Exercise 3.1 (a) Show that any conic in $\mathbf{A}^2$ is isomorphic either to $\mathbf{A}^1$ or $\mathbf{A}^1 - \{0\}$ (cf. Ex. 1.1)

(b) Show that $\mathbf{A}^1$ is not isomorphic to any proper open subset of itself. (This result is generalized by (Ex. 6.7) below.)

(c) Any conic in $\mathbf{P}^2$ is isomorphic to $\mathbf{P}^1$.

(d) We will see later (Ex. 4.8) that any two curves are homeomorphic. But show now that $\mathbf{A}^2$ is not even homeomorphic to $\mathbf{P}^2$.

(e) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

(a) For exercise 1.1 and (3.7) we set that any conic is isomorphic to $y = x^2$ or $xy = 1$, while the former is isomorphic to $\mathbf{A}^1$ again by (3.7). The latter is isomorphic to $\mathbf{A}^1 - \{0\}$ through the map $(x, x^{-1}) \leftrightarrow x$, which is easily verified to be a morphism in both directions.

(b) The only possible regular functions on $\mathbf{A}^1$ are the polynomials ($\mathcal{O}(\mathbf{A}^1) = A(\mathbf{A}^1) = k[x]$), therefore any nonconstant regular function on $\mathbf{A}^1$ must have a zero point. But it is easy to construct nonconstant regular functions (even polynomials) on proper open subsets of $\mathbf{A}^1$ such that it does not vanish anywhere.

(c) Any homogeneous polynomial of degree 2 in three variables corresponds to a bilinear form which can be orthogonally diagonalized to the form $f = f_0^2 + f_1^2 + f_2^2$ (the coefficients are absorbed into the polynomials), where f_i have degree at most 1. Irreducibility of f tells us that none of f_i is zero, hence all of them have degree 1. The change of basis matrix (from (x, y, z) to (f_0, f_1, f_2)) is clearly invertible, so every conic is isomorphic to the conic $x^2 + y^2 + z^2 = 0$. Conversely, every conic is isomorphic to any else, such as $xy = z^2$. The map $Z(xy - z^2) \leftrightarrow \mathbf{P}^1, (x, y, z) \leftrightarrow (y, z)$ is easily verified to be an isomorphism.

(d) The topological fact "Any two irreducible one-dimensional closed subset of X intersects" holds on $\mathbf{P}^2$ (exercise 3.7a) but not on $\mathbf{A}^2$.

(e) Let this variety be X. We have $A(X) = \mathcal{O}(X) = k$ hence X has to be only one point.

Exercise 3.2 A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

(a) For example, let $\varphi : \mathbf{A}^1 \to \mathbf{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbf{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

(b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbf{A}^1 \to \mathbf{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism*.

(a) The proper closed subsets of $\mathbf{A}^1$ and the curve $C : y^2 = x^3$ are both finite points. Hence the bijection φ is bicontinuous. However, $\mathcal{O}(C) = A(C) = k[x, y]/(x^3 - y^2) = k[x, x^{\frac{3}{2}}]$, which is not a polynomial ring in one variable.[1]

(b) The p-th root is unique in k, hence φ is bijective; the proper closed sets in both sides are finite sets, so φ is bicontinuous. But taking p-th root is not a rational function, hence the map φ^{-1} is not a morphism.

[1] The quickest way to see this may be noticing the fact that it is not UFD.

Exercise 3.3 (a) Let $\varphi : X \to Y$ be a morphism. Then for each $P \in X$, φ induces a homomorphism of local rings $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$.

(b) Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism, and the induced map φ_P^* on local rings is an isomorphism, for all $P \in X$.

(c) Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is *injective* for all $P \in X$.

(a) $\varphi_P^* : \langle U, f \rangle \mapsto \langle \varphi^{-1}(U), f \circ \varphi \rangle$.

(b) Notice that φ is a morphism is equivalent to saying that φ is continuous, and that $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}, \langle U, f \rangle \mapsto \langle \varphi^{-1}(U), f \circ \varphi \rangle$ is well defined. The same goes for φ^{-1}.

(c) If $\langle \varphi^{-1}(U), f \circ \varphi \rangle = \langle \varphi^{-1}(V), g \circ \varphi \rangle$, then $f \circ \varphi = g \circ \varphi$ on $\varphi^{-1}(U) \cap \varphi^{-1}(V) = \varphi^{-1}(U \cap V)$, hence $f = g$ on $\varphi(\varphi^{-1}(U \cap V))$. If $\varphi(X)$ is dense in Y, then $\varphi(\varphi^{-1}(U \cap V)) = U \cap V \cap \varphi(X)$ is dense in $U \cap V$, hence $f = g$ on $U \cap V$, i.e. $\langle U, f \rangle = \langle V, g \rangle$.

Exercise 3.4 Show that the d-uple embedding of $\mathbf{P}^n$ (Ex. 2.12) is an isomorphism onto its image.

We have already proven that it is a homeomorphism in exercise 2.12. It is also obviously a morphism. It remains to prove that its inverse is also a morphism, i.e. any rational function f/g with $\deg f = \deg g$ can be written as f'/g' where f', g' are polynomials with variables $M_0, \ldots, M_N$, and $\deg f' = \deg g'$. We take $f' = fg^{d-1}$ and $g' = g^d$, making their degrees (with repect to x_i) multiples of d, so that they are also polynomials of the M_j.

Exercise 3.5 By abuse of language, we will say that a variety "is affine" if it is isomorphic to an affine variety. If $H \subseteq \mathbf{P}^n$ is any hypersurface, show that $\mathbf{P}^n - H$ is affine. [*Hint*: Let H have degree d. Then consider the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$ and use the fact that $\mathbf{P}^N$ minus a hyperplane is affine.]

Following the hint, let $H = Z(f)$ with f homogeneous having degree d and consider the d-uple embedding of $\mathbf{P}^n$. The variety $\mathbf{P}^n - H$ is isomorphic to its image, which is the intersection of $\mathbf{P}^N - Z(f)$ (which is isomorphic to $\mathbf{A}^N$ because f is linear in $M_0, \ldots, M_N$, $Z(f)$ is a hyperplane) with the image of $\mathbf{P}^n$ (which is closed), hence is affine.

Exercise 3.6 There are quasi-affine varieties that are not affine. For example, show that $X = \mathbf{A}^2 - \{(0,0)\}$ is not affine. [*Hint*: Show that $\mathcal{O}(X) \cong k[x,y]$ and use (3.5). See (III, Ex. 4.3) for another proof.]

We know that if variety Y is dense in variety X, then $\mathcal{O}(X) \subseteq \mathcal{O}(Y)$ because every regular function on X is regular on Y and if two regular functions are the same on Y then they must be the same on X. We also know that $Z_1 = \mathbf{A}^2 - \{(0,t) : t \in k\}$ is affine because it is isomorphic to $Z(x_1x_2 - 1)$ in $\mathbf{A}^3$, through $(x,y) \mapsto (x, x^{-1}, y)$. Under this isomorphism we know $\mathcal{O}(Z_1) = k[x, x^{-1}, y]$, with $\mathcal{O}(\mathbf{A}^2) = k[x,y]$ regarded as its subring. Similarly for $Z_2 = \mathbf{A}^2 - \{(t,0) : t \in k\}$ we have $\mathcal{O}(Z_2) = k[x, y, y^{-1}]$.

We have $\mathcal{O}(\mathbf{A}^2) \subseteq \mathcal{O}(X) \subseteq \mathcal{O}(Z_1) \cap \mathcal{O}(Z_2) = k[x,y]$, hence $\mathcal{O}(X) = k[x,y]$. If it is affine, it must be isomorphic to $\mathbf{A}^2$. The injection $X \hookrightarrow \mathbf{A}^2$ corresponds to $\mathrm{id}_{k[x,y]}$ under the map of (3.5). The composition $\mathbf{A}^2 \cong X \hookrightarrow \mathbf{A}^2$ corresponds to an invertible map of $\mathcal{O}(\mathbf{A}^2)$, but this map is not invertible, contradiction.

Exercise 3.7 (a) Show that any two curves in $\mathbf{P}^2$ have a nonempty intersection.

(b) More generally, show that if $Y \subseteq \mathbf{P}^n$ is a projective variety of dimension ≥ 1, and if H is a hypersurface, then $Y \cap H \neq \varnothing$. [*Hint*: Use (Ex. 3.5) and (Ex. 3.1e). See (7.2) for a generalization]

(a) See (b).

(b) If so, then Y is closed in $\mathbf{P}^n - H$, which is affine, making Y an affine variety, but it is also projective. Hence by (3.1e) it must be only one point, contradiction.

Exercise 3.8 Let H_i and H_j be the hyperplanes in $\mathbf{P}^n$ defined by $x_i = 0$ and $x_j = 0$ with $i \neq j$. Show that any regular function on $\mathbf{P}^n - (H_i \cap H_j)$ is constant. (This gives an alternate proof of (3.4a) in the case $Y = \mathbf{P}^n$.)

Under the isomorphism $\mathbf{P}^n - H_i \cong \mathbf{A}^n$ we know $\mathcal{O}(\mathbf{P}^n - H_i) = k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}]$ ($x_0, \ldots, x_n$ are coordinates on $\mathbf{P}^n$), whose elements are of the form $f/x_i^{\deg f}$ where f is homogeneous. A regular function on $\mathbf{P}^n - (H_i \cap H_j)$ must be a regular function on both $\mathbf{P}^n - H_i$ and $\mathbf{P}^n - H_j$, hence must be a constant.

Exercise 3.9 The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbf{P}^1$, and let Y be the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$. Then $X \cong Y$(Ex. 3.4). But show that $S(X) \ncong S(Y)$.

We have $S(X) = k[x, y]$ and $S(Y) = k[x, y, z]/(y^2 - xz)$. They are not isomorphic because the latter is not UFD.

Exercise 3.10 *Subvarieties.* A subset of a topological space is *locally closed* if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also a quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the *induced structure* on Y, and we call Y a *subvariety* of X.

Now let $\varphi : X \to Y$ be a morphism, let $X' \subseteq X$ and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\varphi(X') \subseteq Y'$. Show that $\varphi|_{X'} : X' \to Y'$ is a morphism.

The injection $X' \hookrightarrow X$ is obviously a morphism so without loss of generality we can assume $X' = X$. Fix an open subset U of Y', a regular function $f : U \to k$, and a point $P \in \varphi^{-1}(U)$. By definition there is an open set $V' \subseteq U$ of Y' containing $\varphi(P)$ such that f can be represented as a rational function g/h on V'. Let $V' = V \cap Y'$ where $V \subseteq Y$ is open.

The set U_0 where h is nonzero is open in Y, and $V' \subseteq U_0 \cap Y'$. Thus g/h is a regular function on U_0. We know φ is a morphism, so $g/h \circ \varphi : \varphi^{-1}(U_0) \to k$ is regular. Hence there is an open set $W \subseteq \varphi^{-1}(U_0)$ of X containing P such that $g/h \circ \varphi$ is a rational function on W. On the open set $W \cap \varphi^{-1}(V) \subseteq \varphi^{-1}(V')$ we have $f \circ \varphi = g/h \circ \varphi$ is rational.

The above argument holds for all $P \in \varphi^{-1}(U)$ hence $f \circ \varphi : \varphi^{-1}(U) \to k$ is regular; as $\langle U, f \rangle$ is arbitrary, we have $\varphi|_X$ is a morphism.

Exercise 3.11 Let X be any variety and let $P \in X$. Show there is a 1-1 correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

We first prove that for an irreducible topological space X, its open subspace U and a point $P \in U$, the closed irreducible subsets of X containing P are in 1-1 correspondence with the closed irreducible closed subsets of U containing P. For a closed irreducible subset $C \subseteq X$ we have $C \cap U$ is a closed irreducible subset of U, while for a closed irreducible subset $C' \subseteq U$ we have its closure $\overline{C'}$ is a closed irreducible subset of X. As U is dense in X, these operations are inverses of each other.

We then prove that for an affine (or projective) variety X and its open set U, for $P \in U$ we have $\mathcal{O}_{P,X} = \mathcal{O}_{P,U}$. We define homomorphisms $\varphi : \mathcal{O}_{P,X} \to \mathcal{O}_{P,U}, \langle V, f\rangle \mapsto \langle V \cap U, f\rangle$ and $\psi : \mathcal{O}_{P,U} \to \mathcal{O}_{P,X}, \langle V, f\rangle \mapsto \langle V, f\rangle$. It is easy to verify they are indeed well defined and are inverses of each other.

In the main problem, if X is locally closed in a projective space, then intersection with one of the open sets $U_i : x_i \neq 0$ reduces the claim to (quasi-)affine varieties, which can then be reduced to the case of affine varieties. By (3.2c) this claim follows by facts about localization.

Exercise 3.12 If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$. [*Hint*: Reduce to the affine case and use (3.2c).]

The reduction process is identical to exercise 3.11 (the dimension remains the same), and (3.2c) finishes.

Exercise 3.13 *The Local Ring of a Subvariety.* Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $\langle U, f\rangle$ where $U \subseteq X$ is open, $U \cap Y \neq \varnothing$, and f is a regular function on U. We say $\langle U, f\rangle$ is equivalent to $\langle V, g\rangle$, if $f = g$ on $U \cap V$. Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim X - \dim Y$. It is the *local ring* of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$, and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

If $f(U \cap Y) \neq \{0\}$, then $\langle U - f^{-1}(0), 1/f\rangle$ is an inverse of $\langle U, f\rangle$. Thus all the non-invertible elements of $\mathcal{O}_{Y,X}$ are those $\langle U, f\rangle$ with $f(U \cap Y) = \{0\}$, which forms an ideal $\mathfrak{m}$, hence maximal, which means $\mathcal{O}_{Y,X}$ is local. The homomorphism $\mathcal{O}_{Y,X} \to K(Y)$ which maps $\langle U, f\rangle$ to $\langle U \cap Y, f\rangle$ has kernel $\mathfrak{m}$. This map is surjective because every element in $K(Y)$ can be identified as a local rational function, and rational functions are easy to extend to some open set of X. Hence the residue field of $\mathcal{O}_{Y,X}$ is exactly $K(Y)$.

Pick an arbitrary $P \in Y$. We define a prime ideal $\mathfrak{p}_Y = \{\langle U, f\rangle \in \mathcal{O}_{P,X} : f(U \cap Y) = \{0\}\}$. The restriction map $\mathcal{O}_{P,X} \to \mathcal{O}_{P,Y}$ has kernel $\mathfrak{p}_Y$ and is surjective due to the exact same reason as above, hence $\mathcal{O}_{P,X}/\mathfrak{p}_Y \cong \mathcal{O}_{P,Y}$. The map $(\mathcal{O}_{P,X})_{\mathfrak{p}_Y} \to \mathcal{O}_{Y,X}, \langle U, f\rangle/\langle V, g\rangle \mapsto \langle U \cap (V - g^{-1}(0)), f/g\rangle$ has trivial kernel and is surjective because a regular function is locally a quotient of two polynomials. Hence it is an isomorphism. Thus

$$\dim \mathcal{O}_{Y,X} = \operatorname{height} \mathfrak{p}_Y = \dim \mathcal{O}_{P,X} - \dim \mathcal{O}_{P,Y} = \dim X - \dim Y$$

Exercise 3.14 *Projection from a Point.* Let $\mathbf{P}^n$ be a hyperplane in $\mathbf{P}^{n+1}$ and let $P \in \mathbf{P}^{n+1} - \mathbf{P}^n$. Define a mapping $\varphi : \mathbf{P}^{n+1} - \{P\} \to \mathbf{P}^n$ by $\varphi(Q) =$ the intersection of the unique line containing P and Q with $\mathbf{P}^n$.

(a) Show that φ is a morphism.

(b) Let $Y \subseteq \mathbf{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbf{P}^1$ (Ex. 2.12). It t, u are the homogeneous coordinates on $\mathbf{P}^1$, we say that Y is the curve given parametrically by $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$. Let $P = (0, 0, 1, 0)$, and let $\mathbf{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

(a) By composing with a linear morphism we can assume the hyperplane $\mathbf{P}^n$ is given by $x_0 = 0$. Let P have homogeneous coordinates $(y_0, y_1, \ldots, y_{n+1})$, then φ is given by

$$(z_0, \ldots, z_{n+1}) \mapsto (0, y_0z_1 - z_0y_1, \ldots, y_0z_{n+1} - z_0y_{n+1})$$

which is obviously a morphism.

(b) The image of point (t^3, t^2u, tu^2, u^3) is $(t^3, t^2u, 0, u^3)$. Its equation is $x^2z = y^3$.

Exercise 3.15 *Products of Affine Varieties.* Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbf{A}^{n+m}$ with its induced topology is irreducible. [*Hint*: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X | x \times Y \subseteq Z_i\}$, $i = 1, 2$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the *product* of X and Y. Note that its topology is in general not equal to the product topology.

(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties, i.e., show (i) the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and (ii) given a variety Z, and the morphisms $Z \to X, Z \to Y$, there is a unique morphism $Z \to X \times Y$ making a commutative diagram

$$\begin{tikzcd}
Z \arrow[rr, dashed] \arrow[dr] \arrow[drr] & & X \times Y \arrow[dl] \arrow[d] \\
& X & Y
\end{tikzcd}$$

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

(a) Notice that we have homeomorphisms $f_x : x \times Y \to Y$ and $f_y : X \times y \to X$. Following the hint, we define $X_i = \{x \in X | x \times Y \subseteq Z_i\}$, $i = 1, 2$. For any $x \in X$, the sets $(x \times Y) \cap Z_1$ and $(x \times Y) \cap Z_2$ are closed sets whose union is Y, thus by irreducibility of Y we know one of them must be Y itself. Hence $X_1 \cup X_2 = X$. We have $X_1 = \bigcap_{y \in Y} f_y(Z_1 \cap (X \times y))$, which is closed because it is an intersection of closed sets. Similarly X_2 is closed, hence $X = X_1$ or X_2, thus $X \times Y = Z_1$ or Z_2.

(b) We identify $k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m]$ with $k[x_1, \dots, x_{n+m}]$ by $y_i = x_{n+i}$. Let $\mathfrak{a}$ be the ideal $I(X) \otimes_k k[y_1, \dots, y_m] + k[x_1, \dots, x_n] \otimes_k I(Y)$, then $Z(\mathfrak{a}) = X \times Y$. However we have

$$\frac{k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m]}{\mathfrak{a}} \cong \frac{k[x_1, \dots, x_n]}{I(X)} \otimes_k \frac{k[y_1, \dots, y_m]}{I(Y)}$$

The right hand side is integral hence $\mathfrak{a}$ is prime; thus $I(X \times Y) = \sqrt{\mathfrak{a}} = \mathfrak{a}$ and $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) The pullback of any rational function along a projection is a rational function in the first n (or last m) variables, hence the projections are morphisms. For morphisms $f : Z \to X$ and $g : Z \to Y$, the unique map $(f, g) : Z \to X \times Y, z \mapsto (f(z), g(z))$ satisfies the commutative diagram. It is a morphism because f and g both have locally rational coordinates, hence (f, g) also has locally rational coordinates.

(d) By (b) we shall only prove that $\dim A \otimes_k B = \dim A + \dim B$ for finitely generated k-algebras A and B. Noether normalization shows that there are polynomial subalgebras $A_0 \subseteq A$ and $B_0 \subseteq B$ such that A and B are integral over them. $A \otimes_k B$ is integral over the polynomial subalgebra $A_0 \otimes_k B_0$ because any $a \otimes 1$ and $1 \otimes b$ is integral over it, and the sum and product of integral elements are still integral. Thus $\dim A \otimes_k B = \dim A_0 \otimes_k B_0 = \dim A_0 + \dim B_0 = \dim A + \dim B$, where the middle equality is obvious for polynomial algebras.

Exercise 3.16 *Products of Quasi-Projective Varieties.* Use the Segre embedding (Ex. 2.14) to identify $\mathbf{P}^n \times \mathbf{P}^m$ with its image and hence give it a structure of projective variety. Now for any two quasi-projective varieties $X \subseteq \mathbf{P}^n$ and $Y \subseteq \mathbf{P}^m$, consider $X \times Y \subseteq \mathbf{P}^n \times \mathbf{P}^m$.

(a) Show that $X \times Y$ is a quasi-projective variety.

(b) If X, Y are both projective, show that $X \times Y$ is projective.

(c) Show that $X \times Y$ is a product in the category of varieties.

(a) Let $X = O_X \cap C_X$ and $Y = O_Y \cap C_Y$ where O_X, O_Y are open and C_X, C_Y are closed. Then $X \times Y = (O_X \times O_Y) \cap (C_X \times C_Y)$, which is locally closed in the product topology. The Segre embedding gives $\mathbf{P}^n \times \mathbf{P}^m$ a finer topology than the product topology[2], so $X \times Y$ is still locally closed in the variety $\mathbf{P}^n \times \mathbf{P}^m$. Then we prove $\overline{X \times Y} = \overline{X} \times \overline{Y}$; the inclusion $\subseteq$ is obvious. We notice that $X \times y$ is homeomorphic to X, hence if a closed set $C \supseteq X \times Y$, then $C \cap (X \times y) \supseteq \overline{X} \times y$; thus $\overline{X \times Y} \supseteq \overline{X} \times Y$, by taking another closure we get $\overline{X \times Y} \supseteq \overline{X} \times \overline{Y}$.
We only need to prove the irreducibility of $\overline{X} \times \overline{Y}$, which is proven in (b).

(b) The Segre embedding is finer than the product topology hence $X \times Y$ is closed. Irreducibility of $X \times Y$ is proven exactly like the affine case, exercise 3.15 (a).

(c) Fix an open set $U \subseteq X$, a regular function $f : U \to k$ and $P \in U$. By definition there is an open set $V_P \subseteq U$ such that $f = g/h$ on V_P where g, h are homogeneous polynomials of the same degree. The map $f \circ \pi_X : U \times Y \to k$ is regular, because any $Q \in Y$ is inside one of $U_i = \{y_i \neq 0\}$, hence in the open set $V_p \times U_i \ni (P, Q)$ we have $f \circ \pi = g(y_i x_0, \ldots, y_i x_n)/h(y_i x_0, \ldots, y_i x_n)$ which is rational in z_i. By definition we know π_X (similarly π_Y) is a morphism.
For morphisms $f : Z \to X$ and $g : Z \to Y$ the unique map $(f, g) : Z \to X \times Y, z \mapsto (f(z), g(z))$ satisfies the commutative diagram, where $X \times Y$ is given the subspace topology. We shall only prove that it is a morphism. This is because f and g have locally rational coordinates (as $\frac{x_i}{x_j} \circ f$ is rational for all i, in the open set where $x_j \circ f$ is nonzero we can assume $x_j \circ f = 1$ by homogeneity), hence (f, g) has locally rational coordinates, and composing with the Segre embedding does not change this fact.

Exercise 3.17 *Normal Varieties.* A variety Y is *normal at a point* $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is *normal* if it is normal at every point.

(a) Show that every conic in $\mathbf{P}^2$ is normal.

(b) Show that the quadric surfaces Q_1, Q_2 in $\mathbf{P}^3$ given by equations $Q_1 : xy = zw$; $Q_2 : xy = z^2$ are normal (cf. (II. Ex. 6.4) for the latter.)

(c) Show that the cuspidal cubic $y^2 = x^3$ in $\mathbf{A}^2$ is not normal.

(d) If Y is affine, then Y is normal $\Leftrightarrow A(Y)$ is integrally closed.

(e) Let Y be an affine variety. Show that there is a normal affine variety $\widetilde{Y}$, and a morphism $\pi : \widetilde{Y} \to Y$, with the property that whenever Z is a normal variety, and $\varphi : Z \to Y$ is a *dominant* morphism (i.e., $\varphi(Z)$ is dense in Y), then there is a unique morphism $\theta : Z \to Y$ such that $\varphi = \pi \circ \theta$. $\widetilde{Y}$ is called the *normalization* of Y. You will need (3.9A) above.

[2] e.g. for a closed set $Z(f) \subseteq \mathbf{P}^n$ with f homogeneous, the set $Z(f) \times \mathbf{P}^m$ is the zero set of $m+1$ polynomials $f(z_{0j}, z_{1j}, \ldots, z_{nj})$ where $z_{ij} = x_i y_j$, because at least one of y_j is nonzero. The argument is the same for sets of the form $\mathbf{P}^n \times Z(g)$, and can be naturally extended to the zero sets of more than one polynomials. As they form a basis of closed sets in the product topology, every closed set is still closed in the subspace topology.

(a) Normality is conserved under isomorphism, but every conic in $\mathbf{P}^2$ is isomorphic to $\mathbf{P}^1$ (exercise 3.1 (c)), which is normal.

(b) To prove such projective varieties X are normal, we can take its affine cover $U_i \cap X$ and show this affine variety is normal. For Q_1, every intersection with U_i is isomorphic to the affine variety $Z(z - xy) \in \mathbf{A}^3$. Its coordinate ring is $k[x, y]$ hence isomorphic to $\mathbf{A}^2$, which is normal. For Q_2, we have $U_x \cap Q_2, U_y \cap Q_2$ are both isomorphic to $Z(y - x^2) \in \mathbf{A}^2$ which is normal. The intersection $U_w \cap Q_2$ is isomorphic to $Z(xy - z^2) \in \mathbf{A}^3$ which has coordinate ring $A = k[x, y, z]/(xy - z^2)$. It suffices to prove A is integrally closed, as this poroperty is preserved under localization (integral closure is preserved under localization, see Atiyah-Macdonald Proposition 5.12).
The localization $A_x = k[x, x^{-1}, z] \cong k[x, z]_x$ and $A_y \cong k[y, z]_y$ are integrally closed. We have $A = A_x \cap A_y \subset \operatorname{Frac} A$: if $f/x^n = g/y^m$, then $x^n g - y^m f = (xy - z^2)h$ as polynomials for some h. Letting $y = z = 0$ we find $y|g$, thus $y|h$; we can then lower the exponents of m recursively, hence $y^m|g$ and similarly $x^n|f$. Thus $A = A_x \cap A_y$ is integrally closed.

(c) This variety has coordinate ring $k[x, y]/(y^2 - x^3) \cong k[t^2, t^3]$, which is not integrally closed because t is a root of $x^2 - t^2 = 0$ but is not in the coordinate ring. Thus it is not normal, cf. (d) below.

(d) Integrally closed is a local property, cf. Atiyah-Macdonald Proposition 5.13.

(e) By (3.8) we take $\pi : \widetilde{Y} \to Y$ to be the morphism corresponding to the inclusion $A(Y) \hookrightarrow A'$ where A' is the integral closure of $A(Y)$ in $\operatorname{Frac} A(Y)$. The case where Z is quasi-affine can be reduced to the case of Z affine as it is dense in its closure. The case where Z is (quasi-)projective can be reduced to (quasi-)affine, as Z always has an affine cover[3]. Thus we focus on the case where Z is an affine variety.
The affine variety case translates to the algebra fact below: if a homomorphism $f : A \to B$ between finitely generated integral k-algebras can be extended to a k-algebra homomorphism $\operatorname{Frac} A \to \operatorname{Frac} B$ (which is the condition of φ being dominant, cf. (4.4)), and if B is integrally closed, then f can be extended to $f' : A' \to B$, where A' is the integral closure of A in $\operatorname{Frac} A$. Extendability is the consequence of integrality, and uniqueness is the consequence of injectivity of f, being a restriction of a field inclusion.

Exercise 3.18 *Projectively normal varieties.* A projective variety $Y \subseteq \mathbf{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

(a) If Y is projectively normal, then Y is normal.

(b) There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic curve in $\mathbf{P}^3$ given parametrically by $(x, y, z, w) = (t^4, t^3u, tu^3, u^4)$. Then Y is normal but not projectively normal. See (III, Ex. 5.6) for more examples.

(c) Show that the twisted quartic curve Y above is isomorphic to $\mathbf{P}^1$, which is projectively normal. Thus projective normality depends on the embedding.

(a) We know from exercise 2.6 that $A(U_i \cap Y)$ can be identified with the subring of elements with degree 0 in $S(Y)_{x_i}$. The latter is integrally closed, and thus $A(U_i \cap Y)$ must be because the root of a polynomial with degree 0 coefficients must also has degree 0.

[3]Let Z_i cover Z where Z_i is (quasi-)affine. Then we apply the claim to get $Z_i \to \widetilde{Y}$ and notice that these maps agree on $Z_i \cap Z_j$ due to the exact same claim on $Z_i \cap Z_j$.

(b) $S(Y) \cong k[t^4, t^3u, tu^3, u^4]$. The element t^2u^2 is in its integral closure but not in itself, hence $S(Y)$ is not integrally closed. Normality comes from (c) below.

(c) The isomorphism is $(t, u) \mapsto (t^4, t^3u, tu^3, u^4)$, whose inverse can be locally written in rational functions of the coordinates hence is a morphism.

Exercise 3.19 *Automorphisms of* $\mathbf{A}^n$. Let $\varphi : \mathbf{A}^n \to \mathbf{A}^n$ be a morphism of $\mathbf{A}^n$ to $\mathbf{A}^n$ given by n polynomials $f_1, \dots, f_n$ of n variables $x_1, \dots, x_n$. Let $J = \det |\partial f_i / \partial x_j|$ be the *Jacobian* polynomial of φ.

(a) If φ is an isomorphism (in which case we call φ an *automorphism* of $\mathbf{A}^n$) show that J is a nonzero constant polynomial.

(a) The fact that φ is an isomorphism says that $x_1, \dots, x_n$ can be written as polynomials in $f_1, \dots, f_n$. We have $\det |\partial f_i / \partial x_j| \det |\partial x_i / \partial f_j| = 1$ as the matrices are inverses of each other; hence the two determinants are invertible polynomials in the x_i, i.e. a constant.

Exercise 3.20 Let Y be a variety of dimension ≥ 2, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show this would be false for $\dim Y = 1$.
See (III, Ex. 3.5) for generalization.

(a) Any prime ideal $\mathfrak{p}$ of $\mathcal{O}_P$ that is not maximal corresponds to a subvariety $V_\mathfrak{p}$ of dimension at least 1 in Y containing P, hence contains at least one other point Q. f is regular on $V_\mathfrak{p} - P$, hence there is an open set $Q \in U \subseteq V_\mathfrak{p}$ such that $f = g/h$ on U. Now h can't vanish on U hence $f \in (\mathcal{O}_P)_\mathfrak{p}$. As $\mathcal{O}_P$ is integrally closed, we have $\mathcal{O}_P = \bigcap_{\text{height}\, \mathfrak{p}=1} (\mathcal{O}_P)_\mathfrak{p}$ (Matsumura Theorem 11.5), hence $f \in \mathcal{O}_P$ which assures the extendability.

(b) $f : \mathbf{A}^1 - \{0\} \to k, x \mapsto 1/x$ cannot be extended to $x = 0$.

Exercise 3.21 *Group Varieties.* A group variety consists of a variety Y together with a morphism $\mu : Y \times Y \to Y$, such that the set of points with the operation given by μ is a group, and such that the inverse map $y \mapsto y^{-1}$ is also a morphism of $Y \to Y$.

(a) The *additive group* $\mathbf{G}_a$ is given by the variety $\mathbf{A}^1$ and the morphism $\mu : \mathbf{A}^2 \to \mathbf{A}^1$ defined by $\mu(a, b) = a + b$. Show it is a group variety.

(b) The *multiplicative group* $\mathbf{G}_m$ is given by the variety $\mathbf{A}^1 - \{0\}$ and the morphism $\mu(a, b) = ab$. Show it is a group variety.

(c) If G is a group variety, and X is any variety, show that the set $\mathrm{Hom}(X, G)$ has a natural group structure.

(d) For any variety X, show that $\mathrm{Hom}(X, \mathbf{G}_a)$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.

(e) For any variety X, show that $\mathrm{Hom}(X, \mathbf{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.

(a) (b) Trivial verification of the axioms.

(c) It arises from abstract nonsense that the Hom set to the group objects in a category has a natural group structure.

(d) The morphisms to $\mathbf{A}^1$ are exactly the regular functions.

(e) The morphisms to $\mathbf{A}^1 - \{0\}$ are the nonzero regular functions, and their inverse are also regular, hence they are units in $\mathcal{O}(X)$.