2 = 0$ minus the pont $(1, 0, -1)$. Then show that Y is an irreducible nonsingular curve. It is called the *elliptic quartic curve* in $\mathbf{P}^3$. Since it is defined by two equations it is another example of a complete intersection (Ex. 2.17).

Let $Z \subseteq \mathbf{P}^2$ be the curve defined by $y^2z - x^3 + xz^2 = 0$. φ induces a morphism $Y - P \to \mathbf{P}^3 : (x, y, z, w) \mapsto (x, y, z)$ which has image contained in $Z - \{(1, 0, -1)\}$[1]. The inverse $Z - \{(1, 0, -1)\} \to Y - P$ is given by $(x, y, z) \mapsto (x, y, z, \frac{yz}{x+z})$.

By exercise 5.8 it is easy to see that Y is nonsingular. $Y - P$ is dense in Y, but it is isomorphic to $Z - \{(1, 0, -1)\}$ which is irreducible, hence Y is irreducible.

Exercise 5.12 *Quadric Hypersurfaces*. Assume $\operatorname{char} k \neq 2$, and let f be a homogeneous polynomial of degree 2 in $x_0, \ldots, x_n$.

(a) Show that after a suitable linear change of variables, f can be brought into the form $f = x_0^2 + \cdots + x_r^2$ for some $0 \leq r \leq n$.

(b) Show that f is irreducible if and only if $r \geq 2$.

[1]In fact this need $\operatorname{char} k \neq 2$, otherwise the point $(1, 0, -1)$ is the same as $(1, 0, 1)$, which is in the image.

(c) Assume $r \geq 2$, and let Q be the quadric hypersurface in $\mathbf{P}^n$ defined by f. Show that the singular locus $Z = \operatorname{Sing} Q$ of Q is a linear variety (Ex. 2.11) of dimension $n-r-1$. In particular, Q is nonsingular if and only if $r = n$.

(a) This comes from linear algebra that a bilinear form is always similar to some $x_0^2 + \cdots + x_r^2$. Here the coefficients are absorbed into the square as k is algebraically closed.

(b) When $r = 1$ it can be factorized into $(x_0 + \sqrt{-1}x_1)(x_0 - \sqrt{-1}x_1)$. If $r \geq 2$ and f is not irreducible, then it have to be factorized into two linear factors, plugging in $x_3 = \cdots = x_r = 0$ we know that $f = 2$ is reducible. But when $r = 2$, f is irreducible by exercise 5.9.

(c) By a linear transformation we assume $f = x_0^2 + \cdots + x_r^2$, then the singular locus $Z = \{x_0 = \cdots = x_r = 0\} \subseteq \mathbf{P}^n$ which is a $n-r-1$-dimensional linear variety.

Exercise 5.13 It is a fact that any regular local ring is an integrally closed domain (Matsumura [2, Th. 36, p. 121]). Thus we see from (5.3) that any variety has a nonempty open subset of normal points (Ex. 3.17). In this exercise, show directly (without using (5.3)) that the set of nonnormal points of a variety is a proper closed subset (you will need the finiteness of integral closure: see (3.9A)).

We need a lemma: for an irreducible space X and its open subsets U and V, if a set S satisfy that $S \cap U$ and $S \cap V$ are closed sets in U and V respectively, then $S \cap (U \cup V)$ is closed in $U \cup V$. This is because the condition gives $S \cap U = \overline{S} \cap U$ and $S \cap V = \overline{S} \cap V$, so $S \cap (U \cup V) = \overline{S} \cap (U \cup V)$ a closed set.

This lemma enables us to assume the variety is affine. Denote its coordinate ring A, and $\mathfrak{m}_P$ for the maximal ideal of A corresponding to point P. As A modules we have $\overline{A_{\mathfrak{m}_P}} = \overline{A}_{\mathfrak{m}_P}$, where $\overline{\cdot}$ stands for integral closure; hence P is normal if and only if $A_{\mathfrak{m}_P} = \overline{A}_{\mathfrak{m}_P}$. By (3.9A) we have $\overline{A} = A[h_1, \ldots, h_n]$ with $h_i \in \operatorname{Frac} A$, so P is normal $\Leftrightarrow h_i \in A_{\mathfrak{m}_P}$ for all i.

Hence we shall only prove that the points P with $h \notin A_{\mathfrak{m}_P}$ is a closed set. In fact, it is the zero set of $\{f \in A : hf \in A\}$ (the possible denominators of h). The union of these proper closed sets corresponding to h_i is still a proper closed set.

Exercise 5.14 *Analytically Isomorphic Singularities.*

(a) If $P \in Y$ and $Q \in Z$ are analytically isomorphic plane curve singularities, show that the multiplicities $\mu_P(Y)$ and $\mu_Q(Z)$ are the same (Ex. 5.3).

(b) Generalize the example in the text (5.6.3) to show that if $f = f_r + f_{r+1} + \cdots \in k[[x,y]]$, and if the leading form f_r of f factors as $f_r = g_s h_t$, where $g_s h_t$ are homogeneous of degrees s and t respectively, and have no common linear factor, then there are formal power series

$$g = g_s + g_{s+1} + \cdots$$
$$h = h_t + h_{t+1} + \cdots$$

in $k[[x,y]]$ such that $f = gh$.

(c) Let Y be defined by the equation $f(x,y) = 0$ in $\mathbf{A}^2$, and let $P = (0,0)$ be a point of multiplicity r on Y, so that when f is expanded as a polynomial in x and y, we have $f = f_r +$ higher terms. We say that P is an *ordinary r-fold point* if f_r is a product of r *distinct* linear factors. Show that any two ordinary double points are analytically isomorphic. Ditto for ordinary triple points. But show that there is a one-parameter family of mutually nonisomorphic ordinary 4-fold points.

(d) Assume $\operatorname{char} k \neq 2$. Show that any double point of a plane curve is analytically isomorphic to the singularity at $(0,0)$ of the curve $y^2 = x^r$, for a uniquely determined $r \geq 2$. If $r = 2$ it is a node (Ex. 5.6). If $r = 3$ we call it a *cusp*; if $r = 4$ a *tacnode*. See (V, 3.9.5) for further discussion.

(a) If $\mathfrak{m}(= (x,y))$ is the maximal ideal of $\widehat{\mathcal{O}_P}$, then $\mu_P(X)$ is the largest integer r such that $\dim_k \widehat{\mathcal{O}_P}/\mathfrak{m}^r = \binom{r+1}{2}$. Then the claim is clear.

(b) We prove that if g_s and h_t have no common linear factors, then the ideal they generate contains all the homogeneous polynomials of degree $s+t-1$. Let

$$g_s = \sum_{0\leq i\leq s} a_i x^i y^{s-i}, h_t = \sum_{0\leq j\leq t} b_j x^j y^{t-j}, p = \sum_{0\leq i\leq t-1} v_i x^i y^{t-1-i}, q = \sum_{0\leq j\leq s-1} v_{t+i} x^i y^{s-1-i}$$

then the equation $g_s p + h_s q = \sum_{0\leq k\leq s+t-1} c_k x^k y^{s+t-1-k}$ turns into the matrix equation

$$\begin{pmatrix} v_0 & v_1 & \cdots & v_{s+t-1}\end{pmatrix}\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_s & & & \\ & a_0 & a_1 & a_2 & \cdots & a_s & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & a_0 & a_1 & a_2 & \cdots & a_s \\ b_0 & b_1 & b_2 & \cdots & b_t & & & \\ & b_0 & b_1 & b_2 & \cdots & b_t & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & b_0 & b_1 & b_2 & \cdots & b_t \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{s+t-1}\end{pmatrix}$$

the middle matrix is the Sylvester matrix of the polynomials $g_s(1,y)$ and $h_t(1,y)$. The conditions ensure that these polynomials are coprime, so the Sylvester matrix is invertible, hence this equation always has a solution. Therefore from g_s and h_t we can produce any homogeneous polynomial of degree $s+t-1$ (and of course higher). Thus we can determine the wanted $g_{s+1}, h_{t+1}, g_{s+2}, h_{t+2}, \ldots$ step by step.

(c) If P is an ordinary double point, then $f = gh$ where g and h have different degree 1 parts. Therefore there is an automorphism of $k[[x,y]]$ sending x to g and y to h, i.e. any ordinary double point is analytically isomorphic to the point $(0,0)$ at the curve $xy = 0$. The same holds for triple ordinary points; we can write f as pqr where p,q,r have different degree 1 parts, therefore under an automorphism of $k[[x,y]]$ we may assume $f = xy(x+y) + f_4 + \cdots$, and we consider an automorphism

$$\begin{aligned} x &\mapsto x + g_2 + \cdots \\ y &\mapsto y + h_2 + \cdots \end{aligned}$$

such that $xy(x+y) \mapsto f$; this automorphism exists because x^2+2xy, y^2+2xy are coprime, e.g. ths says $(y^2+2xy)g_2 + (x^2+2xy)h_2 = f_4$, etc.
Denote $f_a = xy(x+y)(x+ay)$. For generic a_1 and a_2, we prove that the 4-fold points $(0,0,0)$ at f_{a_1}, f_{a_2} are not isomorphic. If so, consider an isomorphism from $k[[x,y]]/(f_{a_1})$ to $k[[x,y]]/(f_{a_2})$, under it x,y maps to $p,q \in k[[x,y]]$. The power series p,q must generate the maximal ideal in $k[[x,y]]$, hence they have linearly independent degree 1 parts. We need $pq(p+q)(p+a_1q)$ to be a multiple of $xy(x+y)(x+a_2y)$, which is impossible once we consider the degree 1 parts of p,q and do a simple casework (generic a_1, a_2).

(d) If the degree 2 part of f factors into two different linear factors, then it is an ordinary double point and is analytically isomorphic to $y^2 = x^2$. Otherwise we can assume $f = y^2 + f_3 + f_4 + \cdots$, where f_k is homogeneous polynomial in x, y of degree k. We can take out the part of f_k where the degree of y is ≥ 2 and write

$$f = y^2(1 + \cdots) + g_3(x) + g_2'(x)y + g_4(x) + g_3'(x)y + \cdots$$

The $1+\cdots$ part has a square root in $k[[x, y]]$ so we assume $f = y^2 + g_3(x) + g_2'(x)y + \cdots$; therefore we can further let $y \mapsto y - 1/2(g_2'(x) + g_3'(x) + \cdots)$, so that $f = y^2 + h(x)$ where h is purely a polynomial in x. Taking r to be the lowest degree of h, there is an automorphism of $k[[x, y]]$ taking y to $y(h(x)/x^r)^{-1/2}$, hence quotienting f is the same as quotienting $y^2 - x^r$.
If $r < s$ and $k[[x, y]]/(y^2 - x^r) \cong k[[x, y]]/(y^2 - x^s)$, say under this isomorphism x, y gets mapped to p, q, then p, q have linearly independent degree 1 parts p_1, q_1 and $q^2 - p^r$ is a multiple of $y^2 - x^s$. Up to a constant factor, we assume

$$q^2 - p^r = (y^2 - x^s)(1 + h_1 + h_2 + \cdots)\ , \quad q = y + q_2 + q_3 + \cdots$$

from this we easily deduce that $y|q_n$ for $2 \leq n \leq r - 1$, hence considering the r-th degree we have $y|p_1^r$, which is impossible. Uniqueness follows.

Exercise 5.15 *Families of Plane Curves.* A homogeneous polynomial f of degree d in three variables x, y, z has $\binom{d+2}{2}$ coefficients. Let these coefficients represent a point in $\mathbf{P}^N$, where $N = \binom{d+2}{2} - 1 = \frac{1}{2}d(d+3)$.

(a) Show that this gives a correspondence between points of $\mathbf{P}^N$ and algebraic sets in $\mathbf{P}^2$ which can be defined by an equation of degree d. The correspondence is 1-1 except in some cases where f has a multiple factor.

(b) Show under this correspondence that the (irreducible) nonsingular curves of degree d correspond 1-1 to the points of a nonempty Zariski-open subset of $\mathbf{P}^N$. [*Hints*: (1) Use elimination theory (5.7A) applied to the homogeneous polynomials $\partial f/\partial x_0, \ldots, \partial f/\partial x_n$; (2) use the previous (Ex. 5.5, 5.8, 5.9) above.]

(a) This is obviously a correspondence. For f that does not have a multiple factor, if a degree d polynomial g has $Z(g) = Z(f)$, then $g \in \sqrt{(f)}$, i.e. $f|g^n$; splitting f into its irreducible factors, we immediately see $f|g$, hence f and g correspond to the same point in $\mathbf{P}^N$.

(b) Nonemptiness cones from exercise 5.5. By exercise 5.8, 5.9, f defines an irreducible nonsingular curve if and only if f and all the $\partial f/\partial x_i$ do not have a common zero (different from $(0, \ldots, 0)$), by (5.7A) we see that the points in $\mathbf{P}^N$ corresponding to a irreducible nonsingular curve is closed. Hence its compliment is open. The correspondence is 1-1 as such f cannot have a multiple factor.