Exercise 1.1 (a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

(c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

(a) We have $A(Y) = k[x, y]/(y - x^2) \cong k[x, x^2] \cong k[x]$.

(b) Similarly we have $A(Z) = k[x, y]/(xy - 1) \cong k[x, x^{-1}]$ which is not a polynomial ring in one variable.[1]

(c) We consider a quadratic polynomial $f = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$. We prove that there exists suitable x', y' which are degree one polynomials in x, y, such that $Z(f) = Z(x'y' - 1)$ or $Z(x'^2 - y')$, and that $x', y', 1$ are linearly independent. The second condition guarantees $k[x', y'] = k[x, y]$, and the first condition gives the desired result. We divide into three cases:

(i) $A = C = 0$, then $B \neq 0$. We have $f = 2B(x + E/B)(y + D/B) + (F - 2DE/B)$. We take $x' = 2B(x + E/B)/(F - 2DE/B)$ and $y' = y + D/B$; the denominator of x' is nonzero because f is irreducible.

(ii) $B^2 = AC$, then $Ax^2 + 2Bxy + Cy^2 = x'^2$ for a linear polynomial x'. Hence $f = x'^2 + (2Dx + 2Ey + F)$. We know that x', $y' = 2Dx + 2Ey + F$ and 1 are linearly independent, otherwise f would not be irreducble.

(iii) $B^2 \neq AC$, without loss of generality $A \neq 0$. Then we could write f in the form $f = g_0(x, y)^2 - g_1(y)^2 - g_2$ where g_1 is a nonconstant polynomial in y and g_2 is a nonzero constant. We take $x' = (g_0 + g_1)/\sqrt{g_2}$ and $y' = (g_0 - g_1)/\sqrt{g_2}$.

Exercise 1.2 *The Twisted Cubic Curve.* Let $Y \subseteq \mathbf{A}^3$ be the set $Y = \{(t, t^2, t^3) | t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t$, $y = t^2$, $z = t^3$.

For $f \in k[x, y, z]$ that $f(t, t^2, t^3) = 0$ for arbitrary t, we regard f as an element of $k[x, y][z]$, modulo it with $z - x^3$ and get $f = f_1 \cdot (z - x^3) + f_1'$ where $f_1 \in k[x, y][z]$ and $f_1' \in k[x, y]$. The same process with $f_1' \in k[x][y]$ gives us $f_1' = f_2 \cdot (y - x^2) + f_2'$ where $f_2 \in k[x, y]$ and $f_2' \in k[x]$. Now $f(t, t^2, t^3) = 0\ \forall t$ says $f_2'(t) = 0\ \forall t$, i.e. $f_2' = 0$. Thus $f \in (z - x^3, y - x^2)$.

It is then easy to see $I(Y) = (z - x^3, y - x^2)$. Hence $A(Y) = k[x, y, z]/(z - x^3, y - x^2) \cong k[x, x^2, x^3] = k[x]$ and $\dim Y = \dim A(Y) = 1$.

Exercise 1.3 Let Y be the algebraic set in $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

The irreducible components are $\{(0, 0, t) | t \in k\}$, $\{(0, t, 0) | t \in k\}$, and $\{(t, t^2, 0) | t \in k\}$, corresponding to prime ideals (x, y), (x, z), $(x^2 - y, z)$. These are indeed prime ideals because the quotient of $k[x, y, z]$ on them are all isomorphic to $k[x]$, which is integral.

[1]This is because in $k[\alpha]$ the only invertible elements are the nonzero elements of k; but $x \in k[x, x^{-1}]$ is invertible.

Exercise 1.4 If we identify $\mathbf{A}^2$ with $\mathbf{A}^1 \times \mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbf{A}^1$.

The Zariski topology of $\mathbf{A}^1$ is not Hausdorff, so the diagonal $\{(t,t)|t \in \mathbf{A}^1\} \subseteq \mathbf{A}^1 \times \mathbf{A}^1$ (using the product topology) is not closed. But the diagonal is indeed closed in the Zariski topology of $\mathbf{A}^2$, being the zero set of the polynomial $y - x$.

Exercise 1.5 Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbf{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

A finitely generated k-algebra is naturally of the form $k[x_1, \dots, x_n]/I$ for some n and ideal $I \subseteq k[x_1, \dots, x_n]$. Hence it is an affine coordinate ring of an algebraic set iff I is radical, iff B has no nilpotents.

Exercise 1.6 Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Suppose A is a nonempty open subset of an irreducible topological space X. As X cannot be written as a union of proper closed subsets, any two nonempty open subsets must intersect, making A dense. The irreducibility of A inherits that of X naturally. For the second claim, see Atiyah-Macdonald Exercise 20 i) of Chapter 1.

Exercise 1.7
(a) Show that the following conditions are equivalent for a topological space X: (i) X is noetherian; (ii) every nonempty family of closed subsets has a minimal element; (iii) X satisfies the ascending chain condition for open subsets; (iv) every nonempty family of open subsets has a maximal element.
(b) A noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.
(c) Any subset of a noetherian topological space is noetherian in its induced topology.
(d) A neotherian space which is also Hausdorff must be a finite set with the discrete topology.

(a) Essentially Atiyah-Macdonald Proposition 6.1.

(b) (c) Atiyah-Macdonald Exercise 5 of Chapter 6.

(d) Noetherian spaces are the disjoint union of its irreducible components, but irreducible Hausdorff spaces can only be singletons $\{\cdot\}$.

Exercise 1.8 Let Y be an affine variety of dimension r in $\mathbf{A}^n$. Let H be a hypersurface in $\mathbf{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$. (See (7.1) for a generalization.)

Denote $\mathfrak{a} = I(Y)$ and $I(H) = (f)$ where $f \in A = k[x_1, \dots, x_n]$ is irreducible. The condition that $Y \not\subseteq H$ says $f \notin \mathfrak{a}$.

Notice that $Y \cap H = Z((\mathfrak{a}, f))$, hence $A(Y \cap H) = A/\sqrt{(\mathfrak{a}, f)}$, which is isomorphic to $A/(\mathfrak{a})$ quotienting the image of (f) and then quotienting its nilpotents. An irreducible component V of $Y \cap H$ corresponds to a minimal prime ideal of $A(Y \cap H)$, which in turn correspond to a minimal prime ideal $\mathfrak{p}_V$ of $A/(\mathfrak{a})$ containing (f). As prime ideals all contain the nilpotents, we have

$$\dim V = \dim A(V) = \dim(A/(\mathfrak{a}))/\mathfrak{p}_V = \dim Z(\mathfrak{a}) - 1 = r - 1$$

as height $\mathfrak{p}_V = 1$ from minimality.

Exercise 1.9 Let $\mathfrak{a} \subseteq A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

It suffices to prove that every minimal prime ideal containing $\mathfrak{a}$ has height $\leq r$, which is equivalent to saying $\dim A/\mathfrak{a} \geq \dim A - r$. This is proven by induction on r, as Krull's Hauptidealsatz implies $\dim A/(f) \geq \dim A - 1$ for any non-unit $f \in A$.

Exercise 1.10 (a) If Y is any subset of a topological space X, then $\dim Y \leq \dim X$.

(b) If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.

(c) Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.

(d) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.

(e) Give an example of a noetherian topological space of infinite dimension.

(a) A chain $Z_0 \subset Z_1 \subset \cdots \subset Z_{\dim Y}$ of irreducible closed subsets of Y give rise to a chain of irreducible closed subsets $\overline{Z_0} \subseteq \cdots \subseteq \overline{Z_{\dim Y}}$ of X, cf. exercise 1.6. The equalities can never hold because $\overline{Z_i} \cap Y = Z_i \neq Z_{i+1} = \overline{Z_{i+1}} \cap Y$. By definition we have $\dim Y \leq \dim X$.

(b) For any chain $Z_0 \subset \cdots \subset Z_n$ of irreduible closed subsets of X, we can pick one U_i such that $U_i \cap Z_0 \neq \varnothing$ (otherwise not covered). If $U_i \cap Z_i = U_i \cap Z_{i+1}$, then $(X \backslash Z_i) \cap X_{i+1}$ and $U_i \cap Z_{i+1}$ are nonempty open subsets of Z_{i+1} which have no intersection, contradicting irreducibility; thus the chain restricts to a chain of irreducible closed subsets of U_i, which forces $\dim X \leq \sup \dim U_i$. The other direction is (a).

(c) We pick $X = \{x, y\}$ where the closed sets are $\varnothing, \{y\}, X$. The open dense subset $\{x\}$ has dimension 1 but $\dim X = 2$.

(d) Irreducible closed subsets of Y are also irreducible closed subsets of X, therefore we can add X to the end of any chain of irreducible closed subsets of Y making $\dim Y < \dim X$ unless $Y = X$.

(e) We have $\dim \mathbf{A}^n = n$. Consider the space $\coprod_{i=1}^{\infty} \mathbf{A}^i$, the disjoint union of $\mathbf{A}^i$, with its closed sets defined to be the finite unions of the closed sets of some $\mathbf{A}^i$. The chains of irreducible closed subsets can be of arbitrary length.

Exercise 1.11 Let $Y \subseteq \mathbf{A}^3$ be the curve given parametrically by $x = t^3$, $y = t^4$, $z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection*—cf.(Ex. 2.17).

It is natural to get an identification $A(Y) \cong k[x, y, z, t]/(x - t^3, y - t^4, z - t^5) \cong k[t^3, t^4, t^5, t] \cong k[t]$ which is integral, so $I(Y)$ is prime. Also height $I(Y) = \dim k[x, y, z] - \dim k[t] = 2$.

In the ideal $I(Y)$, there is no degree 1 polynomials. A simple analysis show that $f_1 = xz - y^2$ is the only degree 2 polynomial in $I(Y)$ (up to a constant factor). For degree 3, we have the polynomials $f_2 = x^3 - yz$, $f_3 = x^2 y - z^2$. If $I(Y)$ can be generated by 2 elements, one of them must be a multiple of f_1; but neither of f_2, f_3 can be reduced to degree 2 or below by multiples of f_1, so the other element must be of degree 3. Then notice that no linear combination of f_2, f_3 is a multiple of f_1, contradiction.

Exercise 1.12 Give an example of an irreducible polynomial $f \in \mathbf{R}[x, y]$, whose zero set $Z(f)$ in $\mathbf{A}^2_{\mathbf{R}}$ is not irreducible (cf. 1.4.2).

Take $f = x^2 + y^2 + 1$; $\varnothing$ is not considered irreducible.